Introduction to Optimization

Assignment 1

Problems 1 to 7 are individual work. Problem 8 is a group project.

1. Let
$$\boldsymbol{A} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}$$
Determine $\boldsymbol{AB}$. What do you observe?

2. Let
$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
Determine $\boldsymbol{AB}$ and $\boldsymbol{BA}$. What do you observe?

3. Let
$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 & 3 \\ 4 & -5 & 7 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{pmatrix}$$
   (a) Determine $\boldsymbol{AB}$ and $\boldsymbol{BA}$. What do you observe?
   (b) Determine $(\boldsymbol{AB})'$, $\boldsymbol{B}'$, $\boldsymbol{A}'$ and $\boldsymbol{B}'\boldsymbol{A}'$. What do you observe?

4. Let
$$\boldsymbol{A} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$
   (a) Determine $\boldsymbol{A}^{-1}$.
   (b) Determine $\boldsymbol{A}'$, $(\boldsymbol{A}')^{-1}$ and $(\boldsymbol{A}^{-1})'$. What do you observe?

5. Consider the following linear equation with unknown variables $x_1, x_2, \cdots, x_n$,
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$
Express the equation in matrix formulation, i.e., find out $\boldsymbol{A}$, $\boldsymbol{x}$ and $\boldsymbol{b}$ such that $\boldsymbol{Ax} = \boldsymbol{b}$.

6. A linear function, $f(\boldsymbol{x}) : \Re^n \mapsto \Re$ is one that satisfies the following properties:

   - Additivity: For any $\boldsymbol{x}, \boldsymbol{y} \in \Re^n$, $f(\boldsymbol{x}+\boldsymbol{y}) = f(\boldsymbol{x}) + f(\boldsymbol{y})$
   - Homogeneity: For any $a \in \Re$, $f(a\boldsymbol{x}) = af(\boldsymbol{x})$

   Given a vector, $\boldsymbol{a} \in \Re^n$. and define a function $g(\boldsymbol{x}) : \Re^n \mapsto \Re$ as

   $$g(\boldsymbol{x}) = \boldsymbol{a}'\boldsymbol{x}.$$

   (a) Show that $g(\boldsymbol{x})$ is a linear function.

   (b) Show that $g(\boldsymbol{x}) + b$ is not a linear function if $b \neq 0$.

   (c) (Optional) Show that any linear function can be expressed as a g function. Hint: Observe that

   $$\boldsymbol{x} = \boldsymbol{I}\boldsymbol{x} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} x_1 + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} x_2 + \cdots + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} x_n.$$

7. Define the set

   $$X = \left\{\boldsymbol{x} \in \Re^2 \ : \ \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\right\}.$$

   (a) What are the possible size of matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$?

   (b) Draw the set X for

   $$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -1 \end{pmatrix}, \boldsymbol{b} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}.$$

   (c) Provide possible values of $\boldsymbol{A}$ and $\boldsymbol{b}$ to describe the feasible region of a square with corners at $(0,0), (1,-1), (1,1), (2,0)$.

8. Recall the transportation problem considered in class. We next consider a variant of the transportation problem.

   Consider a dairy company, MyCow, based in the fictional country of Wakanda. MyCow runs a single production facility, lets call it the "hub", based out of the capital city of Wakanda. MyCow then uses multiple trucks to serve stores in Wakanda.

   Suppose there are 20 stores in Wakanda - S1 .. S20. Every store has a weekly demand of milk, that needs to be supplied during the week, using one or more trips. MyCow supplies milk using pre-determined routes, where each route is run by 1 truck on each day. For example, if there is a route "Hub to S5 to S8", then a truck can run on this route each day (Sunday....Saturday) carrying milk and dropping them off at S5 and S8. Each truck has a capacity, C = 20. So in total, the truck can be run on all 7 days and supply 700 litres of milk to S5 and S8. If the demand in S5 and S8 is less than this, we need not run the truck on all 7 days.

Every store also has a maximum amount of milk that it can receive on any given day - for example, if store S5 has a weekly demand of 40 lts and can receive a maximum of 9 lts on a given day, then we need to schedule at least 5 trips to deliver milk to S5 on 5 different days. The cost of running a truck each day depends on the quantity transferred on the truck - from 0 for not running to $100 for a full truck.

Formulate an optimization problem to minimize the total cost of the operation, while making sure all the demand is served in a week, truck capacity is respected and store "delivery accepting capacity" on each day is respected. The demands, delivery accepting capacity, and routes data will be given in separate text files. You need to also create code to read the text files to load data.