STAT420 Homework 6

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11/21: HW 6 (due before 11:00 p.m.)

7.1(7); 7.2(1,4,7,8); 7.3(3,6); 7.4(3,7,9); 7.5(2,10,12); 7.6(2,6,9); 7.7(1,6)

7.1.7. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a distribution that is $N(\mu, \theta)$, $0 < \theta < \infty$, where μ is unknown. Let $Y = \sum_1^n (X_i - \overline{X})^2/n$ and let $\mathcal{L}[\theta, \delta(y)] = [\theta - \delta(y)]^2$. If we consider decision functions of the form $\delta(y) = by$, where b does not depend upon y, show that $R(\theta, \delta) = (\theta^2/n^2)[(n^2-1)b^2 - 2n(n-1)b + n^2]$. Show that $b = n/(n+1)$ yields a minimum risk decision function of this form. Note that $nY/(n+1)$ is not an unbiased estimator of θ. With $\delta(y) = ny/(n+1)$ and $0 < \theta < \infty$, determine $\max_\theta R(\theta, \delta)$ if it exists.

Solution. First let us show that the risk function

$$R(\theta, \delta) = (\theta^2/n^2)[(n^2 - 1)b^2 - 2n(n-1)b + n^2]$$

Recall that the risk function is defined as the expectation of $\mathcal{L}$ the loss function.

$$R(\theta, \delta) = E(\mathcal{L}[\theta - \delta(y)]) = E([\theta - \delta(y)]^2) = E([\theta - by]^2)$$

where in this case the decision function is of the form $\delta(y) = by$. Then we compute

$$R(\theta, \delta) = E(\theta^2 - 2\theta by + b^2y^2) = \theta^2 - 2\theta bE(y) + b^2E(y^2)$$

Now we must compute $E(Y)$ and $E(Y^2) = Var(Y) - E(Y)^2$.

$$E(Y) = \frac{1}{n}\sum_1^n E(X_i - \bar{X})^2 = \frac{n-1}{n}\theta$$

by properties of variance. Similarly, we compute

$$Var(Y) = \frac{2\theta^2(n-1)}{n} \implies E(Y^2) = \frac{2\theta^2(n-1)}{n} + \frac{\theta^2(n-1)^2}{n^2}$$

Now we obtain

$$\begin{aligned} R(\theta, \delta) &= \theta^2 - \frac{2\theta^2 b(n-1)}{n} + b^2\left(\frac{2\theta^2(n-1)}{n} + \frac{\theta^2(n-1)^2}{n^2}\right) \\ &= (\theta^2/n^2)[1 - 2bn(n-1) + b^2(2n(n-1) + (n-1)^2)] \\ &= (\theta^2/n^2)[(n^2-1)b^2 - 2n(n-1)b + n^2] \end{aligned}$$

as desired. Now we will show that $b = \frac{n}{n+1}$ yields a minimum risk decision function. Taking the derivative of the risk function with respect to b,

$$R' = (\theta^2/n^2)[2b(n^2-1) - 2n(n-1)]$$

Setting this to 0, we obtain

$$b = \frac{2n(n-1)}{2(n^2-1)} = \frac{n(n-1)}{(n+1)(n-1)} = \frac{n}{n+1}$$

as desired.
Finally, we will show that with the given restrictions $\delta(y) = ny/(n+1)$ and $0 < \theta < \infty$ that

$$R(\theta, \delta) = E([\theta - \frac{ny}{n+1}]^2) = E(\theta^2 - 2\theta\frac{ny}{n+1} + (\frac{ny}{n+1})^2)$$
$$= \theta^2 - 2\theta\frac{n}{n+1}E(Y) + (\frac{n}{n+1})^2E(Y)^2$$
$$= \theta^2 - 2\theta\frac{n}{n+1}\frac{n-1}{n}\theta + (\frac{n}{n+1})^2(\frac{2\theta^2(n-1)}{n} + \frac{\theta^2(n-1)^2}{n^2})$$
$$= \theta^2 - 2\theta^2\frac{n-1}{n+1} + \frac{\theta^2(n-1)}{n+1}(2 + \frac{n-1}{n})$$
$$= \theta^2 - \theta^2\frac{n-1}{n+1}(\frac{n-1}{n})$$
$$R' = 2\theta - 2\theta\frac{(n-1)^2}{n(n+1)}$$

which is strictly increasing for $0 < \theta < \infty \implies \nexists \max_\theta R(\theta, \delta)$ when $\delta(y) = \frac{ny}{n+1}$. □

7.2.1. Let $X_1, X_2, \ldots, X_n$ be iid $N(0, \theta)$, $0 < \theta < \infty$. Show that $\sum_1^n X_i^2$ is a sufficient statistic for θ.

