

MATH410: Advanced Calculus I

JAMES ZHANG*

April 24, 2024

These are my notes for UMD's MATH410: Advanced Calculus I. These notes are taken live in class ("live- \TeX -ed). This course is taught by Lecturer Anna Szczekutowicz.

Contents

1	The Completeness Axiom	4
1.1	Archimedean Property	5
2	Sequences	6
2.1	Convergence	7
2.2	Boundedness	14
2.3	Sequential Compactness	17
3	Continuous Functions	19
3.1	Continuity Basics	19
3.2	Extreme Value Theorem	21
3.3	Intermediate Value Theorem	23
3.4	Uniform Continuity	24
3.5	Epsilon-Delta Criterion	26
3.6	Images, Inverses, Monotone Functions	29
3.7	Limits	32
4	Differentiation	34
4.1	Basic Differentiation Rules	34
4.2	Differentiating Inverses and Compositions	38
4.3	Rolle's Theorem and Mean Value Theorem	40
4.4	Cauchy Mean Value Theorem	43
5	Differential Equations	45
6	Integration	45
6.1	Darboux Sums and Refinement Lemma	45
6.2	Integrable and Archimedes-Riemann Theorem	48
6.3	Additivity, Monotonicity, Linearity	52
6.3.1	Additivity	52

*Email: jzhang72@terpmail.umd.edu

6.3.2	Monotonicity	53
6.3.3	Linearity	54
6.4	Continuity and Integrability	57
6.5	The First Fundamental Theorem of Calculus (FTC1)	59
6.6	FTC2 and Differentiating Integrals	61
7	Integration Techniques	64
8	Approximation by Taylor Polynomials	64
8.1	Taylor Polynomials	64
8.2	Lagrange Remainder (Error) Theorem	66
8.3	Convergence of Taylor Polynomials (n to infty)	67
8.4	Skip (Thanks Eileen)	69
8.5	Cauchy Integral Remainder Theorem	69
8.6	Skip 8.6	72

This section covers the foundation of analysis, which is just the set of real numbers. It covers basic definitions such as $\in, \notin, \emptyset, \subseteq, =, \cap, \cup, \setminus$, so for example

Definition 0.1. Intersection of A and B is $C = A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Some quantifiers include $\forall, \exists, \exists!$ and some number sets include $\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^C$.

Definition 0.2. The real numbers \mathbb{R} satisfies 3 groups of axioms: Refer to the notes on Canvas for the Consequences of all of the following axioms.

1. Field $(+, *)$
 - Commutativity of Addition
 - Associativity
 - Additive Identity
 - Additive Inverse
 - Commutativity of Multiplication
 - Associativity of Multiplication
 - Multiplicative Identity
 - Multiplicative Inverse
 - Distributive Property

The set of integers \mathbb{Z} is not a field because it fails under the multiplicative inverse.

2. Positivity

There is a subset of \mathbb{R} denoted by \mathcal{P} , called the set of positive numbers for which:

- If x and y are positive, then $x + y$ and xy are both positive.
- For each $x \in \mathbb{R}$, exactly one of the following 3 alternatives is true: $x \in \mathcal{P}$, $-x \in \mathcal{P}$, or $x = 0$

3. Completeness

Definition 0.3. Absolute value is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Definition 0.4. Triangle Inequality is $\forall a, b \in \mathbb{R}, |a + b| \leq |a| + |b|$

Proof. Assume without loss of generality, $a \geq b$. We will proceed with proof by cases.

Case 1: Assume $a \geq b \geq 0$. Then $|a + b| = a + b$ by the definition of absolute value since $a \geq 0, b \geq 0 \implies |a + b| = a + b = |a| + |b|$.

Case 2: Now assume $a \geq 0 \geq b$ and $a + b \geq 0$. Note since $b \leq 0$ then $b \leq |b|$. Then

$$|a + b| = a + b \leq |a| + |b|$$

by the definition of absolute value and our above note.

Case 3: Now consider $a \geq 0 \geq b$ and $a + b < 0$. So

$$|a + b| = -(a + b) = -a - b \leq |a| + |b|$$

Case 4: Now consider $0 \geq a \geq b$ so $a + b < 0$. Therefore,

$$|a + b| = -(a + b) = -a - b = |a| + |b|$$

□

§1 The Completeness Axiom

Definition 1.1. A subset S of \mathbb{R} is said to be **bounded above** if $\exists r \in \mathbb{R}$ such that $s \leq r \forall s \in S$

The definition of **bounded below** is similar.

Definition 1.2. The least upper bound, if it exists, is called the **supremum** of S . We denote it as the "sup" of S . Similarly, the largest lower bound is called the **infimum** and is denoted as the "inf" of S .

Definition 1.3. Let $S \subseteq \mathbb{R}$ where $S \neq \emptyset$. If S has a largest (smallest), the element is a max (min).

Example 1.4

Find the sup of $(0, 1)$ and prove it.

Proof. Let us prove that the $\sup(0, 1) = 1$. First, let us show that we have an upperbound. If $x \in (0, 1)$, then $x < 1$. By definition of upperbound, 1 is an upper bound. Note that we can find many other upper bounds.

On the contrary, assume $x < 1$ is an upper bound. Now consider the average $\frac{1+x}{2}$.

$$\frac{x+1}{2} < \frac{1+1}{2} = 1$$

Therefore, we have showed that $0 < \frac{x+1}{2} < 1 \implies \frac{x+1}{2} \in (0, 1)$. But, $\frac{1+x}{2} > \frac{x+x}{2} \implies \frac{1+x}{2} > x$. This is a contradiction. Since x is an upper bound, and we found $\frac{1+x}{2} \in (0, 1)$ where $\frac{1+x}{2} > x$, so x is not a supremum.

□

Theorem 1.5

Suppose $S \subseteq \mathbb{R}, S \neq \emptyset$ that is bounded above. Then a supremum exists. Every nonempty subset S of \mathbb{R} that is bounded below has a lower bound.

Note 1.6. Let c be a positive number then $\exists!$ a positive number whose square is c . $x^2 = c, x > 0$ has a unique solution and this gives us the notion of square root.

§1.1 Archimedean Property

Definition 1.7. The **Archimedean Property** is a result of the completeness axiom. Suppose there is a small $\epsilon > 0$ and c is an arbitrary large number.

1. $\exists n \in \mathbb{N}$ such that $c < n$, which just means that you can always find a natural number than any large number
2. $\exists m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$, which just means you can always find smaller rational numbers.

Proof. We will proceed by contradiction. Assume that \exists an upper bound c for the \mathbb{N} . So there is no $n \in \mathbb{N}$ s.t. $c < n$. Since \mathbb{N} is bounded above, and the \mathbb{N} is nonempty, the supremum exists (Completeness Axiom). Let $s = \sup \mathbb{N}$. Consider $s - 1$ and $s - 1 < s = \sup \mathbb{N}$, which is the least upper bound, so $s - 1$ is not an upper bound. So $\exists n \in \mathbb{N}$ such that $s - 1 < n \implies s < n + 1$. But $s = \sup \mathbb{N}$, the least upper bound, this is a contradiction since it is less than $(n + 1) \in \mathbb{N}$.

For part b , use $c = \frac{1}{\epsilon}$ and use part a . □

Note 1.8. Some of the following are results from the Archimedean Property.

Theorem 1.9

For all $n \in \mathbb{Z}$, there is no integer in $(n, n + 1)$ (an open interval).

Theorem 1.10

If S is a nonempty subset of \mathbb{Z} that is bounded above, then it has a max.

Theorem 1.11

* For every $c \in \mathbb{R}$, $\exists! n \in \mathbb{Z}$ in $[c, c + 1)$

Definition 1.12. A subset $S \subseteq \mathbb{R}$ is said to be **dense in \mathbb{R}** if for every $a, b \in \mathbb{R}$ with $a < b$, then there is a $s \in S$ s.t. $s \in (a, b)$.

Theorem 1.13

\mathbb{Q} is dense in \mathbb{R} . Reminder that $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$

Proof. Suppose we have arbitrary $a, b \in \mathbb{R}$ and $a < b$. We want to find $\frac{m}{n} \in (a, b)$. By multiplication, we can say we want $na < m < nb$. We want an integer m between na and nb . We can write this as

$$nb - na > 1 \implies n(b - a) > 1 \implies n > \frac{1}{b - a}$$

By part *a* of the Archimedean Property, let $c = \frac{1}{b-a}$, and we know that there exists some $n \in \mathbb{N}$ such that $n > c$. Since $a < b$, and $b - a > 0$, multiply

$$n > \frac{1}{b - a}$$

$$n(b - a) > 1$$

$$nb - na > 1$$

$$nb - 1 > na \implies na < nb - 1$$

By previous (*), $\exists m \in \mathbb{Z}$ s.t. $m \in [nb - 1, nb)$. Therefore, $nb - 1 \leq m < nb$. Therefore,

$$na < nb - 1 \leq m < nb \implies na < m < nb \implies a < \frac{m}{n} < b$$

and so we have found that there exists $m \in \mathbb{Z}, n \in \mathbb{N}$ such that $\frac{m}{n} \in (a, b)$ for all $a, b \in \mathbb{R}$ and $a < b$. Therefore, the rational numbers are dense in the real numbers. \square

§2 Sequences

Definition 2.1. A **sequence** of \mathbb{R} is a real-valued function whose domain is \mathbb{N} . $f : \mathbb{N} \rightarrow \mathbb{R}$ (a list of numbers indexed by \mathbb{N})

Example 2.2

A sequence of odd integers could be $a_1 = 1, a_2 = 3, a_3 = 5, \dots, a_n = 2n - 1$ which can be

$$\{1, 3, 5, \dots\} = \{a_n\}_{n=1}^{\infty} = \{2n - 1\}_{n=1}^{\infty}$$

Example 2.3

$$\left\{\frac{1}{n}\right\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \implies \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$$

§2.1 Convergence

Definition 2.4. A sequence $\{a_n\}$ is said to **converge** to a number L if $\forall \epsilon > 0$, \exists an index N s.t. \forall indices $n \geq N$ we have

$$|a_n - L| < \epsilon \implies \text{Notation: } \lim_{n \rightarrow \infty} a_n = L$$

Example 2.5

Suppose we have the sequence $\{\frac{(-1)^n}{n}\}$ and we WTS

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

Think of the problem as someone gives you a small $\epsilon \implies$ you have to find N , which we call the **threshold**, such that for every sequence value after the threshold is in the ϵ -tube.

For example, $\epsilon = \frac{1}{2} \implies N = 3, \epsilon = \frac{1}{4} \implies N = 5$.

Above $L = 0$, sketch: we want

$$|a_n - L| < \epsilon \implies \left| \frac{(-1)^n}{n} - 0 \right| < \epsilon \implies \left| \frac{1}{n} \right| < \epsilon \implies \frac{1}{\epsilon} < n$$

so choose $N = \frac{1}{\epsilon} < n$

Proof. Let $\epsilon > 0$ be given. By Archimedian Property, $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then if $n \geq N$

$$\left| \frac{(-1)^n}{n} - 0 \right| = \left| \frac{(-1)^n}{n} \right| = \left| \frac{1}{n} \right| = \frac{1}{n}$$

From here, we need to relate n to N and then we can relate N to ϵ . Note that $n \geq N \implies \frac{1}{N} \geq \frac{1}{n}$ by algebra. Therefore,

$$\frac{1}{n} \leq \frac{1}{N} < \epsilon$$

by our choice of N . Therefore,

$$\frac{1}{n} < \epsilon$$

and so we are done since we have shown that

$$\left| \frac{(-1)^n}{n} - 0 \right| < \epsilon$$

□

Example 2.6

Given $\{\frac{n^2-2n}{n^2+1}\}$, prove that this sequence $\lim_{n \rightarrow \infty} \frac{n^2-2n}{n^2+1} = 1$.

Some sketch work: we want to show that $|\frac{n^2-2n}{n^2+1} - 1| < \epsilon$

$$|\frac{n^2-2n}{n^2+1} - 1| = |\frac{n^2-2n}{n^2+1} - \frac{n^2+1}{n^2+1}| = |\frac{-2n-1}{n^2+1}| = |\frac{2n+1}{n^2+1}|$$

Note that both the numerator and denominator are both always positive, so we can consider. Now let us use the \leq operator to simplify and have one singular 'n'.

$$\frac{2n+1}{n^2+1} \leq \frac{2n+1}{n^2} \leq \frac{2n+n}{n^2} = \frac{3n}{n^2} = \frac{3}{n}$$

Recall that $n \geq N \implies \frac{1}{N} \geq \frac{1}{n} \implies \frac{1}{n} \leq \frac{1}{N}$ So we'd choose N to get rid of 3 and introduce ϵ .

Proof. Let $\epsilon > 0$. By A.P., $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \frac{\epsilon}{3}$. For $n \geq N$, then

$$|\frac{n^2-2n}{n^2+1} - 1| = \dots = \frac{2n+1}{n^2+1} < \dots \leq \frac{3}{n} \leq \frac{3}{N} = 3 * \frac{1}{N} < 3 * \frac{\epsilon}{3} = \epsilon$$

Therefore, we have shown that

$$|a_n - L| < \epsilon \implies \lim_{n \rightarrow \infty} \frac{n^2-2n}{n^2+1} = 1$$

□

Theorem 2.7

The Sum Property states that if

$$\lim_{n \rightarrow \infty} a_n = a \text{ and } \lim_{n \rightarrow \infty} b_n = b$$

then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = a + b$$

Some sketch work before the proof:

We want to show that $|a_n + b_n - (a + b)| < \epsilon$. Note that we can group terms together $|(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$ by the Triangle Inequality. It is known that these two terms converge. Therefore, we can choose ϵ s such that

$$|a_n - a| + |b_n - b| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

Proof.

Let $\epsilon > 0$. Since the sequences $\{a_n\}$ and $\{b_n\}$ converge to a and b , respectively, by the Archimedian Principle, $\exists N_1, N_2 \in \mathbb{N}$ such that $\frac{1}{N_1} < \frac{\epsilon}{2}$ and $\frac{1}{N_2} < \frac{\epsilon}{2}$. Choose $N = \max(N_1, N_2)$, which represents the numerically larger threshold. For all $n \geq N$, we show

$$\begin{aligned} |a_n + b_n - (a + b)| &= |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore, we have shown that $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$ □

Lemma 2.8

The Comparison Lemma (C.L.)

Let $\{a_n\}$ converge to a . Then $\{b_n\}$ converges to b if $\exists c \in \mathbb{R}^+$ and $N \in \mathbb{N}$ such that

$$\forall n \geq N, |b_n - b| \leq c|a_n - a|$$

Proof. Let $\epsilon > 0$. Since a_n converges to a , $\exists N_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{c}$, $\forall n \geq N_1$. By the Archimedian Principle, $\exists N_2 \in \mathbb{N}$ such that $\frac{1}{N_2} < \epsilon$. Choose $N = \max(N_1, N_2)$ and if $n \geq N$, then

$$\begin{aligned} |b_n - b| &\leq c|a_n - a| < c * \frac{\epsilon}{c} = \epsilon \\ \implies |b_n - b| &< \epsilon \end{aligned}$$

□

Lemma 2.9

Suppose the $\lim_{n \rightarrow \infty} a_n = a$, then for $c \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n = ca$$

Proof. Use the Comparison Lemma (above). Note that $|ca_n - ca| = |c(a_n - a)| = |c||a_n - a|$ which satisfies $|b_n - b| \leq c|a_n - a| \implies \{b_n\} = \{ca_n\} \implies b = ca$. \square

Lemma 2.10

The following is a useful property (*)

$$\lim_{n \rightarrow \infty} a_n = a \text{ iff } \lim_{n \rightarrow \infty} (a_n - a) = 0$$

Lemma 2.11

Suppose $\lim_{n \rightarrow \infty} a_n = 0$ and $\lim_{n \rightarrow \infty} b_n = 0$ then $\lim_{n \rightarrow \infty} a_n b_n = 0$.

