MATH410: Homework 2

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February 16, 2024

1. Suppose that the sequence $\{a_n\}$ converges to ℓ and that the sequence $\{b_n\}$ has the property that there is an index N such that
$$a_n = b_n \quad \text{for all indices } n \geq N.$$
Show that $\{b_n\}$ also converges to ℓ.

Sketch: we want to show that $\{b_n\}$ converges to l. Therefore, we need to show the definition of convergence such that given an $\epsilon > 0$, $|b_n - l| < \epsilon$ for all $n \geq N, N \in \mathbb{N}$. Let $\epsilon > 0$. By the definition of convergence, $\exists N_1 \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon \ \forall \ n \geq N_1$$

Choose $N_2 = \max(N, N_1)$ where N is the given index in the problem description and N_1 is the index that satisfies the definition of convergence given any positive ϵ.

Proof. By the definition of convergence, let $\epsilon > 0$ be given, and so $\exists \ N_1 \in \mathbb{N}$ such that

$$|a_n - l| < \epsilon \ \forall \ n \geq N_1$$

Let us choose $N_2 = \max(N, N_1)$. Since the above is true for all $n \geq N_1$, it must also be true $n \geq \max(N, N_1)$. Therefore ,

$$|a_n - l| < \epsilon \ \forall \ n \geq N_2$$

Moreover, for all $n \geq N_2$, we know that $a_n = b_n$. By direct substitution into the absolute value,

$$|b_n - l| < \epsilon \ \forall \ n \geq N_2$$

Thus, given any $\epsilon > 0$, there exists a threshold N_2 such that the definition of convergence is satisfied and so $\lim_{n\to\infty} b_n = l$ as desired. $\square$

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2. Prove the following:

   (a) $\lim_{n\to\infty}\left[n^3 - 4n^2 - 100n\right] = \infty$

   (b) $\lim_{n\to\infty}\left[\sqrt{n} - \frac{1}{n^2} + 4\right] = \infty$

a. Sketch: we want to show that given some $M > 0$, $a_n > M\ \forall n \geq N, N \in \mathbb{N}$.

$$n^3 - 4n^2 - 100n > n^2 - 4n - 100 > n^2 - 15n - 100 = (n-20)(n+5) > M$$

Choose $N > \max(M, 20)$.

Proof. Let $M > 0$ be given. By the Archimedian Principle, $\exists\ N \in \mathbb{N}$ such that $N > \max(M, 20)$.

Now observe the following operations

$$n^3 - 4n^2 - 100n > n^2 - 4n - 100$$

where the right side of the inequality was obtained by dividing all terms by n.

$$n^2 - 4n - 100 > n^2 - 15n - 100 = (n-20)(n+5)$$

which also must be true since $n > 0$, and the equality came by factoring. Now note that $N > \max(M, 20)$ and $n \geq N$. Therefore,

$$(n-20)(n+5) \geq (N-20)(N+5) > (M-20)(M+5) > M$$

Thus, we have shown the definition of divergence because for any given $M > 0$, we can find a threshold N such that

$$n^3 - 4n^2 - 100n > M\ \forall\ n \geq N$$

which implies that

$$n^3 - 4n^2 - 100 \to \infty \text{ as } n \to \infty \implies \lim_{n\to\infty} n^3 - 4n^2 - 100n = \infty$$

□

b. Sketch: we WTS to show that for $M > 0$, then $a_n > M\ \forall\ n \geq N, N \in \mathbb{N}$.

$$\sqrt{n} - \frac{1}{n^2} + 4 > \sqrt{n} - \frac{1}{n^2} \geq \sqrt{n} - \frac{1}{n} = (n^{\frac{1}{4}} + \frac{1}{\sqrt{n}})(n^{\frac{1}{4}} - \frac{1}{\sqrt{n}})$$

Choose $N > \max(M^4,\ 1)$.

Proof. Let $M > 0$ be given. By A.P., $\exists\ N \in \mathbb{N}$ such that $N > \max(M^4, 1)$. Note that

$$\sqrt{n} - \frac{1}{n^2} + 4 > \sqrt{n} - \frac{1}{n^2}\ \forall\ n$$

Furthermore, note that $\frac{1}{n^2} \leq \frac{1}{n}\ \forall n$. Therefore, $-\frac{1}{n^2} \geq -\frac{1}{n}\ \forall n$ as well, by mutliplying both sides by -1 and subsequently reversing the direction of the inequality. Thus,

