Introduction to Optimization
Lecture 4

More Gurobi

and

Understanding Geometry

Agenda

- More Problems on Gurobi
  - Scheduling Problem
  - Revenue Management
  - Investment Optimization

- Geometry and Theory of Linear Optimization

Scheduling Problem

- Hospital want to make weekly shift for its nurses
- d_j: demand for nurses in day j, $j \in \{1, \ldots, 7\}$
- Every nurse works 5 days in a row
- Objective: hire minimum number of nurses

Scheduling Problem

Mon	Tue	Wed	Thu	Fri	Sat	Sun
x_1	x_1	x_1	x_1	x_1		
	x_2	x_2	x_2	x_2	x_2	
		x_3	x_3	x_3	x_3	x_3
x_4			x_4	x_4	x_4	x_4
x_5	x_5			x_5	x_5	x_5
x_6	x_6	x_6			x_6	x_6
x_7	x_7	x_7	x_7			x_7

Scheduling Problem

x_i: # of nurses starting their week on day i.

$$
\begin{array}{rl}
\min & x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \\
\text{s.t} & x_1 + x_4 + x_5 + x_6 + x_7 \geq d_1 \\
& x_1 + x_2 + x_5 + x_6 + x_7 \geq d_2 \\
& x_1 + x_2 + x_3 + x_6 + x_7 \geq d_3 \\
& x_1 + x_2 + x_3 + x_4 + x_7 \geq d_4 \\
& x_1 + x_2 + x_3 + x_4 + x_5 \geq d_5 \\
& x_2 + x_3 + x_4 + x_5 + x_6 \geq d_6 \\
& x_3 + x_4 + x_5 + x_6 + x_7 \geq d_7 \\
& x_i \geq 0
\end{array}
$$

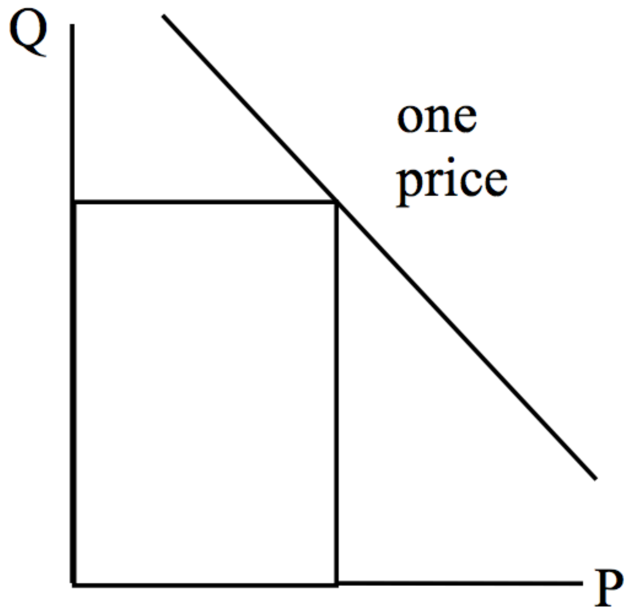
Revenue Management

- The Airline industry in US
  - Deregulation in 1978.
    - Prior to Deregulation: carriers only allowed to certain routes. Hence airlines such as Northwest, Eastern Southwest etc.
    - Fares determined by Civil Aeronautics Board (CAB) based on mileage and other costs - (CAB no longer exists)
  - Post Deregulation
    - anyone can fly anywhere
    - fares determined by carrier and the market

Revenue Management

- Economics
  - Huge sunk and fixed costs
  - Very low variable costs per passenger
    - $10 per passenger or less
  - Strong economically competitive environment
  - Near perfect information and negligible cost of information
  - Highly perishable inventory
  - **Result:** Multiple fares

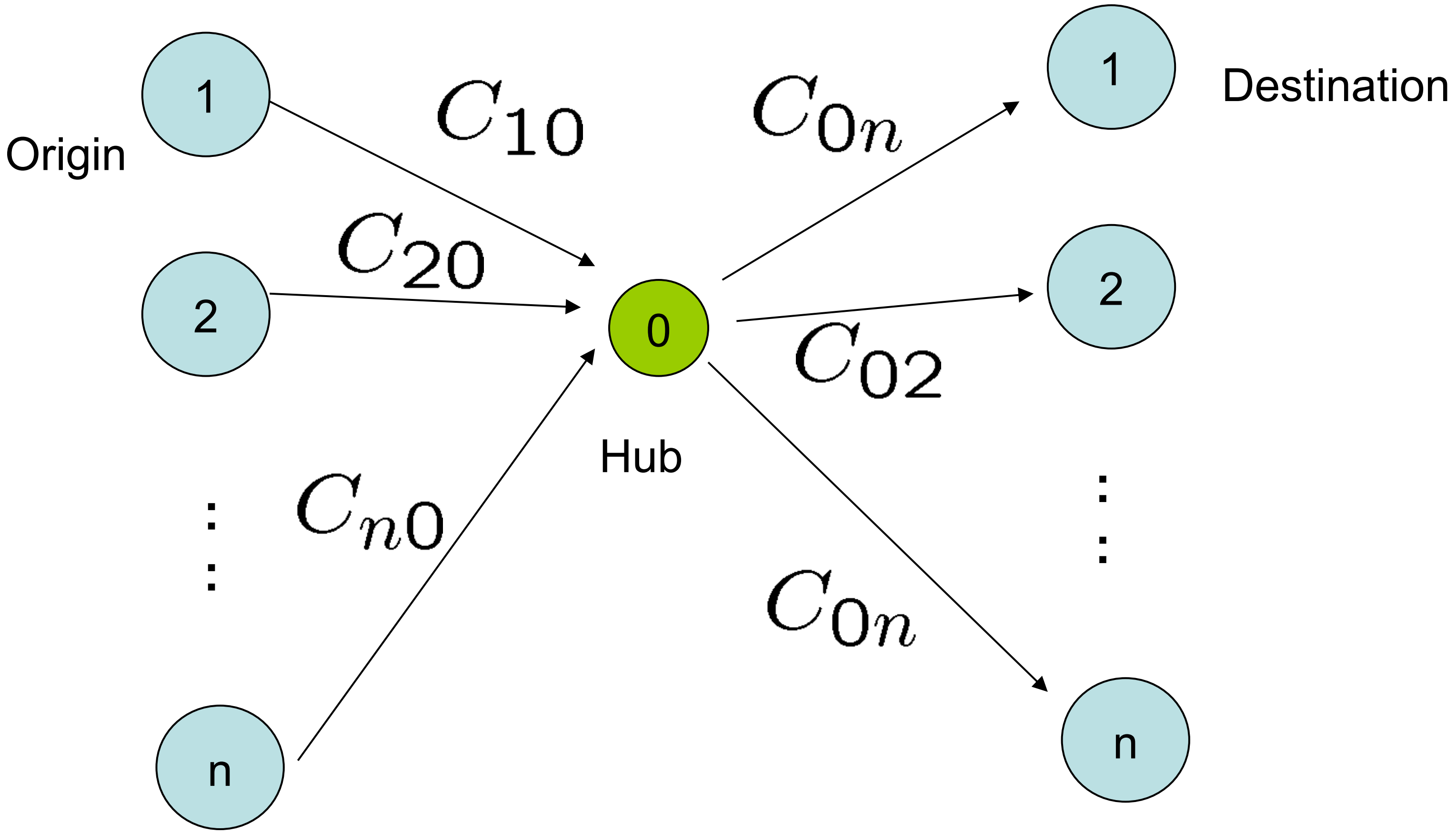
Multiple fare classes: a monopolist's perspective



- The two fare model presumes that customers are willing to pay the higher price, even if the lower price is available. How did airlines achieve this?