Proof. Since $\lim_{n \rightarrow \infty} a_n = 0$ and $\sqrt{\epsilon} > 0$,

$$\exists N_1 \in \mathbb{N} \text{ s.t. } |a_n| < \sqrt{\epsilon} \forall n \geq N_1$$

Since $\lim_{n \rightarrow \infty} b_n = 0$ and $\sqrt{\epsilon} > 0$,

$$\exists N_2 \in \mathbb{N} \text{ s.t. } |b_n| < \sqrt{\epsilon} \forall n \geq N_2$$

Let $N = \max(N_1, N_2)$. Then if $n \geq N$,

$$|a_n b_n - 0| = |a_n b_n| = |a_n| * |b_n| < \sqrt{\epsilon} * \sqrt{\epsilon} = \epsilon$$

\square

Theorem 2.12

The Product Property states that if $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$ then

$$\lim_{n \rightarrow \infty} a_n b_n = ab$$

Proof. Define $\alpha_n = a_n - a$ and $\beta_n = b_n - b$. Using the * property above, since $\lim_{n \rightarrow \infty} a_n = a \implies \lim_{n \rightarrow \infty} (a_n - a) = \lim_{n \rightarrow \infty} \alpha_n = 0$ and then the same for b such that $\lim_{n \rightarrow \infty} \beta_n = 0$.

Note that

$$a_n b_n - ab = (\alpha_n + a)(\beta_n + b) - ab$$

and then by Distributive property we get

$$= \alpha_n \beta_n + \alpha_n b + a \beta_n + ab - ab = \alpha_n \beta_n + \alpha_n b + a \beta_n$$

So using the previous lemma,

$$\lim_{n \rightarrow \infty} (a_n b_n - ab) = \lim_{n \rightarrow \infty} (\alpha_n \beta_n + b \alpha_n + a \beta_n) = \lim_{n \rightarrow \infty} (\alpha_n \beta_n) + b \lim_{n \rightarrow \infty} \alpha_n + a \lim_{n \rightarrow \infty} \beta_n$$

From above, the last two terms are 0 and by the previous lemma, the first term is 0. Therefore,

$$\lim_{n \rightarrow \infty} (a_n b_n - ab) \text{ iff } \lim_{n \rightarrow \infty} (a_n b_n) = ab$$

□

Definition 2.13. A sequence **diverges** to $\infty, (-\infty)$ if

$$\forall M > (<)0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, n \geq N \text{ then } a_n > (<)M$$

Example 2.14

Prove that $\lim_{n \rightarrow \infty} (n^2 - 4n) = \infty$

Sketch: we want $a_n > M \implies n^2 - 4n > M \implies n(n - 4) > M$

Proof. Let $M > 0$ be given. By A.P., $\exists N \in \mathbb{N}$ s.t. $N > \max(M, 4)$. If $n \geq N$, then $n^2 - 4n = n(n - 4) \geq N(N - 4) > M$

Thus,

$$n^2 - 4n \rightarrow \infty \text{ as } n \rightarrow \infty$$

□

Example 2.15

Prove that $(-1)^n$ does not converge.

Proof. On the contrary, suppose $(-1)^n$ converges to a . Let $\epsilon = 1$. In the definition of convergence, then $\exists N \in \mathbb{N}$ if $n \geq N$ then

$$|(-1)^n - a| < 1$$

For $n = 2N$, meaning some even number, we get $|(-1)^n - a| = |1 - a| < 1$

Now for $n = 2N + 1$, we get $|(-1)^{2N+1} - a| = |1 + a| < 1$

Note that $|1 - a| < 1$ and $|1 + a| < 1$ so therefore

$$|1 - a| + |1 + a| < 2$$

But note, using the Triangle Inequality in the reverse direction, note that $2 = |1 - a + 1 + a| \leq |1 - a| + |1 + a| < 1 + 1 = 2$. Therefore, we've shown that $2 < 2$ which is a contradiction and therefore, $(-1)^n$ does not converge. \square

Lemma 2.16

Suppose the sequence $\{b_n\}$ of nonzero numbers converges to $b \neq 0$. Then $\{\frac{1}{b_n}\}$ converges to $\frac{1}{b}$.

Sketch: Use the Comparison Lemma to find $c \in \mathbb{R}^+$ and $N_1 \in \mathbb{N}$ such that

$$|\frac{1}{b_n} - \frac{1}{b}| < c|b_n - b|$$

We just have to find c and N_1 .

Proof. Note that

$$|\frac{1}{b_n} - \frac{1}{b}| = |\frac{b - b_n}{bb_n}| = \frac{1}{|b||b_n|}|b_n - b|$$

We want $\frac{1}{|b||b_n|}$ to be c , but this must be a single constant and not dependent on n . We want to find index N_1 such that

$$|b_n| > \frac{|b|}{2} \quad \forall n \geq N_1$$

$$\frac{1}{|b_n|} < \frac{2}{|b|}$$

If we can find N_1 then $|\frac{1}{b_n} - \frac{1}{b}| \leq \frac{2}{|b|^2}|b_n - b|$ and the term $\frac{2}{|b|^2}$ becomes our c and we can apply the Comparison Lemma, so we need N_1 to make the above true. Let $\epsilon = \frac{b}{2}$. By definition of $\{b_n\}$ converging to b , we can choose N_1 such that $|b_n - b| < \epsilon \quad \forall n \geq N_1$.

$$|b_n - b| < \frac{|b|}{2}$$

$$-\epsilon < b_n - b < \epsilon$$

$$b - \epsilon < b_n < b + \epsilon$$

Check $b > 0, b < 0$ since $\epsilon = \frac{|b|}{2}$. When $b > 0, \epsilon = \frac{b}{2}$ so

$$b_n \in (b - \frac{b}{2}, b + \frac{b}{2}) = (\frac{b}{2}, \frac{3b}{2})$$

so $b_n > \frac{b}{2}$. When $b < 0 \dots$ So $|b_n| > \frac{|b|}{2}$ and this N_1 works and apply the Comparison Lemma. \square

Theorem 2.17

Let $\lim_{n \rightarrow \infty} a_n = a$, $\lim_{n \rightarrow \infty} b_n = b$, and $b_n \neq 0 \forall n$ and $b \neq 0$ then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}$$

Proof.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} a_n * \frac{1}{b_n} = \lim_{n \rightarrow \infty} a_n * \lim_{n \rightarrow \infty} \frac{1}{b_n} = \frac{a}{b}$$

□

§2.2 Boundedness

Definition 2.18. A sequence $\{a_n\}$ is **bounded** if $\exists M \in \mathbb{R}$ such that $|a_n| \leq M \forall n$.

Theorem 2.19

Every convergent sequence is bounded.

- If convergent \implies bounded.
- If it is unbounded, then it diverges.

Proof. Let $\lim_{n \rightarrow \infty} a_n = a$ and take $\epsilon = 1$. Using the definition of convergence, $\exists N \in \mathbb{N}$ s.t.

$$|a_n - a| < 1 \forall n \geq N$$

then $|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a| \forall n \geq N$ by the Triangle Inequality and then the definition of the converging sequence. However, we need to show that this is true (bounded by a constant) for all n , not just for all $n \geq N$.

Define $M = \max(1 + |a|, |a_1|, \dots, |a_{N-1}|)$. Note that there the $N - 1$ terms are finite and so a max exists. Then

$$|a_n| \leq M \forall n$$

and so $\{a_n\}$ is bounded. □

Remark 2.20. Recall that a set $S \subset \mathbb{R}$ is dense in \mathbb{R} if every open set $(a, b) \in \mathbb{R}$ contains a point $s \in S$.

Definition 2.21. A set of numbers $\{x_n\}$ is in a set S provided that $x_n \in S \forall n$.

Lemma 2.22

A set S is **dense** in \mathbb{R} if and only if every $x \in \mathbb{R}$ is a limit of a sequence of a sequence in S .

Proof.

\Rightarrow Let $S \subset \mathbb{R}$ be dense in \mathbb{R} . Fix $x \in \mathbb{R}$ and let n be an index. Since S is dense, there is an element in S in $(x, x + \frac{1}{n})$. For each n , this defines $\{s_n\}$ with

$$s \in (x, x + \frac{1}{n})$$

$$x < s < x + \frac{1}{n}$$

$$|s_n - x| < \frac{1}{n} \quad \forall n$$

$$|s_n - x| < 1|\frac{1}{n} - 0|$$

Use the Comparison Lemma since $\{\frac{1}{n}\}$ converges to 0. So, $\{s_n\}$ converges to x .

\Leftarrow Let S have the property that every number in \mathbb{R} is the limit of a sequence in S . We want to show that any open interval in \mathbb{R} contains a point $s \in S$. Consider an open interval $(a, b) \in \mathbb{R}$. Consider $\frac{a+b}{2} = s \in \mathbb{R}$. By assumption, $\exists \{s_n\}$ of points in S s.t. $\lim_{n \rightarrow \infty} s_n = s$. Define $\epsilon = \frac{b-a}{2} > 0$. By definition of convergence, $\exists N$ s.t. $|s_n - s| < \epsilon \quad \forall n \in \mathbb{N}$.

$$-\epsilon < s_n - s < \epsilon$$

$$s - \epsilon < s_n < s + \epsilon$$

$$\frac{a+b}{2} - \frac{b-a}{2} < s_n < \frac{a+b}{2} + \frac{b-a}{2}$$

$$a < s_n < b$$

The point $s_N \in S$ and $s_n \in (a, b)$ so S is dense in \mathbb{R} . □

Definition 2.23. The **sequential density of \mathbb{Q}** states that every \mathbb{R} is the limit of a sequence in \mathbb{Q} .

Theorem 2.24

Let $\{c_n\} \in [a, b]$ and $\lim_{n \rightarrow \infty} c_n = c$ then $c \in [a, b]$ also.

Definition 2.25. $S \subset \mathbb{R}$ is said to be **closed** (set) if $\{a_n\}$ is a sequence in S that converges to a , then $a \in S$ also.

Example 2.26

$(0, 1]$ not closed since $\{\frac{1}{n} \in (0, 1]\}$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ but $0 \notin (0, 1]$.

Example 2.27

\mathbb{Q} is not closed since we can find $\{r_n\} \in \mathbb{Q}$ that converge to π but $\pi \notin \mathbb{Q}$.

Definition 2.28. A $\{a_n\}$ is said to be **monotonically increasing (decreasing)** if $a_{n+1} \geq (\leq) a_n \forall n$

Note 2.29. If a sequence is monotone, then it is either monotonically increasing or decreasing.

Theorem 2.30

Monotone Convergence Theorem (MCT) states that a monotone sequence converges if and only if it is bounded. Moreover, the bounded monotone $\{a_n\}$ converges to the

1. $\sup\{a_n \mid n \in \mathbb{N}\}$ if monotone increasing
2. $\inf\{a_n \mid n \in \mathbb{N}\}$ if monotone decreasing

Proof.

\Rightarrow Note that we already showed that convergent sequences are bounded.

\Leftarrow We want to show that our sequence converges to either the \inf, \sup depending on if its either increasing or decreasing. Now assume that our sequence is bounded. Define $S = \{a_n \mid n \in \mathbb{N}\}$ and S is bounded by assumption. Since S is nonempty and bounded above, S has $\sup S = l$ by the Completeness Axiom. Claim $\lim_{n \rightarrow \infty} a_n = l$. Let $\epsilon > 0$ be given, and we want to show the usual definition of convergence.

Note that

$$\begin{aligned} |a_n - l| &< \epsilon \\ -\epsilon &< a_n - l < \epsilon \\ l - \epsilon &< a_n < l + \epsilon \quad \forall n \geq N \end{aligned}$$

But l is an upper bound for $S \Rightarrow a_n \leq l < l + \epsilon \forall n$.

On the other hand, since l is the least upper bound for S , $l - \epsilon$ is not an upper bound for S . So, $\exists N$ such that $l - \epsilon < a_N$.

Since a_n is monotonically increasing. $l - \epsilon < a_N \leq a_n \forall n \geq N$. Thus, we have $N \in \mathbb{N}$ such that $\forall n \geq N$ we have $|a_n - l| < \epsilon$, as desired. \square

Remark 2.31. The formula for a finite geometric sum is $S_n = \sum_{k=1}^n r^k$ where $r \neq 1, r < 1$.

$$S_n = \frac{r - r^{n+1}}{1 - r}$$

Example 2.32

Consider $S_n = \sum_{k=1}^n \frac{1}{k} \cdot \frac{1}{2^k}$

$$k = 1 \implies s_1 = \frac{1}{2}$$

$$k = 2 \implies s_2 = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2^2} = \frac{5}{8}$$

$$k = 3 \implies \frac{1}{2} + \frac{1}{8} + \frac{1}{3} \cdot \frac{1}{2^3}$$

$$\vdots$$

so this sequence is monotonically increasing. Is it bounded

$$S_n = \sum_{k=1}^n \frac{1}{k} \cdot \frac{1}{2^k} \leq \sum_{k=1}^n \frac{1}{2^k} = \frac{\frac{1}{2} - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 1$$

Theorem 2.33

The Nested Interval Theorem. Suppose that $I_n = [a_n, b_n]$ is a sequence of intervals, for which $I_{n+1} \subset I_n \forall n$. Then the intersection of those intervals is a nonempty closed interval

$$\cap_{i=1}^{\infty} I_n = [a, b]$$

where $a = \sup a_n, b = \inf b_n$. Furthermore, if $\lim_{n \rightarrow \infty} a_n - b_n = 0$ then $\cap_{i=1}^{\infty} I_n$ contains a single point.

Proof.

\Leftarrow Let $X \in \cap_{i=1}^{\infty} I_n$. So for all $n \in \mathbb{N}, x \in I_n$ by definition of intersection. Therefore,

$$a_n \leq x \leq b_n \forall n$$

Note that x is an upper bound for a_n . So, by definition of sup, $a = \sup a_n \leq x$.

$$a \leq x \leq b \implies x \in [a, b]$$

\implies The reverse direction is similar. □

§2.3 Sequential Compactness

Definition 2.34. Consider a sequence $\{a_n\}$ and let $\{n_k\}$ be a sequence of \mathbb{N} that is strictly increasing. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k} \forall k$ is a **subsequence**.

Note 2.35. Note that a sequence may not converge, but it may be possible to find a subsequence that does.

Theorem 2.36

Let $\{a_n\}$ converges to a . Then every subsequence of $\{a_n\}$ also converges to the same limit a .

Theorem 2.37

Every sequence (does not need to converge) has a monotone increasing or decreasing subsequence.

Proof. Consider $\{a_n\}$. We all an index a **peak index** for $\{a_n\}$ if

$$a_n \leq a_m \quad \forall n \geq m$$

You can have two cases: infinitely many peak indices, or a finite number of peak indices.

Suppose there are finite number of peak indices. Then we choose N such that there are no more peak indices. Since N is not a peak index, $\exists n_1 \in \mathbb{N}$ such that $n_1 > N$ with $a_N \leq a_{n_1}$

$$\vdots$$

Continue for $n_k \implies \exists n_{k+1} \in \mathbb{N}$ with $n_{k+1} \geq n_k$ with $a_{n_k} \leq a_{n_{k+1}}$

$$a_N \leq a_{n_1} \leq \cdots \leq a_{n_k} \leq a_{n_{k+1}}$$

which is a monotonically increasing subsequence.

In the case of infinitely many peak indices, $m_1 < m_2 < m_3 < \cdots < \text{peak indices}$. Since m_1 is a peak index. Then $m_1 < m_2 \implies a_{m_1} > a_{m_2}$.

$$\vdots$$

We'll get a monotonically decreasing subsequence. □

Theorem 2.38

Every bounded sequence has a convergent subsequence.

Proof. Let $\{a_n\}$ be bounded. By the previous theorem, $\{a_n\}$ has a monotone subsequence. Since $\{a_n\}$ is bounded, $\{a_{n_k}\}$ is bounded also. By MCT, $\{a_{n_k}\}$ converges since it is monotone and bounded. □

Definition 2.39. A $S \subset \mathbb{R}$ is said to be **compact (or sequentially compact)** if every sequence in S has a convergent subsequence converging to a point in S . For a set to not be compact, we find a sequence in S that has no convergence subsequence that converges to a point in S .

Example 2.40

$[1, \infty)$ is not compact. Consider $a_n = n, a_n \rightarrow \infty$ by Archimedian Principle. Then every subsequence of n_k also diverges to ∞ . Thus, $\{a_n\}$ has no subsequence that converges.

Example 2.41

$(0, 1]$ is not compact. Let $a_n = \frac{1}{n}, a_n \rightarrow 0, n \rightarrow \infty$, so every subsequence converges to 0 also. But $0 \notin (0, 1]$ so it is not compact.

Theorem 2.42

The Sequentially Compactness Theorem (SCT) states that every interval $[a, b]$ such that $a, b \in \mathbb{R}$ is sequentially compact.

