

# MATH410: Advanced Calculus I

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This section covers the foundation of analysis, which is just the set of real numbers. It covers basic definitions such as  $\in, \notin, \emptyset, \subseteq, =, \cap, \cup, \setminus$ , so for example

**Definition 0.1. Intersection** of  $A$  and  $B$  is  $C = A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Some quantifiers include  $\forall, \exists, \exists!$  and some number sets include  $\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^C$ .

**Definition 0.2.** The real numbers  $\mathbb{R}$  satisfies 3 groups of axioms: Refer to the notes on Canvas for the Consequences of all of the following axioms.

1. Field  $(+, *)$ 
  - Commutativity of Addition
  - Associativity
  - Additive Identity
  - Additive Inverse
  - Commutativity of Multiplication
  - Associativity of Multiplication
  - Multiplicative Identity
  - Multiplicative Inverse
  - Distributive Property

The set of integers  $\mathbb{Z}$  is not a field because it fails under the multiplicative inverse.

2. Positivity

There is a subset of  $\mathbb{R}$  denoted by  $\mathcal{P}$ , called the set of positive numbers for which:

- If  $x$  and  $y$  are positive, then  $x + y$  and  $xy$  are both positive.
- For each  $x \in \mathbb{R}$ , exactly one of the following 3 alternatives is true:  $x \in \mathcal{P}$ ,  $-x \in \mathcal{P}$ , or  $x = 0$

3. Completeness

**Definition 0.3. Absolute value** is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

**Definition 0.4. Triangle Inequality** is  $\forall a, b \in \mathbb{R}, |a + b| \leq |a| + |b|$

*Proof.* Assume without loss of generality,  $a \geq b$ . We will proceed with proof by cases.

Case 1: Assume  $a \geq b \geq 0$ . Then  $|a + b| = a + b$  by the definition of absolute value since  $a \geq 0, b \geq 0 \implies |a + b| = a + b = |a| + |b|$ .

Case 2: Now assume  $a \geq 0 \geq b$  and  $a + b \geq 0$ . Note since  $b \leq 0$  then  $b \leq |b|$ . Then

$$|a + b| = a + b \leq |a| + |b|$$

by the definition of absolute value and our above note.

Case 3: Now consider  $a \geq 0 \geq b$  and  $a + b < 0$ . So

$$|a + b| = -(a + b) = -a - b \leq |a| + |b|$$

Case 4: Now consider  $0 \geq a \geq b$  so  $a + b < 0$ . Therefore,

$$|a + b| = -(a + b) = -a - b = |a| + |b|$$

□

## §1 The Completeness Axiom

**Definition 1.1.** A subset  $S$  of  $\mathbb{R}$  is said to be **bounded above** if  $\exists r \in \mathbb{R}$  such that  $s \leq r \forall s \in S$

The definition of **bounded below** is similar.

**Definition 1.2.** The least upper bound, if it exists, is called the **supremum** of  $S$ . We denote it as the "sup" of  $S$ . Similarly, the largest lower bound is called the **infimum** and is denoted as the "inf" of  $S$ .

**Definition 1.3.** Let  $S \subseteq \mathbb{R}$  where  $S \neq \emptyset$ . If  $S$  has a largest (smallest), the element is a max (min).

### Example 1.4

Find the sup of  $(0, 1)$  and prove it.

*Proof.* Let us prove that the  $\sup(0, 1) = 1$ . First, let us show that we have an upperbound. If  $x \in (0, 1)$ , then  $x < 1$ . By definition of upperbound, 1 is an upper bound. Note that we can find many other upper bounds.

On the contrary, assume  $x < 1$  is an upper bound. Now consider the average  $\frac{1+x}{2}$ .

$$\frac{x+1}{2} < \frac{1+1}{2} = 1$$

Therefore, we have showed that  $0 < \frac{x+1}{2} < 1 \implies \frac{x+1}{2} \in (0, 1)$ . But,  $\frac{1+x}{2} > x \implies \frac{1+x}{2} > x$ . This is a contradiction. Since  $x$  is an upper bound, and we found  $\frac{1+x}{2} \in (0, 1)$  where  $\frac{1+x}{2} > x$ , so  $x$  is not a supremum.

□

### Theorem 1.5

Suppose  $S \subseteq \mathbb{R}, S \neq \emptyset$  that is bounded above. Then a supremum exists. Every nonempty subset  $S$  of  $\mathbb{R}$  that is bounded below has a lower bound.

**Note 1.6.** Let  $c$  be a positive number then  $\exists!$  a positive number whose square is  $c$ .  $x^2 = c, x > 0$  has a unique solution and this gives us the notion of square root.

## §1.1 Archimedian Property

**Definition 1.7.** The **Archimedian Property** is a result of the completeness axiom. Suppose there is a small  $\epsilon > 0$  and  $c$  is an arbitrary large number.

1.  $\exists n \in \mathbb{N}$  such that  $c < n$ , which just means that you can always find a natural number than any large number
2.  $\exists m \in \mathbb{N}$  such that  $\frac{1}{m} < \epsilon$ , which just means you can always find smaller rational numbers.

*Proof.* We will proceed by contradiction. Assume that  $\exists$  an upper bound  $c$  for the  $\mathbb{N}$ . So there is no  $n \in \mathbb{N}$  s.t.  $c < n$ . Since  $\mathbb{N}$  is bounded above, and the  $\mathbb{N}$  is nonempty, the supremum exists (Completeness Axiom). Let  $s = \sup \mathbb{N}$ . Consider  $s - 1$  and  $s - 1 < s = \sup \mathbb{N}$ , which is the least upper bound, so  $s - 1$  is not an upper bound. So  $\exists n \in \mathbb{N}$  such that  $s - 1 < n \implies s < n + 1$ . But  $s = \sup \mathbb{N}$ , the least upper bound, this is a contradiction since it is less than  $(n + 1) \in \mathbb{N}$ .

For part  $b$ , use  $c = \frac{1}{\epsilon}$  and use part  $a$ . □

**Note 1.8.** Some of the following are results from the Archimedian Property.

### Theorem 1.9

For all  $n \in \mathbb{Z}$ , there is no integer in  $(n, n + 1)$  (an open interval).

### Theorem 1.10

If  $S$  is a nonempty subset of  $\mathbb{Z}$  that is bounded above, then it has a max.

### Theorem 1.11

\* For every  $c \in \mathbb{R}$ ,  $\exists! n \in \mathbb{Z}$  in  $[c, c + 1)$

**Definition 1.12.** A subset  $S \subseteq \mathbb{R}$  is said to be **dense in  $\mathbb{R}$**  if for every  $a, b \in \mathbb{R}$  with  $a < b$ , then there is a  $s \in S$  s.t.  $s \in (a, b)$ .

**Theorem 1.13**

$\mathbb{Q}$  is dense in  $\mathbb{R}$ . Reminder that  $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$

*Proof.* Suppose we have arbitrary  $a, b \in \mathbb{R}$  and  $a < b$ . We want to find  $\frac{m}{n} \in (a, b)$ . By multiplication, we can say we want  $na < m < nb$ . We want an integer  $m$  between  $na$  and  $nb$ . We can write this as

$$nb - na > 1 \implies n(b - a) > 1 \implies n > \frac{1}{b - a}$$

By part *a* of the Archimedean Property, let  $c = \frac{1}{b-a}$ , and we know that there exists some  $n \in \mathbb{N}$  such that  $n > c$ . Since  $a < b$ , and  $b - a > 0$ , multiply

$$n > \frac{1}{b - a}$$

$$n(b - a) > 1$$

$$nb - na > 1$$

$$nb - 1 > na \implies na < nb - 1$$

By previous (\*),  $\exists m \in \mathbb{Z}$  s.t.  $m \in [nb - 1, nb)$ . Therefore,  $nb - 1 \leq m < nb$ . Therefore,

$$na < nb - 1 \leq m < nb \implies na < m < nb \implies a < \frac{m}{n} < b$$

and so we have found that there exists  $m \in \mathbb{Z}, n \in \mathbb{N}$  such that  $\frac{m}{n} \in (a, b)$  for all  $a, b \in \mathbb{R}$  and  $a < b$ . Therefore, the rational numbers are dense in the real numbers.  $\square$

## §2 Sequences

**Definition 2.1.** A **sequence** of  $\mathbb{R}$  is a real-valued function whose domain is  $\mathbb{N}$ .  $f : \mathbb{N} \rightarrow \mathbb{R}$  (a list of numbers indexed by  $\mathbb{N}$ )

**Example 2.2**

A sequence of odd integers could be  $a_1 = 1, a_2 = 3, a_3 = 5, \dots, a_n = 2n - 1$  which can be

$$\{1, 3, 5, \dots\} = \{a_n\}_{n=1}^{\infty} = \{2n - 1\}_{n=1}^{\infty}$$

**Example 2.3**

$$\left\{\frac{1}{n}\right\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \implies \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$$

## §2.1 Convergence

**Definition 2.4.** A sequence  $\{a_n\}$  is said to **converge** to a number  $L$  if  $\forall \epsilon > 0$ ,  $\exists$  an index  $N$  s.t.  $\forall$  indices  $n \geq N$  we have

$$|a_n - L| < \epsilon \implies \text{Notation: } \lim_{n \rightarrow \infty} a_n = L$$

### Example 2.5

Suppose we have the sequence  $\{\frac{(-1)^n}{n}\}$  and we WTS

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

Think of the problem as someone gives you a small  $\epsilon \implies$  you have to find  $N$ , which we call the **threshold**, such that for every sequence value after the threshold is in the  $\epsilon$ -tube.

