MATH401 Homework 11

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1. **(Must do all computations by hand.)** Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 4 & 4 \end{bmatrix}$, which has SVD

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{66} & 1/\sqrt{2} & 4/\sqrt{33} \\ 1/\sqrt{66} & -1/\sqrt{2} & 4/\sqrt{33} \\ 8/\sqrt{66} & 0 & -1/\sqrt{33} \end{bmatrix} \begin{bmatrix} \sqrt{33} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}.$$

(a) Compute the Frobenius norm $\|A\|_F$ by using the six entries of A.

(b) Compute the Frobenius norm $\|A\|_F$ by using singular values of A.

(c) Without computing it, determine the error in approximation $\|A - A_1\|_F$, where A_1 is the best rank 1 approximation to A.

(d) Compute A_1 exactly.

Solution.

a. The norm can be calculated as the square root of the squares of the entries so we get

$$\|A\|_F = \sqrt{1^2 + 1^2 + 4^2 + 4^2} = \sqrt{34}$$

b. Computing it using singular values we get

$$\|A\|_F = \sqrt{\sqrt{33}^2 + 1^2} = \sqrt{34}$$

c. 1 because the norm of the left singular vector associated with s_2 is 1

$$\sqrt{(\frac{1}{\sqrt{2}})^2 + (-\frac{1}{\sqrt{2}})^2} = 1$$

d.

$$U \begin{bmatrix} \sqrt{33} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1/\sqrt{66} & 1/\sqrt{2} & 4/\sqrt{33} \\ 1/\sqrt{66} & -1/\sqrt{2} & 4/\sqrt{33} \\ 8/\sqrt{66} & 0 & -1/\sqrt{33} \end{bmatrix} \begin{bmatrix} \sqrt{33/2} & \sqrt{33/2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \\ 4 & 4 \end{bmatrix}$$

□

2. Find the smallest positive integer k such that the rank k approximation A_k of $A = \begin{bmatrix} 1.05 & 1.96 & 2.96 & 3.96 \\ 5.04 & 6.04 & 6.85 & 8.07 \\ 8.98 & 9.94 & 11.07 & 11.91 \\ 13.01 & 13.95 & 15.02 & 15.99 \end{bmatrix}$ satisfies $\|A - A_k\|_F < 1$. Give k, A_k, and the error $\|A - A_k\|_F$.

Solution.
Using MatLab to set singular values to 0, beginning from the bottom right of Σ, we find that A_2 satisfies $\|A - A_2\|_F < 1$ and that $\|A - A_1\|_F > 1$, showing that $k = 2$ is the smallest k.

$$k = 2,\ A_2 = \begin{bmatrix} 1.0492 & 1.9905 & 2.9080 & 3.9829 \\ 5.0295 & 5.9875 & 6.9702 & 8.0112 \\ 9.0038 & 9.9554 & 10.9808 & 11.9611 \\ 12.9976 & 13.9572 & 15.0395 & 15.9755 \end{bmatrix},\ \|A - A_2\|_F = 0.1897 < 1$$

□

3. The linear system

$$\begin{cases} 3.62x_1 + 0.51x_2 - 3.37x_3 + 1.72x_4 - 0.08x_5 = 8.77 \\ 2.48x_1 + 2.25x_2 - 1.36x_3 + 3.07x_4 - 0.37x_5 = 11.70 \\ 0.64x_1 + 0.44x_2 - 0.42x_3 + 0.65x_4 - 0.07x_5 = 2.60 \\ 1.29x_1 + 0.79x_2 - 0.89x_3 + 1.22x_4 - 0.13x_5 = 4.93 \end{cases}$$

is supposed to have more than one free variable, but the coefficients have been rounded, and so some information was lost. (Note that if we RREF the augmented matrix, we find only 1 free variable.) Use an appropriate low rank approximation on the augmented matrix to best recover the solution to the original system before the coefficients were rounded. Present your solution in parametric vector form. Your decision as to which approximation is appropriate should be based on the sizes of the singular values. (Note that if you round the entries of your low rank approximation to the nearest hundredth, you should recover the coefficients written above.)

Solution.
Using MatLab, we find a low rank approximation A_2 by setting s_3 and s_4 to 0. This approximation of the augmented matrix has more than one free variable.

$$x_1 = -1.0011x_3 + 0.3349x_4 + 0.0013x_5 + 2.0004$$

$$x_2 = 0.5x_3 + 0.9953x_4 - 0.1658x_5 + 2.9931$$

□

Exercise 9.3. For each of the following matrices find the SVD and identify the direction in which the column data is most spread out, as well as the variance and proportion of total variance in that direction.

