

MATH410: Advanced Calculus I

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These are my notes for UMD's MATH410: Advanced Calculus I. These notes are taken live in class ("live- \TeX -ed). This course is taught by Lecturer Anna Szczekutowicz.

Contents

1	Set Theory Preliminaries	2
2	The Completeness Axiom	3
2.1	Archimedean Property	4
3	Sequences	5
3.1	Convergence	6
3.2	Boundedness	13
3.3	Sequential Compactness	16
4	Continuous Functions	18
4.1	Continuity Basics	18
4.2	Extreme Value Theorem	20
4.3	Intermediate Value Theorem	22

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§1 Set Theory Preliminaries

This section covers the foundation of analysis, which is just the set of real numbers. It covers basic definitions such as $\in, \notin, \emptyset, \subseteq, =, \cap, \cup, \setminus$, so for example

Definition 1.1. **Intersection** of A and B is $C = A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

Some quantifiers include $\forall, \exists, \exists!$ and some number sets include $\mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{Q}^C$.

Definition 1.2. The real numbers \mathbb{R} satisfies 3 groups of axioms: Refer to the notes on Canvas for the Consequences of all of the following axioms.

1. Field $(+, *)$
 - Commutativity of Addition
 - Associativity
 - Additive Identity
 - Additive Inverse
 - Commutativity of Multiplication
 - Associativity of Multiplication
 - Multiplicative Identity
 - Multiplicative Inverse
 - Distributive Property

The set of integers \mathbb{Z} is not a field because it fails under the multiplicative inverse.

2. Positivity

There is a subset of \mathbb{R} denoted by \mathcal{P} , called the set of positive numbers for which:

- If x and y are positive, then $x + y$ and xy are both positive.
- For each $x \in \mathbb{R}$, exactly one of the following 3 alternatives is true: $x \in \mathcal{P}$, $-x \in \mathcal{P}$, or $x = 0$

3. Completeness

Definition 1.3. **Absolute value** is defined as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Definition 1.4. **Triangle Inequality** is $\forall a, b \in \mathbb{R}, |a + b| \leq |a| + |b|$

Proof. Assume without loss of generality, $a \geq b$. We will proceed with proof by cases.

Case 1: Assume $a \geq b \geq 0$. Then $|a + b| = a + b$ by the definition of absolute value since $a \geq 0, b \geq 0 \implies |a + b| = a + b = |a| + |b|$.

Case 2: Now assume $a \geq 0 \geq b$ and $a + b \geq 0$. Note since $b \leq 0$ then $b \leq |b|$. Then

$$|a + b| = a + b \leq |a| + |b|$$

by the definition of absolute value and our above note.

Case 3: Now consider $a \geq 0 \geq b$ and $a + b < 0$. So

$$|a + b| = -(a + b) = -a - b \leq |a| + |b|$$

Case 4: Now consider $0 \geq a \geq b$ so $a + b < 0$. Therefore,

$$|a + b| = -(a + b) = -a - b = |a| + |b|$$

□

§2 The Completeness Axiom

Definition 2.1. A subset S of \mathbb{R} is said to be **bounded above** if $\exists r \in \mathbb{R}$ such that $s \leq r \forall s \in S$

The definition of **bounded below** is similar.

Definition 2.2. The least upper bound, if it exists, is called the **supremum** of S . We denote it as the "sup" of S . Similarly, the largest lower bound is called the **infimum** and is denoted as the "inf" of S .

Definition 2.3. Let $S \subseteq \mathbb{R}$ where $S \neq \emptyset$. If S has a largest (smallest), the element is a max (min).

Example 2.4

Find the sup of $(0, 1)$ and prove it.

Proof. Let us prove that the $\sup(0, 1) = 1$. First, let us show that we have an upperbound. If $x \in (0, 1)$, then $x \leq 1$. By definition of upperbound, 1 is an upper bound. Note that we can find many other upper bounds.

On the contrary, assume $x < 1$ is an upper bound. Now consider the average $\frac{1+x}{2}$.

$$\frac{x+1}{2} < \frac{1+1}{2} = 1$$

Therefore, we have showed that $0 < \frac{x+1}{2} < 1 \implies \frac{x+1}{2} \in (0, 1)$. But, $\frac{1+x}{2} > \frac{x+x}{2} \implies \frac{1+x}{2} > x$. This is a contradiction. Since x is an upper bound, and we found $\frac{1+x}{2} \in (0, 1)$ where $\frac{1+x}{2} > x$, so x is not a supremum.

□

Theorem 2.5

Suppose $S \in \mathbb{R}, S \neq \emptyset$ that is bounded above. Then a supremum exists. Every nonempty subset S of \mathbb{R} that is bounded below has a lower bound.

Note 2.6. Let c be a positive number then $\exists!$ a positive number whose square is c . $x^2 = c, x > 0$ has a unique solution and this gives us the notion of square root.

§2.1 Archimedian Property

Definition 2.7. The **Archimedian Property** is a result of the completeness axiom. Suppose there is a small $\epsilon > 0$ and c is an arbitrary large number.

1. $\exists n \in \mathbb{N}$ such that $c < n$, which just means that you can always find a natural number than any large number
2. $\exists m \in \mathbb{N}$ such that $\frac{1}{m} < \epsilon$, which just means you can always find smaller rational numbers.