Proof. Let $\{a_n\}$ be in $[a, b]$. So, $a \leq a_n \leq b \forall n$. By a previous theorem, since $\{a_n\}$ is bounded, there exists a convergent subsequence $\{a_{n_k}\}$. Assume $\{a_{n_k}\} \rightarrow l$. Since $a \leq a_n \leq b \forall n$, then

$$a \leq a_{n_k} \leq b \forall n$$

so $l \in [a, b]$ as desired. Therefore, $\{a_n\}$ has a convergent subsequence whose limit is in the interval $[a, b]$, so it is sequentially compact. \square

Theorem 2.43

Bolzano Weirstrass Theorem: If $S \subset \mathbb{R}$, the following are equivalent

$$S \text{ is closed and bounded} \iff S \text{ is compact}$$

§3 Continuous Functions

§3.1 Continuity Basics

Note 3.1. Before $f : \mathbb{N} \rightarrow \mathbb{R}$ but now $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$. $f(x)$ is the value the function assigns to x .

Definition 3.2. A function $f : D \rightarrow \mathbb{R}$ is said to be **continuous at a point** x_0 if whenever $\{x_n\}_{n=1}^{\infty}$ converges to $x_0 \in D$, the image sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$.

Definition 3.3. A function $f : D \rightarrow \mathbb{R}$ is **continuous** if f is continuous at every point in D .

Example 3.4

Consider $f(x) = x^2 + 7x - 3$. We want to show f is continuous. Select $x_0 \in \mathbb{R}$ and let $\{x_n\} \rightarrow x_0 \implies \lim_{n \rightarrow \infty} x_n = x_0$. We want to show that

$$\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$$

Note that

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_n^2 + 7x_n - 3$$

by definition of f .

$$= \lim_{n \rightarrow \infty} x_n^2 + 7 \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} 3$$

by properties of sequences.

$$= x_0^2 + 7x_0 - 3 = f(x_0)$$

by the definition of f

Remark 3.5. Given $f : D \rightarrow \mathbb{R}, g : D \rightarrow \mathbb{R}$ are continuous, then

$$f \pm g, fg, \frac{f}{g} (g \neq 0)$$

are continuous and this follows directly from convergent sequence properties from the previous chapter. Thus, polynomials are continuous.

Example 3.6

Consider Dirichlet's function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Note that f is defined on \mathbb{R} but it is discontinuous at $x_0 \in \mathbb{R}$.

Proof. Let $x_0 \in \mathbb{R}$. By sequential density of the \mathbb{Q} and \mathbb{Q}^c , we can find

$$\{u_n\} \rightarrow x_0, u_n \in \mathbb{Q} \forall n$$

$$\{v_n\} \rightarrow x_0, v_n \in \mathbb{Q}^c \forall n$$

Since $f(u_n) = 1 \forall n$ and $f(v_n) = 0 \forall n$, then

$$\{f(u_n)\} \rightarrow 1 \text{ but } \{f(v_n)\} \rightarrow 0$$

Therefore, $\lim_{n \rightarrow \infty} f(u_n) = 1 \neq 0 = \lim_{n \rightarrow \infty} f(v_n)$ but $\{u_n\} \rightarrow x_0$ and $\{v_n\} \rightarrow x_0$ but we cannot have 2 function values for x_0 . \square

Definition 3.7. Suppose $f : D \rightarrow \mathbb{R}$ and $g : U \rightarrow \mathbb{R}$ such that $f(D) \subset U$ then we define

$$(g \circ f)(x) = g(f(x)) \quad \forall x$$

Theorem 3.8

Let $f : D \rightarrow \mathbb{R}, g : U \rightarrow \mathbb{R}$ and $f(D) \subset U$. Let f be continuous at x_0 and g be continuous at $f(x_0)$. Then $(g \circ f) : D \rightarrow \mathbb{R}$ is continuous at x_0 .

Proof. Suppose $\{x_n\} \in D$ converges to x_0 . Since f is continuous, then $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$.

$$\{f(x_n)\} \xrightarrow{n \rightarrow \infty} f(x_0)$$

Since g is continuous at $f(x_0)$, then $\lim_{n \rightarrow \infty} g(f(x_n)) = g(f(x_0))$. Therefore, $(g \circ f)(x)$ is continuous at x_0 since

$$\{g(f(x_n))\} \xrightarrow{n \rightarrow \infty} g(f(x_0))$$

\implies we can combine continuous functions and remain continuous □

§3.2 Extreme Value Theorem

Definition 3.9. $f : D \rightarrow \mathbb{R}$ attains a **maximum (minimum)** value if there is

$$x_0 \in D \text{ s.t. } f(x_0) \geq (\leq) f(x) \quad \forall x \in D$$

Remark 3.10. Recall that a nonempty set has a maximum if it is bounded above and contains its supremum ie. the supremum is in the set.

\implies Now $f : D \rightarrow \mathbb{R}$ has a maximum when the image $f(D)$ is bounded above and the supremum of the image is a functional value.

Example 3.11

$f : (0, 1) \rightarrow \mathbb{R}$ where $f(x) = 2x$. Note that the supremum of the image is 2, but 2 is not a functional value. Therefore, this function does not have a max.

Theorem 3.12

The **Extreme Value Theorem** states that a continuous function on a closed and bounded interval $f : [a, b] \rightarrow \mathbb{R}$ attains both a maximum and a minimum. Sketch: Note that we want to show that $f(D)$ is bounded above. See lemma below. After that, we need to show that the supremum is a functional value.

Lemma 3.13

Assume on the contrary that given $f : [a, b] \rightarrow \mathbb{R}$ is continuous, assume there is no M such that

$$f(x) \leq M \quad \forall x \in [a, b]$$

There is $x \in [a, b]$ at which $f(x) > n$, $\forall n$. For each n this creates a sequence $\{x_n\}$ in $[a, b]$ with $f(x) > n \quad \forall n$. $\{x_n\}$ may or may not converge. By Sequential Compactness Theorem, choose $\{x_{n_k}\}$ subsequence that converges to $x_0 \in [a, b]$. Since f is continuous at x_0 , $\{f(x_{n_k})\} \rightarrow f(x_0)$, but every convergent sequence is bounded by a theorem, so $\{f(x_{n_k})\}$ is bounded. Therefore, we have a contradiction since $f(x_{n_k}) > n_k \geq k \quad \forall k \in \mathbb{N}$. So $f : [a, b] \rightarrow \mathbb{R}$ is bounded above.

Proof. Define $S = f([a, b])$, all of the image values. By the lemma above, S is bounded. Note S is nonempty and bounded, thus by the Completeness Axiom, $c := \sup(S)$ exists. Note that we want to find $x_0 \in [a, b]$ such that $f(x_0) = c$, as this would show that the supremum is a functional value. Consider

$$c - \frac{1}{n} < c \quad \forall n$$

Note that $c - \frac{1}{n}$ is not an upper bound since c is the least upper bound. So, we can find a point $x \in [a, b]$ such that

$$c - \frac{1}{n} < f(x) < c$$

Label point x_n to create a sequence $\{x_n\}$

$$c - \frac{1}{n} < f(x_n) < c \quad \forall n$$

Since $\{\frac{1}{n}\} \rightarrow 0$ as $n \rightarrow \infty$, then $\{f(x_n)\} \rightarrow c$ by the Squeeze Theorem, as desired. Note by the Sequential Compactness Theorem, there exists a subsequence $\{x_{n_k}\}$ that converges to x_0 . Since f is continuous at x_0 , then $\{f(x_{n_k})\} \rightarrow f(x_0)$. Recall that $\{f(x_{n_k})\}$ is a subsequence of $\{f(x_n)\}$ that converges to c , and any subsequence must also converge to the same value as the full sequence. Therefore, $f(x_0) = c$. Therefore, the supremum exists and is a functional value, so we attain a max at x_0 . \square

§3.3 Intermediate Value Theorem

Theorem 3.14

The Intermediate Value Theorem state that suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous, let $c \in \mathbb{R}$ between $f(a)$ and $f(b)$. Then there exists $x_0 \in (a, b)$ such that $f(x_0) = c$.

Proof. Without loss of generality, suppose $f(a) < c < f(b)$. Recursively define a sequence of nested intervals starting at $[a, b]$ and converging to $x_0 \in (a, b)$ with $f(x) = c$. We WTS $f(x_0) = c$ by letting $a_1 = a, b_1 = b \forall n$.

$\forall n$ define $[a_n, b_n]$ by considering the midpoint $m_n = \frac{a_n + b_n}{2}$. Let us consider some cases.

\Rightarrow If $f(m_n) \leq c$, define $a_{n+1} = m_n$ and $b_{n+1} = b_n$.

\Leftarrow If $f(m_n) > c$, define $a_{n+1} = a_n$ and $b_{n+1} = m_n$.

Note that $a \leq a_n \leq a_{n+1} < b_{n+1} < b_n \leq b$ and $f(a_{n+1}) \leq c$ and $f(b_{n+1}) > c$ by definition. Now, we want to show that

$$\lim_{n \rightarrow \infty} (b_n - a_n) = 0$$

in order to apply the Nested Interval Theorem.

\vdots

So $b_n - a_n = \frac{b-a}{2^{n-1}} \forall n \xrightarrow{n \rightarrow \infty} 0$. So $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$. Thus by Nested Interval Theorem, $\exists x_0 \in (a, b)$ where $\{a_n\} \rightarrow x_0$ and $\{b_n\} \rightarrow x_0$. Since f is continuous at x_0 , then $\{f(a_n)\} \rightarrow f(x_0)$ and $\{f(b_n)\} \rightarrow f(x_0)$. Since $f(a_n) \leq c \forall n \Rightarrow f(x_0) \leq c$ and $f(b_n) \geq c \forall n \Rightarrow f(x_0) \geq c$. Thus, the only this is true is $f(x_0) = c$, as desired. \square

Example 3.15

Suppose we have $h(x) = x^5 + x + 1 = 0$. $h(x)$ is a polynomial so it is continuous. Verify that a solution exists. Let us test some points.

$$h(0) = 0 + 0 + 1 = 1$$

$$h(1) = 1 + 1 + 1 = 3$$

$$h(-1) = -1 - 1 + 1 = -1$$

By the Intermediate Value Theorem, there exists $x_0 \in (-1, 0)$ such that $x_0^5 + x_0 + 1 = 0$.

Example 3.16

$x^2 = c, c > 0$. Verify that a solution exists.

Proof. Consider $f : [0, c + 1] \rightarrow \mathbb{R}$. $f(x) = x^2, 0 \leq x \leq c + 1$. Testing points

$$f(0) = 0^2 = 0 < c$$

$$f(c + 1) = c^2 + 2c + 1 > c$$

Since x^2 it is continuous. By IVT, there exists $x_0 \in (0, c + 1)$ such that $x_0^2 = c$. \square

§3.4 Uniform Continuity

Definition 3.17. A function $f : D \rightarrow \mathbb{R}$ is said to be **uniformly continuous** if for $\{u_n\}$ and $\{v_n\}$ in D with $\lim_{n \rightarrow \infty} u_n - v_n = 0$ then $\lim_{n \rightarrow \infty} f(u_n) - f(v_n) = 0$.

Note 3.18. It doesn't make sense to say f is uniformly continuous at a singular point. Further note that there is no requirement for $\{u_n\}$ and $\{v_n\}$ to converge.

Remark 3.19. Uniform continuity is on an interval.

Example 3.20

$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 3x$ is uniformly continuous.

Proof. Let $\{u_n\}$ and $\{v_n\}$ be in \mathbb{R} and $\{u_n - v_n\} \rightarrow 0$. Then

$$\{f(u_n) - f(v_n)\} = \{3u_n - 3v_n\} = \{3(u_n - v_n)\} \rightarrow 3 * 0$$

as needed. \square

Example 3.21

$f(x) = x^2$ is not uniformly continuous on $f : \mathbb{R} \rightarrow \mathbb{R}$. To do this, we must find a pair of sequences that doesn't work.

Proof. Let $\{u_n\} = \{n + \frac{1}{n}\}$ and $\{v_n\} = \{n\}$. Note that $\{u_n - v_n\} \rightarrow 0$ but

$$\{f(u_n) - f(v_n)\} = \{f(n + \frac{1}{n}) - f(n)\} = \{(n + \frac{1}{n})^2 - n^2\} = \{2 + \frac{1}{n^2}\} \rightarrow 2 \neq 0$$

Therefore, f is not uniformly continuous on \mathbb{R} . \square

Example 3.22

Consider $f : (0, 2) \rightarrow \mathbb{R}$ and $f(x) = \frac{1}{x}$. This is not uniformly continuous since there is a vertical asymptote at $x = 0$.

Proof. Let $\{u_n\} = \frac{1}{n}$ and $\{v_n\} = \frac{2}{n}$. Note that $\{u_n - v_n\} \rightarrow 0$ but

$$\{f(u_n) - f(v_n)\} = \left\{f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right)\right\} = \left\{n - \frac{n}{2}\right\} = \left\{\frac{n}{2}\right\} \rightarrow \infty$$

□

But now consider $f : (2, 3) \rightarrow \mathbb{R}$, $f(x) = \frac{1}{x}$. This is uniformly continuous.

Proof. Suppose $\{u_n - v_n\} \rightarrow 0$ for $\{u_n\}$ and $\{v_n\}$ in $(2, 3)$.

$$|f(u_n) - f(v_n)| = \left| \frac{1}{u_n} - \frac{1}{v_n} \right| = \left| \frac{u_n - v_n}{u_n v_n} \right|$$

We need to bound the product $u_n v_n$. Note that $u_n > 2, v_n > 2 \implies \frac{1}{u_n} < \frac{1}{2}, \frac{1}{v_n} < \frac{1}{2}$, so

$$< \frac{|u_n - v_n|}{2 * 2}$$

so $|f(u_n) - f(v_n)| \leq \frac{1}{4}|u_n - v_n|$ and so by Comparison Lemma, $\{f(u_n) - f(v_n)\} \rightarrow 0$. Note that this would work for domains $(0.00000001, \infty)$. □

Note 3.23. If $f : D \rightarrow \mathbb{R}$ is uniformly continuous then it is continuous, but not all continuous functions are uniformly continuous. The go-to example is $f(x) = x^2$ on \mathbb{R} .

Theorem 3.24

Every continuous function on a closed bounded interval $f : [a, b] \rightarrow \mathbb{R}$ is uniformly continuous.

Remark 3.25. For example, $f(x) = x^2$ on $[a, b]$ is uniformly continuous.

Proof. Let $\{u_n\}, \{v_n\} \subset [a, b]$ with $\lim_{n \rightarrow \infty} (u_n - v_n) = 0$. We WTS that $\lim_{n \rightarrow \infty} (f(u_n) - f(v_n)) = 0$. By contradiction, assume that $\{f(u_n) - f(v_n)\} \not\rightarrow 0$. Therefore,

$$\exists \epsilon > 0 \text{ s.t. } \forall N \in \mathbb{N}, \text{ there is } n \geq N$$

with

$$|f(u_n) - f(v_n)| \geq \epsilon$$

Let us create a subsequence

$$n_1 \geq N = 1 \text{ s.t. } |f(u_{n_1}) - f(v_{n_1})| \geq \epsilon$$

$$n_2 \geq n_1 + 1 \cdots |f(u_{n_2}) - f(v_{n_2})| \geq \epsilon$$

$$n_3 \geq n_2 + 1 \cdots |f(u_{n_3}) - f(v_{n_3})| \geq \epsilon$$

So $\{f(u_{n_k}) - f(v_{n_k})\}$ is a subsequence with $\{f(u_{n_k}) - f(v_{n_k})\} \geq \epsilon \forall n_k$. Because $\{u_n\}$ is a sequence in $[a, b]$, we can use Sequential Compactness to find a subsequence $\{u_{m_k}\}$ that converges to some $x_0 \in [a, b]$. Since f is continuous, then $\lim_{k \rightarrow \infty} f(u_{m_k}) = f(x_0)$. Since $\lim_{k \rightarrow \infty} (u_n - v_n) = 0 \implies \lim_{k \rightarrow \infty} (u_{m_k} - v_{m_k}) = 0$ by a theorem. Thus,

$$\lim_{k \rightarrow \infty} v_{m_k} = \lim_{k \rightarrow \infty} u_{m_k} - \lim_{k \rightarrow \infty} (u_{m_k} - v_{m_k}) = x_0 - 0 \implies \{v_{m_k}\} \rightarrow x_0$$

Therefore,

$$\lim_{k \rightarrow \infty} (f(u_{m_k}) - f(v_{m_k})) = f(x_0) - f(x_0) = 0$$

But this is a contradiction that

$$\forall n_k, |f(u_{n_k}) - f(v_{n_k})| \geq \epsilon$$

and so therefore, $\{f(u_n) - f(v_n) \rightarrow 0\}$ as desired. \square

§3.5 Epsilon-Delta Criterion

Definition 3.26. A function $f : D \rightarrow \mathbb{R}$ is said to satisfy the $\epsilon - \delta$ **criterion** at $x_0 \in D$ if $\forall \epsilon > 0, \exists \delta > 0$ so that

$$\forall x \in D \text{ and } |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

Note 3.27. δ depends on ϵ and maybe x_0 . For uniform continuity, however, δ cannot depend on location, so δ will not depend on x_0 in the case of uniform continuity.

