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STAT420 Homework 3

5.1-3) Note that the definition of convergence on probability is

$$\lim_{n\to\infty} P(|X_n - X| \geq \varepsilon) = 0 \quad \text{e.c.} \quad \lim_{n\to\infty} (P|X_n - X| < \varepsilon) = 1$$

and Chebyshev's Inequality is for $\varepsilon > 0$

$$P(|X - E(X)| \geq \varepsilon) \leq \frac{Var(X)}{\varepsilon^2}$$

Let $W_n = X$ and substitute so we get

$$P(|W_n - E(W_n)| \geq \varepsilon) \leq \frac{Var(W_n)}{\varepsilon^2}$$

$$P(|W_n - \mu| \geq \varepsilon) \leq \frac{b}{n^p \varepsilon^2}$$

Note that $\lim_{n\to\infty} P(|W_n - \mu| \geq \varepsilon) \leq \frac{b}{n^p \varepsilon^2} = 0$. Thus, we have shown that $W_n \xrightarrow{P} \mu$.

$\square$

5.1-7) $f(x) = \begin{cases} e^{-(x-\theta)} & x > \theta, -\infty < \theta < \infty \\ 0 & \text{elsewhere} \end{cases}$ shifted potential $Y_n = \min\{X_1, \dots, X_n\}$

Prove $Y_n \xrightarrow{P} \theta$ by first obtaining the cdf of Y_n

Note that for $x < \theta$, $F_{X_i}(x) = 0$. For $x > \theta$

$$F_{Y_n}(x) = P(Y_n \leq x) = P(\min\{X_1, \dots, X_n\} \leq x)$$

$$= 1 - P(\min\{X_1, \dots, X_n\} > x)$$

$$= 1 - P(X_1 > x) \cdot \dots \cdot P(X_n > x)$$

Note that $P(X_1 > x) = 1 - F_{X_1}(x)$. Let us solve for $F_{X_1}(x)$.

$$F_{X_1}(x) = \int_\theta^x e^{-(t-\theta)}\,dt = \int_\theta^x e^\theta e^{-t}\,dt = e^\theta(-e^{-t})\Big|_\theta^x$$

$$= 1 - e^{\theta - x}$$

By substitution,

$$F_{Y_n}(x) = 1 - (1 - e^{\theta - x})^n$$

Now let us prove $Y_n \xrightarrow{P} \theta$. We seek to show that

$$\lim_{n\to\infty} P(|Y_n - \theta| \leq \varepsilon)$$

Note that $Y_n > \theta$, so we get $\lim_{n\to\infty} P(Y_n - \theta \leq \varepsilon) = \lim_{n\to\infty} P(Y_n \leq \varepsilon + \theta)$.

Using the cdf of Y_n, we get

$$\lim_{n\to\infty} F_{Y_n}(\varepsilon + \theta) = \lim_{n\to\infty} 1 - (1 - e^{\theta - \varepsilon - \theta})^n = 1 - (1 - e^{-\varepsilon})^\infty \to 1$$

Thus, we've shown $Y_n \xrightarrow{P} \theta$

$\square$

$-\frac{n^2}{2t}\left(\frac{t}{2}\right)$

5.1.9) For 5.1.7, obtain mean of Y_n. Is Y_n an unbiased estimator of θ. Obtain an unbiased estimator of θ based on Y_n.

Solution:

We know $F_{Y_n}(x) = 1-(1-e^{\theta - x})^n$ so we can find the pdf.

$$f_{Y_n}(x) = n \cdot (1-e^{\theta-x})^{n-1} \cdot -e^{\theta} = -ne^{\theta}(1-e^{\theta-x})^{n-1}, \quad x>\theta$$

0 elsewhere

$$E(\cancel{X}) = \int_\theta^x -ne^{\theta}(1-e^{\theta-x})^{n-1} \cdot \cancel{x\,dx}\; x$$

$$E(Y_n) = \int_\theta^{\infty} xne^{-(x-\theta)} e^{-(x-\theta)(n-1)}$$

$$= \int_0^\infty xne^{-(x-\theta)n}\,dx$$

$$= n\int_0^\infty xe^{-(x-\theta)n}\,dx$$

$u = x \qquad dv = e^{-(x-\theta)n} = e^{-nx-n\theta}$

$du = dx \qquad v = -e^{-n(x-\theta)}$

$$n\left(-xe^{-n(x-\theta)}\right)\Big|_\theta^\infty + \int_\theta^\infty e^{n(x-\theta)}\,dx \Rightarrow \int_\theta^\infty e^{nx} \cdot e^{-n\theta}$$

$$n(\theta) - (ne^{-n\theta}e^{nx})\Big|_\theta^\infty = n\theta - n = \theta + \frac{1}{n}$$

Since $E(Y_n) \neq \theta$, it is not a biased estimator of θ, but $Y_n - \frac{1}{n}$ is.

5.2.1) Let $\bar{X}_n$ denote mean of sample size n from distribution that is $N(u, \sigma^2)$. Find limiting distribution of $\bar{X}_n$.

Solution:

We know that $\bar{X}_n \sim N(u, \frac{\sigma^2}{n})$. Thus, we know

$$f_{\bar{X}_n} = \frac{1}{\sigma\sqrt{\frac{2\pi}{n}}} e^{-n(x-u)^2/2\sigma^2}$$

~~Make the change of vars $v=\sqrt{n}w$ and so we get the~~

~~$\lim_{n\to\infty} F_n(x) = \begin{cases} 0 & x<0 \\ \frac{1}{2} & x=0 \\ 1 & x>0 \end{cases} \qquad F(x) = \begin{cases} 0, & x<0 \\ 1, & x\geq 0 \end{cases}$~~

~~By weak~~ BY WEAK LAW OF LARGE NUMBERS,

~~Thus, we can write the cdf as~~

$$\cancel{\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{1}{\sigma\sqrt{\frac{2\pi}{n}}}e^{-n(x-u)^2/2\sigma^2}}$$

~~and when we take the limit $n\to\infty$ it becomes obvious (through some symmetry about $x=0$) that the limit becomes a degenerate distribution at 0.~~

we know that $\bar{X}_n \sim N(u, \frac{\sigma^2}{n}) \xrightarrow{P} u$ and so it also converges in distribution to u, by a theorem.