$$\sqrt{n} - \frac{1}{n^2} \geq \sqrt{n} - \frac{1}{n}$$

Recall that $n \geq N$ and so

$$\sqrt{n} - \frac{1}{n} \geq \sqrt{N} - \frac{1}{N} = (N^{\frac{1}{4}} + \frac{1}{\sqrt{N}})(N^{\frac{1}{4}} - \frac{1}{\sqrt{N}})$$

$$> (M + \frac{1}{M^2})(M - \frac{1}{M^2}) > M$$

Note that this last inequality is always true, since we choose $N > \max(M^4, 1)$. Therefore, the second term $(M - \frac{1}{M^2}) > 1$ for all of our choices of $n \geq N$. Thus, we've shown that for any given $M > 0$, we can find a threshold N such that

$$\sqrt{n} - \frac{1}{n^2} + 4 > M \ \forall \ n \implies \lim_{n\to\infty} \sqrt{n} - \frac{1}{n^2} + 4 = \infty$$

□

3. For a sequence $\{a_n\}$ of positive numbers show that $\lim_{n\to\infty} a_n = \infty$ if and only if $\lim_{n\to\infty} \left[\frac{1}{a_n}\right] = 0$

Proof.
$\Longrightarrow$ Suppose we are given that the sequence $\{a_n\}$ diverges such that $\lim_{n\to\infty} a_n = \infty$. By the definition of divergence, for any given $M > 0$, there exists a threshold $N \in \mathbb{N}$ such that for all $n \geq N$, $a_n > M$. We want to show that $\frac{1}{\{a_n\}}$ converges to 0.

$$a_n > M \ \forall n \geq N$$

where both $a_n, M > 0 \ \forall \ n$. Therefore, still for all $n \geq N$,

$$\frac{1}{a_n} < \frac{1}{M}$$
$$-\frac{1}{M} < \frac{1}{a_n} < \frac{1}{M}$$
$$-\frac{1}{M} < \frac{1}{a_n} - 0 < \frac{1}{M}$$
$$|\frac{1}{a_n} - 0| \ < \frac{1}{M}$$

Therefore, our value of $l = 0$ as desired and our $\epsilon = \frac{1}{M}$ and there is one to one mapping from values of M to values of ϵ. Note that any $\epsilon > 0$ has a corresponding value of M such that $\frac{1}{M} = \epsilon$. Thus, for any $\epsilon > 0$, there exists an N (the same threshold used to show that $\{a_n\}$ diverges) such that

$$|\frac{1}{a_n} - 0| < \frac{1}{M} = \epsilon \ \forall \ n \geq N \implies \lim_{n\to\infty} \frac{1}{a_n} = 0$$

$\Longleftarrow$ The reverse direction is similar. Suppose we are given that the sequence $\{\frac{1}{a_n}\}$ converges to 0. Therefore, by the definition of convergence, for any given $\epsilon > 0$, there exists a threshold $N \in \mathbb{N}$ such that

$$|\frac{1}{a_n} - 0| < \epsilon \ \forall \ n \geq N$$

We can algebraically manipulate the above such that, while still for all $n \geq N$,

$$-\epsilon < \frac{1}{a_n} < \epsilon$$

Note that $\frac{1}{a_n} > 0 \ \forall \ n$ is given since a_n is a positive sequence.

$$\frac{1}{a_n} < \epsilon$$
$$a_n > \frac{1}{\epsilon}$$

Let $M = \frac{1}{\epsilon}$ and once again notice the one to one correspondance. Therefore, for any $M > 0$, we can use the same N threshold that is given to show that

$$a_n > M = \frac{1}{\epsilon} \ \forall \ n \geq N \implies \lim_{n\to\infty} a_n = \infty$$

as desired. $\square$

4. For each of the following statements, determine whether it is true or false and justify your answer.

   (a) Every bounded sequence converges.

   (b) A convergent sequence of positive numbers has a positive limit.

   (c) The sequence $\{n^2+1\}$ converges.

   (d) A convergent sequence of rational numbers has a rational limit.

   (e) The limit of a convergent sequence in the interval (a,b) also belongs to (a,b).

a. False. Not every bounded sequence converges. For example, the sequence $\{(-1)^n\}$ is bounded above by 1 and bounded below by -1, but it never converges.

b. False. Consider the sequence $\{\frac{1}{n}\}$ which converges to 0, which is not positive. $0 \not> 0$.

c. False. Let $M > 0$. By A.P, $\exists\ N \in \mathbb{N}$ such that $N > \sqrt{M}$. Note that $n \geq N$. Therefore, $n^2+1 \geq N^2+1 > M+1 > M \implies \{n^2+1\}$ diverges.

d. False. Consider a sequence such that begins as $\{3, 3.1, 3.14, 3.141, 3.14159, \cdots\}$ such that each term adds on the next digit of π. Clearly, all of these numbers are rational, as they can be expressed in decimal, and thus fraction, form, but the limit is $\pi \neq \mathbb{Q}$.

e. False. Consider the sequence $\{\frac{1}{n+1}\}\ \forall\ n$. Note that $a_n \in (0,1)\ \forall\ n$, yet $\lim\limits_{n\to\infty} \frac{1}{n+1} = 0$, and $0 \notin (0,1)$, so this statement is false.

