STAT420 Homework 4

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4.2.6. Let $\overline{X}$ be the mean of a random sample of size n from a distribution that is $N(\mu, 9)$. Find n such that $P(\overline{X} - 1 < \mu < \overline{X} + 1) = 0.90$, approximately.

Solution. Note that altough the sample is iid from a normal distribution, we cannot use the method of finding the Confidence Interval for μ Under Normality mentioned in the textbook because the t-distribution's degrees of freedom is dependent on the sample size, which we desire. Instead, we will proceed by using the properties of the standard normal distribution since $\bar{X}$ is approximately normal by the Central Limit Theorem. By Theorem 5.2.1 in Professor Xu's notes, $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$. From Equation 4.2.5 in the textbook,

$$1 - \alpha \approx P_\mu(\bar{X} - z_{\alpha/2}\frac{S}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{S}{\sqrt{n}})$$

Recall that we seek n such that

$$P(\bar{X} - 1 < \mu < \bar{X} + 1) = 0.9$$

It becomes obvious that $z_{\alpha/2}\frac{S}{\sqrt{n}} = 1$. Furthermore, note that $1-\alpha = 0.9 \implies \alpha = 0.1$. Using a standard normal table, $P(X \leq x) = 0.05 \implies |x| = 1.6448 \implies z_{0.05} = 1.6448$. Furthermore, we know $S = 3$, since we're given $\sigma^2 = 9$. We now have

$$1.6448(\frac{3}{\sqrt{n}}) = 1 \implies \sqrt{n} \approx 4.9334 \implies n \approx 25$$

$\square$

4.2.7. Let a random sample of size 17 from the normal distribution $N(\mu, \sigma^2)$ yield $\overline{x} = 4.7$ and $s^2 = 5.76$. Determine a 90% confidence interval for μ.

Solution. Let us compute the 90% confidence interval for μ using the method under normality.
By Equation 2.2 in the textbook,

$$1 - \alpha = P_\mu(\bar{X} - t_{\alpha/2,n-1}\frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2,n-1}\frac{S}{\sqrt{n}}) = 0.9$$

and so the confidence interval is given as

$$(\bar{x} - t_{\alpha/2,n-1}s/\sqrt{n}, \bar{x} + t_{\alpha/2,n-1}s\sqrt{n})$$

Every variable in the above expression is known besides $t_{\alpha/2,n-1}$ which we can trivially compute as $t_{0.05,16} = 1.746$. Now plugging in all values, we get

$$(4.7 - 1.746(\frac{2.4}{\sqrt{17}}), 4.7 + 1.746(\frac{2.4}{\sqrt{17}})) = (3.6837, 5.7163)$$

$\square$

4.2.22. Let two independent random variables, Y_1 and Y_2, with binomial distributions that have parameters $n_1 = n_2 = 100$, p_1, and p_2, respectively, be observed to be equal to $y_1 = 50$ and $y_2 = 40$. Determine an approximate 90% confidence interval for $p_1 - p_2$.

Solution. Recall that for a confidence interval for difference in proportions that the $(1-\alpha)100\%$ interval is given by

$$\hat{p_1} - \hat{p_2} \pm z_{\alpha/2}\sqrt{\frac{\hat{p_1}(1-\hat{p_1})}{n_1} + \frac{\hat{p_2}(1-\hat{p_2})}{n_2}}$$

Let $\hat{p_1} = \frac{50}{100} = 0.5$ and $\hat{p_2} = \frac{40}{100} = 0.4$ such that our confidence interval becomes

$$(0.1 - 1.6448\sqrt{\frac{0.5^2}{100} + \frac{0.4(0.6)}{100}}, 0.1 + 1.6448\sqrt{\frac{0.5^2}{100} + \frac{0.4(0.6)}{100}}) = (-0.0151, 0.2151)$$

□

6.1.4. Suppose $X_1, \ldots, X_n$ are iid with pdf $f(x;\theta) = 2x/\theta^2,\ \ 0 < x \leq \theta$, zero elsewhere. Note this is a nonregular case. Find:

(a) The mle $\hat{\theta}$ for θ.

(b) The constant c so that $E(c\hat{\theta}) = \theta$.

(c) The mle for the median of the distribution. Show that it is a consistent estimator.

