MATH410: Homework 8

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1. For each of the following statements, determine whether it is true or false and justify your answer.

   (a) If $f : [a,b] \to \mathbb{R}$ is integrable and $\displaystyle\int_a^b f = 0$, then $f(x) = 0$ for all x in $[a,b]$.

   (b) If $f : [a,b] \to \mathbb{R}$ is integrable, then $f : [a,b] \to \mathbb{R}$ is continuous.

   (c) If $f : [a,b] \to \mathbb{R}$ is integrable and $f(x) \geq 0$ for all x in $[a,b]$, then $\displaystyle\int_a^b f \geq 0$.

   (d) A continuous function $f : (a,b) \to \mathbb{R}$ defined on an open interval (a,b) is bounded.

   (e) A continuous function $f : [a,b] \to \mathbb{R}$ defined on a closed interval $[a,b]$ is bounded.

Solution.

a. False. Recall in the Homework 7 problem 5a, we proved that $f(x) = x$ is integrable and that $\int_a^b x dx = \frac{b^2-a^2}{2}$. Let $a = -1, b = 1$ such that we want to compute $\int_{-1}^1 x dx = \frac{1^2-(-1)^2}{2} = 0$. However, clearly, $f(x) = x \neq 0 \; \forall \; x \in (a,b)$. Therefore, this statement is false.

b. False. Step functions are integrable but not continuous. Recall that we did a sketch of a proof in class showing that every step function is integral. For the partition points inside one region, $M_i = m_i$. For the subintervals at the gaps/jumps, there are finitely many and they can still be bounded using M_i, m_i. Therefore, $\lim\limits_{n\to\infty}(U(f,P_n) - L(f,P_n)) = 0$ and so by the AR theorem, step functions are integrable. Now we will show that a step function is not continuous. Consider a generic step function $f : [a,b] \to \mathbb{R}$ with one "step."

$$f(x) = \begin{cases} c, & x \geq x_0 \\ d, & x < x_0 \end{cases}$$

for some arbitrary $c, d, x_0 \in \mathbb{R}, c \neq d, x_0 \in (a,b)$. Consider a sequence $\{u_n\} \to x_0$ from the right and a sequence $\{v_n\} \to x_0$ from the left. Observe that $\{f(u_n)\} \to c$ and $\{f(v_n)\} \to d$. Therefore, we have found a pair of sequences that converge to x_0, whose image sequences do not converge to $f(x_0)$. Therefore, step functions are integrable but not continous.

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c. True. Suppose $f : [a, b] \to \mathbb{R}$ is integrable and $f(x) \geq 0 \ \forall \ x \in [a, b]$. By the AR Theorem, specifically the "moreover" part, recall that

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n)$$

for some partition P_n on $[a, b]$. Note that

$$\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

by definition of Darboux Upper Sums. $M_i \geq 0 \ \forall \ i$ because $f(x) \geq 0 \ \forall \ x$. Further note that $x_i - x_{i-1} > 0 \ \forall \ i$ by properties of partitions. Therefore,

$$M_i(x_i - x_{i-1}) \geq 0 \ \forall \ i$$

Therefore, the sum and then limit of n positive terms must also be positive.

$$\int_a^b f = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \geq 0 \implies \int_a^b f \geq 0$$

d. False. Define $f : (0, 1) \to \mathbb{R}$ such that $f(x) = \frac{1}{x} \ \forall \ x \in (0, 1)$. By limit properties,

$$\lim_{x\to 0} \frac{1}{x} = \infty$$

and so f is not bounded. Alternatively, suppose on the contary that f was bounded, so $\exists M \in \mathbb{R}^+$ such that $|f(x)| = |\frac{1}{x}| \leq M \ \forall \ x \in (0, 1)$. Consider $x_0 = \frac{1}{M+1} \in (0, 1)$ because $M > 0$. Then

$$|f(x_0)| = |\frac{1}{\frac{1}{M+1}}| = |M + 1| = M + 1 \leq M$$

is a contradiction, and so $f(x) = \frac{1}{x}$ is not bounded on $(0, 1)$ and we have proved this statement false by counter example.

e. True. Since f is continuous on $[a, b]$, f must attain all values from $f(a)$ to $f(b)$ and note that these image values are themselves an interval, $f([a, b])$. Note that this interval is closed, and recall that all closed intervals attain a maximum and a minimum, which we will denote M and m, respectively. Let $L = \max(|M|, |m|)$. By definition of max and min,

$$m \leq f(x) \leq M \ \forall \ x \in [a, b]$$

By substitution for L,

$$-L \leq f(x) \leq L \ \forall \ x \in [a, b]$$

$$|f(x)| \leq L \ \forall \ x \in [a, b]$$

and so f is bounded.