Revenue Management

- Origin, Destination and Hub



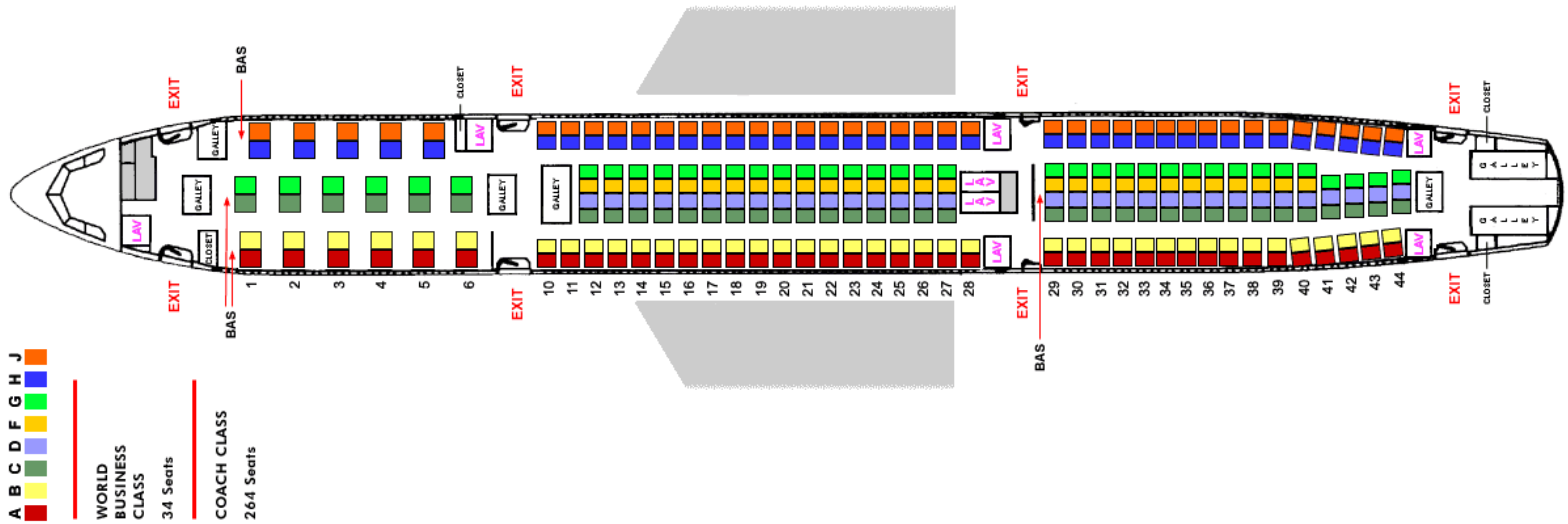
Revenue Management

- 2 class (for simplicity) Q and Y

Revenues: r_{ij}^{Q}, r_{ij}^{Y}

Capacities: $C_{i0}, C_{0j}, \; i = 1, \ldots, n, j = 1, \ldots, n$

Expected Demand: D_{ij}^{Q}, D_{ij}^{Y}



Revenue Management

- The right question asked… how many class Q and Y customers should we accept in order to maximize revenue?

Revenue Management

Q_{ij} : Num. of Q class customers to accept from i to j

Y_{ij} : Num. of Y class customers to accept from i to j

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \sum_{j=1}^{n} r_{ij}^{Q} Q_{ij} + r_{ij}^{Y} Y_{ij} & \\
\text{s.t.} \quad & \sum_{j=1}^{n} (Q_{ij} + Y_{ij}) \leq C_{i0} & i = 1, \ldots, n \\
& \sum_{i=1}^{n} (Q_{ij} + Y_{ij}) \leq C_{0j} & j = 1, \ldots, n \\
& 0 \leq Q_{ij} \leq D_{ij}^{Q}, 0 \leq Y_{ij} \leq D_{ij}^{Y} &
\end{aligned}
$$

Revenue Management

“We estimate that RM has generated $1.4 billion in incremental revenue for American Airlines in the last three years alone. This is not a one-time benefit. We expect RM to generate at least $500 million annually for the foreseeable future…”

Robert Crandall, former CEO of American Airlines

Investment under Taxation

- You purchased s_i shares of stock i at price p_i, i=1,…,n
- Current price of stock i is q_i
- You expect that the price of stock i one year later will be r_i
- You pay a capital gain tax at the rate of 30% on any capital gains at the time of sale
- You want to raise K amount of cash after taxes
- You pay 1% in transaction costs
- Example: You sell 1,000 shares at $50 per share; you have bought them at $30; Net Cash is,

$$50 \times 1,000 - 0.3 \times (50 - 30) \times 1,000 - \\ 0.01 \times 50 \times 1,000 = \$43,500.$$

Investment under Taxation

- Objective: Maximize the expected return (next year)
- Constraints: Able to raise the fund K.

Investment under Taxation

Let x_i be the amount of share i to sell.

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{N} r_i (s_i - x_i) \\
\text{s.t.} \quad & \sum_{i=1}^{N} (q_i x_i - 0.3 \max\{q_i - p_i, 0\} x_i - 0.01 q_i x_i) \geq K \\
& 0 \leq x_i \leq s_i \quad \forall i \in \{1, \ldots, N\}
\end{aligned}
$$

File Input/Output

- CSV file (comma-separated-values)
  - Data separated by commas
  - Format supported in EXCEL
- E.g., data.csv in Lecture 2

start node	end node	link length
1	2	2.0
1	3	4.5
2	3	6.0
2	4	3.0
3	4	5.0
1	2	2.0
1	3	4.5
2	3	6.0
2	4	3.0
3	4	5.0

File Input/Output

- Read CSV data as Dictionary

```
from gurobipy import *
from math import sqrt
import pandas as pd
import numpy as np
import matplotlib
import matplotlib.pyplot as plt
import xlrd
```

```
# read data
data1 = pd.read_csv('./Lectures notes/Lecture 4/Code/data1.csv')
print(data1)
header = data1.columns
data = data1.values
stock, buy_price, cur_share, cur_price, exp_price = multidict({item[0]: (item[1], item[2], item[3], item[4])
                                                               for item in data })
print(stock, buy_price)
```

```
  Stock  Buying Price  Current Share  Current Price  Expected Future Price
0    S1           1.2           1000            2.1                    2.0
1    S2           2.1           1000            3.2                    3.7
2    S3           3.2           1000            4.1                    5.2
3    S4           4.1           1000            5.1                    7.1
4    S5           4.5           1000            6.7                    9.1
['S1', 'S2', 'S3', 'S4', 'S5'] {'S1': 1.2, 'S2': 2.1, 'S3': 3.2, 'S4': 4.1, 'S5': 4.5}
```

Investment on Bonds

- Invest amount \$K on N bonds over T periods.
- Cash earns a fixed return per year
- Each bond pays an interest rate that compounds each year, and pays the principal plus compounded interest at the end of a maturity period.
- Each bond has a maximum invest limit.
- Goal is to maximize the final wealth.

Investment on Bonds

- Example: K=$1M, N=4, T=5, Cash interest rate = 2%

Bond	Available Year	Maturity Period	Annual Interest Rate	Limit
1	1	4	3%	1,000,000
2	5	1	4%	200,000
3	2	4	6%	500,000
4	2	3	6%	200,000

Investment on Bonds

- Hint: Cash is available every year and has a maturity period of 1 year and a interest rate of 2%.

<table>
<tr><th></th><th>Year 1</th><th>Year 2</th><th>Year 3</th><th>Year 4</th><th>Year 5</th></tr>
<tr><td>Cash 2%</td><td>Y[1]</td><td>Y[2]</td><td>Y[3]</td><td>Y[4]</td><td>Y[5]</td></tr>
<tr><td>B1 3%</td><td colspan="4">X[1]</td><td></td></tr>
<tr><td>B2 4%</td><td></td><td></td><td></td><td></td><td>X[2]</td></tr>
<tr><td>B3 6%</td><td></td><td colspan="4">X[3]</td></tr>
<tr><td>B4 6%</td><td></td><td colspan="3">X[4]</td><td></td></tr>
</table>

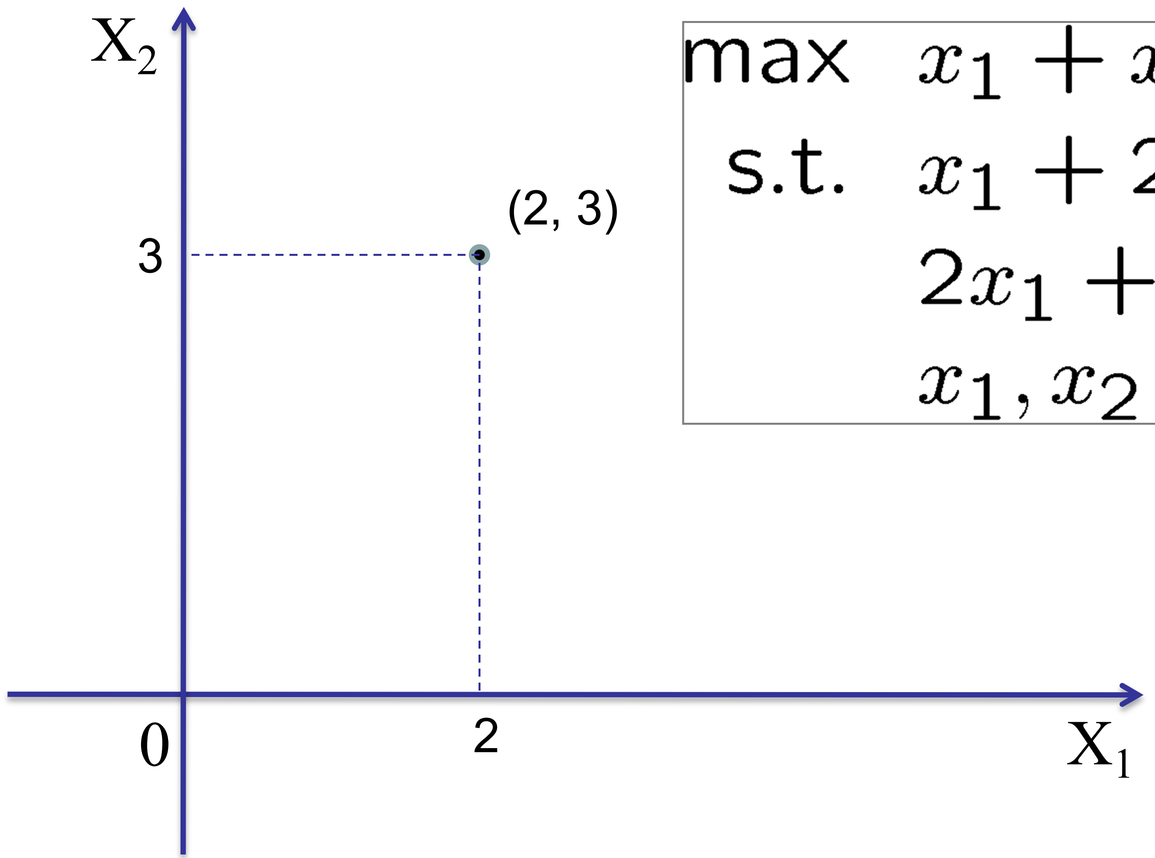
Investment on Bonds

Let y_t be cash investment at the beginning of the tth year and and x_i be the investment in the ith bond.