Example 3.28

$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 3x$. Prove it satisfies $\epsilon - \delta$ criteria at $x_0 = 2$.

Sketch. Given $|x - 2| < \delta$. How do we show that $|f(x) - f(2)| < \epsilon$.

$$|3x - 6| \implies |3(x - 2)| < 3\delta$$

so we take $\delta = \frac{\epsilon}{3}$. □

Proof. Let $\epsilon > 0$ be given. Let $x_0 = 2$ and let $\delta = \frac{\epsilon}{3}$. Then if $|x - 2| < \delta$ then

$$|f(x) - f(x_0)| = |3x - 6| = 3|x - 2| < 3\delta = 3 * \frac{\epsilon}{3} = \epsilon$$

□

Example 3.29

$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$ at any x_0 . Show $\epsilon - \delta$ criterion.

Sketch. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$

$$|x^2 - x_0^2| = |x - x_0||x + x_0| \leq \delta|x + x_0|$$

Note the absolute value term is constant, but x could be large, so we need to bound it. Let $\delta \leq 1$. What happens to $|x + x_0|$ in this case, let's try and relate it to $|x - x_0|$.

$$\begin{aligned} |x + x_0| &= |x - x_0 + x_0 + x_0| \leq |x - x_0| + |x_0 + x_0| = |x - x_0| + 2|x_0| \\ &\leq \delta + 2|x_0| \leq 1 + 2|x_0| \end{aligned}$$

which is a constant as desired. \square

Proof. Let $\epsilon > 0$ and $x_0 \in \mathbb{R}$. Let $\delta = \min(1, \frac{\epsilon}{1+2|x_0|})$. Note that $\epsilon > 0$ and $1 + 2|x_0| > 0$ and so we confirm $\delta > 0$. Thus,

$$\delta \leq 1 \text{ and } \delta \leq \frac{\epsilon}{1 + 2|x_0|}$$

Then

$$|x + x_0| = |x - x_0 + x_0 + x_0| \leq |x - x_0| + 2|x_0| \leq \delta + 2|x_0| \leq 1 + 2|x_0|$$

since $|x - x_0| < \delta$. Thus,

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |(x - x_0)(x + x_0)| < \delta|x + x_0| \leq \delta(1 + 2|x_0|)$$

Recall that $\delta \leq \frac{\epsilon}{1+2|x_0|}$ and so

$$\delta(1 + 2|x_0|) \leq \frac{\epsilon}{1 + 2|x_0|}(1 + 2|x_0|) = \epsilon \implies |f(x) - f(x_0)| < \epsilon$$

\square

Theorem 3.30

Given $f : D \rightarrow \mathbb{R}, x_0 \in D$, f is continuous at x_0 iff f satisfies the $\epsilon - \delta$ criteria at x_0 . Note that here δ depends on ϵ and can depend on x_0 because we are talking about **continuity**.

Definition 3.31. We say $f : D \rightarrow \mathbb{R}$ satisfies the $\epsilon - \delta$ **criterion on D** if

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } \forall u, v \in D, \text{ if } |u - v| < \delta \implies |f(u) - f(v)| < \epsilon$$

Note here that δ can only depend on ϵ and not x_0 .

Theorem 3.32

Given $f : D \rightarrow \mathbb{R}$, f is uniformly continuous on D iff f satisfies $\epsilon - \delta$ criteria on D , and here, note that δ can only depend on ϵ because we are talking about **uniform continuity**.

§3.6 Images, Inverses, Monotone Functions

Definition 3.33. $f : D \rightarrow \mathbb{R}$ is called **monotonically increasing (decreasing)** if

$$\forall u, v \in D, u < v \implies f(u) \leq (\geq) f(v)$$

If "strictly", then the operators become $<$ and $>$ respectively.

Definition 3.34. $f : D \rightarrow \mathbb{R}$ is called **one-to-one (1-1)** when $f(u) = f(v) \implies u = v$.

Definition 3.35. When f is 1-1, its inverse, denoted $f^{-1}(x)$ is a function from $f(D)$ to D satisfying $f(x) = y \leftrightarrow f^{-1}(y) = x$

- $f^{-1}(f(x)) = x \forall x \in D$
- $f(f^{-1}(y)) = y \forall y \in f(D)$

Theorem 3.36

Any strictly monotone function $f : D \rightarrow \mathbb{R}$ is 1-1 and thus has an inverse.

Proof. WLOG, suppose f is strictly increasing and $f(u) = f(v)$. To show 1-1, we WTS $u = v$ for $u, v \in D$. By contradiction, if $u < v$, since f is strictly monotone increasing, then $f(u) < f(v)$. If $u > v \implies f(u) > f(v)$ by definition of strictly monotonically increasing function. Therefore, $u = v$, and so $f(u) = f(v) \implies u = v$ and so f is 1-1. \square

Example 3.37

Prove that the inverse of $f(x) = x^3$ is continuous.

Proof. Note that f is a polynomial and thus continuous. f is strictly increasing.

$$u < v \implies u^3 < v^3 = u * u * u < v * v * v$$

by properties of inequalities. By a previous theorem, since f is strictly increasing, f has an inverse. Let $x_0 \in \mathbb{R}$, let $\{x_n\} \in \mathbb{R}$ such that $\{x_n\} \rightarrow x_0$. We WTS that $f^{-1}(x_n) \rightarrow f^{-1}(x_0)$.

For notation: label $y_n = f^{-1}(x_n), y_0 = f^{-1}(x_0)$. Therefore

$$x_n = f(y_n) = y_n^3$$

$$x_0 = f(y_0) = y_0^3$$

Since $x_n \rightarrow x_0$, then $y_n^3 \rightarrow y_0^3$. We WTS $y_n \rightarrow y_0$. Let $\epsilon > 0$. Let $\delta = \min((y_0 + \epsilon)^3 - (y_0)^3, y_0^3 - (y_0 - \epsilon)^3)$. Since $\epsilon > 0$, it is easy to show that $\delta > 0$. Since

$$y_n^3 \rightarrow y_0^3, \exists N \text{ s.t. } \forall n \geq N, |y_n^3 - y_0^3| < \delta$$

We know this is true for all ϵ , so therefore we can let $\epsilon = \delta$.

$$-\delta < y_n^3 - y_0^3 < \delta$$

$$y_0^3 - \delta < y_n^3 < \delta + y_0^3$$

$$y_0^3 - (y_0^3 - (y_0 - \epsilon)^3) < y_n^3 < (y_0 + \epsilon)^3 - y_0^3 + y_0^3$$

$$(y_0 - \epsilon)^3 < y_n^3 < (y_0 + \epsilon)^3$$

$$y_0 - \epsilon < y_n < y_0 + \epsilon$$

$$|y_n - y_0| < \epsilon$$

and so $y_n \rightarrow y_0$ or $f^{-1}(x_n) \rightarrow f^{-1}(x_0)$ by definition of y_n, y_0 and so $f^{-1}(x)$ is continuous. \square

Theorem 3.38

Let $f : D \rightarrow \mathbb{R}$ is monotone. If its image is an interval, then f is continuous.

Proof. Let $x_0 \in D$ and $\{x_n\} \in D$ with $x_n \rightarrow x_0$. Suppose on the contrary that $f(x_n) \not\rightarrow f(x_0)$. Then $\exists \epsilon > 0$ and subsequence of x_n such that

$$|f(x_{n_k}) - f(x_0)| \geq \epsilon$$

Assume WLOG that f is increasing.

Case 1: If the absolute value is positive

$$f(x_{n_k}) - f(x_0) \geq \epsilon$$

$$f(x_{n_k}) \geq \epsilon + f(x_0)$$

$$f(x_{n_k}) \geq \epsilon + f(x_0) > \frac{\epsilon}{2} + f(x_0) > f(x_0)$$

Since the image of f is an interval (all points in between). So $\exists c \in D$ such that

$$f(c) = f(x_0) + \frac{\epsilon}{2}$$

$$f(x_{n_k}) > f(c) > f(x_0)$$

And since f is strictly monotone increasing, so $x_{n_k} > c > x_0$

$$|x_{n_k} - x_0| > |c - x_0| > 0$$

Note that $c - x_0$ is a constant, and so $x_{n_k} \not\rightarrow x_0$.

Case 2: If the absolute value is negative

$$f(x_0) - f(x_{n_k}) \geq \epsilon$$

$$f(x_0) - \epsilon \geq f(x_{n_k})$$

$$f(x_0) > f(x_0) - \frac{\epsilon}{2} > f(x_{n_k})$$

$$\exists c_2 \in D \text{ such that } f(c_2) = f(x_0) - \frac{\epsilon}{2}$$

Since f is strictly monotonically increasing, we know

$$x_0 > c_2 > x_{n_k}$$

$$|x_{n_k} - x_0| > |x_0 - c_2| > 0$$

Therefore, combining the conclusions from the two cases:

$$x_{n_k} > \min(|x_0 - c|, |x_0 - c_2|) > 0$$

and so therefore, $|x_{n_k}| \not\rightarrow x_0$ which is a contradiction. Therefore, f is continuous. \square

Theorem 3.39

Suppose I is an interval and $f : I \rightarrow \mathbb{R}$ is monotone. Then f is continuous if and only if its image is an interval.

Proof. Omitted. This proof follows from the IVT and the previous theorem. \square

Theorem 3.40

Let $f : I \rightarrow \mathbb{R}$, I is an interval, be strictly monotone. Then its inverse $f^{-1} : f(I) \rightarrow \mathbb{R}$ is continuous. Similar to the x^3 example above.

Example 3.41

$f : [0, \infty) \rightarrow \mathbb{R}$ with $f(x) = x^n$ is strictly increasing, so inverse is continuous. Notation: negative integer n : $x^n = \frac{1}{x^{-n}}$

- $x^n * x^m = x^{n+m}$

- $(x^n)^m = x^{nm}$

$$y^{\frac{1}{n}} = f^{-1}(y^n) \quad \forall y \geq 0, \text{ "nth root of } y\text{"}$$

Definition 3.42. For $x > 0$ and $r \in \mathbb{Q}$ with $r = \frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{N}$, we define

$$x^r = (x^m)^{\frac{1}{n}}$$

Remark 3.43. Let $r \in \mathbb{Q}$ and define $f(x) = x^r \quad \forall x \geq 0$. Then $f : [0, \infty) \rightarrow \mathbb{R}$ is continuous.

§3.7 Limits

Note 3.44. Note that before $\lim_{n \rightarrow \infty} a_n = a$ for sequences but now $\lim_{x \rightarrow a} f(x) = L$

Definition 3.45. We say $x_0 \in \mathbb{R}$ is a **limit point** of D if $\exists \{x_n\} \in (D - \{x_0\})$ and $\{x_n\} \rightarrow x_0$.

Example 3.46

For $(0, 1)$, the numbers 0 and 1 are limit points.

Definition 3.47. Given $f : D \rightarrow \mathbb{R}$ and limit point x_0 , we write

$$\lim_{x \rightarrow x_0} f(x) = l$$

if whenever $\{x_n\} \in (D - \{x_0\})$ with $x_n \rightarrow x_0$ has $\lim_{n \rightarrow \infty} f(x_n) = l$

Remark 3.48. A function is continuous at x_0 if and only if $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

Example 3.49

$\lim_{x \rightarrow 2} \sqrt{\frac{3x+3}{x^3-4}}$. Note that there are no denominator issues at $x = 2$.

Solution. Note that numerator and denominator are both continuous, and so the quotient continuous as well because the denominator is also not 0. Further note that \sqrt{x} is continuous because it is the inverse of a strictly monotone function (on the domain $[0, \infty)$). Compositions of continuous functions are continuous at $x = 2$. So

$$\lim_{x \rightarrow 2} \sqrt{\frac{3x+3}{x^3-4}} = \sqrt{\frac{3(2)+3}{2^3-4}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

□

Example 3.50

$\lim_{x \rightarrow 1} \frac{x^2-1}{x-1}$.

Note that we cannot use the quotient property like above. Let $\{x_n\} \rightarrow 1$ with $x_n \neq 1$.

$$\frac{x_n^2 - 1}{x_n - 1} = \frac{(x_n - 1)(x_n + 1)}{x_n - 1} = x_n + 1$$

So therefore

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{n \rightarrow \infty} \frac{x_n^2 - 1}{x_n - 1} = \lim_{n \rightarrow \infty} x_n + 1 = 1 + 1 = 2$$

Theorem 3.51

$f : D \rightarrow \mathbb{R}$ and $g : D \rightarrow \mathbb{R}$, $x_0 \in \mathbb{R}$ is a limit point. Let $\lim_{x \rightarrow x_0} f(x) = A$ and $\lim_{x \rightarrow x_0} g(x) = B$ and $c \in \mathbb{R}$. Then

- i. $\lim_{x \rightarrow x_0} (f(x) \pm g(x)) = A \pm B$
- ii. $\lim_{x \rightarrow x_0} (f(x)g(x)) = A \cdot B$
- iii. $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{A}{B}, g(x) \neq 0, B \neq 0$
- iv. $\lim_{x \rightarrow x_0} cf(x) = cA$

These follow directly from properties of sequences. Similarly for compositions. $f : D \rightarrow \mathbb{R}, g : U \rightarrow \mathbb{R}, x_0$ is a limit point with $\lim_{x \rightarrow x_0} f(x) = y_0$ and $\lim_{y \rightarrow y_0} g(y) = l$ and $f(D - \{x_0\}) \subset U - \{y_0\}$. Then

$$\lim_{x \rightarrow x_0} (g \circ f)(x) = l$$

We will see limits later on in Differentiation.

§4 Differentiation

§4.1 Basic Differentiation Rules

Remark 4.1. High level: to find the tangent line, take a sequence of secant lines closer and closer towards x

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = m_0 \quad (\text{slope})$$

Definition 4.2. For $x_0 \in \mathbb{R}$, the open interval $I = (a, b)$ that contains x_0 is called a **neighborhood** of x_0 .

Definition 4.3. $f : I \rightarrow \mathbb{R}$ is said to be **differentiable at x_0** if

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} := f'(x_0)$$

exists and we denote it by $f'(x_0)$, the **derivative** of f at x_0 .

Remark 4.4. If f is differentiable at every point in I , f is **differentiable** and $f' : I \rightarrow \mathbb{R}$ is called the **derivative**.

Example 4.5

$f(x) = mx + b$. Find f' .

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{mx + b - mx_0 - b}{x - x_0} = m$$

Example 4.6

$f(x) = x^2$. Find f' .

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \rightarrow x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = 2x_0$$

Note 4.7.

$$(x^2 - x_0^2) = (x - x_0)(x + x_0)$$

$$(x^3 - x_0^3) = (x - x_0)(x^2 + xx_0 + x_0^2)$$

$$(x^4 - x_0^4) = (x - x_0)(x^3 + x^2x_0 + xx_0^2 + x_0^3)$$

Notice the pattern. Binomial Expansion. Note that you can prove this general pattern using induction.

Example 4.8

$f(x) = x^n, n \in \mathbb{N}$. Find f' . **Power Rule.**

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \rightarrow x_0} \frac{(x - x_0)(\cdots)}{x - x_0}$$

where $x \neq x_0$.

$$\begin{aligned} &= x_0^{n-1} + x_0^{n-2}x_0 + x_0^{n-3}x_0^2 + \cdots + x_0^{n-1} \\ &= nx_0^{n-1} \end{aligned}$$

Theorem 4.9

If $f : I \rightarrow \mathbb{R}$ is differentiable at x_0 , f is continuous at x_0 .

Proof. Since f is differentiable at x_0 :

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and we have $\lim_{x \rightarrow x_0} (x - x_0) = 0$. We WTS that $\lim_{x \rightarrow x_0} (f(x) - f(x_0)) = 0$. Thus,

$$\lim_{x \rightarrow x_0} (f(x) - f(x_0)) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} * (x - x_0) = f'(x_0) * 0 = 0$$

as needed, so f is continuous at x_0 . □

Note 4.10. Differentiability implies continuity, but continuity doesn't imply differentiability, and the classical example to show this is $f(x) = |x|$.