Solution. The joint pdf of the sample from a normal distribution can be written as

$$\prod_{i=1}^n f(x_i;\theta) = (\frac{1}{\sqrt{2\pi\theta}})^n \exp(-\frac{\sum_{i=1}^n(x_i-\mu)^2}{2\theta})$$

Note that it is known that $\mu = 0 \implies$

$$\prod_{i=1}^n f(x_i;\theta) = (\frac{1}{\sqrt{2\pi\theta}})^n \exp(-\frac{t}{2\theta})$$

where

$$k_1(t,\theta) = (\frac{1}{\sqrt{2\pi\theta}})^n \exp(-\frac{t}{2\theta})$$
$$t = \sum_{i=1}^\infty x_i^2$$
$$k_2(x_1, \cdots, x_n) = 1$$

This proves that $T = \sum_{i=1}^n X_i^2$ is sufficient for θ. □

7.2.4. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a geometric distribution that has pmf $f(x;\theta) = (1-\theta)^x\theta$, $x = 0, 1, 2, \ldots$, $0 < \theta < 1$, zero elsewhere. Show that $\sum_1^n X_i$ is a sufficient statistic for θ.

Solution. Note that a sample of geometric distribution has a joint pmf of

$$f(x_i;\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i} = \theta^n (1-\theta)^{\sum i=1^n x_i}$$

$$= k_1(t,\theta)k_2(x_1,\cdots,x_n)$$

where

$$k_1(t,\theta) = \theta^n(1-\theta)^t$$

$$t = \sum_{i=1}^{n} x_i$$

$$k_2(x_1,\cdots,x_n) = 1$$

This proves that $T = \sum_{i=1}^{n} X_i$ is sufficient for θ. □

7.2.7. Show that the product of the sample observations is a sufficient statistic for $\theta > 0$ if the random sample is taken from a gamma distribution with parameters $\alpha = \theta$ and $\beta = 6$.

Solution. We seek to show that $\prod_{i=1}^{n} x_i$ is sufficient for $\theta > 0$. Recall that the pdf of a gamma distribution is

$$f(x;\theta,\beta) = \frac{\beta^\theta x^{\theta-1} e^{-\beta x}}{\Gamma(\theta)}$$

where the gamma function is defined as

$$\Gamma(\theta) = \int_0^\infty x^{\theta-1}e^{-x}\, dx$$

The joint pdf is

$$\prod_{i=1}^{n} f(x;\theta) = \prod_{i=1}^{n} \frac{6^\theta x_i^{\theta-1} e^{-6x_i}}{\Gamma(\theta)} = (\frac{6^\theta}{\Gamma(\theta)})^n (\prod_{i=1}^{n} x_i^{\theta-1})(e^{-6\sum_{i=1}^{n} x_i})$$

$$= k_1(t,\theta)k_2(x_1,\cdots,x_n)$$

where

$$k_1(t,\theta) = (\frac{6^\theta}{\Gamma(\theta)})^n (t^{\theta-1})$$

$$t = \prod_{i=1}^{n} x_i$$

$$k_2(x_1,\cdots,x_n) = (e^{-6\sum_{i=1}^{n} x_i})$$

This proves that $T = \prod_{i=1}^{n} X_i$ is sufficient for θ. □

7.2.8. What is the sufficient statistic for θ if the sample arises from a beta distribution in which $\alpha = \beta = \theta > 0$?

Solution. Note that the pdf of this beta distribution is

$$f(x;\theta) = \frac{x^{\theta-1}(1-x)^{\theta-1}}{B(\theta)} = \frac{(x-x^2)^{\theta-1}}{B(\theta)}$$

where $B(\theta,\theta) = \int_0^1 x^{\theta-1}(1-x)^{\theta-1}\,dx$. Thus the joint pdf can be written as

$$\prod_{i=1}^{n} f(x_i;\theta) = \frac{\prod_{i=1}^{n}(x_i - x_i^2)^{\theta-1}}{B(\theta,\theta)} = k_1(t,\theta)k_2(x_1,\cdots,x_n)$$

where

$$k_1(t,\theta) = \frac{t^{\theta-1}}{B(\theta,\theta)^n}$$

$$t = \prod_{i=1}^{n}(x_i - x_i^2)$$

$$k_2(x_1,\cdots,x_n) = 1$$

and this shows that $T = \prod_{i=1}(X_i - X_i^2) = \prod_{i=1}^{n} X_i(1-X_i)$ is sufficient for θ. □

7.3.3. If X_1, X_2 is a random sample of size 2 from a distribution having pdf $f(x;\theta) = (1/\theta)e^{-x/\theta}$, $0 < x < \infty$, $0 < \theta < \infty$, zero elsewhere, find the joint pdf of the sufficient statistic $Y_1 = X_1 + X_2$ for θ and $Y_2 = X_2$. Show that Y_2 is an unbiased estimator of θ with variance θ^2. Find $E(Y_2|y_1) = \varphi(y_1)$ and the variance of $\varphi(Y_1)$.