For example,  $\epsilon = \frac{1}{2} \implies N = 3, \epsilon = \frac{1}{4} \implies N = 5$ .

Above  $L = 0$ , sketch: we want

$$|a_n - L| < \epsilon \implies \left| \frac{(-1)^n}{n} - 0 \right| < \epsilon \implies \left| \frac{1}{n} \right| < \epsilon \implies \frac{1}{\epsilon} < n$$

so choose  $N = \frac{1}{\epsilon} < n$

*Proof.* Let  $\epsilon > 0$  be given. By Archimedian Property,  $\exists N \in \mathbb{N}$  such that  $\frac{1}{N} < \epsilon$ . Then if  $n \geq N$

$$\left| \frac{(-1)^n}{n} - 0 \right| = \left| \frac{(-1)^n}{n} \right| = \left| \frac{1}{n} \right| = \frac{1}{n}$$

From here, we need to relate  $n$  to  $N$  and then we can relate  $N$  to  $\epsilon$ . Note that  $n \geq N \implies \frac{1}{N} \geq \frac{1}{n}$  by algebra. Therefore,

$$\frac{1}{n} \leq \frac{1}{N} < \epsilon$$

by our choice of  $N$ . Therefore,

$$\frac{1}{n} < \epsilon$$

and so we are done since we have shown that

$$\left| \frac{(-1)^n}{n} - 0 \right| < \epsilon$$

□

**Example 2.6**

Given  $\{\frac{n^2-2n}{n^2+1}\}$ , prove that this sequence  $\lim_{n \rightarrow \infty} \frac{n^2-2n}{n^2+1} = 1$ .

Some sketch work: we want to show that  $|\frac{n^2-2n}{n^2+1} - 1| < \epsilon$

$$|\frac{n^2-2n}{n^2+1} - 1| = |\frac{n^2-2n}{n^2+1} - \frac{n^2+1}{n^2+1}| = |\frac{-2n-1}{n^2+1}| = |\frac{2n+1}{n^2+1}|$$

Note that both the numerator and denominator are both always positive, so we can consider. Now let us use the  $\leq$  operator to simplify and have one singular 'n'.

$$\frac{2n+1}{n^2+1} \leq \frac{2n+1}{n^2} \leq \frac{2n+n}{n^2} = \frac{3n}{n^2} = \frac{3}{n}$$

Recall that  $n \geq N \implies \frac{1}{N} \geq \frac{1}{n} \implies \frac{1}{n} \leq \frac{1}{N}$  So we'd choose  $N$  to get rid of 3 and introduce  $\epsilon$ .

*Proof.* Let  $\epsilon > 0$ . By A.P.,  $\exists N \in \mathbb{N}$  s.t.  $\frac{1}{N} < \frac{\epsilon}{3}$ . For  $n \geq N$ , then

$$|\frac{n^2-2n}{n^2+1} - 1| = \dots = \frac{2n+1}{n^2+1} < \dots \leq \frac{3}{n} \leq \frac{3}{N} = 3 * \frac{1}{N} < 3 * \frac{\epsilon}{3} = \epsilon$$

Therefore, we have shown that

$$|a_n - L| < \epsilon \implies \lim_{n \rightarrow \infty} \frac{n^2-2n}{n^2+1} = 1$$

□

**Theorem 2.7**

**The Sum Property** states that if

$$\lim_{n \rightarrow \infty} a_n = a \text{ and } \lim_{n \rightarrow \infty} b_n = b$$

then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = a + b$$

Some sketch work before the proof:

We want to show that  $|a_n + b_n - (a + b)| < \epsilon$ . Note that we can group terms together  $|(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$  by the Triangle Inequality. It is known that these two terms converge. Therefore, we can choose  $\epsilon$ s such that

$$|a_n - a| + |b_n - b| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

*Proof.*

Let  $\epsilon > 0$ . Since the sequences  $\{a_n\}$  and  $\{b_n\}$  converge to  $a$  and  $b$ , respectively, by the Archimedian Principle,  $\exists N_1, N_2 \in \mathbb{N}$  such that  $\frac{1}{N_1} < \frac{\epsilon}{2}$  and  $\frac{1}{N_2} < \frac{\epsilon}{2}$ . Choose  $N = \max(N_1, N_2)$ , which represents the numerically larger threshold. For all  $n \geq N$ , we show

$$\begin{aligned} |a_n + b_n - (a + b)| &= |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore, we have shown that  $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$  □

**Lemma 2.8**

**The Comparison Lemma (C.L.)**

Let  $\{a_n\}$  converge to  $a$ . Then  $\{b_n\}$  converges to  $b$  if  $\exists c \in \mathbb{R}^+$  and  $N \in \mathbb{N}$  such that

$$\forall n \geq N, |b_n - b| \leq c|a_n - a|$$

*Proof.* Let  $\epsilon > 0$ . Since  $a_n$  converges to  $a$ ,  $\exists N_1 \in \mathbb{N}$  such that  $|a_n - a| < \frac{\epsilon}{c}$ ,  $\forall n \geq N_1$ . By the Archimedian Principle,  $\exists N_2 \in \mathbb{N}$  such that  $\frac{1}{N_2} < \epsilon$ . Choose  $N = \max(N_1, N_2)$  and if  $n \geq N$ , then

$$\begin{aligned} |b_n - b| &\leq c|a_n - a| < c * \frac{\epsilon}{c} = \epsilon \\ \implies |b_n - b| &< \epsilon \end{aligned}$$

□



**Lemma 2.9**

Suppose the  $\lim_{n \rightarrow \infty} a_n = a$ , then for  $c \in \mathbb{R}$ ,

$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n = ca$$

*Proof.* Use the Comparison Lemma (above). Note that  $|ca_n - ca| = |c(a_n - a)| = |c||a_n - a|$  which satisfies  $|b_n - b| \leq c|a_n - a| \implies \{b_n\} = \{ca_n\} \implies b = ca$ .  $\square$

**Lemma 2.10**

The following is a useful property (\*)

$$\lim_{n \rightarrow \infty} a_n = a \text{ iff } \lim_{n \rightarrow \infty} (a_n - a) = 0$$

**Lemma 2.11**

Suppose  $\lim_{n \rightarrow \infty} a_n = 0$  and  $\lim_{n \rightarrow \infty} b_n = 0$  then  $\lim_{n \rightarrow \infty} a_n b_n = 0$ .

*Proof.* Since  $\lim_{n \rightarrow \infty} a_n = 0$  and  $\sqrt{\epsilon} > 0$ ,

$$\exists N_1 \in \mathbb{N} \text{ s.t. } |a_n| < \sqrt{\epsilon} \forall n \geq N_1$$

Since  $\lim_{n \rightarrow \infty} b_n = 0$  and  $\sqrt{\epsilon} > 0$ ,

$$\exists N_2 \in \mathbb{N} \text{ s.t. } |b_n| < \sqrt{\epsilon} \forall n \geq N_2$$

Let  $N = \max(N_1, N_2)$ . Then if  $n \geq N$ ,

$$|a_n b_n - 0| = |a_n b_n| = |a_n| * |b_n| < \sqrt{\epsilon} * \sqrt{\epsilon} = \epsilon$$

$\square$

**Theorem 2.12**

**The Product Property** states that if  $\lim_{n \rightarrow \infty} a_n = a$  and  $\lim_{n \rightarrow \infty} b_n = b$  then

$$\lim_{n \rightarrow \infty} a_n b_n = ab$$

*Proof.* Define  $\alpha_n = a_n - a$  and  $\beta_n = b_n - b$ . Using the \* property above, since  $\lim_{n \rightarrow \infty} a_n = a \implies \lim_{n \rightarrow \infty} (a_n - a) = \lim_{n \rightarrow \infty} \alpha_n = 0$  and then the same for  $b$  such that  $\lim_{n \rightarrow \infty} \beta_n = 0$ .

