DSC3214

Introduction to Optimization

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Semester II

NUS
National University
of Singapore

About me

- Bachelors in Computer Science
  - From IIT Chennai
- Masters and PhD from MIT
  - Operations Research
  - Economics
- Research
  - Analytics
  - Optimization under Uncertainty
  - Stochastic Optimization

Agenda

- What is this course about?
  - Analytics and Optimization
- Administrative issues
- Introduction

What is Analytics?

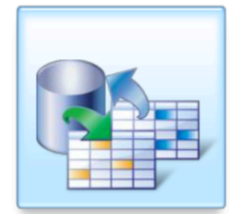
Analytics is the scientific process of deriving insights from data in order to make decisions

Data

Analyze →

Insight

Decide →



Action

Descriptive Analytics
What has happened?

Predictive Analytics
What will happen?

Prescriptive Analytics
What should we do?

→ *Business Value*

Levels of analytics

Prescriptive analytics

What is the best strategy

Predictive analytics

What will happen

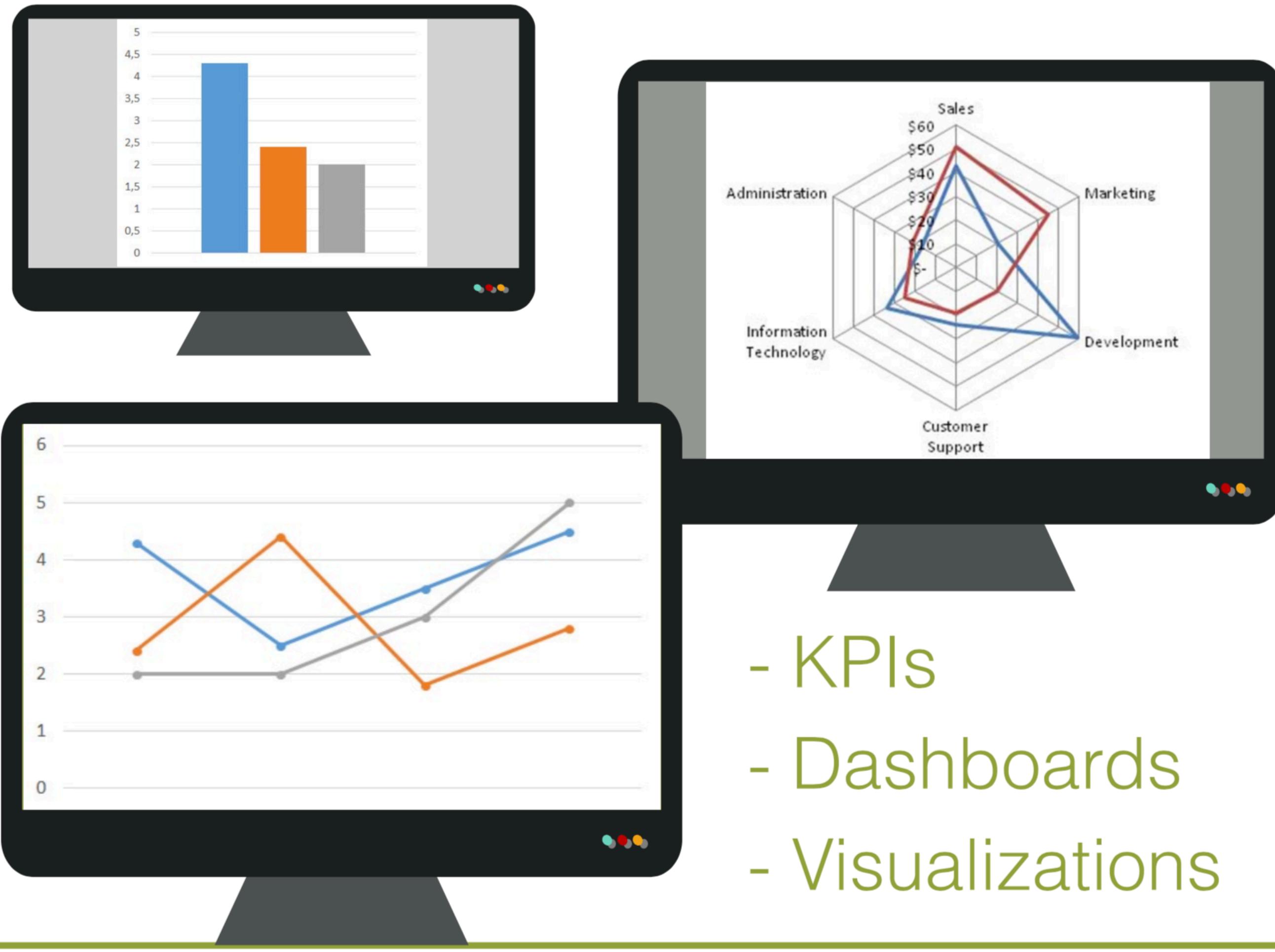
Descriptive analytics

What is happening

Descriptive analytics

What is happening

Describe the data to get useful info

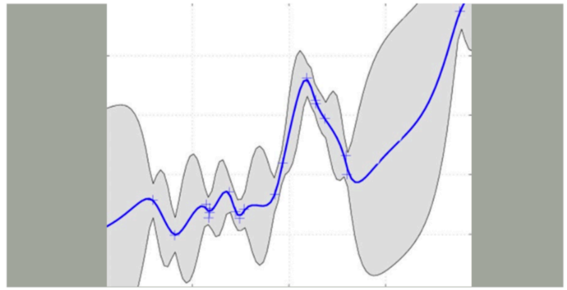


- KPIs
- Dashboards
- Visualizations

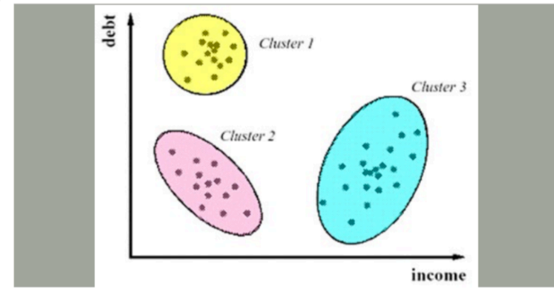
Predictive analytics

