MATH410: Advanced Calculus I

James Zhang*

May 12, 2024

These are my notes for UMD's MATH410: Advanced Calculus I. These notes are taken live in class ("live-TEX"-ed). This course is taught by Lecturer Anna Szczekutowicz.

Contents

*Email: jzhang72@terpmail.umd.edu

This section covers the foundation of analysis, which is just the set of real numbers. It covers basic definitions such as $\in, \notin, \emptyset, \subseteq, =, \cap, \cup, \setminus$, so for example

Definition 0.1. **Intersection** of A and B is $C = A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Some quantifiers include $\forall, \exists, \exists!$ and some number sets include $\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^C$.

Definition 0.2. The real numbers $\mathbb{R}$ satisfies 3 groups of axioms: Refer to the notes on Canvas for the Consequences of all of the following axioms.

1. Field $(+, *)$
   - Commutativity of Addition
   - Associativity
   - Additive Identity
   - Additive Inverse
   - Commutativty of Multiplcation
   - Associativity of Multiplication
   - Multiplicative Identity
   - Multiplicative Inverse
   - Distributive Property

   The set of integers $\mathbb{Z}$ is not a field because it fails under the multiplicative inverse.
2. Positivity

   There is a subset of $\mathbb{R}$ denoted by $\mathcal{P}$, called the set of positive numbers for which:
   - If x and y are positive, then $x + y$ and xy are both positive.
   - For each $x \in \mathbb{R}$, eaxctly one of the following 3 alternatives is true: $x \in \mathcal{P}$, $-x \in \mathcal{P}$, or $x = 0$
3. Completeness

Definition 0.3. **Absolute value** is defined as

$$|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Definition 0.4. **Triangle Inequality** is $\forall\ a, b \in \mathbb{R}, |a + b| \leq |a| + |b|$

Proof. Assume without loss of generality, $a \geq b$. We will proceed with proof by cases.
Case 1: Assume $a \geq b \geq 0$. Then $|a + b| = a + b$ by the definition of absolute value since $a \geq 0, b \geq 0 \implies |a + b| = a + b = |a| + |b|$.
Case 2: Now assume $a \geq 0 \geq b$ and $a + b \geq 0$. Note since $b \leq 0$ then $b \leq |b|$. Then

$$|a + b| = a + b \leq |a| + |b|$$

by the definition of absolute value and our above note.
Case 3: Now consider $a \geq 0 \geq b$ and $a + b < 0$. So

$$|a + b| = -(a + b) = -a - b \leq |a| + |b|$$

Case 4: Now consider $0 \geq a \geq b$ so $a + b < 0$. Therefore,

$$|a + b| = -(a + b) = -a + -b = |a| + |b|$$

□

§1 The Completeness Axiom

Definition 1.1. A subset S of $\mathbb{R}$ is said to be **bounded above** if $\exists\ r \in \mathbb{R}$ such that $s \leq r\ \forall\ s \in S$
The definition of **bounded below** is similar.

Definition 1.2. The least upper bound, if it exists, is called the **supremum** of S. We denote it as the "sup" of S. Similarly, the largest lower bound is called the **infemum** and is denoted as the "inf" of S.

Definition 1.3. Let $S \subseteq R$ where $S \neq \emptyset$. If S has a largest (smallest), the element is a max (min).

Example 1.4

Find the sup of $(0, 1)$ and prove it.

Proof. Let us prove that the $sup(0, 1) = 1$. First, let us show that we have an upperbound. If $x \in (0, 1)$, then $x \leq 1$. By definition of upperbound, 1 is an upper bound. Note that we can find many other upper bounds.
On the contrary, assume $x < 1$ is an upper bound. Now consider the average $\frac{1+x}{2}$.

$$\frac{x+1}{2} < \frac{1+1}{2} = 1$$

Therefore, we have showed that $0 < \frac{x+1}{2} < 1 \implies \frac{x+1}{2}(0, 1)$. But, $\frac{1+x}{2} > \frac{x+x}{2} \implies \frac{1+x}{2} > x$. This is a contradiction. Since x is an upper bound, and we found $\frac{1+x}{2} \in (0, 1)$ where $\frac{1+x}{2} > x$, so x is not a supremum.

□

Theorem 1.5

Suppose $S \in \mathbb{R}, S \neq \emptyset$ that is bounded above. Then a supremum exists. Every nonsempty subset S of $\mathbb{R}$ that is bounded below has a lower bound.

Note 1.6. Let c be a positive number then $\exists!$ a positive number whose square is c. $x^2 = c, x > 0$ has a unique solution and this gives us the notion of square root.

§1.1 Archimedian Property

Definition 1.7. The **Archimedian Property** is a result of the completeness axiom. Suppose there is a small $\epsilon > 0$ and c is an arbitrary large number.

1. $\exists\ n \in \mathbb{N}$ such that $c < n$, which just means that you can always find a natural number than any large number

2. $\exists\ m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$, which just means you can always find smaller rational numbers.

Proof. We will proceed by contradiction. Assume that $\exists$ an upper bound c for the $\mathbb{N}$. So there is no $n \in \mathbb{N}$ s.t. $c < n$. Since $\mathbb{N}$ is bounded above, and the $\mathbb{N}$ is nonempty, the supremum exists (Completeness Axiom). Let $s = \sup \mathbb{N}$. Consider $s - 1$ and $s - 1 < s = \sup \mathbb{N}$, which is the least upper bound, so $s - 1$ is not an upper bound. So $\exists n \in \mathbb{N}$ such that $s - 1 < n \implies s < n + 1$. But $s = \sup \mathbb{N}$, the least upper bound, this is a contradiction since it is less than $(n + 1) \in \mathbb{N}$.
For part b, use $c = \frac{1}{\epsilon}$ and use part a. □

Note 1.8. Some of the following are results from the Archimedian Property.

Theorem 1.9
For all $n \in \mathbb{Z}$, there is no integer in $(n, n + 1)$ (an open interval).

Theorem 1.10
If S is a nonempty subset of $\mathbb{Z}$ that is bounded above, then it has a max.

Theorem 1.11
$*$ For every $c \in \mathbb{R}$, $\exists!\ n \in \mathbb{Z}$ in $[c, c + 1)$

Definition 1.12. A subset $S \subseteq \mathbb{R}$ is said to be **dense in $\mathbb{R}$** if for every $a, b \in \mathbb{R}$ with $a < b$, then there is a $s \in S$ s.t. $s \in (a, b)$.

Theorem 1.13

$\mathbb{Q}$ is dense in $\mathbb{R}$. Reminder that $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$

Proof. Suppose we have arbitrary $a, b \in \mathbb{R}$ and $a < b$. We want to find $\frac{m}{n} \in (a, b)$. By multiplcation, we can say we want $na < m < nb$. We want an integer m between na and nb. We can write this as

$$nb - na > 1 \implies n(b-a) > 1 \implies n > \frac{1}{b-a}$$

By part a of the Archimedian Property, let $c = \frac{1}{b-a}$, and we know that there exists some $n \in \mathbb{N}$ such that $n > c$. Since $a < b$, and $b - a > 0$, multiply

$$n > \frac{1}{b-a}$$

$$n(b-a) > 1$$

$$nb - na > 1$$

$$nb - 1 > na \implies na < nb - 1$$

By previous $(*)$, $\exists\, m \in \mathbb{Z}$ s.t. $m \in [nb-1, nb)$. Therefore, $nb - 1 \leq m < nb$. Therefore,

$$na < nb - 1 \leq m < nb \implies na < m < nb \implies a < \frac{m}{n} < b$$

and so we have found that there exists $m \in \mathbb{Z}, n \in \mathbb{N}$ such that $\frac{m}{n} \in (a, b)$ for all $a, b \in \mathbb{R}$ and $a < b$. Therefore, the rational numbers are dense in the real numbers. $\square$

§2 Sequences

Definition 2.1. A **sequence** of $\mathbb{R}$ is a real-valued function whose domain is $\mathbb{N}$. $f : \mathbb{N} \to \mathbb{R}$ (a list of numbers indiced by $\mathbb{N}$)

Example 2.2

A sequence of odd integers could be $a_1 = 1, a_2 = 3, a_3 = 5, \ldots, a_n = 2n - 1$ which can be

$$\{1, 3, 5, \cdots\} = \{a_n\}_{n=1}^{\infty} = \{2n-1\}_{n=1}^{\infty}$$

Example 2.3

$$\{\frac{1}{n}\} = \{\frac{1}{n}\}_{n=1}^{\infty} \implies \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$$

§2.1 Convergence

Definition 2.4. A sequence $\{a_n\}$ is said to **converge** to a number L if $\forall \epsilon > 0$, $\exists$ an index N s.t. $\forall$ indices $n \geq N$ we have

$$|a_n - L| < \epsilon \implies \text{Notation: } \lim_{n\to\infty} a_n = L$$

Example 2.5

Suppose we have the sequence $\{\frac{(-1)^n}{n}\}$ and we WTS

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

Think of the problem as someone gives you a small $\epsilon \implies$ you have to find N, which we call the **threshold**, such that for every sequence value after the threshold is in the $\epsilon-$tube.
For example, $\epsilon = \frac{1}{2} \implies N = 3, \epsilon = \frac{1}{4} \implies N = 5$.

Above $L = 0$, sketch: we want

$$|a_n - L| < \epsilon \implies |\frac{(-1)^n}{n} - 0| < \epsilon \implies |\frac{1}{n}| < \epsilon \implies \frac{1}{\epsilon} < n$$

so choose $N = \frac{1}{\epsilon} < n$

Proof. Let $\epsilon > 0$ be given. By Archimedian Property, $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then if $n \geq N$

$$|\frac{(-1)^n}{n} - 0| = |\frac{(-1)^n}{n}| = |\frac{1}{n}| = \frac{1}{n}$$

From here, we need to relate n to N and then we can relate N to ϵ. Note that $n \geq N \implies \frac{1}{N} \geq \frac{1}{n}$ by algebra. Therefore,

$$\frac{1}{n} \leq \frac{1}{N} < \epsilon$$

by our choice of N. Therefore,

$$\frac{1}{n} < \epsilon$$

and so we are done since we have shown that

$$|\frac{(-1)^n}{n} < 0| < \epsilon$$

□

Example 2.6

Given $\{\frac{n^2-2n}{n^2+1}\}$, prove that this sequence $\lim_{n\to\infty}\frac{n^2-2n}{n^2+1} = 1$.

Some sketch work: we want to show that $|\frac{n^2-2n}{n^2+1} - 1| < \epsilon$

$$|\frac{n^2-2n}{n^2+1} - 1| = |\frac{n^2-2n}{n^2+1} - \frac{n^2+1}{n^2+1}| = |\frac{-2n-1}{n^2+1}| = |\frac{2n+1}{n^2+1}|$$

Note that both the numerator and denominator are both always positive, so we can consider. Now let us use the $\leq$ operator to simplify and have one singular 'n'.

$$\frac{2n+1}{n^2+1} \leq \frac{2n+1}{n^2} \leq \frac{2n+n}{n^2} = \frac{3n}{n^2} = \frac{3}{n}$$

Recall that $n \geq N \implies \frac{1}{N} \geq \frac{1}{n} \implies \frac{1}{n} \leq \frac{1}{N}$ So we'd choose N to get rid of 3 and introduce ϵ.

Proof. Let $\epsilon > 0$. By A.P., $\exists\ N \in \mathbb{N}$ s.t. $\frac{1}{N} < \frac{\epsilon}{3}$. For $n \geq N$, then

$$|\frac{n^2-2n}{n^2+1} - 1| = \cdots = \frac{2n+1}{n^2+1} < \cdots \leq \frac{3}{n} \leq \frac{3}{N} = 3 * \frac{1}{N} < 3 * \frac{\epsilon}{3} = \epsilon$$

Therefore, we have shown that

$$|a_n - L| < \epsilon \implies \lim_{n\to\infty}\frac{n^2-2n}{n^2+1} = 1$$

□

Theorem 2.7

The Sum Property states that if

$$\lim_{n\to\infty} a_n = a \text{ and } \lim_{n\to\infty} b_n = b$$

then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a + b$$

Some sketch work before the proof:
We want to show that $|a_n + b_n - (a+b)| < \epsilon$. Note that we can group terms together $|(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$ by the Triangle Inequality. It is known that these two terms converge. Therefore, we can choose ϵs such that

$$|a_n - a| + |b_n - b| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

Proof.
Let $\epsilon > 0$. Since the sequences $\{a_n\}$ and $\{b_n\}$ converge to a and b, respectively, by the Archimedian Principle, $\exists N_1, N_2 \in \mathbb{N}$ such that $\frac{1}{N_1} < \frac{\epsilon}{2}$ and $\frac{1}{N_2} < \frac{\epsilon}{2}$. Choose $N = \max(N_1, N_2)$, which represents the numerically larger threshold. For all $n \geq N$, we show

$$|a_n + b_n - (a+b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore, we have shown that $\lim_{n\to\infty}(a_n + b_n) = a + b$ □

Lemma 2.8

The Comparison Lemma (C.L.)

Let $\{a_n\}$ converge to a. Then $\{b_n\}$ converges to b if $\exists\ c \in \mathbb{R}^+$ and $N \in \mathbb{N}$ such that

$$\forall\ n \geq N,\ |b_n - b| \leq c|a_n - a|$$

Proof. Let $\epsilon > 0$. Since a_n converges to a, $\exists\ N_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{c},\ \ \forall\ n \geq N_1$. By the Archimedian Principle, $\exists\ N_2 \in \mathbb{N}$ such that $\frac{1}{N_2} < \epsilon$. Choose $N = \max(N_1, N_2)$ and if $n \geq N$, then

$$|b_n - b| \leq c|a_n - a| < c * \frac{\epsilon}{c} = \epsilon$$

$$\implies |b_n - b| < \epsilon$$

□

Lemma 2.9

Suppose the $\lim_{n\to\infty} a_n = a$, then for $c \in \mathbb{R}$,

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n = ca$$

Proof. Use the Comparison Lemma (above). Note that $|ca_n - ca| = |c(a_n - a)| = |c||a_n - a|$ which satisfies $|b_n - b| \leq c|a_n - a| \implies \{b_n\} = \{ca_n\} \implies b = ca$. $\square$

Lemma 2.10

The following is a useful property (*)

$$\lim_{n\to\infty} a_n = a \underset{\leftrightarrow}{\text{iff}} \lim_{n\to\infty} (a_n - a) = 0$$

Lemma 2.11

Suppose $\lim_{n\to\infty} a_n = 0$ and $\lim_{n\to\infty} b_n = 0$ then $\lim_{n\to\infty} a_n b_n = 0$.

Proof. Since $\lim_{n\to\infty} a_n = 0$ and $\sqrt{\epsilon} > 0$,

$$\exists\ N_1 \in \mathbb{N} \text{ s.t. } |a_n| < \sqrt{\epsilon}\ \forall\ n \geq N_1$$

Since $\lim_{n\to\infty} b_n = 0$ and $\sqrt{\epsilon} > 0$,

$$\exists\ N_1 \in \mathbb{N} \text{ s.t. } |b_n| < \sqrt{\epsilon}\ \forall\ n \geq N_2$$

Let $N = \max(N_1, N_2)$. Then if $n \geq N$,

$$|a_n b_n - 0| = |a_n b_n| = |a_n| * |b_n| < \sqrt{\epsilon} * \sqrt{\epsilon} = \epsilon$$

$\square$

Theorem 2.12

The Product Property states that if $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then $\lim_{n\to\infty} a_n b_n = ab$

Proof. Define $\alpha_n = a_n - a$ and $\beta_n = b_n - b$. Using the * property above, since $\lim_{n\to\infty} a_n = a \implies \lim_{n\to\infty}(a_n - a) = \lim_{n\to\infty} \alpha_n = 0$ and then the same for b such that $\lim_{n\to\infty} \beta_n = 0$.

Note that

$$a_n b_n - ab = (\alpha_n + a)(\beta_n + b) - ab$$

and then by Distributive property we get

$$= \alpha_n\beta_n + \alpha_n b + a\beta_n + ab - ab = \alpha_n\beta_n + \alpha_n b + a\beta_n$$

So using the previous lemma,

$$\lim_{n\to\infty}(a_n b_n - ab) = \lim_{n\to\infty}(\alpha_n\beta_n + b\alpha_n + a\beta_n) = \lim_{n\to\infty}(\alpha_n\beta_n) + b\lim_{n\to\infty}\alpha_n + a\lim_{n\to\infty}\beta_n$$

From above, the last two terms are 0 and by the previous lemma, the first term is 0. Therefore,

$$\lim_{n\to\infty}(a_n b_n - ab) \underset{\leftrightarrow}{\text{iff}} \lim_{n\to\infty}(a_n b_n) = ab$$

$\square$

Definition 2.13. A sequence **diverges** to $\infty, (-\infty)$ if

$$\forall\ M > (<)0, \exists\ N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, n \geq N \text{ then } a_n > (<)M$$

Example 2.14

Prove that $\lim_{n\to\infty}(n^2 - 4n) = \infty$

Sketch: we want $a_n > M \implies n^2 - 4n > M \implies n(n-4) > M$

Proof. Let $M > 0$ be given. By A.P., $\exists\ N \in \mathbb{N}$ s.t. $N > \max(M, 4)$. If $n \geq N$, then $n^2 - 4n = n(n-4) \geq N(N-4) > M$

Thus,

$$n^2 - 4n \to \infty \text{ as } n \to \infty$$

$\square$

Example 2.15

Prove that $(-1)^n$ does not converge.

Proof. On the contrary, suppose $(-1)^n$ converges to a. Let $\epsilon = 1$. In the definition of convergence, then $\exists\ N \in \mathbb{N}$ if $n \geq N$ then

$$|(-1)^n - a| < 1$$

For $n = 2N$, meaning some even number, we get $|(-1)^n - a| = |1 - a| < 1$
Now for $n = 2N + 1$, we get $|(-1)^{2N+1} - a| = |1 + a| < 1$
Note that $|1 - a| < 1$ and $|1 + a| < 1$ so therefore

$$|1 - a| + |1 + a| < 2$$

But note, using the Triangle Inequality in the reverse direction, note that $2 = |1 - a + 1 + a| \leq |1 - a| + |1 + a| < 1 + 1 = 2$. Therefore, we've shown that $2 < 2$ which is a contradiction and therefore, $(-1)^n$ does not converge. $\square$

Lemma 2.16

Suppose the sequence $\{b_n\}$ of nonzero numbers converges to $b \neq 0$. Then $\{\frac{1}{b_n}\}$ converges to $\frac{1}{b}$.
Sketch: Use the Comparison Lemma to find $c \in \mathbb{R}^+$ and $N_1 \in \mathbb{N}$ such that

$$|\frac{1}{b_n} - \frac{1}{b}| < c|b_n - b|$$

We just have to find c and N_1.