Note that

$$a_n b_n - ab = (\alpha_n + a)(\beta_n + b) - ab$$

and then by Distributive property we get

$$= \alpha_n \beta_n + \alpha_n b + a \beta_n + ab - ab = \alpha_n \beta_n + \alpha_n b + a \beta_n$$

So using the previous lemma,

$$\lim_{n \rightarrow \infty} (a_n b_n - ab) = \lim_{n \rightarrow \infty} (\alpha_n \beta_n + b \alpha_n + a \beta_n) = \lim_{n \rightarrow \infty} (\alpha_n \beta_n) + b \lim_{n \rightarrow \infty} \alpha_n + a \lim_{n \rightarrow \infty} \beta_n$$

From above, the last two terms are 0 and by the previous lemma, the first term is 0. Therefore,

$$\lim_{n \rightarrow \infty} (a_n b_n - ab) \iff \lim_{n \rightarrow \infty} (a_n b_n) = ab$$

□

**Definition 2.13.** A sequence **diverges** to  $\infty, (-\infty)$  if

$$\forall M > (<)0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, n \geq N \text{ then } a_n > (<)M$$

**Example 2.14**

Prove that  $\lim_{n \rightarrow \infty} (n^2 - 4n) = \infty$

Sketch: we want  $a_n > M \implies n^2 - 4n > M \implies n(n - 4) > M$

*Proof.* Let  $M > 0$  be given. By A.P.,  $\exists N \in \mathbb{N}$  s.t.  $N > \max(M, 4)$ . If  $n \geq N$ , then  $n^2 - 4n = n(n - 4) \geq N(N - 4) > M$

Thus,

$$n^2 - 4n \rightarrow \infty \text{ as } n \rightarrow \infty$$

□

**Example 2.15**

Prove that  $(-1)^n$  does not converge.

*Proof.* On the contrary, suppose  $(-1)^n$  converges to  $a$ . Let  $\epsilon = 1$ . In the definition of convergence, then  $\exists N \in \mathbb{N}$  if  $n \geq N$  then

$$|(-1)^n - a| < 1$$

For  $n = 2N$ , meaning some even number, we get  $|(-1)^n - a| = |1 - a| < 1$

Now for  $n = 2N + 1$ , we get  $|(-1)^{2N+1} - a| = |1 + a| < 1$

Note that  $|1 - a| < 1$  and  $|1 + a| < 1$  so therefore

$$|1 - a| + |1 + a| < 2$$

But note, using the Triangle Inequality in the reverse direction, note that  $2 = |1 - a + 1 + a| \leq |1 - a| + |1 + a| < 1 + 1 = 2$ . Therefore, we've shown that  $2 < 2$  which is a contradiction and therefore,  $(-1)^n$  does not converge.  $\square$

**Lemma 2.16**

Suppose the sequence  $\{b_n\}$  of nonzero numbers converges to  $b \neq 0$ . Then  $\{\frac{1}{b_n}\}$  converges to  $\frac{1}{b}$ .

Sketch: Use the Comparison Lemma to find  $c \in \mathbb{R}^+$  and  $N_1 \in \mathbb{N}$  such that

$$|\frac{1}{b_n} - \frac{1}{b}| < c|b_n - b|$$

We just have to find  $c$  and  $N_1$ .

*Proof.* Note that

$$|\frac{1}{b_n} - \frac{1}{b}| = |\frac{b - b_n}{bb_n}| = \frac{1}{|b||b_n|}|b_n - b|$$

We want  $\frac{1}{|b||b_n|}$  to be  $c$ , but this must be a single constant and not dependent on  $n$ . We want to find index  $N_1$  such that

$$|b_n| > \frac{|b|}{2} \quad \forall n \geq N_1$$

$$\frac{1}{|b_n|} < \frac{2}{|b|}$$

If we can find  $N_1$  then  $|\frac{1}{b_n} - \frac{1}{b}| \leq \frac{2}{|b|^2}|b_n - b|$  and the term  $\frac{2}{|b|^2}$  becomes our  $c$  and we can apply the Comparison Lemma, so we need  $N_1$  to make the above true. Let  $\epsilon = \frac{b}{2}$ . By definition of  $\{b_n\}$  converging to  $b$ , we can choose  $N_1$  such that  $|b_n - b| < \epsilon \quad \forall n \geq N_1$ .

$$|b_n - b| < \frac{|b|}{2}$$

$$-\epsilon < b_n - b < \epsilon$$

$$b - \epsilon < b_n < b + \epsilon$$

Check  $b > 0, b < 0$  since  $\epsilon = \frac{|b|}{2}$ . When  $b > 0, \epsilon = \frac{b}{2}$  so

$$b_n \in (b - \frac{b}{2}, b + \frac{b}{2}) = (\frac{b}{2}, \frac{3b}{2})$$

so  $b_n > \frac{b}{2}$ . When  $b < 0 \dots$  So  $|b_n| > \frac{|b|}{2}$  and this  $N_1$  works and apply the Comparison Lemma.  $\square$

**Theorem 2.17**

Let  $\lim_{n \rightarrow \infty} a_n = a$ ,  $\lim_{n \rightarrow \infty} b_n = b$ , and  $b_n \neq 0 \forall n$  and  $b \neq 0$  then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}$$

*Proof.*

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} a_n * \frac{1}{b_n} = \lim_{n \rightarrow \infty} a_n * \lim_{n \rightarrow \infty} \frac{1}{b_n} = \frac{a}{b}$$

□

**§2.2 Boundedness**

**Definition 2.18.** A sequence  $\{a_n\}$  is **bounded** if  $\exists M \in \mathbb{R}$  such that  $|a_n| \leq M \forall n$ .

**Theorem 2.19**

Every convergent sequence is bounded.

- If convergent  $\implies$  bounded.
- If it is unbounded, then it diverges.

*Proof.* Let  $\lim_{n \rightarrow \infty} a_n = a$  and take  $\epsilon = 1$ . Using the definition of convergence,  $\exists N \in \mathbb{N}$  s.t.

$$|a_n - a| < 1 \forall n \geq N$$

then  $|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a| \forall n \geq N$  by the Triangle Inequality and then the definition of the converging sequence. However, we need to show that this is true (bounded by a constant) for all  $n$ , not just for all  $n \geq N$ .

Define  $M = \max(1 + |a|, |a_1|, \dots, |a_{N-1}|)$ . Note that there the  $N - 1$  terms are finite and so a max exists. Then

$$|a_n| \leq M \forall n$$

and so  $\{a_n\}$  is bounded. □

**Remark 2.20.** Recall that a set  $S \subset \mathbb{R}$  is dense in  $\mathbb{R}$  if every open set  $(a, b) \in \mathbb{R}$  contains a point  $s \in S$ .

**Definition 2.21.** A set of numbers  $\{x_n\}$  is in a set  $S$  provided that  $x_n \in S \forall n$ .

**Lemma 2.22**

A set  $S$  is **dense** in  $\mathbb{R}$  if and only if every  $x \in \mathbb{R}$  is a limit of a sequence of a sequence in  $S$ .

*Proof.*

$\Rightarrow$  Let  $S \subset \mathbb{R}$  be dense in  $\mathbb{R}$ . Fix  $x \in \mathbb{R}$  and let  $n$  be an index. Since  $S$  is dense, there is an element in  $S$  in  $(x, x + \frac{1}{n})$ . For each  $n$ , this defines  $\{s_n\}$  with

$$s \in (x, x + \frac{1}{n})$$

$$x < s < x + \frac{1}{n}$$

$$|s_n - x| < \frac{1}{n} \quad \forall n$$

$$|s_n - x| < 1|\frac{1}{n} - 0|$$

Use the Comparison Lemma since  $\{\frac{1}{n}\}$  converges to 0. So,  $\{s_n\}$  converges to  $x$ .

$\Leftarrow$  Let  $S$  have the property that every number in  $\mathbb{R}$  is the limit of a sequence in  $S$ . We want to show that any open interval in  $\mathbb{R}$  contains a point  $s \in S$ . Consider an open interval  $(a, b) \in \mathbb{R}$ . Consider  $\frac{a+b}{2} = s \in \mathbb{R}$ . By assumption,  $\exists \{s_n\}$  of points in  $S$  s.t.  $\lim_{n \rightarrow \infty} s_n = s$ . Define  $\epsilon = \frac{b-a}{2} > 0$ . By definition of convergence,  $\exists N$  s.t.  $|s_n - s| < \epsilon \quad \forall n \in \mathbb{N}$ .

$$-\epsilon < s_n - s < \epsilon$$

$$s - \epsilon < s_n < s + \epsilon$$

$$\frac{a+b}{2} - \frac{b-a}{2} < s_n < \frac{a+b}{2} + \frac{b-a}{2}$$

$$a < s_n < b$$

The point  $s_N \in S$  and  $s_n \in (a, b)$  so  $S$  is dense in  $\mathbb{R}$ . □

**Definition 2.23.** The **sequential density of  $\mathbb{Q}$**  states that every  $\mathbb{R}$  is the limit of a sequence in  $\mathbb{Q}$ .

