# MATH410: Homework 7

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1. Suppose that the bounded function $f : [a, b] \to \mathbb{R}$ has the property that for each rational number $x$ in the interval $[a, b]$, $f(x) = 0$. Prove that

$$\underline{\int_a^b} f \leq 0 \leq \overline{\int_a^b} f$$

*Proof.*
Let $P$ be a partition on $[a, b]$. By definition of Lower and Upper Integrals, $\int_{\underline{a}}^{b} f = \sup\{L(f, P) \mid \text{P partition on [a,b]}\}$ and $\int_a^{\bar{b}} f = \inf\{U(f, P) \mid \text{P partition on [a,b]}\}$. First let us show that $\int_{\underline{a}}^{b} f \leq 0$. By definition of Darboux Lower Sum,

$$\int_{\underline{a}}^{b} f = \sup\{\sum_{i=1}^{n} m_i(x_i - x_{i-1})\}$$

where $m_i := \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$. Recall that $\mathbb{Q}$ and $\mathbb{Q}^c$ are dense in each closed interval $[x_{i-1}, x_i]$. Thus, there exists some $x \in [x_{i-1}, x_i]$ such that $x \in \mathbb{Q} \implies f(x) = 0$. Therefore, $m_i \leq 0 \ \forall\ i$. Since $x_i - x_{i-1} > 0$, then

$$m_i(x_i - x_{i-1}) \leq 0 \ \forall\ i \implies \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \leq 0 \ \forall \text{ partitions}$$

$$\implies \sup\{\sum m_i(x_i - x_{i-1}) \mid P \text{ partition on } [a, b]\} \leq 0 \implies \int_{\underline{a}}^{b} f \leq 0$$

Now we will show that $0 \leq \int_a^{\bar{b}} f$.

$$\int_a^{\bar{b}} f = \inf\{\sum_{i=1}^{n} M_i(x_i - x_{i-1})\}$$

By density of $\mathbb{Q}$ and $\mathbb{Q}, \exists x \in [x_{i-1}, x_i]$ such that $f(x) = 0 \implies M_i = \sup\{f(x) \mid x \in [x_{i-1}, x_i]\} \geq 0 \ \forall\ i$. Therefore,

$$M_i(x_i - x_{i-1}) \geq 0 \ \forall\ i \implies \int_a^{\bar{b}} f \geq 0$$

$$\int_{\underline{a}}^{b} f \leq 0 \leq \int_a^{\bar{b}} f$$

as desired.

□

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2. Suppose that the bounded function $f : [a, b] \to \mathbb{R}$ has the property that

$$f(x) \geq 0 \quad \text{for all } x \text{ in } [a, b].$$

Prove that $\underline{\int_a^b} f \geq 0$.

*Proof.*
Let $P$ be a partition on $[a, b]$. First observe that

$$\underline{\int_a^b} f = \sup\{L(f, P) \mid \text{P partition on [a,b]}\}$$

and by substitution using the definition of Darboux Lower Sum,

$$\underline{\int_a^b} f = \sup\{\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \mid P \text{ partition on } [a, b]\}$$

where $m_i = \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$. Since $f(x) \geq 0 \ \forall \ x \in [a, b] \implies f(x) \geq 0 \ \forall \ x \in [x_{i-1}, x_i]$. Therefore, by definition of inf being the largest lower bound, $m_i \geq 0 \ \forall \ i$. Since $x_i - x_{i-1} > 0$ by property of partition, $m_i(x_i - x_{i-1}) \geq 0 \ \forall \ i$ which implies that $\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \geq 0$ for all partitions $P$. Thus,

$$\underline{\int_a^b} f \geq 0$$

as desired.

$\square$

3. Suppose that the two bounded functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ have the property that

$$g(x) \le f(x) \quad \text{for all } x \text{ in } [a, b].$$

(a) For $P$ a partition of $[a, b]$, show that $L(g, P) \le L(f, P)$.

(b) Use part (a) to show that $\underline{\int_a^b} g \le \underline{\int_a^b} f$.