(a) $A = \begin{bmatrix} 5.0 & 6.0 & 5.0 & 5.0 & 6.0 \\ 6.0 & 5.0 & 5.0 & 6.0 & 5.0 \\ 5.0 & 6.0 & 5.0 & 6.0 & 5.0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 0 & 1.0 & 1.0 & 2.0 & 2.0 \\ 1.0 & 2.0 & 2.0 & 0 & 1.0 \\ 7.0 & 8.0 & 8.0 & 8.0 & 9.0 \\ 8.0 & 9.0 & 9.0 & 8.0 & 7.0 \\ 7.0 & 7.0 & 7.0 & 9.0 & 7.0 \end{bmatrix}$

(c) $A = \begin{bmatrix} 4.0 & 5.0 & 0 & 0 & 0 \\ 5.0 & 5.0 & 0 & 0 & 0 \\ 0 & 1.0 & 7.0 & 8.0 & 8.0 \\ 0 & 0 & 8.0 & 7.0 & 7.0 \end{bmatrix}$

(d) $A = \begin{bmatrix} 2.0 & 3.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 3.0 & 3.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.0 & 5.0 & 5.0 & 5.0 \end{bmatrix}$

Solution.

a.

$$\text{Direction} = \begin{bmatrix} -0.5769 \\ -0.5769 \\ -0.5782 \end{bmatrix}, \ \text{Variance} = 438.3338, \ \text{Proportion} = 0.994$$

b.

$$\text{Direction} = \begin{bmatrix} -0.0887 \\ -0.0871 \\ -0.5817 \\ -0.5964 \\ -0.5389 \end{bmatrix}, \text{ Variance} = 946.2677, \text{ Proportion} = 0.9878$$

c.

$$\text{Direction} = \begin{bmatrix} -0.0144 \\ -0.0148 \\ -0.7238 \\ -0.6897 \end{bmatrix}, \text{ Variance} = 338.2961, \text{ Proportion} = 0.7849$$

d.

$$\text{Direction} = \begin{bmatrix} 0 \\ -0.0503 \\ 0 \\ -0.9987 \end{bmatrix}, \text{ Variance} = 100.2519, \text{ Proportion} = 0.7595$$

□

Exercise 9.4. Given the matrix

$$A = \begin{bmatrix} 2.0 & 3.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0 & 0 & 0 & 0 \\ 3.0 & 3.0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.0 & 5.0 & 5.0 & 5.0 \end{bmatrix}$$

(a) Find the singular value decomposition of A.

(b) Find an approximation A' to A preserving the largest two singular values only. What proportion of the total variance is preserved?

Solution.

a. The SVD of A is

$$U = \begin{bmatrix} 0 & -0.6464 & 0 & 0.7630 \\ -0.0503 & 0 & -0.9987 & 0 \\ -0.9987 & 0 & -0.0503 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 10.0126 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5.5414 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.8649 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5414 & 0 & 0 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 0 & -0.6464 & 0 & -0.7630 & 0 & 0 \\ 0 & -0.763 & 0 & 0.6464 & 0 & 0 \\ -0.5038 & 0 & -0.8638 & 0 & 0 & 0 \\ -0.4987 & 0 & 0.2908 & 0 & -0.5774 & -0.5774 \\ -0.4987 & 0 & 0.2908 & 0 & 0.7887 & -0.2113 \\ -0.4987 & 0 & 0.2908 & 0 & -0.2113 & 0.7887 \end{bmatrix}$$

The matrices were too big for me to write them all on one line.

b. The matrix approximation

$$A' = \begin{bmatrix} 2.3152 & 2.7330 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.2538 & 0.2512 & 0.2512 & 0.2512 \\ 2.733 & 3.2262 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5.0376 & 4.9873 & 4.9873 & 4.9873 \end{bmatrix}$$

The percentage of variance preserved is 0.9921.

□

Exercise 9.8. Consider the points stored in the columns of this matrix:

$$\begin{bmatrix} 1.9 & 0.9 & 2.6 & 6.1 & 3.4 & -0.5 & 4.5 & 6.0 \\ 5.3 & 11.0 & 3.9 & 7.0 & 8.2 & 9.2 & 12.0 & 2.6 \\ -0.5 & 1.6 & -1.6 & -13.0 & -10.0 & 3.3 & -18.0 & -8.8 \end{bmatrix}$$

These points mostly lie close to a plane but not through the origin.

(a) Find the mean of all the points. This is basically where all the points are centered.

(b) Center the points around the origin by subtracting this mean from each point.

(c) Find the two directions with the most variance and use these to find a normal vector for the plane.

(d) Find the equation of the plane using this normal vector and also the mean.

(e) Using this equation predict x so that $(x, 1, -2)$ is on the plane, predict y so that $(0, y, 10)$ is on the plane, and predict z so that $(3, 4, z)$ is on the plane.