Theorem 4.11

If $f : I \rightarrow \mathbb{R}, g : I \rightarrow \mathbb{R}$, both differentiable at x_0 then

a. $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{(f \pm g)(x) - (f \pm g)(x_0)}{x - x_0} &= \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \pm \lim_{x \rightarrow x_0} \frac{g(x) - g(x_0)}{x - x_0} \\ &= f'(x_0) \pm g'(x_0) \end{aligned}$$

b. $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

$$\begin{aligned} \lim_{x \rightarrow x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} &= \lim_{x \rightarrow x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} g(x) \frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \rightarrow x_0} f(x_0) \frac{g(x) - g(x_0)}{x - x_0} \end{aligned}$$

Note that since g is differentiable at x_0 , g is continuous at x_0 , and so $\lim_{x \rightarrow x_0} g(x) = g(x_0)$. Therefore, we get

$$= g(x_0)f'(x_0) + f(x_0)g'(x_0)$$

c. $(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$

Before quotient rule, we will prove $(\frac{1}{g})' = -\frac{g'(x_0)}{(g(x_0))^2}$

$$\lim_{x \rightarrow x_0} \frac{(\frac{1}{g})(x) - (\frac{1}{g})(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x - x_0} = \lim_{x \rightarrow x_0} \frac{\frac{g(x_0) - g(x)}{g(x)g(x_0)}}{x - x_0}$$

Note that g is differentiable at x_0 , so it is continuous at x_0 , and so $\lim_{x \rightarrow x_0} g(x) = g(x_0)$

$$\lim_{x \rightarrow x_0} -\frac{1}{g(x_0)g(x)} \frac{g(x) - g(x_0)}{x - x_0} = -\frac{1}{g(x_0)^2} g'(x_0)$$

Now for the quotient rule, observe that

$$(\frac{f(x_0)}{g(x_0)})' = (\frac{1}{g(x_0)} \cdot f(x_0))'$$

Using above and the product rule, we get

$$-\frac{1}{g(x_0)^2} g'(x_0)f(x_0) + f'(x_0)\frac{1}{g(x_0)} = \frac{f'(x_0)g(x_0) - g'(x_0)f(x_0)}{g(x_0)^2}$$

Note 4.12. Power rule works for negative powers too. We know that $f(x) = x^n, n \in \mathbb{N}$ s.t. $f'(x) = nx^{n-1}$.

Let $g(x) = x^n = \frac{1}{x^{-n}}, n < 0$. So,

$$\left(\frac{1}{x^{-n}}\right)' \stackrel{(*)}{=} -\frac{(x^{-n})'}{(x^{-n})^2} = -\frac{(-nx^{-n-1})}{x^{-n}x^{-n}} = nx^{n-1}$$

§4.2 Differentiating Inverses and Compositions

Example 4.13

$f : [0, \infty) \rightarrow \mathbb{R}$ such that $f(x) = x^2$ and therefore $f^{-1}(y) = \sqrt{y}$.

Look at the point $x = 3, y = 9, f'(x) = 2x, f'(3) = 6$. Is the derivative of the inverse at $y = 9$ equal to $\frac{1}{6}$. Yes!

$$\lim_{y \rightarrow 9} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \rightarrow 9} \frac{\sqrt{y} - 3}{y - 9} = \lim_{y \rightarrow 9} \frac{1}{\sqrt{y} + 3} = \frac{1}{6}$$

as desired.

Theorem 4.14

Let $f : I \rightarrow \mathbb{R}$ be strictly monotone and continuous. Suppose f is differentiable at x_0 and $f'(x_0) \neq 0$. Define $J = f(I)$. Then $f^{-1} : J \rightarrow \mathbb{R}$ is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Proof. Note that J is also a neighborhood of $y_0 = f(x_0)$ by IVT. For $y \in J, y \neq y_0$, define $f^{-1}(y) = x$. Then

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}}$$

Note that f^{-1} is differentiable, and so it is continuous, and so

$$\lim_{y \rightarrow y_0} f^{-1}(y) = f^{-1}(y_0) = x_0$$

Now applying the limits, we get

$$\lim_{y \rightarrow y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \rightarrow y_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}$$

Therefore, f^{-1} is differentiable at y_0 and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

□

Corollary 4.15

For functions in general, suppose $f : I \rightarrow \mathbb{R}$ is strictly monotone and differentiable and $f'(x) \neq 0 \forall x$. Define $J = f(I)$. Then $f^{-1} : J \rightarrow \mathbb{R}$ is differentiable and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \quad \forall x \in J$$

Take $x = f(f^{-1}(x))$ and $f^{-1}(x) \rightarrow x_0$.

Lemma 4.16

For $n \in \mathbb{N}$, $g(x) = x^{1/n}$, $g : (0, \infty) \rightarrow \mathbb{R}$. Claim g is differentiable and $g'(x) = \frac{1}{n}x^{1/n-1}$.

Sketch of a Proof. Suppose $f(x) = x^n$ and so we know the inverse is $g(x) = x^{1/n}$. We know $f'(x) = nx^{n-1}$, $n \in \mathbb{N}$. From the corollary above,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n(x^{1-1/n})} = \frac{1}{n}x^{\frac{1}{n}-1}$$

as desired. □

Theorem 4.17

The **Chain Rule**. Let I be a neighborhood of x_0 , $f : I \rightarrow \mathbb{R}$ differentiable at x_0 , J is an open interval such that $f(I) \subset J$, $g : J \rightarrow \mathbb{R}$ is differentiable at $f(x_0)$. Then

$$(g \circ f) : I \rightarrow \mathbb{R} \text{ is differentiable at } x_0$$

and

$$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$$

Proof. Proof omitted in class. □

Example 4.18

Let $r = \frac{m}{n}$, $m \in \mathbb{Z}$, $n \in \mathbb{N}$ and define $f(x) = x^m$ and $g(x) = x^{1/n}$ and so $f'(x) = mx^{m-1}$ and $g'(x) = \frac{1}{n}x^{1/n-1}$. Let $h(x) = f(g(x))$. Then using the Chain Rule,

$$\begin{aligned} h'(x) &= g'(f(x))f'(x) = \frac{1}{n}(x^m)^{1/n-1} \cdot mx^{m-1} \\ &= \frac{m}{n}x^{\frac{m}{n}-m+m-1} = \frac{m}{n}x^{m/n-1} = rx^{r-1} \end{aligned}$$

as needed.

§4.3 Rolle's Theorem and Mean Value Theorem

Definition 4.19. $x_0 \in D$ of $f : D \rightarrow \mathbb{R}$ is said to be a **local max (min)** if \exists a neighborhood I of x_0 for which $f(x_0) \geq f(x)$ ($f(x_0) \leq f(x)$) $\forall x \in I \cap D$

Lemma 4.20

Suppose $f : I \rightarrow \mathbb{R}$ is differentiable at x_0 . If x_0 is either a max or min of f , then $f'(x_0) = 0$.

Proof. Let x_0 be a max WLOG. Then $f(x) \leq f(x_0) \forall x$ by definition of max at x . Consider $x < x_0$ in $x \in I$. Then

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

Note that the numerator is negative and the denominator is negative. Therefore, the entire expression is positive. Now if $x > x_0$, then

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

But in order for the derivative (limit) to exist, then for all sequences, the image sequences must converge to the same value. Therefore,

$$\begin{aligned} f'(x_0) &= \lim_{x \rightarrow x_0, x < x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \\ &= \lim_{x \rightarrow x_0, x > x_0} \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \\ &\implies f'(x_0) = 0 \end{aligned}$$

in order for the derivative to exist. □

Theorem 4.21

Rolle's Theorem says suppose there is a function $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $f : (a, b) \rightarrow \mathbb{R}$ is differentiable, and $f(a) = f(b)$, then

$$\exists x_0 \in (a, b) \text{ such that } f'(x_0) = 0$$

Proof. Let $f(a) = f(b)$. Since $f : [a, b] \rightarrow \mathbb{R}$ is continuity, apply the EVT, so f attains max and min value on $[a, b]$. If both the min/max occur at endpoints, then the function f must be constant, and so $f'(x) = 0$ at every point x in (a, b) . Otherwise, the min/max are in $I = (a, b)$ and apply the previous lemma. □

Theorem 4.22

The **Mean Value Theorem (MVT)** states that suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $f : (a, b) \rightarrow \mathbb{R}$ is differentiable. Then

$$\exists x_0 \in (a, b) \text{ such that } f'(x_0) = \frac{f(b) - f(a)}{b - a} \text{ (slope)}$$

Proof. Let $m = \frac{f(b) - f(a)}{b - a}$. For any m , apply Rolle's Theorem to $h : [a, b] \rightarrow \mathbb{R}$ defined by $h(x) = f(x) - mx \implies h'(x) = f'(x) - m$. Note that h is continuous on $[a, b]$ since $f(x)$ and $-mx$ are continuous (cont + cont = cont) from chapter 3. Similarly, h is differentiable on (a, b) since f and $-mx$ are diff (diff + diff = diff) from chapter 4. Now we need to check if $h(a) = h(b)$.

$$\begin{aligned} h(a) &= f(a) - ma = f(a) - \left(\frac{f(b) - f(a)}{b - a}\right)a = \frac{f(a)(b - a) - f(b)a + f(a)a}{b - a} \\ &= \frac{f(a)b - f(a)a - f(b)a + f(a)a}{b - a} = \frac{f(a)b - f(b)a}{b - a} \end{aligned}$$

Similarly, $h(b) = \frac{f(a)b - f(b)a}{b - a}$, it is the same algebra. Therefore, $h(a) = h(b)$ and so we can apply Rolle's Theorem and so $\exists x_0 \in (a, b)$ with $h'(x_0) = 0$. Thus,

$$f'(x_0) - m = 0 \implies f'(x_0) = m = \frac{f(b) - f(a)}{b - a}$$

as desired. □

Example 4.23

Prove that $x^3 + 3x + 1$ has a unique solution.

Solution. Let $f(x) = x^3 + 3x + 1$. Note that f is continuous because it is a polynomial, and f is differentiable. Note that $f(0) = 1 > 0$ and $f(-1) = -3 < 0$. Since $0 \in (-3, 1)$, by the IVT, $\exists x_0 \in (-1, 0)$ such that $f(x) = 0$. Is it unique? Assume not and assume there are 2 solutions such that

$$f(a) = 0 = f(b)$$

By Rolle's Theorem $\exists c \in (a, b)$ such that $f'(c) = 0$. But $f'(x) = 3x^2 + 3$ and so

$$f'(c) = 3c^2 + 3 = 0 \implies 3c^2 = -3 \implies c^2 = -1$$

which is not a real number, so it is a contradiction and therefore there must only be one solution. □

Remark 4.24. The MVT is useful when you have information about a derivative.

Definition 4.25. Identity Criterion: a function $f : D \rightarrow \mathbb{R}$ is said to be **constant** if $\exists c \in \mathbb{R}$ s.t. $f(x) = c \forall x \in D$.

Lemma 4.26

Let $f : I \rightarrow \mathbb{R}$ be differentiable. Then f is constant if and only if $f'(x) = 0 \forall x \in I$.

Proof.

\Rightarrow Let f be constant such that $f(x) = c, c \in \mathbb{R}, \forall x \in I$ by definition. Then $f'(x) = 0 \forall x \in I$ by derivative rules. Done.

\Leftarrow . Let $f'(x) = 0 \forall x \in I$. Choose $x_0 \in I$ and define $c := f(x_0)$. We WTS that $f(x) = c \forall x \in I$. Let $x \in I$ with $x < x_0$. Recall that differentiability implies continuity, so $f : [x, x_0] \rightarrow \mathbb{R}$ is continuous and $f : (x, x_0) \rightarrow \mathbb{R}$ is differentiable. By MVT, then $\exists z \in (x, x_0)$ with $f'(z) = \frac{f(x_0) - f(x)}{x_0 - x}$. But $f'(z) = 0$ since $f'(x) = 0 \forall x \in I$ by assumption.

$$f'(z) = 0 \implies f(x_0) - f(x) = 0 \implies c = f(x_0) = f(x)$$

$$\implies c = f(x) \forall x \in I, x < x_0$$

The same argument applies for $(x_0, x]$. Therefore, $f(x) = c \forall x \in I$. \square

Definition 4.27. The Identity Criterion (differ by a constant). Let $g : I \rightarrow \mathbb{R}, h : I \rightarrow \mathbb{R}$ both be differentiable. Then g, h **differ by a constant** if and only if $g'(x) = h'(x) \forall x \in I$.

$$\exists c \text{ s.t. } g(x) = h(x) + c$$

Proof. Define $f(x) = g(x) - h(x), f : I \rightarrow \mathbb{R}$ and $f'(x) = g'(x) - h'(x)$. Using the previous lemma,

$$f \text{ constant} \iff f'(x) = 0$$

$$f(x) = c \forall x \in I \iff g'(x) - h'(x) = 0$$

$$\iff g(x) - h(x) = c \text{ by def'n of } f$$

$$\iff g(x) = h(x) + c$$

Note that this gives us antiderivatives. \square

Definition 4.28. The Criterion for Strict Monotonicity. Let $f : I \rightarrow \mathbb{R}$ be differentiable. Suppose $f'(x) > 0 \forall x \in I$. Then $f : I \rightarrow \mathbb{R}$ is **strictly increasing**.

Proof. Let $u < v$ with $u, v \in I$. We WTS that $f(u) < f(v)$. Suppose $f'(x) > 0$. Apply the MVT to $f : [u, v] \rightarrow \mathbb{R}$ and choose $x_0 \in (u, v)$ at which

$$f'(x_0) = \frac{f(v) - f(u)}{v - u} > 0$$

since $f'(x) > 0 \forall x$. Since $u < v \implies v - u > 0$, and so $f(v) - f(u) > 0 \implies f(u) < f(v)$ as needed. Similar process can be shown for $f'(x) < 0$ for f strictly decreasing. \square

Remark 4.29. Knowing $f'(x_0) = 0$ does not guarantee a local min/max. Consider $f = x^3$ at $x = 0$, $f'(0) = 0$ but it is not a local min/max.

Theorem 4.30

Suppose $f : I \rightarrow \mathbb{R}$ has 2 derivatives f', f'' and $f'(x_0) = 0$. Then if

- i. $f''(x_0) > 0 \implies$ concave up, so x_0 is a local min of f
- ii. $f''(x_0) < 0 \implies$ concave down, so x_0 is a local max of f

§4.4 Cauchy Mean Value Theorem**Theorem 4.31**

Cauchy Mean Value Theorem (CMVT). Let $f : [a, b] \rightarrow \mathbb{R}, g : [a, b] \rightarrow \mathbb{R}$ continuous. Let $f : (a, b) \rightarrow \mathbb{R}, g : (a, b) \rightarrow \mathbb{R}$ be differentiable. $g'(x) \neq 0 \forall x \in (a, b)$. Then

$$\exists x_0 \in (a, b) \text{ s.t. } \frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. Let $m := \frac{f(b)-f(a)}{g(b)-g(a)}$. Define $h : [a, b] \rightarrow \mathbb{R}$ as $h(x) = f(x) - mg(x)$. Note that h is continuous on $[a, b]$ and differentiable on (a, b) because f and g are. Let us check that $h(a) = h(b)$.

$$\begin{aligned} h(a) &= f(a) - mg(a) = f(a) - \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right)g(a) \\ &= \frac{f(a)g(b) - f(a)g(a) - f(b)g(a) + f(a)g(a)}{g(b) - g(a)} \\ &= \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a)} = h(b) \end{aligned}$$

where you can do similar algebra for $g(b)$ to check. Therefore, by Rolle's Theorem, $\exists x_0 \in (a, b)$ with $h'(x_0) = 0$ but

$$h'(x_0) = f'(x_0) - mg'(x_0) = 0$$

$$f'(x_0) = mg'(x_0)$$

$$\frac{f'(x_0)}{g'(x_0)} = m = \frac{f(b) - f(a)}{g(b) - g(a)}$$

This is useful for the approximation of f using polynomials. Taylor series. We will see these after integrals. \square

Theorem 4.32

This is an application of CMVT: Let $n \in \mathbb{N}$ and $f : I \rightarrow \mathbb{R}$ have n derivatives. Suppose at some $x_0 \in I$,

$$f(x_0) = f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$$

Then $\forall x \in I$ with $x \neq x_0 \exists z \in (x, x_0) \cup (x_0, x)$ such that

$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n$$

Proof. Let $n \in \mathbb{N}$ and $f : I \rightarrow \mathbb{R}$ have n derivatives. Suppose at some $x_0 \in I$

$$f(x_0) = f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$$

Let $g(x) = (x - x_0)^n$. Note that

$$g'(x) = n(x - x_0)^{n-1}$$

$$\vdots$$

$$g^{(n)}(x) = n(n-1)(n-2) \cdots 2 \cdot 1 = n!$$

Using the CMVT for f, g on $[x, x_0]$ (or $[x_0, x]$) since f is differentiable and therefore continuous, and $g(x) = (x - x_0)^n$ is polynomial so differentiable and continuous,

$$\exists c_1 \in (x, x_0) \text{ s.t. } \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c_1)}{g'(c_1)}$$

However, note that $f(x_0) = 0$ by assumption and $g(x_0) = (x_0 - x_0)^n = 0$ by definition of g . So the above becomes

$$\frac{f(x)}{g(x)} = \frac{f'(c_1)}{g'(c_1)}$$

Repeating the process

$$\frac{f(x)}{g(x)} = \frac{f'(c_1)}{g'(c_1)} = \frac{f'(c_1) - f'(x_0)}{g'(c_1) - g'(x_0)} = \frac{f''(c_2)}{g''(c_2)}$$

for some $c_2 \in [c_1, x_0]$ and then continue iterating such that

$$\frac{f(x)}{g(x)} = \frac{f^{(n)}(c_n)}{n!} \implies f(x) = \frac{f^{(n)}(c_n)}{n!}g(x)$$

□

§5 Differential Equations

Skip this section.

