MATH410: Homework 5

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1. For each of the following statements, determine whether it is true or false and justify your answer.
   (a) A monotone function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one.
   (b) A strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ is one-to-one.
   (c) A strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ is continuous.
   (d) A one-to-one function $f : \mathbb{R} \to \mathbb{R}$ is monotone.

Proof.

a. False. WLOG, suppose f is monotone increasing. Then it follows that $f(u) \leq f(v)\ \forall\ u, v \in \mathbb{R}, u < v$. Consider the function $f(x) = 1\ \forall\ x \in \mathbb{R}$. Note that this function is monotone increasing. Assume on the contrary now that f is also one-to-one. Note that $f(1) = f(2)$. By definition of one-to-one $1 = 2$ but this is obviously a contradiction, so $f : \mathbb{R} \to \mathbb{R}$ is not one-to-one.

b. True. Assume on the contrary that $f : \mathbb{R} \to \mathbb{R}$ is not one-to-one. By definition of strictly increasing, $f(u) < f(v)\ \forall\ u, v \in \mathbb{R}, u < v$. Since f is not one-to-one, there exists $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = f(x_2)$. Choose these points. Let $x_1 < x_2$ WLOG. However, by definition of strictly increasing, $f(x_1) < f(x_2)$ which is a contradiction, so f is one-to-one.

c. False. Consider the piecewise function

$$f(x) = \begin{cases} x, x \geq 0 \\ x - 1, x < 0 \end{cases}$$

Let us show that f is strictly increasing by considering $x_1, x_2 \in \mathbb{R}$ and then considering cases. WLOG, let $x_1 < x_2$. For each case, we WTS that $x_1 < x_2 \implies f(x_1) < x_2$.

   a. $0 \leq x_1 < x_2 \implies x_1 < x_2$ which is true.
   b. $x_1 < 0 \leq x_2 \implies x_1 - 1 < x_2$ which is true.
   c. $0 < x_1 < x_2 \implies x_1 - 1 < x_2 - 1 \implies x_1 < x_2$ which is true.

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Therefore, f is strictly increasing. Now we will show that f is not continuous. Let us define two sequences $\{u_n\} = \{\frac{1}{n}\}$ and $\{v_n\} = \{-\frac{1}{n}\}$ whose limits are both 0. Note that $\{f(u_n)\}_{n=1}^{\infty} = 0 \neq \{f(v_n)\}_{n=1}\infty = -1$. Therefore, for all sequences that converge to $x_0 = 0$, not all image sequences converge to the functional value $f(x_0) = 0$, so f is not continuous.

d. False. Consider the function

$$f(x) = \begin{cases} x, x \in \mathbb{Q} \\ -x, x \notin \mathbb{Q} \end{cases}$$

Let us prove that this function is one-to-one. Assume on the contrary that f is not one-to-one. Since f is not one-to-one, there exists $x_1 \neq x_2$ such that $f(x_1) = f(x_2)$. However, $f(x_1) = f(x_2) \in \mathbb{Q} \implies x_1 = x_2$ and similarly for $\mathbb{Q}^c$. Thus, f is one-to-one. Now we will show that f is not monotone. Consider $x_1 = 0, x_2 = \pi$. $0 < \pi \implies f(0) < f(\pi) \implies 0 < -\pi$ which is a contradiction.

$\square$

2. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be odd provided that $f(-x) = -f(x)$ for all x. Show that if $f : \mathbb{R} \to \mathbb{R}$ is odd and the restriction of this function to the interval $[0, \infty)$ is strictly increasing, then $f : \mathbb{R} \to \mathbb{R}$ itself is strictly increasing.

Proof. Let f be odd and suppose that $f : [0, \infty)$ is strictly increasing. We WTS that $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing, or that for all $x_1, x_2 \in \mathbb{R}, x_1 < x_2$ then $f(x_1) < f(x_2)$. Consider $x_1, x_2 \in \mathbb{R}$ such that $0 \leq x_1 < x_2$. Then by definition of strictly increasing, $f(x_1) < f(x_2) \implies -f(x_1) > -f(x_2)$ by multiplying the inequality by -1. By definition of odd, $f(-x_1) = -f(x_1)$ and $f(-x_2) = -f(x_2)$. Therefore, by substitution, $f(-x_1) > f(-x_2)$ where $x_1 < x_2 \implies -x_1 > -x_2$. Note that $x_1, x_2 \in [0, \infty) \implies -x_1, -x_2 \in (-\infty, 0]$ and that $(-\infty, 0] \cap [0, \infty) = \mathbb{R}$. Thus, we have shown that $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing. $\square$

3. Prove that

   (a) $\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = 4$

   (b) $\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \frac{1}{2}$

Proof.

