MATH410: Homework 1

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1.

Solution. Like the hint says, rather than look considering eight separate cases, we will apply the Triangle Inequality twice.
Note that
$$|a+b+c| = |(a+b)+c|$$
such that $a+b \in \mathbb{R}$ if $a, b \in \mathbb{R}$ and this is by the Positivity Axiom of the Real Numbers, $\mathbb{R}$. Thus, since $(a+b), c \in \mathbb{R}$, we can apply the Triangle Inequality and obtain
$$|(a+b)+c| \leq |a+b| + |c|$$
Observe the term $|a+b|$ term. Since $a, b \in \mathbb{R}$, we can apply the Triangle Inequality once more to get
$$|a+b| + |c| \leq |a| + |b| + |c|$$
and thus we've shown that
$$|a+b+c| \leq |a| + |b| + |c|$$
as desired. Now for the inductive part of the prove, we wish to prove that
$$|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n| \ \forall\ n \in \mathbb{N}, a_i \in \mathbb{R}$$
Base cases: $n = 1 \implies |a_1| \leq |a_1|$ and $n = 2 \implies |a_1 + a_2| \leq |a_1| + |a_2|$ by definition of Triangle Inequality.
Inductive hypotheses: let us assume that $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n| \ \forall\ n \in \mathbb{N}, a_i \in \mathbb{R}$.
Inductive step: Now we will show that $|a_1 + \cdots + a_n + a_{n+1}| \leq |a_1| + \cdots + |a_n| + |a_{n+1}|$. Starting with the left side of this inequality.
$$|a_1 + \cdots + a_n + a_{n+1}| = |(a_1 + \cdots + a_n) + a_{n+1}|$$
Note that $(a_1 + \cdots + a_n) \in \mathbb{R}$ by the Positivity Axiom of $\mathbb{R}$ and $a_{n+1} \in \mathbb{R}$. Therefore, we can apply the Triangle Inequality to get
$$|(a_1 + \cdots + a_n) + a_{n+1}| \leq |a_1 + \cdots + a_n| + |a_{n+1}|$$
By our Inductive step, we know that $|a_1 + \cdots + a_n| \leq |a_1| + \cdots + |a_n|$ so
$$|a_1 + \cdots + a_n| + |a_{n+1}| \leq |a_1| + \cdots + |a_n| + |a_{n_1}|$$
and thus we have showed that
$$|a_1 + \cdots + a_n + a_{n+1}| \leq |a_1| + \cdots + |a_n| + |a_{n+1}|$$
and this completes the proof. $\square$

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2.

Solution.

a. $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

An example of an upper bound of this set is 2. An example of a lower bound of this set is -1. The supremum of this set is 1. The infemum of this set is 0.

b. $\{1 - \frac{1}{3^n} \mid n \in \mathbb{N}\}$

Example upper bound is 2. Example lower bound is -2. Supremum is 1. Infemum is $\frac{2}{3}$ if we don't consider 0 to be in $\mathbb{N}$. If it is, then the infemum is 0.

c. $\{\cos(\frac{n\pi}{3}) \mid n \in \mathbb{N}\}$

An upper bound is 2. An lower bound is -2. The supremum is 1, and the infemum is -1.

$\square$

3.

Proof. Let us consider a bounded, nonempty set of real numbers S such that $\inf S = \sup S$. On the contrary, assume S contains 2 or more numbers. Let us denote two arbitrary elements of the set as $a, b \in \mathbb{R}$ such that $a \neq b$, otherwise they are the same element in the set. Without Loss of Generality, let us say that $a < b$. By the definition of bounded, $\exists r_1, r_2$ such that $r_1 \leq a < b \leq r_2$. Therefore, $r_1 < r_2$. Note that the infemum and supremum are strict bounds on the set S. Let $r_1 = \inf S$ and $r_2 = \sup S$.
By our assumption, $\inf S = \sup S \implies r_1 = r_2$, which is a contradiction since we previously showed that $r_1 < r_2$. Thereofre, S must only contain one number.