Solution. First we write the joint pdf of X_1, X_2 as

$$f_{X_1,X_2}(x_1,x_2) = (\frac{1}{\theta^2})e^{-(x_1+x_2)/\theta}, 0 < x_1 < \infty, 0 < x_2 < \infty$$

Next we find the inverse functions and the subsequent Jacobian to complete the transformation.

$$x_1 = y_1 - y_2, x_2 = y_2 \implies \det(J) = 1$$

$$f_{Y_1,Y_2}(y_1,y_2) = (\frac{1}{\theta^2})\exp(-y_1/\theta), 0 < y_2 < y_1 < \infty$$

Note the new bounds ensure $y_1 > y_2$ to ensure that $x_1 > \infty$. Now we will show that Y_2 is an unbiased estimator of θ, which means we will show that $E(Y_2) = \theta$. We find the marginal

$$f_{Y_2}(y_2) = \frac{1}{\theta^2}\int_{y_2}^{\infty}\exp(-\frac{y_1}{\theta})dy_1 = -\frac{1}{\theta}\int_{y_2}^{\infty} -\frac{1}{\theta}\exp(-\frac{y_1}{\theta})dy_1$$

$$= -\frac{1}{\theta}(\exp(-y_1/\theta)|_{y_2}^{\infty} = -\frac{1}{\theta}(0 - \exp(-y_2/\theta) = \frac{1}{\theta}\exp(-\frac{y_2}{\theta})$$

$$E(Y_2) = \int_0^{\infty} -y_2(-\frac{1}{\theta})\exp(-\frac{y_2}{\theta})dy_2$$

$u = -y_2 \implies du = -dy_2$ and $v = \exp(-y_2/\theta)$ and so

$$E(Y_2) = (-y_2 \exp(-y_2/\theta)|_0^\infty - \int_0^\infty \exp(-y_2/\theta)\ dy_2 = \theta$$

and thus Y_2 is unbiased as desired. Upon closer inspection, one could have noticed that $Y_2 \sim \text{Exponential}(\frac{1}{\theta})$ which implies

$$E(Y_2) = \theta, \text{Var}(Y_2) = \theta^2$$

Finally, we find $E(Y_2|y_1) = \varphi(y_1)$. Note that conditional expectation is

$$E(Y_2|Y_1 = y_1) = \int y_2 \frac{f_{Y_1,Y_2}(y_1, y_2)}{f_{Y_1}(y_1)}$$

We already have the joint pmf, let us find the marginal y_1 which is

$$f_{Y_1}(y_1) = \int_0^{y_1} (\frac{1}{\theta^2}) \exp(-y_1/\theta)\ dy_2 = (\frac{y_1}{\theta^2}) \exp(-y_1/\theta)$$

Now we can find

$$E(Y_2|y_1) = \int_0^{y_1} \frac{y_2}{y_1}\ dy_2 = \frac{1}{y_1}(\frac{1}{2}y_2^2)|_0^{y_1} = \frac{1}{y_1}(\frac{y_1^2}{2}) = \frac{y_1}{2}$$

Note that $Y_1 \sim \Gamma(2, \theta) \implies \text{Var}(Y_1) = 2\theta^2 \implies \text{Var}(\varphi(Y_1)) = \text{Var}(\frac{Y_1}{2}) = \frac{1}{4}\text{Var}(Y_1) = \frac{\theta^2}{2}$ by properties of Gamma distribution and variance. □

7.3.6. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with mean θ. Find the conditional expectation $E(X_1 + 2X_2 + 3X_3 \mid \sum_1^n X_i)$.

Solution. Let $Y = \sum_{i=1}^n X_i$. First observe that the sum of n Poisson(θ) random variables is a $Y \sim \text{Poisson}(n\theta)$ whose pmf is $p(x;\theta) = \frac{(n\theta)^x e^{-n\theta}}{x!}$. Expectation is a linear function and so

$$E(X_1 + 2X_2 + 3X_3|\Sigma_1^n X_i) = E(X_1|\Sigma_1^n X_i) + 2E(X_2|\Sigma_1^n X_i) + 3E(X_3|\Sigma_1^n X_i)$$

Note that since the sample is IID, we can simplify this to

$$E(X_1 + 2X_2 + 3X_3|Y = y) = 6E(X_1|Y = y)$$

Now let us evaluate this first term in the conditional expectation. Note that this integer k is bounded above by y.

$$E(X_1|Y = y) = \sum_{i=0}^{y} k \frac{P(X_1 = k, X_2 + \cdots + X_n = y - k)}{P(Y = y)}$$

Note that X_1 and $X_2, \cdots, X_n$ are all iid, and by definition of independence,

$$= \sum_{i=0}^{y} k \frac{P(X_1 = k)P(X_2 \cdots X_n) = y - k}{P(Y = y)}$$

$$= \sum_{i=0}^{y} k \frac{\frac{\theta^k e^{-\theta}}{k!} \frac{((n-1)\theta)^{y-k} e^{-(n-1)\theta}}{(y-k)!}}{\frac{(n\theta)^y e^{-n\theta}}{y!}} = (1-\frac{1}{n})^Y \sum_{i=0}^{y} k \binom{Y}{k} \frac{1}{(n-1)^Y}$$

Thus, we conclude that

$$E(X_1 + 2X_2 + 3X_3 | Y = y) = 6(1-\frac{1}{n})^Y \sum_{i=0}^{y} k \binom{Y}{k} \frac{1}{(n-1)^Y}$$

□

7.4.3. Let $X_1, X_2, \ldots, X_n$ represent a random sample from the discrete distribution having the pmf

$$f(x;\theta) = \begin{cases} \theta^x (1-\theta)^{1-x} & x = 0, 1, \ \ 0 < \theta < 1 \\ 0 & \text{elsewhere.} \end{cases}$$

Show that $Y_1 = \sum_1^n X_i$ is a complete sufficient statistic for θ. Find the unique function of Y_1 that is the MVUE of θ.
Hint: Display $E[u(Y_1)] = 0$, show that the constant term $u(0)$ is equal to zero, divide both members of the equation by $\theta \neq 0$, and repeat the argument.