Solution.

a. The log of the likelihood simplifies to

$$l(\theta) = \sum_{i=1}^{n} \log f(x_i;\theta) = \sum_{i=1}^{n} \log \frac{2x_i}{\theta^2} = \sum_{i=1}^{n} \log 2x_i - \log \theta^2 \ = -2n \log \theta + \sum_{i=1}^{n} \log 2x_i$$

Taking the first partial derivative, we obtain

$$l'(\theta) = -\frac{2n}{\theta}$$

Since the partial derivative is strictly negative, the likelihood function is a strictly decreasing function. Observe that the derivative is closest to 0 - therefore meaning a critical point of the likelihood function - when θ is as large as possible. Therefore the maximum likelihood estimator is

$$\hat{\theta} = \max(X_i) = Y_n$$

the largest order statistics of the random sample.

b. By substitution and linearity of expectation, we obtain

$$E(c\hat{\theta}) = E(cY_n) = cE(Y_n) = \theta \implies c = \frac{\theta}{E(Y_n)}$$

Now let us find the pdf of Y_n. By a theorem, the pdf of the largest order statistic $f(y_n)$ is

$$f_{y_n}(y) = nF(y)^{n-1} * f(y)$$

Note that we are given $f(y)$, and $F(y) = \int f(y)dy = \frac{y^2}{\theta^2}$. Plugging the cdf and pdf into the above formula, we obtain

$$f_{y_n} = n(\frac{y^2}{\theta^2})^{n-1} * \frac{2y}{\theta^2} = 2n\frac{y^{n-1}}{\theta^n}$$

Thus we can now take the expected value as

$$E(\hat{\theta}) = E(Y_n) = \int_0^\theta yf(y)\ dy = \int_0^\theta 2n\frac{y^n}{\theta^n}\ dy = \frac{2n}{\theta^n}(\frac{1}{n+1}y^{n+1})|_0^\theta$$

$$E(\hat{\theta}) = \frac{2n}{\theta^n}(\frac{\theta^{n+1}}{n+1}) = \frac{2n\theta}{n+1} \implies c = \frac{n+1}{2n\theta}$$

c. Let us define m to be argmax(Median($f(x;\theta)$) or in other words, the value of x such that $f(x)$ is the median of the distribution. Seeking m, we compute

$$\int_0^m \frac{2x}{\theta^2}\ dx = \frac{1}{2} \implies \frac{1}{\theta^2}m^2 = \frac{1}{2} \implies m = \frac{\theta}{\sqrt{2}}$$

It is obvious that we should use our best estimate of $\theta = \hat{\theta}$ to find our best estimate of m. Thus, we can say

$$\hat{m} = \frac{\hat{\theta}}{\sqrt{2}} = \frac{Y_n}{\sqrt{2}}$$

Now let us show that this estimate is consistent. By definition, an estimate T_n where T is a statistic is a consistent estimator of θ if and only if $T_n \xrightarrow{P} \theta$. Observe that it is sufficient to show $Y_n \xrightarrow{P} \theta \implies \frac{Y_n}{\sqrt{2}} \xrightarrow{P} \frac{\theta}{\sqrt{2}}$, which we will do now. By the definition of convergence in probability, we must show

$$\lim_{n\to\infty} P(|Y_n - \theta| \geq \epsilon)$$

Note that

$$P(|Y_n - \theta| \geq \epsilon) = P(Y_n \geq \epsilon + \theta) + P(Y_n \leq \theta - \epsilon)$$

Now note that Y_n is bounded from 0 to θ, so the first expression is 0.

$$P(|Y_n - \theta| \geq \epsilon) = P(Y_n \leq \theta - \epsilon) = P(X_i \leq \theta - \epsilon)^n$$

due to independence. Note that if $\epsilon \geq \theta$, then $P(X_1 \leq \theta - \epsilon)$ is trivially 0. Now consider when $\theta - \epsilon > 0$.

$$P(X_i \leq \theta - \epsilon) = \int_0^{\theta-\epsilon} \frac{2x}{\theta^2}\, dx = \frac{1}{\theta^2}(x^2)|_0^{\theta-\epsilon} = \frac{1}{\theta^2}(\theta^2 - 2\epsilon\theta + \epsilon^2) = 1 - \frac{2\epsilon}{\theta} + \frac{\epsilon^2}{\theta^2}$$

Now it becomes clear that if we choose any small ϵ such that the quantity $1 - \frac{2\epsilon}{\theta} + \frac{\epsilon^2}{\theta^2} \in [0, 1)$, then we have that

$$\lim_{n\to\infty} P(X_i \leq \theta - \epsilon)^n = \lim_{n\to\infty}(1 - \frac{2\epsilon}{\theta} + \frac{\epsilon^2}{\theta^2})^n = 0$$

so $Y_n \xrightarrow{P} \theta \implies \frac{Y_n}{\sqrt{2}} \xrightarrow{P} \frac{\theta}{\sqrt{2}} \implies \hat{m} = \frac{Y_n}{\sqrt{2}}$ is a consistent estimator of the median of the distribution.

