MATH410: Homework 9

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1. Calculate the following derivatives:

   (a) $\frac{d}{dx}\left(\int_0^x x^2t^2dt\right)$

   (b) $\frac{d}{dx}\left(\int_1^{e^x} \ln t dt\right)$

   (c) $\frac{d}{dx}\left(\int_{-x}^x e^{t^2} dt\right)$

Proof. a. $\frac{d}{dx}\left(\int_0^x x^2t^2dt\right) = \frac{5x^4}{3}$

x^2t^2 is a polynomial and so it is continuous. Note that x^2 is a constant respect to the integral. Thus,

$$\frac{d}{dx}(\int_0^x x^2t^2dt) = \frac{d}{dx}(x^2 * \int_0^x t^2dt)$$

By product rule and the Fundamental Theorem of Calculus 2,

$$= x^2(x^2) + 2x\int_0^x t^2dt = x^4 + 2x\left(\frac{t^3}{3}\right)\Big|_0^x = x^4 + 2x(\frac{x^3}{3}) = x^4 + \frac{2x^4}{3} = \frac{5x^4}{3}$$

b. $\frac{d}{dx}(\int_1^{e^x} \ln tdt) = xe^x$

$\ln t$ is continuous on $[1, \infty)$. Using the Corollary following the Fundamental Theorem of Calculus 2 involving the Chain Rule,

$$\frac{d}{dx}(\int_1^{e^x} \ln tdt) = \ln(e^x) * e^x = xe^x$$

c. $\frac{d}{dx}(\int_{-x}^x e^{t^2}dt) =$

e^{t^2} is continuous on $[-x, x]$. By additivity of integrals and derivatives,

$$\frac{d}{dx}(\int_{-x}^x e^{t^2}dt) = \frac{d}{dx}(\int_{-x}^0 e^{t^2}dt) + \frac{d}{dx}(\int_0^x e^{t^2}dt)$$

$$= -\frac{d}{dx}(\int_0^{-x} e^{t^2}dt) + \frac{d}{dx}(\int_0^x e^{t^2}dt)$$

Now we can use FTC2 and chain rule for the first term

$$-e^{(-x)^2} * -1 + e^{x^2} = 2e^{x^2}$$

$\square$

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2. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ is differentiable. Define the function $H : \mathbb{R} \to \mathbb{R}$ by

$$H(x) = \int_{-x}^{x} [f(t) + f(-t)]dt \quad \text{for all } x.$$

Find $H''(x)$.

Proof. $f : \mathbb{R} \to \mathbb{R}$ being differentiable implies continuity. By additivity of integrals,

$$H'(x) = \frac{d}{dx}(\int_{-x}^{0}(f(t) + f(-t)) + \int_{0}^{x}(f(t) + f(-t))dt)$$

By additivity of derivatives,

$$H'(x) = \frac{d}{dx}(\int_{-x}^{0} f(t) + f(-t)dt) + \frac{d}{dx}(\int_{0}^{x} f(t) + f(-t)dt)$$

$$H'(x) = -\frac{d}{dx}(\int_{0}^{-x} f(t) + f(-t)dt) + \frac{d}{dx}(\int_{0}^{x} f(t) + f(-t)dt)$$

$$H'(x) = f(-x) + f(x) + f(x) + f(-x) = 2f(x) + 2f(-x)$$

by applying the Fundamental Theorem of Calculus 2. Now taking the second derivative,

$$H''(x) = 2f'(x) + 2f'(-x)$$

□

3. Suppose that the function $f : \mathbb{R} \to \mathbb{R}$ has a continuous second derivative. Prove that

$$f(x) = f(0) + f'(0)x + \int_0^x (x-t)f''(t)dt \quad \text{for all } x.$$

(Hint: Use the Identity Criterion: Let I be an open interval and let $g : I \to \mathbb{R}$ and $h : I \to \mathbb{R}$ be differentiable. Then these functions differ by a constant if and only if $g'(x) = h'(x)$ for all x in I.)