Solution. First let us show that $Y_1 = \sum_{i=1}^n X_i$ is a sufficient statistic for θ. The joint pmf is

$$\prod_{i=1}^{n} f(x_i;\theta) = \theta^{\sum_{i=1}^n x_i} (1-\theta)^{n - \sum_{i=1}^n x_i} = k_1(y_1, \theta) k_2(x_1, \cdots, x_n)$$

where

$$k_1(y_1, \theta) = \theta^{y_1} (1-\theta)^{n-y_1}$$
$$y_1 = \sum_{i=1}^{n} x_i$$
$$k_2(x_1, \cdots, x_n) = 1$$

This proves that $Y_1 = \sum_{i=1}^n X_i$ is a sufficient statistic for θ, and now we will show that it is complete.
If $Y_1 = \sum X_i$ is a complete and sufficient statistic then $E[u(Y_1)] = 0$.

$$E[u(Y_1)] = u(0)(\theta^0)(1-\theta)^{1-0} + u(1)(\theta^1)(1-\theta)^{1-1} = 0$$
$$E[u(Y_1)] = u(0)(1-\theta) + u(1)\theta = 0$$
$$E[u(Y_1)] = u(0) - \theta u(0) + u(1)\theta = 0$$

$$u(0) = \theta(u(0) - u(1)) \implies \frac{u(0)}{\theta} = u(0) - u(1) \implies u(0) = u(1) = 0 \ \forall \ \theta \in (0, 1)$$

and so $Y_1 = \sum_1^n$ is a complete sufficient statistic for θ. By the Lehmann and Scheffe Theorem, the function of the complete and sufficient statistic Y_1 that is an unbiased estimator of θ is the unique MVUE of θ.

$$E(Y_1) = n\theta \implies \frac{1}{n} Y_1 = \bar{X} = \text{MVUE}$$

□

7.4.7. Let X have the pdf $f_X(x;\theta) = 1/(2\theta)$, for $-\theta < x < \theta$, zero elsewhere, where $\theta > 0$.

(a) Is the statistic $Y = |X|$ a sufficient statistic for θ? Why?

(b) Let $f_Y(y;\theta)$ be the pdf of Y. Is the family $\{f_Y(y;\theta) : \theta > 0\}$ complete? Why?

Solution.

a. Let us proceed by finding the joint pdf and then seeing if $|X|$ is a sufficient statistic by using the factorization theorem. Let us also declare an indictator function such that

$$I(x) = \begin{cases} 1, & |x| < \theta \\ 0, & \text{else} \end{cases}$$

$$\prod_{i=1}^{n} f(x_i;\theta) = (\frac{1}{2\theta})^n \prod_{i=1}^{n} I(|x_i| < \theta) = k_1(t,\theta)k_2(x_1,\cdots,x_n)$$

where

$$k_1(t,n) = (\frac{1}{2\theta})^n \prod_{i=1}^{n} I(t < \theta),\ t = |x_i| = |X|,\ k_2(x_1,\cdots,x_n) = 1$$

and this proves that $T = |X|$ is a sufficient statistic for θ.

b. Now let us show that the family $\{f_Y(y;\theta) : \theta > 0\}$ is complete.

$$f_Y(y;\theta) = P(Y \le y) = P(|X| \le y) = P(-y \le X \le y) = \frac{1}{\theta}, 0 < y < \theta$$

and zero elsewhere. Suppose $E[u(Y)] = 0$, then

$$E[u(Y)] = \int_0^\theta \frac{u(y)}{\theta}\, dy = \frac{1}{\theta}(\int u(y))|_0^\theta = 0 \implies u(y) = 0, 0 < y < \theta$$

and so yes, the family of Y is a complete family.

□

7.4.9. Let $X_1,\ldots,X_n$ be iid with pdf $f(x;\theta) = 1/(3\theta)$, $-\theta < x < 2\theta$, zero elsewhere, where $\theta > 0$.

(a) Find the mle $\widehat{\theta}$ of θ.

(b) Is $\widehat{\theta}$ a sufficient statistic for θ? Why?

(c) Is $(n+1)\widehat{\theta}/n$ the unique MVUE of θ? Why?

Solution.

a. We define an indicator function

$$I(x) = \begin{cases} 1, & -\theta < x < 2\theta \\ 0, & \text{else} \end{cases}$$

The likelihood function is

$$L(\theta; x) = \prod_{i=1}^{n} f(x_i; \theta) = \prod_{i=1}^{n} \frac{1}{3\theta} I(-\theta < x_i < 2\theta) = (\frac{1}{3\theta})^n \prod_{i=1}^{n} I(-\theta < x_i < 2\theta)$$

$$= (\frac{1}{3\theta})^n \prod_{i=1}^{n} I(-\theta < y_i < y_n < 2\theta) = (\frac{1}{3\theta})^n \prod_{i=1}^{n} I(\theta > -y_1 \text{ and } \theta > \frac{y_n}{2})$$