Proof. Note that

$$|\frac{1}{b_n} - \frac{1}{b}| = |\frac{b - b_n}{bb_n}| = \frac{1}{|b||b_n|}|b_n - b|$$

We want $\frac{1}{|b||b_n|}$ to be c, but this must be a single constant and not dependent on n. We want to find index N_1 such that

$$|b_n| > \frac{|b|}{2} \; \forall \; n \geq N_1$$

$$\frac{1}{|b_n|} < \frac{2}{|b|}$$

If we can find N_1 then $|\frac{1}{b_n} - \frac{1}{b}| \leq \frac{2}{|b|^2}|b_n - b|$ and the term $\frac{2}{|b|^2}$ becomes our c and we can apply the Comparison Lemma, so we need N_1 to make the above true.
Let $\epsilon = \frac{b}{2}$. By definition of $\{b_n\}$ converging to b, we can choose N_1 such that $|b_n - b| < \epsilon \; \forall \; n \geq N_1$.

$$|b_n - b| < \frac{|b|}{2}$$

$$-\epsilon < b_n - b < \epsilon$$

$$b - \epsilon < b_n < b + \epsilon$$

Check $b > 0, b < 0$ since $\epsilon = \frac{|b|}{2}$. When $b > 0, \epsilon = \frac{b}{2}$ so

$$b_n \in (b - \frac{b}{2}, b + \frac{b}{2}) = (\frac{b}{2}, \frac{3b}{2})$$

so $b_n > \frac{b}{2}$. When $b < 0$ ...So $|b_n| > \frac{|b|}{2}$ and this N_1 works and apply the Comparison Lemma. $\square$

Theorem 2.17

Let $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, and $b_n \neq 0 \forall\ n$ and $b \neq 0$ then

$$\frac{a_n}{b_n}_{n\to\infty} = \frac{a}{b}$$

Proof.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n * \frac{1}{b_n} = \lim_{n\to\infty} a_n * \lim_{n\to\infty} \frac{1}{b_n} = \frac{a}{b}$$

□

§2.2 Boundedness

Definition 2.18. A sequence $\{a_n\}$ is **bounded** if $\exists\ M \in \mathbb{R}$ such that $|a_n| \leq M\ \forall\ n$.

Theorem 2.19

Every convergent sequence is bounded.

- If convergent $\implies$ bounded.
- If it is unbounded, then it diverges.

Proof. Let $\lim_{n\to\infty} a_n = a$ and take $\epsilon = 1$. Using the definition of convergence, $\exists\ N \in \mathbb{N}$ s.t.

$$|a_n - a| > 1\ \forall\ n \geq N$$

then $|a_n| = |a_n - a + a| \leq |a_n - a| + |a| \leq 1 + |a|\ \forall\ n \geq N$ by the Triangle Inequality and then the definition of the converging sequence. However, we need to show that this is true (bounded by a constant) for all n, not just for all $n \geq N$.

Define $M = \max(1 + |a|, |a_1|, \ldots, |a_{N-1}|)$. Note that there the $N - 1$ terms are finite and so a max exists. Then

$$|a_n| \leq M\ \forall\ n$$

and so $\{a_n\}$ is bounded. □

Remark 2.20. Recall that a set $S \subset \mathbb{R}$ is dense in $\mathbb{R}$ if every open set $(a, b) \in \mathbb{R}$ contains a point $s \in S$.

Definition 2.21. A set of numbers $\{x_n\}$ is in a set S provided that $x_n \in S\ \forall\ n$.

Lemma 2.22

A set S is **dense** in $\mathbb{R}$ if and only if every $x \in \mathbb{R}$ is a limit of a sequence of a sequence in S.

Proof.
$\Longrightarrow$ Let $S \subset \mathbb{R}$ be dense in $\mathbb{R}$. Fix $x \in \mathbb{R}$ and let n be an index. Since S is dense, there is an element in S in $(x, x + \frac{1}{n})$. For each n, this defines $\{s_n\}$ with

$$s \in (x, x + \frac{1}{n})$$

$$x < s < x + \frac{1}{n}$$

$$|s_n - x| < \frac{1}{n} \ \forall \ n$$

$$|s_n - x| < 1|\frac{1}{n} - 0|$$

Use the Comparison Lemma since $\{\frac{1}{n}\}$ converges to 0. So, $\{s_n\}$ converges to x.

$\Longleftarrow$ Let S have the property that every number in $\mathbb{R}$ is the limit of a sequence in S. We want to show that any open interval in $\mathbb{R}$ contains a point $s \in S$. Consider an open interval $(a, b) \in \mathbb{R}$. Consider $\frac{a+b}{2} = s \in \mathbb{R}$. By assumption, $\exists \{s_n\}$ of points in S s.t. $\lim\limits_{n \to \infty} s_n = s$. Define $\epsilon = \frac{b-a}{2} > 0$. By definition of convergence, $\exists \ N$ s.t. $|s_n - s| < \epsilon \ \forall \ n \in \mathbb{N}$.

$$-\epsilon < s_n - s < \epsilon$$

$$s - \epsilon < s_n < s + \epsilon$$

$$\frac{a+b}{2} - \frac{b-a}{2} < s_n < \frac{a+b}{2} + \frac{b-a}{2}$$

$$a < s_n < b$$

The point $s_N \in S$ and $s_n \in (a, b)$ so S is dense in $\mathbb{R}$.

$\square$

Definition 2.23. The **sequential density of $\mathbb{Q}$** states that every $\mathbb{R}$ is the likmit of a sequence in $\mathbb{Q}$.

Theorem 2.24

Let $\{c_n\} \in [a, b]$ and $\lim\limits_{n \to \infty} c_n = c$ then $c \in [a, b]$ also.

Definition 2.25. $S \subset \mathbb{R}$ is said to be **closed** (set) if $\{a_n\}$ is a sequence in S that converges to a, then $a \in S$ also.

Example 2.26

$(0, 1]$ not closed since $\{\frac{1}{n} \in (0, 1]\}$ and $\lim_{n\to\infty} \frac{1}{n} = 0$ but $0 \notin (0, 1]$.

Example 2.27

$\mathbb{Q}$ is not closed since we can find $\{r_n\} \in \mathbb{Q}$ that converge to π but $\pi \notin \mathbb{Q}$.

Definition 2.28. A $\{a_n\}$ is said to be **monotonically increasing (decreasing)** if $a_{n+1} \geq (\leq) a_n \ \forall \ n$

Note 2.29. If a sequence is monotone, then it is either monotonically increasing or decreasing.

Theorem 2.30

Monotone Convergence Theorem (MCT) states that a monotone sequence converges if and only if it is bounded. Moreover, the bounded monotone $\{a_n\}$ converges to the

1. $\sup\{a_n \mid n \in \mathbb{N}\}$ if monotone increasing
2. $\inf\{a_n \mid n \in \mathbb{N}\}$ if monotone decreasing

Proof.
$\implies$ Note that we already showed that convergent sequences are bounded.

$\impliedby$ We want to show that our sequence converges to either the inf, sup depending on if its either increasing or decreasing. Now assume that our sequence is bounded. Define $S = \{a_n \mid n \in \mathbb{N}\}$ and S is bounded by assumption. Since S is nonempty and bounded above, S has $\sup S = l$ by the Completeness Axiom. Claim $\lim_{n\to\infty} a_n = l$. Let $\epsilon > 0$ be given, and we want to show the usual definition of convergence.
Note that

$$|a_n - l| < \epsilon$$

$$-\epsilon < a_n - l < \epsilon$$

$$l - \epsilon < a_n < l + \epsilon \ \forall \ n \geq N$$

But l is an upper bound for $S \implies a_n \leq l < l + \epsilon \ \forall \ n$.
On the other hand, since l is the least upper bound for S, $l - \epsilon$ is not an upper bound for S. So, $\exists \ N$ such that $l - \epsilon < a_N$.
Since a_n is monotonically increasing. $l - \epsilon < a_N \leq a_n \ \forall n \geq N$. Thus, we have $N \in \mathbb{N}$ such that $\forall n \geq N$ we have $|a_n - l| < \epsilon$, as desired. □

Remark 2.31. The formula for a finite geometric sum is $S_n = \sum_{k=1}^{n} r^k$ where $r \neq 1, r < 1$.

$$S_n = \frac{r - r^{n+1}}{1 - r}$$

Example 2.32

Consider $S_n = \sum_{k=1}^{n} \frac{1}{k} \cdot \frac{1}{2^k}$

$$k = 1 \implies s_1 = \frac{1}{2}$$

$$k = 2 \implies s_2 = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2^2} = \frac{5}{8}$$

$$k = 3 \implies \frac{1}{2} + \frac{1}{8} + \frac{1}{3} \cdot \frac{1}{2^3}$$

$$\vdots$$

so this sequence is monotonically increasing. Is it bounded

$$S_n = \sum_{k=1}^{n} \frac{1}{k} \cdot \frac{1}{2^k} \leq \sum_{k=1}^{n} \frac{1}{2^k} = \frac{\frac{1}{2} - \frac{1}{2}^{n+1}}{1 - \frac{1}{2}} = 1$$

Theorem 2.33

The Nested Interval Theorem. Suppose that $I_n = [a_n, b_n]$ is a sequence of intervals, for which $I_{n+1} \subset I_n \ \forall \ n$. Then the intersection of those intervals is a nonempty closed interval

$$\cap_{i=1}^{\infty} I_n = [a, b]$$

where $a = \sup a_n, b = \inf b_n$. Furthermore, if $\lim_{n\to\infty} a_n - b_n = 0$ then $\cap_{i=1}^{\infty} I_n$ contains a single point.

Proof.

$\Longleftarrow$ Let $X \in \cap_{i=1}^{\infty} I_n$. So for all $n \in \mathbb{N}, x \in I_n$ by definition of intersection. Therefore,

$$a_n \leq x \leq b_n \ \forall \ n$$

Note that xx is an upper bound for a_n. So, by definition of sup, $a = \sup a_n \leq x$.

$$a \leq x \leq b \implies x \in [a, b]$$

$\Longrightarrow$ The reverse direction is similar. □

§2.3 Sequential Compactness

Definition 2.34. Consider a sequence $\{a_n\}$ and let $\{n_k\}$ be a sequence of $\mathbb{N}$ that is strictly increasing. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k} \ \forall \ k$ is a **subsequence**.

Note 2.35. Note that a sequence may not converge, but it may be possible to find a subsequence that does.

Theorem 2.36

Let $\{a_n\}$ converges to a. Then every subsequence of $\{a_n\}$ also converges to the same limit a.

Theorem 2.37

Every sequence (does not need to converge) has a monotone increasing or decreasing subsequence.

Proof. Consider $\{a_n\}$. We all an index a **peak index** for $\{a_n\}$ if

$$a_n \leq a_m \ \forall \ n \geq m$$

You can have two cases: infinitely many peak indices, or a finite number of peak indices.
Suppose there are finite number of peak indices. Then we choose N such that there are no more peak indices. Since N is not a peak index, $\exists \ n_1 \in \mathbb{N}$ such that $n_1 > N$ with $a_N \leq a_{n_1}$

$$\vdots$$

Continue for $n_k \implies \exists \ n_{k+1} \in \mathbb{N}$ with $n_{k+1} \geq n_k$ with $a_{n_k} \leq a_{n_{k+1}}$

$$a_N \leq a_{n_1} \leq \cdots \leq a_{n_k} \leq a_{n_{k+1}}$$

which is a monotonically increasing subsequence.
In the case of infinitely many peak indices, $m_1 < m_2 < m_3 < \cdots <$ peak indices. Since m_1 is a peak index. Then $m_1 < m_2 \implies a_{m_1} > a_{m_2}$.

$$\vdots$$

We'll get a monotonically decreasing subsequence. □

Theorem 2.38

Every bounded sequence has a convergent subsequence.

Proof. Let $\{a_n\}$ be bounded. By the previous theorem, $\{a_n\}$ has a monotone subsequence. Since $\{a_n\}$ is boundeed, $\{a_{n_k}\}$ is bounded also. By MCT, $\{a_{n_k}\}$ converges since it is monotone and bounded. □

Definition 2.39. A $S \subset \mathbb{R}$ is said to be **compact (or sequentially compact)** if every sequence in S has a convergent subsequence converging to a point in S. For a set to not be compact, we find a sequence in S that has no convergence subsequence that converges to a point in S.

Example 2.40

$[1, \infty)$ is not compact. Consider $a_n = n, a_n \to \infty$ by Archimedian Principle. Then every subsequence of n_k also diverges to ∞. Thus, $\{a_n\}$ has no subsequence that converges.

Example 2.41

$(0, 1]$ is not compact. Let $a_n = \frac{1}{n}, a_n \to 0, n \to \infty$, so every subsequence converges to 0 also. But $0 \notin (0, 1]$ so it is not compact.

Theorem 2.42

The Sequentially Compactness Theorem (SCT) states that every interval $[a, b]$ such that $a, b \in \mathbb{R}$ is sequentially compact.

Proof. Let $\{a_n\}$ be in $[a, b]$. So, $a \leq a_n \leq b \ \forall \ n$. By a previous theorem, since $\{a_n\}$ is bounded, there exists a convergent subsequence $\{a_{n_k}\}$. Assume $\{a_{n_k}\} \to l$. Since $a \leq a_n \leq b \ \forall \ n$, then

$$a \leq a_{n_k} \leq b \ \forall \ n$$

so $l \in [a, b]$ as desired. Therefore, $\{a_n\}$ has a convergent subsequence whose limit is in the interval $[a, b]$, so it is sequentially compact. $\square$

Theorem 2.43

Bolzano Weirstrass Theorem: If $S \subset \mathbb{R}$, the following are equivalent

$$S \text{ is closed and bounded } \iff S \text{ is compact}$$

§3 Continuous Functions

§3.1 Continuity Basics

Note 3.1. Before $f : \mathbb{N} \to \mathbb{R}$ but now $f : D \subset \mathbb{R} \to \mathbb{R}$. $f(x)$ is the value the function assigns to x.

Definition 3.2. A function $f : D \to \mathbb{R}$ is said to be **continuous at a point** x_0 if whenerver $\{x_n\}_{n=1}^{\infty}$ converges to $x_0 \in D$, the image sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$.

Definition 3.3. A function $f : D \to \mathbb{R}$ is **continuous** if f is continuous at every point in D.

Example 3.4

Consider $f(x) = x^2 + 7x - 3$. We want to show f is continuous. Select $x_0 \in \mathbb{R}$ and let $\{x_n\} \to x_0 \implies \lim_{n\to\infty} x_n = x_0$. We want to show that

$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

Note that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n^2 + 7x_n - 3$$

by definition of f.

$$= \lim_{n\to\infty} x_n^2 + 7\lim_{n\to\infty} x_n + \lim_{n\to\infty} 3$$

by properties of sequences.

$$= x_0^2 + 7x_0 - 3 = f(x_0)$$

by the definition of f

Remark 3.5. Given $f : D \to \mathbb{R}, g : D \to \mathbb{R}$ are continuous, then

$$f \pm g, fg, \frac{f}{g}(g \neq 0)$$

are continuous and this follows directly from convergent sequence properties from the previous chapter. Thus, polynomials are continuous.

Example 3.6

Consider Dirichlet's function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = \begin{cases} 1 \text{ if x is rational} \\ 0 \text{ if x is irrational} \end{cases}$$

Note that f is defined on $\mathbb{R}$ but it is discontinuous at $x_0 \in \mathbb{R}$.

Proof. Let $x_0 \in \mathbb{R}$. By sequential density of the $\mathbb{Q}$ and $\mathbb{Q}^c$, we can find

$$\{u_n\} \to x_0, u_n \in \mathbb{Q}\ \forall n$$

$$\{v_n\} \to x_0, v_n \in \mathbb{Q}^c\ \forall\ n$$

Since $f(u_n) = 1\ \forall\ n$ and $f(v_n) = 0\ \forall\ n$, then

$$\{f(u_n)\} \to 1 \text{ but } \{f(v_n)\} \to 0$$

Therefore, $\lim_{n\to\infty} f(u_n) = 1 \neq 0 = \lim_{n\to\infty} f(v_n)$ but $\{u_n\} \to x_0$ and $\{v_n\} \to x_0$ but we cannot have 2 function values for x_0. □

Definition 3.7. Suppose $f : D \to \mathbb{R}$ and $g : U \to \mathbb{R}$ such that $f(D) \subset U$ then we define

$$(g \circ f)(x) = g(f(x)) \; \forall \; x$$

Theorem 3.8

Let $f : D \to \mathbb{R}, g : U \to \mathbb{R}$ and $f(D) \subset U$. Let f be continuous at x_0 and g be continuous at $f(x_0)$. Then $(g \circ f) : D \to \mathbb{R}$ is continuous at x_0.

Proof. Suppose $\{x_0\} \in D$ converges to x_0. Since f is continuous, then $\lim_{n\to\infty} f(x_n) = f(x_0)$.

$$\{f(x_n)\} \underset{n\to\infty}{\to} f(x_0)$$

Since g is continuous at $f(x_0)$, then $\lim_{n\to\infty} g(f(x_n)) = g(f(x_0))$. Therefore, $(g\circ f)(x)$ is continuous at x_0 since

$$\{g(f(x_n))\} \underset{n\to\infty}{\to} g(f(x_0))$$

$\Longrightarrow$ we can combine continuous functions and remain continuous □

§3.2 Extreme Value Theorem

Definition 3.9. $f : D \to \mathbb{R}$ attains a **maximum (minimum)** value if there is

$$x_0 \in D \text{ s.t. } f(x_0) \geq (\leq) f(x) \; \forall x \in D$$

Remark 3.10. Recall that a nonempty set has a maximum if it is bounded above and contains its supremum ie. the supremum is in the set.

$\Longrightarrow$ Now $f : D \to \mathbb{R}$ has a maximum when the image $f(D)$ is bounded above and the supremum of the image is a functional value.

Example 3.11

$f : (0, 1) \to \mathbb{R}$ where $f(x) = 2x$. Note that the supremum of the image is 2, but 2 is not a functional value. Therefore, this function does not have a max.

Theorem 3.12

The **Extreme Value Theorem** states that a continuous function on a closed and bounded interval $f : [a, b] \to \mathbb{R}$ attains both a maximum and a minimum. Sketch: Note that we want to show that $f(D)$ is bounded above. See lemma below. After that, we need to show that the supremum is a functional value.