Proof. First, let's consider the integral term, and apply distributive property and additivity of integrals such that we obtain

$$\int_0^x (x-t)f''(t) = \int_0^x xf''(t)dt - \int_0^x tf''(t)dt$$

Observe that

$$\frac{d}{dx}(\int_0^x xf''(t)dt) = \frac{d}{dx}(x * \int_0^x f''(t)dt) = xf''(x) + \int_0^x f''(t)dt$$

$$\frac{d}{dx}(\int_0^x tf''(t)dt) = xf''(x)$$

which implies that $\frac{d}{dx}(\int_0^x (x-t)f''(t)dt) = \int_0^x f''(t)dt$. By the Fundamental Theorem of Calculus 1,

$$\int_0^x f''(t) = f'(x) - f'(0)$$

which implies that

$$\frac{d}{dx}(\int_0^x (x-t)f''(t)dt) = f'(x) - f'(0)$$

Let $h, g : I \to \mathbb{R}$ where $h(x) = f(x) - f'(0)x$, $g(x) = \int_0^x (x-t)f''(t)dt$, and $I = (-\infty, \infty)$. Observe that

$$g'(x) = f'(x) - f'(0) = h'(x)$$

which by the Identity Criterion, shows that h, g differ by a constant for all $c \in \mathbb{R} \; \forall \; x$, or mathematically, $h(x) = c + g(x) \; \forall \; x$. Since this is true for all x, let $x = 0 \implies f(0) - f'(0)(0) = c + \int_0^0 (0-t)f''(t)dt \implies c = f(0)$. Therefore, we conclude that

$$h(x) = f(0) + g(x) \; \forall \; x$$

$$f(x) - f'(0)x = f(0) + \int_0^x (x-t)f''(t)dt \; \forall \; x$$

$$f(x) = f(0) + f'(0)x + \int_0^x (x-t)f''(t)dt \; \forall \; x$$

as desired. $\square$

4. Let the function $f : [a,b] \to \mathbb{R}$ be continuous. Suppose that the function $F : [a,b] \to \mathbb{R}$ is continuous, that $F : (a,b) \to \mathbb{R}$ is differentiable, and that $F'(x) = f(x)$ for all x in (a,b). Use the Second Fundamental Theorem to prove that

$$\frac{d}{dx}\left[F(x) - \int_a^x f\right] = 0 \quad \text{for all } x \text{ in } (a,b)$$

and from this derive a new proof of the First Fundamental Theorem.

Proof. By additivity of derivatives,

$$\frac{d}{dx}\left[F(x) - \int_a^x f\right] = \frac{d}{dx}(F(x)) - \frac{d}{dx}(\int_a^x f) = F'(x) - \frac{d}{dx}(\int_a^x f)$$

By the given, which states that $F'(x) = f(x)\ \forall\ x \in (a,b)$. Note that $F(x) - \int_a^b f$ is continuous, since both terms themselves are continuous so the difference is continuous. By the Fundamental Theorem of Calculus 2, we get

$$\frac{d}{dx}\left[F(x) - \int_a^x f\right] = f(x) - f(x) = 0\ \forall\ x \in (a,b)$$

as desired. Note that this is

$$\frac{d}{dx}(F(x) - \int_a^x f) = 0 \implies f(x) = \frac{d}{dx}(\int_a^x f(t)dt) = \frac{d}{dx}(\int_a^x F'(t)dt)$$

Now lets use this to construct a new proof for the First Fundamental Theorem, which states that if F is continous on $[a,b]$ and differentiable on (a,b) and $F' : (a,b) \to \mathbb{R}$ is continuous and bounded then

$$\int_a^b F'(x)dx = F(b) - F(a)$$

Let $g(x) = \int_a^x F'(t)dt$. Note that $g'(x) = F'(x)$. Therefore, g and F differ by a constant.

$$g(x) = c + F(x)$$

$$g(a) = \int_a^a F'(t)dt = 0 \implies g(a) = c + F(a) \implies c = -F(a)$$

Therefore,

$$g(x) - F(x) = -F(a)\ \forall\ x \in [a,b]$$

$$g(b) - F(b) = -F(a) \implies g(b) = F(b) - F(a) \implies \int_a^b F'(x)dx = F(b) - F(a)$$

as desired.

$\square$

5. For each of the following pairs of functions, determine its highest order of contact at the indicated point:

    (a) $f(x) = x^2$ and $g(x) = \sin x$ for all $x; x_0 = 0$.

    (b) $f(x) = e^{x^2}$ and $g(x) = 1 + 2x^2$ for all $x; x_0 = 0$.