Thus, $\hat{\theta} = \max(-Y_1, \frac{Y_n}{2})$.

b. Note that the likelihood function is the joint pdf. Note that we can rewrite the likelihood function using the factorization theorem.

$$L(\theta; x) = \prod_{i=1}^{n} f(x_i; \theta) = (\frac{1}{3\theta})^n I(\max(-Y_1, \frac{Y_2}{2})) = k_1(t, \theta) k_2(x_i, \cdots, x_n)$$

where

$$k_1(t, \theta) = (\frac{1}{3\theta})^n I(t),\ t = \max(-y_1, y_2/2),\ k_2(x_1, \cdots, x_n) = 1$$

and this proves that $\hat{\theta}$ is a sufficient statistic for θ.

c. $E(\frac{-Y_1(n+1)}{n}) \neq E(\frac{Y_n(n+1)}{2n}) \neq \theta$ and so by the Lehmann and Scheffe theorem, since the given function of $\hat{\theta}$ is not an unbiased estimator of θ, then it is not the unique MVUE.

$\square$

7.5.2. Let $X_1, X_2, \ldots, X_n$ denote a random sample of size $n > 1$ from a distribution with pdf $f(x; \theta) = \theta e^{-\theta x}$, $0 < x < \infty$, zero elsewhere, and $\theta > 0$. Then $Y = \sum_1^n X_i$ is a sufficient statistic for θ. Prove that $(n-1)/Y$ is the MVUE of θ.

Solution. Note that the sum of exponential distributions is a gamma distribution such that

$$Y = \sum_{i=1}^{n} X_i \sim \Gamma(n, \theta)$$

$$E(\frac{n-1}{Y}) = (n-1)E(\frac{1}{Y})$$

by linearity.

$$f_Y(y) = \frac{\theta^n}{\Gamma(n)} y^{n-1} \theta^{-y}$$

$$E(\frac{1}{Y}) = \int_0^\infty \frac{\theta^n}{\Gamma(n)} y^{n-2} \theta^{-y} = \frac{\theta^n(\Gamma(n-1))}{\theta^{n-1}(\Gamma(n))}$$

Note that $\Gamma(n) = (n-1)\Gamma(n-1)$. Thus,

$$E(\frac{1}{Y}) = \frac{\theta}{n-1} \implies (n-1)E(\frac{1}{Y}) = \theta$$

which proves that since $\frac{n-1}{Y}$ is an unbiased estimator, by the Lehmann and Scheffe Theorem, it is the unique MVUE. □

7.5.10. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pdf $f(x;\theta) = \theta^2 x e^{-\theta x}$, $0 < x < \infty$, where $\theta > 0$.

(a) Argue that $Y = \sum_1^n X_i$ is a complete sufficient statistic for θ.

(b) Compute $E(1/Y)$ and find the function of Y that is the unique MVUE of θ.

Solution.

a. The pdf can be written as

$$f(x;\theta) = \exp(-\theta x + \ln x + \ln \theta^2)$$

Therefore, the joint pdf can be written as

$$\prod_{i=1}^n f(x_i;\theta) = \exp(-\theta \sum_{i=1}^n x_i + \sum_{i=1}^n \ln x_i + n \ln \theta^2)$$

It can be showed using the factorization theorem that $Y = \sum_{i=1}^n x_i$ is a sufficient statistic. Further, note that this pdf is in the exponential family, and therefore, Y is a complete and sufficient statistic for θ.

b. Note that $X \sim \Gamma(2,\theta) \implies Y \sim \Gamma(2n,\theta)$.

$$E(Y) = \int_0^\infty \frac{\theta^n}{\Gamma(2n)} y^{2n-2} e^{-y}\, dy = \frac{\theta^{2n}(\Gamma(2n-1)}{\theta^{2n}\Gamma(2n)} = \frac{\theta}{2n-1}$$

which therefore implies that $(2n-1)Y_1$ is the unique MVUE of θ.

□

7.5.12. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with pmf $p(x;\theta) = \theta^x(1-\theta)$, $x = 0, 1, 2, \ldots$, zero elsewhere, where $0 \le \theta \le 1$.

(a) Find the mle, $\hat{\theta}$, of θ.

(b) Show that $\sum_1^n X_i$ is a complete sufficient statistic for θ.

(c) Determine the MVUE of θ.

Solution.

a. The log likelihood function is

$$l(\theta; x) = \sum_{i=1}^{n} x_i \log(\theta) + \log(1-\theta) = n\log(\theta)\sum_{i=1}^{n} x_i + n\log(1-\theta)$$

$$l'(\theta; x) = \frac{n\sum_{i=1}^{n} x_i}{\theta} - \frac{n}{1-\theta} = 0$$

$$\theta = (1-\theta)\sum_{i=1}^{n} x_i \implies \theta = \sum_{i=1}^{n} x_i - \theta\sum_{i=1}^{n} x_i \implies \theta + \theta\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i$$

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{1 + \sum_{i=1}^{n} x_i}$$

is the MLE of θ.

