MATH410: Homework 3

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1. **Show that a strictly increasing sequence has no peak indices.**

Proof.
On the contrary, assume that a strictly increasing sequence $\{a_n\}$ has a peak index a_m, for some $m \in \mathbb{N}$. By definition of strictly increasing, $a_{n+1} \geq a_n \ \forall\ n \in \mathbb{N}$. Since this works for all n, note that by definition of strictly increasing, $a_{m+1} \geq m$. However, note that by definition of peak index

$$a_m > a_j \ \forall\ j \geq m \implies a_m > a_{m+1}$$

Thus, we've reached a contradiction since $a_{m+1} \geq m$ and $a_{m+1} < m$ obviously cannot both be simulatenously true, and so a strictly increasing sequence has no peak indices. $\square$

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2. Prove that a sequence $\{a_n\}$ does not converge to the number a if and only if there is some $\epsilon > 0$ and a subsequence $\{a_{n_k}\}$ such that

$$|a_{n_k} - a| \geq \epsilon \quad \text{for every index } k.$$

Proof.
$\Longrightarrow$ Suppose we are given that the sequence $\{a_n\}$ does not converge to the number a. Therefore, by the definition of convergence, given some $\epsilon > 0$, we are not always guaranteed to be able to find $N \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon \ \forall \ n \geq N$$

Choose one of these examples of ϵ. Let us define the monotonically increasing sequence $\{n_k\} = \{i \mid |a_i - a| \geq \epsilon, i \geq N, i \in \mathbb{N}\}$, or in plain English, all indices $i \geq N$ such that $|a_i - a| \geq \epsilon$. Therefore, we have constructed a subsequence $\{a_{n_k}\}$ such that

$$|a_{n_k} - a| \geq \epsilon \ \forall \ k$$

as desired.

$\Longleftarrow$ The reverse direction is similar. Given $\epsilon > 0$ and some subsequence $\{a_{n_k}\}$,

$$|a_{n_k} - a| \geq \epsilon \ \forall \ k$$

We WTS that $\{a_n\}$ does not converge to a. Suppose on the contrary, $\{a_n\}$ did converge to a. By definition of convergence, given any $\epsilon > 0$, we can find threshold $N \in \mathbb{N}$ such that

$$|a_n - a| < \epsilon \ \forall \ n \geq N$$

Note that this definition should work for all $\epsilon > 0$. Therefore, let us choose the same ϵ as in the given statement. Note that $\{n_k\}$ is a monotonically increasing infinite sequence of natural numbers by definition of subsequence. Therefore, there exists some index in $\{n_k\}$ that is greater than or equal to our threshold N. Let us denote this index x. Therefore, at index x, we have that $|a_x - a| \geq \epsilon$ by the given and $|a_x - a| < \epsilon$ by the definition of convergence. Clearly, this is a contradiction, and so $\{a_n\}$ cannot converge to a.

$\square$

3. For each of the following statements, determine whether it is true or false and justify your answer.
   (a) A subsequence of a bounded sequence is bounded.
   (b) A subsequence of a monotone sequence is monotone.
   (c) A subsequence of a convergent sequence is convergent.
   (d) A sequence converges if it has a convergent subsequence.

Solution.

a. True. Assume on the contradiction that a bounded sequence $\{a_n\}$ had an unbounded subsequence $\{a_{n_k}\}$. By definition of bounded, $\exists\ M \in \mathbb{R}$ such that $|a_n| < M\ \forall\ n \implies -M < a_n < M\ \forall\ n$. If subsequence $\{a_{n_k}\}$ is unbounded then there some there exists some index $x \in \{n_k\}$ such that $|a_x| \geq M$ because no scalar, not even M, is greater than the absolute value of all elements in the subsequence. However a_x is in the subsequence and also the sequence, so we have $|a_x| < M$ and $|a_x| \geq M$ simulatenously, which must be a contradiction, and so $\{a_{n_k}\}$ must also be bounded.

b. True. On the contrary, suppose that the subsequence $\{x_{n_k}\}$ of a monotone sequence $\{x_n\}$ was not monotone. WLOG assume the sequence is monotone increasing. By definition of monotone increasing, $x_{n+1} \geq x_n\ \forall\ n$. Consider any sequence of indices $\{n_k\}$. Since the subsequence is not monotone increasing, there must be some $x_{k'} < x_k, k', k \in \mathbb{N}, k' > k$. However, note that $x_{k'}$ and x_k are also in the original sequence, which is monotone increasing and therefore implies that $x_{k'} \geq x_k$. Thus, we have a contradiction, and so the subsequence of any monotone sequence is also monotone.

c. True. By a theorem presented in class, if a sequence $\{x_n\}$ converges to a, then every subsequence $\{x_{n_k}\}$ also converges to a.

d. False. Consider the sequence $\{a_n\} = \{\frac{1}{n}\}$ and the subsequent subsequence $\{a_{n_k}\} = \{\frac{1}{n} \mid n \text{ is even}\}$. Note that $\{a_{n_k}\}$ converges to 1, but that the original subsequence we know does not converge. Therefore, we've found a counterexample.