What will happen

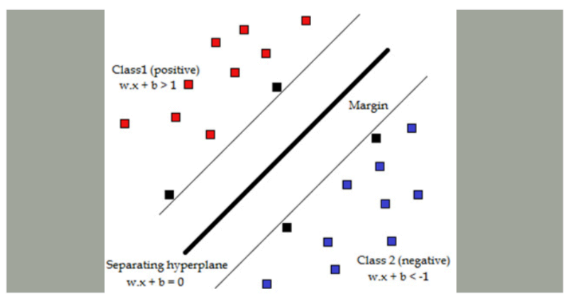
Estimate data we don't have



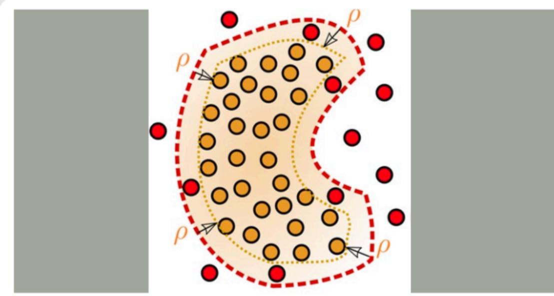
Regression



Clustering



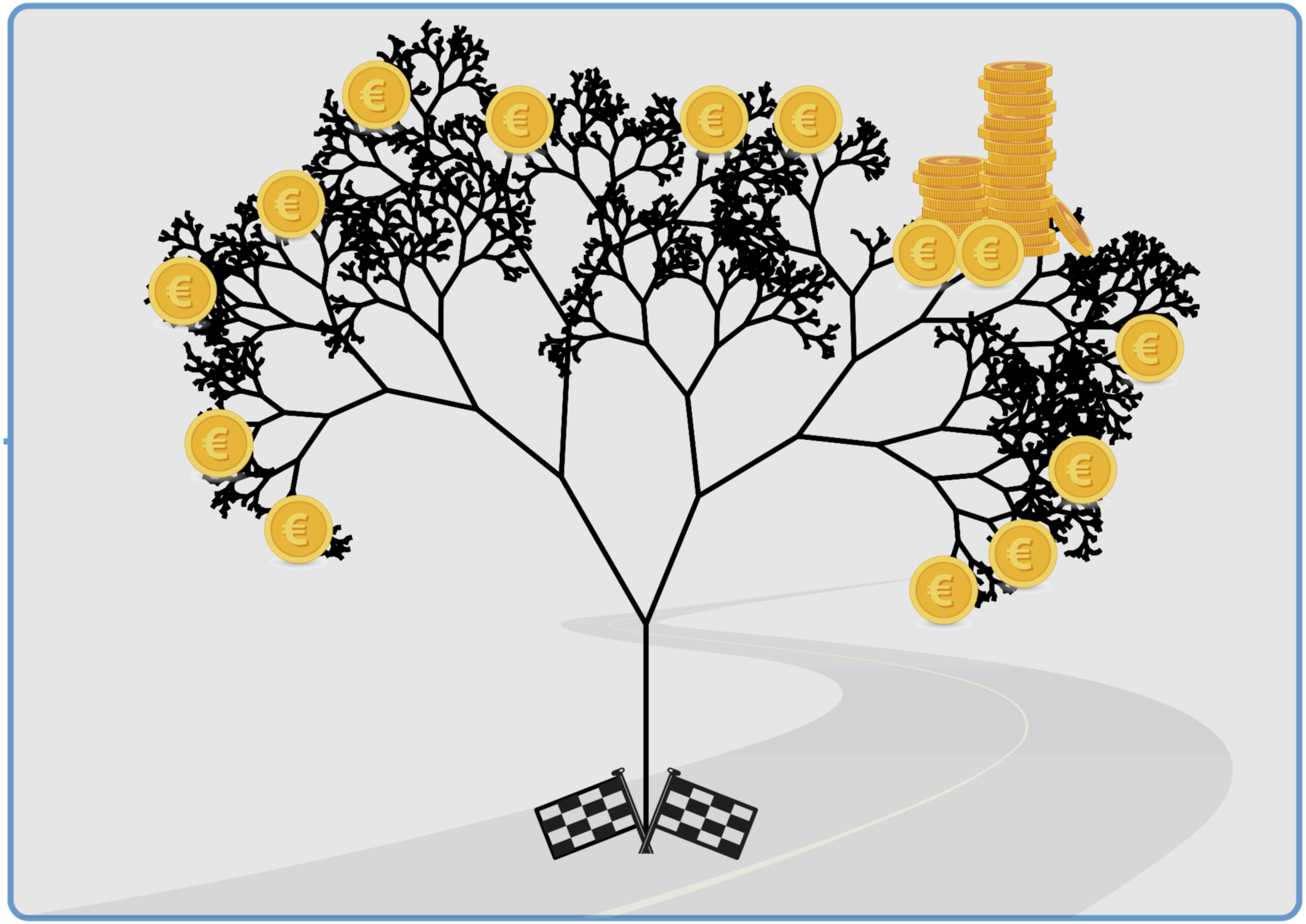
Classification



Anomaly detection

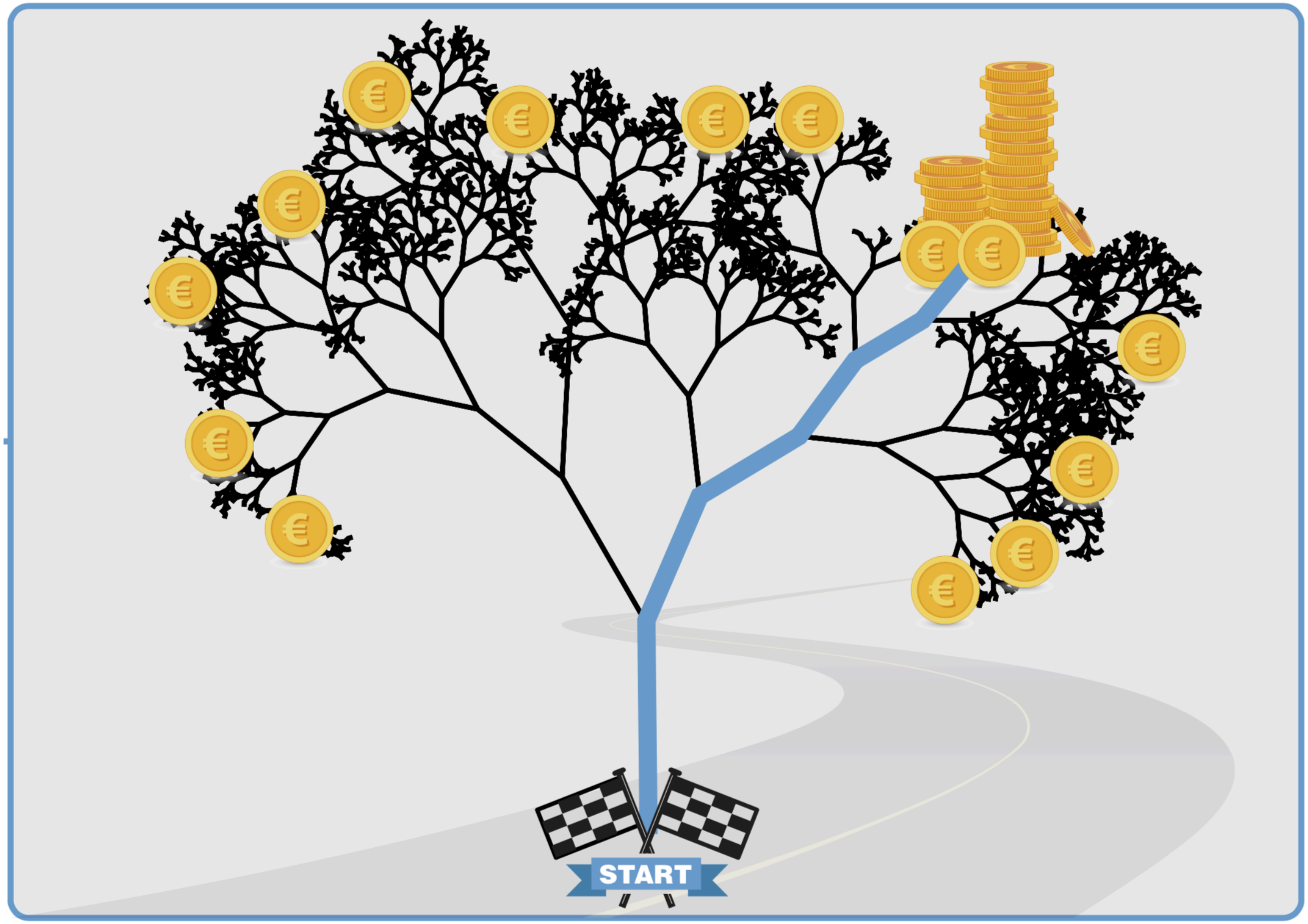
Prescriptive analytics

What is the best strategy

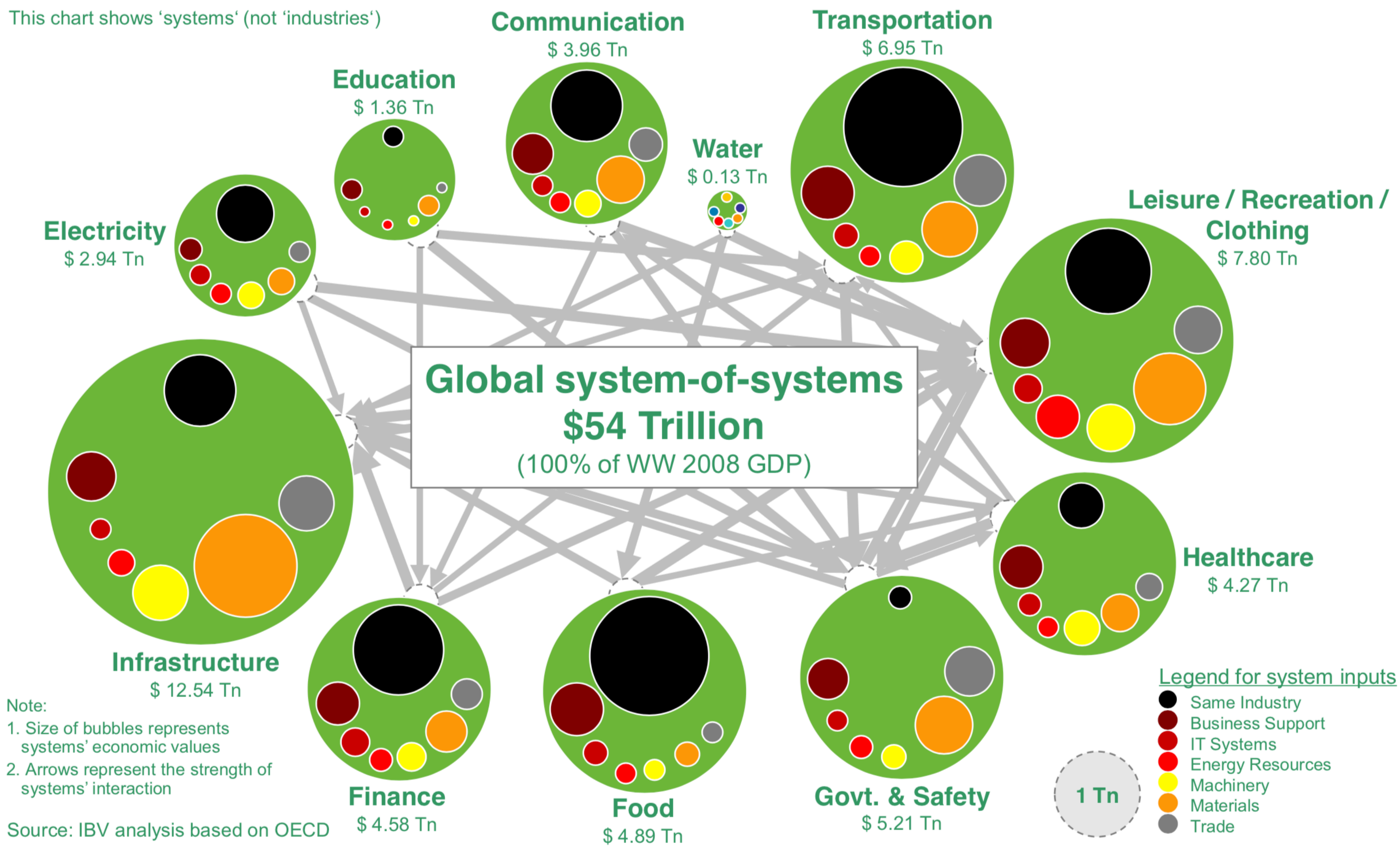


Prescriptive analytics

What is the best strategy



Global Economy

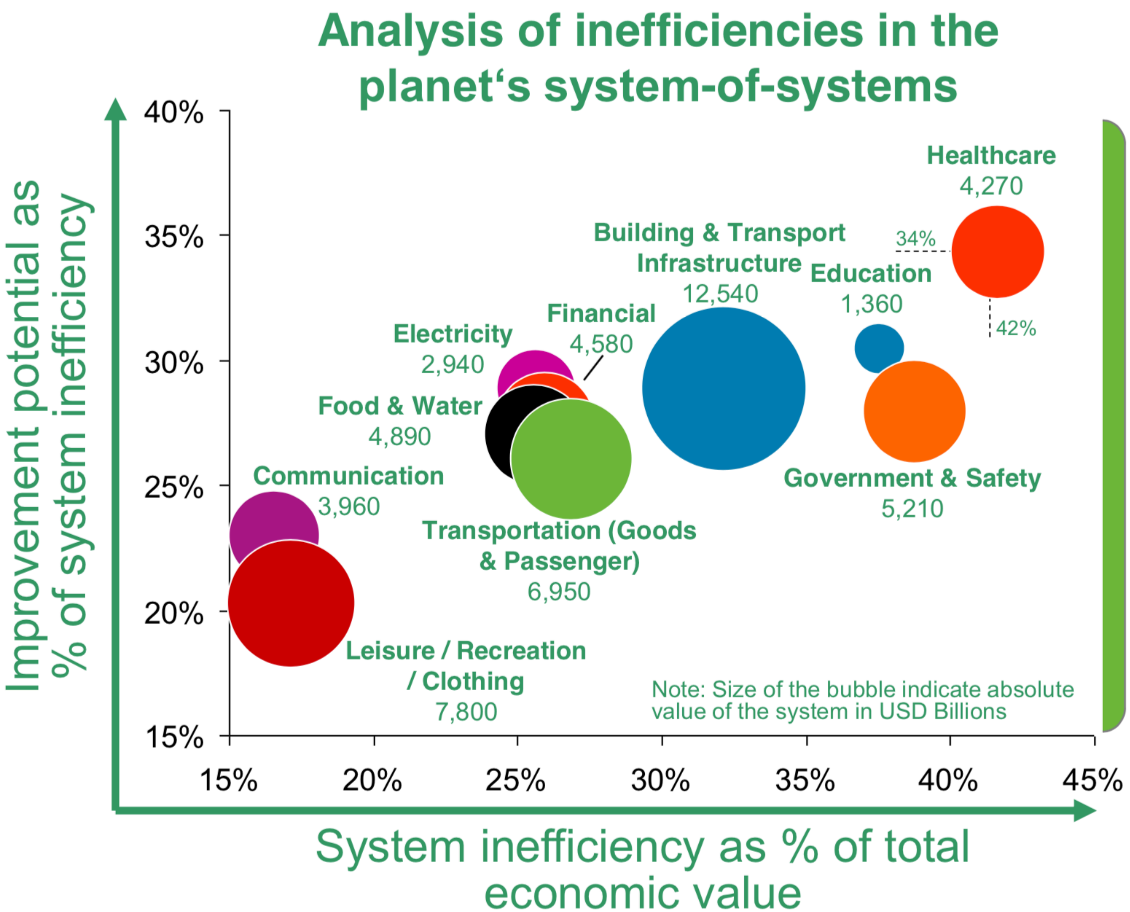


Note:
1. Size of bubbles represents systems' economic values
2. Arrows represent the strength of systems' interaction

Source: IBV analysis based on OECD

Optimizing the Global Economy

Economists estimate, that all systems carry inefficiencies of up to $15 Tn, of which $4 Tn could be eliminated



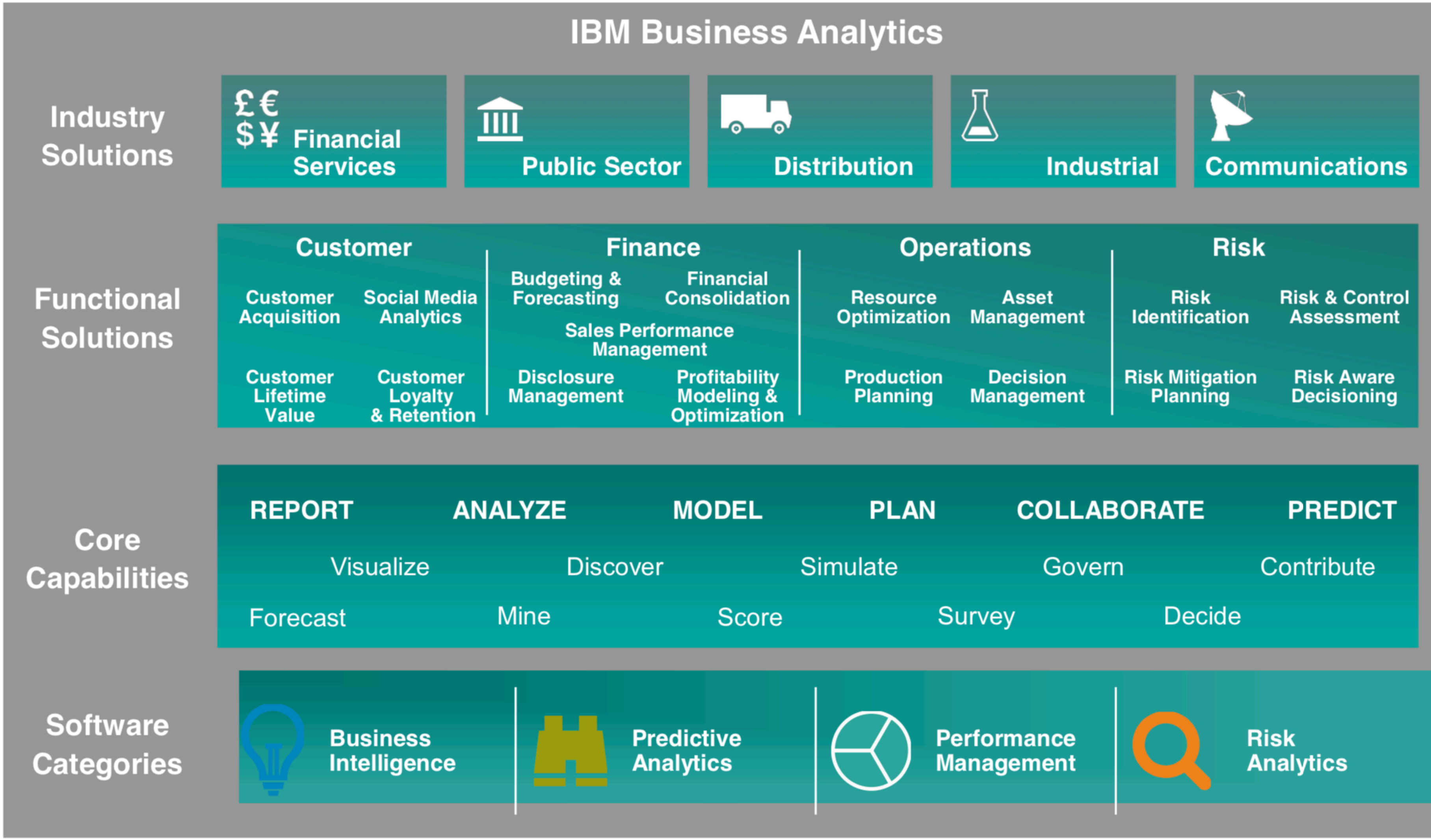
Global economic value of

System-of-systems	**$54 Trillion** 100% of WW 2008 GDP
Inefficiencies	**$15 Trillion** 28% of WW 2008 GDP
Improvement potential	**$4 Trillion** 7% of WW 2008 GDP

How to read the chart:
For example, the Healthcare system's value is $4,270B. It carries an estimated inefficiency of 42%. From that level of 42% inefficiency, economists estimate that ~34% can be eliminated (= 34% x 42%).

