MATH410: Homework 4

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1. Define $f(x) = mx + b$ for all x. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

Proof. Let us define two sequences $\{u_n\}$ and $\{v_n\}$ in $\mathbb{R}$ with $\lim_{n\to\infty}(u_n - v_n) = 0$. We want to show then that $\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0$, and this would satisfy the definition of uniform continuity. Note that

$$\{f(u_n) - f(v_n)\} = \{mu_n + b - mv_n - b\}$$

by definition of f. Now observe that

$$\{mu_n + b - mv_n - b\} = \{m(u_n - v_n)\} \to m * 0 = 0$$

Therefore, f is uniformly continuous on $\mathbb{R}$ as desired.

□

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2. Prove the following:

   (a) If the function $f : D \to \mathbb{R}$ is uniformly continuous and α is any number, show that the function $\alpha f : D \to \mathbb{R}$ also is uniformly continuous.

   (b) If $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are uniformly continuous, then so is the sum $f + g : D \to \mathbb{R}$

Proof.

a. We're given that $f : D \to \mathbb{R}$ is uniformly continuous, and so by definition, we know for any two sequences $\{u_n\}, \{v_n\} \in \mathbb{R}$ such that $\{u_n - v_n\} \to 0 \implies \{f(u_n) - f(v_n)\} \to 0$. We WTS that $\{\alpha f(u_n) - \alpha f(v_n)\} \to 0$, too.

$$\{\alpha f(u_n) - \alpha f(v_n)\} = \{\alpha(f(u_n) - f(v_n))\} \to \alpha * 0 = 0$$

as desired, and so $\alpha f : D \to \mathbb{R}$ is also uniformly continuous.

b. Both f, g are uniformly continuous so for any two sequences, $\{u_n\}, \{v_n\} \in D$ that converge to 0, then their image sequences also converge to 0. Now consider the function $f + g : D \to \mathbb{R}$. Let us denote this new function h. For the same two sequences $\{u_n\}, \{v_n\}$ that converge to 0, we WTS that

$$\{h(u_n) - h(v_n)\} \to 0$$

Note that

$$\{h(u_n) - h(v_n)\} = \{f(u_n) + g(u_n) - f(v_n) - g(v_n)\}$$

by definition of h.

$$= \{(f(u_n) - f(v_n)) + (g(u_n) - g(v_n))\} \to 0 + 0 = 0$$

Therefore, the function $f + g : D \to \mathbb{R}$ is also uniformly continuous, as desired.

□

3. Define $f(x) = x^3$ for all x. Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous.

Proof. To show that f is not uniformly continuous, we have to find a pair of sequences that doesn't work. Let $\{u_n\} = \{n + \frac{1}{n}\}$ and let $\{v_n\} = \{n\}$. Note that $\{u_n - v_n\} \to 0$ but

$$\{f(u_n) - f(v_n)\} = \{(n + \frac{1}{n})^3 - n^3\}$$

by definition of f. Note that

$$(n + \frac{1}{n})^3 = (n + \frac{1}{n})(n^2 + 2 + \frac{1}{n^2}) = n^3 + 2n + \frac{1}{n} + n + \frac{2}{n} + \frac{1}{n^3} = n^3 + 3n + \frac{3}{n} + \frac{1}{n^3}$$

$$= \{n^3 + 3n + \frac{3}{n} + \frac{1}{n^3} - n^3\} = \{3n + \frac{3}{n} + \frac{1}{n^3}\} \to \infty \neq 0$$

by limit rules. Therefore, $f : \mathbb{R} \to \mathbb{R}$ is not uniformly continuous on $\mathbb{R}$.

$\square$

4. A function $f : D \to \mathbb{R}$ is called a Lipschitz function if there is some nonnegative number C such that

$$|f(u) - f(v)| \leq C|u - v| \quad \text{for all points } u \text{ and } v \text{ in } D.$$

(a) Prove that if $f : D \to \mathbb{R}$ is a Lipschitz function, then it is uniformly continuous. (Using sequences)

(b) Show that a Lipschitz function satisfies the $\epsilon - \delta$ criterion on D. (Using the $\epsilon - \delta$ criterion definition)

Proof.