Lemma 3.13

Assume on the contrary that given $f : [a, b] \to \mathbb{R}$ is continuous, assume there is no M such that

$$f(x) \le M \; \forall \; x \in [a, b]$$

There is $x \in [a, b]$ at which $f(x) > n$, $\forall \; n$. For each n this creates a sequence $\{x_n\}$ in $[a, b]$ with $f(x) > n \; \forall \; n$. $\{x_n\}$ may or may not converge. By Sequential Compactness Theorem, choose $\{x_{n_k}\}$ subsequence that converges to $x_0 \in [a, b]$. Since f is continuous at x_0, $\{f(x_{n_k})\} \to f(x_0)$, but every convergent sequence is bounded by a theorem, so $\{f(x_{n_k})\}$ is bounded. Therefore, we have a contradiction since $f(x_{n_k}) > n_k \ge k \; \forall k \; \in \mathbb{N}$. So $f : [a, b] \to \mathbb{R}$ is bounded above.

Proof. Define $S = f([a, b])$, all of the image values. By the lemma above, S is bounded. Note S is nonempty and bounded, thus by the Completeness Axiom, $c := \sup(S)$ exists. Note that we want to find $x_0 \in [a, b]$ such that $f(x_0) = c$, as this would show that the supremum is a functional value. Consider

$$c - \frac{1}{n} < c \; \forall \; n$$

Note that $c - \frac{1}{n}$ is not an upper bound since c is the least upper bound. So, we can find a point $x \in [a, b]$ such that

$$c - \frac{1}{n} < f(x) < c$$

Label point x_n to create a sequence $\{x_n\}$

$$c - \frac{1}{n} < f(x_n) < c \; \forall \; n$$

Since $\{\frac{1}{n}\} \to 0$ as $n \to \infty$, then $\{f(x_n)\} \to c$ by the Squeeze Theorem, as desired. Note by the Sequential Compactness Theorem, there exists a subsequence $\{x_{n_k}\}$ that converges to x_0. Since f is continuous at x_0, then $\{f(x_{n_k})\} \to f(x_0)$. Recall that $\{f(x_{n_k})\}$ is a subsequence of $\{f(x_n)\}$ that converges to c, and any subsequence must also converge to the same value as the full sequence. Therefore, $f(x_0) = c$. Therefore, the supremum exists and is a functional value, so we attain a max at x_0. $\square$

§3.3 Intermediate Value Theorem

Theorem 3.14

The Intermediate Value Theorem state that suppose $f : [a, b] \to \mathbb{R}$ is continuous, let $c \in \mathbb{R}$ between $f(a)$ and $f(b)$. Then there exists $x_0 \in (a, b)$ such that $f(x_0) = c$.

Proof. Without loss of generality, suppose $f(a) < c < f(b)$. Recursively define a sequence of nested intervals starting at $[a, b]$ and converging to $x_0 \in (a, b)$ with $f(x) = c$. We WTS $f(x_0) = c$ by letting $a_1 = a, b_1 = b \; \forall \; n$.
$\forall \; n$ define $[a_n, b_n]$ by considering the midpoint $m_n = \frac{a_n + b_n}{2}$. Let us consider some cases.
$\implies$ If $f(m_n) \leq c$, define $a_{n+1} = m_n$ and $b_{n+1} = b_n$.
$\impliedby$ If $f(m_n) > c$, define $a_{n+1} = a_n$ and $b_{n+1} = m_n$.
Note that $a \leq a_n \leq a_{n+1} < b_{n+1} < b_n \leq b$ and $f(a_{n+1}) \leq c$ and $f(b_{n+1}) > c$ by definition. Now, we want to show that

$$\lim_{n\to\infty} (b_n - a_n) = 0$$

in order to apply the Nested Interval Theorem.

$$\vdots$$

So $b_n - a_n = \frac{b-a}{2^{n-1}} \; \forall \; n \underset{n\to\infty}{\to} 0$. So $\lim_{n\to\infty} (b_n - a_n) = 0$. Thus by Nested Interval Theorem, $\exists \; x_0 \in (a, b)$ where $\{a_n\} \to x_0$ and $\{b_n\} \to x_0$. Since f is continuous at x_0, then $\{f(a_n)\} \to f(x_0)$ and $\{f(b_n) \to f(x_0)\}$. Since $f(a_n) \leq c \; \forall \; n \implies f(x_0) \leq c$ and $f(b_n) \geq c \; \forall \; n \implies f(x_0) \geq c$. Thus, the only this is true is $f(x_0) = c$, as desired. □

Example 3.15

Suppose we have $h(x) = x^5 + x + 1 = 0$. $h(x)$ is a polynomial so it is continuous. Verify that a solution exists. Let us test some points.

$$h(0) = 0 + 0 + 1 = 1$$

$$h(1) = 1 + 1 + 1 = 3$$

$$h(-1) = -1 - 1 + 1 = -1$$

By the Intermediate Value Theorem, there exists $x_0 \in (-1, 0)$ such that $x_0^5 + x_0 + 1 = 0$.

Example 3.16

$x^2 = c, c > 0$. Verify that a solution exists.

Proof. Consider $f : [0, c+1] \to \mathbb{R}$. $f(x) = x^2, 0 \leq x \leq c+1$. Testing points

$$f(0) = 0^2 = 0 < c$$

$$f(c+1) = c^2 + 2c + 1 > c$$

Since x^2 it is continuous. By IVT, there exists $x_0 \in (0, c+1)$ such that $x_0^2 = c$. □

§3.4 Uniform Continuity

Definition 3.17. A function $f : D \to \mathbb{R}$ is said to be **uniformly continuous** if for $\{u_n\}$ and $\{v_n\}$ in D with $\lim\limits_{n\to\infty} u_n - v_n = 0$ then $\lim\limits_{n\to\infty} f(u_n) - f(v_n) = 0$.

Note 3.18. It doesn't make sense to say f is uniformly continuous at a singular point. Further note that there is no requirement for $\{u_n\}$ and $\{v_n\}$ to converge.

Remark 3.19. Uniform continuity is on an interval.

Example 3.20

$f : \mathbb{R} \to \mathbb{R}, f(x) = 3x$ is uniformly continuous.

Proof. Let $\{u_n\}$ and $\{v_n\}$ be in $\mathbb{R}$ and $\{u_n - v_n\} \to 0$. Then

$$\{f(u_n) - f(v_n)\} = \{3u_n - 3v_n\} = \{3(u_n - v_n)\} \to 3 * 0$$

as needed. □

Example 3.21

$f(x) = x^2$ is not uniformly continuous on $f : \mathbb{R} \to \mathbb{R}$. To do this, we must find a pair of sequences that doesn't work.

Proof. Let $\{u_n\} = \{n + \frac{1}{n}\}$ and $\{v_n\} = \{n\}$. Note that $\{u_n - v_n\} \to 0$ but

$$\{f(u_n) - f(v_n)\} = \{f(n + \frac{1}{n}) - f(n)\} = \{(n + \frac{1}{n})^2 - n^2\} = \{2 + \frac{1}{n^2}\} \to 2 \neq 0$$

Therefore, f is not uniformly continuous on $\mathbb{R}$. □

Example 3.22

Consider $f : (0, 2) \to \mathbb{R}$ and $f(x) = \frac{1}{x}$. This is not uniformly continuous since there is a vertical asymptote at $x = 0$.

Proof. Let $\{u_n\} = \frac{1}{n}$ and $\{v_n\} = \frac{2}{n}$. Note that $\{u_n - v_n\} \to 0$ but

$$\{f(u_n) - f(v_n)\} = \{f(\frac{1}{n}) - f(\frac{2}{n})\} = \{n - \frac{n}{2}\} = \{\frac{n}{2}\} \to \infty$$

□

But now consider $f : (2, 3) \to \mathbb{R}, f(x) = \frac{1}{x}$. This is uniformly continuous.

Proof. Suppose $\{u_n - v_n\} \to 0$ for $\{u_n\}$ and $\{v_n\}$ in $(2, 3)$.

$$|f(u_n) - f(v_n)| = |\frac{1}{u_n} - \frac{1}{v_n}| = |\frac{u_n - v_n}{u_n v_n}|$$

We need to bound the product $u_n v_n$. Note that $u_n > 2, v_n > 2 \implies \frac{1}{u_n} < \frac{1}{2}, \frac{1}{v_n} < \frac{1}{2}$, so

$$< \frac{|u_n - v_n|}{2 * 2}$$

so $|f(u_n) - f(v_n)| \leq \frac{1}{4}|u_n - v_n|$ and so by Comparison Lemma, $\{f(u_n) - f(v_n)\} \to 0$. Note that this would work for domains $(0.00000001, \infty)$. □

Note 3.23. If $f : D \to \mathbb{R}$ is uniformly continuous then it is continuous, but not all continuous functions are uniformly continuous. The go-to example is $f(x) = x^2$ on $\mathbb{R}$.

Theorem 3.24

Every continuous function on a closed bounded interval $f : [a, b] \to \mathbb{R}$ is uniformly continuous.

Remark 3.25. For example, $f(x) = x^2$ on $[a, b]$ is uniformly continuous.

Proof. Let $\{u_n\}, \{v_n\} \subset [a, b]$ with $\lim_{n\to\infty}(u_n - v_n) = 0$. We WTS that $\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0$. By contradiction, assume that $\{f(u_n) - f(v_n)\} \not\to 0$. Therefore,

$$\exists\ \epsilon > 0 \text{ s.t. } \quad \forall\ N \in \mathbb{N}, \text{ there is } n \geq N$$

with

$$|f(u_n) - f(v_n)| \geq \epsilon$$

Let us create a subsequence

$$n_1 \geq N = 1 \text{ s.t. } |f(u_{n_1}) - f(v_{n_1})| \geq \epsilon$$

$$n_2 \geq n_1 + 1 \cdots |f(u_{n_2}) - f(v_{n_2})| \geq \epsilon$$

$$n_3 \geq n_2 + 1 \cdots |f(u_{n_3}) - f(v_{n_3})| \geq \epsilon$$

So $\{f(u_{n_k}) - f(v_{n_k})\}$ is a subsequence with $\{f(u_{n_k}) - f(v_{n_k})\} \geq \epsilon\ \forall n_k$. Because $\{u_n\}$ is a sequence in $[a, b]$, we can use Sequential Compactness to find a subsequence $\{u_{m_k}\}$ that converges to some $x_0 \in [a, b]$. Since f is continuous, then $\lim_{k\to\infty} f(u_{m_k}) = f(x_0)$. Since $\lim_{k\to\infty}(u_n - v_n) = 0 \implies \lim_{k\to\infty}(u_{m_k} - v_{m_k}) = 0$ by a theorem. Thus,

$$\lim_{k\to\infty} v_{m_k} = \lim_{k\to\infty} u_{m_k} - \lim_{k\to\infty}(u_{m_k} - v_{m_k}) = x_0 - 0 \implies \{v_{m_k}\} \to x_0$$

Therefore,

$$\lim_{k\to\infty}(f(u_{m_k}) - f(v_{m_k})) = f(x_0) - f(x_0) = 0$$

But this is a contradiction that

$$\forall\ n_k,\ |f(u_{n_k}) - f(v_{n_k})| \geq \epsilon$$

and so therefore, $\{f(u_n) - f(v_n) \to 0\}$ as desired. $\square$

§3.5 Epsilon-Delta Criterion

Definition 3.26. A function $f : D \to \mathbb{R}$ is said to satisfy the **$\epsilon - \delta$ criterion** at $x_0 \in D$ if $\forall\ \epsilon > 0,\ \exists\ \delta > 0$ so that

$$\forall\ x \in D \text{ and } |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

Note 3.27. δ depends on ϵ and maybe x_0. For uniform continuity, however, δ cannot depend on location, so δ will not depend on x_0 in the case of uniform continuity.

Example 3.28

$f : \mathbb{R} \to \mathbb{R}, f(x) = 3x$. Prove it satisfies $\epsilon - \delta$ criteria at $x_0 = 2$.

Sketch. Given $|x - 2| < \delta$. How do we show that $|f(x) - f(2)| < \epsilon$.

$$|3x - 6| \implies |3(x - 2)| < 3\delta$$

so we take $\delta = \frac{\epsilon}{3}$. □

Proof. Let $\epsilon > 0$ be given. Let $x_0 = 2$ and let $\delta = \frac{\epsilon}{3}$. Then if $|x - 2| < \delta$ then

$$|f(x) - f(x_0)| = |3x - 6| = 3|x - 2| < 3\delta = 3 * \frac{\epsilon}{3} = \epsilon$$

□

Example 3.29

$f : \mathbb{R} \to \mathbb{R}, f(x) = x^2$ at any x_0. Show $\epsilon - \delta$ criterion.

Sketch. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$

$$|x^2 - x_0^2| = |x - x_0||x + x_0| \leq \delta|x + x_0|$$

Note the absolute value term is constant, but x could be large, so we need to bound it. Let $\delta \leq 1$. What happens to $|x + x_0|$ in this case, let's try and relate it to $|x - x_0|$.

$$|x + x_0| = |x - x_0 + x_0 + x_0| \leq |x - x_0| + |x_0 + x_0| = |x - x_0| + 2|x_0|$$

$$\leq \delta + 2|x_0| \leq 1 + 2|x_0|$$

which is a constant as desired. □

Proof. Let $\epsilon > 0$ and $x_0 \in \mathbb{R}$. Let $\delta = \min(1, \frac{\epsilon}{1+2|x_0|})$. Note that $\epsilon > 0$ and $1 + 2|x_0| > 0$ and so we confirm $\delta > 0$. Thus,

$$\delta \leq 1 \text{ and } \delta \leq \frac{\epsilon}{1 + 2|x_0|}$$

Then

$$|x + x_0| = |x - x_0 + x_0 + x_0| \leq |x - x_0| + 2|x_0| \leq \delta + |2x_0| \leq 1 + 2|x_0|$$

since $|x - x_0| < \delta$. Thus,

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |(x - x_0)(x + x_0)| < \delta|x + x_0| \leq \delta(1 + 2|x_0|)$$

Recall that $\delta \leq \frac{\epsilon}{1+2|x_0|}$ and so

$$\delta(1 + 2|x_0|) \leq \frac{\epsilon}{1 + 2|x_0|}(1 + 2|x_0|) = \epsilon \implies |f(x) - f(x_0)| < \epsilon$$

□

Theorem 3.30

Given $f : D \to \mathbb{R}, x_0 \in D$, f is continuous at x_0 iff f satisfies the $\epsilon - \delta$ criteria at x_0. Note that here δ depends on ϵ and can depend on x_0 because we are talking about **continuity**.

Definition 3.31. We say $f : D \to \mathbb{R}$ satisifes the **$\epsilon - \delta$ criterion on D** if

$$\forall\ \epsilon > 0\ \exists\ \delta > 0 \text{ s.t. } \ \forall\ u, v \in D, \text{ if } |u - v| < \delta \implies |f(u) - f(v)| < \epsilon$$

Note here that δ can only depend on ϵ and not x_0.

Theorem 3.32

Given $f : D \to \mathbb{R}$, f is uniformly continuous on D iff f satisfies $\epsilon - \delta$ critera on D, and here, note that δ can only depend on ϵ because we are talking about **uniform continuity**.

§3.6 Images, Inverses, Monotone Functions

Definition 3.33. $f : D \to \mathbb{R}$ is called **monotonically increasing (decreasing)** if

$$\forall\, u, v \in D, u < v \implies f(u) \leq (\geq) f(v)$$

If "strictly", then the operators become $<$ and $>$ respectively.

Definition 3.34. $f : D \to \mathbb{R}$ is called **one-to-one (1-1)** when $f(u) = f(v) \implies u = v$.

Definition 3.35. When f is $1-1$, its inverse, denoted $f^{-1}(x)$ is a function from $f(D)$ to D satisfying $f(x) = y \leftrightarrow f^{-1}(y) = x$

- $f^{-1}(f(x)) = y \;\forall\, x \in D$
- $f(f^{-1}(x)) - y \;\forall\, y \in f(D)$

Theorem 3.36

Any strictly monotone function $f : D \to \mathbb{R}$ is $1-1$ and thus has an inverse.

Proof. WLOG, suppose f is strictly increasing and $f(u) = f(v)$. To show $1-1$, we WTS $u = v$ for $u, v \in D$. By contradiction, if $u < v$, since f is strictly monotone increasing, then $f(u) < f(v)$. If $u > v \implies f(u) > f(v)$ by definition of strictly monotonically increasing function. Therefore, $u = v$, and so $f(u) = f(v) \implies u = v$ and so f is 1-1. □

Example 3.37

Prove that the inverse of $f(x) = x^3$ is continuous.

Proof. Note that f is a polynomial and thus continuous. f is strictly increasing.

$$u < v \implies u^3 < v^3 = u * u * u < v * v * v$$

by properties of inequalities. By a previous theorem, since f is strictly increasing, f has an inverse. Let $x_0 \in \mathbb{R}$, let $\{x_n\} \in \mathbb{R}$ such that $\{x_n\} \to x_0$. We WTS that $f^{-1}(x_n) \to f^{-1}(x_0)$.

For notation: label $y_n = f^{-1}(x_n), y_0 = f^{-1}(x_0)$. Therefore

$$x_n = f(y_n) = y_n^3$$

$$x_0 = f(y_0) = y_0^3$$

Since $x_n \to x_0$, then $y_n^3 \to y_0^3$. We WTS $y_n \to y_0$. Let $\epsilon > 0$. Let $\delta = \min((y_0 + \epsilon)^3 - (y_0)^3, y_0^3 - (y_0 - \epsilon)^3)$. Since $\epsilon > 0$, it is easy to show that $\delta > 0$. Since

$$y_n^3 \to y_0^3, \ \exists\ N \text{ s.t. } \ \forall\ n \geq N, |y_n^3 - y_0^3| < \delta$$

We know this is true for all ϵ, so therefore we can let $\epsilon = \delta$.

$$-\delta < y_n^3 - y_0^3 < \delta$$

$$y_0^3 - \delta < y_n^3 < \delta + y_0^3$$
$$y_0^3 - (y_0^3 - (y_0 - \epsilon)^3) < y_n^3 < (y_0 + \epsilon)^3 - y_0^3 + y_0^3$$
$$(y_0 - \epsilon)^3 < y_n^3 < (y_0 + \epsilon)^3$$
$$y_0 - \epsilon < y_n < y_0 + \epsilon$$
$$|y_n - y_0| < \epsilon$$

and so $y_n \to y_0$ or $f^{-1}(x_n) \to f^{-1}(x_0)$ by definition of y_n, y_0 and so $f^{-1}(x)$ is continuous. $\square$

Theorem 3.38

Let $f : D \to \mathbb{R}$ is monotone. If its image is an interval, then f is continuous.