Proof. We will proceed by contradiction. Assume that \exists an upper bound c for the \mathbb{N} . So there is no $n \in \mathbb{N}$ s.t. $c < n$. Since \mathbb{N} is bounded above, and the \mathbb{N} is nonempty, the supremum exists (Completeness Axiom). Let $s = \sup \mathbb{N}$. Consider $s - 1$ and $s - 1 < s = \sup \mathbb{N}$, which is the least upper bound, so $s - 1$ is not an upper bound. So $\exists n \in \mathbb{N}$ such that $s - 1 < n \implies s < n + 1$. But $s = \sup \mathbb{N}$, the least upper bound, this is a contradiction since it is less than $(n + 1) \in \mathbb{N}$.

For part b , use $c = \frac{1}{\epsilon}$ and use part a . □

Note 2.8. Some of the following are results from the Archimedian Property.

Theorem 2.9

For all $n \in \mathbb{Z}$, there is no integer in $(n, n + 1)$ (an open interval).

Theorem 2.10

If S is a nonempty subset of \mathbb{Z} that is bounded above, then it has a max.

Theorem 2.11

* For every $c \in \mathbb{R}$, $\exists! n \in \mathbb{Z}$ in $[c, c + 1)$

Definition 2.12. A subset $S \subseteq \mathbb{R}$ is said to be **dense** in \mathbb{R} if for every $a, b \in \mathbb{R}$ with $a < b$, then there is a $s \in S$ s.t. $s \in (a, b)$.

Theorem 2.13

\mathbb{Q} is dense in \mathbb{R} . Reminder that $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$

Proof. Suppose we have arbitrary $a, b \in \mathbb{R}$ and $a < b$. We want to find $\frac{m}{n} \in (a, b)$. By multiplication, we can say we want $na < m < nb$. We want an integer m between na and nb . We can write this as

$$nb - na > 1 \implies n(b - a) > 1 \implies n > \frac{1}{b - a}$$

By part *a* of the Archimedean Property, let $c = \frac{1}{b-a}$, and we know that there exists some $n \in \mathbb{N}$ such that $n > c$. Since $a < b$, and $b - a > 0$, multiply

$$n > \frac{1}{b - a}$$

$$n(b - a) > 1$$

$$nb - na > 1$$

$$nb - 1 > na \implies na < nb - 1$$

By previous (*), $\exists m \in \mathbb{Z}$ s.t. $m \in [nb - 1, nb)$. Therefore, $nb - 1 \leq m < nb$. Therefore,

$$na \leq nb - 1 \leq m < nb \implies na < m < nb \implies a < \frac{m}{n} < b$$

and so we have found that there exists $m \in \mathbb{Z}, n \in \mathbb{N}$ such that $\frac{m}{n} \in (a, b)$ for all $a, b \in \mathbb{R}$ and $a < b$. Therefore, the rational numbers are dense in the real numbers. \square

§3 Sequences

Definition 3.1. A **sequence** of \mathbb{R} is a real-valued function whose domain is \mathbb{N} . $f : \mathbb{N} \rightarrow \mathbb{R}$ (a list of numbers indexed by \mathbb{N})

Example 3.2

A sequence of odd integers could be $a_1 = 1, a_2 = 3, a_3 = 5, \dots, a_n = 2n - 1$ which can be

$$\{1, 3, 5, \dots\} = \{a_n\}_{n=1}^{\infty} = \{2n - 1\}_{n=1}^{\infty}$$

Example 3.3

$$\left\{\frac{1}{n}\right\} = \left\{\frac{1}{n}\right\}_{n=1}^{\infty} \implies \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$$

§3.1 Convergence

Definition 3.4. A sequence $\{a_n\}$ is said to **converge** to a number L if $\forall \epsilon > 0$, \exists an index N s.t. \forall indices $n \geq N$ we have

$$|a_n - L| < \epsilon \implies \text{Notation: } \lim_{n \rightarrow \infty} a_n = L$$

Example 3.5

Suppose we have the sequence $\{\frac{(-1)^n}{n}\}$ and we WTS

$$\lim_{n \rightarrow \infty} \frac{(-1)^n}{n} = 0$$

Think of the problem as someone gives you a small $\epsilon \implies$ you have to find N , which we call the **threshold**, such that for every sequence value after the threshold is in the ϵ -tube.

For example, $\epsilon = \frac{1}{2} \implies N = 3, \epsilon = \frac{1}{4} \implies N = 5$.

Above $L = 0$, sketch: we want

$$|a_n - L| < \epsilon \implies \left| \frac{(-1)^n}{n} - 0 \right| < \epsilon \implies \left| \frac{1}{n} \right| < \epsilon \implies \frac{1}{\epsilon} < n$$

so choose $N = \frac{1}{\epsilon} < n$

Proof. Let $\epsilon > 0$ be given. By Archimedian Property, $\exists N \in \mathbb{N}$ such that $\frac{1}{N} < \epsilon$. Then if $n \geq N$

$$\left| \frac{(-1)^n}{n} - 0 \right| = \left| \frac{(-1)^n}{n} \right| = \left| \frac{1}{n} \right| = \frac{1}{n}$$

From here, we need to relate n to N and then we can relate N to ϵ . Note that $n \geq N \implies \frac{1}{N} \geq \frac{1}{n}$ by algebra. Therefore,

$$\frac{1}{n} \leq \frac{1}{N} < \epsilon$$

by our choice of N . Therefore,

$$\frac{1}{n} < \epsilon$$

and so we are done since we have shown that

$$\left| \frac{(-1)^n}{n} - 0 \right| < \epsilon$$

□

Example 3.6

Given $\{\frac{n^2-2n}{n^2+1}\}$, prove that this sequence $\lim_{n \rightarrow \infty} \frac{n^2-2n}{n^2+1} = 1$.