a. Let $f : D \to \mathbb{R}$ be a Lipschitz function. Note that $|f(u) - f(v)| \leq C|u - v|$ for all points $u, v \in D$. If this works for all points in D, then we can construct sequences $\{u_n\}, \{v_n\} \in D$ such that the above still holds true for all points in the sequences. Thus, we have

$$|f(u_n) - f(v_n)| \leq C|u_n - v_n| \; \forall \; n \qquad (*)$$

Recall the Comparison Lemma which states that if $\{a_n\} \to a$, then sequence $\{b_n\} \to b$ for all n greater than some threshold N if $\exists \; c \in \mathbb{R}^+$ such that

$$|b_n - b| \leq c|a_n - a|$$

Note that we can rewrite $(*)$ as

$$|(f(u_n) - f(v_n)) - 0| \leq C|(u_n - v_n) - 0| \; \forall \; n$$

Since we've constructed $\{u_n\}, \{v_n\} \in D$ arbitarily, we've shown that for any two sequences such that $\{u_n - v_n\} \to 0$ then $\{f(u_n) - f(v_n)\} \to 0$, too, and so we've shown that all Lipschitz functions are uniform continuous.

b. A function f satisfies $\epsilon - \delta$ criterion on D if $\forall \; x_0 \in D$ then given some $\epsilon > 0, \exists \; \delta > 0$ such that

$$\forall \; x_0 \in D, |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{C}$. Observe that $\delta > 0$ because both C, ϵ are nonnegative. Choose any $x_0 \in D$ and let $v = x_0$. Then for all other points $x \in D$, we have that

$$|f(x) - f(x_0)| \leq C|x - x_0|$$

by definition of the Lipschitz function. We're given that $|x - x_0| < \delta \implies C|x - x_0| < C\delta$. Therefore,

$$|f(x) - f(x_0)| \leq C|x - x_0| < C\delta = C * \frac{\epsilon}{C} = \epsilon$$

Therefore, $|f(x) - f(x_0)| < \epsilon$, and so Lipschitz functions satisfies the $\epsilon - \delta$ criterion on D.

$\square$

5. (a) Provide an example illustrating that it is not necessarily the case that if $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are each uniformly continuous, then so is the product $fg : D \to \mathbb{R}$.
   (b) Suppose that the functions $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are uniformly continuous and bounded. Prove that the product $fg : D \to \mathbb{R}$ also is uniformly continuous.

Proof.

a. Let $f, g : \mathbb{R} \to \mathbb{R}$ and $f(x) = g(x) = x$. Note that both $f(x)$ and $g(x)$ are both uniformly continuous because they are linear. See problem 1 on this homework, and fix $m = 1, b = 0$. Their product, however, $h(x) = x^2$ is not uniformly continuous, as we have proved in class, but I will prove it again here. Consider $\{u_n\} = \{n + \frac{1}{n}\}$ and $\{v_n\} = \{n\}$. Note that $\{u_n - v_n\} \to 0$. However,

$$\{h(u_n) - h(v_n)\} = \{n^2 + 2 + \frac{1}{n^2} - n^2\} \to 2 \neq 0$$

and so it is not necessarily the case that if $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are both uniformly continuous, then their product is also uniformly continuous.

b. Suppose that $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ are both uniformly continuous and bounded. Then we know for any two sequences $\{u_n\}, \{v_n\} \in D$ that

$$\{u_n - v_n\} \to 0 \implies \{f(u_n) - f(v_n)\} \to 0 \text{ and } \{g(u_n) - g(v_n)\} \to 0$$

Furthermore, since f, g are both bounded, then $\exists\ L, M \in \mathbb{R}^+$ such that

$$|f(x)| \leq L \text{ and } |g(x)| \leq M\ \forall\ x \in D$$

Therefore, note that the product

$$|fg(x)| \leq L * M\ \forall\ x \in D$$

Let us use the Comparison Lemma to show that $\{fg(u_n) - fg(v_n)\} \to 0$.

$$|fg(u_n) - fg(v_n)| = |f(u_n) * g(u_n) - f(v_n) * g(v_n)|$$

by the definition of product. Note that this most recent expression is bounded by

$$\leq |L * g(u_n) + L * g(v_n)| = L|g(u_n) + g(v_n)|$$

and recall that $L > 0$ so we can pull it out of the absolute value.

$$= L|g(u_n) + g(v_n) - g(v_n) + g(v_n)| = L|g(u_n) - g(v_n) + 2g(v_n)|$$

by adding and subtracting a $g(v_n)$ from inside the absolute value.