Proof. Let $x_0 \in D$ and $\{x_n\} \in D$ with $x_n \to x_0$. Suppose on the contrary that $f(x_n) \not\to f(x_0)$. Then $\exists \epsilon > 0$ and subsequence of x_n such that

$$|f(x_{n_k}) - f(x_0)| \geq \epsilon$$

Assume WLOG that f is increasing.
Case 1: If the absolute value is positive

$$f(x_{n_k}) - f(x_0) \geq \epsilon$$

$$f(x_{n_k}) \geq \epsilon + f(x_0)$$

$$f(x_{n_k}) \geq \epsilon + f(x_0) > \frac{\epsilon}{2} + f(x_0) > f(x_0)$$

Since the image of f is an interval (all points in between). So $\exists\ c \in D$ such that

$$f(c) = f(x_0) + \frac{\epsilon}{2}$$

$$f(x_{n_k}) > f(c) > f(x_0)$$

And since f is strictly monotone increasing, so $x_{n_k} > c > x_0$

$$|x_{n_k} - x_0| > |c - x_0| > 0$$

Note that $c - x_0$ is a constant, and so $x_{n_k} \not\to x_0$.
Case 2: If the absolute value is negative

$$f(x_0) - f(x_{n_k}) \geq \epsilon$$

$$f(x_0) - \epsilon \geq f(x_{n_k})$$

$$f(x_0) > f(x_0) - \frac{\epsilon}{2} > f(x_{n_k})$$

$$\exists\ c_2 \in D \text{ such that } f(c_2) = f(x_0) - \frac{\epsilon}{2}$$

Since f is strictly monotonically increasing, we know

$$x_0 > c_2 > x_{n_k}$$

$$|x_{n_k} - x_0| > |x_0 - c_2| > 0$$

Therefore, combining the conclusions from the two cases:

$$x_{n_k} > \min(|x_0 - c|, |x_0 - c_2|) > 0$$

and so therefore, $|x_{n_k}| \not\to x_0$ which is a contradiction. Therefore, f is continuous. $\square$

Theorem 3.39

Suppose I is an interval and $f : I \to \mathbb{R}$ is monotone. Then f is continuous if and only if its image is an interval.

Proof. Omitted. This proof follows from the IVT and the previous theorem. $\square$

Theorem 3.40

Let $f : I \to \mathbb{R}$, I is an interval, be strictly monotone. Then its inverse $f^{-1} : f(I) \to \mathbb{R}$ is continuous. Similar to the x^3 example above.

Example 3.41

$f : [0, \infty) \to \mathbb{R}$ with $f(x) = x^n$ is strictly increasing, so inverse is continuous. Notation: negative integer n: $x^n = \frac{1}{x^{-n}}$

- $x^n * x^m = x^{n+m}$
- $(x^n)^m) = x^{nm}$

$y^{\frac{1}{n}} = f^{-1}(y^n)\ \forall\ y \geq 0$, "nth root of y".

Definition 3.42. For $x > 0$ and $r \in \mathbb{Q}$ with $r = \frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{N}$, we define

$$x^r = (x^m)^{\frac{1}{n}}$$

Remark 3.43. Let $r \in \mathbb{Q}$ and define $f(x) = x^r\ \forall\ x \geq 0$. Then $f : [0, \infty) \to \mathbb{R}$ is continuous.

§3.7 Limits

Note 3.44. Note that before $\lim_{n\to\infty} a_n = a$ for sequences but now $\lim_{x\to a} f(x) = L$

Definition 3.45. We say $x_0 \in \mathbb{R}$ is a **limit point** of D if $\exists\ \{x_n\} \in (D - \{x_0\})$ and $\{x_n\} \to x_0$.

Example 3.46

For $(0, 1)$, the numbers 0 and 1 are limit points.

Definition 3.47. Given $f : D \to \mathbb{R}$ and limit point x_0, we write

$$\lim_{x\to x_0} f(x) = l$$

if whenever $\{x_n\} \in (D - \{x_0\})$ with $x_n \to x_0$ has $\lim_{n\to\infty} f(x_n) = l$

Remark 3.48. A function is continuous at x_0 if and only if $\lim_{x\to x_0} f(x) = f(x_0)$

Example 3.49

$\lim_{x\to 2}\sqrt{\frac{3x+3}{x^3-4}}$. Note that there are no denominator issues at $x=2$.

Solution. Note that numerator and denominator are both continuous, and so the quotient continuous as well because the denominator is also not 0. Further note that $\sqrt{x}$ is continuous because it is the inverse of a strictly monotone function (on the domain $[0,\infty)$). Compositions of continuous functions are continuous at $x=2$. So

$$\lim_{x\to 2} = \sqrt{\frac{3x+3}{x^3-4}} = \sqrt{\frac{3(2)+3}{2^3-4}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

□

Example 3.50

$\lim_{x\to 1}\frac{x^2-1}{x-1}$.

Note that we cannot use the quotient property like above. Let $\{x_n\}\to 1$ with $x_n\neq 1$.

$$\frac{x_n^2-1}{x_n-1} = \frac{(x_n-1)(x_n+1)}{x_n-1} = x_n+1$$

So therefore

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{n\to\infty}\frac{x_n^2-1}{x_n-1} = \lim_{n\to\infty} x_n+1 = 1+1 = 2$$

Theorem 3.51

$f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}, x_0 \in \mathbb{R}$ is a limit point. Let $\lim_{x \to x_0} f(x) = A$ and $\lim_{x \to x_0} g(x) = B$ and $c \in \mathbb{R}$. Then

i. $\lim_{x \to x_0} (f(x) \pm g(x)) = A \pm B$

ii. $\lim_{x \to x_0} (f(x)g(x)) = A \cdot B$

iii. $\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{A}{B}, g(x) \neq 0, B \neq 0$

iv. $\lim_{x \to x_0} cf(x) = cA$

These follow directly from properties of sequences. Similarly for compositions. $f : D \to \mathbb{R}, g : U \to \mathbb{R}, x_0$ is a limit point with $\lim_{x \to x_0} f(x) = y_0$ and $\lim_{y \to y_0} g(y) = l$ and $f(D - \{x_0\}) \subset U - \{y_0\}$. Then

$$\lim_{x \to x_0} (g \circ f)(x) = l$$

We will see limits later on in Differentiation.

§4 Differentiation

§4.1 Basic Differentiation Rules

Remark 4.1. High level: to find the tangent line, take a sequence of secant lines closer and closer towards x

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = m_0 \quad \text{(slope)}$$

Definition 4.2. For $x_0 \in \mathbb{R}$, the open interval $I = (a, b)$ that contains x_0 is called a **neighborhood** of x_0.

Definition 4.3. $f : I \to \mathbb{R}$ is said to be **differentiable at x_0** if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} := f'(x_0)$$

exists and we denote it by $f'(x_0)$, the **derivative** of f at x_0.

Remark 4.4. If f is differentiable at every point in I, f is **differentiable** and $f' : I \to \mathbb{R}$ is called the **derivative**.

Example 4.5

$f(x) = mx + b$. Find f'.

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{mx + b - mx_0 - b}{x - x_0} = m$$

Example 4.6

$f(x) = x^2$. Find f'.

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = 2x_0$$

Note 4.7.

$$(x^2 - x_0^2) = (x - x_0)(x + x_0)$$
$$(x^3 - x_0^3) = (x - x_0)(x^2 + xx_0 + x_0^2)$$
$$(x^4 - x_0^4) = (x - x_0)(x^3 + x^2x_0 + xx_0^2 + x_0^3)$$

Notice the pattern. Binomial Expansion. Note that you can prove this general pattern using induction.

Example 4.8

$f(x) = x^n, n \in \mathbb{N}$. Find f'. **Power Rule**.

$$f'(x_0) = \lim_{x \to x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0)(\cdots)}{x - x_0}$$

where $x \neq x_0$.

$$= x_0^{n-1} + x_0^{n-2}x_0 + x_0^{n-3}x_0^2 + \cdots + x_0^{n-1}$$
$$= nx_0^{n-1}$$

Theorem 4.9

If $f : I \to \mathbb{R}$ is differentiable at x_0, f is continuous at x_0.

Proof. Since f is differentiable at x_0:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists and we have $\lim_{x \to \infty} (x - x_0) = 0$. We WTS that $\lim_{x \to \infty} (f(x) - f(x_0)) = 0$. Thus,

$$\lim_{x \to x_0} (f(x) - f(x_0)) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} * (x - x_0) = f'(x_0) * 0 = 0$$

as needed, so f is continuous at x_0. □

Note 4.10. Differentiability implies continuity, but continuity doesn't imply differentiability, and the classical example to show this is $f(x) = |x|$.

Theorem 4.11

If $f : I \to \mathbb{R}, g : I \to \mathbb{R}$, both differentiable at x_0 then

a. $(f \pm g)'(x_0) = f'(x_0) + \pm g'(x_0)$

$$\lim_{x \to x_0} \frac{(f \pm g)(x_0) - (f \pm g)(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \pm \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$
$$= f'(x_0) \pm g'(x_0)$$

b. $(fg')(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$

$$\begin{aligned} \lim_{x \to x_0} \frac{(fg)(x) - (fg)(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x) + f(x_0)g(x) - f(x_0)g(x_0)}{x - x_0} \\ &= \lim_{x \to x_0} g(x)\frac{f(x) - f(x_0)}{x - x_0} + \lim_{x \to x_0} f(x_0)\frac{g(x) - g(x_0)}{x - x_0} \end{aligned}$$

Note that since g is differentiable at x_0, g is continuous at x_0, and so $\lim_{x \to x_0} g(x) = g(x_0)$. Therefore, we get

$$= g(x_0)f'(x_0) + f(x_0)g'(x_0)$$

c. $(\frac{f}{g})'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}$

Before quotient rule, we wil prove $(\frac{1}{g})' = -\frac{g'(x_0)}{(g(x_0))^2}$

$$\lim_{x \to x_0} \frac{(\frac{1}{g})(x) - (\frac{1}{g})(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x - x_0} = \lim_{x \to x_0} \frac{\frac{g(x_0) - g(x)}{g(x_0)g(x)}}{x - x_0}$$

Note that g is differentiable at x_0, so it is continuous at x_0, and so $\lim_{x \to x_0} g(x) = g(x_0)$

$$\lim_{x \to x_0} -\frac{1}{g(x_0)g(x)} \frac{g(x) - g(x_0)}{x - x_0} = -\frac{1}{g(x_0)^2} g'(x_0)$$

Now for the quotient rule, observe that

$$(\frac{f(x_0)}{g(x_0)})' = (\frac{1}{g(x_0)} \cdot f(x_0))'$$

Using above and the product rule, we get

$$-\frac{1}{g(x_0)^2} g'(x_0)f(x_0) + f'(x_0)\frac{1}{g(x_0)} = \frac{f'(x_0)g(x_0) - g'(x_0)f(x_0)}{g(x_0)^2}$$

Note 4.12. Power rule works for negative powers too. We know thart $f(x) = x^n, n \in \mathbb{N}$ s.t. $f'(x) = nx^{n-1}$.
Let $g(x) = x^n = \frac{1}{x^{-n}}, n < 0$. So,

$$(\frac{1}{x^{-n}})' \underset{(*)}{=} -\frac{(x^{-n})'}{(x^{-n})^2} = -\frac{(-nx^{-n-1})}{x^{-n}x^{-n}} = nx^{n-1}$$

§4.2 Differentiating Inverses and Compositions

Example 4.13

$f : [0, \infty) \to \mathbb{R}$ such that $f(x) = x^2$ and therefore $f^{-1}(y) = \sqrt{y}$.
Look at the point $x = 3, y = 9$, $f'(x) = 2x, f'(3) = 6$. Is the derivative of the inverse at $y = 9$ equal to $\frac{1}{6}$. Yes!

$$\lim_{y \to 9} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \to 9} \frac{\sqrt{y} - 3}{y - 9} = \lim_{y \to 9} \frac{1}{\sqrt{y} + 3} = \frac{1}{6}$$

as desired.

Theorem 4.14

Let $f : I \to \mathbb{R}$ be strictly monotone and continuous. Suppose f is differentiable at x_0 and $f'(x_0) \neq 0$. Define $J = f(I)$. Then $f^{-1} : J \to \mathbb{R}$ is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

Proof. Note that J is also a neighborhood of $y_0 = f(x_0)$ by IVT. For $y \in J, y \neq y_0$, define $f^{-1}(y) = x$. Then

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}$$

Note that f^{-1} is differentiable, and so it is continuous, and so

$$\lim_{y \to y_0} f^{-1}(y) = f^{-1}(y_0) = x_0$$

Now applying the limits, we get

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}} = \frac{1}{f'(x_0)}$$

Therefore, f^{-1} is differentiable at y_0 and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

□

Corollary 4.15

For functions in general, suppose $f : I \to \mathbb{R}$ is strictly monotone and differentiable and $f'(x) \neq 0 \; \forall \; x$. Define $J = f(I)$. Then $f^{-1} : J \to \mathbb{R}$ is differentiable and

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} \; \forall \; x \in J$$

Take $x = f(f(^{-1})(x))$ and $f^{-1}(x) \to x_0$.

Lemma 4.16

For $n \in \mathbb{N}, g(x) = x^{1/n}, g : (0, \infty) \to \mathbb{R}$. Claim g is differentiable and $g'(x) = \frac{1}{n}x^{1/n-1}$.

Sketch of a Proof. Suppose $f(x) = x^n$ and so we know the inverse is $g(x) = x^{1/n}$. We know $f'(x) = nx^{x-1}, n \in \mathbb{N}$. From the corollary above,

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{n(x^{1/n})^{n-1}} = \frac{1}{n(x^{1-1/n})} = \frac{1}{n}x^{\frac{1}{n}-1}$$

as desired. □

Theorem 4.17

The **Chain Rule**. Let I be a neighborhood of x_0, $f : I \to \mathbb{R}$ differentiable at x_0, J is an open interval such that $f(I) \subset J$, $g : J \to \mathbb{R}$ is differentiable at $f(x_0)$. Then

$$(g \circ f) : I \to \mathbb{R} \text{ is differentiable at } x_0$$

and

$$(g \circ f)'(x_0) = g'(f(x_0)) \; \cdot f'(x_0)$$

Proof. Proof omitted in class. □

Example 4.18

Let $r = \frac{m}{n}, m \in \mathbb{Z}, n \in \mathbb{N}$ and define $f(x) = x^m$ and $g(x) = x^{1/n}$ and so $f'(x) = mx^{m-1}$ and $g'(x) = \frac{1}{n}x^{1/n-1}$. Let $h(x) = f(g(x))$. Then using the Chain Rule,

$$h'(x) = g'(f(x))f'(x) = \frac{1}{n}(x^m)^{1/n-1} \cdot mx^{m-1}$$
$$= \frac{m}{n}x^{\frac{m}{n}-m+m-1} = \frac{m}{n}x^{m/n-1} = rx^{r-1}$$

as needed.

§4.3 Rolle's Theorem and Mean Value Theorem

Definition 4.19. $x_0 \in D$ of $f : D \to \mathbb{R}$ is said to be a **local max (min)** if $\exists$ a neighborhood I of x_0 for which $f(x_0) \geq f(x)$ $(f(x_0) \leq f(x))$ $\forall\ x \in I \cap D$

Lemma 4.20

Suppose $f : I \to \mathbb{R}$ is differentiable at x_0. If x_0 is either a max or min of f, then $f'(x_0) = 0$.

Proof. Let x_0 be a max WLOG. Then $f(x) \leq f(x_0)$ $\forall\ x$ by definition of max at x. Consider $x < x_0$ in $x \in I$. Then

$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

Note that the numerator is negative and the denominator is negative. Therefore, the entire expression is positive. Now if $x > x_0$, then

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

But in order for the derivative (limit) to exist, then for all sequences, the image sequences must converge to the same value. Therefore,

$$f'(x_0) = \lim_{x \to x_0, x < x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0$$

$$= \lim_{x \to x_0, x > x_0} \frac{f(x) - f(x_0)}{x - x_0} \leq 0$$

$$\implies f'(x_0) = 0$$

in order for the derivative to exist. □

Theorem 4.21

Rolle's Theorem says suppose there is a function $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ is differentiable, and $f(a) = f(b)$, then

$$\exists\ x_0 \in (a, b) \text{ such that } f'(x_0) = 0$$

Proof. Let $f(a) = f(b)$. Since $f : [a, b] \to \mathbb{R}$ is continuity, apply the EVT, so f attains max and min value on $[a, b]$. If both the min/max occur at endpoints, then the function f must be constant, and so $f'(x) = 0$ at every point x in (a, b). Otherwise, the min/max are in $I = (a, b)$ and apply the previous lemma. □

Theorem 4.22

The **Mean Value Theorem (MVT)** states that suppose $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ is differentiable. Then

$$\exists\ x_0 \in (a, b) \text{ such that } f'(x_0) = \frac{f(b) - f(a)}{b - a} \text{ (slope)}$$

Proof. Let $m = \frac{f(b)-f(a)}{b-a}$. For any m, apply Rolle's Theorem to $h : [a, b] \to \mathbb{R}$ defined by $h(x) = f(x) - mx \implies h'(x) = f'(x) - m$. Note that h is continuous on $[a, b]$ since $f(x)$ and $-mx$ are continuous (cont + cont = cont) from chapter 3. Similarly, h is differnetiable on (a, b) since f and $-mx$ are diff (diff + diff = diff) from chapter 4. Now we need to check if $h(a) = h(b)$.

$$h(a) = f(a) - ma = f(a) - (\frac{f(b) - f(a)}{b - a})a = \frac{f(a)(b - a) - f(b)a + f(a)a}{b - a}$$

$$= \frac{f(a)b - f(a)a - f(b)a + f(a)a}{b - a} = \frac{f(a)b - f(b)a}{b - a}$$

Similarly, $h(b) = \frac{f(a)b-f(b)a}{b-a}$, it is the same algebra. Therefore, $h(a) = h(b)$ and so we can apply Rolle's Theorem and so $\exists\ x_0 \in (a, b)$ with $h'(x_0) = 0$. Thus,

$$f'(x_0) - m = 0 \implies f'(x_0) = m = \frac{f(b) - f(a)}{b - a}$$

as desired. □

Example 4.23

Prove that $x^3 + 3x + 1$ has a unique solution.

Solution. Let $f(x) = x^3 + 3x + 1$. Note that f is continuous because it is a polynomial, and f is differentiable. Note that $f(0) = 1 > 0$ and $f(-1) = -3 < 0$. Since $0 \in (-3, 1)$, by the IVT, $\exists x_0 \in (-1, 0)$ such that $f(x) = 0$. Is it unique? Assume not and assume there are 2 solutions such that

$$f(a) = 0 = f(b)$$

By Rolle's Theorem $\exists c \in (a, b)$ such that $f'(c) = 0$ But $f'(x) = 3x^2 + 3$ and so

$$f'(c) = 3c^2 + 3 = 0 \implies 3c^2 = -3 \implies c^2 = -1$$

which is not a real number, so it is a contradiction and therefore there must only be one solution. □

Remark 4.24. The MVT is useful when you have information about a derivative.