5.2.2) $Z_n = n(Y_1 - \theta)$, $Y_1 = \min\{X_1, \dots, X_n\}$, $f(x) = e^{-(x-\theta)}$

We have already found the cdf of Y_1 to be

$$F_{Y_1}(x) = 1 - e^{n(\theta - x)}$$

SOLUTION HERE

exponential

Therefore

$$F_{Z_n}(x) = P(Z_n \le x) = P(n(Y_1-\theta) \le x) = P(Y_1 \le \tfrac{x}{n} + \theta)$$

$$F_{Z_n}(x) = F_{Y_1}(\tfrac{x}{n}+\theta) = 1 - e^{n(\theta - \frac{x}{n} - \theta)} = 1 - e^{-x}$$

~~Therefore the limiting distribution of Z_n is~~

$1 - e^{n(\theta - x)}$

$\lim_{n\to\infty} e^{-n}$

e^{-x}

$$\lim_{n\to\infty} 1 - e^{-x} = 1 - e^{-x} \Rightarrow \text{the limiting distribution is } \boxed{\text{exponential}}$$

5.2.8) $Z_n \sim \chi^2(n)$ and $W_n = \frac{Z_n}{n^2}$. Find limiting distribution of W_n.

Solution:

$$M_{W_n}(t) = E(e^{t\frac{Z_n}{n^2}}) = (1 - \tfrac{2t}{n^2})^{-n/2}, \quad t < \tfrac{n^2}{2}$$

$$\text{Now } \lim_{n\to\infty}(1 - \tfrac{2t}{n^2})^{-n/2} = 1 \Rightarrow \text{degenerate distribution } 0$$

5.2.12 on next page

~~5.2.12) $Z_n \sim$ Poisson($\mu = n$). Show limiting distribution of $Y_n = (Z_n - n)/\sqrt{n}$ is normal w/ mean 0, var 1~~

$\cancel{M_{Y_n}(t) = E(e^{t(\frac{Z_n - n}{\sqrt{n}})})}$

$\cancel{= -\frac{tn}{\sqrt{n}} E(e^{tZ_n/\sqrt{n}})}$

$\cancel{m_{Y_n}(t) = e^{(\frac{t^2}{2})}}$

$\cancel{= -\frac{tn}{\sqrt{n}} \cdot e^{(\frac{n}{\sqrt{n}}(e^t - 1))}}$

$\cancel{= -\frac{t}{\sqrt{n}} \cdot e^{(\sqrt{n}(e^t - 1))}}$ $\rightarrow t = \frac{t}{\sqrt{n}}$

$\cancel{= -\frac{tn}{\sqrt{n}} E(e^{t\frac{Z_n}{\sqrt{n}}})}$

$\cancel{= -\frac{tn}{\sqrt{n}} e^{\mu\frac{t}{\sqrt{n}} + \frac{t^2}{n}}}$

$\cancel{= -\frac{tn}{\sqrt{n}} \cdot e^{(\mu(e^{\frac{t}{\sqrt{n}}} - 1))}}$

$\cancel{= -\frac{t}{\sqrt{n}} e^{(\mu(e^{t/\sqrt{n}} - 1))}}$

5.3.1) We can use the CLT. ~~mean of~~ $\chi^2(50)$ are $\mu = 50$ and $\sigma^2 = 100$. The SE of the sample mean is

$$SE = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{100}}{\sqrt{100}} = 1$$

Now $\bar{X} \sim N(50, \sigma^2)$ with SE=1. To use CLT,

$$Z_{lower} = \frac{49 - 50}{1} = -1, \quad Z_{upper} = \frac{51 - 50}{1} = 1$$

$$P(49 < \bar{X} < 51) = P(-1 < Z < 1) = 0.8413 - 0.1587$$

$$= \boxed{0.6826}$$

5.3.3) Similar to the previous problem. $Y \sim b(72, \frac{1}{3})$

$\Rightarrow \mu = 24$ and $\sigma^2 = 16$.

$$Z_{lower} = \frac{22 - 24}{\sqrt{16}} = -\frac{1}{2}$$

$$Z_{upper} = \frac{28 - 24}{\sqrt{16}} = 1$$

$P(-\frac{1}{2} < Z < 1) = 0.8413 - 0.3085$

$= 0.5328$

5.2.12) $M_{Y_n}(t) = E\left(e^{t\left(\frac{Z_n - n}{\sqrt{n}}\right)}\right) = \sum_{k=0}^{\infty} \frac{(ne^{t/\sqrt{n}})^k}{k!}$

Note that this is simply an expansion for $\exp(ne^{t/\sqrt{n}} - n)$ and so we have

$$M_{Y_n}(t) = e^{n(e^{\frac{t}{\sqrt{n}}} - 1)}$$

$\lim_{n \to \infty} e^{n(e^{\frac{t}{\sqrt{n}}} - 1)} = e^{\frac{t^2}{2}}$ which is the mgf of a standard normal distribution $\square$

5.3.11) $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$ for large n, find approximate distribution $u(\bar{X}) = \bar{X}^3$.