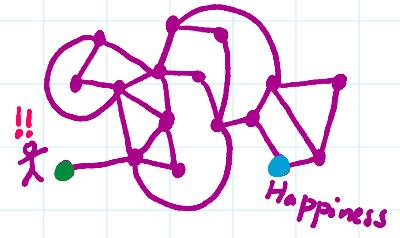


SHORTEST PATH ALGORITHM (FOR UNWEIGHTED GRAPHS)

① Intro: Sps we have a graph and choose two vertices s, t . How can we find the shortest path (least #edges) from s to t ?



② Idea: We start at s . First check all vertices adj. to s . If we find t , great. If not, check the vertices adj. to those! If we find t , great; if not, etc.
As we go we'll:

- label the predecessor of each vertex to be the one which was one edge closer and adj. to it.
- label #edges from starting vertex.

Note: once we visit/tag/label a vertex, never visit it again!

Guarantee: $s, t \in \text{graph}$

graph will be unweighted, undirected, simple, connected

③ More Formally: we do the following.

Initialize two lists: $d = \text{list of } V \text{ distances, all } = \infty \text{ except } d[s] = 0$
 $p = \text{list of } V \text{ predecessors, all } = \text{NULL}$

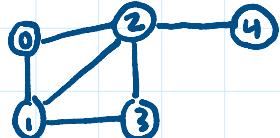
Create a queue \emptyset and push s onto it.

Then until we find t we:

$x = Q.\text{dequeue}$. If $x = t$, done
look at each y adj. to x w/ $dist = \infty$:
assign it a distance: $d[y] = d[x] + 1$
assign it a predecessor: $p[y] = x$
add y to end of Q .

typically we check adj. vertices in some sensible order, like increasing-

ex



Find shortest path from $s=1$ to $t=4$.

$$d = [\infty, 0, \infty, \infty, \infty]$$

$$p = [\text{NULL}, N, N, N, N]$$

$$Q = [1]$$

1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13. 14. 15. 16. 17. 18. 19. 20. 21. 22. 23. 24. 25. 26. 27. 28. 29. 30. 31. 32. 33. 34. 35. 36. 37. 38. 39. 40. 41. 42. 43. 44. 45. 46. 47. 48. 49. 50. 51. 52. 53. 54. 55. 56. 57. 58. 59. 60. 61. 62. 63. 64. 65. 66. 67. 68. 69. 70. 71. 72. 73. 74. 75. 76. 77. 78. 79. 80. 81. 82. 83. 84. 85. 86. 87. 88. 89. 90. 91. 92. 93. 94. 95. 96. 97. 98. 99. 100. 101. 102. 103. 104. 105. 106. 107. 108. 109. 110. 111. 112. 113. 114. 115. 116. 117. 118. 119. 120. 121. 122. 123. 124. 125. 126. 127. 128. 129. 130. 131. 132. 133. 134. 135. 136. 137. 138. 139. 140. 141. 142. 143. 144. 145. 146. 147. 148. 149. 150. 151. 152. 153. 154. 155. 156. 157. 158. 159. 160. 161. 162. 163. 164. 165. 166. 167. 168. 169. 170. 171. 172. 173. 174. 175. 176. 177. 178. 179. 180. 181. 182. 183. 184. 185. 186. 187. 188. 189. 190. 191. 192. 193. 194. 195. 196. 197. 198. 199. 200. 201. 202. 203. 204. 205. 206. 207. 208. 209. 210. 211. 212. 213. 214. 215. 216. 217. 218. 219. 220. 221. 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1018. 1019. 1020. 1021. 1022. 1023. 1024. 1025. 1026. 1027. 1028. 1029. 1030. 1031. 1032. 1033. 1034. 1035. 1036. 1037. 1038. 1039. 1040. 1041. 1042. 1043. 1044. 1045. 1046. 1047. 1048. 1049. 1050. 1051. 1052. 1053. 1054. 1055. 1056. 1057. 1058. 1059. 1060. 1061. 1062. 1063. 1064. 1065. 1066. 1067. 1068. 1069. 1070. 1071. 1072. 1073. 1074. 1075. 1076. 1077. 1078. 1079. 1080. 1081. 1082. 1083. 1084. 1085. 1086. 1087. 1088. 1089. 1090. 1091. 1092. 1093. 1094. 1095. 1096. 1097. 1098. 1099. 1100. 1101. 1102. 1103. 1104. 1105. 1106. 1107. 1108. 1109. 1110. 1111. 1112. 1113. 1114. 1115. 1116. 1117. 1118. 1119. 1120. 1121. 1122. 1123. 1124. 1125. 1126. 1127. 1128. 1129. 1130. 1131. 1132. 1133. 1134. 1135. 1136. 1137. 1138. 1139. 1140. 1141. 1142. 1143. 1144. 1145. 1146. 1147. 1148. 1149. 1150. 1151. 1152. 1153. 1154. 1155. 1156. 1157. 1158. 1159. 1160. 1161. 1162. 1163. 1164. 1165. 1166. 1167. 1168. 1169. 1170. 1171. 1172. 1173. 1174. 1175. 1176. 1177. 1178. 1179. 1180. 1181. 1182. 1183. 1184. 1185. 1186. 1187. 1188. 1189. 1190. 1191. 1192. 1193. 1194. 1195. 1196. 1197. 1198. 1199. 1200. 1201. 1202. 1203. 1204. 1205. 1206. 1207. 1208. 1209. 1210. 1211. 1212. 1213. 1214. 1215. 1216. 1217. 1218. 1219. 1220. 1221. 1222. 1223. 1224. 1225. 1226. 1227. 1228. 1229. 1230. 1231. 1232. 1233. 1234. 1235. 1236. 1237. 1238. 1239. 1230. 1231. 1232. 1233. 1234. 1235. 1236. 1237. 1238. 1239. 1240. 1241. 1242. 1243. 1244. 1245. 1246. 1247. 1248. 1249. 1240. 1241. 1242. 1243. 1244. 1245. 1246. 1247. 1248. 1249. 1250. 1251. 1252. 1253. 1254. 1255. 1256. 1257. 1258. 1259. 1250. 1251. 1252. 1253. 1254. 1255. 1256. 1257. 1258. 1259. 1260. 1261. 1262. 1263. 1264. 1265. 1266. 1267. 1268. 1269. 1260. 1261. 1262. 1263. 1264. 1265. 1266. 1267. 1268. 1269. 1270. 1271. 1272. 1273. 1274. 1275. 1276. 1277. 1278. 1279. 1270. 1271. 1272. 1273. 1274. 1275. 1276. 1277. 1278. 1279. 1280. 1281. 1282. 1283. 1284. 1285. 1286. 1287. 1288. 1289. 1280. 1281. 1282. 1283. 1284. 1285. 1286. 1287. 1288. 1289. 1290. 1291. 1292. 1293. 1294. 1295. 1296. 1297. 1298. 1299. 1300. 1301. 1302. 1303. 1304. 1305. 1306. 1307. 1308. 1309. 1300. 1301. 1302. 1303. 1304. 1305. 1306. 1307. 1308.