$\square$

2. Define

$$f(x) \equiv \begin{cases} x & \text{if the point } x \text{ in } [0,1] \text{ is rational} \\ -x & \text{if the point } x \text{ in } [0,1] \text{ is irrational.} \end{cases}$$

Prove that the function $f : [0,1] \to \mathbb{R}$ is not integrable.

Proof.
Let $P = \{0, x_1 \ldots, x_{n-1}, 1\}$ be a partition of $[0,1]$. Since $\mathbb{Q}$ and $\mathbb{Q}^C$ are dense in each $[x_{i-1}, x_i]$, there exists a rational number and irrational number in each $[x_{i-1}, x_i]$. Note that to show that $f : [0,1] \to \mathbb{R}$ is not integrable, we want to show that there does not exist a sequence of partitions P_n such that $\lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$. Observe that

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$
$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

where $M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$ and $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$. Note that $M_i = x_i \; \forall \; i$. To show this, assume on the contrary that there exists some index value $x_0 \in (x_{i-1}, x_i)$ such that $M_i = f(x_0) = x_0$ is the sup of the interval. Now consider the new interval $[x_0, x_i]$. By density of $\mathbb{Q}$ once more, there exists a rational number in this interval. Denote this index x_1. Note that $x_1 > x_0 \implies f(x_1) > f(x_0)$ which contradicts that x_0 is the sup of the interval. Therefore, x_i is the sup. Similarly, we can show that $m_i = -x_{i-1} \; \forall \; i$. Therefore,

$$L(f,P) = \sum_{i=1}^{n} -x_{i-1}(x_i - x_{i-1}) = (x_0^2 - x_0x_1) + (x_1^2 - x_1x_2) + \cdots + (x_{n-1}^2 - x_{n-1}x_n)$$

$$U(f,P) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}) = (x_1^2 - x_0x_1) + (x_2^2 - x_1x_2) + \cdots + (x_n^2 - x_{n-1}x_n)$$

Taking their elementwise differences, observe that

$$U(f,P) - L(f,P) = (x_1^2 - x_0^2) + (x_2^2 - x_1^2) + \cdots + (x_n^2 - x_{n-1}^2) = x_n^2 - x_0^2$$

However, $x_0 = 0$ and $x_n = 1$ by construction for our partition P. Therefore,

$$U(f,P) - L(f,P) = 1^2 - 0^2 = 1 \implies \lim_{n\to\infty}(U(f,P) - L(f,P)) = 1 \neq 0$$

and so therefore, f is not integrable.

$\square$

3. Suppose that the continuous function $f : [a, b] \to \mathbb{R}$ has $\displaystyle\int_a^b f = 0$. Prove that there is some point x_0 in the interval $[a, b]$ at which $f(x_0) = 0$.

Proof.
First, let us prove the Mean Value Theorem for Integrals (I don't think we have covered this in class). The MVT for Integrals states that if $f : [a, b] \to \mathbb{R}$ is continuous on (a, b) then there exists at least one point $x_0 \in (a, b)$ such that $f(x_0) = \frac{1}{b-a}\int_a^b f(x)dx$. To prove this, first apply Extreme Value theorem such that f attains a maximum and a minimum, M and m respectively. Therefore,

$$m(b-a) \leq \int_a^b f \leq M(b-a)$$

Dividing both sides by $b - a$,

$$m \leq \frac{1}{b-a}\int_a^b f \leq M$$

Since f is continuous, by the Intermediate Value Theorem, f attains all values between m and M, and so therefore,

$$\exists\ x_0 \in (a, b) \text{ s.t. } f(x_0) = \frac{1}{b-a}\int_a^b f$$

and the proof is done. In the context of this problem, note that we are given that $\int_a^b f = 0$. Substituting this into the above,

$$\exists\ x_0 \in (a, b) \text{ s.t. } f(x_0) = \frac{1}{b-a}(0) \implies \exists\ x_0 \in (a, b) \text{ s.t. } f(x_0) = 0$$

as desired.

$\square$