MATH463: Complex Variables

James Zhang[*]

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These are my notes for UMD's MATH463: Complex Variables. These notes are taken live in class ("live-TEX"-ed). This course is taught by Professor Antoine Mellet.

Contents

[*]Email: jzhang72@terpmail.umd.edu

§1 Preliminaries

§1.1 Definition of Complex Numbers

Definition 1.1 (Complex numbers)**.** A **complex number** takes the form $z = x + iy$ where $x, y \in \mathbb{R}$ and the "x" is the real part $= Re(z)$ and the "y" is the imaginary part $= Im(z)$. We denote $\mathcal{C}$ to be the set of complex numbers.

Note 1.2. Note that $i^2 = -1$.

Note 1.3.

$$z_1 = z_2 \in \mathcal{C} \iff \begin{cases} Re(z_1) = Re(z_2) \\ Im(z_1) = Im(z_2) \end{cases}$$

Definition 1.4 (Pure Imaginary Number)**.** $Im(z) = 0 \implies z$ is real and $Re(z) = 0 \implies z$ is a **pure imaginary number**.

Note 1.5. $z = x + iy \in \mathcal{C} \iff (x, y) \in \mathbb{R}^2$ and we draw this as the **complex plane**. We call the x-axis the "real axis" and the y-axis the "imaginary axis."

Example 1.6

Denote the set S to be

$$S = \{z \in \mathcal{C} \mid Re(z) = 2\} = \{z = 2 + iy \mid y \in \mathbb{R}\}$$

which graphically is a vertical line at $x = 2$.

§1.2 Algebra

Definition 1.7 (Sum, Product of Complex Numbers)**.** Let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathcal{C}$ then we can define the sum

$$z_1 \pm z_2 = (x_1 \pm x_2) + i(y_1 \pm y_2)$$

$$\begin{aligned} z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + ix_1y_2 + ix_2y_1 - y_1y_2 \\ \implies z_1z_2 = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \\ z_1^2 = z_1z_1 = x_1^2 - y_1^2 + 2ix_1y_1 \end{aligned}$$

We can continue to get more powers.

Definition 1.8 (Inverse)**.** Given $z = x + iy$, we want to find $w = u + iv$ such that $zw = 1$, so $w = \frac{1}{z}$.

$$zw = xu - yv + i(xv + yu) = 1 + i0 \implies \begin{cases} xu - yv = 1 \\ yu + xv = 0 \end{cases}$$

If you know x and y, then this is just a linear system with 2 unknowns. We can rewrite this as

$$\begin{bmatrix} x & -y \\ y & x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Therefore, this system has a unique soltion if and only if the determinant of this matrix $= x^2 + y^2 \neq 0 \implies x \neq 0 \,\&y \ \neq 0$.
Thus, z has a unique inverse if and only if $z \neq 0 + i0$ and

$$u = \frac{x}{x + 2 + y^2} v = -\frac{y}{x^2 + y^2}$$

So,

$$\frac{1}{z} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

Definition 1.9 (Fraction)**.** Following inverses, fractions naturally follow. For example, $\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2}$

Definition 1.10 (Projection)**.** $(z_1 z_2)^{-1} = z_1^{-1} z_2^{-1}$

§1.3 Vectors and Modulus

Definition 1.11 (Vector Representations)**.** For example, if $z_1 = 3 - 2i$ then in vector representation this appears as the vector $\begin{bmatrix} 3 & -2 \end{bmatrix}' \in \mathbb{R}^2$

Definition 1.12 (Modulus)**.** The modulus of $z = x + iy$ is the norm of the vector $\begin{bmatrix} x & y \end{bmatrix}' \implies |z| = \sqrt{x^2 + y^2} \in \mathbb{R}, |z| \geq 0$

Note 1.13. $|z| = 0$ if and only if $z = 0 + i0$

Note 1.14. If $y = 0 \implies z = x \implies |x| = \sqrt{x^2} = |x|$. Thus, when $y = 0$, the modulus of z is the absolute value of x

Example 1.15
$|3 - 2i| = \sqrt{3^2 + 2^2} = \sqrt{13}$

Example 1.16
$|i| = \sqrt{0^2 + i^2} = 1$

Remark 1.17. Observe that

$$|z| = \sqrt{x^2 + y^2} \geq \sqrt{x^2} = |Re(z)|$$

$$|z| = \sqrt{x^2 + y^2} \geq \sqrt{y^2} = |Im(z)|$$

Definition 1.18 (Distance)**.** The distance between $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = |(x_2 - x_1) + i(y_2 - y_1)| = |z_2 - z_1|$$

Definition 1.19 (Circles)**.** A cricle centered at $z_0 = x_0 + iy_0$ with radius R can be expressed as the set

$$C = \{z \in \mathcal{C} \mid |z - z_0| = R\} = \{z \in \mathcal{C} \mid (x - x_0)^2 + (y - y_0)^2 = R^2\}$$

and so we recover the classical definition of a circle.