$\square$

4a. $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$
Sketch: above, we want to show that $|\frac{1}{\sqrt{n}} - 0| < \epsilon$

$$|\frac{1}{\sqrt{n}}| = \frac{1}{\sqrt{n}} < \epsilon \implies \frac{1}{\epsilon^2} < n$$

Thus, let $N = \frac{1}{\epsilon^2} < n$.

Proof. Let $\epsilon > 0$ be given. By A.P., $\exists\ N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon^2$

$$|\frac{1}{\sqrt{n}} - 0| = |\frac{1}{\sqrt{n}}| = \frac{1}{\sqrt{n}}$$

since square root of a real number is positive. From here, we need to relate n to N and then N to ϵ. Note that $n \geq N$ implies $\frac{1}{n} \leq \frac{1}{N} \implies \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}}$ and $\frac{1}{N} < \epsilon^2 \implies \frac{1}{\sqrt{N}} < \epsilon$ Thus,

$$\frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \epsilon$$

Therefore, we've shown that

$$|\frac{1}{\sqrt{n}} - 0| < \epsilon\ \forall\ n \geq N \implies \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

□

4b. $\lim_{n\to\infty} \frac{1}{n+5} = 0$
Sketch: we want to show that $|\frac{1}{n+5} - 0| < \epsilon$

$$|\frac{1}{n+5} - 0| = |\frac{1}{n+5}|$$

Note that the denominator will always be postive, so

$$= \frac{1}{n+5} < \epsilon \implies \frac{1}{\epsilon} < n+5 \implies \frac{1}{\epsilon} - 5 < \frac{1}{\epsilon} < n$$

Note that we can get rid of the minus 5 because $\frac{1}{\epsilon} > 0$, and for any number $a \in \mathbb{R}^+$, $a - 5 < a$. Let us choose $N = \frac{1}{\epsilon} < n$.

Proof. Let $\epsilon > 0$ be given. By A.P., $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon \implies$

$$|\frac{1}{n+5} - 0| = |\frac{1}{n+5}| = \frac{1}{n+5} < \frac{1}{n}$$

Recall that $n \geq N$ and so $\frac{1}{n} \leq \frac{1}{N}$

$$\frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Therefore, we've shown that

$$|\frac{1}{n+5} - 0| < \epsilon\ \forall\ n \geq N \implies \lim_{n\to\infty} \frac{1}{n+5} = 0$$

as desired. □

5a. Sketch: From calculus, we know the limit is 1, but we will prove it rigorously. We want to show that $|\frac{n^2}{n^2+n} - 1| < \epsilon$.

$$|\frac{n^2}{n^2+n} - 1| = |\frac{n^2}{n^2+n} - \frac{n^2+n}{n^2+n}| = |-\frac{n}{n^2+n}|$$

Both the numerator and denomintor will always be positive, so

$$|-\frac{n}{n^2+n}| = \frac{n}{n^2+n} < \frac{n}{n^2} = \frac{1}{n} < \epsilon \implies \frac{1}{\epsilon} < n$$

Thus let us choose $N = \frac{1}{\epsilon}$.

Proof. Let $\epsilon > 0$ be given. By A.P., $\exists\ N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ which implies that

$$|\frac{n^2}{n^2+n} - 1| = |-\frac{n}{n^2+n}| = \frac{n}{n^2+n} < \frac{n}{n^2} = \frac{1}{n}$$

Look at the above sketch for more detail and for additional logic. Now, recall that $n \geq N$ which implies that

$$\frac{1}{n} \leq \frac{1}{N} < \epsilon$$

Therefore,

$$|\frac{n^2}{n^2+n} - 1| < \epsilon\ \forall\ n \geq \frac{1}{\epsilon} \implies \lim_{n\to\infty} \frac{n^2}{n^2+n} = 1$$

as desired.