Some sketch work: we want to show that $|\frac{n^2-2n}{n^2+1} - 1| < \epsilon$

$$|\frac{n^2-2n}{n^2+1} - 1| = |\frac{n^2-2n}{n^2+1} - \frac{n^2+1}{n^2+1}| = |\frac{-2n-1}{n^2+1}| = |\frac{2n+1}{n^2+1}|$$

Note that both the numerator and denominator are both always positive, so we can consider. Now let us use the \leq operator to simplify and have one singular 'n'.

$$\frac{2n+1}{n^2+1} \leq \frac{2n+1}{n^2} \leq \frac{2n+n}{n^2} = \frac{3n}{n^2} = \frac{3}{n}$$

Recall that $n \geq N \implies \frac{1}{N} \geq \frac{1}{n} \implies \frac{1}{n} \leq \frac{1}{N}$ So we'd choose N to get rid of 3 and introduce ϵ .

Proof. Let $\epsilon > 0$. By A.P., $\exists N \in \mathbb{N}$ s.t. $\frac{1}{N} < \frac{\epsilon}{3}$. For $n \geq N$, then

$$|\frac{n^2-2n}{n^2+1} - 1| = \dots = \frac{2n+1}{n^2+1} < \dots \leq \frac{3}{n} \leq \frac{3}{N} = 3 * \frac{1}{N} < 3 * \frac{\epsilon}{3} = \epsilon$$

Therefore, we have shown that

$$|a_n - L| < \epsilon \implies \lim_{n \rightarrow \infty} \frac{n^2-2n}{n^2+1} = 1$$

□

Theorem 3.7

The Sum Property states that if

$$\lim_{n \rightarrow \infty} a_n = a \text{ and } \lim_{n \rightarrow \infty} b_n = b$$

then

$$\lim_{n \rightarrow \infty} (a_n + b_n) = \lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} b_n = a + b$$

Some sketch work before the proof:

We want to show that $|a_n + b_n - (a + b)| < \epsilon$. Note that we can group terms together $|(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b|$ by the Triangle Inequality. It is known that these two terms converge. Therefore, we can choose ϵ s such that

$$|a_n - a| + |b_n - b| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

Proof.

Let $\epsilon > 0$. Since the sequences $\{a_n\}$ and $\{b_n\}$ converge to a and b , respectively, by the Archimedean Principle, $\exists N_1, N_2 \in \mathbb{N}$ such that $\frac{1}{N_1} < \frac{\epsilon}{2}$ and $\frac{1}{N_2} < \frac{\epsilon}{2}$. Choose $N = \max(N_1, N_2)$, which represents the numerically larger threshold. For all $n \geq N$, we show

$$\begin{aligned} |a_n + b_n - (a + b)| &= |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Therefore, we have shown that $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$ □

Lemma 3.8

The Comparison Lemma (C.L.)

Let $\{a_n\}$ converge to a . Then $\{b_n\}$ converges to b if $\exists c \in \mathbb{R}^+$ and $N \in \mathbb{N}$ such that

$$\forall n \geq N, |b_n - b| \leq c|a_n - a|$$

Proof. Let $\epsilon > 0$. Since a_n converges to a , $\exists N_1 \in \mathbb{N}$ such that $|a_n - a| < \frac{\epsilon}{c}$, $\forall n \geq N_1$. By the Archimedean Principle, $\exists N_2 \in \mathbb{N}$ such that $\frac{1}{N_2} < \epsilon$. Choose $N = \max(N_1, N_2)$ and if $n \geq N$, then

$$\begin{aligned} |b_n - b| &\leq c|a_n - a| < c * \frac{\epsilon}{c} = \epsilon \\ \implies |b_n - b| &< \epsilon \end{aligned}$$

□

Lemma 3.9

Suppose the $\lim_{n \rightarrow \infty} a_n = a$, then for $c \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} ca_n = c \lim_{n \rightarrow \infty} a_n = ca$$

Proof. Use the Comparison Lemma (above). Note that $|ca_n - ca| = |c(a_n - a)| = |c||a_n - a|$ which satisfies $|b_n - b| \leq c|a_n - a| \implies \{b_n\} = \{ca_n\} \implies b = ca$. \square

Lemma 3.10 *Proof.* The following is a useful property (*)

$$\lim_{n \rightarrow \infty} a_n = a \text{ iff } \lim_{\leftrightarrow n \rightarrow \infty} (a_n - a) = 0$$

 \square **Lemma 3.11**

Suppose $\lim_{n \rightarrow \infty} a_n = 0$ and $\lim_{n \rightarrow \infty} b_n = 0$ then $\lim_{n \rightarrow \infty} a_n b_n = 0$.