§6 Integration

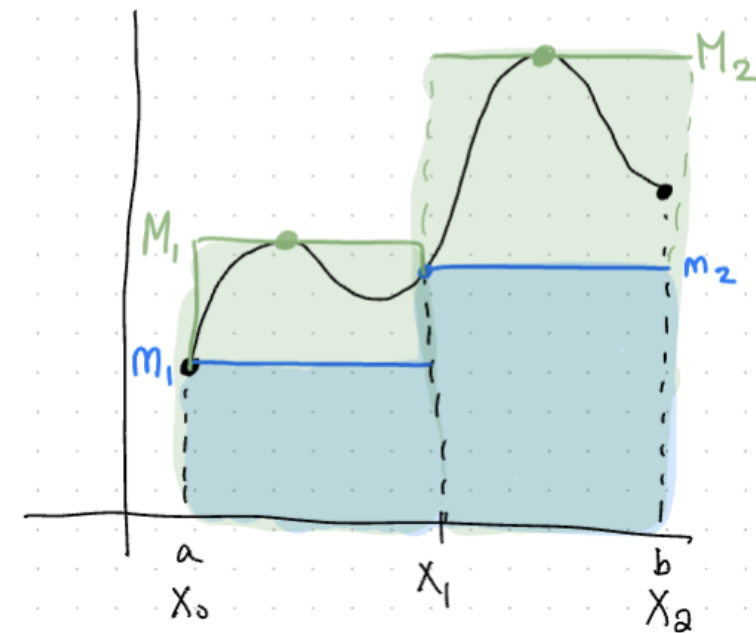
§6.1 Darboux Sums and Refinement Lemma

Remark 6.1. Unless stated otherwise, in this chapter, $I = [a, b]$ and $f : [a, b] \rightarrow \mathbb{R}$ is bounded.

Definition 6.2. Let $a < b, a, b \in \mathbb{R}$ and

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Then $P = \{x_0, x_1, x_2, \dots, x_n\}$ is called a **partition** on $[a, b]$. For all $i \geq 0$, x_i is called a **partition point** and $[x_{i-1}, x_i]$ is a **partition interval**.



Definition 6.3. Let

$$m_1 := \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

$$m_2 := \sup\{f(x) \mid x \in [x_{i-1}, x_i]\}$$

We define **Darboux Lower/Upper Sums** of f on P as

$$L(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1}) \text{ blue above}$$

$$U(f, P) = \sum_{i=1}^n M_i(x_i - x_{i-1}) \text{ green above}$$

Note that these are just the sums of areas of rectangles.

Note 6.4. Note that $m_i \leq M_i$ by definition of $\inf \leq \sup$. Therefore, $L(f, P) \leq U(f, P) \forall$ partitions of $[a, b]$. The goal is to obtain

$$L(f, P) \leq \int_a^b f \leq U(f, P)$$

Note 6.5. Given $P = \{x_0, \dots, x_n\}$ on $[a, b]$, the length of $[a, b]$ is the sum of all of the lengths of partition intervals

$$b - a = \sum_{i=1}^n (x_i - x_{i-1})$$

Definition 6.6. Given a partition of P of $[a, b]$, another partition P^* of $[a, b]$ is called a **refinement** of P if each partition point of P is also a partition point of P^* . $P \subseteq P^*$.

Lemma 6.7

The Refinement Lemma states that given partition P , if P^* is a refinement of P , then

$$L(f, P) \leq L(f, P^*) \text{ and } U(f, P^*) \leq U(f, P)$$

Proof. Let $P = \{x_0, \dots, x_n\}$. Assume P^* is a refinement with exactly one additional point compared to P and label it z . Note that you can iterate this process for more additional points.

$$P^* = \{x_0, \dots, x_{k-1}, z, z_k, \dots, x_n\}, P^* = P \cup \{z\}$$

Let $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$ and $L(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1})$ by definition. Observe that

$$L(f, P^*) = \sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) + A(z - x_{k-1}) + B(x_k - z) + \sum_{i=k+1}^n m_i(x_i - x_{i-1})$$

where $A = \inf\{f(x) \mid x \in [x_{k-1}, z]\}$ and $B = \inf\{f(x) \mid x \in [z, x_k]\}$. Then

$$L(f, P^*) - L(f, P) = A(z - x_{k-1}) + B(x_k - z) - m_k(x_k - x_{k-1})$$

and we want to show that this is ≥ 0 . Note that if $x \in [x_{k-1}, z]$ or $x \in [z, x_k]$ then $x \in [x_{k-1}, x_k] \implies f(x)m_k$ by definition of \inf . Therefore, m_k is a lower bound for $\{f(x) \mid x \in [x_{k-1}, z]\}$. Therefore, $m_k \leq A$ and $m_k \leq B$.

$$L(f, P^*) - L(f, P) = A(z - x_{k-1}) + B(x_k - z) - m_k(x_k - x_{k-1})$$

$$\geq m_k(z - x_{k-1}) + m_k(x_k - z) - m_k(x_k - x_{k-1}) = 0$$

$$L(f, P^*) - L(f, P) \geq 0 \implies L(f, P^*) \geq L(f, P)$$

and similarly for Darboux Upper Sums. □

Corollary 6.8

Let P, Q be partitions of $[a, b]$ then $L(f, P) \leq U(f, Q)$.

Proof. Consider $P \cup Q$ refinement. Then use the refinement lemma which gives us

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q)$$

Therefore, $L(f, P) \leq U(f, Q)$ as desired. \square

Definition 6.9. For $f : [a, b] \rightarrow \mathbb{R}$ such that f is bounded, then the **Lower Integral** is defined as

$$\int_a^b f := \sup\{L(f, P) \mid P \text{ partition on } [a, b]\}$$

and the **Upper Integral** is defined as

$$\int_a^{\bar{b}} f := \inf\{U(f, P) \mid P \text{ partition on } [a, b]\}$$

Lemma 6.10

Note that $\int_a^b f \leq \int_a^{\bar{b}} f$ using lower/upper sum properties.

Example 6.11

Let $f : [a, b] \rightarrow \mathbb{R}$ such that $f(x) = c$. Note that by geometry, the area is $c(b - a)$. Both upper and lower integrals = $c(b - a)$. Since by definition $m_i = c, M_i = c \forall i$. So, by the sums formula,

$$c(b - a) = c \sum_{i=1}^n (x_i - x_{i-1}) = \sum_{i=1}^n c(x_i - x_{i-1}) = L(f, P) = U(f, P)$$

So

$$\int_a^b f = c(b - a) = \int_a^{\bar{b}} f$$

Example 6.12

Let

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases}$$

Note that this is Dirichlet's function. Let $P = \{x_0, \dots, x_n\}$. Since \mathbb{Q} and \mathbb{Q}^c are dense in each $[x_{i-1}, x_i]$, then \exists rational and irrational number in each $[x_{i-1}, x_i]$ such that $m_i = 0, M_i = 1 \forall i$. Therefore,

$$L(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1}) = 0$$

$$\begin{aligned} U(f, P) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) = \sum_{i=1}^n 1(x_i - x_{i-1}) \\ &= (x_1 - x_0) + (x_2 - x_1) + \dots + (x_n - x_{n-1}) = x_n - x_0 = b - a \end{aligned}$$

Therefore,

$$\int_a^{\bar{b}} f = \inf\{b - a\} = b - a \text{ and } \int_{\underline{a}}^b f = \sup\{0\} = 0$$

§6.2 Integrable and Archimedes-Riemann Theorem

Definition 6.13. Let $f : [a, b] \rightarrow \mathbb{R}$ be bounded. Then we say f is **integrable** on $[a, b]$ if $\int_a^b f = \int_a^{\bar{b}} f$. We denote the **integral** of f

$$\int_{\underline{a}}^b f = \int_a^{\bar{b}} f = \int_a^b f$$

Recall that

$$L(f, P) \leq \int_{\underline{a}}^b f \leq \int_a^{\bar{b}} f \leq U(f, P)$$

by definition of sup, properties of Darboux sums, and then by definition of inf. As a consequence, if we rearrange the above, then

$$0 \leq \int_a^{\bar{b}} f - \int_{\underline{a}}^b f \leq U(f, P) - L(f, P)$$

$$0 \leq U(f, P) - \int_a^{\bar{b}} f \leq U(f, P) - L(f, P)$$

$$0 \leq \int_{\underline{a}}^b f - L(f, P) \leq U(f, P) - L(f, P)$$

Theorem 6.14

The **Archimedes-Riemann Theorem** states that suppose $f : [a, b] \rightarrow \mathbb{R}$ is bounded. Then f is **integrable** on $[a, b]$ if and only if \exists sequence $\{P_n\}$ of partition on $[a, b]$ such that

$$\lim_{n \rightarrow \infty} (U(f, P_n) - L(f, P_n)) = 0$$

Moreover, for any sequence of partitions

$$\lim_{n \rightarrow \infty} L(f, P_n) = \int_a^b f = \lim_{n \rightarrow \infty} U(f, P_n)$$

Proof.

\Rightarrow Suppose f is integrable. Therefore, $\int_a^b f = \int_a^{\bar{b}} f = \int_a^b f \ \forall n \in \mathbb{N}$. Note that $(\int_a^b f) - \frac{1}{n}$ is not an upper bound for $L(f, P)$ since $\int_a^{\bar{b}} f = \sup\{L(f, P) \mid P \text{ partition on } [a, b]\}$. Therefore, \exists partition Q_n of $[a, b]$ such that

$$(\int_a^b f) - \frac{1}{n} < L(f, Q_n)$$

Similarly, \exists partition R_n of $[a, b]$ such that

$$(\int_a^{\bar{b}} f) + \frac{1}{n} > U(f, R_n)$$

Let $P_n = Q_n \cup R_n$ be a refinement. Then

$$L(f, P_n) \geq L(f, Q_n) > \int_a^b f - \frac{1}{n}$$

by the Refinement Lemma and then by our definition of Q_n . Similarly, Equivalently, if we multiply by -1 , then see that $-L(f, P_n) \leq -(\int_a^{\bar{b}} f - \frac{1}{n})$

$$U(f, P_n) \leq U(f, R_n) < \int_a^{\bar{b}} f + \frac{1}{n}$$

But, note that f is integrable by assumption, which means $\int_a^b f = \int_a^{\bar{b}} f = \int_a^b f$. So,

$$\begin{aligned} 0 \leq U(f, P_n) - L(f, P_n) &\leq \int_a^b f + \frac{1}{n} - (\int_a^b f - \frac{1}{n}) \\ &= \frac{2}{n} \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

By The Comparison Lemma, $\lim_{n \rightarrow \infty} (U(f, P_n) - L(f, P_n)) = 0$. □

Theorem 6.15

The **Archimedes-Riemann Theorem** left direction proof.

Proof.

\Leftarrow . Suppose $\lim_{n \rightarrow \infty} (U(f, P_n) - L(f, P_n)) = 0$ for some sequence of partitions P_n .

We want to show that f is integrable, which means showing $\int_a^b f = \int_a^{\bar{b}} f$. But recall that

$$0 \leq \int_a^{\bar{b}} f - \int_a^b f \leq U(f, P_n) - L(f, P_n)$$

by the first consequence in the definition of integrable. By the Comparison Lemma,

$$\int_a^{\bar{b}} f - \int_a^b f = 0 \implies \int_a^{\bar{b}} f = \int_a^b f \implies f \text{ is integrable}$$

Moreover,

$$0 \leq U(f, P_n) - \int_a^{\bar{b}} f \leq U(f, P_n) - L(f, P_n) \text{ by (2)}$$

$$\implies \lim_{n \rightarrow \infty} U(f, P_n) = \int_a^{\bar{b}} f = \int_a^b f$$

Similarly, using (3),

$$\lim_{n \rightarrow \infty} L(f, P_n) = \int_a^b f = \int_a^{\bar{b}} f$$

□

Definition 6.16. $\{P_n\}$ is said to be an **Archimedian sequence of partitions (Asop)** for f on $[a, b]$ if

$$\lim_{n \rightarrow \infty} (U(f, P_n) - L(f, P_n)) = 0$$

Example 6.17

Prove $f(x) = x$ is integrable over $[0, 1]$.

Proof. Let P_n be the n th regular partition. P_n is regular is $x_i = a + i\frac{b-a}{n} = a + i\Delta x$. (This just means that each subinterval is the same length).

$$P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$$

Note that $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} \implies m_i = \inf\{f(x) \mid x \in [\frac{i-1}{n}, \frac{i}{n}]\} = f(\frac{i-1}{n}) = \frac{i-1}{n}$, which is just the left endpoint of the subinterval. Similarly, $M_i = \frac{i}{n}$, the right endpoint of the interval. Therefore, by definition of Darboux Upper and Lower Sums,

$$\begin{aligned} U(f, P_n) - L(f, P_n) &= \sum_{i=1}^n M_i(x_i - x_{i-1}) - \sum_{i=1}^n m_i(x_i - x_{i-1}) \\ &= \sum_{i=1}^n \left(\frac{i}{n} * \frac{1}{n} - \left(\frac{i-1}{n} * \frac{1}{n} \right) \right) = \sum_{i=1}^n \frac{1}{n^2} = \frac{1}{n} \end{aligned}$$

and it is known that $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$. Therefore,

$$\lim_{n \rightarrow \infty} (U(f, P_n) - L(f, P_n)) = 0$$

Thus, P_n is an Asop and using the Archimedian Theorem, $f(x) = x$ is integrable. We can find the integral

$$\int_0^1 f(x) dx = \int_0^1 x dx$$

Using the "Moreover" part of the AR Theorem,

$$\begin{aligned} \int_0^1 x dx &= \lim_{n \rightarrow \infty} U(f, P_n) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i}{n} * \frac{1}{n} \\ &= \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i=1}^n i = \lim_{n \rightarrow \infty} \frac{1}{n^2} \left(\frac{n^2 + n}{2} \right) = \frac{1}{2} \end{aligned}$$

□

Theorem 6.18

Every monotone function $f : [a, b] \rightarrow \mathbb{R}$ is integrable.

Proof. WLOG, suppose f is monotone increasing. Let P_n be a regular partition. Since f is monotone increasing,

$$m_i = f(x_{i-1}) \leq f(x) \leq f(x_i) = M_i, x \in [x_{i-1}, x_i]$$

Then,

$$\begin{aligned} U(f, P_n) - L(f, P_n) &= \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1}) \\ &= \frac{b-a}{n} \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \end{aligned}$$

Note that this is a telescope and so

$$= \frac{b-a}{n} (f(x_n) - f(x_0)) = \frac{b-a}{n} (f(b) - f(a))$$

Let $c = (b-a)(f(b) - f(a)) \in \mathbb{R}$. Then

$$\lim_{n \rightarrow \infty} (U(f, P_n) - L(f, P_n)) = \lim_{n \rightarrow \infty} \frac{1}{n} (c) = 0$$

Thus, P_n is Asop and by the AR Theorem, f is integrable. □

Theorem 6.19

Every step function $f : [a, b] \rightarrow \mathbb{R}$ is integrable.