Source: IBM economists survey 2009; n= 480

Example.


IBM Business Analytics
Industry Solutions
Financial Services
Public Sector
Distribution
Industrial
Communications
Functional Solutions
Customer
Customer Acquisition
Social Media Analytics
Customer Lifetime Value
Customer Loyalty & Retention
Finance
Budgeting & Forecasting
Financial Consolidation
Sales Performance Management
Disclosure Management
Profitability Modeling & Optimization
Operations
Resource Optimization
Asset Management
Production Planning
Decision Management
Risk
Risk Identification
Risk & Control Assessment
Risk Mitigation Planning
Risk Aware Decisioning
Core Capabilities
REPORT
ANALYZE
MODEL
PLAN
COLLABORATE
PREDICT
Visualize
Discover
Simulate
Govern
Contribute
Forecast
Mine
Score
Survey
Decide
Software Categories
Business Intelligence
Predictive Analytics
Performance Management
Risk Analytics

Textbook and Reference

- **Text:**
  - Introduction to Operations Research, Hillier and Lieberman (2015)
  - Applied Mathematical Programming, Bradley, Hax and Magnanti (1977)
  - Introduction to Linear Optimization, Bertsimas and Tsitsiklis (1997)
  - Mostly, lecture notes
- **Software:**
  - Python (Anaconda) and Gurobi
  - Excel Solver

Optimization Software

V · T · E	Mathematical optimization software [hide]
Data formats	LP · MPS · nl · OptML · OSiL · sol · xMPS
Modeling tools	AIMMS · AMPL · APMonitor · CMPL · CVX · CVXOPT · CVXPY · ECLiPSe-CLP · GEKKO · GAMS · GNU MathProg · JuMP · LINDO · OPL · MPL · OptimJ · PICOS · PuLP · Pyomo · ROML · TOMLAB · Xpress-Mosel · YALMIP · ZIMPL
LP, MILP* solvers	ABACUS* · APOPT* · Artelys Knitro* · BCP* · BDMLP · BPMPD · BPOPT · CLP · CBC* · CPLEX* · CSDP · DSDP · FortMP* · GCG* · GIPALS32 · GLPK/GLPSOL* · Gurobi* · HOPDM · LINDO* · Lp_solve · LOQO · MINOS · MINTO* · MOSEK* · OOPS · OOQP · PCx · QSopt · SAS/OR* · SCIP* · SoPlex · SOPT-IP* · Sulum Optimization Tools* · SYMPHONY* · XA* · Xpress-Optimizer*
QP, MIQP* solvers	APOPT* · Artelys Knitro* · BPMPD · BPOPT · BQPD · CBC* · CLP · CPLEX* · FortMP* · GloMIQO* · Gurobi* · IPOPT · LINDO* · LSSOL · LOQO · MINOS · MOSEK* · OOPS · OOQP · OSQP · QPOPT · QPSOL · SCIP* · XA Quadratic Solver · Xpress-Optimizer*
QCP, MIQCP* solvers	APOPT* · Artelys Knitro* · BPMPD · BPOPT · CPLEX* · GloMIQO* · Gurobi* · IPOPT · LINDO* · LOQO · MINOS · MOSEK* · SCIP* · Xpress-Optimizer* · Xpress-SLP*
SOCP, MISOCP* solvers	CPLEX* · DSDP · Gurobi* · LINDO* · LOQO · MOSEK* · SCIP* · SDPT3 · SeDuMi · Xpress-Optimizer*
SDP, MISDP* solvers	CSDP · DSDP · MOSEK · PENBMI · PENSDP · SCIP-SDP* · SDPA · SDPT3 · SeDuMi
NLP, MINLP* solvers	ALGENCAN · AlphaECP* · ANTIGONE* · AOA* · APOPT* · Artelys Knitro* · BARON* · Bonmin* · BPOPT · CONOPT · Couenne* · DICOPT* · FilMINT* · FilterSQP · Galahad library · ipfilter · IPOPT · LANCELOT · LINDO* · LOQO · LRAMBO · MIDACO* · MILANO* · MINLP BB* · MINOS · Minotaur* · MISQP* · NLPQLP · NPSOL · OQNLP* · PATHNLP · PENNON · SBB* · SCIP* · SNOPT* · SQPlab · WORHP · Xpress-SLP*
GO solvers	BARON · Couenne* · LINDO · SCIP
CP solvers	Artelys Kalis · Choco · Comet · CPLEX CP Optimizer · Gecode · Google CP Solver · JaCoP · OscaR
Metaheuristic solvers	OptaPlanner · LocalSolver

What You Should Expect

- **Mathematical Thinking**
  - Understand the basic logic behind algorithms
  - Understand the general principles in optimization
  - Judgement on the difficulty of a problem; be able to assess the quality of a solution
- **Analytics Skills**
  - Building up models
  - Picking the right tools
  - Implementing in Software

What I Expect

- Prerequisite
  - Linear algebra
- Assessment
  - Team Term Paper 30%
  - Team Term Paper Presentation 5%
  - Individual Assignments 30%
  - In-Class Quizzes 30%
  - Class Participation 5%

Term Paper

- Apply the analytical techniques to a business problem of your choosing
  - Motivation and statement of the problem
  - Methodology and processes of solving the problem
  - Analysis and interpretation of the results

Course Content

- Theory and application of linear programming*
- Integer programming and network flow*
- Dynamic programming*
- Heuristic design

Homework 0

- Download and install python (Anaconda distribution) and gurobi (academic license)
  - See the note and Gurobi optimizer quickstart guide
- Form a team of around 6 members

Operations Research in Practice

In a Nutshell

- **Maximize/Minimize …**
- **Subject to …**

Areas of Operations Research

- Manufacturing
- Transportation
- Airline, Hotel and Retail

Synonyms and overlaps: Management Science, Decision Science, Industrial Engineering, Operations Management.

Areas of Operations Research

DiDi
JD.COM
UBER
THE UNIVERSITY OF ARIZONA College of Engineering
杉数科技 Cardinal Operations

FICO
GUROBI OPTIMIZATION
wayfair a zillion things home
SF TECHNOLOGY 顺 丰 科 技

悠桦林 YOUHUALIN
IBM
AIMMS
AMPL
BAYER

sas
OPERATIONS RESEARCH AND INDUSTRIAL ENGINEERING UTORIE COCKRELL SCHOOL OF ENGINEERING THE UNIVERSITY OF TEXAS AT AUSTIN
Carnegie Mellon Tepper SCHOOL OF BUSINESS
GAMS
PennState

UT DALLAS Naveen Jindal School of Management
lyft

Ford

MAR 17, 2011

FORD OPENS FLEXIBLE, GREEN MICHIGAN ASSEMBLY PLANT WITH PRODUCTION OF ALL-NEW FORD FOCUS

- Transformed Michigan Assembly Plant (MAP), which will produce the fuel-efficient new global Ford Focus for North American customers, features flexible manufacturing, environmentally friendly practices and a highly trained work force
- A $550 million investment transformed the plant, creating new benchmarks for flexible manufacturing. Multiple models to run down the same production line, making Michigan Assembly the world's first plant to build gasoline-powered, battery electric, hybrid electric and plug-in hybrid electric vehicles on the same line
- MAP also features one of Michigan's largest solar-powered generation systems and electric vehicle charging stations

Ford Motor Company celebrated the launch of the 2012 Ford Focus with employees, dealers, suppliers, media and other invited guests at the Michigan Assembly Plant.

Click here to download related images.

Ford

FIESTA
Starting MSRP $14,090[1]

FOCUS
Starting MSRP $17,225[1]

FUSION
Starting MSRP $22,120[1]

MUSTANG
Starting MSRP $24,645[1]

C-MAX
Starting MSRP $24,170[1]

TAURUS
Starting MSRP $27,110[1]

United States - Chicago Assembly Plant +

United States - Dearborn Truck Plant +

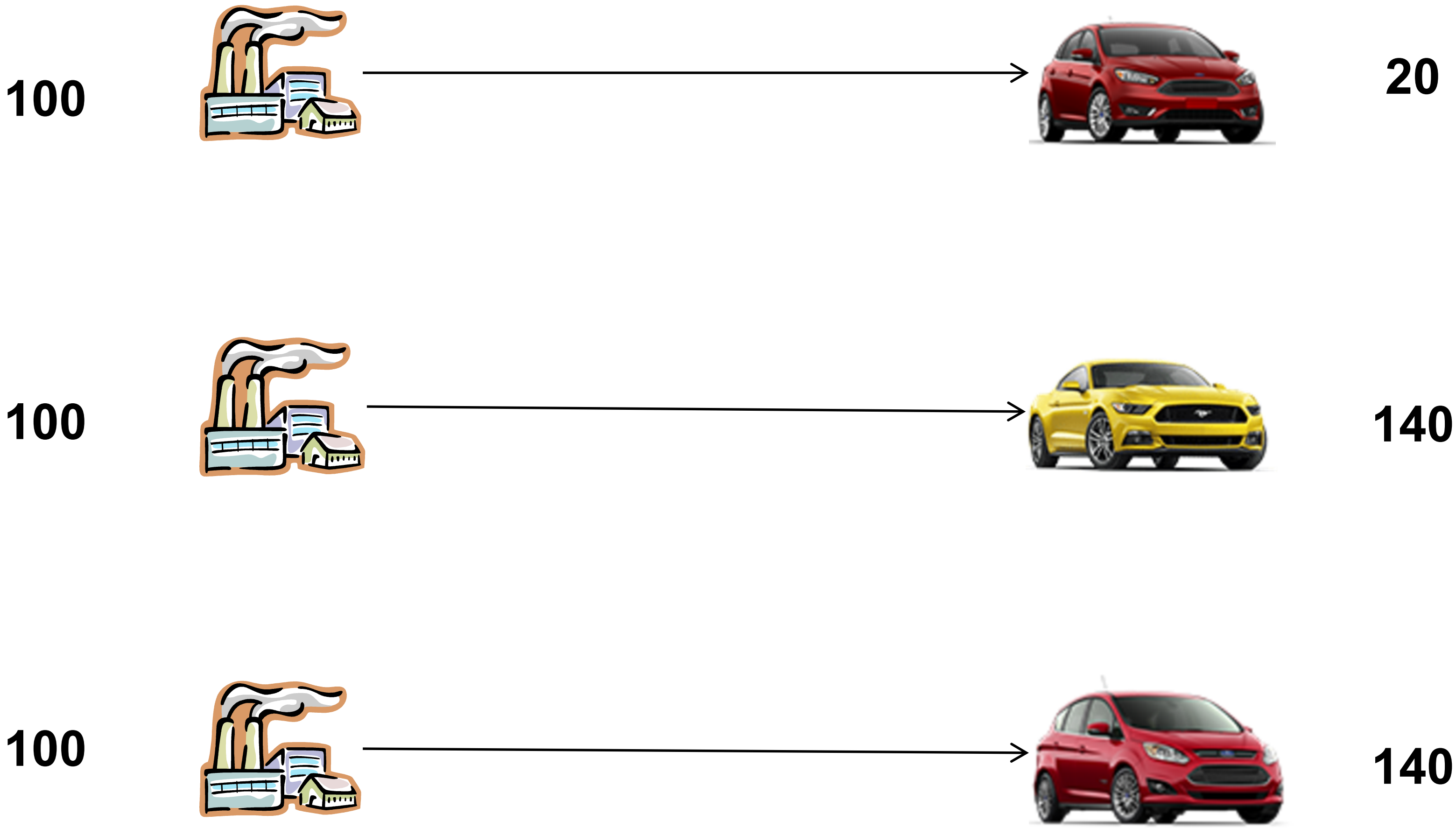
United States - Kansas City Assembly Plant +