Proof. Since $\lim_{n \rightarrow \infty} a_n = 0$ and $\sqrt{\epsilon} > 0$,

$$\exists N_1 \in \mathbb{N} \text{ s.t. } |a_n| < \sqrt{\epsilon} \forall n \geq N_1$$

Since $\lim_{n \rightarrow \infty} b_n = 0$ and $\sqrt{\epsilon} > 0$,

$$\exists N_2 \in \mathbb{N} \text{ s.t. } |b_n| < \sqrt{\epsilon} \forall n \geq N_2$$

Let $N = \max(N_1, N_2)$. Then if $n \geq N$,

$$|a_n b_n - 0| = |a_n b_n| = |a_n| * |b_n| < \sqrt{\epsilon} * \sqrt{\epsilon} = \epsilon$$

 \square

Theorem 3.12

The Product Property states that if $\lim_{n \rightarrow \infty} a_n = a$ and $\lim_{n \rightarrow \infty} b_n = b$ then $\lim_{n \rightarrow \infty} a_n b_n = ab$

Proof. Define $\alpha_n = a_n - a$ and $\beta_n = b_n - b$. Using the * property above, since $\lim_{n \rightarrow \infty} a_n = a \implies \lim_{n \rightarrow \infty} (a_n - a) = \lim_{n \rightarrow \infty} \alpha_n = 0$ and then the same for b such that $\lim_{n \rightarrow \infty} \beta_n = 0$.

Note that

$$a_n b_n - ab = (\alpha_n + a)(\beta_n + b) - ab$$

and then by Distributive property we get

$$= \alpha_n \beta_n + \alpha_n b + a \beta_n + ab - ab = \alpha_n \beta_n + \alpha_n b + a \beta_n$$

So using the previous lemma,

$$\lim_{n \rightarrow \infty} (a_n b_n - ab) = \lim_{n \rightarrow \infty} (\alpha_n \beta_n + b \alpha_n + a \beta_n) = \lim_{n \rightarrow \infty} (\alpha_n \beta_n) + b \lim_{n \rightarrow \infty} \alpha_n + a \lim_{n \rightarrow \infty} \beta_n$$

From above, the last two terms are 0 and by the previous lemma, the first term is 0. Therefore,

$$\lim_{n \rightarrow \infty} (a_n b_n - ab) \text{ iff } \lim_{n \rightarrow \infty} (a_n b_n) = ab$$

□

Definition 3.13. A sequence **diverges** to $\infty, (-\infty)$ if

$$\forall M > (<)0, \exists N \in \mathbb{N} \text{ s.t. } \forall n \in \mathbb{N}, n \geq N \text{ then } a_n > (<)M$$

Example 3.14

Prove that $\lim_{n \rightarrow \infty} (n^2 - 4n) = \infty$

Sketch: we want $a_n > M \implies n^2 - 4n > M \implies n(n - 4) > M$

Proof. Let $M > 0$ be given. By A.P., $\exists N \in \mathbb{N}$ s.t. $N > \max(M, 4)$. If $n \geq N$, then $n^2 - 4n = n(n - 4) \geq N(N - 4) > M$

Thus,

$$n^2 - 4n \rightarrow \infty \text{ as } n \rightarrow \infty$$

□

Example 3.15

Prove that $(-1)^n$ does not converge.

Proof. On the contrary, suppose $(-1)^n$ converges to a . Let $\epsilon = 1$. In the definition of convergence, then $\exists N \in \mathbb{N}$ if $n \geq N$ then

$$|(-1)^n - a| < 1$$

For $n = 2N$, meaning some even number, we get $|(-1)^n - a| = |1 - a| < 1$

Now for $n = 2N + 1$, we get $|(-1)^{2N+1} - a| = |1 + a| < 1$

Note that $|1 - a| < 1$ and $|1 + a| < 1$ so therefore

$$|1 - a| + |1 + a| < 2$$

But note, using the Triangle Inequality in the reverse direction, note that $2 = |1 - a + 1 + a| \leq |1 - a| + |1 + a| < 1 + 1 = 2$. Therefore, we've shown that $2 < 2$ which is a contradiction and therefore, $(-1)^n$ does not converge. \square

Lemma 3.16

Suppose the sequence $\{b_n\}$ of nonzero numbers converges to $b \neq 0$. Then $\{\frac{1}{b_n}\}$ converges to $\frac{1}{b}$.

Sketch: Use the Comparison Lemma to find $c \in \mathbb{R}^+$ and $N_1 \in \mathbb{N}$ such that

$$|\frac{1}{b_n} - \frac{1}{b}| < c|b_n - b|$$

We just have to find c and N_1 .

Proof. Note that

$$|\frac{1}{b_n} - \frac{1}{b}| = |\frac{b - b_n}{bb_n}| = \frac{1}{|b||b_n|}|b_n - b|$$

We want $\frac{1}{|b||b_n|}$ to be c , but this must be a single constant and not dependent on n . We want to find index N_1 such that

$$|b_n| > \frac{|b|}{2} \quad \forall n \geq N_1$$

$$\frac{1}{|b_n|} < \frac{2}{|b|}$$

If we can find N_1 then $|\frac{1}{b_n} - \frac{1}{b}| \leq \frac{2}{|b|^2}|b_n - b|$ and the term $\frac{2}{|b|^2}$ becomes our c and we can apply the Comparison Lemma, so we need N_1 to make the above true. Let $\epsilon = \frac{b}{2}$. By definition of $\{b_n\}$ converging to b , we can choose N_1 such that $|b_n - b| < \epsilon \quad \forall n \geq N_1$.