□

6.1.6. Suppose $X_1, X_2, \ldots, X_n$ are iid with pdf $f(x;\theta) = (1/\theta)e^{-x/\theta},\ 0 < x < \infty$, zero elsewhere. Find the mle of $P(X \leq 2)$ and show that it is consistent.

Solution. The log of the likelihood simplifies to

$$l(\theta) = \sum_{i=1}^{n} \log f(x_i;\theta) = \sum_{i=1}^{n} \log \frac{1}{\theta} e^{-x_i/\theta} = \sum_{i=0}^{n} -\log\theta + \log e^{-x_i/\theta}$$

$$l(\theta) = -n\log\theta - \sum_{i=0}^{n} \frac{x_i}{\theta}$$

Taking the partial derivative with respect to θ, we obtain

$$l'(\theta) = -\frac{n}{\theta} + \sum_{i=0}^{n} \frac{x_i}{\theta^2}$$

Setting $l'(\theta) = 0$ and solving this equation for θ yields

$$\sum_{i=0}^{n} \frac{x_i}{\theta^2} = \frac{n}{\theta} \implies \sum_{i=0}^{n} \frac{x_i}{\theta^2} = \frac{n\theta}{\theta^2} \implies \sum_{i=0}^{n} x_i = n\theta \implies \theta = \frac{1}{n}\sum_{i=0}^{n} x_i = \bar{X}$$

is the maximum likelihood estimator of θ. To find the MLE of $P(X \leq 2)$ we compute

$$P(X \leq 2) = \int_0^2 \frac{1}{\theta} e^{-x/\theta}\, dx = (-e^{-x/\theta})|_0^2 = -e^{-2/\theta} + 1$$

Therefore, by substitution of the MLE of θ, the MLE of $P(X \leq 2)$ is

$$1 - e^{-2/\bar{X}}$$

□

6.2.8. Let X be $N(0, \theta)$, $0 < \theta < \infty$.

(a) Find the Fisher information $I(\theta)$.

(b) If $X_1, X_2, \ldots, X_n$ is a random sample from this distribution, show that the mle of θ is an efficient estimator of θ.

(c) What is the asymptotic distribution of $\sqrt{n}(\widehat{\theta} - \theta)$?

Solution.

a. Recall that the Fisher Information $I(\theta)$ can be computed as

$$I(\theta) = E(\frac{\partial \log f(X;\theta)}{\partial \theta})^2 = -E(\frac{\partial^2 \log f(X;\theta)}{\partial \theta^2})$$

although the latter formula is usually computationally simpler. We can get

$$\log f(X;\theta) = \log \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta} = -\frac{1}{2}\log(2\pi\theta) - \frac{x^2}{2\theta}$$

$$\frac{\partial \log f(X;\theta)}{\partial \theta} = -\frac{1}{2\theta} + \frac{x^2}{2\theta^2}$$

$$\frac{\partial^2 \log f(X;\theta)}{\partial \theta^2} = \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}$$

Therefore, by substitution and then by linearity of expectation,

$$I(\theta) = -E(\frac{1}{2\theta^2} - \frac{x^2}{\theta^3}) = -\frac{1}{2\theta^2} + \frac{E(X^2)}{\theta^3}$$

Let us find $E(X^2)$. Note that $Var(X) = E(X^2) - E(X)^2 \implies E(X^2) = \theta - 0^2 = \theta$

and so we have

$$I(\theta) = -\frac{1}{2\theta^2} + \frac{1}{\theta^2} = \frac{1}{2\theta^2}$$

b. Let us first compute the MLE of θ and then show that is is an efficient estimator of θ. The log likelihood simplifies to

$$l(\theta) = \sum_{i=1}^{n} \log f(X;\theta) = \sum_{i=1}^{n} -\frac{1}{2}\log(2\pi\theta) - \frac{x_i^2}{2\theta} = -\frac{n}{2}\log(2\pi\theta) - \sum_{i=1}^{n} \frac{x_i}{2\theta}$$

$$l'(\theta) = -\frac{n}{2\theta} + \sum_{i=1}^{n} \frac{x_i^2}{2\theta^2}$$

Setting $l'(\theta) = 0$ and solving for θ yields

$$\frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2 = \frac{n}{2\theta} \implies \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