United States - Kentucky Truck Plant +

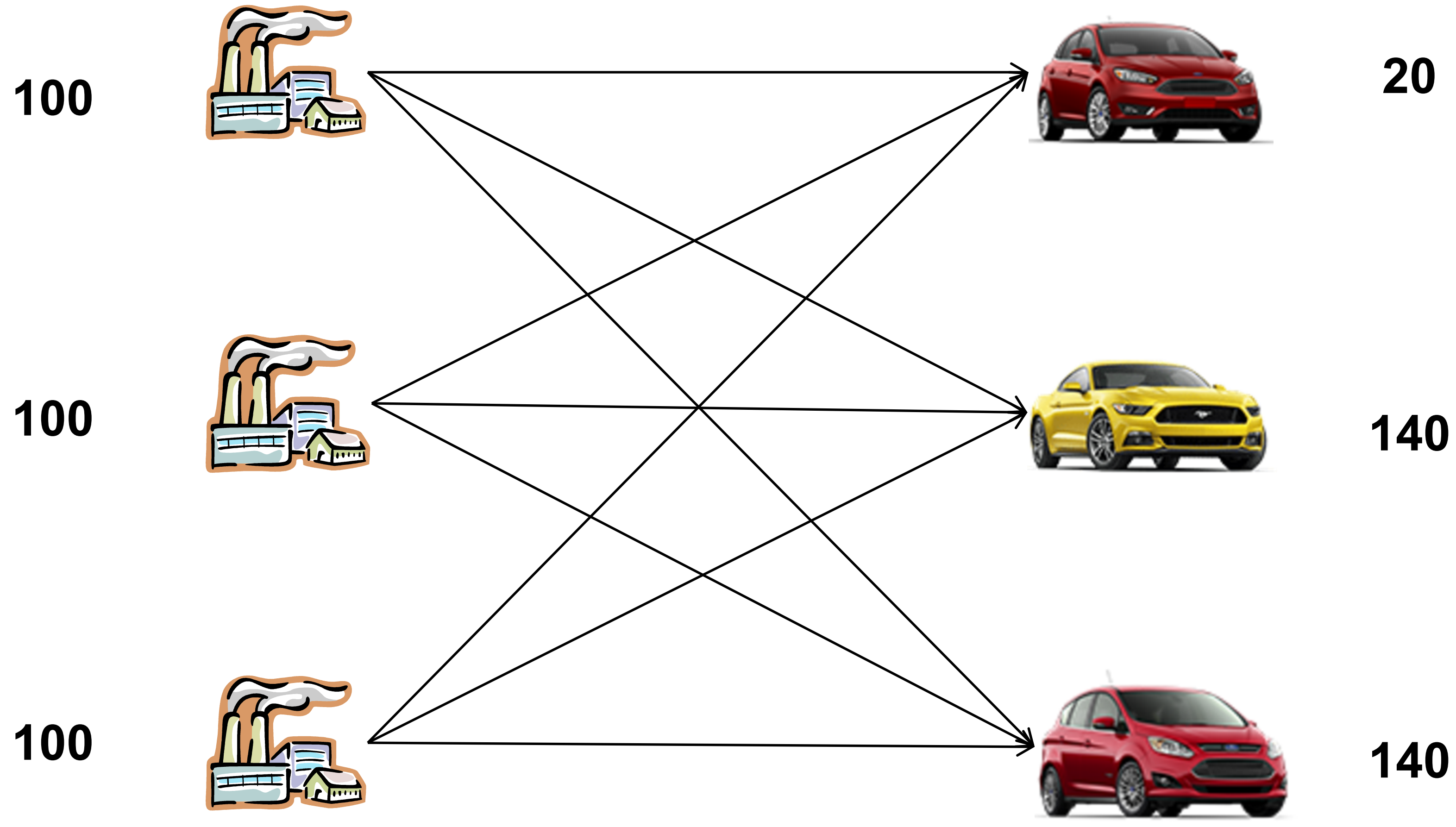
United States - Louisville Assembly Plant +