$$|b_n - b| < \frac{|b|}{2}$$

$$-\epsilon < b_n - b < \epsilon$$

$$b - \epsilon < b_n < b + \epsilon$$

Check $b > 0, b < 0$ since $\epsilon = \frac{|b|}{2}$. When $b > 0, \epsilon = \frac{b}{2}$ so

$$b_n \in (b - \frac{b}{2}, b + \frac{b}{2}) = (\frac{b}{2}, \frac{3b}{2})$$

so $b_n > \frac{b}{2}$. When $b < 0 \dots$ So $|b_n| > \frac{|b|}{2}$ and this N_1 works and apply the Comparison Lemma. \square

Theorem 3.17

Let $\lim_{n \rightarrow \infty} a_n = a$, $\lim_{n \rightarrow \infty} b_n = b$, and $b_n \neq 0 \forall n$ and $b \neq 0$ then

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}$$

Proof.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} a_n * \frac{1}{b_n} = \lim_{n \rightarrow \infty} a_n * \lim_{n \rightarrow \infty} \frac{1}{b_n} = \frac{a}{b}$$

□

§3.2 Boundedness

Definition 3.18. A sequence $\{a_n\}$ is **bounded** if $\exists M \in \mathbb{R}$ such that $|a_n| \leq M \forall n$.

Theorem 3.19

Every convergent sequence is bounded.

- If convergent \implies bounded.
- If it is unbounded, then it diverges.

Proof. Let $\lim_{n \rightarrow \infty} a_n = a$ and take $\epsilon = 1$. Using the definition of convergence, $\exists N \in \mathbb{N}$ s.t.

$$|a_n - a| < 1 \forall n \geq N$$

then $|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < 1 + |a| \forall n \geq N$ by the Triangle Inequality and then the definition of the converging sequence. However, we need to show that this is true (bounded by a constant) for all n , not just for all $n \geq N$.

Define $M = \max(1 + |a|, |a_1|, \dots, |a_{N-1}|)$. Note that there the $N - 1$ terms are finite and so a max exists. Then

$$|a_n| \leq M \forall n$$

and so $\{a_n\}$ is bounded. □

Remark 3.20. Recall that a set $S \subset \mathbb{R}$ is dense in \mathbb{R} if every open set $(a, b) \in \mathbb{R}$ contains a point $s \in S$.

Definition 3.21. A set of numbers $\{x_n\}$ is in a set S provided that $x_n \in S \forall n$.

Lemma 3.22

A set S is **dense** in \mathbb{R} if and only if every $x \in \mathbb{R}$ is a limit of a sequence of a sequence in S .

Proof.

\Rightarrow Let $S \subset \mathbb{R}$ be dense in \mathbb{R} . Fix $x \in \mathbb{R}$ and let n be an index. Since S is dense, there is an element in S in $(x, x + \frac{1}{n})$. For each n , this defines $\{s_n\}$ with

$$s \in (x, x + \frac{1}{n})$$

$$x < s < x + \frac{1}{n}$$

$$|s_n - x| < \frac{1}{n} \quad \forall n$$

$$|s_n - x| < 1|\frac{1}{n} - 0|$$

Use the Comparison Lemma since $\{\frac{1}{n}\}$ converges to 0. So, $\{s_n\}$ converges to x .

\Leftarrow Let S have the property that every number in \mathbb{R} is the limit of a sequence in S . We want to show that any open interval in \mathbb{R} contains a point $s \in S$. Consider an open interval $(a, b) \in \mathbb{R}$. Consider $\frac{a+b}{2} = s \in \mathbb{R}$. By assumption, $\exists \{s_n\}$ of points in S s.t. $\lim_{n \rightarrow \infty} s_n = s$. Define $\epsilon = \frac{b-a}{2} > 0$. By definition of convergence, $\exists N$ s.t. $|s_n - s| < \epsilon \quad \forall n \in \mathbb{N}$.

$$-\epsilon < s_n - s < \epsilon$$

$$s - \epsilon < s_n < s + \epsilon$$

$$\frac{a+b}{2} - \frac{b-a}{2} < s_n < \frac{a+b}{2} + \frac{b-a}{2}$$

$$a < s_n < b$$

The point $s_N \in S$ and $s_n \in (a, b)$ so S is dense in \mathbb{R} . □

Definition 3.23. The **sequential density of \mathbb{Q}** states that every \mathbb{R} is the limit of a sequence in \mathbb{Q} .

Theorem 3.24

Let $\{c_n\} \in [a, b]$ and $\lim_{n \rightarrow \infty} c_n = c$ then $c \in [a, b]$ also.

Definition 3.25. $S \subset \mathbb{R}$ is said to be **closed** (set) if $\{a_n\}$ is a sequence in S that converges to a , then $a \in S$ also.

Example 3.26

$(0, 1]$ not closed since $\{\frac{1}{n} \in (0, 1]\}$ and $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$ but $0 \notin (0, 1]$.

Example 3.27

\mathbb{Q} is not closed since we can find $\{r_n\} \in \mathbb{Q}$ that converge to π but $\pi \notin \mathbb{Q}$.

Definition 3.28. A $\{a_n\}$ is said to be **monotonically increasing (decreasing)** if $a_{n+1} \geq (\leq) a_n \forall n$

Note 3.29. If a sequence is monotone, then it is either monotonically increasing or decreasing.