*Proof.*

a. By definition of Darboux Lower Sums, we WTS that

$$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \sum_{i=1}^{n} p_i(x_i - x_{i-1})$$

or equivalently

$$\sum_{i=1}^{n} (p_i - m_i)(x_i - x_{i-1}) \ge 0$$

where $m_i := \inf\{g(x) \mid x \in [x_{i-1}, x]\}$ and $p_i := \inf\{f(x) \mid x \in [x_{i-1}, x_i]\}$. Since $g(x) \le f(x) \ \forall \ x \in [a, b] \implies g(x) \le f(x) \ \forall \ x \in [x_{i-1}, x_i]$. Since inf is by definition the largest lower bound, intuitively $m_i \le p_i \ \forall \ i \implies p_i - m_i \ge 0$. Since $x_i - x_{i-1} > 0$, then

$$(p_i - m_i)(x_i - x_{i-1}) \ge 0 \ \forall \ i \implies \sum_{i=1}^{n} (p_i - m_i)(x_i - x_{i-1}) \ge 0$$

$$\implies L(g, P) \le L(f, P)$$

b. We WTS that $\underline{\int_a^b} g \le \underline{\int_a^b} f$ or equivalently

$$\sup\{L(g, P) \mid P \text{ partition on } [a, b]\} \le \sup\{L(f, P) \mid P \text{ partition on } [a, b]\}$$

Assume on the contary that

$$\sup\{L(g, P) \mid P \text{ partition on } [a, b]\} > \sup\{L(f, P) \mid P \text{ partition on } [a, b]\}$$

Then there must exist a partition $P^*$ such that

$$L(g, P^*) > \sup\{L(f, P) \mid P \text{ partition on } [a, b]\}$$

However, from (a), recall that we showed that $L(g, P^*) \le L(f, P^*)$ and so we've reached a contradiction. Therefore,

$$\underline{\int_a^b} g \le \underline{\int_a^b} f$$

$\square$

4. Suppose that $f : [a, b] \to \mathbb{R}$ is a bounded function for which there is a partition $P$ of $[a, b]$ with $L(f, P) = U(f, P)$. Prove that $f : [a, b] \to \mathbb{R}$ is constant.

*Proof.*
By definition of Darboux Lower and Upper Sums, we are given that

$$\sum_{i=1}^{n} m_i(x_i - x_{i-1}) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

where $m_i$ is the inf and $M_i$ is the sup of the $ith$ subinterval. Rearranging above we can easily show that

$$\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = 0$$

Note that $x_i > x_{i-1} \; \forall \; 0$. Note that $m_i \leq M_i \; \forall \; i$ by properties of inf and sup. Note that $m_i = M_i \; \forall \; i$, otherwise, $L(f, P) < U(f, P)$ which contradicts a given. Therefore, for all closed subintervals $[x_{i-1}, x_i], m_i = M_i$. For any $f : [a, b] \to \mathbb{R}$, if $\inf(f) = \sup(f) = c, c \in \mathbb{R}$ then $f(x) = c \; \forall \; x \in [a, b]$. Suppose on the contrary there existed an $x_0 \in [a, b]$ such that $f(x) > c$. Then $c$ is not the sup. Similar for the other direction. Thereefore, for any closed interval, if the inf and sup are equal, then all functional values on the closed intervals are a constant.

Note that subintervals are connected and that for any $x_i, x_i \in [x_{i-1}, x_i]$ and $x_i \in [x_i, x_{i+1}]$. Suppose we have two connected subintervals such that $f(x) = c \; \forall \; x \in [x_{i-1}, x_i]$ and $f(x) = d \; \forall \; x \in [x_i, x_{i+1}], c, d \in \mathbb{R}$. Note that $f(x_i) = c = d \implies f(x) = a \; \forall \; x \in [x_{i-1}, x_{i+1}]$. Repeating this process for all guaranteed connected subintervals in the partition yields that $f(x) = c \; \forall \; x \in [a, b]$.

□

5. (a) Use the Archimedes-Riemann Theorem to show that $\int_a^b x dx = \frac{b^2 - a^2}{2}$.

   (b) Use the Archimedes-Riemann Theorem to find the value of:

$$\int_0^1 [4x + 1] dx$$

   (c) Recall that:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

   Use the Archimedes-Riemann Theorem to show that for $0 \leq a < b$,

$$\int_a^b x^2 dx = \frac{b^3 - a^3}{3}$$

.

*Proof.*

a. First let us show that $\int_a^b x\ dx$ is integrable. By the Archimedes-Riemann Theorem, an $f$ that is bounded is integrable if $\exists \{P_n\}$ such that

$$\lim_{n \to \infty} ((U(f, P_n) - L(f, P_n))) = 0$$

Let $P_n$ be the regular partition such that all subintervals are the same length

$$P_n = \{a, a + \frac{b-a}{n}, a + 2\frac{b-a}{n}, \dots, b\} \implies x_i = a + i\frac{b-a}{n}$$