Example 1.20

Define the set $S_1 = \{z \in \mathcal{C} \mid |z| = 2\}$ which is a circle of radius 2 centered at $z_0 = 0$. Now define $S_2 = \{z \in \mathcal{C} \mid |z - (-2 + 3i)| = 2\}$ which is a circle of radius 2 centered at $-2 + 3i$.

Definition 1.21 (Disk)**.** Consider $D = \{z \in \mathcal{C} \mid |z - 1| \leq 3\}$ which is a disk centered at $z_0 = 1, R = 3$.

Theorem 1.22 (Triangle Inequality)

Consider two complex numbers z_1, z_2 then

$$|z_1 + z_2| \leq |z_1| + |z_2| \ \forall \ z_1, z_2 \in \mathcal{C}$$

Proof. Proof ommitted. □

Corollary 1.23 (Consequences of Triangle Inequality)

- $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$
- $|z_1 + z_2| \geq ||z_1| - |z_2||$

Proof. Note that $z_1 = z_1 + z_2 + (-z_2)$ and so

$$|z_1| = |(z_1 + z_2) + (-z_2) \leq |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|$$

a and so we have showed that $|z_1| - |z_2| \leq |z_1 + z_2|$. The absolute value is necessary in case $|z_2| > |z_1|$, in which case we can perform a similar proof by switching the order of z_1 and z_2. □

Example 1.24

Consider the unit circle $\{z \in \mathcal{C} \mid |z| = 1\}$ and the point $z_0 = 2$. The smallest this distance can be is 1, and the largest is 3.

$$|z - z_0| \leq |z| + |-z_0| = 1 + 2 = 3 \text{ by Triangle Inequality}$$

$$|z - z_0| \geq ||z| - |z_0|| = |1 - 2| = 1 \text{ by Corollary}$$

§1.4 Conjugate of a Complex Number

Definition 1.25 (Conjugate of a Complex Number)**.** If $z = x+iy$ then the **conjugate of z** is denoted $\bar{z} = x - iy$.

Note 1.26 (Properties of Conjugates)**.**

- $\overline{z_1 + z_2} = \bar{z_1} + \bar{z_2}$
- $\overline{z_1 z_2} = \bar{z_1} * \bar{z_2}$
- $\overline{(\frac{1}{z_1})} = \frac{1}{\bar{z_1}}$
- $\overline{\overline{z}} = z$

Remark 1.27.

- $z + \overline{z} = x + iy + x - iy = 2x$
- $z - \overline{z} = 2iy$
- $Im(z) = \frac{1}{2i}(z - \bar{z})$
- $z * \bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2 \geq 0$

Note 1.28 (Applications of the Product of a Complex Number and its Conjugate)**.**

1. $\frac{z_1}{z_2} = \frac{z_1\overline{z_2}}{z_2\overline{z_2}} = \frac{z_1\overline{z_2}}{|z_2|^2}$
2. $|z_1 z_2|^2 = z_1 z_2(\overline{z_1 z_2}) = z_1 z_2 \overline{z_1}\,\overline{z_2} = (z_1\overline{z_1})(z_2\overline{z_2}) = |z_1|^2|z_2|^2 \implies |z_1 z_2| = |z_1||z_2|$
3. $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$
4. $|z^2| = |z|^2$ and $|z^n| = |z|^n \; \forall \; n = \pm 1, \pm 2, \cdots$

§1.5 Exponential Form of Complex Numbers

Definition 1.29. Complex numbers $z \neq 0$ also have polar coordinates (r, θ) and the results $x = r\cos\theta \implies \cos\theta = \frac{x}{r}$, $y = r\sin\theta \implies \sin\theta = \frac{y}{r}$, $r = \sqrt{x^2 + y^2}$ still holds. Note that $r\mathbb{R}^+$ and $\theta \in \mathbb{R}$ and is the angle in radians.

Note 1.30. Note that $r = \sqrt{x^2 + y^2} = |z|$ is the modulus of z.

Definition 1.31 (Argument of a Complex Number)**.** θ is called the **argument of z** and is denoted as $\arg(z)$, which is a multivalued function because θ is not uniquely defined. You can think of it as the z here being a set. Note that if θ is an argument, then $\theta + 2k\pi, k = \pm 1, \pm 2, \cdots$ is also still an argument.

Definition 1.32 (Principal Argument)**.** $\text{Arg}(z)$ is the **principal argument**, and it is the only argument of z in the interval $[-\pi, \pi]$.