**Theorem 2.24**

Let  $\{c_n\} \in [a, b]$  and  $\lim_{n \rightarrow \infty} c_n = c$  then  $c \in [a, b]$  also.

**Definition 2.25.**  $S \subset \mathbb{R}$  is said to be **closed** (set) if  $\{a_n\}$  is a sequence in  $S$  that converges to  $a$ , then  $a \in S$  also.

**Example 2.26**

$(0, 1]$  not closed since  $\{\frac{1}{n} \in (0, 1]\}$  and  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$  but  $0 \notin (0, 1]$ .

**Example 2.27**

$\mathbb{Q}$  is not closed since we can find  $\{r_n\} \in \mathbb{Q}$  that converge to  $\pi$  but  $\pi \notin \mathbb{Q}$ .

**Definition 2.28.** A  $\{a_n\}$  is said to be **monotonically increasing (decreasing)** if  $a_{n+1} \geq (\leq) a_n \forall n$

**Note 2.29.** If a sequence is monotone, then it is either monotonically increasing or decreasing.

**Theorem 2.30**

**Monotone Convergence Theorem (MCT)** states that a monotone sequence converges if and only if it is bounded. Moreover, the bounded monotone  $\{a_n\}$  converges to the

1.  $\sup\{a_n \mid n \in \mathbb{N}\}$  if monotone increasing
2.  $\inf\{a_n \mid n \in \mathbb{N}\}$  if monotone decreasing

*Proof.*

$\Rightarrow$  Note that we already showed that convergent sequences are bounded.

$\Leftarrow$  We want to show that our sequence converges to either the  $\inf, \sup$  depending on if its either increasing or decreasing. Now assume that our sequence is bounded. Define  $S = \{a_n \mid n \in \mathbb{N}\}$  and  $S$  is bounded by assumption. Since  $S$  is nonempty and bounded above,  $S$  has  $\sup S = l$  by the Completeness Axiom. Claim  $\lim_{n \rightarrow \infty} a_n = l$ . Let  $\epsilon > 0$  be given, and we want to show the usual definition of convergence.

Note that

$$\begin{aligned} |a_n - l| &< \epsilon \\ -\epsilon &< a_n - l < \epsilon \\ l - \epsilon &< a_n < l + \epsilon \quad \forall n \geq N \end{aligned}$$

But  $l$  is an upper bound for  $S \Rightarrow a_n \leq l < l + \epsilon \forall n$ .

On the other hand, since  $l$  is the least upper bound for  $S$ ,  $l - \epsilon$  is not an upper bound for  $S$ . So,  $\exists N$  such that  $l - \epsilon < a_N$ .

Since  $a_n$  is monotonically increasing.  $l - \epsilon < a_N \leq a_n \forall n \geq N$ . Thus, we have  $N \in \mathbb{N}$  such that  $\forall n \geq N$  we have  $|a_n - l| < \epsilon$ , as desired.  $\square$

**Remark 2.31.** The formula for a finite geometric sum is  $S_n = \sum_{k=1}^n r^k$  where  $r \neq 1, r < 1$ .

$$S_n = \frac{r - r^{n+1}}{1 - r}$$

**Example 2.32**

Consider  $S_n = \sum_{k=1}^n \frac{1}{k} \cdot \frac{1}{2^k}$

$$k = 1 \implies s_1 = \frac{1}{2}$$

$$k = 2 \implies s_2 = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2^2} = \frac{5}{8}$$

$$k = 3 \implies \frac{1}{2} + \frac{1}{8} + \frac{1}{3} \cdot \frac{1}{2^3}$$

$$\vdots$$

so this sequence is monotonically increasing. Is it bounded

$$S_n = \sum_{k=1}^n \frac{1}{k} \cdot \frac{1}{2^k} \leq \sum_{k=1}^n \frac{1}{2^k} = \frac{\frac{1}{2} - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 1$$

**Theorem 2.33**

**The Nested Interval Theorem.** Suppose that  $I_n = [a_n, b_n]$  is a sequence of intervals, for which  $I_{n+1} \subset I_n \forall n$ . Then the intersection of those intervals is a nonempty closed interval

$$\cap_{i=1}^{\infty} I_n = [a, b]$$

where  $a = \sup a_n, b = \inf b_n$ . Furthermore, if  $\lim_{n \rightarrow \infty} a_n - b_n = 0$  then  $\cap_{i=1}^{\infty} I_n$  contains a single point.

*Proof.*

$\Leftarrow$  Let  $X \in \cap_{i=1}^{\infty} I_n$ . So for all  $n \in \mathbb{N}, x \in I_n$  by definition of intersection. Therefore,

$$a_n \leq x \leq b_n \forall n$$

Note that  $x$  is an upper bound for  $a_n$ . So, by definition of sup,  $a = \sup a_n \leq x$ .

$$a \leq x \leq b \implies x \in [a, b]$$

$\implies$  The reverse direction is similar. □

**§2.3 Sequential Compactness**

**Definition 2.34.** Consider a sequence  $\{a_n\}$  and let  $\{n_k\}$  be a sequence of  $\mathbb{N}$  that is strictly increasing. Then the sequence  $\{b_k\}$  defined by  $b_k = a_{n_k} \forall k$  is a **subsequence**.

**Note 2.35.** Note that a sequence may not converge, but it may be possible to find a subsequence that does.



**Theorem 2.36**

Let  $\{a_n\}$  converges to  $a$ . Then every subsequence of  $\{a_n\}$  also converges to the same limit  $a$ .

**Theorem 2.37**

Every sequence (does not need to converge) has a monotone increasing or decreasing subsequence.

*Proof.* Consider  $\{a_n\}$ . We all an index a **peak index** for  $\{a_n\}$  if

$$a_n \leq a_m \quad \forall n \geq m$$

You can have two cases: infinitely many peak indices, or a finite number of peak indices.

Suppose there are finite number of peak indices. Then we choose  $N$  such that there are no more peak indices. Since  $N$  is not a peak index,  $\exists n_1 \in \mathbb{N}$  such that  $n_1 > N$  with  $a_N \leq a_{n_1}$

$$\vdots$$

Continue for  $n_k \implies \exists n_{k+1} \in \mathbb{N}$  with  $n_{k+1} \geq n_k$  with  $a_{n_k} \leq a_{n_{k+1}}$

$$a_N \leq a_{n_1} \leq \cdots \leq a_{n_k} \leq a_{n_{k+1}}$$

which is a monotonically increasing subsequence.

In the case of infinitely many peak indices,  $m_1 < m_2 < m_3 < \cdots < \text{peak indices}$ . Since  $m_1$  is a peak index. Then  $m_1 < m_2 \implies a_{m_1} > a_{m_2}$ .

$$\vdots$$

We'll get a monotonically decreasing subsequence. □

**Theorem 2.38**

Every bounded sequence has a convergent subsequence.

*Proof.* Let  $\{a_n\}$  be bounded. By the previous theorem,  $\{a_n\}$  has a monotone subsequence. Since  $\{a_n\}$  is bounded,  $\{a_{n_k}\}$  is bounded also. By MCT,  $\{a_{n_k}\}$  converges since it is monotone and bounded. □

**Definition 2.39.** A  $S \subset \mathbb{R}$  is said to be **compact (or sequentially compact)** if every sequence in  $S$  has a convergent subsequence converging to a point in  $S$ . For a set to not be compact, we find a sequence in  $S$  that has no convergence subsequence that converges to a point in  $S$ .

**Example 2.40**

$[1, \infty)$  is not compact. Consider  $a_n = n, a_n \rightarrow \infty$  by Archimedian Principle. Then every subsequence of  $n_k$  also diverges to  $\infty$ . Thus,  $\{a_n\}$  has no subsequence that converges.

**Example 2.41**

$(0, 1]$  is not compact. Let  $a_n = \frac{1}{n}, a_n \rightarrow 0, n \rightarrow \infty$ , so every subsequence converges to 0 also. But  $0 \notin (0, 1]$  so it is not compact.

**Theorem 2.42**

**The Sequentially Compactness Theorem (SCT)** states that every interval  $[a, b]$  such that  $a, b \in \mathbb{R}$  is sequentially compact.