□

5b. Sketch: We want to show that $|\frac{\sin n}{n} - 0| < \epsilon$.

$$|\frac{\sin n}{n} - 0| = |\frac{\sin n}{n}| \leq |\frac{1}{n}| = \frac{1}{n} < \epsilon \implies$$

Choose $N = \frac{1}{\epsilon}$

Proof. Let $\epsilon > 0$ be given. By A.P., $\exists\ N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$ and so

$$|\frac{\sin n}{n} - 0| = |\frac{\sin n}{n}| \leq |\frac{1}{n}|$$

since $|\sin n| \leq 1\ \forall\ n$.

$$|\frac{1}{n}| = \frac{1}{n}$$

Recall that $n \geq N \implies \frac{1}{n} \leq \frac{1}{N}$.

$$\frac{1}{n} \leq \frac{1}{N} < \epsilon$$

by our choice of N. Therefore, we have shown that given some $\epsilon > 0$, we can find an $\frac{1}{N} < \epsilon$ such that $\forall\ n \geq N$

$$|\frac{\sin n}{n} - 0| < \epsilon\ \forall\ n \geq N \implies \lim_{n\to\infty} \frac{\sin n}{n} = 0$$

□

6.

Proof.
We are given that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = s$. By the definition of convergence,

$$\forall\ \epsilon > 0\ \exists\ N_1 \in \mathbb{N} \text{ s.t. } \forall\ n \geq N_1, |a_n - s| < \epsilon$$

Therefore,

$$|a_n - s| < \epsilon \implies -\epsilon < a_n - s < \epsilon$$

$\forall\ n \geq N_1$. Similarly, for the same $\epsilon > 0$, we can say that

$$-\epsilon < b_n - s < \epsilon$$

$\forall\ n \geq N_2$. Furthermore, since $a_n \leq s_n \leq b_n\ \forall\ n$, we can subtract s from all terms such that we obtain

$$a_n - s \leq s_n - s \leq b_n - s\ \forall\ n \geq N$$

Choose $N = \max(N_1, N_2)$. Therefore, we can say that

$$-\epsilon < a_n - s \leq s_n - s \leq b_n - s < \epsilon$$

$$-\epsilon < s_n - s < \epsilon$$

$$|s_n - s| < \epsilon$$

for all $n \geq N$, and the last step is by definition of absolute value. Now, by the definition of convergence, since for any $\epsilon > 0$, there exists some $N = \max(N_1, N_2) \in \mathbb{N}$ such that $\forall\ n \geq N$,

$$|s_n - s| < \epsilon \implies \lim_{n\to\infty} a_n = s$$

and this completes the proof.

$\square$

7.

Proof.
Note that this is an "if and only if" statement, so we must prove both directions.
$\Longrightarrow$ Suppose we are given that $\{c_n\}$ converges to c. By the definition of convergence, given $\epsilon > 0$

$$|c_n - c| < \epsilon$$

for all $n \geq N, N \in \mathbb{N}$. Note that we can expand the expression in the absolute value to be

$$|c_n - c - 0| = |(c_n - c) - 0| < \epsilon$$

which satisfies the structure $|a_n - L| < \epsilon$, where here $a_n = c_n - c$ and $L = 0$. Therefore, $\lim\limits_{n\to\infty} c_n - c = 0$. Note that the above is a strict equality, but we can also use the Comparison Lemma since given our $\epsilon > 0$ and our choice of N,

$$|(c_n - c) - 0| \leq 1|c_n - c| \ \forall \ n \geq N$$

Since there exists some $a = 1, a \in \mathbb{R}^+$, then we conclude that $\{c_n - c\}$ converges to 0.

$\Longleftarrow$ Supose we are given instead that $c_n - c$ converges to 0. By the definition of convergnce, given some $\epsilon > 0$, we write that

$$|(c_n - c) - 0| < \epsilon$$

for all $n \geq N, N \in \mathbb{N}$. Simplifying the expression in absolute value, we get

$$|(c_n - c) - 0| = |c_n - c| < \epsilon$$

which again satisfies the definition of convergence where $a_n = c_n$ and $L = c$. Once more, we could have used the Comparison Lemma since

$$|c_n - c| \leq 1|(c_n - c) - 0| \ \forall \ n \geq N$$

since $\exists \ 1 \in \mathbb{R}^+$, then we conclude that $\{c_n\}$ converges to c.

Thus, we have proven that the sequence $\{c_n\}$ converges to c iff the sequence $\{c_n - c\}$ converges to 0, as desired.

$\square$