Theorem 3.30

Monotone Convergence Theorem (MCT) states that a monotone sequence converges if and only if it is bounded. Moreover, the bounded monotone $\{a_n\}$ converges to the

1. $\sup\{a_n \mid n \in \mathbb{N}\}$ if monotone increasing
2. $\inf\{a_n \mid n \in \mathbb{N}\}$ if monotone decreasing

Proof.

\Rightarrow Note that we already showed that convergent sequences are bounded.

\Leftarrow We want to show that our sequence converges to either the \inf, \sup depending on if its either increasing or decreasing. Now assume that our sequence is bounded. Define $S = \{a_n \mid n \in \mathbb{N}\}$ and S is bounded by assumption. Since S is nonempty and bounded above, S has $\sup S = l$ by the Completeness Axiom. Claim $\lim_{n \rightarrow \infty} a_n = l$. Let $\epsilon > 0$ be given, and we want to show the usual definition of convergence.

Note that

$$\begin{aligned} |a_n - l| &< \epsilon \\ -\epsilon &< a_n - l < \epsilon \\ l - \epsilon &< a_n < l + \epsilon \quad \forall n \geq N \end{aligned}$$

But l is an upper bound for $S \Rightarrow a_n \leq l < l + \epsilon \forall n$.

On the other hand, since l is the least upper bound for S , $l - \epsilon$ is not an upper bound for S . So, $\exists N$ such that $l - \epsilon < a_N$.

Since a_n is monotonically increasing. $l - \epsilon < a_N \leq a_n \forall n \geq N$. Thus, we have $N \in \mathbb{N}$ such that $\forall n \geq N$ we have $|a_n - l| < \epsilon$, as desired. \square

Remark 3.31. The formula for a finite geometric sum is $S_n = \sum_{k=1}^n r^k$ where $r \neq 1, r < 1$.

$$S_n = \frac{r - r^{n+1}}{1 - r}$$

Example 3.32

Consider $S_n = \sum_{k=1}^n \frac{1}{k} \cdot \frac{1}{2^k}$

$$k = 1 \implies s_1 = \frac{1}{2}$$

$$k = 2 \implies s_2 = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2^2} = \frac{5}{8}$$

$$k = 3 \implies \frac{1}{2} + \frac{1}{8} + \frac{1}{3} \cdot \frac{1}{2^3}$$

$$\vdots$$

so this sequence is monotonically increasing. Is it bounded

$$S_n = \sum_{k=1}^n \frac{1}{k} \cdot \frac{1}{2^k} \leq \sum_{k=1}^n \frac{1}{2^k} = \frac{\frac{1}{2} - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}} = 1$$

Theorem 3.33

The Nested Interval Theorem. Suppose that $I_n = [a_n, b_n]$ is a sequence of intervals, for which $I_{n+1} \subset I_n \forall n$. Then the intersection of those intervals is a nonempty closed interval

$$\cap_{i=1}^{\infty} I_n = [a, b]$$

where $a = \sup a_n, b = \inf b_n$. Furthermore, if $\lim_{n \rightarrow \infty} a_n - b_n = 0$ then $\cap_{i=1}^{\infty} I_n$ contains a single point.

Proof.

\Leftarrow Let $x \in \cap_{i=1}^{\infty} I_n$. So for all $n \in \mathbb{N}, x \in I_n$ by definition of intersection. Therefore,

$$a_n \leq x \leq b_n \forall n$$

Note that x is an upper bound for a_n . So, by definition of sup, $a = \sup a_n \leq x$.

$$a \leq x \leq b \implies x \in [a, b]$$

\implies The reverse direction is similar. □

§3.3 Sequential Compactness

Definition 3.34. Consider a sequence $\{a_n\}$ and let $\{n_k\}$ be a sequence of \mathbb{N} that is strictly increasing. Then the sequence $\{b_k\}$ defined by $b_k = a_{n_k} \forall k$ is a **subsequence**.

Note 3.35. Note that a sequence may not converge, but it may be possible to find a subsequence that does.

Theorem 3.36

Let $\{a_n\}$ converges to a . Then every subsequence of $\{a_n\}$ also converges to the same limit a .

Theorem 3.37

Every sequence (does not need to converge) has a monotone increasing or decreasing subsequence.

Proof. Consider $\{a_n\}$. We call an index a **peak index** for $\{a_n\}$ if

$$a_n \leq a_m \quad \forall n \geq m$$

You can have two cases: infinitely many peak indices, or a finite number of peak indices.

Suppose there are finite number of peak indices. Then we choose N such that there are no more peak indices. Since N is not a peak index, $\exists n_1 \in \mathbb{N}$ such that $n_1 > N$ with $a_N \leq a_{n_1}$

$$\vdots$$

Continue for $n_k \implies \exists n_{k+1} \in \mathbb{N}$ with $n_{k+1} \geq n_k$ with $a_{n_k} \leq a_{n_{k+1}}$

$$a_N \leq a_{n_1} \leq \cdots \leq a_{n_k} \leq a_{n_{k+1}}$$

which is a monotonically increasing subsequence.

In the case of infinitely many peak indices, $m_1 < m_2 < m_3 < \cdots < \text{peak indices}$. Since m_1 is a peak index. Then $m_1 < m_2 \implies a_{m_1} > a_{m_2}$.

$$\vdots$$

We'll get a monotonically decreasing subsequence. □

Theorem 3.38

Every bounded sequence has a convergent subsequence.

