MATH410: Advanced Calculus I

James Zhang*

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These are my notes for UMD's MATH410: Advanced Calculus I. These notes are taken live in class ("live-TEX"-ed). This course is taught by Lecturer Anna Szczekutowicz.

Contents

*Email: jzhang72@terpmail.umd.edu

§1 Set Theory Preliminaries

This section covers the foundation of analysis, which is just the set of real numbers. It covers basic definitions such as $\in, \notin, \emptyset, \subseteq, =, \cap, \cup, \setminus$, so for example

Definition 1.1. **Intersection** of A and B is $C = A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Some quantifiers include $\forall, \exists, \exists!$ and some number sets include $\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^C$.

Definition 1.2. The real numbers $\mathbb{R}$ satisfies 3 groups of axioms: Refer to the notes on Canvas for the Consequences of all of the following axioms.

1. Field $(+, *)$
   - Commutativity of Addition
   - Associativity
   - Additive Identity
   - Additive Inverse
   - Commutativty of Multiplcation
   - Associativity of Multiplication
   - Multiplicative Identity
   - Multiplicative Inverse
   - Distributive Property

   The set of integers $\mathbb{Z}$ is not a field because it fails under the multiplicative inverse.

2. Positivity

   There is a subset of $\mathbb{R}$ denoted by $\mathcal{P}$, called the set of positive numbers for which:
   - If x and y are positive, then $x + y$ and xy are both positive.
   - For each $x \in \mathbb{R}$, eaxctly one of the following 3 alternatives is true: $x \in \mathcal{P}$, $-x \in \mathcal{P}$, or $x = 0$

3. Completeness

Definition 1.3. **Absolute value** is defined as

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Definition 1.4. **Triangle Inequality** is $\forall\ a, b \in \mathbb{R}, |a + b| \leq |a| + |b|$

Proof. Assume without loss of generality, $a \geq b$. We will proceed with proof by cases.
Case 1: Assume $a \geq b \geq 0$. Then $|a + b| = a + b$ by the definition of absolute value since $a \geq 0, b \geq 0 \implies |a + b| = a + b = |a| + |b|$.

Case 2: Now assume $a \geq 0 \geq b$ and $a + b \geq 0$. Note since $b \leq 0$ then $b \leq |b|$. Then

$$|a + b| = a + b \leq |a| + |b|$$

by the definition of absolute value and our above note.
Case 3: Now consider $a \geq 0 \geq b$ and $a + b < 0$. So

$$|a + b| = -(a + b) = -a - b \leq |a| + |b|$$

Case 4: Now consider $0 \geq a \geq b$ so $a + b < 0$. Therefore,

$$|a + b| = -(a + b) = -a + -b = |a| + |b|$$

□

§2 The Completeness Axiom

Definition 2.1. A subset S of $\mathbb{R}$ is said to be **bounded above** if $\exists\, r \in \mathbb{R}$ such that $s \leq r \;\forall\; s \in S$
The definition of **bounded below** is similar.

Definition 2.2. The least upper bound, if it exists, is called the **supremum** of S. We denote it as the "sup" of S. Similarly, the largest lower bound is called the **infemum** and is denoted as the "inf" of S.

Definition 2.3. Let $S \subseteq R$ where $S \neq \emptyset$. If S has a largest (smallest), the element is a max (min).

Example 2.4

Find the sup of $(0, 1)$ and prove it.

Proof. Let us prove that the $sup(0, 1) = 1$. First, let us show that we have an upperbound. If $x \in (0, 1)$, then $x \leq 1$. By definition of upperbound, 1 is an upper bound. Note that we can find many other upper bounds.
On the contrary, assume $x < 1$ is an upper bound. Now consider the average $\frac{1+x}{2}$.

$$\frac{x+1}{2} < \frac{1+1}{2} = 1$$

Therefore, we have showed that $0 < \frac{x+1}{2} < 1 \implies \frac{x+1}{2}(0, 1)$. But, $\frac{1+x}{2} > \frac{x+x}{2} \implies \frac{1+x}{2} > x$. This is a contradiction. Since x is an upper bound, and we found $\frac{1+x}{2} \in (0, 1)$ where $\frac{1+x}{2} > x$, so x is not a supremum.