a. Note that polynomials are continuous, and so the quotient is continuous. Let $\{x_n\} \to 1$ with $x_n \neq 1$. Then

$$\frac{x_n^4 - 1}{x_n - 1} = \frac{(x_n^2 + 1)(x_n + 1)(x_n - 1)}{(x_n - 1)} = (x_n^2 + 1)(x_n + 1)$$

$$\lim_{x \to 1} \frac{x^4 - 1}{x - 1} = \lim_{n \to \infty} (x_n^2 + 1)(x_n + 1) = 4$$

b. $\sqrt{x}$ is continuous because it is the inverse of a strictly increasing funciton, and the denominator is continuous because it is linear, and so the quotient is continuous. Let $\{x_n\} \to 1$, but $x_n \neq 1$. Then

$$\frac{\sqrt{x_n} - 1}{x - 1} = \frac{\sqrt{x_n} - 1}{(\sqrt{x_n} + 1)(\sqrt{x_n} - 1)} = \frac{1}{\sqrt{x_n + 1}}$$

$$\lim_{x \to 1} \frac{\sqrt{x} - 1}{x - 1} = \lim_{n \to \infty} \frac{\sqrt{x_n} - 1}{x - 1} = \lim_{n \to \infty} \frac{1}{\sqrt{x_n + 1}} = \frac{1}{2}$$

$\square$

4. Find the following limits and prove the limits:

   (a) $\lim_{x\to 0} \frac{1+1/x}{1+1/x^2}$

   (b) $\lim_{x\to 0} \frac{1+1/x^2}{1+1/x}$

   (c) $\lim_{x\to 1} \frac{1+1/(x-1)}{2+1/(x-1)^2}$

Proof.

a. Note that

$$\frac{1+\frac{1}{x}}{1+\frac{1}{x^2}} = (1+\frac{1}{x})(\frac{1}{1+\frac{1}{x^2}}) = (\frac{x+1}{x})(\frac{x^2}{x^2+1}) = \frac{x(x+1)}{x^2+1}$$

Let $\{x_n\} \to 0$ but $x_n \neq 0$. Then

$$\lim_{x\to 0} \frac{1+1/x}{1+1/x^2} = \lim_{n\to\infty} \frac{x_n(x_n+1)}{(x_n^2+1)} = \frac{0(1)}{1} = 0$$

b. Note similar to above that

$$\frac{1+1/x^2}{1+1/x} = (1+1/x^2)(\frac{1}{1+1/x}) = (\frac{x^2+1}{x^2})(\frac{x}{x+1}) = \frac{x^2+1}{x(x+1)}$$

$$= \frac{1}{x^2+x} + \frac{x^2}{x^2+x} = \frac{1}{x^2+x} + \frac{x}{x+1}$$

Let $\{x_n\} \to 0$ but $x_n \neq 0$. Then

$$\lim_{x\to 0} \frac{1+1/x^2}{1+1/x} = \lim_{n\to\infty} \frac{1}{x^2+1} + \lim_{n\to\infty} \frac{x}{x+1} = \infty + 1 = \infty$$

We can break up the above into two limits by limit rules.

c. Note that

$$\frac{1+1/(x-1)}{2+1/(x-1)^2} = \frac{x}{x-1} * \frac{(x-1)^2}{2(x-1)^2+1} = \frac{x(x-1)}{2(x-1)^2+1}$$

Let $\{x_n\} \to 1$ but $x_n \neq 1$. Then

$$\lim_{x\to 1} \frac{1+1/(x-1)}{2+1/(x-1)^2} = \lim_{n\to\infty} \frac{x_n(x_n-1)}{2(x_n-1)^2+1} = \frac{0}{1} = 0$$

$\square$

5. Suppose the function $f : \mathbb{R} \to \mathbb{R}$ has the property that there is some $M > 0$ such that

$$|f(x)| \le M|x|^2 \quad \text{for all } x.$$

Prove that

$$\lim_{x\to 0} f(x) = 0 \quad \text{and} \quad \lim_{x\to 0} \frac{f(x)}{x} = 0$$

Proof.

i. First let us show that $\lim_{x\to 0} f(x) = 0$. Let $\{x_n\} \to 0$ but $x_n \neq 0$. Observe that $|f(x_n)| = |f(x_n) - 0| \le M|x_n|^2 = M|x_n^2| = M|x_n^2 - 0|$. Since $\{x_n\} \to 0$ then $\{x_n^2\} \to 0$ by the product property. Since $M > 0$, by the Comparison Lemma, then $\{f(x_n)\} \to 0 \implies \lim_{n\to\infty} f(x_n) = 0 \implies \lim_{x\to 0} f(x) = 0$.

ii. Now for $\lim_{x\to 0} \frac{f(x)}{x}$, let $\{x_n\} \to 0$ but $x_n \neq 0\ \forall\ n \in \mathbb{N}$. Observe that

$$|f(x_n)| \le M|x_n|^2 \implies |\frac{f(x_n)}{x_n} - 0| \le M|x_n - 0|$$

by dviding both sides by $|x_n|$, and so by the Comparison Lemma once more,

$$\{\frac{f(x_n)}{x_n}\} \to 0 \implies \lim_{n\to\infty} \frac{f(x_n)}{x_n} = 0 \implies \lim_{x\to 0} \frac{f(x)}{x} = 0$$

$\square$

6. For m_1 and m_2 numbers, with $m_1 \neq m_2$, define

$$f(x) = \begin{cases} m_1 x + 4 & \text{if } x \leq 0 \\ m_2 x + 4 & \text{if } x \geq 0 \end{cases}$$

Prove that the function $f : \mathbb{R} \to \mathbb{R}$ is continuous but not differentiable at $x = 0$.