$$\begin{array}{lll}
\max & 1.02y_5 + 1.04x_2 + 1.06^4 x_3 & \\
\text{s.t.} & y_1 + x_1 = 1000000 & : \text{Year } 1 \\
 & y_2 + x_3 + x_4 = 1.02y_1 & : \text{Year } 2 \\
 & y_3 = 1.02y_2 & : \text{Year } 3 \\
 & y_4 = 1.02y_3 & : \text{Year } 4 \\
 & y_5 + x_2 = 1.02y_4 + 1.03^4 x_1 + 1.06^3 x_4 & : \text{Year } 5 \\
 & 0 \leq x_1 \leq 1000000 & \\
 & 0 \leq x_2 \leq 200000 & \\
 & 0 \leq x_3 \leq 500000 & \\
 & 0 \leq x_4 \leq 200000 &
\end{array}$$

Geometry of LOP in 2D

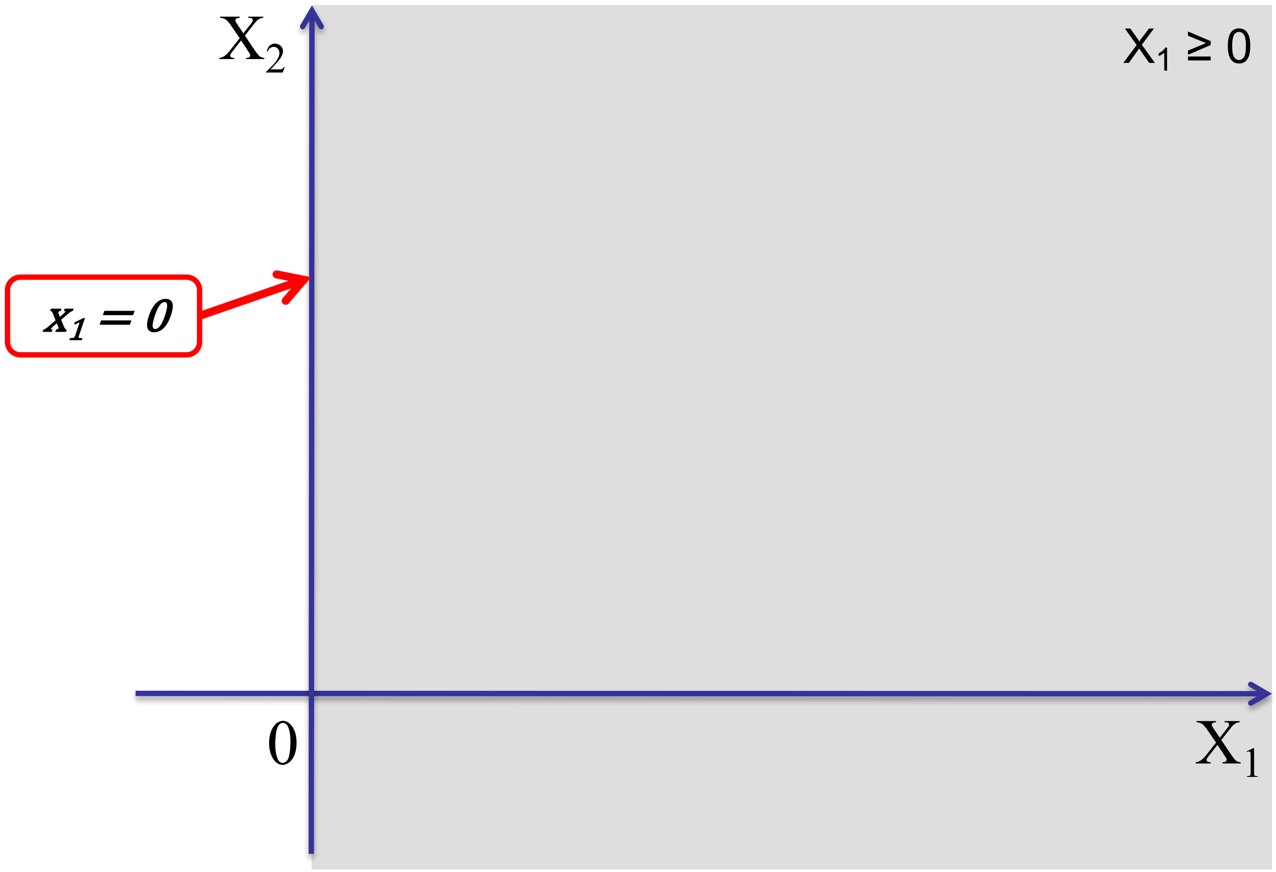
- Each dimension represents a decision variable
- Each point in the space represents a particular solution



$$
\begin{aligned}
\max \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 3 \\
& 2x_1 + x_2 \leq 3 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Geometry of LOP in 2D

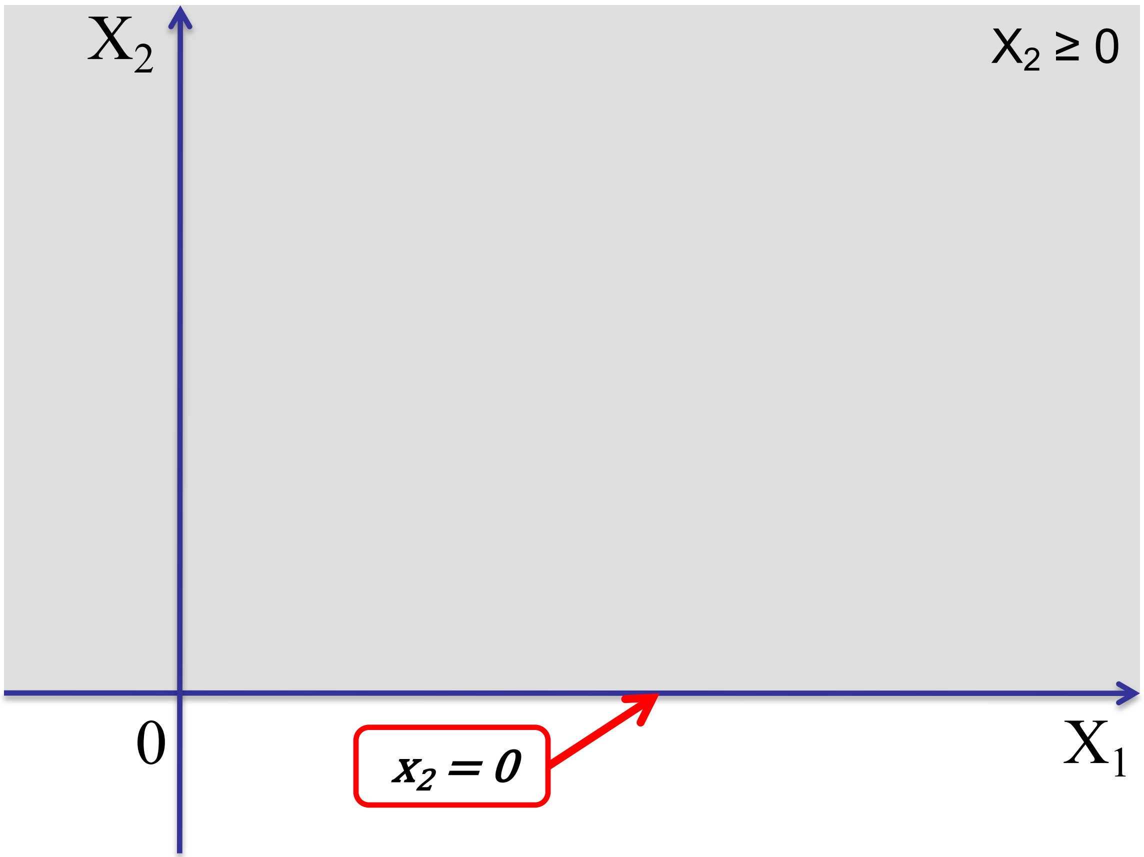
- Constraint: $\boldsymbol{x_1 \geq 0}$
- Inequality constraint $x_1 \geq 0$ defines a half space



$$
\begin{array}{ll}
\max & x_1 + x_2 \\
\text{s.t.} & x_1 + 2x_2 \leq 3 \\
& 2x_1 + x_2 \leq 3 \\
& x_1, x_2 \geq 0
\end{array}
$$