Definition 4.25. **Identity Criterion**: a function $f : D \to \mathbb{R}$ is said to be **constant** if $\exists c \in \mathbb{R}$ s.t. $f(x) = c\ \forall\ x \in D$.

Lemma 4.26

Let $f : I \to \mathbb{R}$ be differentiable. Then f is constant if and only if $f'(x) = 0 \ \forall\ x \in D$.

Proof.
$\implies$ Let f be constant such that $f(x) = c, c \in \mathbb{R}, \forall\ x \in I$ by definition. Then $f'(x) = 0 \ \forall\ x \in I$ by derivatve rules. Done.
$\impliedby$. Let $f'(x) = 0 \ \forall\ x \in I$. Choose $x_0 \in I$ and define $c := f(x_0)$. We WTS that $f(x) = c \ \forall\ x \in I$. Let $x \in I$ with $x < x_0$. Recall that differentiability implies continuity, so $f : [x, x_0] \to \mathbb{R}$ is continuous and $f : (x, x_0) \to \mathbb{R}$ is differentiable. By MVT, then $\exists\ z \in (x, x_0)$ with $f'(z) = \frac{f(x_0)-f(x)}{x_0-x}$. But $f'(z) = 0$ since $f'(x) = 0 \ \forall\ x \in I$ by assumption.

$$f'(z) = 0 \implies f(x_0) - f(x) = 0 \implies c = f(x_0) = f(x)$$

$$\implies c = f(x) \ \forall\ x \in I, x < x_0$$

The same argument applies for $(x_0, x]$. Therefore, $f(x) = c \ \forall\ x \in I$. □

Definition 4.27. The Identity Criterion (differ by a constant). Let $g : I \to \mathbb{R}, h : I \to \mathbb{R}$ both be differentiable. Then g, h **differ by a constant** if and only if $g'(x) = f'(x) \forall\ x \in I$.

$$\exists\ c \text{ s.t. } g(x) = h(x) + c$$

Proof. Define $f(x) = g(x) - h(x), f : I \to \mathbb{R}$ and $f'(x) = g'(x) - g'(x)$. Using the previous lemma,

$$f \text{ constant } \iff f'(x) = 0$$

$$f(x) = c \ \forall\ x \in I \iff g'(x) - h'(x) = 0$$

$$\iff g(x) - h(x) = c \text{ by def'n of f}$$

$$\iff g(x) = h(x) + c$$

Note that this gives us antiderivatives. □

Definition 4.28. The Criterion for Strict Monotonicity. Let $f : I \to \mathbb{R}$ be differentiable. Suppose $f'(x) > 0 \forall\ x \in I$. Then $f : I \to \mathbb{R}$ is **strictly increasing**.

Proof. Let $u < v$ with $u, v \in I$. We WTS that $f(u) < f(v)$. Suppose $f'(x) > 0$. Apply the MVT to $f : [u, v] \to \mathbb{R}$ and choose $x_0 \in (u, v)$ at which

$$f'(x_0) = \frac{f(v) - f(u)}{v - u} > 0$$

since $f'(x) > 0 \ \forall\ x$. Since $u < v \implies v - u > 0$, and so $f(v) - f(u) > 0 \implies f(u) < f(v)$ as needed. Similar process can be shown for $f'(x) < 0$ for f strictly decreasing. □

Remark 4.29. Knowing $f'(x_0) = 0$ does not guarantee a local min/max. Consider $f = x^3$ at $x = 0$, $f'(0) = 0$ but it is not a local min/max.

Theorem 4.30

Suppose $f : I \to \mathbb{R}$ has 2 derivatives f', f'' and $f'(x_0) = 0$. Then if

i. $f''(x_0) > 0 \implies$ concave up, so x_0 is a local min of f

ii. $f''(x_0) < 0 \implies$ concave down, so x_0 is a local max of f

§4.4 Cauchy Mean Value Theorem

Theorem 4.31

Cauchy Mean Value Theorem (CMVT). Let $f : [a, b] \to \mathbb{R}, g : [a, b] \to \mathbb{R}$ continuous. Let $f : (a, b) \to \mathbb{R}, g : (a, b) \to \mathbb{R}$ be differentiable. $g'(x) \neq 0 \ \forall \ x \in (a, b)$. Then

$$\exists \ x_0 \in (a, b) \text{ s.t. } \frac{f'(x_0)}{g'(x_0)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof. Let $m := \frac{f(b)-f(a)}{g(b)-g(a)}$. Define $h : [a, b] \to \mathbb{R}$ as $h(x) = f(x) - mg(x)$. Note that h is continuous on $[a, b]$ and differentiable on (a, b) because f and g are. Let us check that $h(a) = h(b)$.

$$h(a) = f(a) - mg(a) = f(a) - (\frac{f(b) - f(a)}{g(b) - g(a)})g(a)$$

$$= \frac{f(a)g(b) - f(a)g(a) - f(b)g(a) + f(a)g(a)}{g(b) - g(a)}$$

$$= \frac{f(a)g(b) - f(b)g(a)}{g(b) - g(a) = h(b)}$$

where you can do similar algebra for $g(b)$ to check. Therefore, by Rolle's Theorem, $\exists \ x_0 \in (a, b)$ wtih $h'(x_0) = 0$ but

$$h'(x_0) = f'(x_0) - mg'(x_0) = 0$$

$$f'(x_0) = mg'(x_0)$$

$$\frac{f'(x_0)}{g'(x_0)} = m = \frac{f(b) - f(a)}{g(b) - g(a)}$$

This is useful for the approximation of f using polynomials. Taylor series. We will see these after integrals. □

Theorem 4.32

This is an application of CMVT: Let $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ have n derivatives. Suppose at some $x_0 \in I$,

$$f(x_0) = f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$$

Then $\forall\, x \in I$ with $x \neq x_0$ $\exists\, z \in (x, x_0) \cup (x_0, x)$ such that

$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n$$

Proof. Let $n \in \mathbb{N}$ and $f : I \to \mathbb{R}$ have n derivatives. Suppose at some $x_0 \in I$

$$f(x_0) = f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$$

Let $g(x) = (x - x_0)^n$. Note that

$$g'(x) = n(x - x_0)^{n-1}$$

$$\vdots$$

$$g^{(n)}(x) = n(n-1)(n-2)\cdots 2 * 1 = n!$$

Using the CMVT for f, g on $[x, x_0]$ (or $[x_0, x]$) since f is differentiable and therefore continuous, and $g(x) = (x - x_0)^n$ is polynomial so differentiable and continuous,

$$\exists\, c_1 \in (x, x_0) \text{ s.t. } \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(c_1)}{g'(c_1)}$$

However, note that $f(x_0) = 0$ bu assumption and $g(x_0) = (x_0 - x_0)^n = 0$ by definition of g. So the above becomes

$$\frac{f(x)}{g(x)} = \frac{f'(c_1)}{g'(c_1)}$$

Repeating the process

$$\frac{f(x)}{g(x)} = \frac{f'(c_1)}{g'(c_1)} = \frac{f'(c_1) - f'(x_0)}{g'(c_1) - g'(x_0)} = \frac{f''(c_2)}{g''(c_2)}$$

for some $c_2 \in [c_1, x_0]$ and then continue iterating such that

$$\frac{f(x)}{g(x)} = \frac{f^{(n)}(c_n)}{n!} \implies f(x) = \frac{f^{(n)}(c_n)}{n!} g(x)$$

$\square$

§5 Differential Equations

Skip this section.

§6 Integration

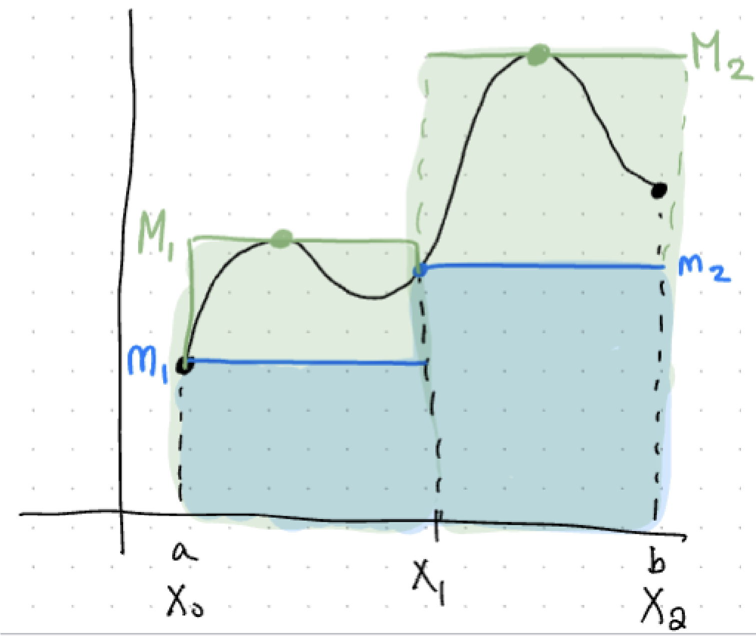
§6.1 Darboux Sums and Refinement Lemma

Remark 6.1. Unless stated otherwise, in this chapter, $I = [a, b]$ and $f : [a, b] \to \mathbb{R}$ is bounded.

Definition 6.2. Let $a < b, a, b \in \mathbb{R}$ and

$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$

Then $P = \{x_1, x_1, x_2, \ldots, x_n\}$ is called a **partition** on $[a, b]$. For all $i \geq 0$, x_i is called a **partition point** and $[x_{i-1}, x_i]$ is a **partition interval**.



Definition 6.3. Let

$$m_1 := \inf\{f(x) \mid x \in [x_{i-1}, x]\}$$

$$m_2 := \sup\{f(x) \mid x \in [x_{i-1}, x]\}$$

We define **Darboux Lower/Upper Sums** of f on P as

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \text{ blue above}$$

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) \text{ green above}$$

Note that these are just the sums of areas of rectangles.

Note 6.4. Note that $m_i \leq M_i$ by definition of inf $\leq$ sup. Therefore, $L(f, P) \leq U(f, P)$ $\forall$ paritions of $[a, b]$. The goal is to obtain

$$L(f, P) \leq \int_a^b f \leq U(f, P)$$

Note 6.5. Given $P = \{x_0, \ldots, x_n\}$ on $[a, b]$, the length of $[a, b]$ is the sum of all of the lengths of partition intervals

$$b - a = \sum_{i=1}^{n}(x_i - x_{i-1})$$

Definition 6.6. Given a partition of P of $[a, b]$, another partition P^* of $[a, b]$ is called a **refinement** of P if each partition point of P is also a partition point of P^*. $P \subseteq P^*$.

Lemma 6.7

The Refinement Lemma states that given parition P, if P^* is a refinement of P, then

$$L(f, P) \leq L(f, P^*) \text{ and } U(f, P^*) \leq U(f, P)$$

Proof. Let $P = \{x_0, \ldots, x_n\}$. Assume P^* is a refinent with exactly one additional point compared to P and label it z. Note that you can iterate this process for more additional points.

$$P^* = \{x_0, \ldots, x_{k-1}, z, z_k, \ldots, x_n\}, P^* = P \cup \{z\}$$

Let $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$ and $L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$ by definition. Observe that

$$L(f, P^*) = \sum_{i=1}^{k-1} m_i(x_i - x_{i-1}) + A(z - x_{k-1}) + B(x_k - z) + \sum_{i=k+1}^{n} m_i(x_i - x_{i-1})$$

where $A = \inf\{f(x) \mid x \in [x_{k-1}, z]\}$ and $B = \inf\{f(x) \mid x \in [z, x_k]\}$. Then

$$L(f, P^*) - L(f, P) = A(z - x_{k-1}) + B(x_k - z) - m_k(x_k - x_{k-1})$$

and we want to show that this is ≥ 0. Note that if $x \in [x_{k-1}, z]$ or $x \in [z, x_k]$ then $x \in [x_{k-1}, x_k] \implies f(x)m_k$ by definition of inf. Therefore, m_k is a lower bound for $\{f(x) \mid x \in [x_{k-1}, z]\}$. Therefore, $m_k \leq A$ and $m_k \leq B$.

$$L(f, P^*) - L(f, P) = A(z - x_{k-1}) + B(x_k - z) - m_k(x_k - x_{k-1})$$

$$\geq m_k(z - x_{k-1}) + m_k(x_k - z) - m_k(x_k - x_{k-1}) = 0$$
$$L(f, P^*) - L(f, P) \geq 0 \implies L(f, P^*) \geq L(f, P)$$

and similarly for Darboux Upper Sums. $\square$

Corollary 6.8

Let P, Q be parit
itions of $[a, b]$ then $L(f, P) \leq U(f, Q)$.

Proof. Consider $P \cup Q$ refinement. Then use the refinement lemma which gives us

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q)$$

Therefore, $L(f, P) \leq U(f, Q)$ as desired. □

Definition 6.9. For $f : [a, b] \to \mathbb{R}$ such that f is bounded, then the **Lower Integral** is defined as

$$\int_{\underline{a}}^{b} f := \sup\{L(f, P) \mid \text{ P partition on } [a, b]\}$$

and the **Upper Intergral** is defined as

$$\int_{a}^{\bar{b}} f := \inf\{U(f, P) \mid \text{P parition on } [a, b]\}$$

Lemma 6.10

Note that $\int_{\underline{a}}^{b} f \leq \int_{a}^{\bar{b}} f$ using lower/upper sum properties.

Example 6.11

Let $f : [a, b] \to \mathbb{R}$ such that $f(x) = c$. Note that by geometry, the area is $c(b-a)$. Both upper and lower integrals $= c(b-a)$. Since by definition $m_i = c$, $M_i = c \; \forall \; i$. So, by the sums formula,

$$c(b-a) = c\sum_{i=1}^{n}(x_i - x_{i-1}) = \sum_{i=1}^{n} c(x_i - x_{i-1}) = L(f, P) = U(f, P)$$

So

$$\int_{\bar{a}}^{b} f = c(b-a) = \int_{a}^{\bar{b}} f$$

Example 6.12

Let

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{Q}^c \end{cases}$$

Note that this is Dirichlet's function. Let $P = \{x_0, \dots, x_n\}$. Since $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in each $[x_{i-1}, x_i]$, then $\exists$ rational and irrational number in each $[x_{i-1}, x_i]$ such that $m_i = 0, M_i = 1 \ \forall \ i$. Therefore,

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) = 0$$

$$U(f,P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = \sum_{i=1}^{n} 1(x_i - x_{i-1})$$

$$= (x_i - x_0) - (x_2 - x_1) \cdots + (x_n - x_{n-1}) = x_n - x_0 = b - a$$

Therefore,

$$\int_a^{\bar{b}} f = \inf\{b-a\} = b - a \text{ and } \int_{\underline{a}}^{b} f = \sup\{0\} = 0$$

§6.2 Integrable and Archimedes-Riemann Theorem

Definition 6.13. Let $f : [a,b] \to \mathbb{R}$ be bounded. Then we say f is **integrable** on $[a,b]$ if $\int_{\underline{a}}^{b} f = \int_a^{\bar{b}} f$. We denote the **integral** of f

$$\int_{\underline{a}}^{b} f = \int_a^{\bar{b}} f = \int_a^b f$$

Recall that

$$L(f,P) \le \int_{\underline{a}}^{b} f \le \int_a^{\bar{b}} f \le U(f,P)$$

by definition of sup, properties of Darboux sums, and then by definition of inf. As a consequence, if we rearrange the above, then

$$0 \le \int_a^{\bar{b}} f - \int_{\underline{a}}^{b} f \le U(f,P) - L(f,P)$$

$$0 \le U(f,P) - \int_a^{\bar{b}} f \le U(f,P) - L(f,P)$$

$$0 \le \int_{\underline{a}}^{b} f - L(f,P) \le U(f,P) - L(f,P)$$

Theorem 6.14

The **Archimedes-Riemann Theorem** states that suppose $f : [a, b] \to \mathbb{R}$ is bounded. Then f is **integrable** on $[a, b]$ if and only if $\exists$ sequence $\{P_n\}$ of partition on $[a, b]$ such that

$$\lim_{n\to\infty} (U(f, P) - L(f, P)) = 0$$

Moreover, for any sequence of partitions

$$\lim_{n\to\infty} L(f, P_n) = \int_a^b f = \lim_{n\to\infty} U(f, P_n)$$

Proof.
$\Longrightarrow$ Suppose f is integrable. Therefore, $\int_{\underline{a}}^b f = \int_a^{\bar{b}} f = \int_a^b f \; \forall \; n \in N$. Note that $(\int_{\underline{a}}^b f) - \frac{1}{n}$ is not an upper bound for $L(f, P)$ since $\int_{\underline{a}}^b f = \sup\{L(f, P) \mid \text{P partition on [a,b]}\}$ Therefore, $\exists$ partition Q_n of $[a, b]$ such that

$$(\int_{\underline{a}}^b f) - \frac{1}{n} < L(f, Q_n)$$

Similarly, $\exists$ partition R_n of $[a, b]$ such that

$$(\int_a^{\bar{b}} f) + \frac{1}{n} > U(f, R_n)$$

Let $P_n = Q_n \cup R_n$ be a refinement. Then

$$L(f, P_n) \geq L(f, Q_n) > \int_{\underline{a}}^b f - \frac{1}{n}$$

by the Refinement Lemma and then by our definition of Q_n. Similarly, Equivalently, if we multiply by -1, then see that $-L(f, P_n) \leq -(\int_a^{\bar{b}} f - \frac{1}{n})$

$$U(f, P_n) \leq U(f, R_n) < \int_a^{\bar{b}} f + \frac{1}{n}$$

But, note that f is integrable by assumption, which means $\int_{\underline{a}}^b f = \int_a^{\bar{b}} f = \int_a^b f$. So,

$$0 \leq U(f, P_n) - L(f, P_n) \leq \int_a^b f + \frac{1}{n} - (\int_a^b f - \frac{1}{n})$$
$$= \frac{2}{n} \to 0 \text{ as } n \to \infty$$

By The Comparison Lemma, $\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0$. $\square$

Theorem 6.15

The **Archimedes-Riemann Theorem** left direction proof.

Proof.
$\Longleftarrow$. Suppose $\lim_{n\to\infty}(U(f,P_n)-U(f,P_n))=0$ for some sequence of partitions P_n. We want to show that f is integrable, which means showing $\underline{\int_a^b} f = \overline{\int_a^b} f$. But recall that

$$0 \le \overline{\int_a^b} f - \underline{\int_a^b} f \le U(f,P_n) - L(f,P_n)$$

by the first consequence in the definition of integrable. By the Comparison Lemma,

$$\overline{\int_a^b} f - \underline{\int_a^b} f = 0 \implies \overline{\int_a^b} f = \underline{\int_a^b} f \implies \text{f is integrable}$$

Moreover,

$$0 \le U(f,P_n) - \overline{\int_a^b} f \le U(f,P_n) - L(f,P_n) \text{ by (2)}$$

$$\implies \lim_{n\to\infty} U(f,P_n) = \overline{\int_a^b} f = \int_a^b f$$

Similarly, using (3),

$$\lim_{n\to\infty} L(f,P_n) = \underline{\int_a^b} f = \int_a^b$$

$\square$

Definition 6.16. $\{P_n\}$ is said to be an **Archimedian sequence of partitions (Asop)** for f on $[a,b]$ if

$$\lim_{n\to\infty}(U(f,P_n)-L(f,P_n))=0$$

Example 6.17

Prove $f(x) = x$ is integrable over $[0, 1]$.