$\square$

4. Define

$$f(x) = \begin{cases} 11 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x \leq 2. \end{cases}$$

At what points is the function $f : [0, 2] \to \mathbb{R}$ continuous? Justify your answer with a proof.

Proof.
I claim the function is continuous at all points in the interval $[0, 1) \cup (1, 2]$. Essentially, f is continuous between 0 and 2 except for at $x = 1$, and we will show this over three parts.

a. Select $x_0 \in [0, 1)$ and let the sequence $\{x_n\} \to x_0 \implies \lim_{n\to\infty} x_n = x_0$. We WTS that $\lim_{n\to\infty} f(x_n) = f(x_0)$. Note that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} 11$ by definition of f. Further note that $\lim_{n\to\infty} 11 = 11 = f(x_0)\ \forall\ x \in [0, 1)$. Therefore, f is continuous on $[0, 1)$.

b. Now we will show that f is discontinuous at $x_0 = 1$. Let $\{x_n\} \to x_0$ such that the sequence approaches x_0 from the right, graphically. Note that $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = x_0 = 1 \neq 11 = f(x_0)$. Therefore, $\lim_{n\to\infty} f(x_n) \neq f(x_0)$ and so f is discontinuous at 1.

c. Finally, select $x_0 \in (1, 2]$ and let $\{x_n\} \to x_0$. Note that $\underset{n\to\infty}{f(x_n)} = \lim_{n\to\infty} x_n = x_0 = f(x_0)\ \forall\ x_0 \in (1, 2]$. Therefore, f is continuous at all points in $(1, 2]$.

Putting everything together, f is continuous on the interval $[0, 1) \cup (1, 2]$, as desired. $\square$

5. Prove that the function $f(x)$ is discontinuous at 0 , where $f(x)=\sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and $f(0)=0$.

Proof.
Assume on the contary that f is continuous at $x_0 = 0$. Therefore, by definition of continuous at a point,

$$\lim_{n\to\infty} f(x_n) = f(x_0) = f(0) = 0$$

Let the given sequence be $\{x_n\} = \{\frac{1}{n\pi}\} \to 0$ as $n \to \infty$ which we have discussed in class many times. Now consider

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \sin(\frac{1}{\frac{1}{n\pi}}) = \lim_{n\to\infty} \sin(n\pi) \neq 0 \ \forall \ n$$

since $\sin(n\pi)$ oscillates. Since the function is never equal to 0 and oscillates, it does not converge to 0, and therefore, we have reached a contradiction, and so f must be discontinuous at $x_0 = 0$. $\square$

6. Let $f(x) = x$ for rational numbers and $f(x) = 0$ for irrational numbers. Define the function $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$
Prove that f is continuous at $x = 0$.

Proof.
Note that f is defined on $\mathbb{R}$ and we WTS that it is continuous at $x = 0$. By Sequential Density of $\mathbb{Q}$ and $\mathbb{Q}^c$, we can find two sequences