Geometry of LOP in 2D

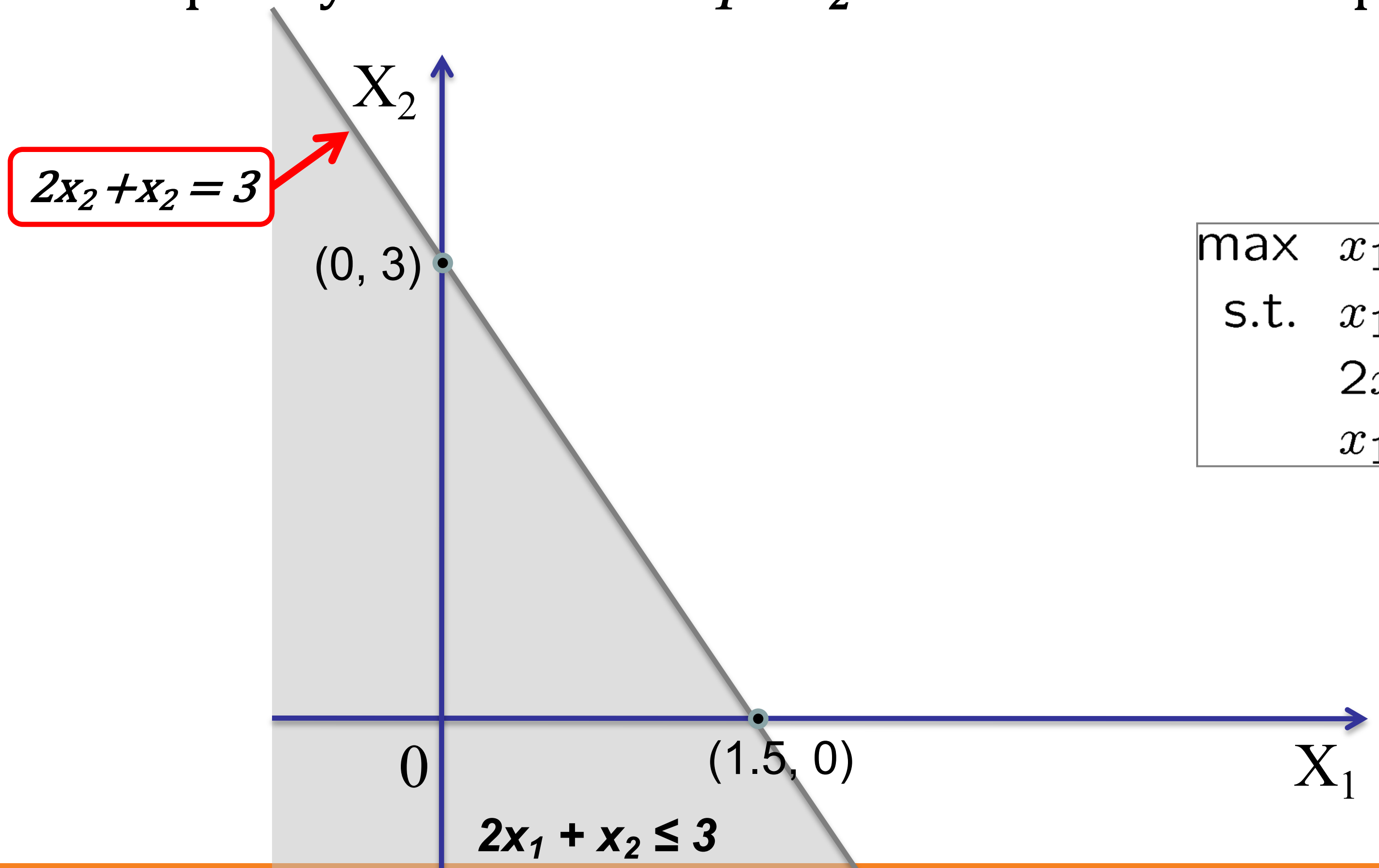
- Constraint: $\boldsymbol{x_2 \geq 0}$
- Inequality constraint $x_2 \geq 0$ defines a half space



$$
\begin{array}{ll}
\max & x_1 + x_2 \\
\text{s.t.} & x_1 + 2x_2 \leq 3 \\
& 2x_1 + x_2 \leq 3 \\
& x_1, x_2 \geq 0
\end{array}
$$

Geometry of LOP in 2D

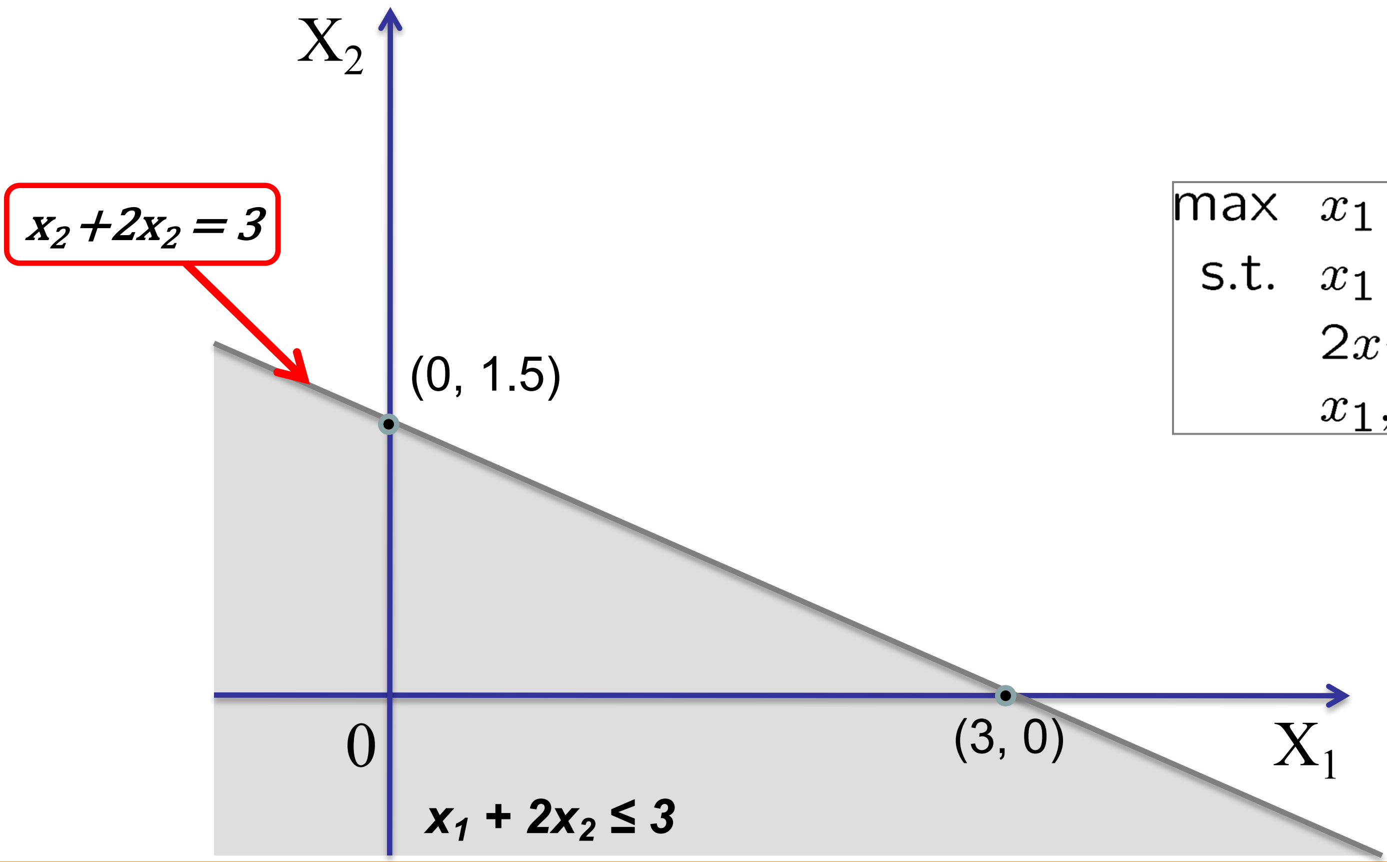
- Constraint: $\boldsymbol{2x_1 + x_2 \leq 3}$
- Inequality constraint $\boldsymbol{2x_1 + x_2 \leq 3}$ defines a half space