$$\leq L|g(u_n) - g(v_n)| + 2L|g(v_n)|$$

by the Triangle Inequality. Now by the definition of bounded,

$$\leq L|g(u_n) - g(v_n)| + 2LM$$

Note that L, M are fixed in the positive real numbers. By A.P., $\exists\ c \in \mathbb{R}$ such that

$$L|g(u_n) - g(v_n)| + 2LM \leq c|g(u_n) - g(v_n)|\ \forall\ n$$

Therefore, by subtracting 0 from the first and last expressions we have

$$|(fg(u_n) - fg(v_n)) - 0| \leq c|(g(u_n) - g(v_n)) - 0|$$

and so by the Comparison Lemma, $\{fg(u_n) - fg(v_n)\} \to 0$ and so $fg : D \to \mathbb{R}$ is uniformly continuous as desired.

$\square$

6. Define $f(x) = \sqrt{x}$ for all $x \geq 0$. Verify the $\epsilon - \delta$ criterion for continuity at $x = 4$ and at $x = 100$. Hint: First show that for $x \geq 0, x_0 > 0$,

$$\left|\sqrt{x} - \sqrt{x_0}\right| \leq |x - x_0| / \sqrt{x_0}.$$

Proof.
First, let us show the given hint. Let $x \geq 0, x_0 > 0$ then

$$|\sqrt{x} - \sqrt{x_0}| = |\frac{x - x_0}{\sqrt{x} + \sqrt{x_0}}| \leq |\frac{x - x_0}{\sqrt{x_0}}| = \frac{1}{\sqrt{x_0}}|x - x_0|$$

i. Let us show that $f(x)$ satisfies $\epsilon - \delta$ criterion at $x = 4$. Let $\epsilon > 0$ be given. Let $\delta = 2\epsilon$. Then if $|x - 4| < \delta$ then

$$|\sqrt{x} - \sqrt{4}| \leq \frac{1}{2}|x - x_0| < \frac{\delta}{2} = \epsilon$$

ii. Now let us show that $f(x)$ satisfies $\epsilon - \delta$ criterion at $x = 100$. Let $\epsilon > 0$ be given. Let $\delta = 10\epsilon$. Then if $|x - 100| < \delta$ then

$$|\sqrt{x} - \sqrt{100}| \leq \frac{1}{10}|x - x_0| < \frac{1}{10}\delta = \epsilon$$

□

7. Define $f(x) = x^3$ for all x. Verify the $\epsilon - \delta$ criterion for continuity at each point x_0.

Proof. Let $\epsilon > 0$ be given. Let $x_0 \in \mathbb{R}$ and let $\delta = \frac{\epsilon}{2|x_0^2|+3|x_0|+1}$. Then if $|x - x_0| < \delta$ then

$$|f(x) - f(x_0)| = |x^3 - x_0^3| = |(x - x_0)(x^2 + xx_0 + x_0^2)| < \delta|x^2 + xx_0 + x_0^2|$$

Note that the absolute value term is a constant, but x could be large, so we need to bound it. Let $\delta \leq 1$. Now we will examine $|x^2 + xx_0 + x_0^2|$ and try and relate it to $|x - x_0|$.

$$|x^2 + xx_0 + x_0^2| = |x^2 + xx_0 + x_0^2 + x_0^2 - x_0^2| = |(x^2 - x_0^2) + xx_0 + 2x_0^2|$$

$$\leq |(x - x_0)(x + x_0)| + |xx_0| + |2x_0^2| < \delta|x + x_0| + |xx_0| + |2x_0^2|$$

Note that it was shown in class in a similar example that $|x + x_0| < 1 + 2|x_0|$ and observe that the $|2x_0^2|$ is a constant already. Thus, we have

$$\leq \delta(1 + 2|x_0|) + |2x_0^2| + |xx_0|$$

Now note that we have to bound $|xx_0| = |x_0||x|$ and relate it to $|x - x_0|$. Note that

$$|x| = |x - x_0 + x_0| \leq |x - x_0| + |x_0| < 1 + |x_0|$$

Thus we get

$$\leq \delta((1 + 2|x_0|) + |2x_0^2| + 1 + |x_0|) = \delta(2 + 3|x_0| + 2|x_0^2|) = \epsilon$$

Therefore, $f(x) = x^3$ satisfies the $\epsilon - \delta$ criterion for continuity at all points $x_0 \in \mathbb{R}$. $\square$