CMSC 351 Fall 2023 Homework 4

Due Wednesday Oct 4, 2023 by 11:59pm on Gradescope.

Directions:

- Homework must be done on printouts of these sheets and then scanned properly, or via latex, or by downloading, writing on the PDF, and uploading.
- Do not use your own blank paper!
- The reason for this is that gradescope will be following this template to locate the answers to the problems so if your answers are organized differently they will not be recognized.
- Tagging is automatic, do not manually tag.

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1. Consider the run of INSERTION SORT below (with missing data). Considering that: We have the final sorted list (after four passes) and we know several things: [10 pts]

   - One pass had exactly 4 shifts.
   - One pass had exactly 3 shifts.
   - One pass had exactly 1 shift.
   - One pass had exactly 0 shift.
   - 78 was never at index 1.

   Fill in all the blanks:

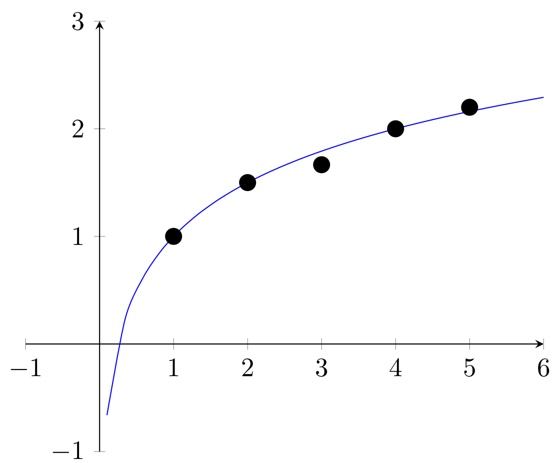
Index	0	1	2	3	4
Original List	50	45	78	34	12
After 1st Pass	45	50	78	34	12
After 2nd Pass	45	50	78	34	12
After 3rd Pass	34	45	50	78	12
After 4th Pass	12	34	45	50	78

2. This is some experimental verification for Binary Search being average case $\Theta(\lg n)$.

   (a) For each of $n = 1, 2, 3, 4, 5$ consider the list $[1, ..., n]$. For each list determine the expected number of iterations of the while— loop when searching for a uniformly randomly chosen integer from amongst $\{1, ..., n\}$. which will of course be found. [10 pts]

n	Average # Iterations
1	1
2	$\frac{3}{2}$
3	$\frac{5}{3}$
4	2
5	$\frac{11}{5}$

   (b) Plot the average number v. n on the axes below. [5 pts]



   (c) Experimentally find constants A and B so that the graph of $y = A + B \lg x$ approximates your points. Then sketch this function on top of your graph above. [5 pts]

A	1
B	0.5

3. Given the recurrence relation: [10 pts]

$$T(n) = 3T(\lfloor n/2 \rfloor) + 2n + 1$$
$$T(0) = 0$$

Calculate $T(n)$ for $n = 1, 2, 3, 4, 5$.

n	1	2	3	4	5
$T(n)$	3	14	16	51	53

4. Consider the following recurrence relation with base case $T(1) = 5$: [10 pts]

$$T(n) = 8T(n/2) + n^3$$

Given the relation, do an expansion (i.e. drill down as many times as you think necessary) to find a pattern and then a non-recursive expression for $T(n)$. You may assume $n = 2^k$ for some $k \in \mathbb{Z}^+$. **Solution:**

Observe that

$$T(n) = 8T(\frac{n}{2}) + n^3$$

Now note that $T(\frac{n}{2}) = 8T(\frac{n}{2^2}) + (\frac{n}{2})^3$ and so we have that

$$T(n) = 8(8(T(\frac{n}{2^2}) + (\frac{n}{2})^3) + n^3 = 8^2T(\frac{n}{2^2}) + 2n^3$$

We can dig down once more because $T(\frac{n}{4}) = 8T(\frac{n}{2^3}) + (\frac{n}{2^2})^3$. By substitution,

$$T(n) = 8(8(8T(\frac{n}{2^3}) + (\frac{n}{2^2})^3) + (\frac{n}{2})^3) + n^3 = 8^3T(\frac{n}{2^3}) + 3n^3$$

In general,

$$T(n) = 8^kT(\frac{n}{2^k}) + kn^3$$

so and this ends when $\frac{n}{2^k} = 1$ so $k = \lg n$ so

$$T(n) = 8^{\lg n}T(1) + \lg n(n^3)$$
$$T(n) = (2^3)^{\lg n}(5) + n^3 \lg n$$
$$T(n) = (2^{\lg n})^3(5) + n^3 \lg n$$
$$T(n) = 5n^3 + n^3 \lg n$$

5. Consider the following recurrence relation with base case $T(1) = 1$: [20 pts]

$$T(n) = aT(n/a) + cn \text{ where } a > 1 \text{ and } c > 0 \text{ are constants}$$

Drill down and express $T(n)$ as a function of $T(n/a^3)$ then as a function of $T(n/a^k)$ where k is a positive integer. Then, derive a non- recursive expression for $T(n)$. Finally, conclude what the Θ of $T(n)$ is. In all of these cases, simplify as much as possible.