$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & x_1 + 2x_2 \leq 3 \\ & 2x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

Geometry of LOP in 2D

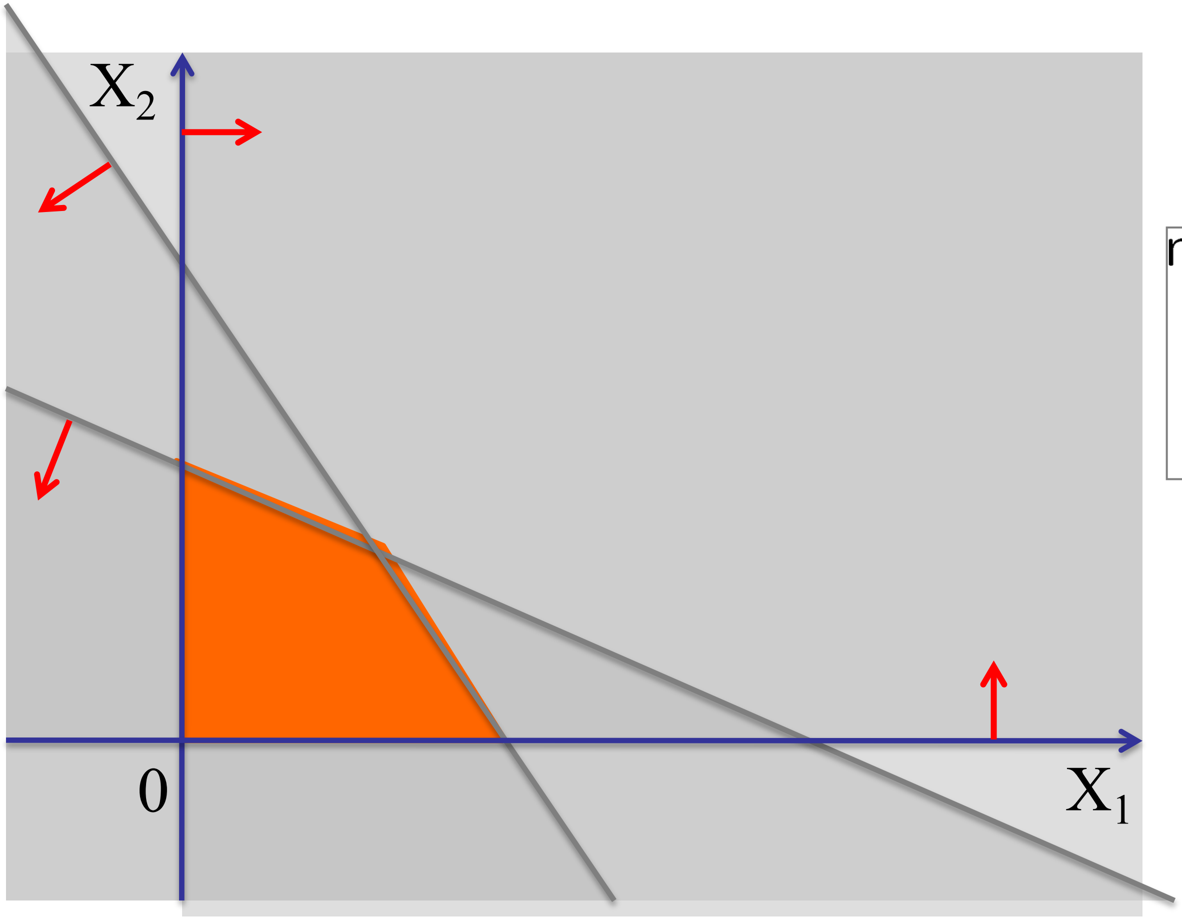
- Constraint: $\boldsymbol{x_1 + 2x_2 \le 3}$
- Inequality constraint $\boldsymbol{x_1 + 2x_2 \le 3}$ defines a half space



$$
\begin{array}{ll}
\max & x_1 + x_2 \\
\text{s.t.} & x_1 + 2x_2 \le 3 \\
& 2x_1 + x_2 \le 3 \\
& x_1, x_2 \ge 0
\end{array}
$$

Geometry of LOP in 2D

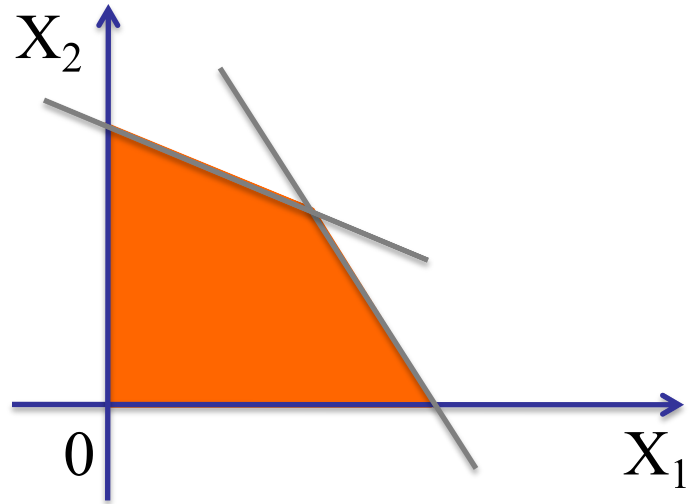
- The intersection of all the constraints ⇒ **Feasible Region**



$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & x_1 + 2x_2 \leq 3 \\ & 2x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

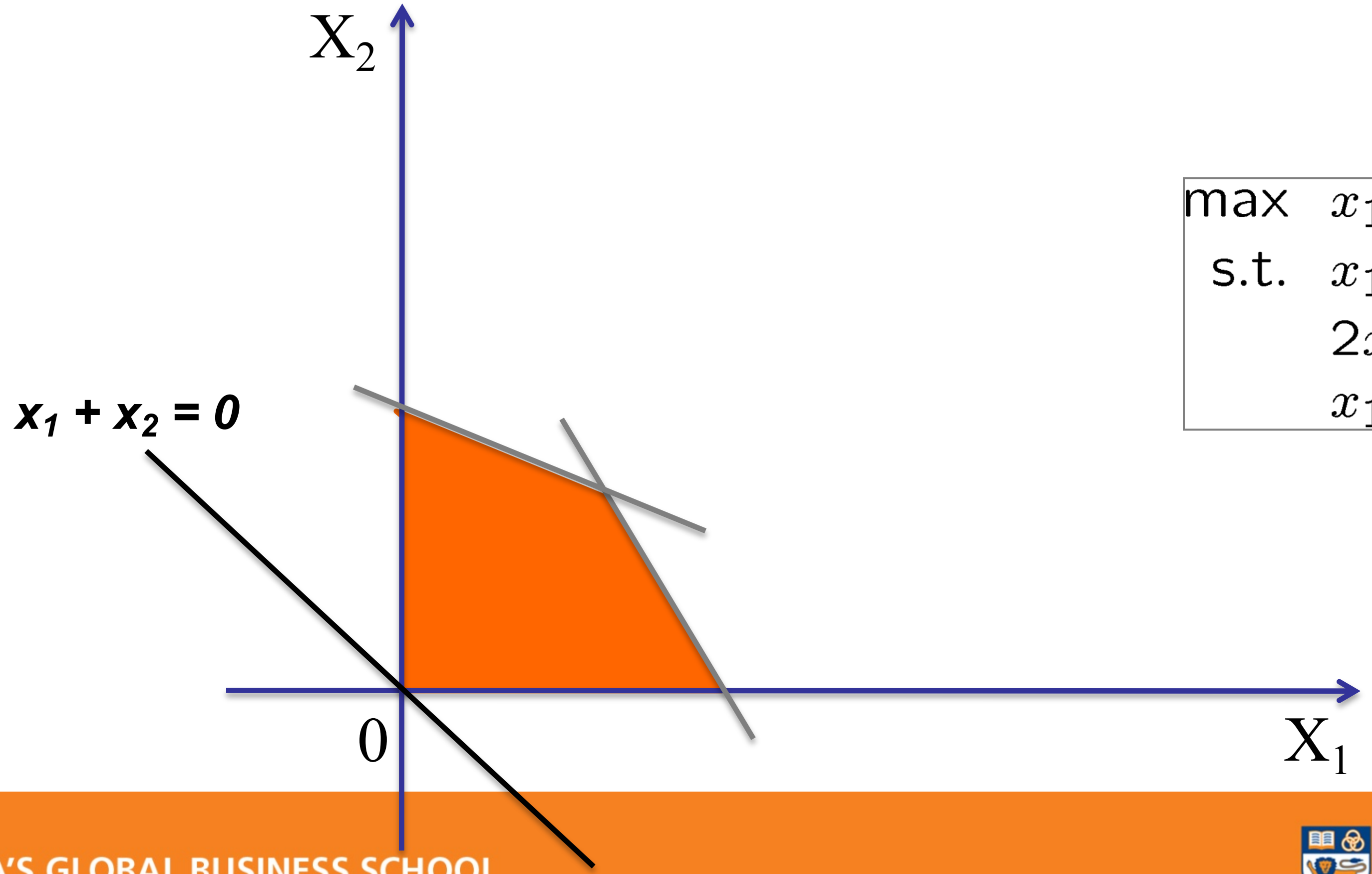
Geometry of LOP in 2D

- **Feasible Region:** The set of all the allowed solutions; a region (polygon) bounded by the constraints
  - Each equality constraint defines a **line**
  - Each inequality constraint defines a **half-space**
- **Extreme Points:** Corner points on the boundary of the feasible region. E.g., (0, 0), (1.5, 0), (0, 1.5), and (1, 1)
- **Infeasible problem:** A problem with an empty feasible region



