MATH463: Complex Variables

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These are my notes for UMD's MATH463: Complex Variables. These notes are taken live in class ("live-TEX"-ed). This course is taught by Professor Antoine Mellet.

Contents

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§1 Introduction

Definition 1.1 (Complex numbers)**.** A **complex number** takes the form $z = x + iy$ where $x, y \in \mathbb{R}$ and the "x" is the real part $= Re(z)$ and the "y" is the imaginary part $= Im(z)$. We denote $\mathcal{C}$ to be the set of complex numbers.

Note 1.2. Note that $i^2 = -1$.

Note 1.3.

$$z_1 = z_2 \in \mathcal{C} \iff \begin{cases} Re(z_1) = Re(z_2) \\ Im(z_1) = Im(z_2) \end{cases}$$

Definition 1.4 (Pure Imaginary Number)**.** $Im(z) = 0 \implies z$ is real and $Re(z) = 0 \implies z$ is a **pure imaginary number**.

Note 1.5. $z = x + iy \in \mathcal{C} \iff (x, y) \in \mathbb{R}^2$ and we draw this as the **complex plane**. We call the x-axis the "real axis" and the y-axis the "imaginary axis."

Example 1.6

Denote the set S to be

$$S = \{z \in \mathcal{C} \mid Re(z) = 2\} = \{z = 2 + iy \mid y \in \mathbb{R}\}$$

which graphically is a vertical line at $x = 2$.

§1.1 Algebra

Definition 1.7 (Sum, Product of Complex Numbers)**.** Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathcal{C}$ then we can define the sum

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

$$\begin{aligned} z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) &= x_1x_2 + ix_1y_2 + ix_2y_1 - y_1y_2 \\ \implies z_1z_2 &= (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \\ z_1^2 &= z_1z_1 = x_1^2 - y_1^2 + 2ix_1y_1 \end{aligned}$$

We can continue to get more powers.

Definition 1.8 (Inverse)**.** Given $z = x + iy$, we want to find $w = u + iv$ such that $zw = 1$, so $w = \frac{1}{z}$.

$$zw = xu - yv + i(xv + yu) = 1 + i0 \implies \begin{cases} xu - yv = 1 \\ yu + xv = 0 \end{cases}$$

If you know x and y, then this is just a linear system with 2 unknowns. We can rewrite this as

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore, this system has a unique soltion if and only if the determinant of this matrix $= x^2 + y^2 \neq 0 \implies x \neq 0 \ \&y \ \neq 0$.
Thus, z has a unique inverse if and only if $z \neq 0 + i0$ and

$$u = \frac{x}{x + 2 + y^2} v = -\frac{y}{x^2 + y^2}$$

So,

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Definition 1.9 (Fraction)**.** Following inverses, fractions naturally follow. For example, $\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2}$

Definition 1.10 (Projection)**.** $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$

§1.2 Vectors and Modulus

Definition 1.11 (Vector Representations)**.** For example, if $z_1 = 3 - 2i$ then in vector representation this appears as the vector $\begin{bmatrix} 3 & -2 \end{bmatrix}' \in \mathbb{R}^2$

Definition 1.12 (Modulus)**.** The modulus of $z = x + iy$ is the norm of the vector $\begin{bmatrix} x & y \end{bmatrix}' \implies |z| = \sqrt{x^2 + y^2} \in \mathbb{R}, |z| \geq 0$

Note 1.13. $|z| = 0$ if and only if $z = 0 + i0$

Note 1.14. If $y = 0 \implies z = x \implies |x| = \sqrt{x^2} = |x|$. Thus, when $y = 0$, the modulus of z is the absolute value of x

Example 1.15
$|3 - 2i| = \sqrt{3^2 + 2^2} = \sqrt{13}$

Example 1.16
$|i| = \sqrt{0^2 + i^2} = 1$

Remark 1.17. Observe that

$$|z| = \sqrt{x^2 + y^2} \geq \sqrt{x^2} = |Re(z)|$$
$$|z| = \sqrt{x^2 + y^2} \geq \sqrt{y^2} = |Im(z)|$$

Definition 1.18 (Distance)**.** The distance between $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |(x_2 - x_1) + i(y_2 - y_1)| = |z_2 - z_1|$$

Definition 1.19 (Circles)**.** A cricle centered at $z_0 = x_0 + iy_0$ with radius R can be expressed as the set

$$C = \{z \in \mathcal{C} \mid |z - z_0| = R\} = \{z \in \mathcal{C} \mid (x - x_0)^2 + (y - y_0)^2 = R^2\}$$

and so we recover the classical definition of a circle.

Example 1.20

Define the set $S_1 = \{z \in \mathcal{C} \mid |z| = 2\}$ which is a circle of radius 2 centered at $z_0 = 0$. Now define $S_2 = \{z \in \mathcal{C} \mid |z - (-2+3i)| = 2\}$ which is a circle of radius 2 centered at $-2+3i$.

Definition 1.21 (Disk)**.** Consider $D = \{z \in \mathcal{C} \mid |z-1| \leq 3\}$ which is a disk centered at $z_0 = 1, R = 3$.

Theorem 1.22 (Triangle Inequality)

Consider two complex numbers z_1, z_2 then

$$|z_1 + z_2| \leq |z_1| + |z_2| \; \forall \; z_1, z_2 \in \mathcal{C}$$

Proof. Proof ommitted. □

Corollary 1.23 (Consequences of Triangle Inequality)

- $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$
- $|z_1 + z_2| \geq ||z_1| - |z_2||$

Proof. Note that $z_1 = z_1 + z_2 + (-z_2)$ and so

$$|z_1| = |(z_1 + z_2) + (-z_2) \leq |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|$$

a and so we have showed that $|z_1| - |z_2| \leq |z_1 + z_2|$. The absolute value is necessary in case $|z_2| > |z_1|$, in which case we can perform a similar proof by switching the order of z_1 and z_2. □

Example 1.24

Consider the unit circle $\{z \in \mathcal{C} \mid |z| = 1\}$ and the point $z_0 = 2$. The smallest this distance can be is 1, and the largest is 3.

$$|z - z_0| \leq |z| + |-z_0| = 1 + 2 = 3 \text{ by Triangle Inequality}$$

$$|z - z_0| \geq ||z| - |z_0|| = |1 - 2| = 1 \text{ by Corollary}$$

§1.3 Conjugate of a Complex Number

Definition 1.25 (Conjugate of a Complex Number)**.** If $z = x + iy$ then the **conjugate of z** is denoted $\bar{z} = x - iy$.

Note 1.26 (Properties of Conjugates)**.**

- $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
- $\overline{z_1 z_2} = \bar{z}_1 * \bar{z}_2$
- $(\frac{\bar{1}}{z_1}) = \frac{1}{\bar{z}_1}$
- $\overline{\overline{z}} = z$

Remark 1.27.

- $z + \overline{z} = x + iy + x - iy = 2x$
- $z - \overline{z} = 2iy$
- $Im(z) = \frac{1}{2i}(z - \bar{z})$
- $z * \bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2 \geq 0$