Proof. Let P_n be the nth regular partition. P_n is regular is $x_i = a+i\frac{b-a}{n} = a+i\Delta x$. (This just means that each subinterval is the same length).

$$P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$$

Note that $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\} \implies m_i = \inf\{f(x) \mid x \in [\frac{i-1}{n}, \frac{i}{n}]\} = f(\frac{i-1}{n}) = \frac{i-1}{n}$, which is just the left endpoint of the subinterval. Simiarly, $M_i = \frac{1}{n}$, the right endpoint of the interval. Therefore, by definition of Darboux Upper and Lower Sums,

$$U(f, P_n) - L(f, P_n) = \sum_{i=1}^{n} M_i(x_{i-1} - x_i) - \sum_{i=1}^{n} m_i(x_{i-1}, x_i)$$

$$= \sum_{i=1}^{n} (\frac{i}{n} * \frac{1}{n} - (\frac{i-1}{n} * \frac{1}{n})) = \sum_{i=1}^{n} \frac{1}{n^2} = \frac{1}{n}$$

and it is known that $\lim_{n\to\infty} \frac{1}{n} = 0$. Therefore,

$$\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0$$

Thus, P_n is an Asop and using the Archimedian Theorem, $f(x) = x$ is integrable. We can find the integral

$$\int_0^1 f(x)dx = \int_0^1 x \, dx$$

Using the "Moreover" part of the AR Theorem,

$$\int_0^1 xdx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{i}{n} * \frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n^2} \sum_{i=1}^{n} i = \lim_{n\to\infty} \frac{1}{n^2} (\frac{n^2+n}{2}) = \frac{1}{2}$$

□

Theorem 6.18

Every monotone function $f : [a, b] \to \mathbb{R}$ is integrable.

Proof. WLOG, suppose f is monotone increasing. Let P_n be a regular partition. Since f is monotone increasing,

$$m_i = f(x_{i-1}) \leq f(x) \leq f(x_i) = M_i, x \in [x_{i-1}, x_i]$$

Then,

$$\begin{aligned} U(f, P_n) - L(f, P_n) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\ &= \frac{b-a}{n} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) \end{aligned}$$

Note that this is a telescope and so

$$= \frac{b-a}{n}(f(x_n) - f(x_0)) = \frac{b-a}{n}(f(b) - f(a))$$

Let $c = (b-a)(f(b) - f(a)) \in \mathbb{R}$. Then

$$\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = \lim_{n\to\infty} \frac{1}{n}(c) = 0$$

Thus, P_n is Asop and by the AR Theorem, f is integrable. □

Theorem 6.19

Every step function $f : [a, b] \to \mathbb{R}$ is integrable.

Sketch. For the partition points inside one region, $M_i = m_i$. For the subintervals at the gaps/jumps, there are finitely many and they can still be bounded using M_i, m_i. □

§6.3 Additivity, Monotonicity, Linearity

§6.3.1 Additivity

Theorem 6.20

Suppose $f : [a, b] \to \mathbb{R}$ is integrable and $c \in (a, b)$. Then f is integrable on $[a, c]$ and $[c, b]$ and $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof. Since f is integrable on $[a, b]$ by the AR Theorem, there exists an Asop $\{P_n\}$ on $[a, b]$ for f. Let $Q_n = P_n \cup \{c\}$. We WTS that Q_n is Asop. By the Refinement Lemma,

$$L(f, P_n) \le L(f, Q_n) \implies -L(f, P_n) \ge -L(f, Q_n)$$

$$U(f, P_n) \ge U(f, Q_n)$$

Therefore,

$$0 \le U(f, Q_n) - L(f, Q_n) \le U(f, P_n) - L(f, P_n)$$

Since P_n is Asop, then $\lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0$ and so by Comparison Lemma, $\lim_{n\to\infty} (U(f, Q_n) - L(f, Q_n)) = 0$ and so Q_n is an Asop for f on $[a, b]$. Observe that

$$Q_n = \{x_0, \ldots, x_k, c, x_{k+1}, \ldots, x_n\}$$

Let $R_n = Q_n \cap [a, c]$ and let $S_n = Q_n \cap [c, b]$. Then,

$$U(f, R_n) = \sum_{i=1}^{k} M_i(x_i - x_{i-1}) + A(c - x_k), \; A = \sup\{f(x) \mid x \in [x_k, c]\}$$

$$U(f, S_n) = B(x_{k+1} - c) + \sum_{i=k+1}^{n} M_i(x_i - x_{i-1}), B = \sup\{f(x) \mid x \in [c, x_{k+1}]\}$$

And so $U(f, Q_n) = U(f, R_n) + U(f, S_n)$ and similarly $L(f, Q_n) = L(f, R_n) + L(f, S_n)$.

$$0 \le U(f, R_n) - L(f, R_n) \le U(f, Q_n) - L(f, Q_n)$$

By CL, $\lim_{n\to\infty}(U(f, R_n) - L(f, R_n)) = 0$ and so f is integrable on $[a, b]$ and similarly for $[c, b]$. and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

□

§6.3.2 Monotonicity

Proof.

Theorem 6.21

If $f, g : [a, b] \to \mathbb{R}$ are bounded with $f(x) \leq g(x) \; \forall \; x \in [a, b]$, then

$$(1) \quad \underline{\int_a^b} f \leq \underline{\int_a^b} g \text{ and } \overline{\int_a^b} f \leq \overline{\int_a^b} g$$

and if f, g are integrable, then

$$(2) \quad \int_a^b f \leq \int_a^b g$$

Sketch. Sine f, g are integrable, use AR Theorem and Refinement Lemma such that $\exists \; \{P_n\}$ on $[a, b]$ such that

$$\lim_{n \to \infty} U(f, P_n) = \int_a^b f \qquad \lim_{n \to \infty} U(g, P_n) = \int_a^b g$$

Since $f(x) \leq g(x) \; \forall \; x \in [a, b]$ then $U(f, P_n) \leq U(f, P_n)$ by using sup. So

$$\int_a^b f = \lim_{n \to \infty} U(f, P_n) \leq \lim_{n \to \infty} U(g, P_n) = \int_a^b g \implies \int_a^b f \leq \int_a^b g$$

as desired. □

□

§6.3.3 Linearity

Theorem 6.22

Suppose $f, g : [a, b] \to \mathbb{R}$ are both integrable. Let $\alpha, \beta \in \mathbb{R}$. Then $(\alpha f + \beta g) : [a, b] \to \mathbb{R}$ is integrable. and

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$$

Proof.
Let us first show that scalar multiplication is satisfied. Let $P = \{x_0, \dots, x_n\}$ be a partition on $[a, b]$ and define $M_i(f), M_i(g), m_i(f), m_i(g)$ as usual.
If $\alpha \geq 0, \alpha \in \mathbb{R}$

$$M_i(\alpha f) = \alpha M_i(f) \qquad m_i(\alpha f) = \alpha m_i(f)$$

by exercising the definition of sup, inf.
If $\alpha < 0, \alpha \in \mathbb{R}$

$$M_i(\alpha f) = \alpha m_i(f) \qquad m_i(\alpha f) = \alpha M_i(f)$$

Let $\alpha \geq 0$. Then

$$\begin{aligned} U(\alpha f, P) - L(\alpha f, P) &= \sum_{i=1}^{n} (M_i(\alpha f) - m_i(\alpha f))(x_i - x_{i-1}) \\ &= \alpha \sum_{i=1}^{n} (M_i(f) - m_i(f))(x_i - x_{i-1}) \\ &= \alpha(U(f, P) - L(f, P)) \end{aligned}$$

Suppose P_n is an ASOP for f on $[a, b]$, since f is integrable take the limit.

$$\lim_{n\to\infty} (U(\alpha f, P_n) - L(\alpha f, P_n)) = \alpha \lim_{n\to\infty} (U(f, P_n) - L(f, P_n)) = 0$$

and so therefore, P_n is ASOP for αf and αf is integrable.
If $\alpha < 0$ then it's similar.

□

Theorem 6.23

Now we will show the second half of the proof, the additivity part.

Proof.
Now for the second part, we WTS that $\int_a^b (f+g) = \int_a^b + \int_a^b g$. Consider

$$(f+g)(x_i) = f(x_i) + g(x_i) \le M_i(f) + M_i(g) \;\forall\; x \in [x_{i-1}, x_i]$$

Starting with

$$M_i(f+g) \le M_i(f) + M_i(g)$$

Multply by $(x_i - x_{i-1})$ and add all of the subintervals and similarly for Darboux Lower Sums to obtain

$$U(f+g,P) \le U(f,P) + U(g,P) \qquad L(f+g,P) \ge L(f,P) + L(g,P)$$

Take $\{P_n\}$ ASOP for f on $[a,b]$ since f, g are integrable by the AR Theorem. Similarly take $\{R_n\}$ ASOP for g on $[a,b]$. Let $Q_n = P_n \cup R_n$ (refinement). Now we want to show that Q_n is an ASOP for both f and g individually. Then

$$0 \le U(f+g,Q_n) - L(f+g,Q_n)$$

$$\begin{aligned} &\le U(f,Q_n) + U(g,Q_n) - (L(f,Q_n) - L(g,Q_n)) \\ &= U(f,Q_n) - L(f,Q_n) + U(g,Q_n) - L(g,Q_n) \end{aligned}$$

By CL, $\lim\limits_{n\to\infty} (U(f+g,Q_n) - L(f+g,Q_n)) = 0 \implies f+g$ is integrable and so

$$\int_a^b f+g = \lim_{n\to\infty} U(f+g,Q_n) \le \lim_{n\to\infty} (U(f,Q_n) + U(g,Q_n)) = \int_a^b f + \int_a^b g$$

But at the same time

$$\int_a^b f+g = \lim_{n\to\infty} L(f+g,Q_n) \ge\le \lim_{n\to\infty} (L(f,Q_n) + L(g,Q_n)) = \int_a^b f + \int_a^b g$$

which implies that $\int_a^b f+g = \int_a^b f + \int_a^b g$ and combining the result from the first part of the proof we obtain $\int_a^b \alpha f + \beta g = \alpha \int_a^b f + \beta \int_a^b g$, as desired. $\square$

Corollary 6.24

Suppose $f : [a, b] \to \mathbb{R}, |f| : [a, b] \to \mathbb{R}$ are integrable. Then

$$|\int_a^b f| \leq \int_a^b |f|$$

Sketch. $\forall\, x \in [a, b]$

$$-|f(x)| \leq f(x) \leq |f(x)|$$

Applying montonicity and linearity,

$$-\int_a^b |f(x)| \leq \int_a^b f(x) \leq \int_a^b |f(x)|$$

which is $|\int_a^b f| \leq \int_a^b |f|$ $\square$

§6.4 Continuity and Integrability

So far we have only proven the following functions are integrable: x, x^2 and monotone functions.

Lemma 6.25

Let $f : [a, b] \to \mathbb{R}$ be continuous and P partition on $[a, b]$. Then $\exists$ a partition interval of P that contains two points u, v with

$$0 \leq U(f, P) - L(f, P) \leq (f(u) - f(v))(b - a)$$

Sketch.
Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$.

- f is continuous on $[a, b]$ so f is continuous on all $[x_{i-1}, x_i]$ $\forall$ i
- By EVT, max and min are attained (stronger than inf and sup)
- Define the min and the max, $f(u_i) = m_i, f(v_i) = M_i$
- Choose i_0 such that $M_{i_0} - m_{i_0} := \max\limits_{1 \leq i \leq n}(M_i - m_i)$. Intuitively, this means to choose the largest difference betwen subinterval max - min.
- Let $u := u_{i_0}$, $v := v_{i_0}$.
- Then $M_i - m_i \leq M_{i_0} - m_{i_0} = f(v) - f(u)$ by definition of max.
- Then $0 \leq U(f, P) - L(f, P) = \sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1})$

$$\leq \sum_{i=1}^{n}(f(v) - f(u))(x_i - x_{i-1}) = (f(v) - f(u))(b - a)$$

So we have found $u, v \in [a, b]$ with

$$0 \leq U - L \leq (f(v) - f(u))(b - a)$$

$\square$

Remark 6.26. Recall that continuity on $[a, b]$ implies uniform continuity. For any $\{u_n\}$ and $\{v_n\}$ in D if $\lim\limits_{n\to\infty}(u_n - v_n) = 0 \implies \lim\limits_{n\to\infty}(f(u_n) - f(v_n)) = 0$. Use the Lemma above for a sequence of partitions and sequence of points $\{u_n\}$ and $\{v_n\}$ to prove that if f is continuous on $[a, b]$ then f is integrable.

Theorem 6.27

A continuous function $f : [a, b] \to \mathbb{R}$ is integrable.

Proof.
Let $\{P_n\}$ be a sequence of regular partitions. For each n, apply the previous lemma:

$$(*) \quad 0 \leq U(f, P_n) - L(f, P_n) \leq (f(v_n) - f(u_n))(b - a)$$

Note that $|u_n - v_n| \leq \frac{b-a}{n}$ since $u_n, v_n \in [x_{i-1}, x_i]$. Take the limit as $n \to \infty$, by CL $\lim\limits_{n\to\infty}(u_n - v_n) = 0$ since $\lim\limits_{n\to\infty} \frac{b-a}{n} = 0$.
Since f is continuous on $[a, b]$, f is uniformly continuous on $[a, b]$

$$\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0 \quad (1)$$

Take $(*)$ and apply a limit

$$0 \leq \lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) \leq \lim_{n\to\infty}(f(v_n) - f(u_n))(b - a)$$

By CL, $\lim\limits_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0$ and so by AR Theorem, f is integrable. □

Theorem 6.28

Suppose $f : [a, b] \to \mathbb{R}$ is bounded and continuous on (a, b). Then $f : [a, b] \to \mathbb{R}$ is integrable and the value of the integral $\int_a^b f$ does not depend on the values of f at the endpoints.

Sketch. Look at the endpoints a, b and take $a_n \to a, b_n \to b$ and measure how big the finite (because bounded) gap of the discontinuity is. Bound the difference of the gap and take $n \to \infty$ for P_n.

$$U(f, P_n) - L(f, P_n) \leq \text{usual } + \text{bound}(a_n - a) + \text{bound}(b_n - b)$$

Note that f does not need continuity at the endponts to be integrable. □

Example 6.29

Consider

$$f(x) = \begin{cases} \sin(\frac{1}{x}) \text{ if } 0 < x \leq 1 \\ 100 \text{ if } x = 0 \end{cases}$$

$f(x) : [0, 1] \to \mathbb{R}$ is bounded since $|f(x)| \leq 100$ and $f : (0, 1) \to \mathbb{R}$ is continuous. So f is integrable by the previous theorem.

§6.5 The First Fundamental Theorem of Calculus (FTC1)

Theorem 6.30

Let $F : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differnetiable on (a,b) and $F' : (a,b) \to \mathbb{R}$ is continuous and bounded, then

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Proof. Assume $F' : (a,b) \to \mathbb{R}$ is continuous and bounded, then $F' : [a,b] \to \mathbb{R}$ is integrable (above, 6.4) since the integral does not depend on endpoints. Let P be a partition of $[a,b]$. Then

$$U(F', P) = \sum_{i=1}^{n} M_i(F')(x_i - x_{i-1})$$

Consider $[x_i, x_{i-1}]$, since F is continuous on $[a,b]$, differentiable on (a,b), we can apply MVT to the partition interval, $\exists\ c_i \in [x_i, x_{i-1}]$, with

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}\ \forall\ i, 1 \leq i \leq n$$

Then use sup to get

$$\frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} = F'(c_i) \leq M_i(F')$$

by the definition of sup. Then multiply by $(x_i - x_{i-1})$ to get

$$\sum_{i=1}^{n}(F(x_i) - F(x_{i-1})) \leq \sum_{i=1}^{n} M_i(F')(x_i - x_{i-1}) = U(F', P)$$

Note that the left side is telescoping, and so we obtain that

$$F(b) - F(a) \leq U(F', P)$$

By property of inf in Upper Integral

$$F(b) = F(a) \leq \int_a^{\bar{b}} F' = \int_a^b F'$$

because F' is integral. Simiarly, it can be shown that $F(b) - F(a) \geq \int_a^b F'$ which implies that $F(b) - F(a) = \int_a^b F'$. $\square$

Note 6.31. Notation: $f : [a,b] \to \mathbb{R}$ is continous and bounded on (a,b), then FTC1 asserts if it possible to find an antiderivative $F : [a,b] \to \mathbb{R}$ for f then the integral is given by

$$\int_a^b f(x)dx = F(b) - F(a)$$

An antiderivatve continuous function F having a derivative on (a, b) such that

$$F'(x) = f(x) \ \forall \ (a, b)$$

Note that FTC1 only gives us some (most) integrals. In order to

§6.6 FTC2 and Differentiating Integrals

Theorem 6.32

The **Mean Value Theorem for Integrals** states that suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then $\exists \ x_0 \in (a, b)$ at which

$$f(x_0) = \frac{1}{b-a} \int_a^b f(x)dx$$

Proof. Use EVT on $f : [a, b] \to \mathbb{R}$ (cont). Therefore, $\exists \ x_m, x_M \in [a, b]$ with

$$f(x_m) \leq f(x) \leq f(x_M) \ \forall \ x \in [a, b]$$

By monotonicity of $\int$:

$$\int_a^b f(x_m) \leq \int_a^b f(x) \leq \int_a^b f(x_M)$$

$$f(x_m) \int_a^b 1 \leq \int_a^b f(x) \leq f(x_M) \int_a^b 1$$

$$f(x_m)(b-a) \leq \int_a^b f(x) \leq f(x_M)(b-a)$$

$$f(x_m) \leq \frac{1}{b-a} \int_a^b f(x) \leq f(x_M)$$

From here, apply the IVT, $\exists \ x_0 \in (x_m, X_M)$ such that

$$\exists x_0 \in (a, b) \text{ s.t. } f(x_0) = \frac{1}{b-a} \int_a^b f$$

□

Definition 6.33. Consider the **area function**

$$f(x) = \int_a^x f(t)dt$$

Input changes the upper bound. Lower bound is fixed. Tells you how much area is under $f(t)$ between a and x.