Sketch. For the partition points inside one region, $M_i = m_i$. For the subintervals at the gaps/jumps, there are finitely many and they can still be bounded using M_i, m_i . □

§6.3 Additivity, Monotonicity, Linearity**§6.3.1 Additivity**

Theorem 6.20

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is integrable and $c \in (a, b)$. Then f is integrable on $[a, c]$ and $[c, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof. Since f is integrable on $[a, b]$ by the AR Theorem, there exists an Asop $\{P_n\}$ on $[a, b]$ for f . Let $Q_n = P_n \cup \{c\}$. We WTS that Q_n is Asop. By the Refinement Lemma,

$$L(f, P_n) \leq L(f, Q_n) \implies -L(f, P_n) \geq -L(f, Q_n)$$

$$U(f, P_n) \geq U(f, Q_n)$$

Therefore,

$$0 \leq U(f, Q_n) - L(f, Q_n) \leq U(f, P_n) - L(f, P_n)$$

Since P_n is Asop, then $\lim_{n \rightarrow \infty} (U(f, P_n) - L(f, P_n)) = 0$ and so by Comparison Lemma, $\lim_{n \rightarrow \infty} (U(f, Q_n) - L(f, Q_n)) = 0$ and so Q_n is an Asop for f on $[a, b]$. Observe that

$$Q_n = \{x_0, \dots, x_k, c, x_{k+1}, \dots, x_n\}$$

Let $R_n = Q_n \cap [a, c]$ and let $S_n = Q_n \cap [c, b]$. Then,

$$U(f, R_n) = \sum_{i=1}^k M_i(x_i - x_{i-1}) + A(c - x_k), \quad A = \sup\{f(x) \mid x \in [x_k, c]\}$$

$$U(f, S_n) = B(x_{k+1} - c) + \sum_{i=k+1}^n M_i(x_i - x_{i-1}), \quad B = \sup\{f(x) \mid x \in [c, x_{k+1}]\}$$

And so $U(f, Q_n) = U(f, R_n) + U(f, S_n)$ and similarly $L(f, Q_n) = L(f, R_n) + L(f, S_n)$.

$$0 \leq U(f, R_n) - L(f, R_n) \leq U(f, Q_n) - L(f, Q_n)$$

By CL, $\lim_{n \rightarrow \infty} (U(f, R_n) - L(f, R_n)) = 0$ and so f is integrable on $[a, c]$ and similarly for $[c, b]$. and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

□

§6.3.2 Monotonicity

Proof.

Theorem 6.21

If $f, g : [a, b] \rightarrow \mathbb{R}$ are bounded with $f(x) \leq g(x) \forall x \in [a, b]$, then

$$(1) \quad \int_a^b f \leq \int_a^b g \text{ and } \int_a^{\bar{b}} f \leq \int_a^{\bar{b}} g$$

and if f, g are integrable, then

$$(2) \quad \int_a^b f \leq \int_a^b g$$

Sketch. Since f, g are integrable, use AR Theorem and Refinement Lemma such that $\exists \{P_n\}$ on $[a, b]$ such that

$$\lim_{n \rightarrow \infty} U(f, P_n) = \int_a^b f \quad \lim_{n \rightarrow \infty} U(g, P_n) = \int_a^b g$$

Since $f(x) \leq g(x) \forall x \in [a, b]$ then $U(f, P_n) \leq U(g, P_n)$ by using sup. So

$$\int_a^b f = \lim_{n \rightarrow \infty} U(f, P_n) \leq \lim_{n \rightarrow \infty} U(g, P_n) = \int_a^b g \implies \int_a^b f \leq \int_a^b g$$

as desired. □

□

§6.3.3 Linearity

Theorem 6.22

Suppose $f, g : [a, b] \rightarrow \mathbb{R}$ are both integrable. Let $\alpha, \beta \in \mathbb{R}$. Then $(\alpha f + \beta g) : [a, b] \rightarrow \mathbb{R}$ is integrable. and

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$$

Proof.

Let us first show that scalar multiplication is satisfied. Let $P = \{x_0, \dots, x_n\}$ be a partition on $[a, b]$ and define $M_i(f), M_i(g), m_i(f), m_i(g)$ as usual.

If $\alpha \geq 0, \alpha \in \mathbb{R}$

$$M_i(\alpha f) = \alpha M_i(f) \qquad m_i(\alpha f) = \alpha m_i(f)$$

by exercising the definition of sup, inf.

If $\alpha < 0, \alpha \in \mathbb{R}$

$$M_i(\alpha f) = \alpha m_i(f) \qquad m_i(\alpha f) = \alpha M_i(f)$$

Let $\alpha \geq 0$. Then

$$\begin{aligned} U(\alpha f, P) - L(\alpha f, P) &= \sum_{i=1}^n (M_i(\alpha f) - m_i(\alpha f))(x_i - x_{i-1}) \\ &= \alpha \sum_{i=1}^n (M_i(f) - m_i(f))(x_i - x_{i-1}) \\ &= \alpha (U(f, P) - L(f, P)) \end{aligned}$$

Suppose P_n is an ASOP for f on $[a, b]$, since f is integrable take the limit.

$$\lim_{n \rightarrow \infty} (U(\alpha f, P_n) - L(\alpha f, P_n)) = \alpha \lim_{n \rightarrow \infty} (U(f, P_n) - L(f, P_n)) = 0$$

and so therefore, P_n is ASOP for αf and αf is integrable.

If $\alpha < 0$ then it's similar.

□

Theorem 6.23

Now we will show the second half of the proof, the additivity part.

Proof.

Now for the second part, we WTS that $\int_a^b (f + g) = \int_a^b f + \int_a^b g$. Consider

$$(f + g)(x_i) = f(x_i) + g(x_i) \leq M_i(f) + M_i(g) \quad \forall x \in [x_{i-1}, x_i]$$

Starting with

$$M_i(f + g) \leq M_i(f) + M_i(g)$$

Multiply by $(x_i - x_{i-1})$ and add all of the subintervals and similarly for Darboux Lower Sums to obtain

$$U(f + g, P) \leq U(f, P) + U(g, P) \quad L(f + g, P) \geq L(f, P) + L(g, P)$$

Take $\{P_n\}$ ASOP for f on $[a, b]$ since f, g are integrable by the AR Theorem. Similarly take $\{R_n\}$ ASOP for g on $[a, b]$. Let $Q_n = P_n \cup R_n$ (refinement). Now we want to show that Q_n is an ASOP for both f and g individually. Then

$$\begin{aligned} 0 &\leq U(f + g, Q_n) - L(f + g, Q_n) \\ &\leq U(f, Q_n) + U(g, Q_n) - (L(f, Q_n) + L(g, Q_n)) \\ &= U(f, Q_n) - L(f, Q_n) + U(g, Q_n) - L(g, Q_n) \end{aligned}$$

By CL, $\lim_{n \rightarrow \infty} (U(f + g, Q_n) - L(f + g, Q_n)) = 0 \implies f + g$ is integrable and so

$$\int_a^b f + g = \lim_{n \rightarrow \infty} U(f + g, Q_n) \leq \lim_{n \rightarrow \infty} (U(f, Q_n) + U(g, Q_n)) = \int_a^b f + \int_a^b g$$

But at the same time

$$\int_a^b f + g = \lim_{n \rightarrow \infty} L(f + g, Q_n) \geq \lim_{n \rightarrow \infty} (L(f, Q_n) + L(g, Q_n)) = \int_a^b f + \int_a^b g$$

which implies that $\int_a^b f + g = \int_a^b f + \int_a^b g$ and combining the result from the first part of the proof we obtain $\int_a^b \alpha f + \beta g = \alpha \int_a^b f + \beta \int_a^b g$, as desired. \square

Corollary 6.24

Suppose $f : [a, b] \rightarrow \mathbb{R}$, $|f| : [a, b] \rightarrow \mathbb{R}$ are integrable. Then

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Sketch. $\forall x \in [a, b]$

$$-|f(x)| \leq f(x) \leq |f(x)|$$

Applying monotonicity and linearity,

$$-\int_a^b |f(x)| \leq \int_a^b f(x) \leq \int_a^b |f(x)|$$

which is $\left| \int_a^b f \right| \leq \int_a^b |f|$

□

§6.4 Continuity and Integrability

So far we have only proven the following functions are integrable: x , x^2 and monotone functions.

Lemma 6.25

Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous and P partition on $[a, b]$. Then \exists a partition interval of P that contains two points u, v with

$$0 \leq U(f, P) - L(f, P) \leq (f(u) - f(v))(b - a)$$

Sketch.

Let $P = \{x_0, \dots, x_n\}$ be a partition of $[a, b]$.

- f is continuous on $[a, b]$ so f is continuous on all $[x_{i-1}, x_i] \forall i$
- By EVT, max and min are attained (stronger than inf and sup)
- Define the min and the max, $f(u_i) = m_i, f(v_i) = M_i$
- Choose i_0 such that $M_{i_0} - m_{i_0} := \max_{1 \leq i \leq n} (M_i - m_i)$. Intuitively, this means to choose the largest difference between subinterval max - min.
- Let $u := u_{i_0}, v := v_{i_0}$.
- Then $M_i - m_i \leq M_{i_0} - m_{i_0} = f(v) - f(u)$ by definition of max.
- Then $0 \leq U(f, P) - L(f, P) = \sum_{i=1}^n (M_i - m_i)(x_i - x_{i-1})$

$$\leq \sum_{i=1}^n (f(v) - f(u))(x_i - x_{i-1}) = (f(v) - f(u))(b - a)$$

So we have found $u, v \in [a, b]$ with

$$0 \leq U - L \leq (f(v) - f(u))(b - a)$$

□

Remark 6.26. Recall that continuity on $[a, b]$ implies uniform continuity. For any $\{u_n\}$ and $\{v_n\}$ in D if $\lim_{n \rightarrow \infty} (u_n - v_n) = 0 \implies \lim_{n \rightarrow \infty} (f(u_n) - f(v_n)) = 0$. Use the Lemma above for a sequence of partitions and sequence of points $\{u_n\}$ and $\{v_n\}$ to prove that if f is continuous on $[a, b]$ then f is integrable.

Theorem 6.27

A continuous function $f : [a, b] \rightarrow \mathbb{R}$ is integrable.

Proof.

Let $\{P_n\}$ be a sequence of regular partitions. For each n , apply the previous lemma:

$$(*) \quad 0 \leq U(f, P_n) - L(f, P_n) \leq (f(v_n) - f(u_n))(b - a)$$

Note that $|u_n - v_n| \leq \frac{b-a}{n}$ since $u_n, v_n \in [x_{i-1}, x_i]$. Take the limit as $n \rightarrow \infty$, by CL $\lim_{n \rightarrow \infty} (u_n - v_n) = 0$ since $\lim_{n \rightarrow \infty} \frac{b-a}{n} = 0$.

Since f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$

$$\lim_{n \rightarrow \infty} (f(u_n) - f(v_n)) = 0 \quad (1)$$

Take $(*)$ and apply a limit

$$0 \leq \lim_{n \rightarrow \infty} (U(f, P_n) - L(f, P_n)) \leq \lim_{n \rightarrow \infty} (f(v_n) - f(u_n))(b - a)$$

By CL, $\lim_{n \rightarrow \infty} (U(f, P_n) - L(f, P_n)) = 0$ and so by AR Theorem, f is integrable. \square

Theorem 6.28

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is bounded and continuous on (a, b) . Then $f : [a, b] \rightarrow \mathbb{R}$ is integrable and the value of the integral $\int_a^b f$ does not depend on the values of f at the endpoints.

Sketch. Look at the endpoints a, b and take $a_n \rightarrow a, b_n \rightarrow b$ and measure how big the finite (because bounded) gap of the discontinuity is. Bound the difference of the gap and take $n \rightarrow \infty$ for P_n .

$$U(f, P_n) - L(f, P_n) \leq \text{usual} + \text{bound}(a_n - a) + \text{bound}(b_n - b)$$

Note that f does not need continuity at the endpoints to be integrable. \square

Example 6.29

Consider

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } 0 < x \leq 1 \\ 100 & \text{if } x = 0 \end{cases}$$

$f(x) : [0, 1] \rightarrow \mathbb{R}$ is bounded since $|f(x)| \leq 100$ and $f : (0, 1) \rightarrow \mathbb{R}$ is continuous. So f is integrable by the previous theorem.

§6.5 The First Fundamental Theorem of Calculus (FTC1)

Theorem 6.30

Let $F : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) and $F' : (a, b) \rightarrow \mathbb{R}$ is continuous and bounded, then

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Proof. Assume $F' : (a, b) \rightarrow \mathbb{R}$ is continuous and bounded, then $F' : [a, b] \rightarrow \mathbb{R}$ is integrable (above, 6.4) since the integral does not depend on endpoints. Let P be a partition of $[a, b]$. Then

$$U(F', P) = \sum_{i=1}^n M_i(F')(x_i - x_{i-1})$$

Consider $[x_i, x_{i-1}]$, since F is continuous on $[a, b]$, differentiable on (a, b) , we can apply MVT to the partition interval, $\exists c_i \in [x_i, x_{i-1}]$, with

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} \quad \forall i, 1 \leq i \leq n$$

Then use sup to get

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i) \leq M_i(F')$$

by the definition of sup. Then multiply by $(x_i - x_{i-1})$ to get

$$\sum_{i=1}^n (F(x_i) - F(x_{i-1})) \leq \sum_{i=1}^n M_i(F')(x_i - x_{i-1}) = U(F', P)$$

Note that the left side is telescoping, and so we obtain that

$$F(b) - F(a) \leq U(F', P)$$

By property of inf in Upper Integral

$$F(b) - F(a) \leq \int_a^b F' = \int_a^b F'$$

because F' is integral. Similarly, it can be shown that $F(b) - F(a) \geq \int_a^b F'$ which implies that $F(b) - F(a) = \int_a^b F'$. \square

Note 6.31. Notation: $f : [a, b] \rightarrow \mathbb{R}$ is continuous and bounded on (a, b) , then FTC1 asserts if it possible to find an antiderivative $F : [a, b] \rightarrow \mathbb{R}$ for f then the integral is given by

$$\int_a^b f(x)dx = F(b) - F(a)$$

An antiderivative continuous function F having a derivative on (a, b) such that

$$F'(x) = f(x) \quad \forall (a, b)$$

Note that FTC1 only gives us some (most) integrals. In order to

§6.6 FTC2 and Differentiating Integrals

Theorem 6.32

The **Mean Value Theorem for Integrals** states that suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Then $\exists x_0 \in (a, b)$ at which

$$f(x_0) = \frac{1}{b-a} \int_a^b f(x) dx$$

Proof. Use EVT on $f : [a, b] \rightarrow \mathbb{R}$ (cont). Therefore, $\exists x_m, x_M \in [a, b]$ with

$$f(x_m) \leq f(x) \leq f(x_M) \quad \forall x \in [a, b]$$

By monotonicity of \int :

$$\int_a^b f(x_m) \leq \int_a^b f(x) \leq \int_a^b f(x_M)$$

$$f(x_m) \int_a^b 1 \leq \int_a^b f(x) \leq f(x_M) \int_a^b 1$$

$$f(x_m)(b-a) \leq \int_a^b f(x) \leq f(x_M)(b-a)$$

$$f(x_m) \leq \frac{1}{b-a} \int_a^b f(x) \leq f(x_M)$$

From here, apply the IVT, $\exists x_0 \in (x_m, x_M)$ such that

$$\exists x_0 \in (a, b) \text{ s.t. } f(x_0) = \frac{1}{b-a} \int_a^b f$$

□

Definition 6.33. Consider the **area function**

$$f(x) = \int_a^x f(t) dt$$

Input changes the upper bound. Lower bound is fixed. Tells you how much area is under $f(t)$ between a and x .

Proposition 6.34

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is integrable. (This could be discontinuous). Define $F(x) = \int_a^x f(t)dt \forall x \in [a, b]$. Then $F : [a, b] \rightarrow \mathbb{R}$ is continuous.

Proof.

