MATH403: Homework 1

James Zhang[*]

September 9, 2024

14. Show that $5n + 3$ and $7n + 4$ are relatively prime for all n.

Proof. Two numbers are relatively prime if the greatest common divisor between then is 1. By Bezout's Lemma, we want to find two integers x, y such that $(5n + 3)x + (7n + 4)y = 1$. Note that we can rewrite this as

$$(5x + 7y)n + (3x + 4y) = 1$$

which yields the following system of equations

$$\begin{cases} 5x + 7y = 0 \\ 3x + 4y = 1 \end{cases}$$

which yields $x = 7, y = -5$. Therefore, since $(7n + 4) * -5 + (5n + 3) * 7 = -1$, we conclude that $GCD(5n + 3, 7n + 4) = 1$ and so $5n + 3$ and $7n + 4$ are relatively prime. $\square$

[*]Email: jzhang72@terpmail.umd.edu

28. Prove that $2^n 3^{2n} - 1$ is always divisible by 17.

Proof. We will prove the following using strong induction.
Base case: $n = 0 \implies 2^n 3^{2n} - 1 = 1 - 1 = 0$ is divisible by 17. If the set of naturals doesn't include 0, $n = 1 = 2^1 3^2 - 1 = 18 - 1 = 17$, which is trivially satisfied by 17.

Inductive hypothesis: assume $2^n 3^{2n} - 1$ is divisible by 17 for all $n \leq k$.
Therefore, for all $n \leq k$, we can write express $2^n 3^{2n} = 17x + 1, x \in \mathbb{Z}$

Inductive case: we seek to show that $2^{k+1} 3^{2(k+1)} - 1$ is divisible by 17. Equivalently, we want to show that there exists a $x \in \mathbb{Z}$ such that $2^{k+1} 3^{2(k+1)} - 1 = 17x$. Note that

$$2^{k+1} 3^{2(k+1)} - 1 = (2 \cdot 3^2)(2^k 3^{2k}) - 1$$

Further note that by our inductive hypothesis, $\exists\ y \in \mathbb{Z}$ such that $2^k 3^{2k} + 1 = 17y$. By direct substitution,

$$18(17y + 1) - 1 = 306y + 18 - 1 = 306y + 17 = 17(18y + 1)$$

Let $x = 18y + 1 \in \mathbb{Z}$ and therefore we have expressed $2^{k+1} 3^{2(k+1)} = 17x$ and so $2^n 3^{2n} - 1$ is always divisible by 17 for all n. $\square$

36. Suppose that there is a statement involving a positive integer parameter n and you have an argument that shows that whenever the statement is true for a particular n it is also true for $n + 2$. What remains to be done to prove the statement is true for every positive integer? Describe a situation in which this strategy would be applicable.

Solution. To prove that the statement is true for every positive integer, you have to explicitly show that it is true for $n = 1$ and $n = 2$ because for these two examples, you cannot use the above argument here because $n - 2$ is -1 and 0, respectively. These numbers are not positive integers, and so the above argument cannot be applied. This idea naturally leads to the mathematical idea of induction, and these base cases provide the foundation for the induction process.

This strategy of inductively proving an statement is particularly useful when the statement involves properties that are either periodic or depend on even or odd integers. It can also be useful for proofs related to sequences defined using recurrence relations. □

48. Let S be the set of real numbers. If $a, b \in S$, define $a \sim b$ if $a - b$ is an integer. Show that $\sim$ is an equivalence relation on S. Describe the equivalence classes of S.

Proof. Let $R \subset \mathbb{R} \times \mathbb{R}$. First, let us show that this relation $\sim$ is an equivalence relation. For any $a \in \mathbb{R}$, $a - a = 0 \in \mathbb{Z}$, so $a \sim a$ for all $a \in \mathbb{R}$ and reflexivity is satisfied. For symmetry, note that if $a \sim b$, then $a - b \in \mathbb{Z}$ but also $b - a = -(a - b) \in \mathbb{Z}$ and so $b \sim a$, so symmetry is satisfied. Finally, suppose $a \sim b$ such that $a - b = m \in \mathbb{Z}$ and $b \sim c = n \in \mathbb{Z}$. Since $(a - c) = (a - b) + (b - c) = m + n \in ZZ$ then $a - c \in \mathbb{Z}$ and so $a \sim c$. In this original relation, the equivalence class for a number $a \in \mathbb{R}$ is

$$[a] = \{x \in \mathbb{R} \mid x \sim a\} = \{x \in \mathbb{R} \mid x - a \in \mathbb{Z}\}$$

Here are some example equivalence classes.
$[0] = [\dots, -1, 0, 1, \dots]$
$[\pi] = [\dots, -\pi, 0, \pi, \dots]$
$[1.2] = [\dots, -2.2, -1.2, -0.2, 0.8, 1.8, 2.8, \dots]$. $\square$

52. Prove that 3, 5, and 7 are the only three consecutive odd integers that are prime.

Proof. Assume on the contrary there exists an odd prime number $n > 3$ such that $n, n+2, n+4$, the 3 consecutive odd numbers starting from n, are all prime. Consider the number n. $n \bmod 3 = 1$ or 2. It cannot be 0 because this would contradict the fact that the number is prime, if it is immediately divisible by 3.

Case 1: Suppose $n \bmod 3 = 1$. Then

$$(n+2) \bmod 3 = (n \bmod 3 + 2 \bmod 3) \bmod 3 = (1+2) \bmod 3 = 0$$

This is a contradiction because $n+2$ is assumed to be prime, but we have shown that $n+2$ is divisible by 3 and therefore not prime.

Case 2: Now suppose $n \bmod 3 = 2$. Then

$$(n+2) \bmod 3 = (n \bmod 3 + 2 \bmod 3) \bmod 3 = (2+2) \bmod 3 = 1$$

Thus, $(n+2) \bmod 3 = 1$ and so 3 does not divide $n+2$. However, now consider $n+4$

$$(n+4 \bmod 3) = (n \bmod 3 + 4 \bmod 3) \bmod 3 = (2+1) \bmod 3 = 0$$

Therefore, 3 divides $n+4$ and so by definition, $n+4$ is not prime, and this is also a contradiction.

Therefore, there does not exist 3 consecutive odd integers starting from $n > 3$ that are all prime, and the only example of 3 consecutive odd integers that are prime is indeed $3, 5, 7$. $\square$