Unites States - Michigan Assembly Plant +

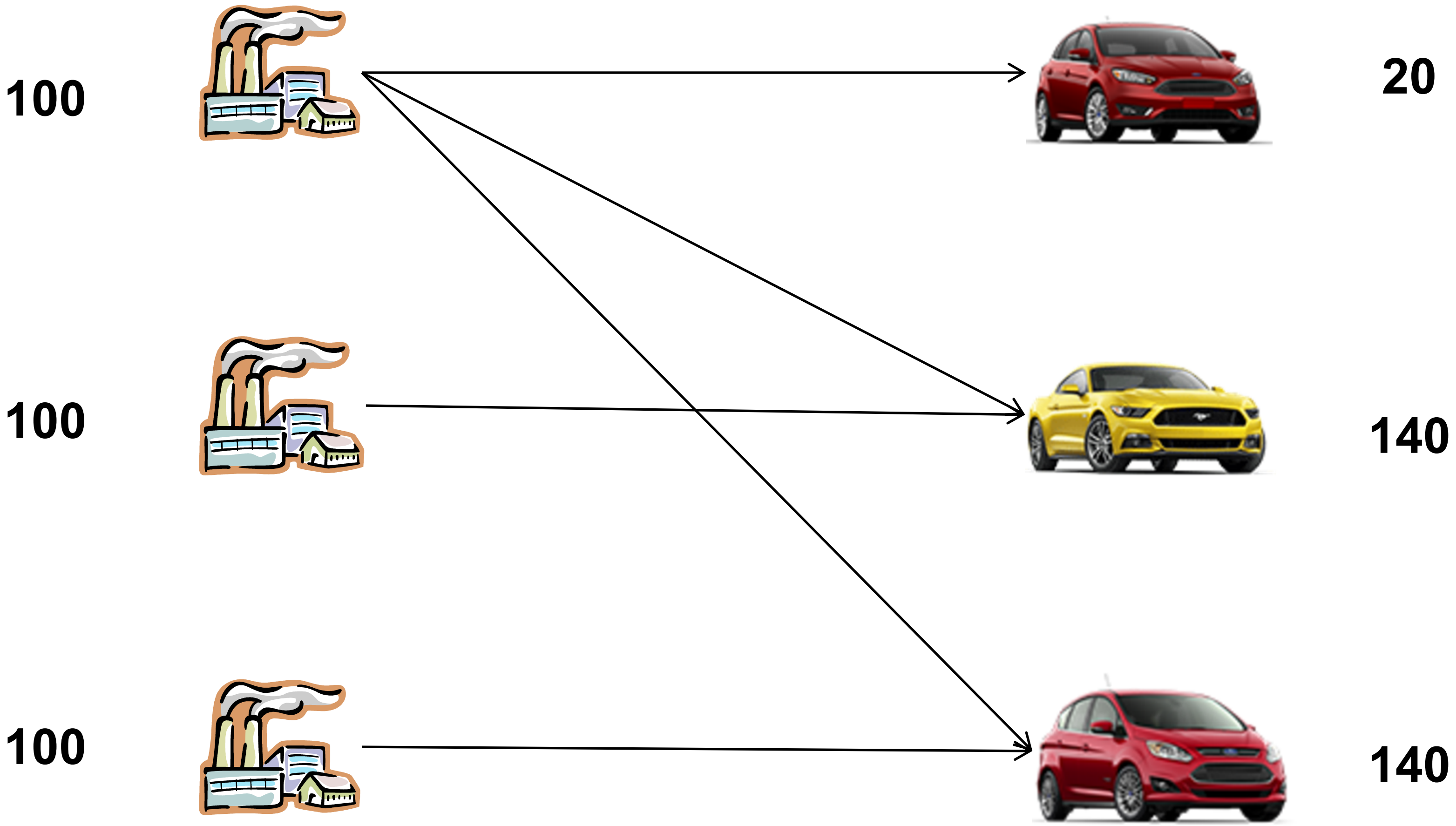
Traditional Dedicated System



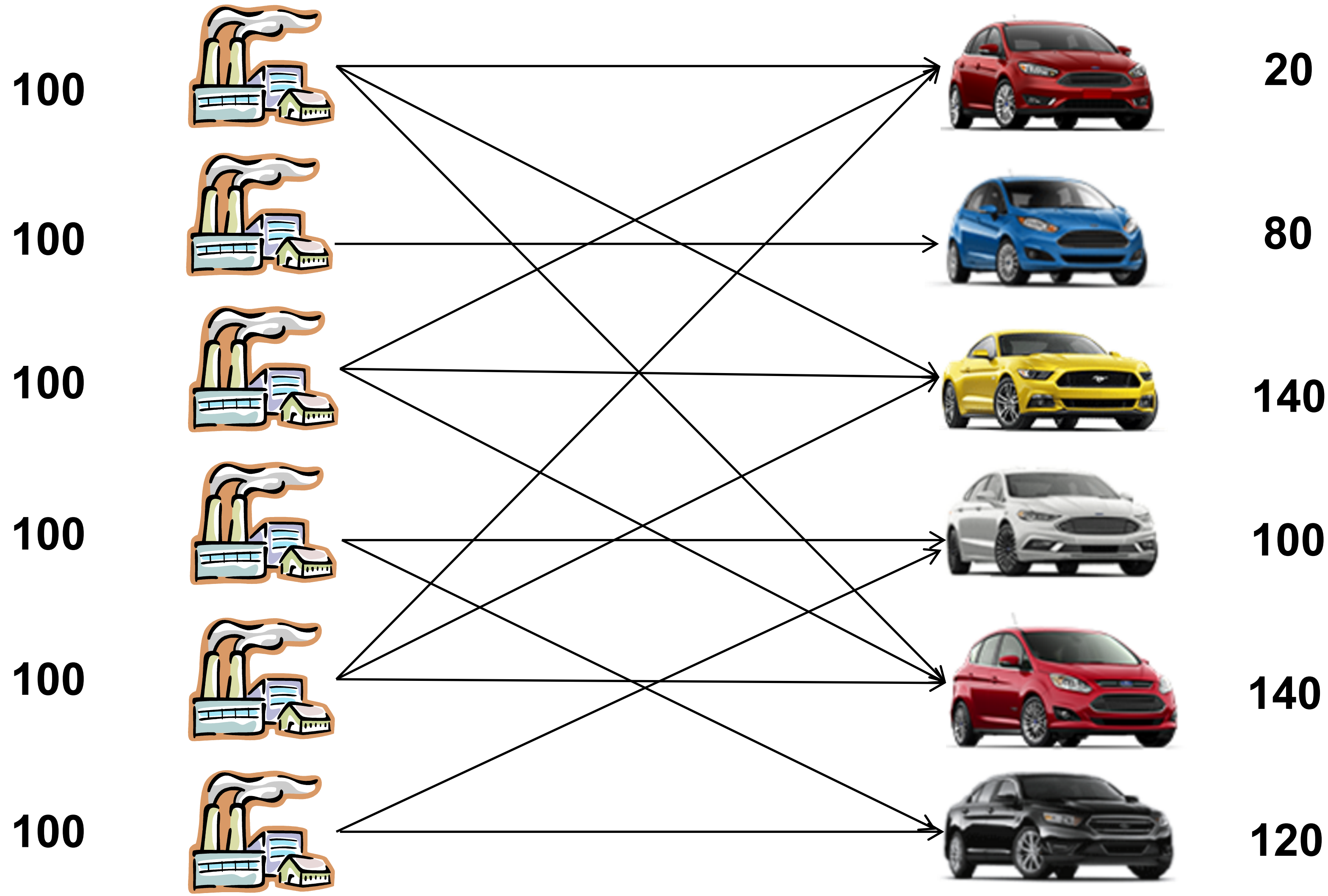
Full Flexible System



If we know demand…



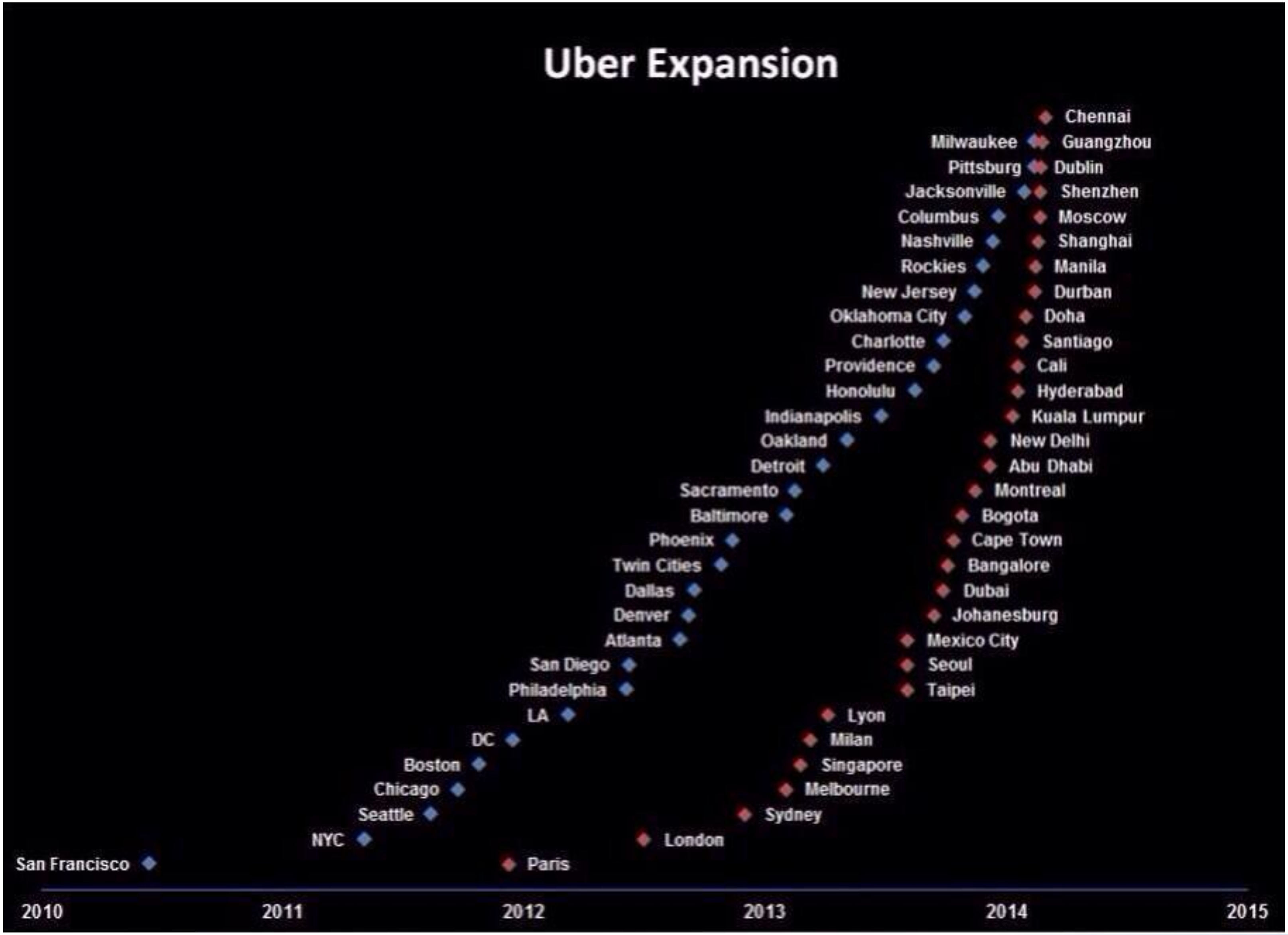
How to Compute the Maximum Sales?



Operations Research Tools

- Given a design, how do we evaluate its performance?
  - Linear programming
  - Maximum flow
  - Simulation

Uber

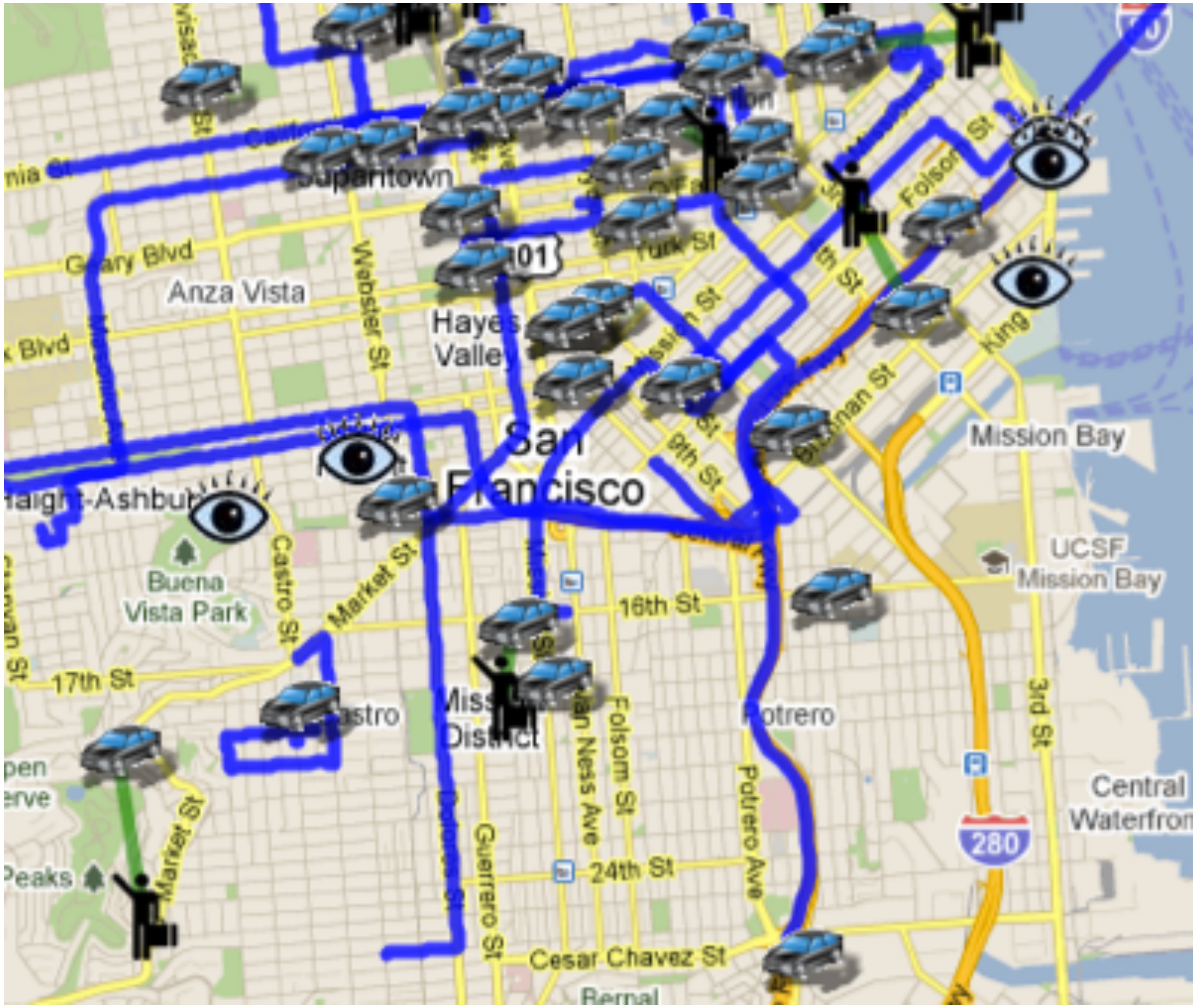

Uber Expansion
Chennai
Milwaukee
Guangzhou
Pittsburg
Dublin
Jacksonville
Shenzhen
Columbus
Moscow
Nashville
Shanghai
Rockies
Manila
New Jersey
Durban
Oklahoma City
Doha
Charlotte
Santiago
Providence
Cali
Honolulu
Hyderabad
Indianapolis
Kuala Lumpur
Oakland
New Delhi
Detroit
Abu Dhabi
Sacramento
Montreal
Baltimore
Bogota
Phoenix
Cape Town
Twin Cities
Bangalore
Dallas
Dubai
Denver
Johanesburg
Atlanta
Mexico City
San Diego
Seoul
Philadelphia
Taipei
LA
Lyon
DC
Milan
Boston
Singapore
Chicago
Melbourne
Seattle
Sydney
NYC
London
San Francisco
Paris
2010
2011
2012
2013
2014
2015

Uber

- **What does Uber offer?**

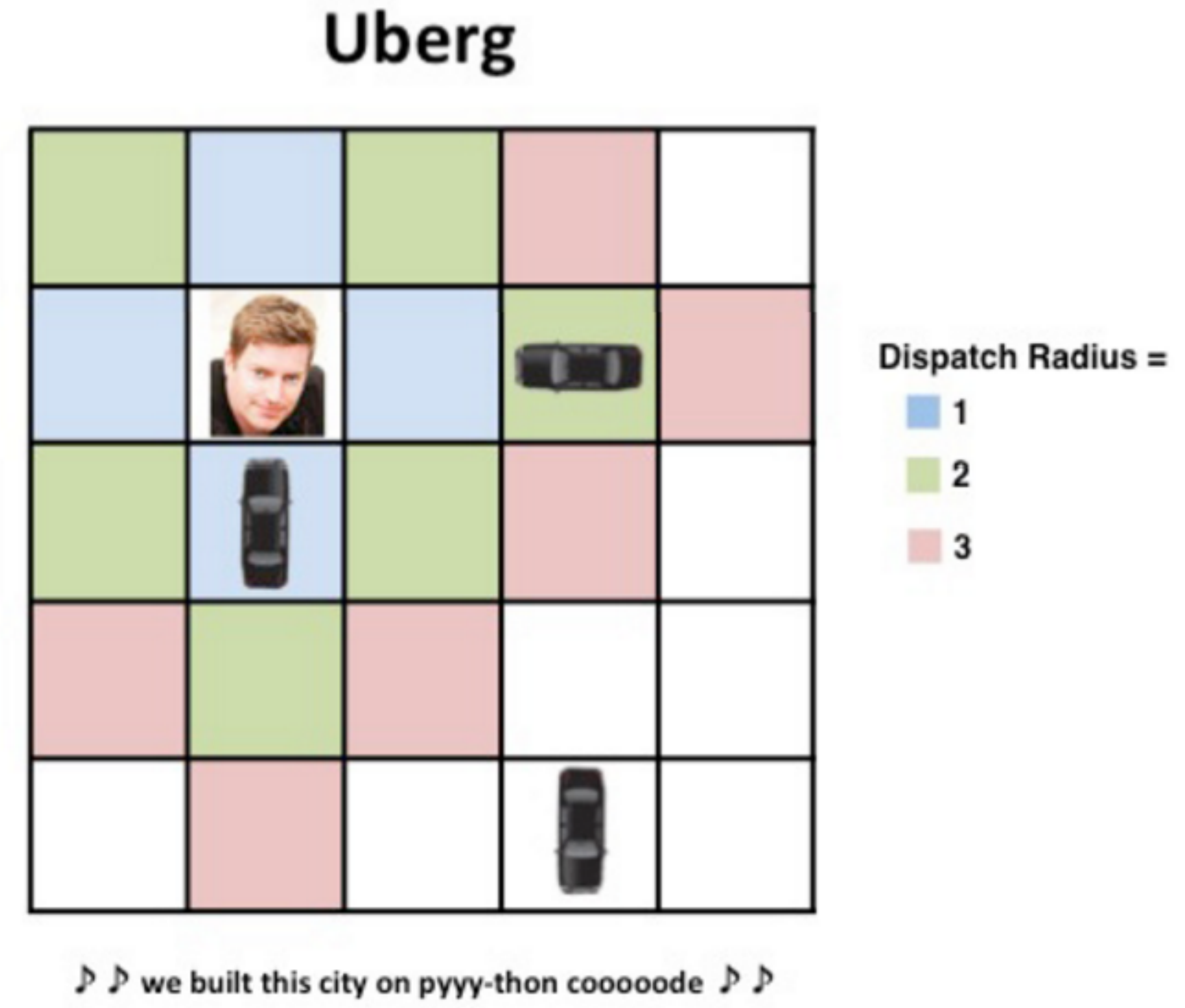
- **Matching Algorithms**
  - Dispatch system
  - Surge pricing
  - Uber POOL

