MATH410: Homework 6

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1. Prove that the following equation has exactly one solution:

$$x^5 + 5x + 1 = 0, \quad -1 < x < 0$$

Proof.
Let $f(x) = x^5 + 5x + 1$. Note that f is continuous and differentiable because it is a polynomial. Note that $f(-1) = -3 < 0$ and $f(0) = 1 > 0$. Note that $0 \in (-3, 1)$. By the IVT, $\exists\ c \in (-1, 0)$ such that $f(c) = 0$. Now I will show that this c is unique. Assume on the contrary that there are two solutions, such that $f(a) = 0 = f(b)$. By Rolle's Theorem, $\exists\ z \in (a, b)$ such that $f'(z) = 0$ but note that

$$f'(x) = 5x^4 + 5$$

and so

$$f'(z) = 5z^4 + 5 = 0 \implies z^4 = -1$$

and so z is not a real number, which is a contradiction. Therefore, the solution c is unique.

□

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2. Suppose that the function $h : \mathbb{R} \to \mathbb{R}$ is strictly monotone differentiable, $h'(x) > 0$ for all x, and $h(\mathbb{R}) = \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and define $g(x) = f\left(h^{-1}(x)\right)$ for all x. Find $g'(x)$.

Proof.
First note that the inverse h^{-1} exists by a theorem because h is strictly monotone, so it is 1-1 and therefore has an inverse. Further note that f and h are differentiable on $\mathbb{R}$ and so we can apply the Chain Rule to find

$$g'(x) = f'(h^{-1}(x)) \cdot (h^{-1})'(x)$$

Now recall that $h'(x) > 0 \ \forall \ x$, and so we can use the corollary in class which states that

$$(h^{-1})'(x) = \frac{1}{h'(h^{-1}(x))}$$

By substitution,

$$g'(x) = \frac{f'(h^{-1}(x))}{h'(h^{-1}(x))}$$

□

3. Let $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be differentiable functions and suppose that

$$g(x)f'(x) = f(x)g'(x) \quad \text{for all } x.$$

If $g(x) \neq 0$ for all x in $\mathbb{R}$, show that there is some c in $\mathbb{R}$ such that $f(x) = cg(x)$ for all x in $\mathbb{R}$.

Proof.
We WTS that $\exists\ c \in \mathbb{R}$ s.t. $f(x) = cg(x)$. Note that in order for the above equation to hold for all x, then if $g(x) \neq 0$, then both $f(x) \neq 0\ \forall\ x$ and $g'(x) \neq x\ \forall\ x$, too. Let $h : \mathbb{R} \to \mathbb{R}$ be $h(x) = \frac{f(x)}{g(x)}$. Therefore, it is sufficient now to show that $h(x) = c\ \forall\ x$, or in other words, we wish to show that h is constant. Note that h is differentiable because f and g are. By the Identity Criterion, h is constant if and only if $h'(x) = 0\ \forall\ x$. Let us compute $h'(x)$.

$$h'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g'(x))^2}$$

by the quotient rule. Note that the numerator is 0 for all x, which is a given and that $g'(x) \neq 0\ \forall\ x$. Therefore,

$$h'(x) = 0\ \forall\ x \implies h(x) = c\ \forall\ x$$

by the Identity Criterion, and so we equivalently $f(x) = cg(x)\ \forall\ x$.

□

4. Let D be the set of nonzero real numbers. Suppose that the functions $g : D \to \mathbb{R}$ and $h : D \to \mathbb{R}$ are differentiable and that
$$g'(x) = h'(x) \quad \text{for all } x \text{ in } D.$$
Do the functions $g : D \to \mathbb{R}$ and $h : D \to \mathbb{R}$ differ by a constant? (Hint: Is D an interval?)

Proof.
Recall the Identity Criterion (Differ by a Constant) which states that given some interval I and functions $g : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$, f and g differ by a constant if and only if $g'(x) = h'(x)\ \forall\ x$. Note that D is not an interval. Let us define $g : D \to \mathbb{R}$ and $h : D \to \mathbb{R}$ such that

$$g(x) = \begin{cases} x+2, & x > 0 \\ x-1, & x < 0 \end{cases} \qquad \text{and } h(x) = x\ \forall\ \in D$$

Note that $g'(x) = 1 = h'(x)\ \forall\ x \in D$. However, it is clear that g and h do not differ by a constant. $g(-1) = -2$ and $h(-1) = -1$ but $g(2) = 4$ and $h(1) = 1$. Thus, we have found a counterexample, and so the functions g and h do not necessarily differ by a constant, as desired.