Therefore,

$$\lim_{n \to \infty} (U(f, P_n) - L(f, P_n)) = \lim_{n \to \infty} \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1})$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} (a + i\frac{b-a}{n} - a - (i-1)\frac{b-a}{n})(\frac{b-a}{n}) = \lim_{n \to \infty} (b-a)^2 \sum_{i=1}^{n} \frac{1}{n^2}$$

$$= \lim_{n \to \infty} \frac{b-a}{n} = 0$$

Therefore, $f(x) = x$ is integrable and by the "moreover" part of the AR Theorem

$$\int_a^b x = \lim_{n \to \infty} U(f, P_n) = \lim_{n \to \infty} \sum_{i=1}^{n} (a + i\frac{b-a}{n})(\frac{b-a}{n})$$

$$= \lim_{n \to \infty} \frac{b-a}{n} (\sum_{i=1}^{n} a + \frac{b-a}{n} \sum_{i=1}^{n} i) = \lim_{n \to \infty} \frac{b-a}{n} (an + \frac{(b-a)(n+1)}{2})$$

$$= ab - a^2 + \lim_{n \to \infty} \frac{(b-a)^2(n+1)}{2n} = ab - a^2 + \lim_{n \to \infty} \frac{(b^2 - 2ab + a^2)(n+1)}{2n}$$

$$= ab - a^2 + \frac{b^2 - 2ab + a^2}{2} = \frac{b^2 - a^2}{2}$$

as desired.

b. Let us show that $f(x) = 4x + 1$ is integrable. Let $P_n$ be a regular partition such that

$$P_n = \{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\} \implies x_i = \frac{i}{n}$$

Observe that

$$\lim_{n\to\infty}(U(f,P_n) - L(f,P_n)) = \lim_{n\to\infty}\sum_{i=1}^{n}(M_i - m_i)\frac{1}{n}$$

Note that the sup and inf are $M_i = \frac{4i}{n} + 1$ and $m_i = \frac{4(i-1)}{n} + 1 \implies M_i - m_i = \frac{4}{n}$. Thus,

$$\lim_{n\to\infty}(U(f,P_n) - L(f,P_n)) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n^2} = \lim_{n\to\infty}\frac{4}{n} = 0$$

which means $f$ is integrable and by the AR theorem

$$\int_0^1 4x + 1dx = \lim_{n\to\infty}U(f,P_n) = \lim_{n\to\infty}\sum_{i=1}^{n}(\frac{4i}{n} + 1)(\frac{1}{n})$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4i}{n^2} + \frac{1}{n} = \lim_{n\to\infty}\frac{4n(n+1)}{2n^2} + 1 = 3$$

c. First let us show that $f(x) = x^2$ is integrable. Let $P_n$ be the regular partition once more such that

$$P_n = \{a, a + \frac{b-a}{n}, a + 2\frac{b-a}{n}, \dots, b\} \implies x_i = a + i\frac{b-a}{n}$$

$$\lim_{n\to\infty}(U(f,P_n) - L(f,P_n)) = \lim_{n\to\infty}\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1})$$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}(M_i - m_i)(\frac{b-a}{n})$$

Now observe that $M_i = (a + i\frac{b-a}{n})^2$ and $m_i = (a + i\frac{b-a}{n} - \frac{b-a}{n})^2$ and this limit can be shown to approach to 0. Furthermore,

$$\int_a^b = \lim_{n\to\infty}U(f,P_n) = \lim_{n\to\infty}\sum_{i=1}^{\infty}(a + i\frac{b-a}{n})^2(\frac{b-a}{n})$$

$$= \frac{b^3 - a^3}{3}$$

□

6. Suppose that the functions $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are integrable. Show that there is a sequence $\{P_n\}$ of partitions of $[a, b]$ that is an Archimedean sequence of partitions for $f$ on $[a, b]$ and also an Archimedean sequence of partitions for $g$ on $[a, b]$. (Hint: Use the Refinement Lemma.)

*Proof.*
Since $f, g$ are integrable, there exists two Asop $P_n$ and $Q_n$ such that

$$\lim_{n\to\infty}(U(f, P_n) - L(f, P_n)) = 0 \text{ and } \lim_{n\to\infty}(U(g, Q_n) - L(g, Q_n)) = 0$$

Now let us define a new sequence $R_n$ that is a refinement of $P_n$ and $Q_n$. Therefore, by the Refinement Lemma,

$$L(f, P_n) \leq L(f, R_n) \text{ and } U(f, R_n) \leq U(f, P_n)$$

$$L(g, Q_n) \leq L(g, R_n) \text{ and } U(g, R_n) \leq U(Q, P_n)$$

Combining some of the above inequalities, note that

$$U(f, R_n) - L(f, R_n) \leq U(f, P_n) - L(f, P_n)$$

$$U(g, R_n) - L(g, R_n) \leq U(g, Q_n) - L(g, Q_n)$$

Note that both right hand sides converge to 0 because $f$ and $g$ are integrable. Therefore, by the Comparison Lemma, the left hand sides also converge to 0, which means $R_n$ is an Asop for both $f$ and $g$ on $[a, b]$.

□