MATH401 Homework 9

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1. Note that a matrix is skew-symmetric if $A^T = -A$

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}^T = -\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

$$\begin{bmatrix} a & e & i & m \\ b & f & j & n \\ c & g & k & o \\ d & h & l & p \end{bmatrix} = \begin{bmatrix} -a & -b & -c & -d \\ -e & -f & -g & -h \\ -i & -j & -k & -l \\ -m & -n & -o & -p \end{bmatrix}$$

This trivially implies $a = 0, f = 0, k = 0, p = 0$, which are all the diagonal elements. We also have

$$b = -e, c = -i, d = -m, g = -j, h = -n, l = -o$$

Thus we have the matrix

$$\begin{bmatrix} 0 & -b & -c & -d \\ b & 0 & -g & -h \\ c & g & 0 & -l \\ d & h & l & 0 \end{bmatrix}$$

is the general form of a 4×4 skew-symmetric matrix.

2. First we must normalize the vector $\mathbf{v}$ such that

$$u = \frac{1}{\sqrt{7^2 + 2^2 + 3^2}}\mathbf{v} = \frac{1}{\sqrt{62}}\begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} \implies A = \frac{1}{\sqrt{62}}\begin{bmatrix} 0 & -3 & -2 \\ 3 & 0 & -7 \\ 2 & 7 & 0 \end{bmatrix}$$

Our rotation matrix is

$$e^{\theta A} = \exp(\frac{2\pi}{7}\frac{1}{\sqrt{62}}\begin{bmatrix} 0 & -3 & -2 \\ 3 & 0 & -7 \\ 2 & 7 & 0 \end{bmatrix}) \approx \begin{bmatrix} 0.9211 & -0.3829 & -0.0711 \\ 0.2129 & 0.6478 & -0.7315 \\ 0.3261 & 0.6586 & 0.6781 \end{bmatrix}$$

$$\begin{bmatrix} 0.9211 & -0.3829 & -0.0711 \\ 0.2129 & 0.6478 & -0.7315 \\ 0.3261 & 0.6586 & 0.6781 \end{bmatrix}\begin{bmatrix} 3.0 \\ 1.5 \\ -2.5 \end{bmatrix} = \begin{bmatrix} 2.3665 \\ 3.4390 \\ 0.2709 \end{bmatrix}$$

$$\begin{bmatrix} 0.9211 & -0.3829 & -0.0711 \\ 0.2129 & 0.6478 & -0.7315 \\ 0.3261 & 0.6586 & 0.6781 \end{bmatrix}^7\begin{bmatrix} 3.0 \\ 1.5 \\ -2.5 \end{bmatrix} = \begin{bmatrix} 3.0 \\ 1.5 \\ -2.5 \end{bmatrix}$$

3. Let us normalize the vector. We get

$$u = \frac{1}{5}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} \implies A = \frac{1}{5}\begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix}$$

$$e^{\theta A} = \exp(\theta \frac{1}{5} \begin{bmatrix} 0 & -3 & 0 \\ 3 & 0 & -4 \\ 0 & 4 & 0 \end{bmatrix})$$

$$e^{\theta A} = \begin{bmatrix} \frac{9}{50}e^{-\theta i} + \frac{9}{50}e^{\theta i} + \frac{16}{25} & -\frac{3i}{10}e^{-\theta i} + \frac{3i}{10}e^{\theta i} & \frac{12}{25} - \frac{6}{25}e^{\theta i} - \frac{6}{25}e^{-\theta i} \\ \frac{3i}{10}e^{-\theta i} - \frac{3i}{10}e^{\theta i} & \frac{1}{2}e^{-\theta i} + \frac{1}{2}e^{\theta i} & -\frac{2i}{5}e^{-\theta i} + \frac{2i}{5}e^{\theta i} \\ \frac{12}{25} - \frac{6}{25}e^{\theta i} - \frac{6}{25}e^{-\theta i} & -\frac{2i}{5}e^{-\theta i} + \frac{2i}{5}e^{\theta i} & \frac{8}{25}e^{-\theta i} + \frac{8}{25}e^{\theta i} + \frac{9}{25} \end{bmatrix}$$

4.

$$e^A = \begin{bmatrix} 0.1544 & 0.5654 & 0.4291 & 0.6873 \\ -0.7896 & -0.2942 & 0.5312 & 0.0877 \\ 0.5802 & -0.6434 & 0.4908 & 0.0925 \\ -0.1266 & -0.4241 & -0.5410 & 0.7151 \end{bmatrix}$$

Note that A doesn't have an eigenvalue of 0. Therefore, the rotation matrix e^A has no eigenvalue $e^0 = 1$. Therefore, there cannot exist a nonzero, fixed vector $\vec{v}$ such that $e^A\vec{v} = \vec{v}$

5.

$$e^B = \begin{bmatrix} -0.5428 & 0.3584 & 0.4662 & -0.2452 & 0.5423 \\ 0.7620 & 0.4617 & 0.0194 & 0.0126 & 0.4535 \\ -0.2838 & 0.5408 & -0.7789 & -0.1377 & -0.0368 \\ 0.1963 & -0.0647 & 0.0622 & -0.9464 & -0.2403 \\ 0.0178 & 0.6014 & 0.4145 & 0.1585 & -0.6642 \end{bmatrix}$$

Yes, there is a nonzero fixed vector since the matrix A does have an eigenvalue 0, and this corresponding eigenvector is

$$v_0 = \begin{bmatrix} 0.3774 \\ 0.8357 \\ 0.1887 \\ -0.0270 \\ 0.3505 \end{bmatrix}$$