Thus, we have found the MLE of θ, but let us show that is is efficient. Recall that a statistic Y is efficient estimator of θ if and only if it attains the Rao-Cramer lower bound, such that $Var(Y) \geq \frac{1}{nI(\theta)}$.

$$Var(\hat{\theta}) = Var(\frac{1}{n}\sum_{i=1}^{n} X_i^2) = \frac{1}{n^2}Var(\sum_{i=1}^{n} X_i^2)$$

by the property of variance $Var(aX) = a^2Var(X)$. Now note that $X \sim N(0,\theta) \implies \frac{X}{\sqrt{\theta}} \sim N(0,1)$, by a theorem. Therefore, $\frac{X^2}{\theta} \sim \chi_n^2$, by the definition of chi squared. Therefore,

$$Var(\sum_{i=1}^{n} \frac{X_i^2}{\theta}) = 2n \implies Var(\sum_{i=1}^{n} X_i^2) = 2n\theta^2$$

By substitution, we have

$$Var(\hat{\theta}) = \frac{2\theta^2}{n} \geq \frac{1}{nI(\theta)} = \frac{2\theta^2}{n}$$

which satisfies the definition of efficient estimator, as desired.

c. By Theorem 5.4.3 in Professor Xu's Notes, the asymptotic distribution of the MLE is given as

$$\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{D} N(0, I(\theta)^{-1}) = N(0, 2\theta^2)$$

□

6.2.10. Let $X_1, X_2, \ldots, X_n$ be a random sample from a $N(0,\theta)$ distribution. We want to estimate the standard deviation $\sqrt{\theta}$. Find the constant c so that $Y = c\sum_{i=1}^{n} |X_i|$ is an unbiased estimator of $\sqrt{\theta}$ and determine its efficiency.

Solution.
First, we will find the constant c such that $Y = c\sum_{i=1}^{n} |X_i|$ is an unbiased estimator of θ. By definition of unbiased estimator, $E(Y) = \sqrt{\theta}$. With this in mind, let us find $E(|X_i|)$, which is the equal for all $i \in [0,n]$ since the sample is iid.

$$E(|X_i|) = 2\int_0^{\infty} \frac{x}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{2\theta}}\ dx = -\frac{2\theta}{\sqrt{2\pi\theta}} \int_0^{\infty} -\frac{x}{\theta} e^{-x^2/2\theta}\ dx$$

$$E(|X_i|) = -\frac{2\theta}{\sqrt{2\pi\theta}} (e^{-x^2/2\theta})|_0^{\infty} = -\frac{2\theta}{\sqrt{2\pi\theta}}(0-1) = \frac{2\theta}{\sqrt{2\pi\theta}} = \sqrt{\frac{2\theta}{\pi}}$$

Therefore, we know $\sum_{i=1}^{n} |X_i| = n\sqrt{2\theta/\pi}$ which implies

$$cn\sqrt{\frac{2\theta}{\pi}} = \sqrt{\theta} \implies c = \frac{1}{n}\sqrt{\frac{\pi}{2}}$$

Now let us determine the efficiency of this unbiased estimator. Note that the efficiency of an unbiased estimator Y of $k(\theta)$ is given as

$$\frac{k'(\theta)^2}{nI(\theta)Var(Y)}$$

Solving for the numerator is simple.

$$k(\theta) = \sqrt{\theta} \implies k'(\theta) = \frac{1}{2\sqrt{\theta}} \implies k'(\theta)^2 = \frac{1}{4\theta}$$

Note that we have already computed the Fisher Information $I(\theta)$ of $N(0, \theta)$ which is $I(\theta) = \frac{1}{2\theta^2}$, so all that we have left is the variance.