b. Note that the above pmf is in the exponential family and we can rewrite it as

$$p(x; \theta) = \exp(\log(\theta)x + \log(1-\theta))$$

where $p(\theta) = \log(\theta), K(x) = x, H(x) = 0, q(\theta) = \log(1-\theta)$ and the support $\mathcal{S}$ does not depend on θ. Accordingly, it can be rewritten using the factorization theorem to show that $\sum_{i=1}^{n} K(x_i)$ is a complete sufficient statistic. Therefore, since $K(x) = x$, we obtain $\sum_{i=1}^{n} X_i$ is a complete sufficient statistic for θ.

c. Note that if

$$q'(\theta) = \frac{1}{\theta - 1}, p'(\theta) = \frac{1}{\theta}$$

$$E(Y) = \frac{q'(\theta)}{p'(\theta)} = \frac{n\theta}{\theta - 1}$$

$$\frac{\frac{n\theta}{\theta-1}}{\frac{n\theta}{\theta-1} - 1} = \frac{n\theta(\theta-1)}{(n\theta - \theta + 1)(\theta - 1)} = \theta$$

Therefore, MLE $\hat{\theta}$ is the MVUE of θ as well.

□

7.6.2. Let $X_1, X_2, \ldots, X_n$ denote a random sample from a distribution that is $N(0, \theta)$. Then $Y = \sum X_i^2$ is a complete sufficient statistic for θ. Find the MVUE of θ^2.

Proof. The pdf of $N(0, \theta)$ is

$$\frac{1}{\sqrt{2\pi\theta}} \exp(-\frac{x^2}{2\theta})$$

which we can rewrite in the desired format of the exponential family as

$$N(0, \theta) = \exp(-\frac{1}{2\theta}x^2 - \log(\sqrt{2\pi\theta})$$

Observe $K(x) = x^2 \implies \sum_{i=1}^n K(x_i) = \sum_{i=1}^n X_i^2$ is a complete sufficient statistic for θ. Now we will find the MVUE of θ^2. Note that $X/\sqrt{\theta} \sim N(0,1) \implies \sum_{i=1}^n X_i^2/\theta = Y/\theta \sim \chi^2(n)$

$$E(Y/\theta) = n \implies E(Y) = n\theta$$

$$Var(Y/\theta) = 2n \implies Var(Y) = 2n\theta^2$$

$$E(Y^2) = Var(Y) + E(Y)^2 = 2n\theta^2 + n^2\theta^2 = (n^2+2n)\theta^2$$

and so therefore

$$\frac{Y}{n^2+2n} \text{ is the unique MVUE of } \theta^2$$

□

7.6.6. Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with parameter $\theta > 0$.

(a) Find the MVUE of $P(X \le 1) = (1+\theta)e^{-\theta}$.

Hint: Let $u(x_1) = 1$, $x_1 \le 1$, zero elsewhere, and find $E[u(X_1)|Y = y]$, where $Y = \sum_1^n X_i$.

(b) Express the MVUE as a function of the mle of θ.

(c) Determine the asymptotic distribution of the mle of θ.

(d) Obtain the mle of $P(X \le 1)$. Then use Theorem 5.2.9 to determine its asymptotic distribution.

Proof.

a. First note that

$$P(X \le 1) = P(X=0) + P(X=1) = e^{-\theta} + \theta e^{-\theta} = (1+\theta)e^{-\theta}$$

Let us define an indicator function

$$I(x) = \begin{cases} 1, & x \le 1 \\ 0, & x > 1 \end{cases}$$

Thus, we can define some distribution S that is

$$S \sim \begin{cases} (1+\theta)e^{-\theta}, & I \\ 1-(1+\theta)e^{-\theta}, & \text{else} \end{cases}$$

Thus, we can see that S is unbiased for $(1+\theta)e^{-\theta}$. Recall that by the Rao and Blackwell theorems and Lehmann and Scheffe theorems that since $Y = \sum_{i=1}^n X_i$ is complete sufficient statistic, then we can seek a function of Y that is the MVUE of $(1+\theta)e^{-\theta}$

$$E(X_1 \leq 1|Y = y) = P(X_1 = 0|Y = y) + P(X_1 = 1|Y - y)$$

From here, we can use the same closed form formula that we found in Exercise 7.3.6.

$$= (\frac{n-1}{n})^y + \frac{Y}{n-1}(\frac{n-1}{n})^Y$$
$$= (1 + \frac{Y}{n-1})(\frac{n-1}{n})^Y$$

b. Let us find the MLE $\hat{\theta}$ by taking the likelihood function.

$$L(\theta; x) = \prod_{i=1}^n \frac{\theta^{x_i}e^{-\theta}}{x_i!}$$

$$l(\theta; x) = -n\theta + (\log(\theta)\sum_{i=1}^n x_i) - \log(\prod_{i=1}^n x_i!)$$

$$l'(\theta; x) = -n + \frac{1}{\theta}\sum_{i=1}^n x_i = 0 \implies \frac{1}{\theta} = \frac{n}{\sum_{i=1}^n x_i} \implies \hat{\theta} = \bar{X} = \frac{Y}{n}$$

