MATH401 Homework 10

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1. **(Must do all computations by hand.)** Orthogonally diagonalize the matrix $A = \begin{bmatrix} 2 & 3 \\ 3 & 2 \end{bmatrix}$.

Solution.
To find the eigenvalues of A,

$$\det(A - \lambda I) = \det(\begin{bmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{bmatrix}) = (2-\lambda)^2 - 9 = 4 - 4\lambda + \lambda^2 - 9$$

$$\implies \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1) \implies \lambda = -1, 5$$

Thus,

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}$$

Let us find corresponding eigenvectors starting with $\lambda_1 = -1$. We solve

$$(A + 1I)v_1 = 0 \implies \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \implies v_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

Now for $\lambda_2 = 5$, we solve

$$(A - 5I)v_2 = 0 \implies \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0 \implies v_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

and so

$$P = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

thus we obtain the diagonalization

$$A = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^{-1}$$

□

2. **(Must do all computations by hand.)** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

   (a) Diagonalize $A = PDP^{-1}$.
   (b) Find the singular value decomposition $A = U\Sigma V^T$.
   (c) Are the singular values of A the same as the eigenvalues of A?
   (d) Compute the pseudoinverse A^+. Present your answer as a simplified 2×2 matrix.

Solution.

a. Let us first find the eigenvalues of A

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ 0 & -\lambda \end{bmatrix} = -\lambda + \lambda^2 = 0 \implies \lambda = 1, 0$$

$$(A - 1\lambda)v_1 = 0 \implies \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} v_1 = 0 \implies v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A - 0\lambda)v_2 = 0 \implies \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} v_2 = 0 \implies v_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

and so we obtain the orthogonal diagonalization

$$A = \begin{bmatrix} 1 & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{bmatrix}^{-1}$$

b. Note that A is $2 \times 2 \implies U, \Sigma, V$ are all 2×2. First we compute AA^T and its corresponding eigenvalues to be

$$AA^T = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \implies \det(AA^T - \lambda I) = \lambda^2 - 2\lambda \implies \lambda = 2, 0$$

Now the same process for $A^T A$

$$A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \implies \det(A^T A - \lambda I) = (1-\lambda)^2 - 1 = \lambda^2 - 2\lambda = 0 \implies \lambda = 2, 0$$

Thus, both eigenvalues are shared and the singular values are

$$\sigma_1 = \sqrt{2}, \sigma_2 = 0$$

We construct

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix}$$

Now we find the unit eigenvectors associated with each of shared eigenvalues to construct U and V

$$(AA^T - 2I)v_1 = 0 \implies \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} v_1 = 0 \implies u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(AA^T - 0I)v_2 = 0 \implies u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$(A^T A - 2I)v_1 = 0 \implies \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} v_1 = 0 \implies v_1 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

$$(A^T A - 0I)v_2 = 0 \implies v_2 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

and so the singualar value decomposition of A is

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T$$

c. No, the singular values of A are not equal to the eigenvalues of A.

d. Let us compute the pseudoinverse of A.

$$A^{+} = U\Sigma^{+}V^{T}$$

$$A^{+} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^{T} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

□

3. In this problem, use MATLAB, but don't directly use the **svd** command. Let $A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{bmatrix}$.

(a) Give the matrix $A^T A$, as well as its eigenvalues and eigenvectors.

(b) Give the matrix AA^T, as well as its eigenvalues and eigenvectors.

(c) Put the information together to give the singular value decomposition $A = U\Sigma V^T$. Double check that it works in MATLAB. (If you have issues, make sure that $A\mathbf{v}_i = \sigma_i \mathbf{u}_i$)

(d) Use your SVD to give the pseudoinverse A^+.

(e) Let $\mathbf{b} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}$. Compute $\mathbf{x} = A^+\mathbf{b}$. What does the vector $\mathbf{x}$ represent? Be as specific as your can. (e.g. does it solve something? is it the unique solution? does it have any special properties?)

Solution.

a. The eigenpairs of $A^T A = \begin{bmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{bmatrix}$ are

$$\{0, \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}\}, \{9, \begin{bmatrix} 0.2357 \\ -0.2357 \\ 0.9428 \end{bmatrix}\}, \{25, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}\}$$

b. The eigenpairs of $AA^T = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ are

$$\{9, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}\}, \{25, \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}\}$$

c. The SVD of A is given by

$$A = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} & 0.2357 & -\frac{2}{3} \\ -\frac{\sqrt{2}}{2} & -0.2357 & \frac{2}{3} \\ 0 & 0.9428 & \frac{1}{3} \end{bmatrix}^{T}$$

d. The pseudoinverse is

$$A^{+} = U\Sigma^{+}V^{T} = \begin{bmatrix} 0.1556 & 0.044 & 0.2222 \\ 0.0444 & 0.1556 & -0.2222 \end{bmatrix}^{T}$$

e. This vector $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is the least squares solution of smallest norm, or essentially the vector that minimizes $||A\mathbf{x} - b||$.

$\square$

4. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \\ 2 & 0 & 1 \end{bmatrix}$.

(a) Use the `svd` command to find the SVD $A = U\Sigma V^T$.

(b) Use the `pinv` command to find A^+.

(c) Let $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Compute $\mathbf{x} = A^+\mathbf{b}$. What does the vector $\mathbf{x}$ represent? Be as specific as your can. (e.g. does it solve something? is it the unique solution? does it have any special properties?)