$T(n)$ as a function of $T(n/a^3) =$	$a^3T(\frac{n}{a^3}) + 3cn$
$T(n)$ as a function of $T(n/a^k) =$	$a^kT(\frac{n}{a^k}) + kcn$
$T(n) =$ (non-recursive)	$cn \log_a n + n$
Θ of $T(n) =$	$n \log_a n$

Solution:

(a) Let us begin by drilling down and solving for $T(\frac{n}{a})$.

$$T(\frac{n}{a}) = aT(\frac{a}{n^2}) + \frac{cn}{a}$$

By substitution back into the definition of T, we get

$$T(n) = a(aT(\frac{a}{n^2}) + \frac{cn}{a}) + cn$$

Repeating this process and solving for $T(\frac{a}{n^2})$, we get

$$T(\frac{n}{a^2}) = aT(\frac{a}{n^3}) + \frac{cn}{a^2}$$

By substitution,

$$T(n) = a(a(aT(\frac{a}{n^3}) + \frac{cn}{a^2})) + \frac{cn}{a}) + cn = a^3T(\frac{n}{a^3}) + 3cn$$

(b) It is easy to generalize the above solution for a to obtain $T(n) = a^kT(\frac{n}{a^k}) + kcn$

(c) Observe that this recursion stops when $\frac{n}{a^k} = 1$ because $T(1) = 1$. $\frac{n}{a^k} = 1 \implies n = a^k \implies k = \log_a n$.

$$T(n) = nT(1) + \log_a ncn = cn \log_a n + n$$

(d) We hypothesize that $T(n) = \theta(n \log_a n)$ and we will use a limit theorem to show this.

$$\lim_{n\to\infty} \frac{cn \log_a n + n}{n \log_a n} = \lim_{n\to\infty} \frac{cn \log_a n}{n \log_a n} + \frac{n}{n \log_a n} = c \neq 0, \infty$$

Therefore, $T(n) = \theta(n \log_a n)$

6. Consider the following recurrence relation with base cases $T(1) = T(0) = 1$: [15 pts]

$$T(n) = T(n-1) + T(n-2) \text{ for } n \geq 2$$

Prove that $T(n)$ is $\Omega(2^{n/2})$. Without loss of generality, you can assume that $n = 2k$ is even.
Solution:

Recall that the definition of Ω is

$$T(n) = \Omega(2^{n/2}) \implies \exists\, c, n_0\ s.t.\ T(n) \geq c2^{n/2}\ \forall\ n \geq n_0$$

Observe that $T(n)$ is a monotonically increasing sequence. Since the i'th term is computed using the (i-1)th term where all terms are positive and greater than or equal to 1. Now let us proceed by forming an inequality chain. Let $n = 2k$, and given the hint, we know $n = 2k$ is even.

$$T(n) = T(2k) = T(2k-1) + T(2k-2)$$

Observe that $T(2k-1) > T(2k-2)$. Let us continue.

$$T(n) > 2T(2k-2) = 2(T(2k-3) + T(2k-4)) > 4T(2k-4) > \cdots > 2^k T(0) = 2^k = 2^{n/2}$$

Thus, we have that

$$T(n) > 2^{n/2}$$

To complete the proof, let us choose $c = 1$ and $n_0 = 2$, such that we obtain

$$T(n) \geq 2^{n/2}\ \forall\ n >= 2 \implies T(n) = \Omega(2^{n/2})$$

as desired.

7. Consider the recurrence relation:

$$T(n) = 2T(n/3) + \sqrt{n} + 1 \text{ with } T(1) = 4$$

(a) Draw the complete tree for $n = 81$ and fill in the values. **Solution:** [10 pts]

$\sqrt{81} + 1$

$\sqrt{27} + 1$ $\quad$ $\sqrt{27} + 1$

$\sqrt{9} + 1$ $\quad$ $\sqrt{9} + 1$ $\quad$ $\sqrt{9} + 1$ $\quad$ $\sqrt{9} + 1$

$\sqrt{3} + 1$ $\quad$ $\sqrt{3} + 1$ $\quad$ $\sqrt{3} + 1$ $\quad$ $\sqrt{3} + 1$ $\quad$ $\sqrt{3} + 1$ $\quad$ $\sqrt{3} + 1$ $\quad$ $\sqrt{3} + 1$ $\quad$ $\sqrt{3} + 1$

1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1

(b) Use the tree in (a) (and parts of it) to find $T(n)$ for $n = 1, 3, 9, 27, 81$. Simpify these. [5 pts]

n	1	3	9	27	81
$T(n)$	4	$\sqrt{3}+9$	$2\sqrt{3}+22$	$7\sqrt{3}+45$	$14\sqrt{3}+100$

Put scratch work below. Scratch work is not graded but note that you should know how to explain your answer because you may be expected to do this on an exam.