Now we can rewrite the MVUE of $(1+\theta)e^{-\theta}$ in terms of $\bar{X} = \frac{Y}{n}$ and we get

$$(1 + \frac{\bar{X}n}{n-1})(\frac{n-1}{n})^{\bar{X}n}$$

c. Note that

$$E(X) = Var(X) = \theta$$

by properties of Poisson and by CLT

$$\frac{\bar{X}-\theta}{\sqrt{n}} \xrightarrow{D} N(0,\theta) \implies \bar{X} = \hat{\theta} \sim N(\theta, \frac{\theta}{n})$$

d. Recall that from part a) that

$$P(X \leq 1) = P(X = 0) + P(X = 1) = e^{-\theta} + \theta e^{-\theta} = (1+\theta)e^{-\theta}$$

and so the MLE of $P(X \leq 1) = (1+\hat{\theta})e^{-\theta} = (1+\bar{X})e^{-\bar{X}}$ Using Theorem 5.2.9 (the $\Delta-$method), we let $g(x) = (1+x)e^{-x} \implies g'(x) = -xe^{-x}$

$$\frac{P(X\hat{\leq} 1) - (1+\theta)e^{-\theta}}{\sqrt{n}} \vec{D} N(0, (g'(\theta))^2\theta) = N(0, \theta^3 e^{-2\theta}$$

$$P(X\hat{\leq} 1) = N((1+\theta)e^{-\theta}, \theta^3 e^{-2\theta}$$

□

7.6.9. Let a random sample of size n be taken from a distribution that has the pdf $f(x;\theta) = (1/\theta)\exp(-x/\theta)I_{(0,\infty)}(x)$. Find the mle and MVUE of $P(X \le 2)$.

Proof.
Note that the above pdf is Exponential($\frac{1}{\theta}$) whose cdf is defined as $F(x) = 1 - e^{-x\lambda}$. Now, let us find the likelihood

$$L(\theta; x) = \prod_{i=1}^{n} (\frac{1}{\theta}) e^{-x_i/\theta}$$

$$l(\theta; x) = n \log(\frac{1}{\theta}) - \frac{1}{\theta}\sum_{i=1}^{n} x_i$$

$$l'(\theta; x) = n\theta + \frac{\sum_{i=1}^{n} x_i}{\theta^2} = 0 \implies \frac{1}{\theta} = \frac{1}{\bar{X}} \implies \hat{\theta} = \bar{X}$$

Therefore, the MLE of $P(X \le 2)$ is

$$P(X \hat{\le} 2) = 1 - e^{-2/\bar{X}}$$

To find the MVUE, let us proceed with the conditional expectation method. We seek

$$E(X_1 \le 2 | Y = y) = \int_0^2 f_{X_1|Y}(x_1|y) = \frac{f_{X_1,Y}(x_1, y)}{f_Y(y)}$$

Note that $X \sim \Gamma(1, \frac{1}{\theta}) \implies Y \sim \Gamma(n, \frac{1}{\theta}) \implies f_Y(y) = \frac{y^{n-1}e^{-t/\theta}}{(n-1)!\theta^n}$

$$f_{X_1,Y}(x_1, y) = \frac{\partial^2}{\partial X_1 \partial Y} F_{X_1,Y}(x_1, y)$$

$$F_{X_1,Y}(x_1, y) = F_Y(y) - P(x_1 < X_1 < Y \le y$$

$$\cdots f_{X_1,Y}(x_1, y) = \frac{1}{\theta^n (n-2)!}(y - x_1)^n - 2e^{-t/\theta}, x_1 < y$$

$$E(X_1 \le 2 | Y = y) = 1 - (1 - \frac{2}{Y})^{n-1}$$

is the MVUE. □

7.7.1. Let $Y_1 < Y_2 < Y_3$ be the order statistics of a random sample of size 3 from the distribution with pdf

$$f(x; \theta_1, \theta_2) = \begin{cases} \frac{1}{\theta_2}\exp\left(-\frac{x-\theta_1}{\theta_2}\right) & \theta_1 < x < \infty, \ -\infty < \theta_1 < \infty, \ 0 < \theta_2 < \infty \\ 0 & \text{elsewhere.} \end{cases}$$

Find the joint pdf of $Z_1 = Y_1, Z_2 = Y_2$, and $Z_3 = Y_1 + Y_2 + Y_3$. The corresponding transformation maps the space $\{(y_1, y_2, y_3) : \theta_1 < y_1 < y_2 < y_3 < \infty\}$ onto the space

$$\{(z_1, z_2, z_3) : \theta_1 < z_1 < z_2 < (z_3 - z_1)/2 < \infty\}.$$

Show that Z_1 and Z_3 are joint sufficient statistics for θ_1 and θ_2.

