MATH410: Homework 3

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1. Show that a strictly increasing sequence has no peak indices.

Proof.
On the contrary, assume that a strictly increasing sequence $\{a_n\}$ has a peak index a_m, for some $m \in \mathbb{N}$. By definition of strictly increasing, $a_{n+1} \geq a_n \ \forall \ n \in \mathbb{N}$. Since this works for all n, note that by definition of strictly increasing, $a_{m+1} \geq m$. However, note that by definition of peak index

$$a_m > a_j \ \forall \ j \geq m \implies a_m > a_{m+1}$$

Thus, we've reached a contradiction since $a_{m+1} \geq m$ and $a_{m+1} < m$ obviously cannot both be simulatenously true. $\square$

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2. Prove that a sequence $\{a_n\}$ does not converge to the number a if and only if there is some $\epsilon > 0$ and a subsequence $\{a_{n_k}\}$ such that

$$|a_{n_k} - a| \geq \epsilon \quad \text{for every index } k.$$

Proof.
$\Longrightarrow$ Suppose we are given that the sequence $\{a_n\}$ does not converge to the number a. Therefore, by the definition of convergence, given some $\epsilon > 0$, we are not always guaranteed to be able to find $N \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon \ \forall \ n \geq N$$

Choose one of these examples of ϵ. Let us define the monotonically increasing sequence $\{n_k\} = \{i \mid |a_i - a| \geq \epsilon, i \geq N, i \in \mathbb{N}\}$, or in plain English, all indices $i \geq N$ such that $|a_i - a| \geq \epsilon$. Therefore, we have constructed a subsequence $\{a_{n_k}\}$ such that

$$|a_{n_k} - a| \geq \epsilon \ \forall \ k$$

as desired.

$\Longleftarrow$ The reverse direction is similar. Given $\epsilon > 0$ and some subsequence $\{a_{n_k}\}$,

$$|a_{n_k} - a| \geq \epsilon \ \forall \ k$$

We WTS that $\{a_n\}$ does not converge to a. Suppose on the contrary, $\{a_n\}$ did converge to a. By definition of convergence, given any $\epsilon > 0$, we can find threshold $N \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon \ \forall \ n \geq N$$

Note that this definition should work for all $\epsilon > 0$. Therefore, let us choose the same ϵ as in the given statement. Note that $\{n_k\}$ is a monotonically increasing infinite sequence of natural numbers by definition of subsequence. Therefore, there exists some index in $\{n_k\}$ that is greater than or equal to our threshold N. Let us denote this index x. Therefore, at index x, we have that $|a_x - a| \geq \epsilon$ by the given and $|a_x - a| < \epsilon$ by the definition of convergence. Clearly, this is a contradiction, and so $\{a_n\}$ cannot converge to a.

□

3. For each of the following statements, determine whether it is true or false and justify your answer.
   (a) A subsequence of a bounded sequence is bounded.
   (b) A subsequence of a monotone sequence is monotone.
   (c) A subsequence of a convergent sequence is convergent.
   (d) A sequence converges if it has a convergent subsequence.

Solution.

a. True. Assume on the contradiction that a bounded sequence $\{a_n\}$ had an unbounded subsequence $\{a_{n_k}\}$. By definition of bounded, $\exists\ M \in \mathbb{R}$ such that $|a_n| < M \ \forall\ n \implies -M < a_n < M \ \forall\ n$. If subsequence $\{a_{n_k}\}$ is unbounded then there some there exists some index $x \in \{n_k\}$ such that $|a_x| \geq M$ because no scalar, not even M, is greater than the absolute value of all elements in the subsequence. However a_x is in the subsequence and also the sequence, so we have $|a_x| < M$ and $|a_x| \geq M$ simulatenously, which must be a contradiction, and so $\{a_{n_k}\}$ must also be bounded.

b. True.

c. True.

d. False. Consider the sequence $\{a_n\} = \{\frac{1}{n}\}$ and the subsequent subsequence $\{a_{n_k}\} = \{\frac{1}{n} \mid n \text{ is even}\}$. Note that $\{a_{n_k}\}$ converges to 1, but that the original subsequence we know does not converge. Therefore, we've found a counterexample.

$\square$

4. Define
$$f(x) = \begin{cases} 11 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x \leq 2. \end{cases}$$
At what points is the function $f : [0, 2] \to \mathbb{R}$ continuous? Justify your answer with a proof.

Proof.

$\square$

5. Prove that the function $f(x)$ is discontinuous at 0 , where $f(x) = \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0) = 0$.

Proof.

□

6. Let $f(x) = x$ for rational numbers and $f(x) = 0$ for irrational numbers. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that f is continuous at $x = 0$.

Proof. $\square$

7. Is it true that if $f : [a, b] \to \mathbb{R}$ has a maximum and minimum value, then f must be continuous? Justify your answer.

Proof.

$\square$

8. Let a and b be real numbers with $a < b$. Find a continuous function $f : (a, b) \to \mathbb{R}$ having an image that is unbounded above. Also, find a continuous function $f : (a, b) \to \mathbb{R}$ having an image that is bounded above but does not attain a maximum value.

Proof.

$\square$