*Proof.* Let  $\{a_n\}$  be in  $[a, b]$ . So,  $a \leq a_n \leq b \forall n$ . By a previous theorem, since  $\{a_n\}$  is bounded, there exists a convergent subsequence  $\{a_{n_k}\}$ . Assume  $\{a_{n_k}\} \rightarrow l$ . Since  $a \leq a_n \leq b \forall n$ , then

$$a \leq a_{n_k} \leq b \forall n$$

so  $l \in [a, b]$  as desired. Therefore,  $\{a_n\}$  has a convergent subsequence whose limit is in the interval  $[a, b]$ , so it is sequentially compact.  $\square$

**Theorem 2.43**

Bolzano Weirstrass Theorem: If  $S \subset \mathbb{R}$ , the following are equivalent

$$S \text{ is closed and bounded} \iff S \text{ is compact}$$

## §3 Continuous Functions

### §3.1 Continuity Basics

**Note 3.1.** Before  $f : \mathbb{N} \rightarrow \mathbb{R}$  but now  $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ .  $f(x)$  is the value the function assigns to  $x$ .

**Definition 3.2.** A function  $f : D \rightarrow \mathbb{R}$  is said to be **continuous at a point**  $x_0$  if whenever  $\{x_n\}_{n=1}^{\infty}$  converges to  $x_0 \in D$ , the image sequence  $\{f(x_n)\}_{n=1}^{\infty}$  converges to  $f(x_0)$ .

**Definition 3.3.** A function  $f : D \rightarrow \mathbb{R}$  is **continuous** if  $f$  is continuous at every point in  $D$ .

**Example 3.4**

Consider  $f(x) = x^2 + 7x - 3$ . We want to show  $f$  is continuous. Select  $x_0 \in \mathbb{R}$  and let  $\{x_n\} \rightarrow x_0 \implies \lim_{n \rightarrow \infty} x_n = x_0$ . We want to show that

$$\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$$

Note that

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_n^2 + 7x_n - 3$$

by definition of  $f$ .

$$= \lim_{n \rightarrow \infty} x_n^2 + 7 \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} 3$$

by properties of sequences.

$$= x_0^2 + 7x_0 - 3 = f(x_0)$$

by the definition of  $f$

**Remark 3.5.** Given  $f : D \rightarrow \mathbb{R}, g : D \rightarrow \mathbb{R}$  are continuous, then

$$f \pm g, fg, \frac{f}{g} (g \neq 0)$$

are continuous and this follows directly from convergent sequence properties from the previous chapter. Thus, polynomials are continuous.

**Example 3.6**

Consider Dirichlet's function  $f : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Note that  $f$  is defined on  $\mathbb{R}$  but it is discontinuous at  $x_0 \in \mathbb{R}$ .

*Proof.* Let  $x_0 \in \mathbb{R}$ . By sequential density of the  $\mathbb{Q}$  and  $\mathbb{Q}^c$ , we can find

$$\{u_n\} \rightarrow x_0, u_n \in \mathbb{Q} \forall n$$

$$\{v_n\} \rightarrow x_0, v_n \in \mathbb{Q}^c \forall n$$

Since  $f(u_n) = 1 \forall n$  and  $f(v_n) = 0 \forall n$ , then

$$\{f(u_n)\} \rightarrow 1 \text{ but } \{f(v_n)\} \rightarrow 0$$

Therefore,  $\lim_{n \rightarrow \infty} f(u_n) = 1 \neq 0 = \lim_{n \rightarrow \infty} f(v_n)$  but  $\{u_n\} \rightarrow x_0$  and  $\{v_n\} \rightarrow x_0$  but we cannot have 2 function values for  $x_0$ .  $\square$

**Definition 3.7.** Suppose  $f : D \rightarrow \mathbb{R}$  and  $g : U \rightarrow \mathbb{R}$  such that  $f(D) \subset U$  then we define

$$(g \circ f)(x) = g(f(x)) \quad \forall x$$

### Theorem 3.8

Let  $f : D \rightarrow \mathbb{R}, g : U \rightarrow \mathbb{R}$  and  $f(D) \subset U$ . Let  $f$  be continuous at  $x_0$  and  $g$  be continuous at  $f(x_0)$ . Then  $(g \circ f) : D \rightarrow \mathbb{R}$  is continuous at  $x_0$ .

*Proof.* Suppose  $\{x_n\} \in D$  converges to  $x_0$ . Since  $f$  is continuous, then  $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$ .

$$\{f(x_n)\} \xrightarrow{n \rightarrow \infty} f(x_0)$$

Since  $g$  is continuous at  $f(x_0)$ , then  $\lim_{n \rightarrow \infty} g(f(x_n)) = g(f(x_0))$ . Therefore,  $(g \circ f)(x)$  is continuous at  $x_0$  since

$$\{g(f(x_n))\} \xrightarrow{n \rightarrow \infty} g(f(x_0))$$

$\implies$  we can combine continuous functions and remain continuous □

## §3.2 Extreme Value Theorem

**Definition 3.9.**  $f : D \rightarrow \mathbb{R}$  attains a **maximum (minimum)** value if there is

$$x_0 \in D \text{ s.t. } f(x_0) \geq (\leq) f(x) \quad \forall x \in D$$

**Remark 3.10.** Recall that a nonempty set has a maximum if it is bounded above and contains its supremum ie. the supremum is in the set.

$\implies$  Now  $f : D \rightarrow \mathbb{R}$  has a maximum when the image  $f(D)$  is bounded above and the supremum of the image is a functional value.

### Example 3.11

$f : (0, 1) \rightarrow \mathbb{R}$  where  $f(x) = 2x$ . Note that the supremum of the image is 2, but 2 is not a functional value. Therefore, this function does not have a max.

**Theorem 3.12**

The **Extreme Value Theorem** states that a continuous function on a closed and bounded interval  $f : [a, b] \rightarrow \mathbb{R}$  attains both a maximum and a minimum. Sketch: Note that we want to show that  $f(D)$  is bounded above. See lemma below. After that, we need to show that the supremum is a functional value.

**Lemma 3.13**

Assume on the contrary that given  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, assume there is no  $M$  such that

$$f(x) \leq M \quad \forall x \in [a, b]$$

There is  $x \in [a, b]$  at which  $f(x) > n$ ,  $\forall n$ . For each  $n$  this creates a sequence  $\{x_n\}$  in  $[a, b]$  with  $f(x) > n \quad \forall n$ .  $\{x_n\}$  may or may not converge. By Sequential Compactness Theorem, choose  $\{x_{n_k}\}$  subsequence that converges to  $x_0 \in [a, b]$ . Since  $f$  is continuous at  $x_0$ ,  $\{f(x_{n_k})\} \rightarrow f(x_0)$ , but every convergent sequence is bounded by a theorem, so  $\{f(x_{n_k})\}$  is bounded. Therefore, we have a contradiction since  $f(x_{n_k}) > n_k \geq k \quad \forall k \in \mathbb{N}$ . So  $f : [a, b] \rightarrow \mathbb{R}$  is bounded above.

*Proof.* Define  $S = f([a, b])$ , all of the image values. By the lemma above,  $S$  is bounded. Note  $S$  is nonempty and bounded, thus by the Completeness Axiom,  $c := \sup(S)$  exists. Note that we want to find  $x_0 \in [a, b]$  such that  $f(x_0) = c$ , as this would show that the supremum is a functional value. Consider

$$c - \frac{1}{n} < c \quad \forall n$$

Note that  $c - \frac{1}{n}$  is not an upper bound since  $c$  is the least upper bound. So, we can find a point  $x \in [a, b]$  such that

$$c - \frac{1}{n} < f(x) < c$$

Label point  $x_n$  to create a sequence  $\{x_n\}$

$$c - \frac{1}{n} < f(x_n) < c \quad \forall n$$

Since  $\{\frac{1}{n}\} \rightarrow 0$  as  $n \rightarrow \infty$ , then  $\{f(x_n)\} \rightarrow c$  by the Squeeze Theorem, as desired. Note by the Sequential Compactness Theorem, there exists a subsequence  $\{x_{n_k}\}$  that converges to  $x_0$ . Since  $f$  is continuous at  $x_0$ , then  $\{f(x_{n_k})\} \rightarrow f(x_0)$ . Recall that  $\{f(x_{n_k})\}$  is a subsequence of  $\{f(x_n)\}$  that converges to  $c$ , and any subsequence must also converge to the same value as the full sequence. Therefore,  $f(x_0) = c$ . Therefore, the supremum exists and is a functional value, so we attain a max at  $x_0$ .  $\square$

### §3.3 Intermediate Value Theorem

#### Theorem 3.14

**The Intermediate Value Theorem** state that suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, let  $c \in \mathbb{R}$  between  $f(a)$  and  $f(b)$ . Then there exists  $x_0 \in (a, b)$  such that  $f(x_0) = c$ .