Proof. First let us show that this function is continuous at 0. We will use the $\epsilon - \delta$ criterion. By a theorem, given $f : D \to \mathbb{R}, x_0 \in D$, if f satisfies the $\epsilon - \delta$ criterion at $x = 0$, then f is continuous at $x = 0$. Thus, let $\epsilon > 0$ be given and let $\delta = \frac{\epsilon}{\max(|m_1|,|m_2|)}$. Note that $\delta > 0$. If $|x - 0| < \delta$ then

$$|f(x) - f(0)| = |m_* x + 4 - 4| = |m_* x| = |m_*||x - 0| < |m_*|\delta =$$

$$|m_*|\frac{\epsilon}{\max(|m_1|, |m_2|)} < \epsilon$$

where $m_* = \begin{cases} m_1, x \leq 0 \\ m_2, x \geq 0 \end{cases}$ and the last step because $\frac{|m_*|}{\max(|m_1|,|m_2|)} \leq 1$ always. Therefore, f is continuous at $x = 0$. Now let us show that f is not differentiable. Consider $\{x_n\} \to 0$ but $x_n \neq 0 \;\forall\; n$.

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \begin{cases} \lim\limits_{n\to\infty} \frac{m_1 x_n + 4 - 4}{x_n} = m_1, x \geq 0 \\ \lim\limits_{n\to\infty} \frac{m_2 x_n + 4 - 4}{x_n} = m_2, x < 0 \end{cases}$$

Since $m_1 \neq m_2$, this limit is not defined, and so f is not differentiable. $\square$

7. Use the definition of derivative to compute the derivative of the following functions at $x = 1$:
   (a) $f(x) = \sqrt{x+1}$ for all $x > 0$.
   (b) $f(x) = x^3 + 2x$ for all x.
   (c) $f(x) = 1/\left(1+x^2\right)$ for all x.

Proof.
For all of these problems, let $\{x_n\} \to 1$ but $x_n \neq 1 \; \forall \; n \in \mathbb{N}$. We seek to find

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

a. Consider

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\sqrt{x+1} - \sqrt{2}}{x - 1} = \lim_{x \to 1} \frac{x + 1 - 2}{(x-1)(\sqrt{x+1} + \sqrt{2})}$$

by multiplying by the conjugate.

$$= \lim_{x \to 1} \frac{1}{\sqrt{x+1} + \sqrt{2}} = \lim_{n \to \infty} \frac{1}{\sqrt{x_n + 1} + \sqrt{2}} = \frac{1}{2\sqrt{2}}$$

b. Next

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{x^3 + 2x - 3}{x - 1} = \lim_{n \to \infty} \frac{(x_n^2 + x_n + 3)(x_n - 1)}{x_n - 1}$$

$$= \lim_{n \to \infty} (x_n^2 + x_n + 3) = 5$$

c. Finally

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\frac{1}{1+x^2} - \frac{1}{2}}{x - 1} = \lim_{x \to 1} \frac{\frac{2-1-x^2}{1+x^2}}{x - 1} = \lim_{x \to 1} \frac{1 - x^2}{(x-1)(1+x^2)}$$

$$\lim_{x \to 1} \frac{(1+x)(1-x)}{(x-1)(1+x^2)} = \lim_{n \to \infty} -1 * \frac{1 + x_n}{1 + x_n^2} = -1$$

$\square$

8. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that there is a bounded sequence $\{x_n\}$ with $x_n \neq x_m$, if $n \neq m$, such that $f(x_n) = 0$ for every index n. Show that there is a point x_0 at which $f(x_0) = 0$ and $f'(x_0) = 0$. (Hint: Use the Sequential Compactness Theorem.)

Proof.
Since f is differentiable, we know that the limit

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists for all $x_0 \in \mathbb{R}$. By the definition of bounded, $\exists M > 0$ such that $|x_n| \leq M \ \forall \ n \in \mathbb{N}$. Furthermore, $f(x_n) = 0 \ \forall \ n$ except for when $n = m$. Consider the interval $I = [a, b], a, b \in \mathbb{R}$ such that $m \notin I$. By the Sequential Compactness Theorem, there exist a subsequence $\{x_{n_k}\}$ that converge to a point in the interval, say x_0, but modify the subsequence such that it never equals 0 for all k. Since $x_0 \neq m \implies f(x_0) = 0$ as desired. Now let us show that the derivative $f'(x_0) = 0$.

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{n\to\infty} \frac{f(x_{n_k}) - f(x_0)}{x_{n_k} - x_0} = \lim_{n\to\infty} \frac{f(x_{n_k}) - 0}{x_{n_k} - x_0}$$

$$\lim_{n\to\infty} \frac{0}{(x_{n_k} - x_0)} = 0 = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

Note that the numerator is, by construction, always precisely equal to 0 since $f(x_{n_k}) = 0 \ \forall \ k$, whereas the denominator approaches 0 but will never actually be 0. $\square$