Solution.

a. The mean is $\begin{bmatrix} 3.1125 \\ 7.4000 \\ -5.8750 \end{bmatrix}$

b.

$$\begin{bmatrix} -1.2125 & -2.2125 & -0.5125 & 2.9875 & 0.2875 & -3.6125 & 1.3875 & 2.8875 \\ -2.1 & 3.6 & -3.5 & -0.4 & 0.8 & 1.8 & 4.6 & -4.8 \\ 5.3750 & 7.4750 & 4.2750 & -7.1250 & -4.1250 & 9.1750 & -12.1250 & -2.9250 \end{bmatrix}$$

c. The two directions with the most variance are $\begin{bmatrix} 0.2533 \\ 0.0538 \\ -0.9659 \end{bmatrix}$ and $\begin{bmatrix} -0.3151 \\ 0.9486 \\ -0.0298 \end{bmatrix}$.

The third normal vector is $\begin{bmatrix} -0.9147 \\ -0.3119 \\ -0.2572 \end{bmatrix}$

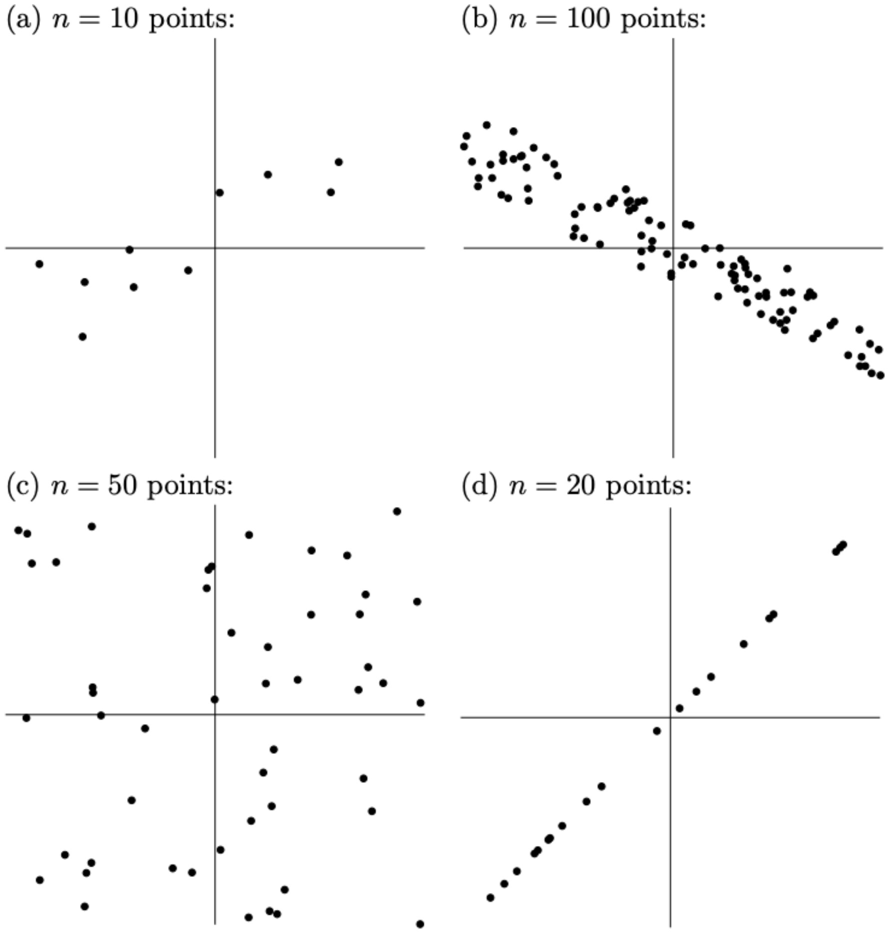
d. The equation for the plane is

$$-0.9147(x - 3.1125) - 0.3119(y - 7.4) - 0.2572(z + 5.8750) = 0$$

e. $x = 4.112, y = 4.288, z = -1.315$

□

Exercise 9.9. Suppose n points are placed as columns in a $2 \times n$ matrix. If $A = U\Sigma V^T$ is the SVD and if the points are plotted below, say as much as you can about the matrices U and Σ



Solution.

a. U has 2 column vectors that are about $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and s_1 is much larger than s_2.

b. U has two column vectors around $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and s_1 is much larger than s_2.

c. No conclusion about either U or Σ can be drawn from this graph

d. U has two column vectors around $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ with s_1 much larger than s_2.

□

Exercise 9.10. Suppose a 100×100 matrix has 100 singular values whose squares add to 9512 and in decreasing order are $\{82.2, 27.6, 23.3, 19.1, 16.3, 8.5, 5.3, ...\}$. How many singular values must be preserved in order to keep 90% of the data variance? How about 80%?

Solution.
For 90%, the first 5 singular values preserve $92.94\% > 90\%$.
For 80%, the first 3 singular values preserve $86.32\% > 80\%$.

□