5. Show that a sequence $\{a_n\}$ is bounded if and only if there is an interval $[c, d]$ such that $\{a_n\}$ is a sequence in $[c, d]$.

Proof.
$\Longleftarrow$ Suppose that a sequence $\{a_n\}$ is bounded. We want to show that there is $[c, d]$ such that $\{a_n\}$ is a sequence in $[c, d]$. By the definition of bounded

$$\exists\ M \in \mathbb{R} \text{ such that } |a_n| \leq M\ \forall\ n \in \mathbb{N}$$

Therefore, for all n, by a property of absolute value,

$$-M \leq a_n \leq M$$

Let $c = -M$ and let $d = M$. By direct substitution,

$$c \leq a_n \leq d$$

$$a_n \in [c, d]\ \forall\ n \in \mathbb{N}$$

Thus, we've shown that there exists the interval $[c, d]$ such that $\{a_n\}$ is a sequence in $[c, d]$.
$\Longrightarrow$ The reverse direction is similar. Suppose that there is an interval $[c, d]$ such that $\{a_n\}$ is a sequence in $[c, d]$. Then we can write

$$c \leq a_n \leq d\ \forall\ n \in \mathbb{R}$$

We want to show that there exists an $M \in \mathbb{R}$ such that $|a_n| \leq M\ \forall\ n$. Thus, let us choose $M = \max(|c|, |d|)$. This ensures that M is at least greater than or equal to both c and d in terms of magnitude. We now write

$$-M \leq c \leq a_n \leq d \leq M\ \forall\ n$$

$$-M \leq a_n \leq M\ \forall\ n$$

$$|a_n| \leq M\ \forall\ n$$

which means the sequence $\{a_n\}$ is bounded, as desired. □

6. Suppose that the sequence $\{a_n\}$ is monotone. Prove that $\{a_n\}$ converges if and only if $\{a_n^2\}$ converges. Show that this result does not hold without the monotonicity assumption.

Proof.
$\Longleftarrow$ Suppose the sequence $\{a_n\}$ converges. This direction is trivial because we can simply use the Product Property. Let us denote that $\{a_n\}$ converges to $a \implies \lim_{n\to\infty} a_n = a$. Then by the Product Property,

$$\lim_{n\to\infty} a_n a_n = \lim_{n\to\infty} a_n^2 = a * a = a^2$$

Therefore, $\{a_n^2\}$ converges to a^2, and so generally, it does converge, as desired.

$\Longrightarrow$ The reverse direction is more challenging, and requires that knowledge sequence $\{a_n\}$ is monotone. As a counterexample without the monotonicity example, consider the sequence $\{(-1)^{2n}\}$ which converges to 1. However, the square root of that sequence, $\{(-1)^n\}$ does not converge as it oscillates between -1 and 1.

Now suppose that $\{a_n^2\}$ converges and we're given that $\{a_n\}$ is monotone. We want to show that $\{a_n\}$ converges. Note that by a theorem, every convergent sequence is bounded. Therefore, $\{a_n^2\}$ is bounded. By the definition of bounded, $\exists\ M \in \mathbb{R}$ such that

$$|a_n^2| \leq M \ \forall\ n$$

We want to show that $\{a_n\}$ is bounded, and then we can use the Monotone Convergence Theorem to prove that the monotone sequence $\{a_n\}$ converges to either the $\sup, \inf$ depending on whether it is monotonically increasing or monotonically decreasing, respectively.

$$\sqrt{|a_n^2|} \leq \sqrt{M} \ \forall\ n$$

Note that the square operator implies that $a_n^2 \geq 0$, and so we can remove the absolute value.

$$\sqrt{a_n^2} \leq \sqrt{M}\ \forall n$$

Now note that the square root ensures that the left hand side is greater than or equal to 0, meaning wrapping it in absolute value signs will not change the result.

$$|\sqrt{a_n^2}| \leq \sqrt{M}\ \forall n$$

$$|a_n| \leq \sqrt{M} \ \forall\ n$$

Thus, $\{a_n\}$ is bounded by $\sqrt{M} \in \mathbb{R}$. Therefore, by the Monotone Convergence Theorem, $\{a_n\}$ converges, as desired.

$\square$