$Q = [1]$

then $x = Q.\text{dequeue} = 1$. Well, $1 \neq t$ so visit all adj. vertices!

We'll do 0, 2, 3. We assign $d[0] = d[2] = d[3] = d[1] + 1 = 0 + 1 = 1$

$$p[0] = p[2] = p[3] = 1$$

so now $d = [1, 0, 1, 1, \infty]$

$$p = [1, \text{NULL}, 1, 1, \text{NULL}]$$

$$Q = [0, 2, 3]$$

wl dist = ∞

then $x = Q.\text{dequeue} = 0$. Well $0 \neq t$ so visit all adj. vertices!

there are none! b/c it's adj. to 1, 2 but $d[1], d[2] \neq \infty$!

so now d, p unchanged but $Q = [2, 3]$

wl dist = ∞

then $x = Q.\text{dequeue} = 2$. Well $2 \neq t$ so visit all adj. vertices

We'll do 4. We assign $d[4] = d[2] + 1 = 1 + 1 = 2$

$$p[4] = 2$$

so now $d = [1, 0, 1, 1, 2]$

$$p = [1, \text{NULL}, 1, 1, 2]$$

$$Q = [3, 4]$$

Note: b/c we have found 4 adj. to 2 and we've assigned d, p we can stop now and we're done. (We don't actually need to wait to dequeue 4.)

For the path, look at $p = [1, \text{NULL}, 1, 1, 2]$

We start at $t = 4$: $p[4] = 2$

$$p[2] = 1 \quad \text{and} \quad p[1] = \text{NULL}$$

so the path backwards is 4, 2, 1.

④ Pseudocode:

```

function shortestpath(G, s, t)
    dist = distance array of size V full of inf
    pred = predecessor array of size V full of NULL
    Q = empty queue
    dist[s] = 0
    Q.push(s)
    while Q is nonempty
        x = Q.pop
        for each infinity vertex y adjacent to x
            dist[y] = dist[x] + 1
            pred[y] = x
            if y == t
                return(pred)
            end
            Q.push(y)
        end
    end

```

$\Theta(V)$ each / together
 $\Theta(V)$ times \rightarrow WC $\Theta(V)$

WORST-CASE:

- ⊗ If G is stored as an adj matrix then this loop iterates exactly V times b/c we must scan row x and check each vertex.
In which case we'd have $T(V) = \Theta(V^2)$

BETTER: Store G as an adj. list.

Then we get the argument from Wednesday:

- Initialization: $\Theta(V)$
- While loop body w/out for loop runs $\Theta(V)$ times at $\Theta(1)$ each so $\Theta(V)$
- For loop body runs twice for each edge at $\Theta(1)$ each time, so $\Theta(2E)$

In total $\Theta(V) + \Theta(V) + \Theta(2E)$ thus

$$T(V, E) = \Theta(V + E).$$

Addendum: For a simple unweighted, connected graph:

- $V \leq 2E$ b/c each edge conn. 2 vert, but vert are prob shared
- $E \leq C(V, 2)$ b/c at most every pair of vertices could be conn. by an edge.
 \uparrow
 $E = \frac{1}{2}V(V-1)$

Thus we can also say $T(E) = O(2E + E) = O(E)$

$$\text{and } T(V) = O(V + \frac{1}{2}V(V-1)) = O(V^2)$$

thus we can also say $T(E) = V(2E + E) = O(E)$
and $T(V) = O(V + \frac{1}{2}V(V-1)) = O(V^2)$