*Proof.* Without loss of generality, suppose  $f(a) < c < f(b)$ . Recursively define a sequence of nested intervals starting at  $[a, b]$  and converging to  $x_0 \in (a, b)$  with  $f(x) = c$ . We WTS  $f(x_0) = c$  by letting  $a_1 = a, b_1 = b \forall n$ .

$\forall n$  define  $[a_n, b_n]$  by considering the midpoint  $m_n = \frac{a_n + b_n}{2}$ . Let us consider some cases.

$\Rightarrow$  If  $f(m_n) \leq c$ , define  $a_{n+1} = m_n$  and  $b_{n+1} = b_n$ .

$\Leftarrow$  If  $f(m_n) > c$ , define  $a_{n+1} = a_n$  and  $b_{n+1} = m_n$ .

Note that  $a \leq a_n \leq a_{n+1} < b_{n+1} < b_n \leq b$  and  $f(a_{n+1}) \leq c$  and  $f(b_{n+1}) > c$  by definition. Now, we want to show that

$$\lim_{n \rightarrow \infty} (b_n - a_n) = 0$$

in order to apply the Nested Interval Theorem.

$\vdots$

So  $b_n - a_n = \frac{b-a}{2^{n-1}} \forall n \xrightarrow{n \rightarrow \infty} 0$ . So  $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$ . Thus by Nested Interval Theorem,  $\exists x_0 \in (a, b)$  where  $\{a_n\} \rightarrow x_0$  and  $\{b_n\} \rightarrow x_0$ . Since  $f$  is continuous at  $x_0$ , then  $\{f(a_n)\} \rightarrow f(x_0)$  and  $\{f(b_n)\} \rightarrow f(x_0)$ . Since  $f(a_n) \leq c \forall n \Rightarrow f(x_0) \leq c$  and  $f(b_n) \geq c \forall n \Rightarrow f(x_0) \geq c$ . Thus, the only this is true is  $f(x_0) = c$ , as desired.  $\square$

#### Example 3.15

Suppose we have  $h(x) = x^5 + x + 1 = 0$ .  $h(x)$  is a polynomial so it is continuous. Verify that a solution exists. Let us test some points.

$$h(0) = 0 + 0 + 1 = 1$$

$$h(1) = 1 + 1 + 1 = 3$$

$$h(-1) = -1 - 1 + 1 = -1$$

By the Intermediate Value Theorem, there exists  $x_0 \in (-1, 0)$  such that  $x_0^5 + x_0 + 1 = 0$ .

**Example 3.16**

$x^2 = c, c > 0$ . Verify that a solution exists.

*Proof.* Consider  $f : [0, c + 1] \rightarrow \mathbb{R}$ .  $f(x) = x^2, 0 \leq x \leq c + 1$ . Testing points

$$f(0) = 0^2 = 0 < c$$

$$f(c + 1) = c^2 + 2c + 1 > c$$

Since  $x^2$  it is continuous. By IVT, there exists  $x_0 \in (0, c + 1)$  such that  $x_0^2 = c$ .  $\square$

**§3.4 Uniform Continuity**

**Definition 3.17.** A function  $f : D \rightarrow \mathbb{R}$  is said to be **uniformly continuous** if for  $\{u_n\}$  and  $\{v_n\}$  in  $D$  with  $\lim_{n \rightarrow \infty} u_n - v_n = 0$  then  $\lim_{n \rightarrow \infty} f(u_n) - f(v_n) = 0$ .

**Note 3.18.** It doesn't make sense to say  $f$  is uniformly continuous at a singular point. Further note that there is no requirement for  $\{u_n\}$  and  $\{v_n\}$  to converge.

**Remark 3.19.** Uniform continuity is on an interval.

**Example 3.20**

$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 3x$  is uniformly continuous.

*Proof.* Let  $\{u_n\}$  and  $\{v_n\}$  be in  $\mathbb{R}$  and  $\{u_n - v_n\} \rightarrow 0$ . Then

$$\{f(u_n) - f(v_n)\} = \{3u_n - 3v_n\} = \{3(u_n - v_n)\} \rightarrow 3 * 0$$

as needed.  $\square$

**Example 3.21**

$f(x) = x^2$  is not uniformly continuous on  $f : \mathbb{R} \rightarrow \mathbb{R}$ . To do this, we must find a pair of sequences that doesn't work.

*Proof.* Let  $\{u_n\} = \{n + \frac{1}{n}\}$  and  $\{v_n\} = \{n\}$ . Note that  $\{u_n - v_n\} \rightarrow 0$  but

$$\{f(u_n) - f(v_n)\} = \{f(n + \frac{1}{n}) - f(n)\} = \{(n + \frac{1}{n})^2 - n^2\} = \{2 + \frac{1}{n^2}\} \rightarrow 2 \neq 0$$

Therefore,  $f$  is not uniformly continuous on  $\mathbb{R}$ .  $\square$

**Example 3.22**

Consider  $f : (0, 2) \rightarrow \mathbb{R}$  and  $f(x) = \frac{1}{x}$ . This is not uniformly continuous since there is a vertical asymptote at  $x = 0$ .

*Proof.* Let  $\{u_n\} = \frac{1}{n}$  and  $\{v_n\} = \frac{2}{n}$ . Note that  $\{u_n - v_n\} \rightarrow 0$  but

$$\{f(u_n) - f(v_n)\} = \left\{f\left(\frac{1}{n}\right) - f\left(\frac{2}{n}\right)\right\} = \left\{n - \frac{n}{2}\right\} = \left\{\frac{n}{2}\right\} \rightarrow \infty$$

□

But now consider  $f : (2, 3) \rightarrow \mathbb{R}$ ,  $f(x) = \frac{1}{x}$ . This is uniformly continuous.

*Proof.* Suppose  $\{u_n - v_n\} \rightarrow 0$  for  $\{u_n\}$  and  $\{v_n\}$  in  $(2, 3)$ .

$$|f(u_n) - f(v_n)| = \left| \frac{1}{u_n} - \frac{1}{v_n} \right| = \left| \frac{u_n - v_n}{u_n v_n} \right|$$

We need to bound the product  $u_n v_n$ . Note that  $u_n > 2, v_n > 2 \implies \frac{1}{u_n} < \frac{1}{2}, \frac{1}{v_n} < \frac{1}{2}$ , so

$$< \frac{|u_n - v_n|}{2 * 2}$$

so  $|f(u_n) - f(v_n)| \leq \frac{1}{4}|u_n - v_n|$  and so by Comparison Lemma,  $\{f(u_n) - f(v_n)\} \rightarrow 0$ . Note that this would work for domains  $(0.00000001, \infty)$ . □

**Note 3.23.** If  $f : D \rightarrow \mathbb{R}$  is uniformly continuous then it is continuous, but not all continuous functions are uniformly continuous. The go-to example is  $f(x) = x^2$  on  $\mathbb{R}$ .



**Theorem 3.24**

Every continuous function on a closed bounded interval  $f : [a, b] \rightarrow \mathbb{R}$  is uniformly continuous.

*Proof.* Let  $\{u_n\}, \{v_n\} \subset [a, b]$  with  $\lim_{n \rightarrow \infty} (u_n - v_n) = 0$ . We WTS that  $\lim_{n \rightarrow \infty} (f(u_n) - f(v_n)) = 0$ . By contradiction, assume that  $\{f(u_n) - f(v_n)\} \not\rightarrow 0$ . Therefore,

$$\exists \epsilon > 0 \text{ s.t. } \forall N \in \mathbb{N}, \exists n \geq N$$

with

$$|f(u_n) - f(v_n)| \geq \epsilon$$

Let us create a subsequence

$$n_1 \geq N = 1 \text{ s.t. } |f(u_{n_1}) - f(v_{n_1})| \geq \epsilon$$

$$n_2 \geq n_1 + 1 \cdots |f(u_{n_2}) - f(v_{n_2})| \geq \epsilon$$

$$n_3 \geq n_2 + 1 \cdots |f(u_{n_3}) - f(v_{n_3})| \geq \epsilon$$

So  $\{f(u_{n_k}) - f(v_{n_k})\}$  is a subsequence with  $\{f(u_{n_k}) - f(v_{n_k})\} \geq \epsilon \forall n_k$ . Because  $\{u_n\}$  is a sequence in  $[a, b]$ , we can use Sequential Compactness to find a subsequence  $\{u_{m_k}\}$ . Since  $f$  is continuous, then  $\lim_{n \rightarrow \infty} f(u_{m_k}) = f(x_0)$ . Since  $\lim_{k \rightarrow \infty} (u_n - v_n) = 0 \implies \lim_{k \rightarrow \infty} (u_{m_k} - v_{m_k}) = 0$  by a theorem. Thus,

$$\lim_{k \rightarrow \infty} v_{m_k} = \lim_{k \rightarrow \infty} u_{m_k} - \lim_{k \rightarrow \infty} (u_{m_k} - v_{m_k}) = x_0 - 0 \implies v_{m_k} \rightarrow x_0$$

Therefore,

$$\lim_{k \rightarrow \infty} (f(u_{m_k}) - f(v_{m_k})) = f(x_0) - f(x_0) = 0$$

But this is a contradiction

$$\forall n_k, |f(u_{n_k}) - f(v_{n_k})|$$

□

**§3.5 Epsilon-Delta Criterion**

**Definition 3.25.** A function  $f : D \rightarrow \mathbb{R}$  is said to satisfy the  $\epsilon - \delta$  criterion at  $x_0 \in D$  if  $\forall \epsilon > 0, \exists \delta > 0$  so that

$$\forall x \in D \text{ and } |x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$$

**Note 3.26.**  $\delta$  depends on  $\epsilon$  and maybe  $x_0$ . For uniform continuity, however,  $\delta$  cannot depend on location, so  $\delta$  will not depend on  $x_0$  in the case of uniform continuity.