Example 1.33

$\text{Arg}(1) = 0$, $\text{Arg}(-1) = \pi$, $\text{Arg}(i) = \frac{\pi}{2}$, $\text{Arg}(-5i) = -\frac{\pi}{2}$

Example 1.34

$\text{Arg}(\sqrt{3} + i) = \frac{\pi}{6}$

Solution. $z = \sqrt{3} + i$. Note that $r = |z| = \sqrt{\sqrt{3}^2 + 1^2} = 2$. Since $\cos\theta = \frac{x}{r} \implies \cos\theta = \frac{\sqrt{3}}{2} = \frac{\pi}{6}$. □

Definition 1.35 (Euler's Formula)**.**

$$e^{i\theta} = \cos\theta + i\sin\theta$$

If z has modulus r, argument θ then

$$z = Re(z) + iIm(z) = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Example 1.36

Some cases of the above formula

- $\sqrt{3} + i = 2e^{i\pi/6}$
- $1 + i = \sqrt{e^{i\pi/4}}$
- $3 = 3e^{i0}$
- $-5 = 5e^{i\pi} \implies -1 = e^{i\pi}$
- $i = e^{i\pi/2}$

Example 1.37

Given the form of a complex number in this form $z = re^{i\theta}$ we can rewrite it using Euler's Formula.

$$z = e^{i\pi/3} = 3\cos\frac{\pi}{3} + i3\sin\frac{\pi}{3} = \frac{3}{2} + i\frac{3\cdot\sqrt{3}}{2}$$

Note 1.38. We can define circles using this form $\{re^{i\theta} \mid \theta \in \mathbb{R}\} = \{r^{i\theta} \mid \theta \in [0, 2\pi]\}$ and this is just a circle centered at the origin with radius r.

Note 1.39. For circles not centered at the origin, say z_0 with radius R, recall that conventionally

$$\begin{aligned} |z - z_0| = R \implies z - z_0 = Re^{i\theta} &\implies z = z_0 + Re^{i\theta} \\ c &= \{z_0 + Re^{i\theta} \mid \theta \in [-\pi, \pi]\} \end{aligned}$$

Note 1.40 (Products in Exponential Form)**.**

$$\begin{aligned} e^{i\theta_i} \cdot e^{i\theta_2} &= (cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1) \\ &= \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = e^{i(\theta_1+\theta_2)} \end{aligned}$$

Note 1.41. If $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{\theta_2}$ not zero, then

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

This shows us the following

$$|z_1 z_2| = |z_1||z_2| \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

Note that this is lowercase arg, and we present the following counterexample. Let $z_1 = -1, z_2 = i, z_1 z_2 = -i$

$$\operatorname{Arg}(z_1) = \pi \qquad \operatorname{Arg}(z_2) = \frac{\pi}{2} \qquad \operatorname{Arg}(z_1 z_2) = -\frac{\pi}{2}$$

Example 1.42

$(1+i)^4 = (\sqrt{2}e^{i\pi/4}) = (\sqrt{2})^4 e^{i4\pi/4} = 4e^{i\pi} = -4$

Theorem 1.43 (De Moivre's Formula)

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

Definition 1.44 (Inverse of z in Exponential Form)**.** Let $z = re^{i\theta}$ and so $z^{-1} = r^{-1}e^{-i\theta}$. We can verify this because

$$(re^{i\theta})(r^{-1}e^{-i\theta}) = (r \cdot r^{-1})e^{i(\theta-\theta)} = 1$$

Example 1.45

$(\sqrt{3}+i)^{-1} = \frac{1}{\sqrt{3}+i} = \frac{1}{2e^{i\pi/6}} = \frac{1}{2}e^{-i\pi/6}$

Definition 1.46 (Fractions)**.** $\frac{z_1}{z_2} = z_1 \cdot \frac{1}{z_2} \implies \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$ This tells us that

$$|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|} \qquad \arg(\frac{z_1}{z_2}) = \arg(z_1) - \arg(z_2)$$

Example 1.47

Find the $\text{Arg}(\frac{-5}{\sqrt{3}+i}) = \frac{5\pi}{6}$

Solution. Recall that

$$\arg(\frac{-5}{\sqrt{3}+i}) = \arg(-5) - \arg(\sqrt{3}+i) = \pi - \frac{\pi}{6} + 2k\pi = \frac{5\pi}{6} + 2k\pi$$

and so the Principal Argument is $\frac{5\pi}{6}$. $\square$

Note 1.48. Note that for all $n \in \mathbb{N}$,

$$z^n = r^n e^{in\theta} \qquad\qquad z^{-n} = r^{-n}e^{-in\theta}$$

and so therefore

$$|z^n| = |z|^n \qquad\qquad \arg(z^n) = n\arg(z) + 2k\pi \text{ for } n = \pm 1, \pm 2, \cdots$$

Definition 1.49 (Conjugate in Exponential Form)**.** Recall that $|z| = |\overline{z}|$ and $\arg(\overline{z}) = -\arg(\overline{z})$ and so therefore if $z = re^{i\theta}$

$$\overline{z} = re^{-i\theta}$$

and $z^{-1} = r^{-1}e^{-i\theta}$