□

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are each differentiable and that

$$\begin{cases} f'(x) = g(x) \quad \text{and} \quad g'(x) = -f(x) \quad \text{for all } x \\ f(0) = 0 \quad \text{and } g(0) = 1. \end{cases}$$

Prove that

$$[f(x)]^2 + [g(x)]^2 = 1 \quad \text{for all } x.$$

Proof.
Let us create a function $h : \mathbb{R} \to \mathbb{R}$ such that $h(x) = |f(x)|^2 + |g(x)|^2 \ \forall \ x$. Note that h is differentiable because f and g are. Now note that $|f(x)|^2 = f(x)$ and $|g(x)|^2 = g(x)^2 \ \forall x$. Therefore, $h(x) = f(x)^2 + g(x)^2$. By the Identity Criterion, h is constant if and only if $h'(x) = 0 \ \forall \ x$. Let us compute $h'(x)$.

$$h'(x) = 2f(x)f'(x) + 2g(x)g'(x) \ \forall \ x$$

by the Chain Rule twice. Let us show that this derivative is 0. Recall that $f'(x) = g(x)$ and $g'(x) = -f(x) \ \forall \ x$ by the given. Note that $g(x)$ and $f'(x)$ are the same sign for all x, and $f(x)$ and $f'(x)$ are different signs for all x.

$$\frac{g(x)}{f'(x)} = \frac{-f(x)}{g'(x)} = 1 \ \forall \ x$$

Therefore, by cross multiplying the above,

$$g(x)g'(x) = -f(x)f'(x) \implies g(x)g'(x) + f(x)f'(x) = 0$$

Multiplying the above by 2 we obtain $h'(x)$

$$h'(x) = 2g(x)g'(x) + 2f(x)f'(x) = 0 \ \forall \ x$$

Therefore, $h'(x) = 0 \ \forall \ x$ and so by the Identity Criterion, $h(x)$ is a constant for all x. Specifically, $h(x) = 1 \ \forall \ x$ because at $x = 0$,

$$h(x) = |f(0)|^2 + |g(x)|^2 = 0 + 1 = 1$$

Therefore, $h(x) = 1 \ \forall \ x$ as desired.

$\square$

6. Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous and that their restrictions to the open interval (a, b) are differentiable. Also suppose that $|f'(x)| \geq |g'(x)| > 0$ for all x in (a, b). Prove that

$$|f(u) - f(v)| \geq |g(u) - g(v)| \quad \text{for all } u, v \text{ in } [a, b].$$

Proof.
Let $u, v \in [a, b]$ such that $u < v$. Note that f and g satisfy the conditions of the Cauchy Mean Value Theorem, which gives us that

$$\exists\, c \in (u, v) \text{ s.t. } \frac{f'(c)}{g'(c)} = \frac{f(u) - f(v)}{g(u) - g(v)}$$

Equivalently, we can wrap both sides of the equation in absolute value and the equality still holds

$$\frac{|f'(c)|}{|g'(c)|} = \frac{|f(u) - f(v)|}{|g(u) - g(v)|}$$

$$|f(u) - f(v)| = \frac{|f'(c)|}{|g'(c)|} |g(u) - g(v)|$$

Note that $|f'(x)| \geq |g'(x)| > 0 \;\forall\, x \implies \frac{|f'(x)|}{|g'(x)|} \geq 1 \;\forall\, x$. Therefore,

$$|f(u) - f(v)| \geq |g(u) - g(v)| \;\forall\, u, v \in [a, b]$$

as desired.

$\square$

7. Suppose $a < b$ are positive real numbers and $f : [a, b] \to \mathbb{R}$ is continuous and its restriction to (a, b) is differentiable. Prove that there is a real number $c \in (a, b)$ for which

$$\frac{af(b) - bf(a)}{a - b} = f(c) - cf'(c).$$

Proof.
First let us define two auxiliary functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ such that $g(x) = \frac{f(x)}{x}$ and $h(x) = \frac{1}{x}$ $\forall$ x. Note that both f, g are continuous on $[a, b]$ and differentiable on (a, b) because f is, and we are told that $0 < a < b$. Therefore, the conditions for the Cauchy Mean Value Theorem are satisfied, and so

$$\exists\ c \in (a, b) \text{ s.t. } \frac{g'(c)}{h'(c)} = \frac{g(b) - g(a)}{h(b) - h(a)}$$

Observe that $g'(x) = \frac{xf'(x) - f(x)}{x^2}$ by the quotient rule and $g'(x) = -\frac{1}{x^2}$ by the power rule. Substituting known values, we get

$$\frac{\frac{1}{c^2}(cf'(c) - f(c))}{-\frac{1}{c^2}} = \frac{\frac{f(b)}{b} - \frac{f(a)}{a}}{\frac{1}{b} - \frac{1}{a}}$$

$$f(c) - c'f(c) = \frac{\frac{1}{ab}(af(b) - bf(a))}{\frac{1}{ab}(a - b)}$$

$$f(c) - cf'(c) = \frac{af(b) - bf(a)}{a - b}$$

as desired.

$\square$