Solution.

a. Using the svd command,

$$A = \begin{bmatrix} -0.8580 & -1.084 & -0.3378 & -0.3693 \\ -0.1419 & 0.1950 & 0.8363 & -0.4924 \\ -0.4031 & 0.6628 & 0.1394 & 0.6155 \\ -0.2822 & -0.7148 & 0.4087 & 0.4924 \end{bmatrix} \begin{bmatrix} 4.3346 & 0 & 0 \\ 0 & 2.5856 & 0 \\ 0 & 0 & 0.7249 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -0.2354 & -0.8511 & 0.4692 \\ -0.6150 & 0.5042 & 0.6062 \\ -0.7526 & -0.1459 & -0.6422 \end{bmatrix}$$

b. Using the pinv command,

$$A^+ = \begin{bmatrix} -0.1364 & 0.4848 & -1.061 & 0.5152 \\ -0.1818 & 0.7576 & 0.3030 & 0.2424 \\ 0.4545 & -0.7273 & -0.0909 & -0.2727 \end{bmatrix}$$

c. This vector $\mathbf{x} = \begin{bmatrix} 0.7576 \\ 1.1212 \\ -0.6364 \end{bmatrix}$ is the least squares solution of smallest norm, or essentially the vector that minimizes

$\square$

5. What are the singular values of an orthogonal matrix? Justify your answer.

Solution.
Consider a matrix A. Singular values of A are computed as the square root of the shared eigenvalues of A^TA and AA^T. Recall that if A is orthogonal, then $A^TA = AA^T = I$. Thus, the eigenvalues of A^TA, AA^T are all 1 because the identity matrix is already diagonal, and since $\sqrt{1} = 1$, the the singular values of A are all equal to 1. $\square$

6. Let $\mathbf{v}$ denote a column vector in $\mathbb{R}^n$, thought of as an $n \times 1$ matrix. Find the singular value of $\mathbf{v}$.

Solution.
Note that singular values are the square roots of shared eigenvalues from v^Tv and vv^T. Note further that v^Tv is of shape 1×1, which is a scalar. Thus, the singular value must be $\sqrt{v^Tv}$. Let us simplify this further. Let $v = (v_1, \cdots, v_n)^T$. Then

$$\sqrt{v^Tv} = \sqrt{v_1^2 + \cdots + v_n^2} = ||v||$$

and this proves that the singular value of v is simply the norm of v.

□

7. Let A be an $m \times n$ matrix. Suppose that $\mathbf{v}$ is an eigenvector for A^TA with eigenvalue λ.
   (a) Show that $\|A\mathbf{v}\|^2 = \lambda \|\mathbf{v}\|^2$. (Hint: use the fact that $\|\mathbf{x}\|^2 = \mathbf{x}^T\mathbf{x}$ for any vector $\mathbf{x}$.)
   (b) Explain why the previous part implies that $\lambda \geq 0$. This proves that all eigenvalues of A^TA are nonnegative.
   (c) As above, we are still assuming that $\mathbf{v}$ is an eigenvector for A^TA with eigenvalue λ. Assume here that $\lambda \neq 0$. Show that $A\mathbf{v}$ is an eigenvector for AA^T with the same eigenvalue λ. (Hint: Just show that it satisfies the eigenvector equation. Remember also that an eigenvector is nonzero by definition.) This proves that every nonzero eigenvalue of A^TA is an eigenvalue of AA^T. A symmetric argument shows that every nonzero eigenvalue of AA^T is an eigenvalue of A^TA. Thus A^TA and AA^T have the same nonzero eigenvalues.

Solution.

a. Let us proceed with a direct proof.

$$||Av||^2 = (Av)^TAv = v^TA^TAv = v^T\lambda v = \lambda v^Tv = \lambda||v||^2$$

as desired.

b. The previous part implies that $\lambda \geq 0$ because the norm operator followed by squaring the resulting value is must be a positive scalar, or be some $n \times m$ matrix or vector containing all positive values. Therefore, λ must be ≥ 0, or else this equality cannot possibly hold.

c. We are assuming that $(A^TA)v = \lambda v$. We seek to show that $(AA^T)(Av) = \lambda(Av)$, which would prove that Av is an eigenvector of AA^T with eigenvalue λ. By direct proof,

$$(AA^T)(Av) = AA^TAv = A(A^TAv) = A\lambda v = \lambda Av = \lambda(Av)$$

as desired.

□

8. Here we'll investigate why $A^+ = (A^TA)^{-1}A^T$ in the special case where the columns of A are linearly independent. Here are some facts we'll use:

- The columns of A are linearly independent if and only if A^TA is invertible.
- A square matrix is invertible if and only if 0 is not an eigenvalue.
- Combining the two above: The columns of A are linearly independent if and only if 0 is not a singular value of A.

For simplicity, we'll assume A is 3×2 with linearly independent columns (the general case is not really any harder.) By the above, we know that A has an SVD

$$A = U \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} V^T, \qquad \sigma_1, \sigma_2 \neq 0.$$

For all parts below, present your answer in a form like what is written on the previous line.

(a) Using the definition, write down A^+.

(b) Multiply A^TA and simplify.

(c) Simplify $(A^TA)^{-1}$.

(d) Multiply $(A^TA)^{-1}A^T$ and simplify. Your answer should be the same as part (a).

Solution.

a.

$$A^+ = V \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \end{bmatrix} U^T$$

b.

$$(U \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix} V^T)^T U(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix})V^T$$

$$V \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{bmatrix} U^T U(\begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix})V^T$$

$$V \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} V^T$$

c.

$$V \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{bmatrix} V^T$$

d.

$$V \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{bmatrix} V^T A^T = V \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{bmatrix} \Sigma^T U^T = V \begin{bmatrix} \frac{1}{\sigma_1} & 0 & 0 \\ 0 & \frac{1}{\sigma_2} & 0 \end{bmatrix} U^T$$

$\square$