Proposition 6.34

Suppose $f : [a, b] \to \mathbb{R}$ is integrable. (This could be discontinuous). Define $F(x) = \int_a^x f(t)dt \ \forall \ x \in [a, b]$. Then $F : [a, b] \to \mathbb{R}$ is continuous.

Proof.
Let $u, v \in [a, b]$ with $u < v$ WLOG. Let $F(v) = \int_a^v f = \int_a^u f + \int_u^v f = F(u) + \int_u^v f$ by additivty. Therefore,

$$F(v) - F(u) = \int_u^v = f \quad (*)$$

Since f is integrable and bounded, choose $M > 0$,

$$-M \leq f(x) \leq M \ \forall \ x \in [a, b]$$

$$-M \leq f(x) \leq M \ \text{ if } u \leq x \leq v$$

By monotone integral theorem,

$$-\int_u^v M \leq \int_u^v f \leq \int_u^v M$$

$$-M(v - u) \leq \int_u^v f \leq M(v - u)$$

$$|F(v) - F(u)| \leq M|v - u| \ \forall \ u, v \in [a, b]$$

Recall this from Homework 4, if Lipschitz continuous, then it is also uniform continuous. $F : [a, b] \to \mathbb{R}$ is uniformly continuous, which implies continuous, and so we are done. □

Remark 6.35. The above proof is more broad than the FTC2, shown below.

Theorem 6.36

The **Fundamental Theorem of Calculus 2** states that suppose $f : [a, b] \to \mathbb{R}$ is continuous. Then,

$$\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)\ \forall\ x \in (a, b)$$

Proof.
Note that $\int_a^b f = -\int_b^a f$. Define $F(x) = \int_a^x\ \forall\ x \in [a, b]$. F is continuous from the proposition above. Let $x_o \in (a, b)$.
We WTS that

$$F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0}$$

Let $x \in (a, b)$ with $x \neq x_0$.
Case 1: If $x > x_0$. Then

$$F(x) - F(x_0) = \int_{x_0}^x f \implies \int_a^x f - \int_a^{x_0} f = -\int_x^a f - \int_a^{x_0} f$$

$$= -(\int_x^a f + \int_a^{x_0} f) \implies -\int_x^{x_0} f = \int_{x_0}^x f$$

Case 2: If $x < x_0$, then

$$F(x) - F(x_0) = \int_{x_0}^x f = \int_a^x f - \int_a^{x_0} f = -\int_x^a f - \int_a^{x_0} f$$

$$-(\int_x^a f + \int_a^{x_0} f) = -\int_x^{x_0} f = \int_{x_0}^x f$$

In both cases, apply MVT for $\int$

$$\exists\ c \in (a, b) \text{ s.t. } f(c(x)) = \frac{1}{x - x_0}\int_{x_0}^x f$$

Rearranging and using what we have above,

$$= \frac{F(x) - F(x_0)}{x - x_0}$$

Applying the limit as $x \to x_0$,

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = \lim_{x \to x_0} f(c(x)) = F'(x_0)$$

by definition of derivative. Since f is continuous at x_0, then $\lim_{x \to x_0} f(c(x)) = f(x_0)$ as desired. $\square$

Corollary 6.37

Some corollary results from FTC2.

1. $f : [a, b] \to \mathbb{R}$ is continuous then $\frac{d}{dx} \int_a^x f = \frac{d}{dx}(-\int_a^x f) = -f(x)$
2. I, J open intervals, $h(I) \subseteq J$. Then,

$$f : I \to \mathbb{R}\text{cont} \qquad\qquad h : J \to \mathbb{R} \text{ diff}$$

Fix a: then $\frac{d}{dx} \int_a^{h(x)} = f(h(x)) * h'(x)$ using the chain rule

$$\int_{x^2}^{-3} \cos(1 - 5t)dt = F(x) \implies F'(x) = -ex^{2x^2} \cos(1 - 5x^2) * 2x$$

§7 Integration Techniques

Skip this section.

§8 Approximation by Taylor Polynomials

§8.1 Taylor Polynomials

Note 8.1. Note that $(e^x)^1 = e^x$.

Definition 8.2. Let I be a neighborhood of x_0, and so we say $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are said to have **contact of order** 0 **at** x_0 if $f(x_0) = g(x_0)$.

$$\vdots$$

They are said to have **contact of order** n **at** x_0 oif $f^{(k)}(x_0) = g^{(k)}(x_0)\ \forall\ k \in [0, n]$.

Note 8.3. Note that in this chapter, $0 \in \mathbb{N}$ because for the above definition, it means just the functional values.

Theorem 8.4

Let I be a neighborhood of x_0. Suppose $f : I \to \mathbb{R}$ has n derivatives, $n \in \mathbb{N}$. Then there is a unique polynomial of degree at most n that has contact of order n with f at x_0. The polynomial is defined as

$$P_n(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

Proof.
Suppose $P_n(x)$ is a general polynomial.

$$P_n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n$$

$P_n(x)$ has degree n. Now let us work to get contact of order at least n with f at $x = x_0$.

$$n = 0 \implies P_0(x) = a_0 \implies p(x_0) = a_0$$

Since we want contact of order 0 with f at x_0, then

$$f(x_0) = p(x_0) = a_0 \implies f(x_0) = a_0$$

$$\begin{aligned} n = 1 &\implies P_n'(x_0) = a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \cdots + na_n(x - x_0)^{n-1} \\ &\implies P_n'(x_0) = a_1 = f'(x_0) \text{ in order to have contact of order 1} \\ n = 2 &\implies P_n''(x) = 2a_2 + 6a_3(x - x_0) + \cdots + n(n-1)a_n x^{n-2} \end{aligned}$$

Now plugging in $x = x_0$, we get

$$\implies P_n''(x) = 2a_2 = f''(x_0) \implies a_2 = \frac{f''(x_0)}{2!}$$

and we start to see the pattern.

$$a_3 = \frac{f^{(3)}(x_0)}{3!}$$

$$\text{Pattern: } a_k = \frac{f^{(k)}(x_0)}{k!}$$

So putting coefficients a_k into $P_n(x)$:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

□

Note 8.5. Note that this was only created for a single point x_0. The construction says nothing about how the polynomial behaves away from x_0.

Example 8.6

Find the nth Taylor Polynomial for $f(x) = e^x$ at $x = 0$

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Solution. Note that $f^{(k)}(0) = 1 \ \forall \ k$. Thus,

$$P_n(x) = \sum_{i=0}^{n} \frac{1}{k!}(x - 0)^k = \sum_{i=0}^{n} \frac{x^k}{k!}$$

□

§8.2 Lagrange Remainder (Error) Theorem

How close is $P_n(x)$ to $f(x)$? What is the error?

Definition 8.7. Define $R_n := f(x) - P_n(x)$ where $R_n : I \to \mathbb{R}$ is the **nth remainder**

Note 8.8. Recall this Corollary from Cauchy MVT in Chapter 4. Suppose $f : I \to \mathbb{R}$ has n derivatives for $f^{(k)}(x_0) = 0 \ \forall \ k, 0 \leq k \leq n - 1$. Then $\forall \ x \neq x_0$, $\exists z \in (x, x_0)$ with

$$f(x) = \frac{f^{(n)}(z)}{n!}(x - x_0)^n$$

Theorem 8.9

Let I be a neighborhood of x_0, and let $\mathbb{N}$ include 0. Suppose $f : I \to \mathbb{R}$ has $n+1$ derivatives. Then for each point $x \neq x_0$ in I, $\exists\, c \in (x, x_0)$

$$f(x) = \sum_{k=1}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

where the summation is the Taylor polynomial and the second term is the $R_n(x)$ remainder.

Proof. Let $P_n(x)$ be the nth Taylor Polynomial for $f(x)$ at x_0.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

Define $R_n(x) := f(x) - P_n(x)\ \forall\ x \in I$. By construction in 8.1, $f(x)$ and $p_n(x)$ have a contact of order n at x_0 so $R_n^{(k)} = 0\ \forall\ k \in [0, n]$. Thus, we can apply the Corollary from the CMVT on R:

$$\forall\ x \neq x_0,\ \exists\ c \text{ between } x_0, x \text{ w/ } R_n(x) = \frac{R^{(n+1)(c)}}{(n+1)!}(x - x_0)^{n+1}$$

But $R^{(n+1)}(c) = f^{(n+1)}(c) - P_n^{(n+1)}(c)$ by definition of R. Note that the second term is the Polynomial degree n so $(n+1)$th derivative is 0.

$$R^{(n+1)}(c) = f^{(n+1)}(c) - 0$$

By substitution into the corollary from CMVT,

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

where $R_n := f - P_n \implies f = P_n + R_n$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$$

□

§8.3 Convergence of Taylor Polynomials (n to infinity)

Definition 8.10. $\{S_n\}$ is a **sequence of partial sums** when $S_n = \sum_{k=0}^{n} a_k$ for a given a_k.

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} = \lim_{n\to\infty} \sum_{k=0}^{n} a_k \text{ if } \{s_n\} \text{ converges}$$

Note that if $\{S_n\}$ does not converge, then $\sum_{k=0}^{\infty} a_k$ diverges.

Definition 8.11. Let $P_n(x)$ be the nth Taylor Polynomial for $f : I \to \mathbb{R}$ then

$$\lim_{n\to\infty} P_n(x) = f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

is called the **Taylor Series Expansion of f at x_0**

Note 8.12. Note that we can assume that $\lim\limits_{n\to\infty} r^n = 0$ if $|r| < 1$.

Lemma 8.13

$\lim\limits_{n\to\infty} \frac{c^n}{n!} = 0, c \in \mathbb{R}$

Proof. Let $c \in \mathbb{R}$ be a constant and choose $k \in \mathbb{N}$ such that $k \geq 2|c| \implies |c| \leq \frac{k}{2}$, which is doable by AP. Apply absolute value for $n \geq K$,

$$0 \leq |\frac{c^n}{n!}| = \frac{|c||c|\cdots|c|}{(1)(2)\cdots(n-1)(n)}$$

$$= \left[\frac{|c||c|\cdots|c|}{1 * 2 * \cdots * k}\right]\left[\frac{|c|\cdots|c|}{(k+1)\cdots n}\right] = \frac{|c|^k}{k!}\frac{|c|^{n-k}}{(k+1)\cdots n} \leq \frac{|c|^k}{k!}\frac{|c|^{n-k}}{k^{n-k}}$$

$$\leq \frac{|c|^k}{k!}\frac{(k/2)^{n-k}}{k^{n-k}} = \frac{|c|^k}{k!2^{n-k}} \leq |c|^k(\frac{1}{2})^{n-k}$$

$$= |c|^k\frac{(1/2)^n}{(1/2)^k} = |c|^k * 2^k * (\frac{1}{2})^n$$

Note that $|c|^k * 2^k$ is a constant wrt n and $(\frac{1}{2})^n \to \infty, n \to \infty$

$$\lim_{n\to\infty}(\frac{1}{2})^n = 0$$

and so since that term is a constant then by CL, $\lim\limits_{n\to\infty} \frac{c^n}{n!} = 0$. ☐

Theorem 8.14

Let I be a neighborhood of x_0 and $f : I \to \mathbb{R}$ have derivative of all orders. Suppose $r, M \in \mathbb{R}^+$ such that $[x_0 - r, x_0 + r] \subseteq I$ and $\forall\, n$ with $x \in [x_0 - r, x_0 + r]$, $|f^{(n)}(x)| \le M^n$ then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \text{ if } |x - x_0| \le r$$

Proof. Let $P_n(x)$ be the nth Taylor Polynomial of f at x_0. We WTS that

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \leftrightarrow \lim_{n\to\infty} P_n \leftrightarrow \lim_{n\to\infty} (f(x) - P_n(x)) \leftrightarrow \lim_{n\to\infty} R_n(x)$$

$$= 0 \leftrightarrow \lim_{n\to\infty} \frac{f^{(n+1)}(z)}{(n+1)!}(x - x_0)^{n+1}$$

by the Lagrange Remainder Theorem, z is strictly between x and x_0. By assumption, $|f^{(n+1)}| \le M^{n+1}$ and from the given $|x - x_0| \le r$

$$\left| \frac{f^{(n+1)}(z)}{(n+1)!}(x - x_0)^{n+1} \right| \le \left| \frac{M^{n+1} r^{n+1}}{(n+1)!} \right| = \left| \frac{(Mr)^{n+1}}{(n+1)!} \right|$$

Note that $\lim_{n\to\infty} \left| \frac{(Mr)^{n+1}}{(n+1)!} \right| = 0$ and so by CL

$$\lim_{n\to\infty} \frac{f^{(n+1)}(z)}{(n+1)!}(x - x_0)^{n+1} = \lim_{n\to\infty} R_n(x) = 0 \implies$$

the Taylor Series expension of f at x_0 converges to $f(x)$ $\square$

§8.4 Skip (Thanks Eileen)

§8.5 Cauchy Integral Remainder Theorem

Previously, $f(x) = P_n(x) + R_n(x)$, where the $R_n(x)$ was the formula using the Lagrange Remainder using unknown c, z. Now, we will use $R_n(x)$ as an integral and approximate it using sums.

Theorem 8.15

Integration by Parts: Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous and have continuous bounded derivatives on (a, b). Then

$$\int_a^b fg' = fg|_a^b - \int_a^b f'g$$

$$udv = uv - \int vdu$$

Proof. From product rule,

$$(fg)' = f'g + fg' \implies fg' = (fg') - f'g$$

$$\int_a^b fg' = \int_a^b (fg)' - \int_a^b f'g$$

Now by FTC1

$$\int_a^b fg' = (fg)|_a^b - \int_a^b f'g$$

□

Theorem 8.16

Cauchy Integral Remainder Theorem: Let I be a neighborhood of $x_0 \in \mathbb{R}$, and let $n \in \mathbb{N} \cup \{0\}$. Suppose $f : I \to \mathbb{R}$ has $n+1$ derivatives and that $f^{(n+1)} : I \to \mathbb{R}$ is continuous. Then for each $x \in I$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt$$

where the first term is the nth Taylor Polynomial and the second term is the **Cauchy Integral Remainder**.

Proof. Basis step: $n = 0$

$$f(x_0)+\frac{1}{0!}\int_{x_0}^{x} f'(t)(x-t)^0 dt = f(x_0)+\int_{x_0}^{x} f'(t)dt = f(x_0)+f(x)-f(x_0) = f(x)$$

Now for $n = 1$ by FTC1: $\int_{x_0}^{x} f'(t)dt = f(x) - f(x_0) \implies f(x) = f(x_0) + \int_{x_0}^{x} f'(t)dt$ and

$$\int_{x_0}^{x} f'(t)dt = \int_{x_0}^{x} -f'(t)\frac{d}{dt}(x-t)dt$$

Let $u = -f'(t)$ and let $du = \frac{d}{dt}(x-t) \implies u = (x-t)$ and so by Integration by Parts

$$= -f'(t)(x-t)\Big|_{t=x_0}^{t=x} - \int_{x_0}^{x} -f''(t)(x-t)dt =$$

$$= -f'(x)(x-x) - (-f'(x_0)(x-x_0)) + \int_{x_0}^{x} f''(t)(x-t)dt$$

Plugging this in for $\int_{x_0}^{x} f'(t)dt$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{1!}\int_{x_0}^{x} f''(t)(x-t)dt$$

Note that the first two terms are terms of $P_n(x)$ and the last term is $R_1(x)$
Inductive hypothesis: $f(x) = P_n(x) + \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt$
Inductive step: Observe that

$$\frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x-t)^n dt = \frac{1}{n}\int_{x_0}^{x} f^{(n+1)}(t)\left[\frac{d}{dt} - \frac{1}{n+1}(x-t)^{n+1}\right]dt$$

$$= -\frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+1)}(t)\frac{d}{dt}(x-t)^{n+1}dt$$

and using IBP, let $u = f^{(n+1)}(t)$ and $g' = \frac{d}{dt}(x-t)^{n+1} \implies g = (x-t)^{n+1}$

$$= (-\frac{1}{(n+1)!})f^{(n+1)}(t)(x-t)^{n+1}\Big|_{x_0}^{x} - \int_{x_0}^{x} f^{(n+2)}(x-t)dt$$

$$\frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1} + \frac{1}{(n+1)!}\int_{x_0}^{x} f^{(n+2)}(x-t)dt \implies P_{n+1}(x) + R_{n+1}(x)$$

∎

§8.6 Skip 8.6

§8.7 The Weierstrass Approximation Theorem

Theorem 8.17

Weierstrass Approximation Theorem states that suppose $f : [a, b] \to \mathbb{R}$ is continous. Then for all $\epsilon > 0$, there is a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $\forall\ x \in [a, b]$, we have

$$|f(x) - p(x)| < \epsilon$$

Note that $p(x)$ may not be a Taylor Polynomial. This works for all x, not just a small neighborhood around x_0.

Proof. Omitted. Not expected to know for the exam. □

§9 Sequences and Series of Functions

§9.1 Sequences and Series of Numbers

Note 9.1. Note that previously a sequence of numbers is convergent if

$$\forall\ \epsilon > 0,\ \exists N \in \mathbb{N} \text{ s.t. } \ \forall n \geq N, |a_n - a| < \epsilon$$

However, we needed to know the limit a beforehand in order to use the definition. Some tools we have to determine convergence

1. Monotone Convergence Theorem: bounded monotone sequence $\implies$ converge

Definition 9.2. A sqeuence $\{a_n\}$ is called a **Cauchy Sequence** if

$$\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N} \text{ s.t. } n, m \in \mathbb{N} \implies |a_n - a_m| < \epsilon$$

Essentially, the terms of the sqeuence get closer to each other after N and we do not need to know $\lim\limits_{n\to\infty} a_n = a$.

Theorem 9.3

If a sequence converges, then it is a Cauchy Sequence.

Proof. Let $\epsilon > 0$. Then $\exists\ N \in \mathbb{N}$ such that $\forall\ n \in \mathbb{N}, |a_n - a| < \frac{\epsilon}{2}$ by definition of convergent sequence, since it works for all ϵ. Thus if $n, m \in \mathbb{N}$, we have

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a - a_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Lemma 9.4
If $\{a_n\}$ is Cauchy, then $\{a_n\}$ is bounded.

Proof. Since $\{a_n\}$ is Cauchy, $\forall\ \epsilon > 0$, $\exists\ N \in \mathbb{N}$ such that $\forall\ m, n \geq NN, |a_n - a_m| < \epsilon$. Let $\epsilon = 1$. So

$$|a_n - a_m| < 1\ \forall\ m, n \in N$$

In particular, $|a_n - a_N| < 1\ \forall\ n \geq N$. Therefore,

$$|a_n| = |a_n - a_N + a_N| \leq |a_n - a_N| + |a_N| < 1 + |a_N|\ \forall\ n \geq N$$

Let $M = \max\{|a_1|, \ldots, |a_{N+1}|, 1 + |a_N|\}$ Then $|a_n| \leq M\ \forall\ n \in \mathbb{N}$, and so $\{a_n\}$ is bounded, as desired. □

Theorem 9.5
If $\{a_n\}$ is Cauchy, then $\{a_n\}$ converges.