**Example 3.27**

$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = 3x$ . Prove it satisfies  $\epsilon - \delta$  criteria at  $x_0 = 2$ .

*Sketch.* Given  $|x - 2| < \delta$ . How do we show that  $|f(x) - f(2)| < \epsilon$ .

$$|3x - 6| \implies |3(x - 2)| < 3\delta$$

so we take  $\delta = \frac{\epsilon}{3}$ . □

*Proof.* Let  $\epsilon > 0$  be given. Let  $x_0 = 2$  and let  $\delta = \frac{\epsilon}{3}$ . Then if  $|x - 2| < \delta$  then

$$|f(x) - f(x_0)| = |3x - 6| = 3|x - 2| < 3\delta = 3 * \frac{\epsilon}{3} = \epsilon$$

□

**Example 3.28**

$f : \mathbb{R} \rightarrow \mathbb{R}, f(x) = x^2$  at any  $x_0$ . Show  $\epsilon - \delta$  criterion.

*Sketch.*  $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$

$$|x^2 - x_0^2| = |x - x_0||x + x_0| \leq \delta|x + x_0|$$

Note the absolute value term is constant, but  $x$  could be large, so we need to bound it. Let  $\delta \leq 1$ . What happens to  $|x + x_0|$  in this case, let's try and relate it to  $|x - x_0|$ .

$$\begin{aligned} |x + x_0| &= |x - x_0 + x_0 + x_0| \leq |x - x_0| + |x_0 + x_0| = |x - x_0| + 2|x_0| \\ &\leq \delta + 2|x_0| \leq 1 + 2|x_0| \end{aligned}$$

which is a constant as desired.  $\square$

*Proof.* Let  $\epsilon > 0$  and  $x_0 \in \mathbb{R}$ . Let  $\delta = \min(1, \frac{\epsilon}{1+2|x_0|})$ . Note that  $\epsilon > 0$  and  $1 + 2|x_0| > 0$  and so we confirm  $\delta > 0$ . Thus,

$$\delta \leq 1 \text{ and } \delta \leq \frac{\epsilon}{1 + 2|x_0|}$$

Then

$$|x + x_0| = |x - x_0 + x_0 + x_0| \leq |x - x_0| + 2|x_0| \leq \delta + 2|x_0| \leq 1 + 2|x_0|$$

since  $|x - x_0| < \delta$ . Thus,

$$|f(x) - f(x_0)| = |x^2 - x_0^2| = |(x - x_0)(x + x_0)| < \delta|x + x_0| \leq \delta(1 + 2|x_0|)$$

Recall that  $\delta \leq \frac{\epsilon}{1+2|x_0|}$  and so

$$\delta(1 + 2|x_0|) \leq \frac{\epsilon}{1 + 2|x_0|}(1 + 2|x_0|) = \epsilon \implies |f(x) - f(x_0)| < \epsilon$$

$\square$

**Theorem 3.29**

Given  $f : D \rightarrow \mathbb{R}, x_0 \in D$ ,  $f$  is continuous at  $x_0$  iff  $f$  satisfies the  $\epsilon - \delta$  criteria at  $x_0$ .

**Definition 3.30.** We say  $f : D \rightarrow \mathbb{R}$  satisfies the  $\epsilon - \delta$  criterion **on**  $D$  if

$$\forall \epsilon > 0 \exists \delta > 0 \text{ s.t. } \forall u, v \in D, \text{ if } |u - v| < \delta \implies |f(u) - f(v)| < \epsilon$$

Note here that  $\delta$  can only depend on  $\epsilon$ .

**Theorem 3.31**

Given  $f : D \rightarrow \mathbb{R}$ ,  $f$  is uniformly continuous on  $D$  iff  $f$  satisfies  $\epsilon - \delta$  criteria on  $D$ .

**§3.6 Images, Inverses, Monotone Functions**

**Definition 3.32.**  $f : D \rightarrow \mathbb{R}$  is called **monotonically increasing (decreasing)** if

$$\forall u, v \in D, u < v \implies f(u) \leq (\geq) f(v)$$

If "strictly", then the operators become  $<$  and  $>$  respectively.

**Definition 3.33.**  $f : D \rightarrow \mathbb{R}$  is called **one-to-one (1-1)** when  $f(u) = f(v) \implies u = v$ .

**Definition 3.34.** When  $f$  is 1-1, its inverse, denoted  $f^{-1}(x)$  is a function from  $f(D)$  to  $D$  satisfying  $f(x) = y \leftrightarrow f^{-1}(y) = x$

- $f^{-1}(f(x)) = x \forall x \in D$
- $f(f^{-1}(y)) = y \forall y \in f(D)$

**Theorem 3.35**

Any strictly monotone function  $f : D \rightarrow \mathbb{R}$  is 1-1 and thus has an inverse.

*Proof.* WLOG, suppose  $f$  is strictly increasing and  $f(u) = f(v)$ . To show 1-1, we WTS  $u = v$  for  $u, v \in D$ . By contradiction, if  $u < v$ , since  $f$  is strictly monotone increasing, then  $f(u) < f(v)$ . This contradicts  $f(u) = f(v)$ . The direction  $u > v$  is very similar.  $\square$

**Example 3.36**

Prove that the inverse of  $f(x) = x^3$  is continuous.

*Proof.* Note that  $f$  is a polynomial and thus continuous.  $f$  is strictly increasing.

$$u < v \implies u^3 < v^3 = u * u * u < v * v * v$$

by properties of inequalities. By a previous theorem, since  $f$  is strictly increasing,  $f$  has an inverse. Let  $x_0 \in \mathbb{R}$ , let  $\{x_n\} \in \mathbb{R}$  such that  $\{x_n\} \rightarrow x_0$ . We WTS that  $f^{-1}(x_n) \rightarrow f^{-1}(x_0)$ .

For notation: label  $y_n = f^{-1}(x_n), y_0 = f^{-1}(x_0) \implies y_n \rightarrow y_0$ . Therefore

$$x_n = f(y_n) = y_n^3$$

$$x_0 = f(y_0) = y_0^3$$

Since  $x_n \rightarrow x_0$ , then  $y_n^3 \rightarrow y_0^3$ . We WTS  $y_n^3 \rightarrow y_0^3$ . Let  $\epsilon > 0$ . Let  $\delta = \min((y_0 + \epsilon)^3 - (y_0)^3, y_0^3 - (y_0 - \epsilon)^3)$ . Since  $\epsilon > 0$ , it is easy to show that  $\delta > 0$ . Since

$$y_n^3 \rightarrow y_0^3, \exists N \text{ s.t. } \forall n \geq N, |y_n^3 - y_0^3| < \delta$$

We know this is true for all  $\epsilon$ , so therefore we can let  $\epsilon = \delta$ .

$$-\delta < y_n^3 - y_0^3 < \delta$$

$$y_0^3 - \delta < y_n^3 < \delta + y_0^3$$

$$y_0^3 - (y_0^3 - (y_0 - \epsilon)^3) < y_n^3 < (y_0 + \epsilon)^3 - y_0^3 + y_0^3$$

$$(y_0 - \epsilon)^3 < y_n^3 < (y_0 + \epsilon)^3$$

$$y_0 - \epsilon < y_n < y_0 + \epsilon$$

$$|y_n - y_0| < \epsilon$$

and so  $y_n \rightarrow y$  or  $f^{-1}(x_n) \rightarrow f^{-1}(x_0)$  and so  $f^{-1}(x)$  is continuous.  $\square$

**Theorem 3.37**

Let  $f : D \rightarrow \mathbb{R}$  is monotone. If its image is an interval, then  $f$  is continuous.