Geometry of LOP in 2D

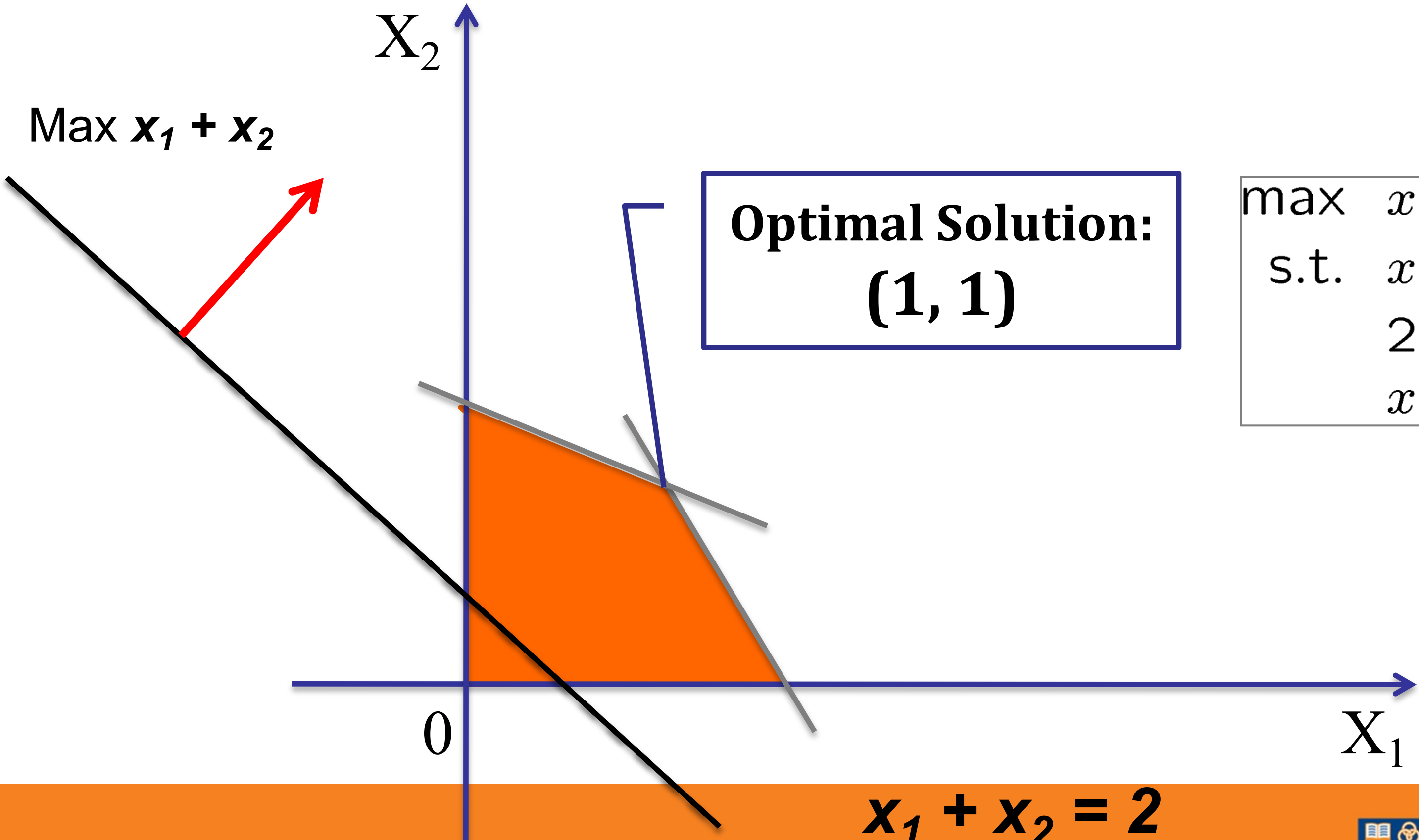
- Objective: ***maximize*** $\boldsymbol{x_1 + x_2}$
- **<u>Isoquant:</u>** A line on which all points have the same objective value; all points are equally good on the objective function.



$$
\begin{array}{ll}
\max & x_1 + x_2 \\
\text{s.t.} & x_1 + 2x_2 \le 3 \\
 & 2x_1 + x_2 \le 3 \\
 & x_1, x_2 \ge 0
\end{array}
$$

Geometry of LOP in 2D

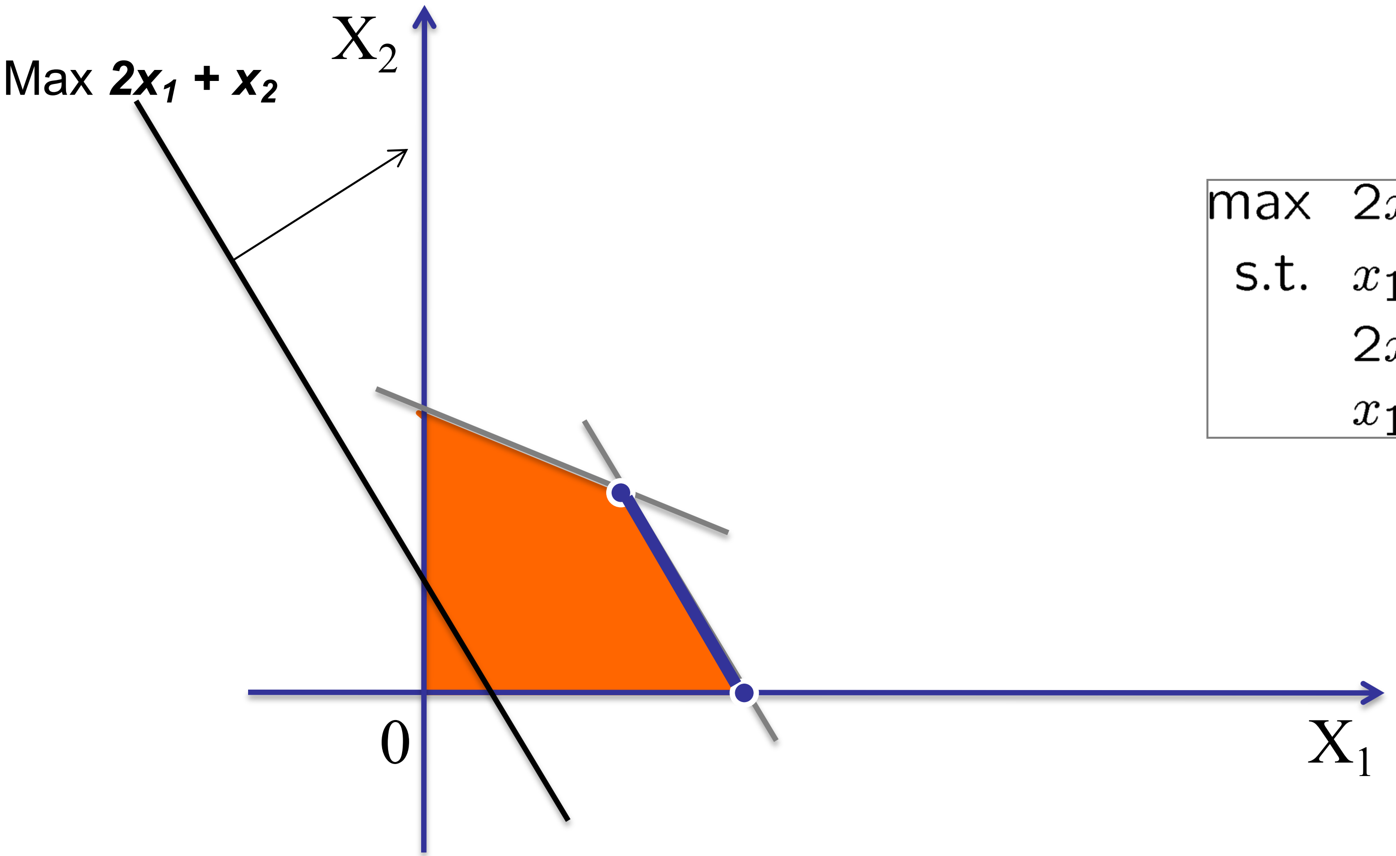
- **<u>Optimal Solution:</u>** The ***best feasible*** solution
- For any feasible LOP with a <u>finite</u> optimal solution, there exists an optimal solution that is an extreme point