Proof. Recall that if I is a neighborhood around x_0, then $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are said to have contact of order n at x_0 if $f^{(k)}(x_0) = g^{(k)}(x_0)\ \forall\ k \in [0, n]$.

a. f and g have contact of order 1 at x_0.

$$k = 0 \implies f(0) = (0)^2 = 0 = \sin(0) = g(0)$$

Note that

$$f'(x) = 2x \implies f'(0) = 0$$
$$g'(x) = \cos(x) \implies g'(0) = \cos(0) = 1$$

Therefore, $g'(0) \neq f'(0)$ and so f and g have contact of order 0 at $x_0 = 0$.

b. f and g have contact of order 0 at x_0

$$k = 0 \implies f(0) = e^{0*0} = e^0 = 1, g(0) = 1 + 2(0)^2 = 1$$

Now note that when $k = 1$

$$f'(x) = 2xe^{x^2} \implies f'(0) = 0$$

$$g'(x) = 4x \implies g'(0) = 0$$

When $k = 2$,

$$f''(x) = (2x)(2xe^{x^2}) + 2e^{x^2} \implies f''(0) = 2$$

by product rule.

$$g''(x) = 4 \implies g''(0) = 4$$

Therefore, f and g have contact of order 1 at x_0.

$\square$

6. Define $f(x) = x^6e^x$ for all x. Find the sixth Taylor polynomial for the function f at $x = 0$.

Proof. Note that the sixth Taylor Polynomial of $f(x)$ is when $x_0 = 0$

$$P_6(x) = \sum_{k=0}^{6} \frac{f^{(k)}(0)}{k!} x^n$$

$$\begin{aligned} f(0) &= 0 \\ f'(x) = x^6e^x + 6x^5e^x &\implies f'(0) = 0 \\ f''(x) = x^6e^x + 6x^5e^x + 6x^5e^x + (6)(5)x^4e^x &\implies f''(0) = 0 \\ &\vdots \\ f^{(5)} \text{ has lowest term of degree } 1 &\implies f^{(5)}(0) = 0 \\ f^{(6)}(x) = x^6e^x + \cdots + 6!e^x &\implies f^{(6)}(0) = 6! \end{aligned}$$

Therefore,

$$P_n(6) = 0 + \cdots + 0 + \frac{6!}{6!}x^6 = x^6$$

$\square$

7. Prove that

$$1 + \frac{x}{3} - \frac{x^2}{9} < (1+x)^{1/3} < 1 + \frac{x}{3} \quad \text{if } x > 0$$

Proof. Let $f : (0, \infty) \to \mathbb{R}$ such that $f(x) = (1+x)^{1/3}$, and note that it is continuous and differentiable because it is a rational function and $1 + x \neq 0 \; \forall \; x > 0$. Observe the following derivatives

$$f'(x) = \frac{1}{3}(1+x)^{-2/3}$$
$$f''(x) = -\frac{2}{9}(1+x)^{-5/3}$$
$$f'''(x) = \frac{10}{27}(1+x)^{-8/3}$$

Consider the Taylor Polynomial expansion of $f(x)$ at $x_0 = 0$

$$P_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + \frac{1}{3}x - \frac{1}{9}x^2$$

Furthermore, consider the Lagrange Remainder where for each $x \neq 0$, $\exists c \in (0, x)$ such that

$$R_2(x) = \frac{f^{(3)}(c)}{3!}x^3 = \frac{5}{81}\frac{x^3}{(1+c)^{8/3}}$$

Note that by the Lagrange Remainder Theorem that

$$f(x) = P_2(x) + R_2(x)$$

However, $R_2(x) > 0$ because $x > 0, c > 0$. Therefore, $f(x) = P_2(x) + R_2(x) > P_2(x) \implies P_2(x) < f(x)$ yields

$$1 + \frac{x}{3} - \frac{x^2}{9} < (1+x)^{1/3} \qquad (*)$$

Now for the other side of the inequality, consider

$$R_1(x) = \frac{f''(c)}{2!}x^2 = -\frac{1}{9}\frac{x^2}{(1+x)^{5/3}} \implies R_1(x) < 0$$

Therefore, $f(x) = P_1(x) + R_1(x) \implies f(x) < P_1(x)$

$$\implies (1+x)^{1/3} < 1 + \frac{x}{3} \qquad (**)$$

Combining $(*)$ and $(**)$, we obtain

$$1 + \frac{x}{3} - \frac{x^2}{9} < (1+x)^{1/3} < 1 + \frac{x}{3}$$

as desired. □