Dispatch System



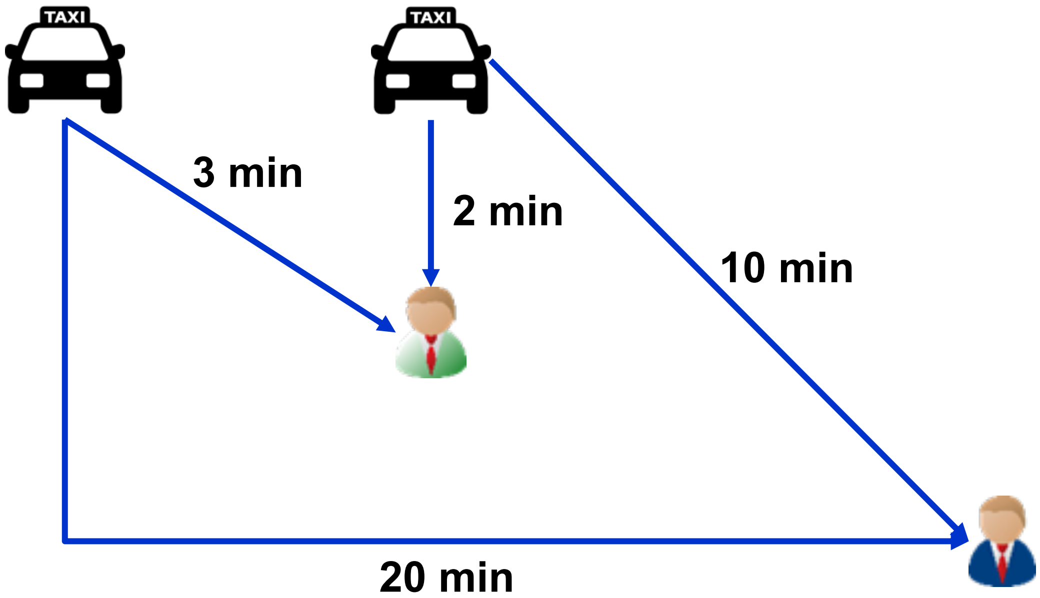
Source: http://www.wired.com/2011/04/app-stars-uber/

Greedy Approach



Source: https://newsroom.uber.com/semi-automated-science-using-an-ai-simulation-framework/

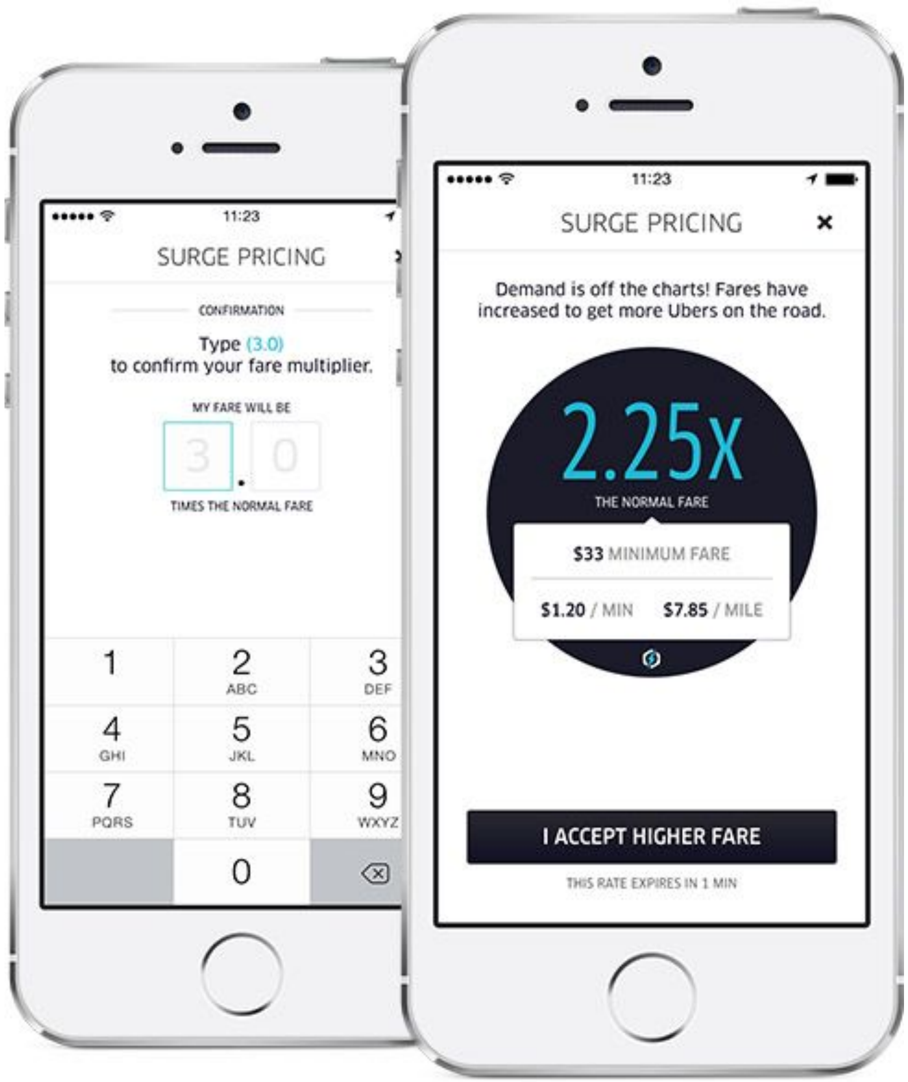
Is Greedy Good?



Operations Research Tools

- Linear programming.
- Shortest Path.
- Assignment Problem.

Surge Pricing

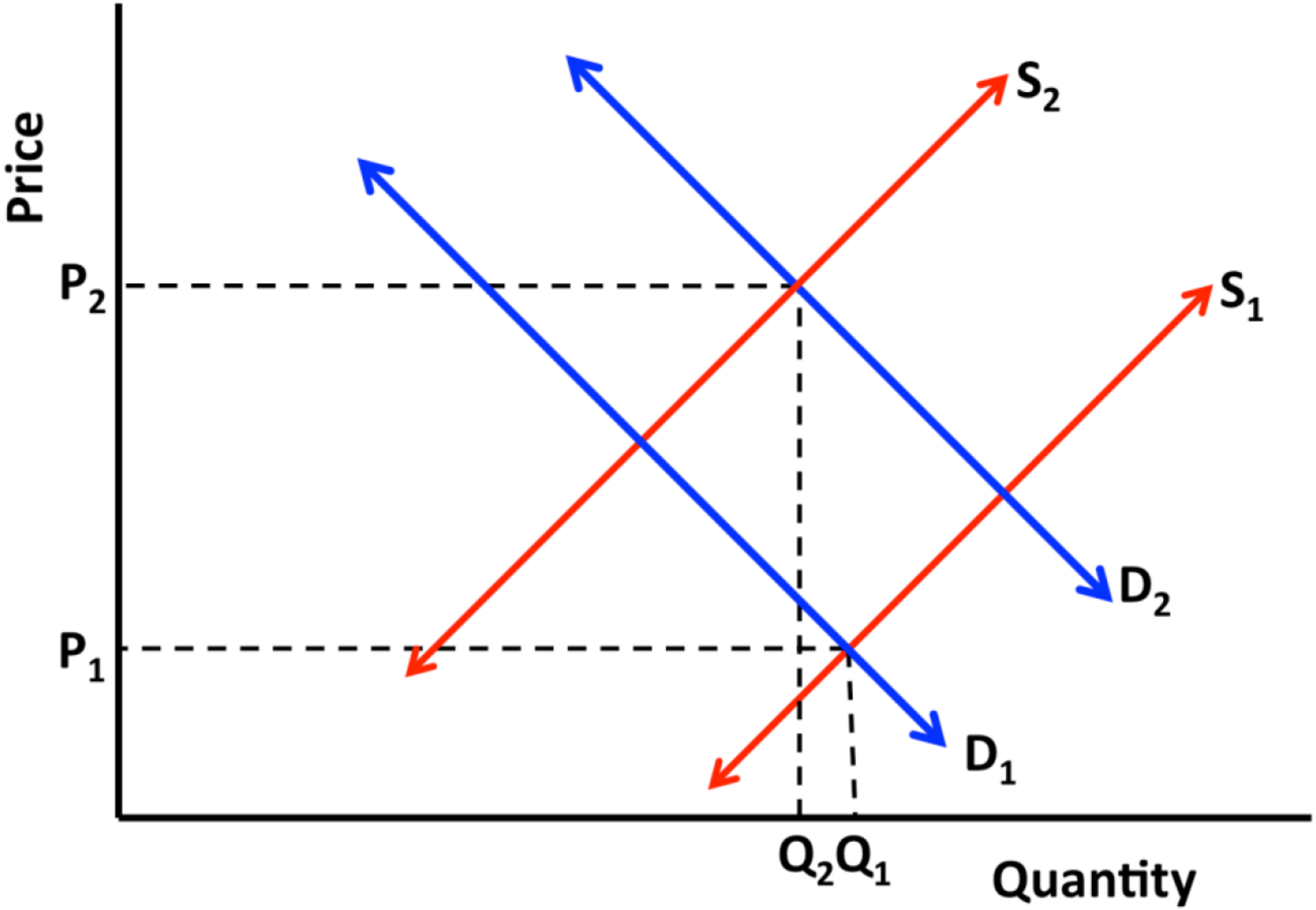


Source: http://www.tnp.sg/news/singapore-news/she-pays-169-27-minute-uber-ride-beach-road-ayer-rajah

Why?

- **Profit Maximization?**
  - **Compare to airline industry**

ECON 1001



Source: https://newsroom.uber.com/guest-post-a-deeper-look-at-ubers-dynamic-pricing-model/

A Model for Supply and Demand

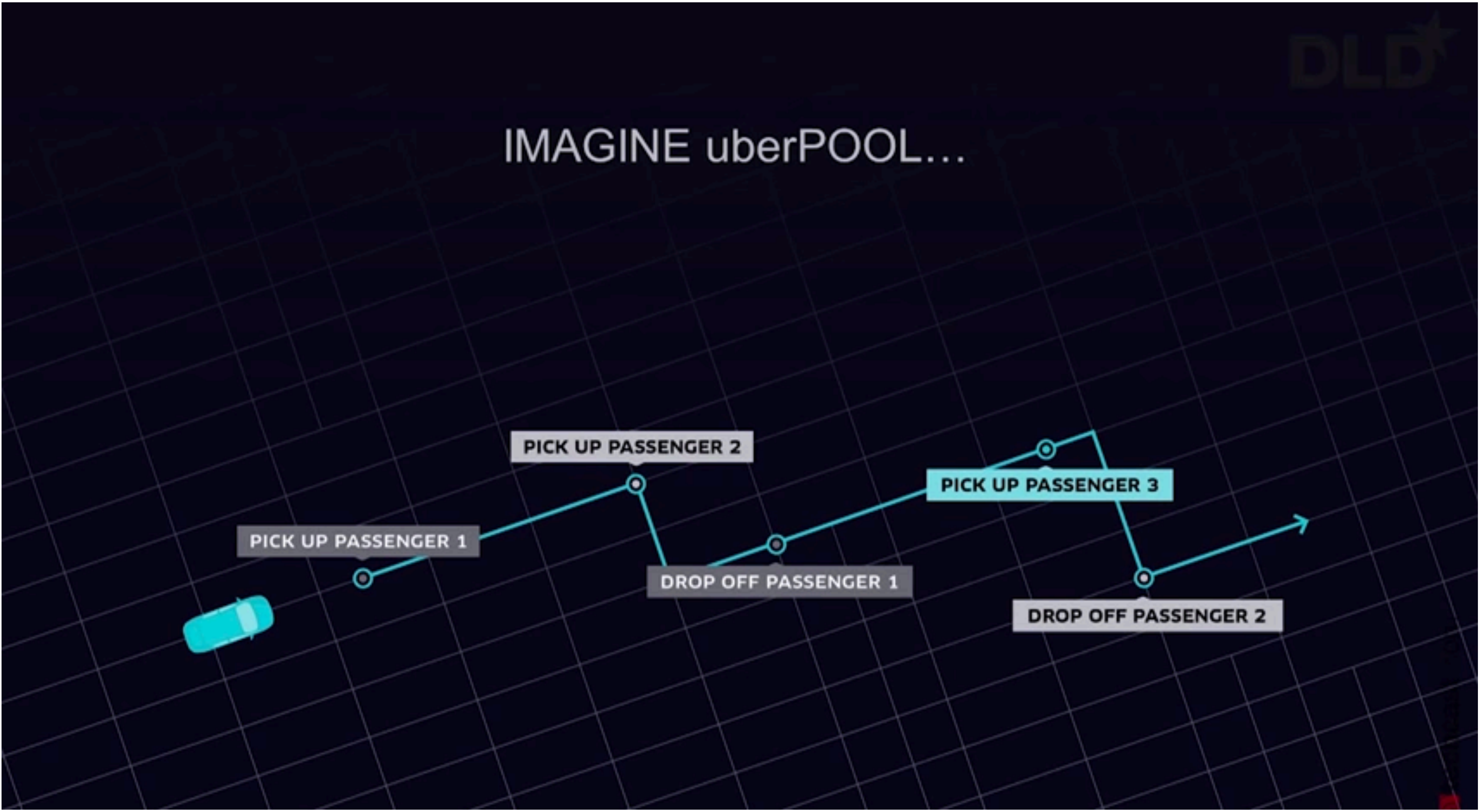
- ***n*** **locations.**
- **Demand at location *i*:** $a_i - b_i p_i$
- **Supply at location *i*:**