Let $u, v \in [a, b]$ with $u < v$ WLOG. Let $F(v) = \int_a^v f = \int_a^u f + \int_u^v f = F(u) + \int_u^v f$ by additivity. Therefore,

$$F(v) - F(u) = \int_u^v f \quad (*)$$

Since f is integrable and bounded, choose $M > 0$,

$$-M \leq f(x) \leq M \quad \forall x \in [a, b]$$

$$-M \leq f(x) \leq M \quad \text{if } u \leq x \leq v$$

By monotone integral theorem,

$$-\int_u^v M \leq \int_u^v f \leq \int_u^v M$$

$$-M(v-u) \leq \int_u^v f \leq M(v-u)$$

$$|F(v) - F(u)| \leq M|v-u| \quad \forall u, v \in [a, b]$$

Recall this from Homework 4, if Lipschitz continuous, then it is also uniform continuous. $F : [a, b] \rightarrow \mathbb{R}$ is uniformly continuous, which implies continuous, and so we are done. \square

Remark 6.35. The above proof is more broad than the FTC2, shown below.

Theorem 6.36

The **Fundamental Theorem of Calculus 2** states that suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous. Then,

$$\frac{d}{dx} \left(\int_a^x f(t) dt \right) = f(x) \quad \forall x \in (a, b)$$

Proof.

Note that $\int_a^b f = -\int_b^a f$. Define $F(x) = \int_a^x f \quad \forall x \in [a, b]$. F is continuous from the proposition above. Let $x_0 \in (a, b)$.

We WTS that

$$F'(x_0) = \lim_{x \rightarrow x_0} \frac{F(x) - F(x_0)}{x - x_0}$$

Let $x \in (a, b)$ with $x \neq x_0$.

Case 1: If $x > x_0$. Then

$$\begin{aligned} F(x) - F(x_0) &= \int_{x_0}^x f \implies \int_a^x f - \int_a^{x_0} f = -\int_x^a f - \int_a^{x_0} f \\ &= -\left(\int_x^a f + \int_a^{x_0} f \right) \implies -\int_x^{x_0} f = \int_{x_0}^x f \end{aligned}$$

Case 2: If $x < x_0$, then

$$\begin{aligned} F(x) - F(x_0) &= \int_{x_0}^x f = \int_a^x f - \int_a^{x_0} f = -\int_x^a f - \int_a^{x_0} f \\ &= -\left(\int_x^a f + \int_a^{x_0} f \right) = -\int_x^{x_0} f = \int_{x_0}^x f \end{aligned}$$

In both cases, apply MVT for \int

$$\exists c \in (a, b) \text{ s.t. } f(c(x)) = \frac{1}{x - x_0} \int_{x_0}^x f$$

Rearranging and using what we have above,

$$= \frac{F(x) - F(x_0)}{x - x_0}$$

Applying the limit as $x \rightarrow x_0$,

$$\lim_{x \rightarrow x_0} \frac{F(x) - F(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} f(c(x)) = F'(x_0)$$

by definition of derivative. Since f is continuous at x_0 , then $\lim_{x \rightarrow x_0} f(c(x)) = f(x_0)$ as desired. \square

Corollary 6.37

Some corollary results from FTC2.

1. $f : [a, b] \rightarrow \mathbb{R}$ is continuous then $\frac{d}{dx} \int_a^x f = \frac{d}{dx} (-\int_a^x f) = -f(x)$
2. I, J open intervals, $h(I) \subseteq J$. Then,

$$f : I \rightarrow \mathbb{R} \text{ cont} \qquad h : J \rightarrow \mathbb{R} \text{ diff}$$

Fix a : then $\frac{d}{dx} \int_a^{h(x)} f = f(h(x)) * h'(x)$ using the chain rule

$$\int_{x^2}^{-3} \cos(1 - 5t) dt = F(x) \implies F'(x) = -e^{x^{2x^2}} \cos(1 - 5x^2) * 2x$$

§7 Integration Techniques

Skip this section.

§8 Approximation by Taylor Polynomials**§8.1 Taylor Polynomials**

Note 8.1. Note that $(e^x)^1 = e^x$.

Definition 8.2. Let I be a neighborhood of x_0 , and so we say $f : I \rightarrow \mathbb{R}$ and $g : I \rightarrow \mathbb{R}$ are said to have **contact of order 0 at x_0** if $f(x_0) = g(x_0)$.

\vdots

They are said to have **contact of order n at x_0** oif $f^{(k)}(x_0) = g^{(k)}(x_0) \forall k \in [0, n]$.

Note 8.3. Note that in this chapter, $0 \in \mathbb{N}$ because for the above definition, it means just the functional values.

Theorem 8.4

Let I be a neighborhood of x_0 . Suppose $f : I \rightarrow \mathbb{R}$ has n derivatives, $n \in \mathbb{N}$. Then there is a unique polynomial of degree at most n that has contact of order n with f at x_0 . The polynomial is defined as

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Proof.

Suppose $P_n(x)$ is a general polynomial.

$$P_n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$P_n(x)$ has degree n . Now let us work to get contact of order at least n with f at $x = x_0$.

$$n = 0 \implies P_0(x) = a_0 \implies p(x_0) = a_0$$

Since we want contact of order 0 with f at x_0 , then

$$f(x_0) = p(x_0) = a_0 \implies f(x_0) = a_0$$

$$n = 1 \implies P'_n(x_0) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1}$$

$$\implies P'_n(x_0) = a_1 = f'(x_0) \text{ in order to have contact of order 1}$$

$$n = 2 \implies P''_n(x) = 2a_2 + 6a_3(x - x_0) + \cdots + n(n - 1)a_n x^{n-2}$$

Now plugging in $x = x_0$, we get

$$\implies P''_n(x) = 2a_2 = f''(x_0) \implies a_2 = \frac{f''(x_0)}{2!}$$

and we start to see the pattern.

$$a_3 = \frac{f^{(3)}(x_0)}{3!}$$

$$\text{Pattern: } a_k = \frac{f^{(k)}(x_0)}{k!}$$

So putting coefficients a_k into $P_n(x)$:

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

□

Note 8.5. Note that this was only created for a single point x_0 . The construction says nothing about how the polynomial behaves away from x_0 .

Example 8.6

Find the n th Taylor Polynomial for $f(x) = e^x$ at $x = 0$

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

Solution. Note that $f^{(k)}(0) = 1 \forall k$. Thus,

$$P_n(x) = \sum_{i=0}^n \frac{1}{k!} (x - 0)^k = \sum_{i=0}^n \frac{x^k}{k!}$$

□

§8.2 Lagrange Remainder (Error) Theorem

How close is $P_n(x)$ to $f(x)$? What is the error?

Definition 8.7. Define $R_n := f(x) - P_n(x)$ where $R_n : I \rightarrow \mathbb{R}$ is the **nth remainder**

Note 8.8. Recall this Corollary from Cauchy MVT in Chapter 4. Suppose $f : I \rightarrow \mathbb{R}$ has n derivatives for $f^{(k)}(x_0) = 0 \forall k, 0 \leq k \leq n-1$. Then $\forall x \neq x_0, \exists z \in (x, x_0)$ with

$$f(x) = \frac{f^{(n)}(z)}{n!} (x - x_0)^n$$

Theorem 8.9

Let I be a neighborhood of x_0 , and let \mathbb{N} include 0. Suppose $f : I \rightarrow \mathbb{R}$ has $n + 1$ derivatives. Then for each point $x \neq x_0$ in I , $\exists c \in (x, x_0)$

$$f(x) = \sum_{k=1}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

where the summation is the Taylor polynomial and the second term is the $R_n(x)$ remainder.

Proof. Let $P_n(x)$ be the n th Taylor Polynomial for $f(x)$ at x_0 .

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

Define $R_n(x) := f(x) - P_n(x) \forall x \in I$. By construction in 8.1, $f(x)$ and $p_n(x)$ have a contact of order n at x_0 so $R_n^{(k)} = 0 \forall k \in [0, n]$. Thus, we can apply the Corollary from the CMVT on R :

$$\forall x \neq x_0, \exists c \text{ between } x_0, x \text{ w/ } R_n(x) = \frac{R^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

But $R^{(n+1)}(c) = f^{(n+1)}(c) - P_n^{(n+1)}(c)$ by definition of R . Note that the second term is the Polynomial degree n so $(n+1)$ th derivative is 0.

$$R^{(n+1)}(c) = f^{(n+1)}(c) - 0$$

By substitution into the corollary from CMVT,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

where $R_n := f - P_n \implies f = P_n + R_n$

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

□

§8.3 Convergence of Taylor Polynomials (n to infy)

Definition 8.10. $\{S_n\}$ is a **sequence of partial sums** when $S_n = \sum_{k=0}^n a_k$ for a given a_k .

$$\sum_{k=0}^{\infty} a_k = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=0}^n a_k \text{ if } \{S_n\} \text{ converges}$$

Note that if $\{S_n\}$ does not converge, then $\sum_{k=0}^{\infty} a_k$ diverges.

Definition 8.11. Let $P_n(x)$ be the n th Taylor Polynomial for $f : I \rightarrow \mathbb{R}$ then

$$\lim_{n \rightarrow \infty} P_n(x) = f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

is called the **Taylor Series Expansion of f at x_0**

Note 8.12. Note that we can assume that $\lim_{n \rightarrow \infty} r^n = 0$ if $|r| < 1$.

Lemma 8.13

$$\lim_{n \rightarrow \infty} \frac{c^n}{n!} = 0, c \in \mathbb{R}$$

Proof. Let $c \in \mathbb{R}$ be a constant and choose $k \in \mathbb{N}$ such that $k \geq 2|c| \implies |c| \leq \frac{k}{2}$, which is doable by AP. Apply absolute value for $n \geq K$,

$$\begin{aligned} 0 \leq \left| \frac{c^n}{n!} \right| &= \frac{|c||c| \cdots |c|}{(1)(2) \cdots (n-1)(n)} \\ &= \left[\frac{|c||c| \cdots |c|}{1 * 2 * \cdots * k} \right] \left[\frac{|c| \cdots |c|}{(k+1) \cdots n} \right] = \frac{|c|^k}{k!} \frac{|c|^{n-k}}{(k+1) \cdots n} \leq \frac{|c|^k}{k!} \frac{|c|^{n-k}}{k^{n-k}} \\ &\leq \frac{|c|^k}{k!} \frac{(k/2)^{n-k}}{k^{n-k}} = \frac{|c|^k}{k! 2^{n-k}} \leq |c|^k \left(\frac{1}{2} \right)^{n-k} \\ &= |c|^k \frac{(1/2)^n}{(1/2)^k} = |c|^k * 2^k * \left(\frac{1}{2} \right)^n \end{aligned}$$

Note that $|c|^k * 2^k$ is a constant wrt n and $(\frac{1}{2})^n \rightarrow 0, n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2} \right)^n = 0$$

and so since that term is a constant then by CL, $\lim_{n \rightarrow \infty} \frac{c^n}{n!} = 0$. □

Theorem 8.14

Let I be a neighborhood of x_0 and $f : I \rightarrow \mathbb{R}$ have derivative of all orders. Suppose $r, M \in \mathbb{R}^+$ such that $[x_0 - r, x_0 + r] \subseteq I$ and $\forall n$ with $x \in [x_0 - r, x_0 + r]$, $|f^{(n)}(x)| \leq M^n$ then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \text{ if } |x - x_0| \leq r$$

Proof. Let $P_n(x)$ be the n th Taylor Polynomial of f at x_0 . We WTS that

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k \leftrightarrow \lim_{n \rightarrow \infty} P_n \leftrightarrow \lim_{n \rightarrow \infty} (f(x) - P_n(x)) \leftrightarrow \lim_{n \rightarrow \infty} R_n(x) \\ &= 0 \leftrightarrow \lim_{n \rightarrow \infty} \frac{f^{(n+1)}(z)}{(n+1)!} (x - x_0)^{n+1} \end{aligned}$$

by the Lagrange Remainder Theorem, z is strictly between x and x_0 . By assumption, $|f^{(n+1)}| \leq M^{n+1}$ and from the given $|x - x_0| \leq r$

$$\left| \frac{f^{(n+1)}(z)}{(n+1)!} (x - x_0)^{n+1} \right| \leq \left| \frac{M^{n+1} r^{n+1}}{(n+1)!} \right| = \left| \frac{(Mr)^{n+1}}{(n+1)!} \right|$$

Note that $\lim_{n \rightarrow \infty} \left| \frac{(Mr)^{n+1}}{(n+1)!} \right| = 0$ and so by CL

$$\lim_{n \rightarrow \infty} \frac{f^{(n+1)}(z)}{(n+1)!} (x - x_0)^{n+1} = \lim_{n \rightarrow \infty} R_n(x) = 0 \implies$$

the Taylor Series expansion of f at x_0 converges to $f(x)$ □

§8.4 Skip (Thanks Eileen)**§8.5 Cauchy Integral Remainder Theorem**

Previously, $f(x) = P_n(x) + R_n(x)$, where the $R_n(x)$ was the formula using the Lagrange Remainder using unknown c, z . Now, we will use $R_n(x)$ as an integral and approximate it using sums.

Theorem 8.15

Integration by Parts: Suppose $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous and have continuous bounded derivatives on (a, b) . Then

$$\int_a^b fg' = fg|_a^b - \int_a^b f'g$$

$$udv = uv - \int vdu$$

Proof. From product rule,

$$(fg)' = f'g + fg' \implies fg' = (fg)' - f'g$$

$$\int_a^b fg' = \int_a^b (fg)' - \int_a^b f'g$$

Now by FTC1

$$\int_a^b fg' = (fg)|_a^b - \int_a^b f'g$$

□

Theorem 8.16

Cauchy Integral Remainder Theorem: Let I be a neighborhood of $x_0 \in \mathbb{R}$, and let $n \in \mathbb{N} \cup \{0\}$. Suppose $f : I \rightarrow \mathbb{R}$ has $n + 1$ derivatives and that $f^{(n+1)} : I \rightarrow \mathbb{R}$ is continuous. Then for each $x \in I$

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{n!} \int_{x_0}^x f^{(n+1)}(t) (x - t)^n dt$$

where the first term is the n th Taylor Polynomial and the second term is the **Cauchy Integral Remainder**.

Proof. Basis step: $n = 0$

$$f(x_0) + \frac{1}{0!} \int_{x_0}^x f'(t) (x - t)^0 dt = f(x_0) + \int_{x_0}^x f'(t) dt = f(x_0) + f(x) - f(x_0) = f(x)$$

Now for $n = 1$ by FTC1: $\int_{x_0}^x f'(t) dt = f(x) - f(x_0) \implies f(x) = f(x_0) + \int_{x_0}^x f'(t) dt$ and

$$\int_{x_0}^x f'(t) dt = \int_{x_0}^x -f'(t) \frac{d}{dt} (x - t) dt$$

Let $u = -f'(t)$ and let $du = \frac{d}{dt} (x - t) \implies u = (x - t)$ and so by Integration by Parts

$$\begin{aligned} &= -f'(t)(x - t) \Big|_{t=x_0}^{t=x} - \int_{x_0}^x -f''(t)(x - t) dt = \\ &= -f'(x)(x - x) - (-f'(x_0)(x - x_0)) + \int_{x_0}^x f''(t)(x - t) dt \end{aligned}$$

Plugging this in for $\int_{x_0}^x f'(t) dt$

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{1!} \int_{x_0}^x f''(t)(x - t) dt$$

Note that the first two terms are terms of $P_n(x)$ and the last term is $R_1(x)$

Inductive hypothesis: $f(x) = P_n(x) + \frac{1}{n!} \int_{x_0}^x f^{(n+1)}(t)(x - t)^n dt$

Inductive step: Observe that

$$\begin{aligned} \frac{1}{n!} \int_{x_0}^x f^{(n+1)}(t)(x - t)^n dt &= \frac{1}{n} \int_{x_0}^x f^{(n+1)}(t) \left[\frac{d}{dt} - \frac{1}{n+1} (x - t)^{n+1} \right] dt \\ &= -\frac{1}{(n+1)!} \int_{x_0}^x f^{(n+1)}(t) \frac{d}{dt} (x - t)^{n+1} dt \end{aligned}$$

and using IBP, let $u = f^{(n+1)}(t)$ and $g' = \frac{d}{dt} (x - t)^{n+1} \implies g = (x - t)^{n+1}$

$$= \left(-\frac{1}{(n+1)!} \right) f^{(n+1)}(t)(x - t)^{n+1} \Big|_{x_0}^x - \int_{x_0}^x f^{(n+2)}(x - t) dt$$

$$\frac{f^{(n+1)}(x_0)}{(n+1)!} (x - x_0)^{n+1} + \frac{1}{(n+1)!} \int_{x_0}^x f^{(n+2)}(x - t) dt \implies P_{n+1}(x) + R_{n+1}(x)$$

□

§8.6 Skip 8.6

§8.7

Definition 8.17.