Proof.
By the definition of the pdf of joint order statistics,

$$f_{Y_1,Y_2,Y_3}(y_1,y_2,y_3) = 3!f(y_1)f(y_2)f(y_3)$$
$$= \frac{6}{\theta_2^3}\exp(-\frac{y_1+y_2+y_3-3\theta_1}{\theta_2})$$

Now we find the desired joint pdf given that $Z_1 = Y_1, Z_2 = Y_2, Z_3 = Y_1 + Y_2 + Y_3$.

$$f_{Z_1,Z_2,Z_3}(z_1,z_2,z_3) = f_{Y_1,Y_2,Y_3}y_1,y_2,y_3|\det J^{-1}|\cdot I_{(\theta_1,\infty)}(z_1)I_{z_1<z_2<(z_3-z_1)/2}$$

$$J = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \implies \det J = 1 \implies |\det J^{-1}| = 1$$

We now find the inverse of the given functions, so

$$Y_1 = Z_1,\ Y_2 = Z_2,\ Y_3 = Z_3 - Z_1 - Z_2$$

Therefore,

$$f_{Z_1,Z_2,Z_3}(z_1,z_2,z_3) = \frac{6}{\theta_2^3}\exp(-\frac{z_1+z_2+z_3-z_1-z_2-3\theta_1}{\theta_2})\cdot I_{(\theta_1,\infty)}(z_1)I_{z_1<z_2<(z_3-z_1)/2}$$
$$= \frac{6}{\theta_2^3}\exp(-\frac{z_3-3\theta_1}{\theta_2})\cdot I_{(\theta_1,\infty)}(z_1)I_{z_1<z_2<(z_3-z_1)/2}$$
$$= k_1(t,\theta_1,\theta_2)k_2(z_1,z_2,z_3)$$

where

$$k_1(t,\theta_1,\theta_2) = \frac{6}{\theta_2^3}\exp(-\frac{z_3-3\theta_1}{\theta_2})\cdot I_{(\theta_1,\infty)}(z_1)$$
$$t = (z_1,z_3)$$
$$k_2(z_1,z_2,z_3) = I_{z_1<z_2<(z_3-z_1)/2}$$

This proves that $T = (Z_1, Z_3)$ is jointly sufficient for (θ_1, θ_2) using the generalized factorization theorem, as desired.

□

7.7.6. Let $X_1, X_2, \ldots, X_n$ be a random sample from the uniform distribution with pdf $f(x;\theta_1,\theta_2) = 1/(2\theta_2),\ \theta_1 - \theta_2 < x < \theta_1 + \theta_2$, where $-\infty < \theta_1 < \infty$ and $\theta_2 > 0$, and the pdf is equal to zero elsewhere.

(a) Show that $Y_1 = \min(X_i)$ and $Y_n = \max(X_i)$, the joint sufficient statistics for θ_1 and θ_2, are complete.

(b) Find the MVUEs of θ_1 and θ_2.

Proof.

a. Note that the joint pdf can be written as

$$\prod_{i=1}^{n} f(x_i;\theta_1,\theta_2) = \frac{1}{(2\theta_2)^n} I(\theta_1-\theta_2 < x_1,\cdots,x_n < \theta_1+\theta_2)$$

$$\prod_{i=1}^{n} f(x_i;\theta_1,\theta_2) = \frac{1}{(2\theta_2)^n} I(Y_1 > \theta_1-\theta_2, Y_n < \theta_1+\theta_2)$$

which we can express in a format that satisfies the joint factorization theorem, proving that $Y_1 = \min(X_i), Y_n = \max(X_i)$ are jointly sufficient for θ_1, θ_2. Now let us prove that these statistics are complete, which is the case if $\exists\ g$ such that

$$E(g(Y_1,Y_2)) = 0$$

$\forall\ \theta_1,\theta_2$. To establish completeness, assume there exists a function g such that:

$$E_{\theta_1,\theta_2}[g(Y_1,Y_2)] = 0 \quad \forall \theta_1,\theta_2.$$

Using a change of variables, let $Z_i = \frac{1}{2}(\theta_2 X_i - \theta_1 + 1) \sim U(0,1)$ iid. Defining $\theta_1 = \min(\theta_1, W_1)$ and $\theta_2 = \max(\theta_2, W_2)$ results in expressions where the distribution of W_1 and W_2 remains independent of $\boldsymbol{\theta}$.

Additionally:

$$f(w_1,w_2) = w_2^n - (w_2-w_1)^n = n(n-1)(w_2-w_1)^{n-2}.$$

The expectation condition leads to an integral that, due to the continuity of g, must be zero almost everywhere unless g is identically zero. Thus, (Y_1, Y_2) form a complete set of statistics.

b. From our previous calculations, we've established that $Y_i = \theta_2(2W_i - 1) + \theta_1$, for $i = 1, 2$. The pdfs are as follows:

$$f_{W_1}(w_1) = n(1-w_1)^{n-1}, \quad f_{W_2}(w_2) = n w_2^{n-1}.$$

Hence,

$$E(W_1) = n\int_0^1 w_1(1-w_1)^{n-1} dw_1 = n\left(\frac{1}{n} - \frac{n+1}{n}\right) = \frac{1}{n+1}.$$

$$E(W_2) = n\int_0^1 w_2^n dw_2 = \frac{n}{n+1}.$$

$$E(Y_1) = -\frac{\theta_2(n-1)}{n+1} + \theta_1, \quad E(Y_2) = \theta_2\frac{\theta_2(n-1)}{n-1} + \theta_1.$$

Consequently,

$$E(\frac{1}{2}(Y_1 + Y_2)) = \theta_1, \quad E(2\frac{n+1}{n-1}(Y_2 - Y_1)) = \theta_2.$$

$\frac{1}{2}(Y_1 + Y_2)$ is the MVUE for θ_1, and $2\frac{n+1}{n-1}(Y_2 - Y_1)$ is the MVUE for θ_2.

$\square$