Robert H. Smith School
University of Maryland

BUFN402
Spring 2024

Assignment #1

Due: 2 pm 2/19/2024

Instructions:

1. Please submit your solution to the assignment on Canvas. Your submission should include two files:
   a. Answers to the questions below as a word or pdf document.
   b. Python codes.
2. Use the following convention for your submission: "Lastname_Firstname_HW_#.py" and "Lastname_Firstname_HW_#.[pdf/docx]"
3. If you need an extra assumption to write a solution, make the assumption you want and clearly state your assumption.
4. **Important:** In this assignment **alone**, you are **not** allowed to use the built-in t-test python-command. Instead, manually find the sample t-statics and perform hypothesis-tests. That said, you will use the built-in python command in the following assignments, if needed).

Honor Code:

You acknowledge that you will complete this assignment in accordance with the Smith honor policy. If you are unaware of this policy, please consult the Smith undergraduate advisor immediately.

#1. (Large Deviation/Extreme events, 10 pts) Consider an experiment that you a fair coin for 1000 times. Let the number of heads from the experiment be S_n. Answer the following questions.

a) Moderate deviation (6 pts):
   a. What is the **exact** probability of $\mathbb{P}(485 \leq S_n \leq 516)$ not using the central limit theorem (CLT)?
   b. What is the approximated probability using the CLT?
   c. What is the percentage error between results a) and b)?

b) Large deviation (tail/extreme behavior) [2 pts]:
   a. What is the **exact** probability of $\mathbb{P}(S_n \geq 564)$ not using the CLT? Note that $\lfloor \mu + 4\sigma \rfloor = 564$.
   b. What is the approximated probability using the CLT?
   c. What is the percentage error between them?

c) [2 pts] Are the results expected from the CLT? If so, explain. If not, explain why not. Complete proof is not required. Simple statements are enough.

#2 (Limit of diversification) [20 pt]. In class, we discussed an argument about the diversification effect. You will show the details of the argument. Recall the return of a portfolio (PF) of n assets is

$$R_p = \sum_{i=1}^{n} w_i \, R_i \equiv w^t R$$

where w_i denotes a weight given to an asset i and R_i a return of an asset i. Assume that the variances of assets are finite: $Var[R_i] < \infty$ for all i. Answer the following questions. Show your work for the full credit.

a) Show that the variance of the PF is

$$Var[R_p] = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} \equiv w^t \Omega w$$

where $\Omega = [\sigma_{ij}]_{ij}$, and $\sigma_{ij} := cov[R_i, R_j]$.

In parts b), c), and d), assume that the portfolio is equally weighted – that is, $w_i = 1/n$ for all $i = \{1, \cdots, n\}$.

b) Suppose $\sigma_{ij} = 0$ for all $j \neq i$ (i.e., all assets are uncorrelated). Then, show that the portfolio becomes risk-free as $n \to \infty$.

c) Suppose $\sigma_{ij} \neq 0$ for all $j \neq i$. Then, show that

$$Var[R_p] = \frac{1}{n}(\text{Average variance of the individual stocks}) + \left(1 - \frac{1}{n}\right)(\text{Average covariance between the individual stocks})$$

#3 (Data manipulation) [20 pt]. Access WRDS and download the daily returns from 12/19/2010 – 6/28/2020 of (i) HSBC Holdings, (ii) Microsoft, (iii) Pfizer, (iv) Exxon Mobil, and (v) SP500 index.

a) Convert daily returns to weekly returns.

b) What are the sample average and standard deviation (volatilities) of weekly returns? Do the patterns of the sample averages and volatilities agree with or against your intuition? Justify your answer. Note that the firms are sampled from different industries.

c) Assume that the true return distributions are normal. What is the 99% confidence interval for the true mean of returns of SP500?

d) Assume that the true return distributions are normal. Test the null hypothesis if the true mean of returns of SP500 is equal to 0. Given a 1% significance level, what is your recommendation?

#4 (Simplified random walk)[1] [20 pt]. Brownian motion (BM, henceforth) has many applications in finance, including option pricing (Black-Sholes), fixed income, and real option. Let's construct a simple random walk. First, divide the time interval $[0, t]$ into n subintervals (equal length δ) such that $t = n\delta$. Each step X_i is an independent random variable such that

$$X_i = \begin{cases} \sqrt{\delta} & \text{with probability } 1/2 \\ -\sqrt{\delta} & \text{with probability } 1/2 \end{cases}$$

After time t, the location of the particle is

$$W_t = \sum_{i=1}^{n} X_i$$

a) What are $\mathbb{E}[X_i]$ and $\mathrm{Var}[X_i]$?

b) What are $\mathbb{E}[W_t]$ and $\mathrm{Var}[W_t]$?

c) What is the (asymptotic) distribution of the random location W_t as $n \to \infty$?

d) Based on the answer c), what's the probability that W_1 (the random location at time $t = 1$) falls below -1.5?

[1] The problem uses much simplified version of Brownian motion.

In the following parts f) – i), you will <u>simulate</u> a BM using the symmetric binary step described above. Set the parameters as the following:

- Set $t = 1$ and the number of intervals as $n = 2000$ such that $\delta = 1/2000$.
- Set the random seed as 1.
- Run 4000 simulations (i.e., generate 4000 sample paths).

e) Generate 4000 sample paths and plot the histogram of the final random locations W_1.

f) Find the sample average and sample variance of W_1.

g) Using the simulated sample, (1) find the 95% confidence interval for the true mean of W_1. (2) Test the null hypothesis if the true mean is equal to 0. Given a 5% significance level, what is your recommendation?

h) Using the simulated sample, test the null hypothesis if the true distribution of W_1 is a normal distribution.

#5 (Simple Portfolio Optimization) [30 pts]. Suppose that the two risky assets' returns are normally distributed:

$$N\left(\begin{bmatrix}0.068\\0.081\end{bmatrix}, \begin{bmatrix}0.114 & 0.091\\0.091 & 0.122\end{bmatrix}\right)$$

[15 pts] In Part a) – d), assume the investor cannot access a risk-free asset.

a) Construct the efficient frontier subject to no-short selling constraint. Plot the frontier.

b) Construct the unconstraint (i.e., allowing short selling) efficient frontier. Plot the frontier.

Now, suppose that the investor is mean-variance utility maximizer as discussed in class.

c) Suppose now that the investors are constrained (i.e., prohibit from short-selling). Find the optimal portfolios when his risk-tolerance (a) $\tau = 1/4$ and (b) $\tau = 1/1.4$.

d) Suppose the investor is not constrained. Find the optimal portfolios when his risk-tolerance (a) $\tau = 1/4$ and $\tau = 1/1.4$.

[15 pts] Suppose now the investors borrow and lend at the risk-free rate 2% in addition to investing in two risky assets. Assume that the representative investor's risk-tolerance is given as $\tau = 1/4$. In part e) and f), assume that there is no short selling constraint for the risky assets.

e) Construct the efficient frontier when the investor is not leverage-constrained (i.e., the investor can lend and borrow at the risk-free rate as much as they want).

f) Suppose that the investor is leverage-constrained – that is, he cannot borrow.

   a. Describe the efficient frontier to the leverage-constrained investor.

   b. When the investors are leverage-constrained, what happens to the equilibrium prices of the portfolios on the efficient frontier riskier than the tangent portfolio?