Proof. Let $\{a_n\}$ be bounded. By the previous theorem, $\{a_n\}$ has a monotone subsequence. Since $\{a_n\}$ is bounded, $\{a_{n_k}\}$ is bounded also. By MCT, $\{a_{n_k}\}$ converges since it is monotone and bounded. □

Definition 3.39. A $S \subset \mathbb{R}$ is said to be **compact (or sequentially compact)** if every sequence in S has a convergent subsequence converging to a point in S . For a set to not be compact, we find a sequence in S that has no convergence subsequence that converges to a point in S .

Example 3.40

$[1, \infty)$ is not compact. Consider $a_n = n, a_n \rightarrow \infty$ by Archimedian Principle. Then every subsequence of n_k also diverges to ∞ . Thus, $\{a_n\}$ has no subsequence that converges.

Example 3.41

$(0, 1]$ is not compact. Let $a_n = \frac{1}{n}, a_n \rightarrow 0, n \rightarrow \infty$, so every subsequence converges to 0 also. But $0 \notin (0, 1]$ so it is not compact.

Theorem 3.42

The Sequentially Compactness Theorem (SCT) states that every interval $[a, b]$ such that $a, b \in \mathbb{R}$ is sequentially compact.

Proof. Let $\{a_n\}$ be in $[a, b]$. So, $a \leq a_n \leq b \forall n$. By a previous theorem, since $\{a_n\}$ is bounded, there exists a convergent subsequence $\{a_{n_k}\}$. Assume $\{a_{n_k}\} \rightarrow l$. Since $a \leq a_n \leq b \forall n$, then

$$a \leq a_{n_k} \leq b \forall n$$

so $l \in [a, b]$ as desired. Therefore, $\{a_n\}$ has a convergent subsequence whose limit is in the interval $[a, b]$, so it is sequentially compact. \square

Theorem 3.43

Bolzano Weirstrass Theorem: If $S \subset \mathbb{R}$, the following are equivalent

$$S \text{ is closed and bounded} \iff S \text{ is compact}$$

§4 Continuous Functions

§4.1 Continuity Basics

Note 4.1. Before $f : \mathbb{N} \rightarrow \mathbb{R}$ but now $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$. $f(x)$ is the value the function assigns to x .

Definition 4.2. A function $f : D \rightarrow \mathbb{R}$ is said to be **continuous at a point** x_0 if whenever $\{x_n\}_{n=1}^{\infty}$ converges to $x_0 \in D$, the image sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(x_0)$.

Definition 4.3. A function $f : D \rightarrow \mathbb{R}$ is **continuous** if f is continuous at every point in D .

Example 4.4

Consider $f(x) = x^2 + 7x - 3$. We want to show f is continuous. Select $x_0 \in \mathbb{R}$ and let $\{x_n\} \rightarrow x_0 \implies \lim_{n \rightarrow \infty} x_n = x_0$. We want to show that

$$\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$$

Note that

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} x_n^2 + 7x_n - 3$$

by definition of f .

$$= \lim_{n \rightarrow \infty} x_n^2 + 7 \lim_{n \rightarrow \infty} x_n + \lim_{n \rightarrow \infty} 3$$

by properties of sequences.

$$= x_0^2 + 7x_0 - 3 = f(x_0)$$

by the definition of f

Remark 4.5. Given $f : D \rightarrow \mathbb{R}, g : D \rightarrow \mathbb{R}$ are continuous, then

$$f \pm g, fg, \frac{f}{g} (g \neq 0)$$

are continuous and this follows directly from convergent sequence properties from the previous chapter. Thus, polynomials are continuous.

Example 4.6

Consider Dirichlet's function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

Note that f is defined on \mathbb{R} but it is discontinuous at $x_0 \in \mathbb{R}$.

Proof. Let $x_0 \in \mathbb{R}$. By sequential density of the \mathbb{Q} and \mathbb{Q}^c , we can find

$$\{u_n\} \rightarrow x_0, u_n \in \mathbb{Q} \forall n$$

$$\{v_n\} \rightarrow x_0, v_n \in \mathbb{Q}^c \forall n$$

Since $f(u_n) = 1 \forall n$ and $f(v_n) = 0 \forall n$, then

$$\{f(u_n)\} \rightarrow 1 \text{ but } \{f(v_n)\} \rightarrow 0$$

Therefore, $\lim_{n \rightarrow \infty} f(u_n) = 1 \neq 0 = \lim_{n \rightarrow \infty} f(v_n)$ but $\{u_n\} \rightarrow x_0$ and $\{v_n\} \rightarrow x_0$ but we cannot have 2 function values for x_0 . \square

Definition 4.7. Suppose $f : D \rightarrow \mathbb{R}$ and $g : U \rightarrow \mathbb{R}$ such that $f(D) \subset U$ then we define

$$(g \circ f)(x) = g(f(x)) \quad \forall x$$

Theorem 4.8

Let $f : D \rightarrow \mathbb{R}, g : U \rightarrow \mathbb{R}$ and $f(D) \subset U$. Let f be continuous at x_0 and g be continuous at $f(x_0)$. Then $(g \circ f) : D \rightarrow \mathbb{R}$ is continuous at x_0 .