*Proof.* Let  $x_0 \in D$  and  $\{x_n\} \in D$  with  $x_n \rightarrow x_0$ . Suppose on the contrary that  $f(x_n) \not\rightarrow f(x_0)$ . Then  $\exists \epsilon > 0$  and subsequence of  $x_n$  such that

$$|f(x_{n_k}) - f(x_0)| \geq \epsilon$$

Assume WLOG that  $f$  is increasing.

Case 1: If the absolute value is positive

$$f(x_{n_k}) - f(x_0) \geq \epsilon$$

$$f(x_{n_k}) \geq \epsilon + f(x_0)$$

$$f(x_{n_k}) \geq \epsilon + f(x_0) > \frac{\epsilon}{2} + f(x_0) > f(x_0)$$

Since the image of  $f$  is an interval (all points in between). So  $\exists c \in D$  such that

$$f(c) = f(x_0) + \frac{\epsilon}{2}$$

$$f(x_{n_k}) > f(c) > f(x_0)$$

And since  $f$  is strictly monotone increasing, so  $x_{n_k} > c > x_0$

$$|x_{n_k} - x_0| > |c - x_0| > 0$$

Note that  $c - x_0$  is a constant.

Case 2: If the absolute value is negative

$$f(x_0) - f(x_{n_k}) \geq \epsilon$$

$$f(x_0) - \epsilon \geq f(x_{n_k})$$

$$f(x_0) > f(x_0) - \frac{\epsilon}{2} > f(x_{n_k})$$

$\vdots$

□

**Theorem 3.38**

Suppose  $I$  is an interval and  $f : I \rightarrow \mathbb{R}$  is monotone. Then  $f$  is continuous if and only if its image is an interval.

**Theorem 3.39**

Let  $f : I \rightarrow \mathbb{R}$ ,  $I$  is an interval, be strictly monotone. Then its inverse  $f^{-1} : f(I) \rightarrow \mathbb{R}$  is continuous. Similar to the  $x^3$  example above.

**Example 3.40**

$f : [0, \infty) \rightarrow \mathbb{R}$  with  $f(x) = x^n$  is strictly increasing, so inverse is continuous.  
Notation: negative integer  $n$ :  $x^n = \frac{1}{x^{-n}}$

$$\bullet x^n * x^m = x^{n+m}$$

$$\bullet (x^n)^m = x^{nm}$$

$$y^{\frac{1}{n}} = f^{-1}(y) \quad \forall y \geq 0.$$

**Definition 3.41.** For  $x > 0$  and  $r \in \mathbb{Q}$  with  $r = \frac{m}{n}$ ,  $m \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ , we define

$$x^r = (x^m)^{\frac{1}{n}}$$

**Remark 3.42.** Let  $r \in \mathbb{Q}$  and define  $f(x) = x^r \quad \forall x \geq 0$ . Then  $f : [0, \infty) \rightarrow \mathbb{R}$  is continuous.

**§3.7 Limits**

**Note 3.43.** Note that before  $\lim_{n \rightarrow \infty} a_n = a$  for sequences but now  $\lim_{x \rightarrow a} f(x) = L$

**Definition 3.44.** We say  $x_0 \in \mathbb{R}$  is a **limit point** of  $D$  if  $\exists \{x_n\} \in (D - \{x_0\})$

**Example 3.45**

For  $(0, 1)$ , the numbers 0 and 1 are limit points.

**Definition 3.46.** Given  $f : D \rightarrow \mathbb{R}$  and limit point  $x_0$ , we write

$$\lim_{x \rightarrow x_0} f(x) = l$$

if whenever  $\{x_n\} \in (D - \{x_0\})$  with  $x_n \rightarrow x_0$  has  $\lim_{n \rightarrow \infty} f(x_n) = l$

**Remark 3.47.** A function is continuous at  $x_0$  if and only if  $\lim_{x \rightarrow x_0} f(x) = f(x_0)$

**Example 3.48**

$\lim_{x \rightarrow 2} \sqrt{\frac{3x+3}{x^3-4}}$ . Note that there are no denominator issues at  $x = 2$ .

*Solution.* Note that numerator and denominator are both continuous, and so the quotient continuous as well because the denominator is also not 0. Further note that  $\sqrt{x}$  is continuous because it is the inverse of a strictly monotone function. Compositions of continuous functions are continuous at  $x = 2$ . So

$$\lim_{x \rightarrow 2} \sqrt{\frac{3x+3}{x^3-4}} = \sqrt{\frac{3(2)+3}{2^3-4}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

□

**Example 3.49**

$\lim_{x \rightarrow 1} \frac{x^2-1}{x-1}$ .

Note that we cannot use the quotient property like above. Let  $\{x_n\} \rightarrow 1$  with  $x_n \neq 1$ .

$$\frac{x_n^2 - 1}{x_n - 1} = \frac{(x_n - 1)(x_n + 1)}{x_n - 1} = x_n + 1$$

So therefore

$$\lim_{n \rightarrow 1} \frac{x^2 - 1}{x - 1} = \lim_{n \rightarrow \infty} \frac{x_n^2 - 1}{x_n - 1} = \lim_{n \rightarrow \infty} x_n + 1 = 1 + 1 = 2$$



**Theorem 3.50**

$f : D \rightarrow \mathbb{R}$  and  $g : D \rightarrow \mathbb{R}$ ,  $x_0 \in \mathbb{R}$  is a limit point. Let  $\lim_{x \rightarrow x_0} f(x) = A$  and  $\lim_{x \rightarrow x_0} g(x) = B$  and  $c \in \mathbb{R}$ . Then

- i.  $\lim_{x \rightarrow x_0} (f(x) \pm g(x)) = A \pm B$
- ii.  $\lim_{x \rightarrow x_0} (f(x)g(x)) = A \cdot B$
- iii.  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = \frac{A}{B}, g(x) \neq 0, B \neq 0$
- iv.  $\lim_{x \rightarrow x_0} cf(x) = cA$

These follow directly from properties of compositions. Similarly for compositions.  $f : D \rightarrow \mathbb{R}, g : U \rightarrow \mathbb{R}, x_0$  is a limit point with  $\lim_{x \rightarrow x_0} f(x) = y_0$  and  $\lim_{y \rightarrow y_0} g(y) = l$  and  $f(D - \{x_0\}) \subset U - \{y_0\}$ . Then

$$\lim_{x \rightarrow x_0} (g \circ f)(x) = l$$

We will see limits later on in Differentiation.

## §4 Differentiation

**Remark 4.1.** High level: to find the tangent line, take a sequence of secant lines closer and closer towards  $x$

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = m_0 \quad (\text{slope})$$

**Definition 4.2.** For  $x_0 \in \mathbb{R}$ , the open interval  $I = (a, b)$  that contains  $x_0$  is called a **neighborhood** of  $x_0$ .

**Definition 4.3.**  $f : I \rightarrow \mathbb{R}$  is said to be **differentiable at  $x_0$**  if

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} := f'(x_0)$$

exists and we denote it by  $f'(x_0)$ , the **derivative** of  $f$  at  $x_0$ .

**Remark 4.4.** If  $f$  is differentiable at every point in  $I$ ,  $f$  is **differentiable** and  $f' : I \rightarrow \mathbb{R}$  is called the **derivative**.

**Example 4.5**

$f(x) = mx + b$ . Find  $f'$ .

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{mx + b - mx_0 - b}{x - x_0} = m$$

**Example 4.6**

$f(x) = x^2$ . Find  $f'$ .

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \rightarrow x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \rightarrow x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} = 2x_0$$

**Note 4.7.**

$$(x^2 - x_0^2) = (x - x_0)(x + x_0)$$

$$(x^3 - x_0^3) = (x - x_0)(x^2 + xx_0 + x_0^2)$$

$$(x^4 - x_0^4) = (x - x_0)(x^3 + x^2x_0 + xx_0^2 + x_0^3)$$

Notice the pattern. Binomial Expansion. Note that you can prove this general pattern using induction.

**Example 4.8**

$f(x) = x^n, n \in \mathbb{N}$ . Find  $f'$ . **Power Rule.**

$$f'(x_0) = \lim_{x \rightarrow x_0} \frac{x^n - x_0^n}{x - x_0} = \lim_{x \rightarrow x_0} \frac{(x - x_0)(\cdots)}{x - x_0}$$

where  $x \neq x_0$ .

$$\begin{aligned} &= x_0^{n-1} + x_0^{n-2}x_0 + x_0^{n-3}x_0^2 + \cdots + x_0^{n-1} \\ &= nx_0^{n-1} \end{aligned}$$