□

Theorem 2.5

Suppose $S \in \mathbb{R}, S \neq \emptyset$ that is bounded above. Then a supremum exists. Every nonsempty subset S of $\mathbb{R}$ that is bounded below has a lower bound.

Note 2.6. Let c be a positive number then $\exists!$ a positive number whose square is c. $x^2 = c, x > 0$ has a unique solution and this gives us the notion of square root.

§2.1 Archimedian Property

Definition 2.7. The **Archimedian Property** is a result of the completeness axiom. Suppose there is a small $\epsilon > 0$ and c is an arbitrary large number.

1. $\exists\ n \in \mathbb{N}$ such that $c < n$, which just means that you can always find a natural number than any large number
2. $\exists\ m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$, which just means you can always find smaller rational numbers.

Proof. We will proceed by contradiction. Assume that $\exists$ an upper bound c for the $\mathbb{N}$. So there is no $n \in \mathbb{N}$ s.t. $c < n$. Since $\mathbb{N}$ is bounded above, and the $\mathbb{N}$ is nonempty, the supremum exists (Completeness Axiom). Let $s = \sup \mathbb{N}$. Consider $s - 1$ and $s - 1 < s = \sup \mathbb{N}$, which is the least upper bound, so $s - 1$ is not an upper bound. So $\exists n \in \mathbb{N}$ such that $s - 1 < n \implies s < n + 1$. But $s = \sup \mathbb{N}$, the least upper bound, this is a contradiction since it is less than $(n + 1) \in \mathbb{N}$.
For part b, use $c = \frac{1}{\epsilon}$ and use part a. $\square$

Note 2.8. Some of the following are results from the Archimedian Property.

Theorem 2.9

For all $n \in \mathbb{Z}$, there is no integer in $(n, n + 1)$ (an open interval).

Theorem 2.10

If S is a nonempty subset of $\mathbb{Z}$ that is bounded above, then it has a max.

Theorem 2.11

$*$ For every $c \in \mathbb{R}$, $\exists!\ n \in \mathbb{Z}$ in $[c, c + 1)$

Definition 2.12. A subset $S \subseteq \mathbb{R}$ is said to be **dense** in $\mathbb{R}$ if for every $a, b \in \mathbb{R}$ with $a < b$, then there is a $s \in S$ s.t. $s \in (a, b)$.

Theorem 2.13

$\mathbb{Q}$ is dense in $\mathbb{R}$. Reminder that $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$

Proof. Suppose we have arbitrary $a, b \in \mathbb{R}$ and $a < b$. We want to find $\frac{m}{n} \in (a, b)$. By multiplcation, we can say we want $na < m < nb$. We want an integer m between na and nb. We can write this as

$$nb - na > 1 \implies n(b-a) > 1 \implies n > \frac{1}{b-a}$$

By part a of the Archimedian Property, let $c = \frac{1}{b-a}$, and we know that there exists some $n \in \mathbb{N}$ such that $n > c$. Since $a < b$, and $b - a > 0$, multiply

$$n > \frac{1}{b-a}$$

$$n(b-a) > 1$$

$$nb - na > 1$$

$$nb - 1 > na \implies na < nb - 1$$

By previous $(*)$, $\exists\ m \in \mathbb{Z}$ s.t. $m \in [nb-1, nb)$. Therefore, $nb - 1 \leq m < nb$. Therefore,

$$na \leq nb - 1 \leq m < nb \implies na < m < nb \implies a < \frac{m}{n} < b$$

and so we have found that there exists $m \in \mathbb{Z}, n \in \mathbb{N}$ such that $\frac{m}{n} \in (a, b)$ for all $a, b \in \mathbb{R}$ and $a < b$. Therefore, the rational numbers are dense in the real numbers. $\square$

§3 Sequences

Definition 3.1. A **sequence** of $\mathbb{R}$ is a real-valued function whose domain is $\mathbb{N}$. $f : \mathbb{N} \to \mathbb{R}$ (a list of numbers indiced by $\mathbb{N}$)