$$Var(c\sum_{i=1}^{n}|X_i|) = c^2Var(\sum_{i=1}^{n}|X_i|)) = c^2nVar(|X_i|) = c^2n(E(|X_i^2|) - E(|X_i|)^2)$$

$$E(|X_i^2|) = 2\int_0^\infty \frac{2x^2}{\sqrt{2\pi\theta}}e^{-x^2/2\theta}\,dx = \theta$$

$$Var(Y) = c^2n(\theta - \frac{2\theta}{\pi}) = c^2n\theta(\frac{\pi-2}{\pi})$$

Finally, the efficiency of Y can be computed as

$$\frac{\frac{\pi}{4\theta}}{n(\frac{1}{2\theta^2})\frac{1}{n^2}(\frac{\pi}{2})n\theta(\pi-2))} = \frac{\pi}{\pi(\pi-2)} = \frac{1}{\pi-2}$$

□

6.4.3. Let $X_1, X_2, \ldots, X_n$ be iid, each with the distribution having pdf $f(x;\theta_1,\theta_2) = (1/\theta_2)e^{-(x-\theta_1)/\theta_2}$, $\theta_1 \le x < \infty$, $-\infty < \theta_2 < \infty$, zero elsewhere. Find the maximum likelihood estimators of θ_1 and θ_2.

Solution.
The log likelihood simplifies to

$$l(\theta_1,\theta_2) = \sum_{i=1}^{n}\log f(x;\theta_1,\theta_2) = \sum_{i=1}^{n}\log\frac{1}{\theta_2}e^{-(x_i-\theta_1)/\theta_2} = \sum_{i=1}^{n} -\log\theta_2 - \frac{x_i-\theta_1}{\theta_2}$$

$$l(\theta_1,\theta_2) = -n\log\theta_2 - \sum_{i=1}^{n}\frac{x_i-\theta_1}{\theta_2}$$

Note that finding the MLE of θ_1 is trivial because L is a strictly decreasing function with respect to θ_1 and θ_2 is constant. Thus, the MLE

$$\hat{\theta_1} = Y_1$$

, the smallest order statistic, and this maximizes L. To find the MLE of θ_2, let us compute

$$\frac{\partial l}{\partial\theta_2} = -\frac{n}{\theta_2} + \sum_{i=1}^{n}\frac{x_i-\theta_1}{\theta_2^2}$$

Setting the partial derivative to 0 and solving for θ_2 yields

$$\sum_{i=1}^{n} \frac{x_i - \theta_1}{\theta_2^2} = \frac{n}{\theta_2} \implies \theta_2 = \frac{1}{n}\sum_{i=1}^{n} x_i - \theta_1 \implies \hat{\theta_2} = \bar{X} - Y_1$$

$\square$

6.4.6. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$.

(a) If the constant b is defined by the equation $P(X \le b) = 0.90$, find the mle of b.

(b) If c is given constant, find the mle of $P(X \le c)$.

Solution.

a.

$$P(X \le b) = 0.90 \implies P(\frac{X-\mu}{\sigma} \le \frac{b-\mu}{\sigma}) = 0.90$$

Using a standard normal distribution table,

$$\frac{b-\mu}{\sigma} = 1.2815 \implies b = 1.2815\sigma + \mu$$

The MLEs of μ and σ are known for a normal distribution, but let us derive it. The log likelihood is

$$l(\mu, \sigma^2) = \sum_{i=1}^{n} \log \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_i-\mu)^2/2\sigma^2} = -\frac{n}{2}\log(2\pi\sigma^2) + \sum_{i=1}^{n} -(\frac{(x-\mu)^2}{2\sigma^2})$$

$$\frac{\partial l}{\partial \mu} = -\frac{1}{2\sigma^2}\sum_{i=1} 2x_i - 2\mu$$

Setting this to 0 and solving for μ we get

$$2n\mu = \sum_{i=1}^{n} 2x_i \implies \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{X}$$

Similarly, for the MLE of σ we take the partial derivative

$$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}(x_i - \mu)^2$$

Setting this to 0 and solving yields

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2 \implies \hat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2} = \sqrt{\frac{n-1}{n}}S$$

Therefore,

$$b = 1.2815\sqrt{\frac{n-1}{n}}S + \bar{X}$$

b. Note that

$$P(X \leq c) = P(\frac{X-\mu}{\sigma} \leq \frac{c-\mu}{\sigma}) = \Phi(\frac{c-\mu}{\sigma})$$

Therefore,

$$P(\hat{X \leq c}) = \Phi(\frac{c-\hat{\mu}}{\hat{\sigma}}) = \Phi(\frac{\sqrt{n}(c-\bar{X})}{S\sqrt{n-1}}))$$

$\square$