$$\{u_n\} \to 0, u_n \in \mathbb{Q} \; \forall \; n$$

$$\{v_n\} \to 0, v_n \in \mathbb{Q}^c \; \forall \; n$$

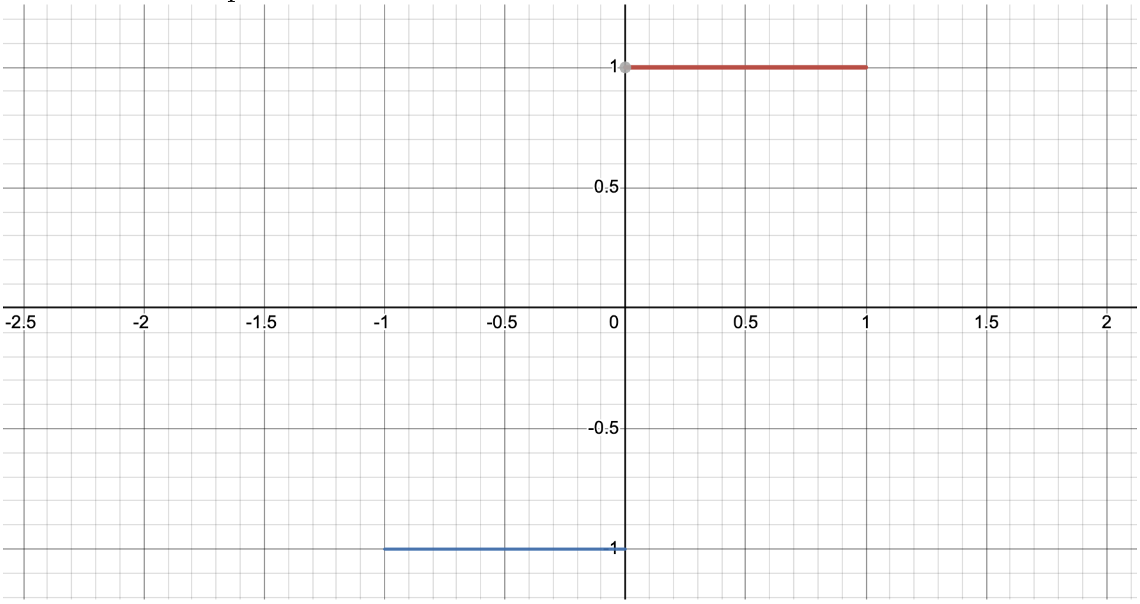
Trivially, since $f(v_n) = 0 \; \forall \; n \implies \lim_{n\to\infty} f(v_n) = f(x) = f(0) = 0$. Now, note that $\lim_{n\to\infty} f(u_n) = \lim_{n\to\infty} u_n = 0$ by construction. Even consider sequences that are mixed, in the sense that they contain elements that are rational and elements that are irrational. All functional values of the irrational numbers are 0, so it suffices to show that functional values of the rationals also converges to 0 as the sequence of rationals converges to 0. Therefore, since all index sequences that converge to $x = 0$ implies that all image sequences converge to $f(x) = f(0) = 0$, then f is continuous at $x = 0$. $\square$

7. Is it true that if $f : [a, b] \to \mathbb{R}$ has a maximum and minimum value, then f must be continuous? Justify your answer.

Proof.
We will show that this statement is false by counterexample. Let $a = -1$ and $b = 1$. Then the function

$$f(x) = \begin{cases} 1, 0 \leq x \leq 1 \\ -1, -1 \leq x < 0 \end{cases}$$

This function is plotted below.



Clearly, the domain of f is $[-1, 1]$, the max of f is 1 and the min of f is -1, and f is not continuous at. We will prove this rigorously. Let $x_0 = 0$ and let there be a sequence $\{x_n\} \to 0$, specifically approaching 0 from the left side graphically. Note that $\lim_{n \to \infty} f(x_n) = -1 \neq f(x_0) = f(0) = 1$. Therefore, f is not continuous at $x_0 = 0$, and we proved the given statemtent false.

$\square$

8. Let a and b be real numbers with $a < b$. Find a continuous function $f : (a,b) \to \mathbb{R}$ having an image that is unbounded above. Also, find a continuous function $f : (a,b) \to \mathbb{R}$ having an image that is bounded above but does not attain a maximum value.

Proof.

a. Consider $f : (a,b) \to \mathbb{R}$ such that $f(x) = \frac{1}{x-a}$. f is continuous because $x \neq a \implies$ the denominator is never 0 and so this function is defined everywhere. Consider the sequence $\{x_n\} \to a$, which exists by density of $\mathbb{R}$. Now consider

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{1}{x_n - a} = \infty$$

Theerefore, f diverges on this interval and is therefore unbounded above.

b. Now consider $f : (a,b) \to \mathbb{R}$ such that $f(x) = x$. Note that f is bounded above by b on the interval (a,b). Suppose on the contrary that f attains a maximum value. By definition of maximum, there is $x_0 \in (a,b)$ such that $f(x_0) \geq f(x)\ \forall\ x \in (a,b)$. Note that $f(x_0) = x_0$. Take the midpoint between x_0 and b, given by $\frac{x_0+b}{2}$. Note that $f(\frac{x_0+b}{2}) = \frac{x_0+b}{2}$. Since b is guaranteed to be larger than x_0, since $x_0 \in (a,b)$, then we know that

$$\frac{x_0+b}{2} > x_0$$

and thus, x_0 cannot be a maximum for f, and so we have our contradiction. Therefore, while f is bounded above, it does not attain a maximum value.

□