Example 3.2

A sequence of odd integers could be $a_1 = 1, a_2 = 3, a_3 = 5, \ldots, a_n = 2n - 1$ which can be

$$\{1, 3, 5, \cdots\} = \{a_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty}$$

Example 3.3

$$\{\frac{1}{n}\} = \{\frac{1}{n}\}_{n=1}^{\infty} \implies \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$$

Definition 3.4. A sequence $\{a_n\}$ is said to **converge** to a number L if $\forall \epsilon > 0$, $\exists$ an index N s.t. $\forall$ indices $n \geq N$ we have

$$|a_n - L| < \epsilon \implies \text{Notation: } \lim_{n\to\infty} a_n = L$$

Example 3.5

Suppose we have the sequence $\{\frac{(-1)^n}{n}\}$ and we WTS

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

Think of the problem as someone gives you a small $\epsilon \implies$ you have to find N, which we call the **threshold**, such that for every sequence value after the threshold is in the $\epsilon-$tube.
For example, $\epsilon = \frac{1}{2} \implies N = 3, \epsilon = \frac{1}{4} \implies N = 5$.

Above $L = 0$, sketch: we want

$$|a_n - L| < \epsilon \implies |\frac{(-1)^n}{n} - 0| < \epsilon \implies |\frac{1}{n}| < \epsilon \implies \frac{1}{\epsilon} < n$$

so choose $N = \frac{1}{\epsilon} < n$

Proof. Let $\epsilon > 0$ be given. By Archimedian Property, $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then if $n \geq N$

$$|\frac{(-1)^n}{n} - 0| = |\frac{(-1)^n}{n}| = |\frac{1}{n}| = \frac{1}{n}$$

From here, we need to relate n to N and then we can relate N to ϵ. Note that $n \geq N \implies \frac{1}{N} \geq \frac{1}{n}$ by algebra. Therefore,

$$\frac{1}{n} \leq \frac{1}{N} < \epsilon$$

by our choice of N. Therefore,

$$\frac{1}{n} < \epsilon$$

and so we are done since we have shown that

$$|\frac{(-1)^n}{n} < 0| < \epsilon$$

$\square$

Example 3.6

Given $\{\frac{n^2-2n}{n^2+1}\}$, prove that this sequence $\lim_{n\to\infty} \frac{n^2-2n}{n^2+1} = 1$.
Some sketch work: we want to show that $|\frac{n^2-2n}{n^2+1} - 1| < \epsilon$

$$|\frac{n^2-2n}{n^2+1} - 1| = |\frac{n^2-2n}{n^2+1} - \frac{n^2+1}{n^2+1}| = |\frac{-2n-1}{n^2+1}| = |\frac{2n+1}{n^2+1}|$$

Note that both the numerator and denominator are both always positive, so we can consider. Now let us use the $\leq$ operator to simplify and have one singular 'n'.

$$\frac{2n+1}{n^2+1} \leq \frac{2n+1}{n^2} \leq \frac{2n+n}{n^2} = \frac{3n}{n^2} = \frac{3}{n}$$

Recall that $n \geq N \implies \frac{1}{N} \geq \frac{1}{n} \implies \frac{1}{n} \leq \frac{1}{N}$ So we'd choose N to get rid of 3 and introduce ϵ.

Proof. Let $\epsilon > 0$. By A.P., $\exists\ N \in \mathbb{N}$ s.t. $\frac{1}{N} < \frac{\epsilon}{3}$. For $n \geq N$, then

$$|\frac{n^2-2n}{n^2+1} - 1| = \cdots = \frac{2n+1}{n^2+1} < \cdots \leq \frac{3}{n} \leq \frac{3}{N} = 3 * \frac{1}{N} < 3 * \frac{\epsilon}{3} = \epsilon$$

Therefore, we have shown that

$$|a_n - L| < \epsilon \implies \lim_{n\to\infty} \frac{n^2-2n}{n^2+1} = 1$$

$\square$