$$c_i + d_i p_i - \sum_{j \neq i} \theta_{ij} p_j$$

Optimization Problem

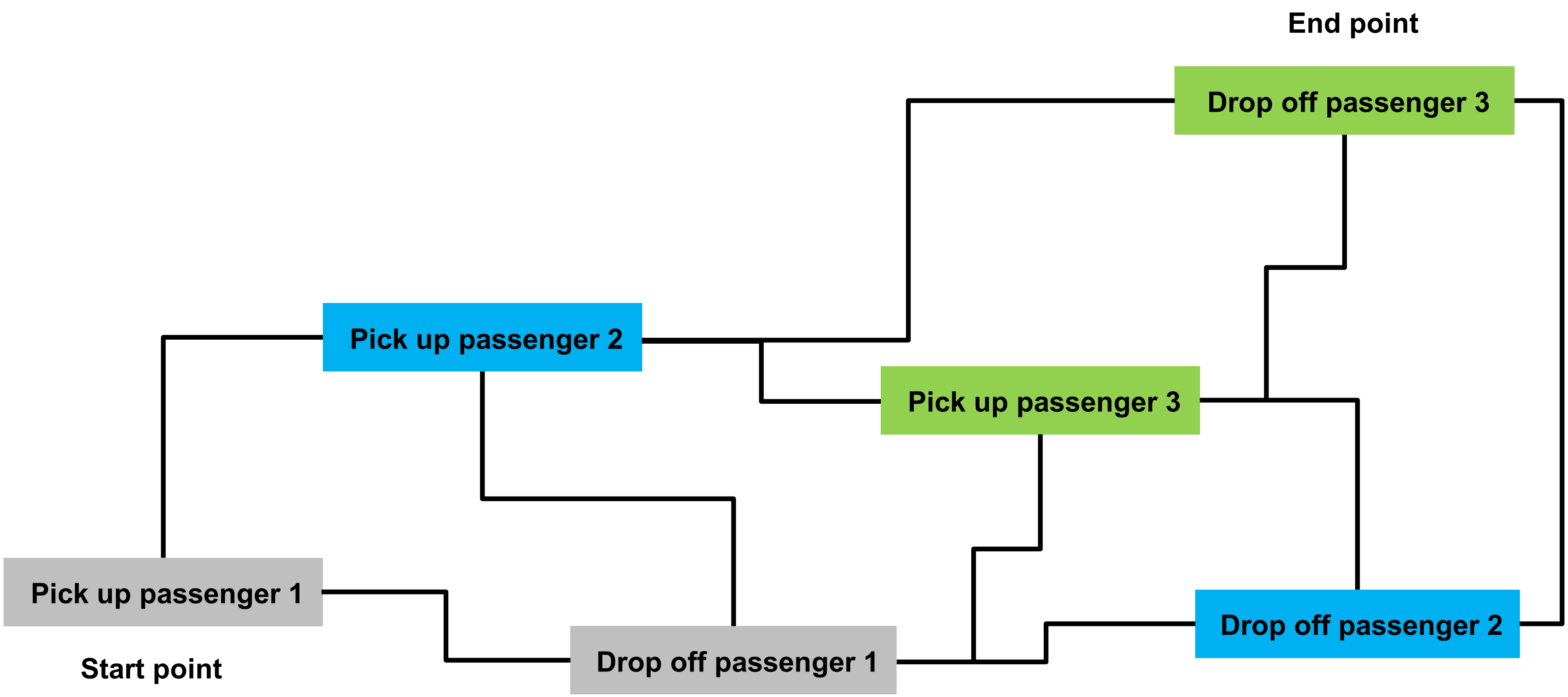
- Non-profit
  - Linear programming (in fact, linear algebra)
- Profit
  - Nonlinear programming

UberPOOL



Source: http://blog.under.com/uberPOOL-2015

Traveling Salesman Problem



Solving TSP

- Linear programming.
- Integer programming.
- Heuristic design.

Rue La La

LAGOS

Closing in 1 day, 07:01:02

First Look: Cool-Weather Style for Men

Closing in 3 days, 07:01:02

Feel Brand New: Crisp Looks for Work

Closing in 1 day, 07:01:02

Source: https://www.ruelala.com/boutique/

Rue La La

John Varvatos Star U.S.A. Linen Moto Jacket
$169.99 ~~$498.00~~
2 LEFT

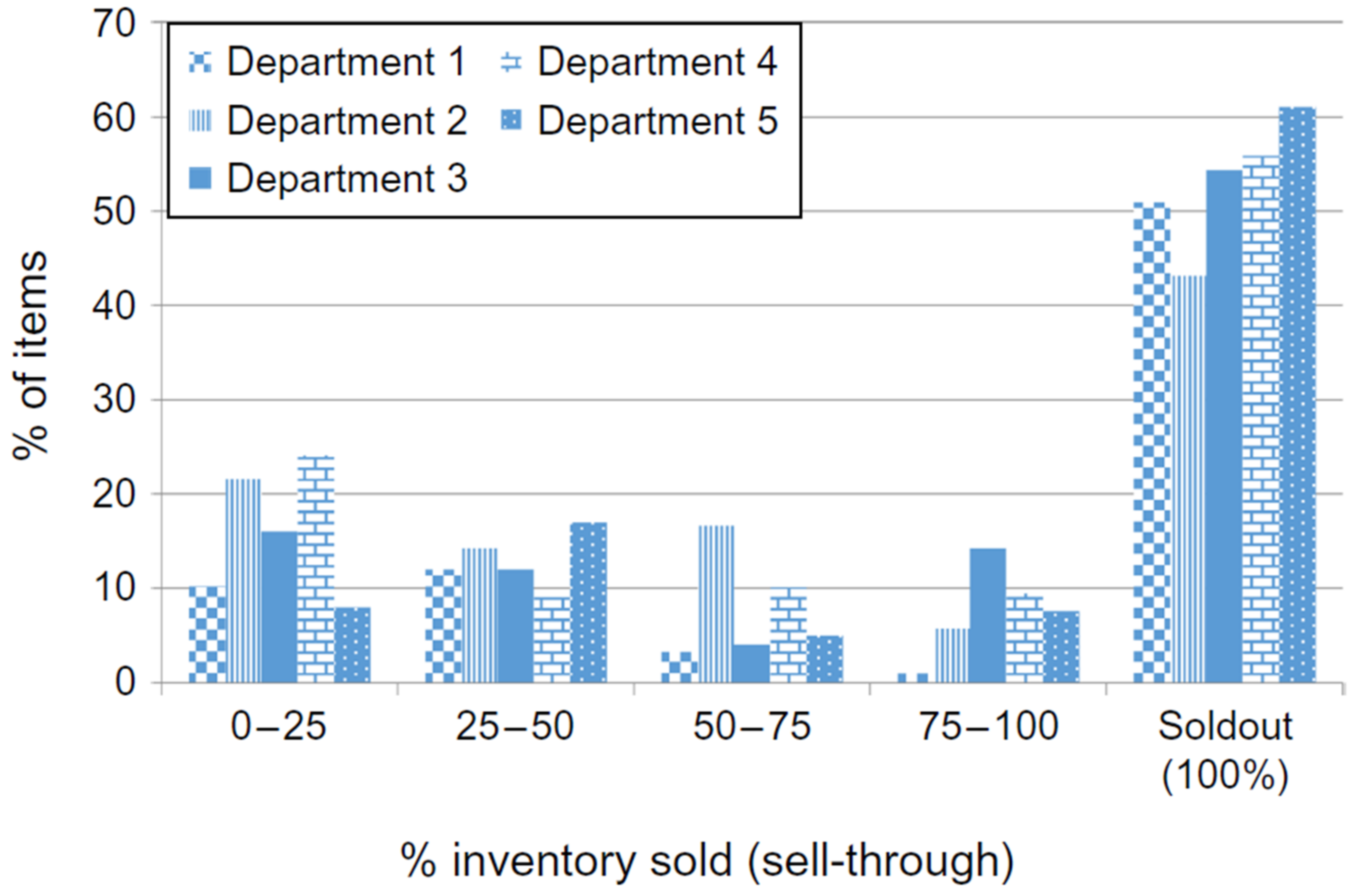
Feldton Japanese Selvedge Jacket
$199.99 ~~$398.00~~

John Varvatos Star U.S.A. Linen-Blend Hooded Pullover
$84.99 ~~$248.00~~

Source: https://www.ruelala.com/boutique/

- **Legacy pricing process**
  - **Percentage markup on cost**
  - **Competitor's price**
  - **Subjective judgement**

Rue La La



Source: Ferreria, Lee, Simchi-Levi (2016), "Analytics for an Online Retailer: Demand Forecasting and Price Optimization", 2016, *MSOM,* Vol. 18, No.1

Price Optimization

- **Methodology**
  - **Integer programming/ linear programming**
- **Experiment**

Category	Estimate of percent increase in revenue (%)	Estimate of 90% confidence interval (%)	Estimate of 95% confidence interval (%)
A	−3.4	[−11.5, 7.7]	[−13.5, 10.1]
B	11.4	[3.9, 19.2]	[1.1, 21.0]
C	12.5	[1.1, 23.4]	[−2.0, 26.6]
D	13.7	[3.4, 22.8]	[0.0, 25.2]
E	23.8	[5.4, 47.6]	[0.0, 56.7]
Overall	9.7	[2.3, 17.8]	[0.0, 20.2]

Source: Ferreria, Lee, Simchi-Levi (2016), "Analytics for an Online Retailer: Demand Forecasting and Price Optimization", 2016, *MSOM,* Vol. 18, No.1

Other Ways of Improving Profit?

- **Markdown Management/Dynamic Pricing**
  - Inventory is determined before the selling season begins
  - Dynamically reduce price over the selling season to boost demand and increase revenue

- **When to decrease and by how much? Tradeoff?**

http://retailer2.net/

Dynamic Pricing

- **Linear programming.**
- **Dynamic programming.**

Linear Optimization Problem (LOP)

- Decision **variables**, denoted by unknowns. Say, x,y,z… (or x_1, x_2, x_3)
- **Data**
- An **objective function**, say 2x+3y-z, to be minimized/maximized
- A set of **constraints**, denoted by equations or inequalities

Linear Optimization Problem (LOP)

- **General Framework**

$$
\begin{array}{rll}
\text{min(max)} & c_1x_1 + \ldots + c_nx_n & \\
\text{s.t.} & a_{11}x_1 + \ldots + a_{1n}x_n = b_1 & \text{Equality constraints} \\
& \vdots & \\
& a_{41}x_1 + \ldots + a_{4n}x_n \leq b_4 & \text{Inequality constraints} \\
& \vdots & \\
& a_{81}x_1 + \ldots + a_{8n}x_n \geq b_8 & \text{Inequality constraints} \\
& \vdots & \\
& x_1, \ldots, x_n \geq 0, & \text{Nonnegative constriants}
\end{array}
$$

Formulation of LOP

- **What is linear?**

Where does LOP arise?

- Management – manufacturing, banking, marketing, transportation, communication, …
- Engineering –design of products, data analysis, …
- Science – medicine, biology, geometry, chemistry, physics,…
- Sudoku

A LOP Example:

- **A furniture company makes tables and chairs**
- **Production require wood, finishing labor and carpentry labor.**

	Desk	Table	Chair	Avail.
Profit	60	30	20	
Wood	8	6	1	48
Finish Hrs	4	2	1.5	20
Carpentry Hrs	2	1.5	0.5	8

LOP Example:

Decision variables:

$x_1 =$ Num. desks, $x_2 =$ Num. tables

$x_3 =$ Num. chairs

$$
\begin{array}{lll}
\max & 60x_1 + 30x_2 + 20x_3 & \\
\text{s.t.} & 8x_1 + 6x_2 + x_3 & \leq 48 \\
& 4x_1 + 2x_2 + 1.5x_3 & \leq 20 \\
& 2x_1 + 1.5x_2 + 0.5x_3 & \leq 8 \\
& x_1, x_2, x_3 & \geq 0,
\end{array}
$$

LOP Models

- We maximize (or minimize) a linear function (called the objective function) of the decision variables.
  - Decision variables must satisfy a set of constraints. Each constraint must be a linear equation (=) or linear inequality (>= or <=).