$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & x_1 + 2x_2 \leq 3 \\ & 2x_1 + x_2 \leq 3 \\ & x_1, x_2 \geq 0 \end{array}$$

Geometry of LOP in 2D

- Optimal solutions may NOT be <u>unique</u>

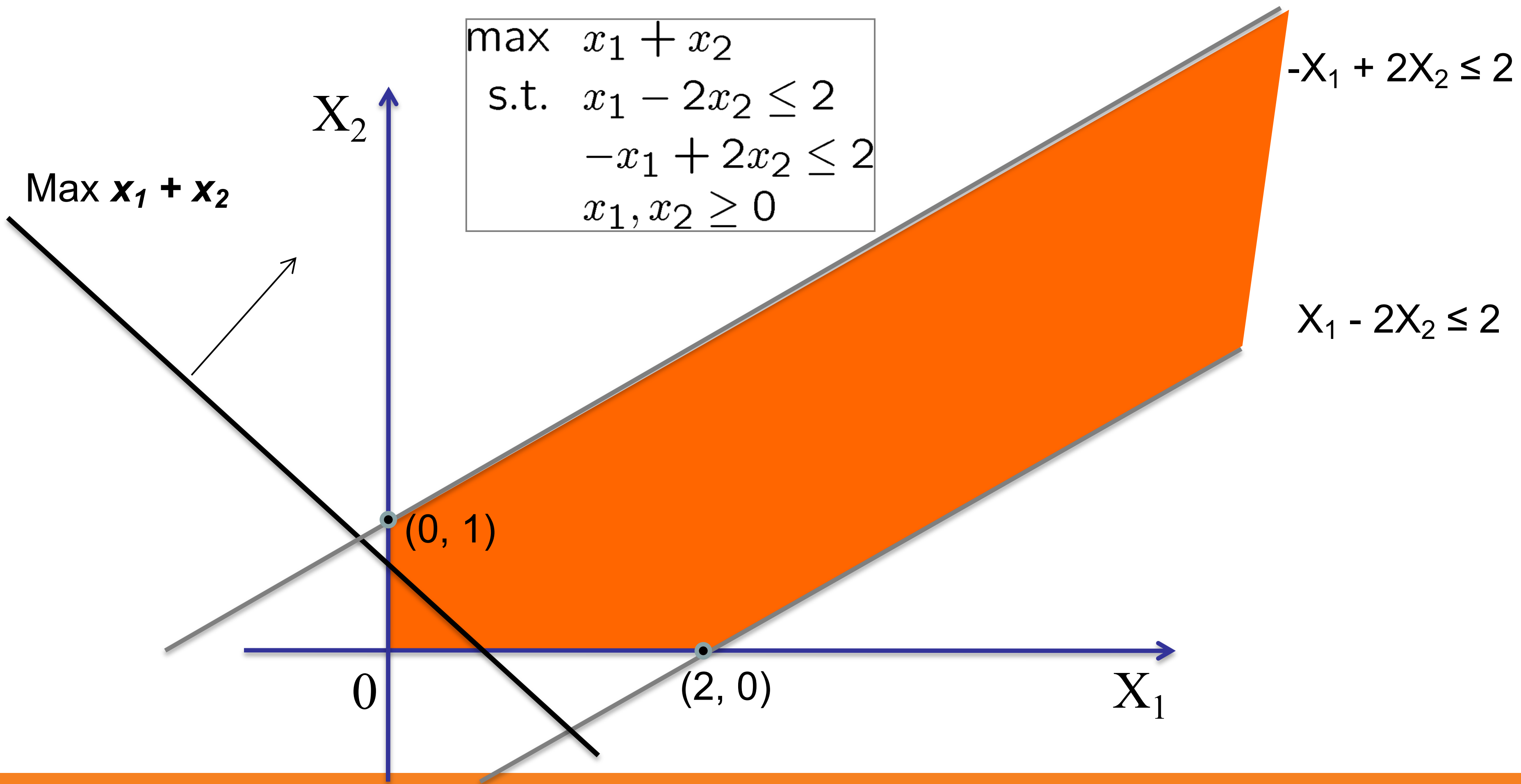


$$
\begin{array}{ll}
\max & 2x_1 + x_2 \\
\text{s.t.} & x_1 + 2x_2 \le 3 \\
 & 2x_1 + x_2 \le 3 \\
 & x_1, x_2 \ge 0
\end{array}
$$

Geometry of LOP in 2D

- Optimal solution may NOT be finite

$$\begin{array}{ll} \max & x_1 + x_2 \\ \text{s.t.} & x_1 - 2x_2 \leq 2 \\ & -x_1 + 2x_2 \leq 2 \\ & x_1, x_2 \geq 0 \end{array}$$



Preliminary insights

- Solutions of LOP
  - Unique optimal solution
  - Multiple optimal solutions
  - The optimal objective is not finite and no feasible solution is optimal
  - Infeasible

Active constraints

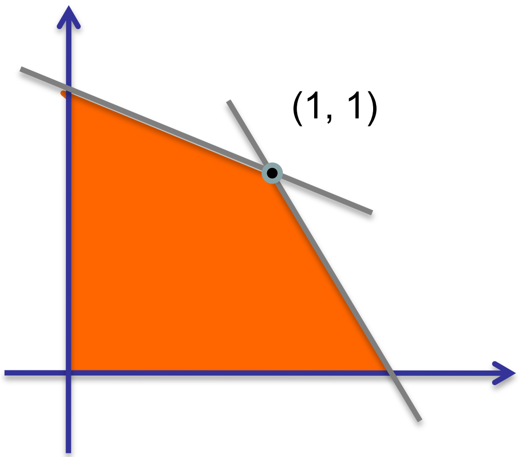
- **<u>Binding (or active) constraints:</u>** The constraints that are satisfied at ***equality*** for a given solution
  - All equality constraints are binding by definition

- **<u>Non-binding (or inactive) constraints</u>** are satisfied at ***strict inequality*** for a given solution

- The inequality level (= RHS – LHS) is known as the **<u>slack</u>**
  - Binding constraints have zero slack by definition

Active constraints

$$\begin{aligned} \text{max} \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \le 3 \\ & 2x_1 + x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$



Constraints	Binding?	Slack
$X_1 + 2X_2 \le 3$		
$2X_1 + X_2 \le 3$		
$X_1 \ge 0$		
$X_2 \ge 0$		

Beyond 2D - Central Problem

$$\begin{aligned} \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \end{aligned}$$

Beyond 2D - Central Problem

$$\max \boldsymbol{c}'\boldsymbol{x} \Leftrightarrow -\min(-\boldsymbol{c}'\boldsymbol{x})$$

$$\begin{array}{lll}
\min & \boldsymbol{c}'\boldsymbol{x} & \\
\text{s.t.} & \boldsymbol{a_i}'\boldsymbol{x} = b_i & i \in M_1 \\
 & \boldsymbol{a_i}'\boldsymbol{x} \leq b_i & i \in M_2 \\
 & \boldsymbol{a_i}'\boldsymbol{x} \geq b_i & i \in M_3 \\
 & x_j \geq 0 & j \in N_1 \\
 & x_j \text{ free} & j \in N_2
\end{array}$$

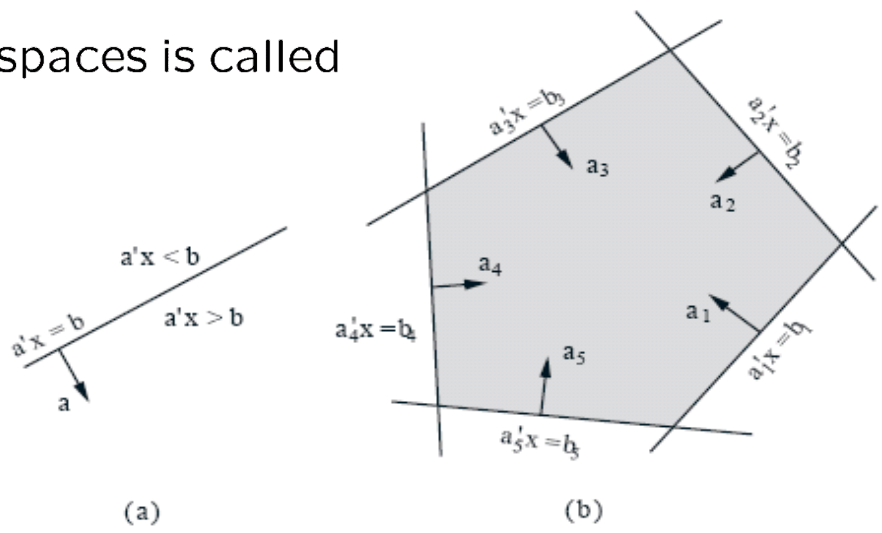
Standard Form

$$\begin{aligned} \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \end{aligned}$$