Proof. If $\{a_n\}$ is Cauchy, then it is bounded by the Lemma above. Since it is bounded, by Sequential Compactness, $\exists\ a_{n_k} \to a, a \in \mathbb{R}$ and a_{n_k} is a monotone convergent subsequence. We WTS that $|a_n - a| < \epsilon$. Let $\epsilon > 0$. Since $\{a_n\}$ is Cauchy, then $\exists\ N_1 \in \mathbb{N}$ such that $\forall\ m, n \geq N_1$, then

$$|a_n - a_m| < \frac{\epsilon}{2}$$

Since $\lim\limits_{n \to \infty} a_{n_k} = a$, $\exists\ N_2 \in \mathbb{N}$ such that $n_K \geq N_2$

$$|a_{n_k} - a| < \frac{\epsilon}{2}$$

Choose $n, k \geq \max\{N_1, N_2\}$, then

$$|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k - a}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

□

Definition 9.6. Given a sequence $\{a_n\}$, we can construct the series (infinite sum) by $\sum_{k=1}^{\infty} a_k$. We define the **nth partial sum** as $S_n = \sum_{k=1}^{n} a_k$. We say that the the series converges when the sequence of partial sums $\{s_n\}$ converges.

$$\lim_{n \to \infty} S_n = s \implies \sum_{k=1}^{\infty} a_k = s$$

Theorem 9.7

Suppose $\sum_{k=1}^{\infty} a_k$ converges to a, and $\sum_{k=1}^{\infty} b_k$ converges to b. Then

$$\sum_{k=1}^{\infty}(\alpha a_k + \beta b_k) \to \alpha a + \beta b, \alpha, \beta \in \mathbb{R}$$

Proof. See Chapter 2, just working with sequences on $\{S_n\}$ partial sums. □

Theorem 9.8

The **Cauchy Convergence Criterion for Series**. A series $\sum_{n=0}^{\infty} a_n$ converges if and only if $\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N},$ s.t. $\forall\ n \geq m \geq N, n, m \in \mathbb{N}$, then

$$|a_{m+1} + \cdots + a_n| < \epsilon$$

Proof. Recall that if a sequence converges if and only if it is Cauchy. If the sequences are partial sums then

$$S_n - S_m = \sum_{k=0}^{n} a_k - \sum_{k=0}^{m} a_k = \sum_{k=m+1}^{n} a_k$$

So, since S_n converges, it is Cauchy, and so

$$|S_n - S_n| = |\sum_{k=m+1}^{n} a_k| < \epsilon$$

□

Theorem 9.9

If $\sum_{n=0}^{\infty} a_n$ converges, then $\lim\limits_{n\to\infty} a_n = 0$. Note that the contrapositive of this is if $\lim\limits_{n\to\infty} a_n \neq 0 \implies \sum_{n=0}^{\infty} a_n$ diverges is known as the **Test for Divergence**. We will prove this in the standard way (not the contrapositive).

Proof. Note that $a_n = s_n - s_{n-1}\ \forall\ n \geq 1$. Suppose $\sum_{n=0}^{\infty} a_n$ converges to $a \in \mathbb{R}$. By definition of partial sums, $s_n \to a$.

$$\lim_{n\to\infty} s_n = a \qquad\qquad \lim_{n\to\infty} s_{n-1} = a$$

And so $\lim\limits_{n\to\infty} a_n = a - a = 0$ □

Note 9.10. Note that the finite geometric sum formula is

$$\sum_{k=0}^{n} r^k = 1 + r + \cdots + r^k = \frac{1 - r^{n-1}}{1 - r}$$

and

$$\lim_{n\to\infty} r^n = 0 \text{ if } |r| < 1$$

Proposition 9.11

For $r \in \mathbb{R}$ with $|r| < 1$ then

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$$

Proof. Consider

$$\lim_{n\to\infty} \sum_{k=0}^{n} r^k = \lim_{n\to\infty} \left(\frac{1-r^{n-1}}{1-r}\right) = \frac{1}{1-r}(1-0) = \frac{1}{1-r}$$

□

Lemma 9.12

Lemma for Comparison Test Proof. $\{a_k\}$ where $a_k \geq 0 \ \forall \ k$, then $\sum_{k=1}^{\infty} a_k$ converges $\iff$ partial sums sequence is bounded.

Proof. Since $a_k \geq 0 \ \forall \ k$, $\sum_{k=1}^{\infty} a_k$ is nonnegative, and thus $\{s_n\}$ is monotonically increasing.

$$\sum_{k=1}^{\infty} a_k \text{ conv } \iff \{s_n\} \text{ conv } \iff \{s_n\} \text{ is bounded}$$

by definition of partial sums, and then convergent sequences are bounded. In the reverse direction, use Monotone Convergence Theorem. □

Theorem 9.13

Comparison Test. $\{a_n\}, \{b_n\}$ sequences with $0 \leq a_k \leq b_k \ \forall \ k$.

- If $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ also converges.
- If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ also diverges.

Proof. $\forall \ n \sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k$ since $a_k \leq b_k \ \forall \ k$.

- If $\sum_{k=1}^{\infty} b_k$ converges, say to b, then

$$\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k \leq \sum_{k=1}^{\infty} b_k = b$$

  So $\sum_{k=1}^{n} a_k$ (partial sums) are bounded by b. So $\sum_{k=1}^{\infty} a_k$ converges by previous lemma.

- Similar.

□

Theorem 9.14

Integral Test. Let $\{a_k\}$ be a sequence of nonnegative numbers and suppose $f : [1, \infty) \to \mathbb{R}$ is continuous and monotonic decreasing, and $f(k) = a_k \ \forall\ k$. Then the series $\sum_{k=1}^{\infty} a_k$ converges if and only if $\{\int_a^n f(x)dx\}_{n=1}^{\infty}$ is bounded.

Proof. f is continuous and thus on each bounded interval, it is integrable. Since f is monotonically decreasing, then $\forall\ k \in \mathbb{N},\ \forall\ x \in [k, k+1]$, we have

$$a_k = f(k) \geq f(x) \geq f(k+1) = a_{k+1}$$

Integrate from k to $k+1$, such that

$$\int_k^{k+1} a_k \geq \int_k^{k+1} f(x)dx \geq \int_k^{k+1} a_{k+1}$$

$$a_k \geq \int_k^{k+1} f(x)dx \geq a_{k+1}$$

Sum up all k such that

$$\sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} \int_k^{k+1} f \geq \sum_{k=2}^{n+1} a_{k+1}$$

$$\sum_{k=1}^{n} a_k \geq \int_1^{n+1} f \geq \sum_{k=2}^{n+1} a_{k+1}$$

Note that $\int_1^n f$ is bounded above and below by partial sums of $\sum_{k=1}^{\infty} a_k$. Then use Comparison Test and note that f is monotonically decreasing.

$$\sum_{k=1}^{\infty} a_k \text{ conv } \iff \{\int_1^n f\} \text{ bdd}$$

□

Theorem 9.15

Alternating Series Test. Let $\{a_k\}$ be monotonically decreasing sequence of nonnegative numbers that converges to 0. Then the series

$$\sum_{k=1}^{\infty} (-1)^k a_k \text{ converges}$$

Proof. Proof omitted. □

Definition 9.16. $\sum_{k=1}^{\infty} a_k$ **absolutely converges** if $\sum_{k=1}^{\infty} |a_k|$ converges.

Theorem 9.17

Absolute Convergence Test. If $\sum_{k=1}^{\infty} |a_k|$ converges then $\sum_{k=1}^{\infty} a_k$ also converges.

Proof. Assume $\sum_{k=1}^{\infty} |a_k|$ converges. We WTS that $\sum_{k=1}^{\infty} a_k$ converges. Note that

$$a_k + |a_k| = \begin{cases} 2|a_k| \text{ if } a_k \geq 0 \\ 0 \text{ else} \end{cases}$$

and further note that

$$0 \leq a_k + |a_k| \leq 2|a_k| \ \forall \ k$$

Since $\sum_{k=1}^{\infty} |a_k|$ converges, then $\sum_{k=1}^{\infty} 2|a_k|$ also converges. So by $(*)$, $\sum(a_k + |a_k|) \leq 2\sum |a_k|$. The latter converges, so by Comparison Test, $\sum(a_k + |a_k|)$ also converges. Combining,

$$\sum a_k = \sum(a_k + |a_k| - |a_k|) = \sum(a_k + |a_k|) - \sum |a_k|$$

and the first term converges by CT and the second term converges by assumption, so by Linearity, $\sum_{k=1}^{\infty} a_k$ converges as well. □

Lemma 9.18

Lemma for Ratio Test Proof. For $\sum_{k=1}^{\infty} a_k$, suppose $\exists \ r \in \mathbb{R}$ with $0 \leq r < 1$, and $\exists \ N \in \mathbb{N}$ such that $|a_{n+1}| \leq r|a_n| \ \forall \ n \geq N$. Then $\sum_{k=1}^{\infty} a_k$ absolutely converges.

Proof. Apply the given iteratively,

$$|a_{N+K}| \leq r^k |a_N|$$

Then the partial sums of $\sum |a_n|$

$$|a_1| + |a_2| + \cdots + |a_{N+K}| \leq |a_1| + |a_2| + \cdots |a_N|(1 + r + r^2 + \cdots + r^k)$$

by factoring out $|a_N|$. Note that

$$\leq |a_1| + \cdots + |a_N| \left(\frac{1 - r^{k+1}}{1 - r} \right)$$

Taking the limit as $k \to \infty$, we get that the partial sums of $\sum |a_n|$ is

$$\leq |a_1| + \cdots + |a_n| \left(\frac{1}{1 - r} \right)$$

Note that the above has no dependence on K, so set it to M. The sequence of partial sums of $\sum |a_k|$ is bounded $\implies \sum_{k=1}^{\infty} |a_k|$ converges. □

Theorem 9.19

Ratio Test. For $\sum_{k=1}^{\infty} a_k$ suppose $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = l$. Then

1. If $l < 1$, then $\sum^{\infty} a_k$ converges absolutely.
2. If $l > 1$, then $\sum^{\infty}$ diverges.

Proof.

1. Consider the case of $l < 1$. Choose N such that $\forall\ n \geq N, \frac{|a_{n+1}|}{|a_n|} < l$. $|a_{n+1}| < l|a_n|$, and note that $l < 1$ and so by the previous theorem $\sum^{n} |a_k|$ converges.
2. The other case is similar.

□

§9.2 Pointwise Convergence of Sequence of Functions

Definition 9.20. From now on, let $D \subseteq \mathbb{R}$. Given $f : D \to \mathbb{R}$ and a sequence of functions $\{f_n : D \to \mathbb{R}\}$, we say $\{f_n\}$ **converges pointwise** to f

$$\forall\ x \in D, \lim_{n\to\infty} f_n(x) = f(x)$$

We write $f_n \xrightarrow{\text{p.w.}} f$.

Example 9.21

$f_n = x^n$ for $0 \leq x \leq 1, n \in \mathbb{N}$.

$$f_1(x) = x \qquad f_2(x) = x^2 \qquad f_3(x) = x^3 \qquad \cdots$$

Note that when $x = 1 \implies \lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1 = 1$. But for $0 \leq x < 1 \implies \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$ Thus, $\{f_n\}$ converges pointwise to f on $[0, 1]$ where

$$f(x) = \begin{cases} 1, x = 1 \\ 0, 0 \leq x < 1 \end{cases}$$

Example 9.22

Let the sequence of functions be $f_n(x) = e^{-x^2}\ \forall\ n \in \mathbb{N}$. What does this sequence of functions converge to?

Solution. When $x = 0 \implies \lim_{n\to\infty} e^{-n(0)^2} = 1$

For when $x \neq 0 \implies \lim_{n\to\infty} e^{-nx^2} = 0$ and so

$$\{f_n(x)\} \xrightarrow{\text{p.w.}} f(x) = \begin{cases} 0, x \neq 0, \\ 1, x = 0 \end{cases}$$

Note that limits do not preserve properties (cont, diff, integrable) of sequences of functions. □

§9.3 Uniform Convergence of a Sequence of Functions

Definition 9.23. We say $f_n : D \to \mathbb{R}$ converges uniformly to $f : D \to \mathbb{R}$ if

$$\epsilon > 0\ \exists\ N \in \mathbb{N} \text{ s.t } \ \forall\ n \geq N \text{ and } \forall\ x \in D \implies |f(x) - f_n(x)| < \epsilon$$

We write $f_n \xrightarrow{u} f$.

Example 9.24

From above, $f_n(x) = x^n$ on $[0, 1]$. We found that

$$f_n \xrightarrow{\text{p.w.}} f = \begin{cases} 1, x = 1 \\ 0, 0 \leq x < 1 \end{cases}$$

Now we will show that $f_n \overset{u}{\not\longrightarrow} f$.

Proof. By contradiction, assume that $f_n \xrightarrow{u} f$ on $[0, 1]$. Let $\epsilon = \frac{1}{2}$, so by def'n of uniform convergence, $\exists\ N \in \mathbb{N}$ such that $\forall\ n \geq N$ and $\forall\ x \in [0, 1]$, we have

$$|f_n - f| < \frac{1}{2}$$

so $x \in [0, 1) \implies |f_n - f| = |x^n - f| < \frac{1}{2}$. But take a sequence of points $x \to 1, x_m = 1 - \frac{1}{m}$ then

$$|x_m^n - 0| < \frac{1}{2}$$

$$|1^n - 0| < \frac{1}{2} \implies |1| < \frac{1}{2}$$

is a contradiction. □

Example 9.25

Suppose $f_n(x) = e^{-nx^2}, D = \mathbb{R}$. We claim

$$f_n \overset{u}{\not\longrightarrow} f = \begin{cases} 0, x \neq 0 \\ 1, x = 0 \end{cases}$$

Proof. By contradiction, assume $f_n \xrightarrow{u} f$. Let $\epsilon = \frac{1}{2}$. $\exists\ N \in \mathbb{N}$ such that $\forall\ n \geq N,\ \forall\ x \in \mathbb{R}$,

$$|f_n(x) - f(x)| < \frac{1}{2}$$

$$|e^{-nx^2} - 0| < \frac{1}{2}$$

For $x \neq 0$, take $x_m = \frac{1}{m}$, so the sequence approaches 0 as $m \to \infty$ Now plugging in x_m we get

$$|e^{-nx_m^2} - 0| < \frac{1}{2} \implies e^{-x_m^2} < \frac{1}{2}$$

Take the limit as $m \to \infty$

$$e^{-\frac{n}{m^2}} < \frac{1}{2} \implies e^0 < \frac{1}{2}$$

which is a contradiction and so $f_n \overset{u}{\not\longrightarrow} f$ $\square$

Example 9.26

Let $f_n : [-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ where $f_n = x^n$. Prove that $f_n \xrightarrow{u} f$ for $f(x) = 0$.

Solution. Let $\epsilon > 0$. We WTS that $|x^n - 0| = |x^n| < \epsilon\ \forall\ x \in [-\frac{1}{2}, \frac{1}{2}]$
Note that $|x^n| < (\frac{1}{2})^n$ since $x \in [-\frac{1}{2}, \frac{1}{2}]$ and $|x| \in [0, \frac{1}{2}]$. However, as

$$n \to \infty, (\frac{1}{2})^n \to 0,\ \exists\ N \in \mathbb{N} \text{ s.t. } \ \forall\ n \geq N, (\frac{1}{2})^n < \epsilon$$

by definition of convergence. So if $n \geq N$ and $\forall\ x \in [-\frac{1}{2}, \frac{1}{2}]$ then

$$|f_n(x) - f(x)| = |x^n - 0| = |x^n| \leq (\frac{1}{2})^n < \epsilon$$

and so therefore, $f_n(x) \xrightarrow{u} 0$ on $[-\frac{1}{2}, \frac{1}{2}]$ both pointwise and uniformly. $\square$

Definition 9.27. A sequence of functions $\{f_n\}, f_n : D \to \mathbb{R}$ is said to be **uniformly Cauchy** if

$$\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N} \text{ s.t. } \ \forall\ n, m \geq N,\ \forall\ x \in D \implies |f_m(x) - f_n(x)| < \epsilon$$

Theorem 9.28

Weierstrass Uniform Convergence Theorem states that $\{f_n : D \to \mathbb{R}\}$ is uniformly convergent if and only if $\{f_n\}$ is uniformly Cauchy.

Proof.
$\Longrightarrow$ Assume $\{f_n\}$ is uniformly convergent. We WTS that $\{f_n\}$ is uniformly Cauchy. Let $\epsilon > 0$. Since $f_n \xrightarrow{u} f$, $\exists\ N \in \mathbb{N}$ such that $\forall\ n \geq N$ and $\forall\ x \in D$, $|f_n(x) - f(x)| < \frac{\epsilon}{2}$. If $m, n \geq N$, then

$$|f_n(x) - f_m(x)| = |f_n - f + f - f_m| \leq |f_n - f| + |f_m - f| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

$\Longleftarrow$ Assume $\{f_n\}$ is uniformly Cauchy. Let $x \in D$. Then

$$\forall\ \epsilon > 0,\ \exists\ N \in \mathbb{N} \text{ s.t.} \forall\ m, n \geq N, |f_n(x) - f_m(x)| < \epsilon$$

Note that $\{f_n(x)\}$ is a sequence of y-vlaues for each x. Cauchy implies convergent for all sequences of numbers. Define $f : D \to \mathbb{R}$ such that $\underset{n\to\infty}{f_n(x)} = f(x)$ (pointwise?). Since $\{f_n\}$ is uniformly Cauchy, $\exists N_2 \in \mathbb{N}$ such that $\forall k, l \geq N_2$, $\forall\ x \in D$

$$|f_k(x) - f_l(x)| < \frac{\epsilon}{2}$$

We WTS that $f_l \xrightarrow{u} f$

$$-\frac{\epsilon}{2} < f_k(x) - f_l(x) < \frac{\epsilon}{2}$$

$$f_l - \frac{\epsilon}{2} < f_k < \frac{\epsilon}{2} + f_l + \frac{\epsilon}{2}$$

Take the limit $\underset{n\to\infty}{f_k} = f$

$$f_l - \frac{\epsilon}{2} \leq f \leq \frac{\epsilon}{2} + f_l$$

$$|f - f_l| \leq \frac{\epsilon}{2} < \epsilon$$

Therefore, $f_l \xrightarrow{u} f$ as needed. $\square$