LOP Models

- Usually, sign restriction is associated with each decision variable. However, a variable may be unrestricted in sign (say, temperature).
- By convention, constraints are written such that all the decision variables are on the LHS, while the constant is on the RHS.

Basic Math Notations

- **A set is a collection of distinct objects.**
  - **Check Wiki "Set"**
- **Examples:**

$$S = \{1, 2, 3\} \qquad 2 \in S, 5 \notin S$$

$$T = \{red, green, blue\}$$

Basic Math Notations

$$U = \{n^2 \ : \ n \text{ is an integer}, 0 \leq n \leq 3\}$$

$$F = [0, 5) \qquad G = [0, 5] \quad H = [0, \infty)$$

Basic Math Notations

- **Membership**

$$S = \{1, 2, 3\} \quad F = [0, 5) \quad G = [0, 5]$$

$$2 \in S, 5 \notin S \qquad 5 \in G, 5 \notin F$$

$$\{1, 2\} \subseteq S \qquad F \subseteq G$$

Basic Math Notations

- **Basic operations**
  - **Union** $A \cup B$
  - **Intersection** $A \cap B$

Basic Math Notations

- **Special Sets**

  - Empty set: $\; = \{\}$
  - Natural numbers: $\mathbb{N} = \{1, 2, 3, \dots\}$
  - Integers: $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
  - Real numbers: $\mathbb{R} = (-\infty, \infty)$
  - Nonnegative Real numbers: $\mathbb{R}_+ = [0, \infty)$

$$U = \{n^2 \;:\; n \text{ is an integer}, 0 \leq n \leq 3\}$$

$$U = \{n^2 \;:\; n \in \mathbb{Z}, n \in [0, 3]\}$$

Basic Math Notations

- **Summation**

$$S = \{1, 2, 3\} \qquad T = \{1, 2\}$$

$$\sum_{i \in S} a_i x_i = a_1 x_1 + a_2 x_2 + a_3 x_3$$

$$\sum_{i \in S} \sum_{j \in T} a_{ij} x_{ij} =$$

Basic Math Notations

$$V = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 6 \\ 4 \end{pmatrix} \right\}$$

$$\sum_{\binom{i}{j} \in V} a_{ij} x_{ij} =$$

Elementary Linear Algebra

- **Scalar**
  - **Variable described by a single number**

$$x \in \mathbb{R} \qquad x = 3$$

- **Vector**
  - **Column of numbers**

$$\boldsymbol{x} \in \mathbb{R}^n \qquad \boldsymbol{x} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ 6 \\ 0 \end{pmatrix} \in \mathbb{R}^5$$

Matrices

- Rectangular display of vectors in rows and columns
  - Defined as rows x columns (R x C)
- Vector is just a n x 1 matrix

$$\boldsymbol{A} \in \mathbb{R}^{m \times n}$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 8 & 4 \\ 7 & 3 & 4 \\ 6 & 4 & 2 \end{pmatrix} \in \mathbb{R}^{4 \times 3}$$

Columns of Matrices

$$\boldsymbol{A} = [\boldsymbol{A}_1 \boldsymbol{A}_2 \ldots \boldsymbol{A}_n] \in \mathbb{R}^{m \times n}$$

$$\boldsymbol{A}_i \in \mathbb{R}^m, i = 1, \ldots, n$$

ith column of matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 8 & 4 \\ 7 & 3 & 4 \\ 6 & 4 & 2 \end{pmatrix} \qquad \boldsymbol{A}_2 =$$

Transposition

- Reflect over its main diagonal:
  - Write the rows of A as the columns of A

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 8 & 4 \\ 7 & 3 & 4 \\ 6 & 4 & 2 \end{pmatrix} \qquad \boldsymbol{A}' =$$

$$(\boldsymbol{A}')' = \boldsymbol{A}$$

Rows of Matrices

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \vdots \\ \boldsymbol{a}_m' \end{bmatrix} \in \mathbb{R}^{m \times n}$$

$$\boldsymbol{a}_i \in \mathbb{R}^n, i = 1, \dots, m$$

ith row of matrix $\boldsymbol{A}$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 6 \\ 2 & 8 & 4 \\ 7 & 3 & 4 \\ 6 & 4 & 2 \end{pmatrix} \qquad \boldsymbol{a}_2 =$$

Matrix Calculations

Addition

- **Commutative: A+B=B+A**
- **Associative: (A+B)+C=A+(B+C)**
- **Size of matrix must agree**

Subtraction

- By adding a negative matrix

$$\begin{pmatrix} 1 & 4 \\ 2 & 8 \\ 7 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 2 \\ -2 & 4 \\ 6 & 3 \end{pmatrix} =$$

Scalar multiplication

- **Scalar x matrix = scalar multiplication**

$$\lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 5 \\ 2 \end{pmatrix} =$$

Scalar multiplication

- **Line segment:**

$$\lambda \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (1-\lambda) \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

Matrix Multiplication

- **When A is a mxn matrix & B is a kxl matrix, AB is only possible if n=k. The result will be an mxl matrix**

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \times \begin{pmatrix} b_{11} & \dots & b_{1l} \\ b_{21} & \dots & b_{2l} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nl} \end{pmatrix}$$

= **m** x **l** matrix

Number of columns in A = Number of rows in B

Matrix Multiplication

- **Multiplication method:**

Sum over product of respective rows and columns

$$\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \times \begin{pmatrix} 2 & 1 \\ 3 & 1 \end{pmatrix} =$$

A **B**

Matrix Multiplication

- **How many operations are needed to multiply 2 matrices:**

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \times \begin{pmatrix} b_{11} & \dots & b_{1l} \\ b_{21} & \dots & b_{2l} \\ \vdots & \ddots & \vdots \\ b_{n1} & \dots & b_{nl} \end{pmatrix}$$

- **Number of multiplication:**

Matrix Multiplication

- Matrix multiplication is NOT commutative
- $AB \neq BA$
- Matrix multiplication IS associative
- $A(BC)=(AB)C$
- Matrix multiplication IS distributive
- $A(B+C)=AB+AC$
- $(A+B)C=AC+BC$

Matrix Multiplication

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$$

$$\boldsymbol{a}'\boldsymbol{x} =$$

Matrix Multiplication

$$\boldsymbol{A} = [\boldsymbol{A}_1 \boldsymbol{A}_2 \ldots \boldsymbol{A}_n] \in \mathbb{R}^{m \times n}$$

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\boldsymbol{A}\boldsymbol{x} =$$

Matrix Multiplication

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1' \\ \boldsymbol{a}_2' \\ \vdots \\ \boldsymbol{a}_m' \end{bmatrix} \in \mathbb{R}^{m \times n} \qquad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\boldsymbol{A}\boldsymbol{x} =$$

Matrix Representation of LOP

- **Formulation**

$$\begin{aligned} \min \quad & 3x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 2 \\ & 2x_1 + x_2 \geq 3 \\ & x_1 \geq 0, x_2 \geq 0 \end{aligned}$$

$$\boldsymbol{c} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

$$\begin{aligned} \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

Identity matrix

- **Is there a matrix which plays a similar role as the number 1 in number multiplication?**
- **Consider the nxn matrix:**

$$\boldsymbol{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

For any nxn matrix $\boldsymbol{A}$, we have $\boldsymbol{A}\,\boldsymbol{I}_n = \boldsymbol{I}_n\,\boldsymbol{A} = \boldsymbol{A}$

For any nxm matrix $\boldsymbol{A}$, we have $\boldsymbol{I}_n\,\boldsymbol{A} = \boldsymbol{A}$, and $\boldsymbol{A}\,\boldsymbol{I}_m = \boldsymbol{A}$

Identity matrix

Worked example

$\boldsymbol{A\,I_3 = A}$

for a 3x3 matrix:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0+0 & 0+2+0 & 0+0+3 \\ 4+0+0 & 0+5+0 & 0+0+6 \\ 7+0+0 & 0+8+0 & 0+0+9 \end{bmatrix}$$

Matrix inverse

- **Definition.** A matrix ***A*** is called **nonsingular** or **invertible** if there exists a matrix ***B*** such that:

$$AB = BA = I_n$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} \frac{2}{3} & \frac{-1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}+\frac{1}{3} & \frac{-1}{3}+\frac{1}{3} \\ \frac{-2}{3}+\frac{2}{3} & \frac{1}{3}+\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Notation.** A common notation for the inverse of a matrix ***A*** is A^{-1}. So:

$$AA^{-1} = A^{-1}A = I_n$$

- The inverse matrix is unique when it exists.

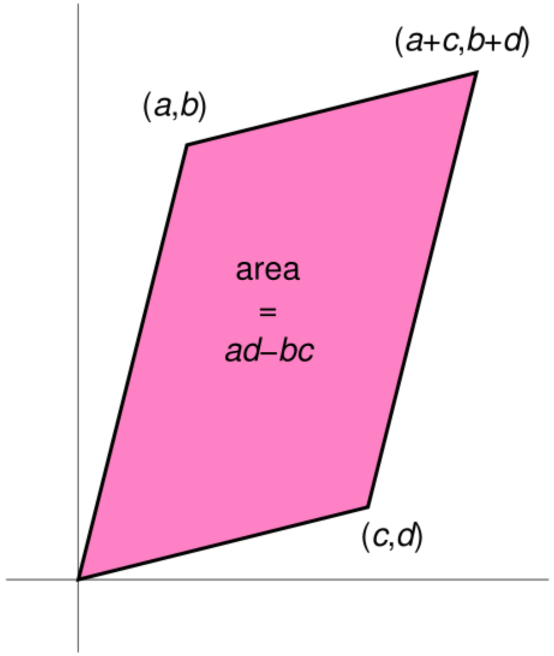
Matrix inverse

- For a $A = \begin{pmatrix} x_{1,1} & \dots & x_{1,j} \\ \vdots & \ddots & \vdots \\ x_{i,1} & \dots & x_{i,j} \end{pmatrix}$
- The inverse mat $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} \mathrm{cof}(A, x_{1,1}) & \dots & \mathrm{cof}(A, x_{1,j}) \\ \vdots & \ddots & \vdots \\ \mathrm{cof}(A, x_{i,1}) & \dots & \mathrm{cof}(A, x_{i,j}) \end{pmatrix}^T$
- E.g.: 2 $\mathbf{A}^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$

Determinant

- Determinants are mathematical objects that are very useful in the analysis and solution of systems of linear equations.
- The **determinant** is a function that associates a scalar det(*A*) to every square matrix *A*.
  - A matrix A has an inverse matrix A^{-1} if and only if **det(A)≠0**.
- Determinants can only be found for square matrices an it is related to the volume of a parallelepiped.

$$\det(A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$



Solving simultaneous equations

For one linear equation *ax=b* where the unknown is *x*, and a and *b* are constants, we have three possibilities:

- If $\boxed{a \neq 0}$ then $x = \frac{b}{a} \equiv a^{-1}b$ thus there is single solution
- If $\boxed{a = 0\,,\; b = 0}$ then the equation $ax = b$ becomes $0 = 0$ and any value of x will do
- If $\boxed{a = 0\,,\; b \neq 0}$ then $ax = b$ becomes $0 = b$ which is a contradiction

Solving simultaneous equations

- **Eg. Two equations, two unknowns**

$$2x_1 + 3x_2 = 5$$

$$x_1 - 2x_2 = -1$$

- **In matrix form**

$$\begin{pmatrix} 2 & 3 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\boldsymbol{Ax} = \boldsymbol{b}$$

Solving simultaneous equations

- **If A is invertible or det(A) is not zero then x is uniquely defined.**

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \Leftrightarrow \boldsymbol{A}^{-1}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b} \Leftrightarrow \boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

Solving simultaneous equations

- If A is not invertible, then it could have more than one solutions…

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 40 \\ 20 \end{pmatrix}$$

Solving simultaneous equations

- **Or no solutions at all…**

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 40 \\ 30 \end{pmatrix}$$

Additional information

- Other important topics on Linear Algebra
  - Positive/Semi-positive definite matrices
  - Eigen values/vectors
- Recommended videos:
  - Introduction to Linear Algebra by Gilbert Strang
  - Lecture notes in YouTube
  - https://www.youtube.com/watch?gl=SG&hl=en-GB&v=ZK3O402wf1c

Next Week

- **We will cover Python:**
  - Python is an easy to learn, powerful programming language. It has efficient high-level data structures and a simple but effective approach to object-oriented programming. Python's elegant syntax and dynamic typing, together with its interpreted nature, make it an ideal language for scripting and rapid application development in many areas on most platforms.
  - IMPORTANT: Install Anaconda for Python and Gurobi.