Proof. Suppose $\{x_n\} \in D$ converges to x_0 . Since f is continuous, then $\lim_{n \rightarrow \infty} f(x_n) = f(x_0)$.

$$\{f(x_n)\} \xrightarrow{n \rightarrow \infty} f(x_0)$$

Since g is continuous at $f(x_0)$, then $\lim_{n \rightarrow \infty} g(f(x_n)) = g(f(x_0))$. Therefore, $(g \circ f)(x)$ is continuous at x_0 since

$$\{g(f(x_n))\} \xrightarrow{n \rightarrow \infty} g(f(x_0))$$

\implies we can combine continuous functions and remain continuous □

§4.2 Extreme Value Theorem

Definition 4.9. $f : D \rightarrow \mathbb{R}$ attains a **maximum (minimum)** value if there is

$$x_0 \in D \text{ s.t. } f(x_0) \geq (\leq) f(x) \quad \forall x \in D$$

Remark 4.10. Recall that a nonempty set has a maximum if it is bounded above and contains its supremum ie. the supremum is in the set.

\implies Now $f : D \rightarrow \mathbb{R}$ has a maximum when the image $f(D)$ is bounded above and the supremum of the image is a functional value.

Example 4.11

$f : (0, 1) \rightarrow \mathbb{R}$ where $f(x) = 2x$. Note that the supremum of the image is 2, but 2 is not a functional value. Therefore, this function does not have a max.

Theorem 4.12

The **Extreme Value Theorem** states that a continuous function on a closed and bounded interval $f : [a, b] \rightarrow \mathbb{R}$ attains both a maximum and a minimum. Sketch: Note that we want to show that $f(D)$ is bounded above. See lemma below. After that, we need to show that the supremum is a functional value.

Lemma 4.13

Assume on the contrary that given $f : [a, b] \rightarrow \mathbb{R}$ is continuous, assume there is no M such that

$$f(x) \leq M \quad \forall x \in [a, b]$$

There is $x \in [a, b]$ at which $f(x) > n$, $\forall n$. For each n this creates a sequence $\{x_n\}$ in $[a, b]$ with $f(x) > n \quad \forall n$. $\{x_n\}$ may or may not converge. By Sequential Compactness Theorem, choose $\{x_{n_k}\}$ subsequence that converges to $x_0 \in [a, b]$. Since f is continuous at x_0 , $\{f(x_{n_k})\} \rightarrow f(x_0)$, but every convergent sequence is bounded by a theorem, so $\{f(x_{n_k})\}$ is bounded. Therefore, we have a contradiction since $f(x_{n_k}) > n_k \geq k \quad \forall k \in \mathbb{N}$. So $f : [a, b] \rightarrow \mathbb{R}$ is bounded above.

Proof. Define $S = f([a, b])$, all of the image values. By the lemma above, S is bounded. Note S is nonempty and bounded, thus by the Completeness Axiom, $c := \sup(S)$ exists. Note that we want to find $x_0 \in [a, b]$ such that $f(x_0) = c$, as this would show that the supremum is a functional value. Consider

$$c - \frac{1}{n} < c \quad \forall n$$

Note that $c - \frac{1}{n}$ is not an upper bound since c is the least upper bound. So, we can find a point $x \in [a, b]$ such that

$$c - \frac{1}{n} < f(x) < c$$

Label point x_n to create a sequence $\{x_n\}$

$$c - \frac{1}{n} < f(x_n) < c \quad \forall n$$

Since $\{\frac{1}{n}\} \rightarrow 0$ as $n \rightarrow \infty$, then $\{f(x_n)\} \rightarrow c$ by the Squeeze Theorem, as desired. Note by the Sequential Compactness Theorem, there exists a subsequence $\{x_{n_k}\}$ that converges to x_0 . Since f is continuous at x_0 , then $\{f(x_{n_k})\} \rightarrow f(x_0)$. Recall that $\{f(x_{n_k})\}$ is a subsequence of $\{f(x_n)\}$ that converges to c , and any subsequence must also converge to the same value as the full sequence. Therefore, $f(x_0) = c$. Therefore, the supremum exists and is a functional value, so we attain a max at x_0 . \square

§4.3 Intermediate Value Theorem

Theorem 4.14

The Intermediate Value Theorem state that suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous, let $c \in \mathbb{R}$ between $f(a)$ and $f(b)$. Then there exists $x_0 \in (a, b)$ such that $f(x_0) = c$.

Proof. Without loss of generality, suppose $f(a) < c < f(b)$. Recursively define a sequence of nested intervals starting at $[a, b]$ and converging to $x_0 \in (a, b)$ with $f(x) = c$. We WTS $f(x_0) = c$ by letting $a_1 = a, b_1 = b \forall n$.

$\forall n$ define $[a_n, b_n]$ by considering the midpoint $m_n = \frac{a_n + b_n}{2}$. Let us consider some cases.

\implies If $f(m_n) \leq c$, define $a_{n+1} = m_n$ and $b_{n+1} = b_n$.

\impliedby If $f(m_n) > c$, define $a_{n+1} = a_n$ and $b_{n+1} = m_n$.

Note that $a \leq a_n \leq a_{n+1} < b_{n+1} < b_n \leq b$ and $f(a_{n+1}) \leq c$ and $f(b_{n+1}) > c$ by definition. Now, we want to show that

$$\lim_{n \rightarrow \infty} (b_n - a_n) = 0$$

in order to apply the Nested Interval Theorem. □