MATH410: Advanced Calculus I

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These are my notes for UMD's MATH410: Advanced Calculus I. These notes are taken live in class ("live-TEX"-ed). This course is taught by Lecturer Anna Szczekutowicz.

Contents

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§1 Set Theory Preliminaries

This section covers the foundation of analysis, which is just the set of real numbers. It covers basic definitions such as $\in, \notin, \emptyset, \subseteq, =, \cap, \cup, \setminus$, so for example

Definition 1.1. **Intersection** of A and B is $C = A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Some quantifiers include $\forall, \exists, \exists!$ and some number sets include $\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^C$.

Definition 1.2. The real numbers $\mathbb{R}$ satisfies 3 groups of axioms: Refer to the notes on Canvas for the Consequences of all of the following axioms.

1. Field $(+, *)$
   - Commutativity of Addition
   - Associativity
   - Additive Identity
   - Additive Inverse
   - Commutativty of Multiplcation
   - Associativity of Multiplication
   - Multiplicative Identity
   - Multiplicative Inverse
   - Distributive Property

   The set of integers $\mathbb{Z}$ is not a field because it fails under the multiplicative inverse.

2. Positivity

   There is a subset of $\mathbb{R}$ denoted by $\mathcal{P}$, called the set of positive numbers for which:
   - If x and y are positive, then $x + y$ and xy are both positive.
   - For each $x \in \mathbb{R}$, eaxctly one of the following 3 alternatives is true: $x \in \mathcal{P}$, $-x \in \mathcal{P}$, or $x = 0$

3. Completeness

Definition 1.3. **Absolute value** is defined as

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Definition 1.4. **Triangle Inequality** is $\forall\ a, b \in \mathbb{R}, |a + b| \leq |a| + |b|$

Proof. Assume without loss of generality, $a \geq b$. We will proceed with proof by cases.
Case 1: Assume $a \geq b \geq 0$. Then $|a + b| = a + b$ by the definition of absolute value since $a \geq 0, b \geq 0 \implies |a + b| = a + b = |a| + |b|$.

Case 2: Now assume $a \geq 0 \geq b$ and $a+b \geq 0$. Note since $b \leq 0$ then $b \leq |b|$. Then

$$|a+b| = a+b \leq |a|+|b|$$

by the definition of absolute value and our above note.
Case 3: Now consider $a \geq 0 \geq b$ and $a+b < 0$. So

$$|a+b| = -(a+b) = -a-b \leq |a|+|b|$$

Case 4: Now consider $0 \geq a \geq b$ so $a+b<0$. Therefore,

$$|a+b| = -(a+b) = -a+-b = |a|+|b|$$

□

§2 The Completeness Axiom

Definition 2.1. A subset S of $\mathbb{R}$ is said to be **bounded above** if $\exists\, r \in \mathbb{R}$ such that $s \leq r \;\forall\; s \in S$
The definition of **bounded below** is similar.

Definition 2.2. The least upper bound, if it exists, is called the **supremum** of S. We denote it as the "sup" of S. Similarly, the largest lower bound is called the **infemum** and is denoted as the "inf" of S.

Definition 2.3. Let $S \subseteq R$ where $S \neq \emptyset$. If S hs a largest (smallest), the element is a max (min).

Example 2.4

Find the sup of $(0,1)$ and prove it.

Proof. Let us prove that the $sup(0,1) = 1$. First, let us show that we have an upperbound. If $x \in (0,1)$, then $x \leq 1$. By definition of upperbound, 1 is an upper bound. Note that we can find many other upper bounds.
On the contrary, assume $x < 1$ is an upper bound. Now consider the average $\frac{1+x}{2}$.

$$\frac{x+1}{2} < \frac{1+1}{2} = 1$$

Therefore, we have showed that $0 < \frac{x+1}{2} < 1 \implies \frac{x+1}{2}(0,1)$. But, $\frac{1+x}{2} > \frac{x+x}{2} \implies \frac{1+x}{2} > x$. This is a contradiction. Since x is an upper bound, and we found $\frac{1+x}{2} \in (0,1)$ where $\frac{1+x}{2} > x$, so x is not a supremum.

□

Theorem 2.5

Suppose $S \in \mathbb{R}, S \neq \emptyset$ that is bounded above. Then a supremum exists. Every nonsempty subset S of $\mathbb{R}$ that is bounded below has a lower bound.

Note 2.6. Let c be a positive number. $\exists!$

Definition 2.7. The **Archimedian Property** is a result of the completeness axiom. Suppose there is a small $\epsilon > 0$ and c is an arbitrary large number.

1. $\exists\ n \in \mathbb{N}$ such that $c < n$, which just means that you can always find a natural number than any large number

2. $\exists\ m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$, which just means you can always find smaller rational numbers.