- Useful in computation
  - The Simplex and interior point method
- Reduction to standard form
  - Elimination of free variables
  - Elimination of inequality constraints

Polyhedra

- Definitions
  - The set $\{x \mid a'x = b\}$ s called a **hyperplane**
  - The set $\{x \mid a'x \geq b\}$ is called **a halfspace**
  - The intersection of many halfspaces is called a polyhedron



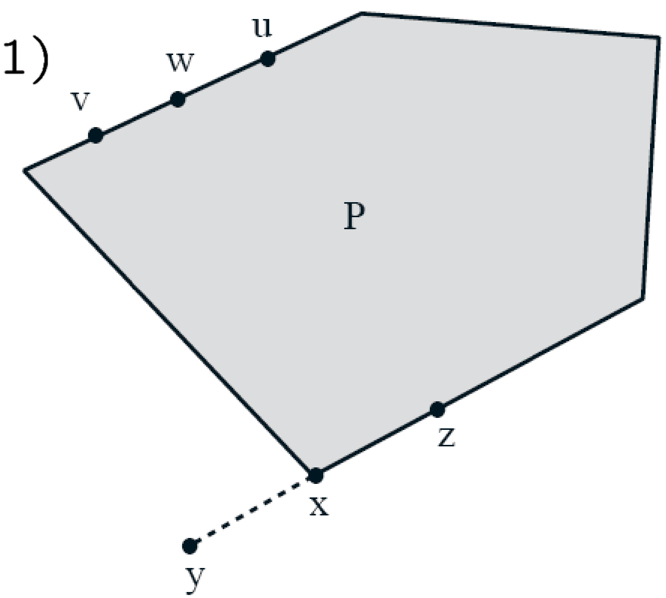
Corners

- Extreme Points

- Polyhedreon $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$

- $x \in P$ is an **extreme point** of P if

$$\nexists \boldsymbol{y}, \boldsymbol{z} \in P \ : \ \boldsymbol{x} = \lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{z}, \lambda \in (0,1)$$
$$\boldsymbol{y}, \boldsymbol{z} \neq \boldsymbol{x}$$

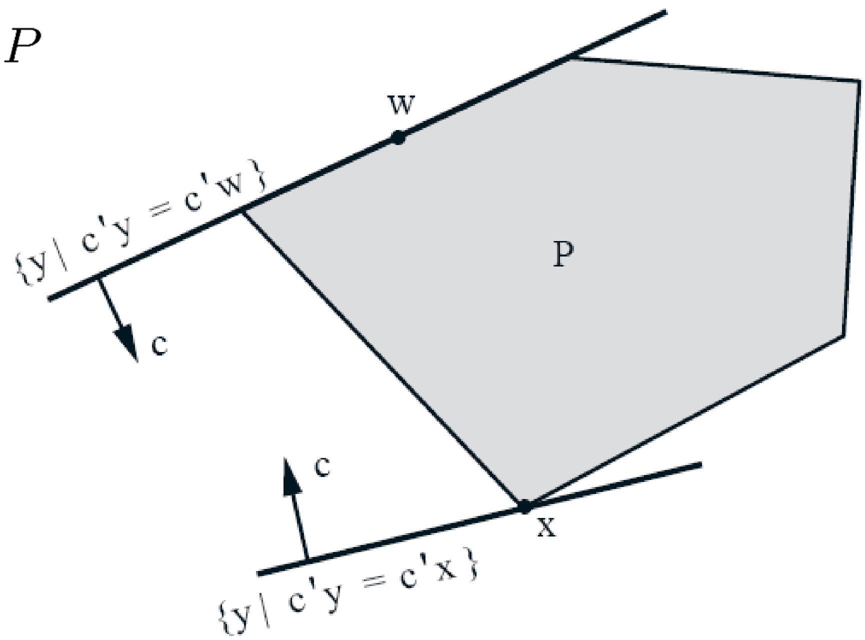


Corners

- Vertex

  - $x \in P$ is an **vertex** of P if $\exists \boldsymbol{c}$ such that $\boldsymbol{x}$ is the unique optimum to

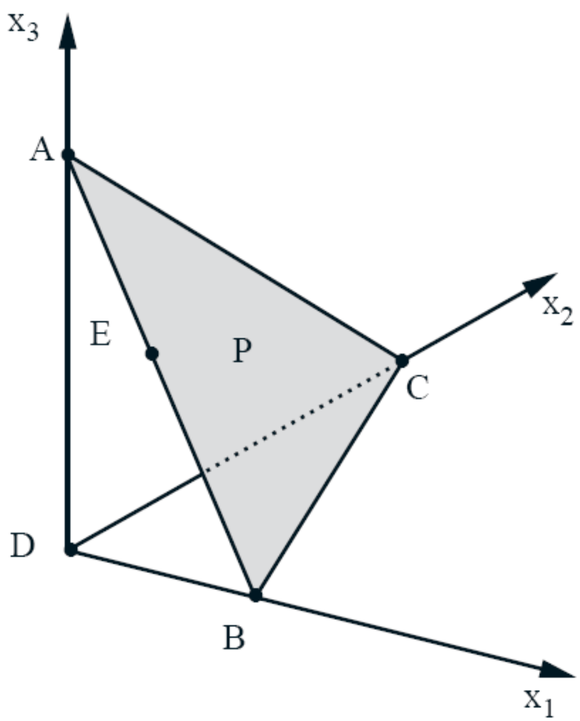
$$\begin{array}{ll} \text{min} & \boldsymbol{c}'\boldsymbol{y} \\ \text{s.t.} & \boldsymbol{y} \in P \end{array}$$



Corners

- Basic Feasible solution

$$P = \{(x_1, x_2, x_3) \mid x_1+x_2+x_3 = 1, x_1, x_2, x_3 \geq 0\}$$



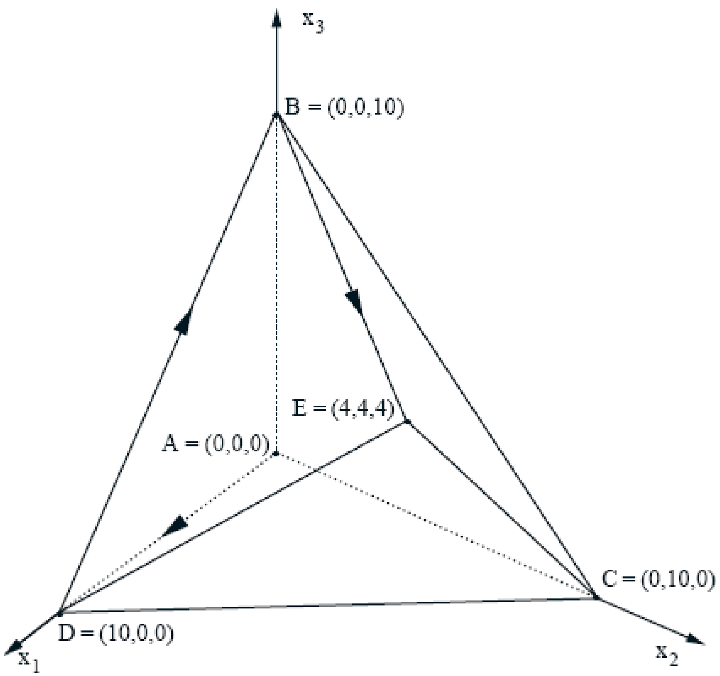
Points A, B, C: 3 constraints active

Point E: 2 constraints active

suppose we add $2x_1 + 2x_2 + 2x_3 = 2$. Then 3 hyperplanes are tight, but the contraints are not linearly independent.

Conceptual Simplex Algorithm

- Start at an extreme point (BFS)
- Visit a neighboring corner that improves objective



BFS of Manufacturing Problem

- BFS has at most m positive productions.

x_j = amount of product j produced

$$
\begin{aligned}
\text{max} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & a_{11}x_1 + \ldots + a_{1n}x_n \leq b_1 \\
& \vdots \\
& a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m \\
& x_1, \ldots, x_n \geq 0
\end{aligned}
$$

BFS of Manufacturing Problem

- Proof
  - Problem has n variables

$x_j =$ amount of product j produced

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & a_{11}x_1 + \ldots + a_{1n}x_n \le b_1 \\
& \vdots \\
& a_{m1}x_1 + \ldots + a_{mn}x_n \le b_m \\
& x_1, \ldots, x_n \ge 0
\end{aligned}
$$

Basic and non-basic variables

- There exists exactly n set of binding constraints to define a BFS.
  - **Non basic variables** correspond to those nonnegative constraints that are in the set of n binding constraints. The rest are called **basic variables**.
    - By definition, non basic variables must have zero values.
    - Basic variables are usually nonzero but they could be zero in degenerate cases.

Degeneracy

- A BFS can have more than n set of binding constraints
  - Degeneracy