Introduction to Optimization
Lecture 6
Sensitivity Analysis and Duality

On Group Project

- Short case study of an optimization problem
- Report - At most 6 pages
  - Introduction
  - Model formulation
  - Analysis of solutions
- Code in Python.
  - Model should be generic
  - Data must be read from a file.
- Presentation on last weel
- Grading
  - Group presentation (20%) (Awarded to Group)
  - Report 20% X Peer review weight (1, .8, .5, .2, 0)
  - Based on peer review weight.

Sensitivity Analysis

- In real applications, you may not be very certain about the data used
- Sensitivity Analysis studies the effect of parameter changes
  - How does the **optimal solution** change with respect to changes in b_i *and* c_j?
  - How does the **optimal objective value** change with respect to changes in b_i *and* c_j?

$$
\begin{array}{lll}
\max & c_1x_1 + c_2x_2 + \cdots + c_nx_n & \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 & : p_1 \\
 & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 & : p_2 \\
 & \vdots & \\
 & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m & : p_m \\
 & x_1, x_2, \ldots, x_n \geq 0 &
\end{array}
$$

Example: A pottery company

- Products: Bowls ($40) and Mugs ($50)

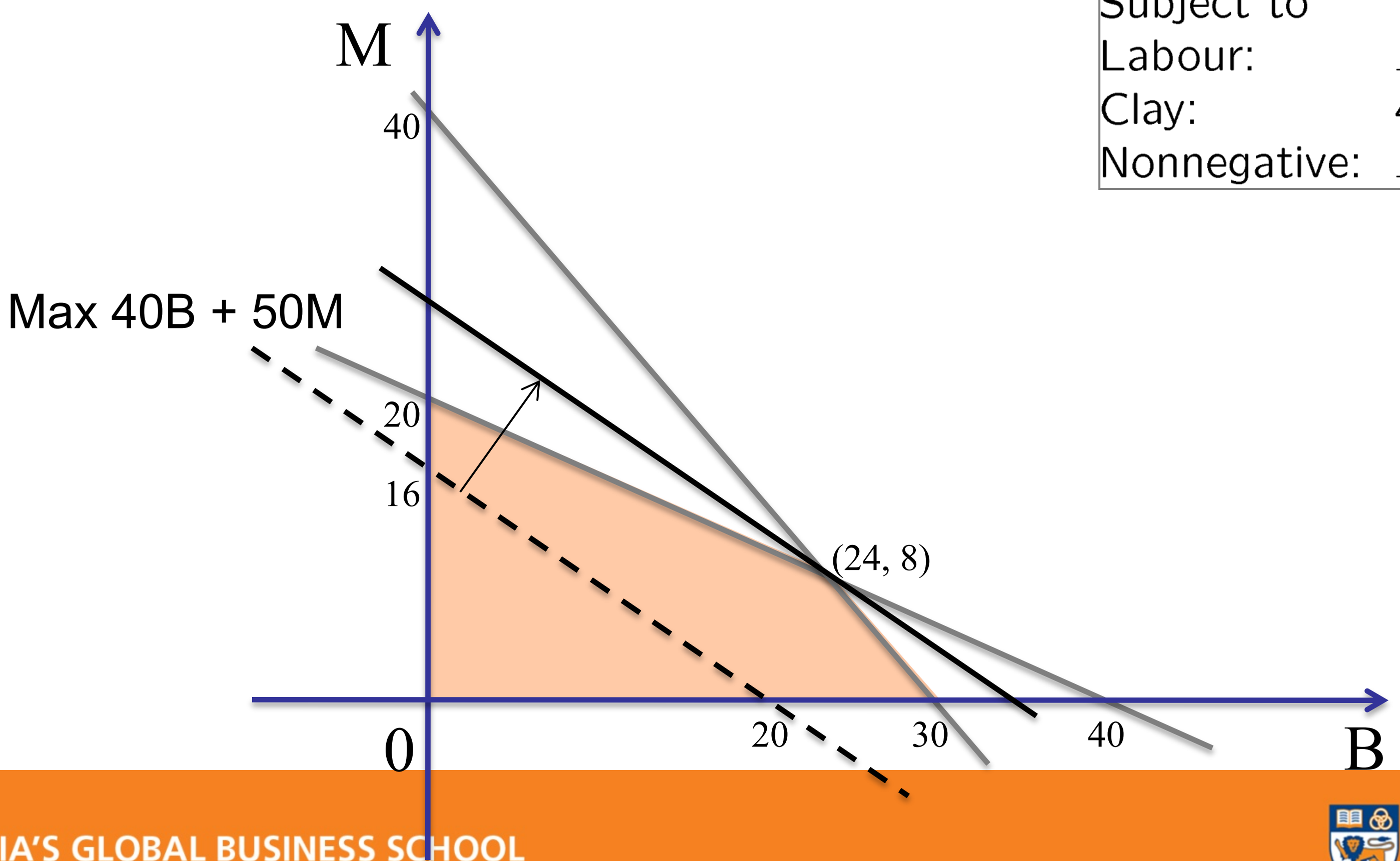
	Bowl (B)	Mug (M)	Capacity
Labour	1	2	40
Clay	4	3	120

$$
\begin{array}{ll}
\text{Maximize} & \overset{c_1}{40}B + \overset{c_2}{50}M \\
\text{Subject to} & \\
\text{Labour:} & \overset{a_{11}}{B} + \overset{a_{12}}{2}M \leq \overset{b_1}{40} \\
\text{Clay:} & \overset{a_{21}}{4}B + \overset{a_{22}}{3}M \leq \overset{b_2}{120} \\
\text{Nonnegative:} & B, M \geq 0
\end{array}
$$

Changes in Objective Function Coefficients (c_j)

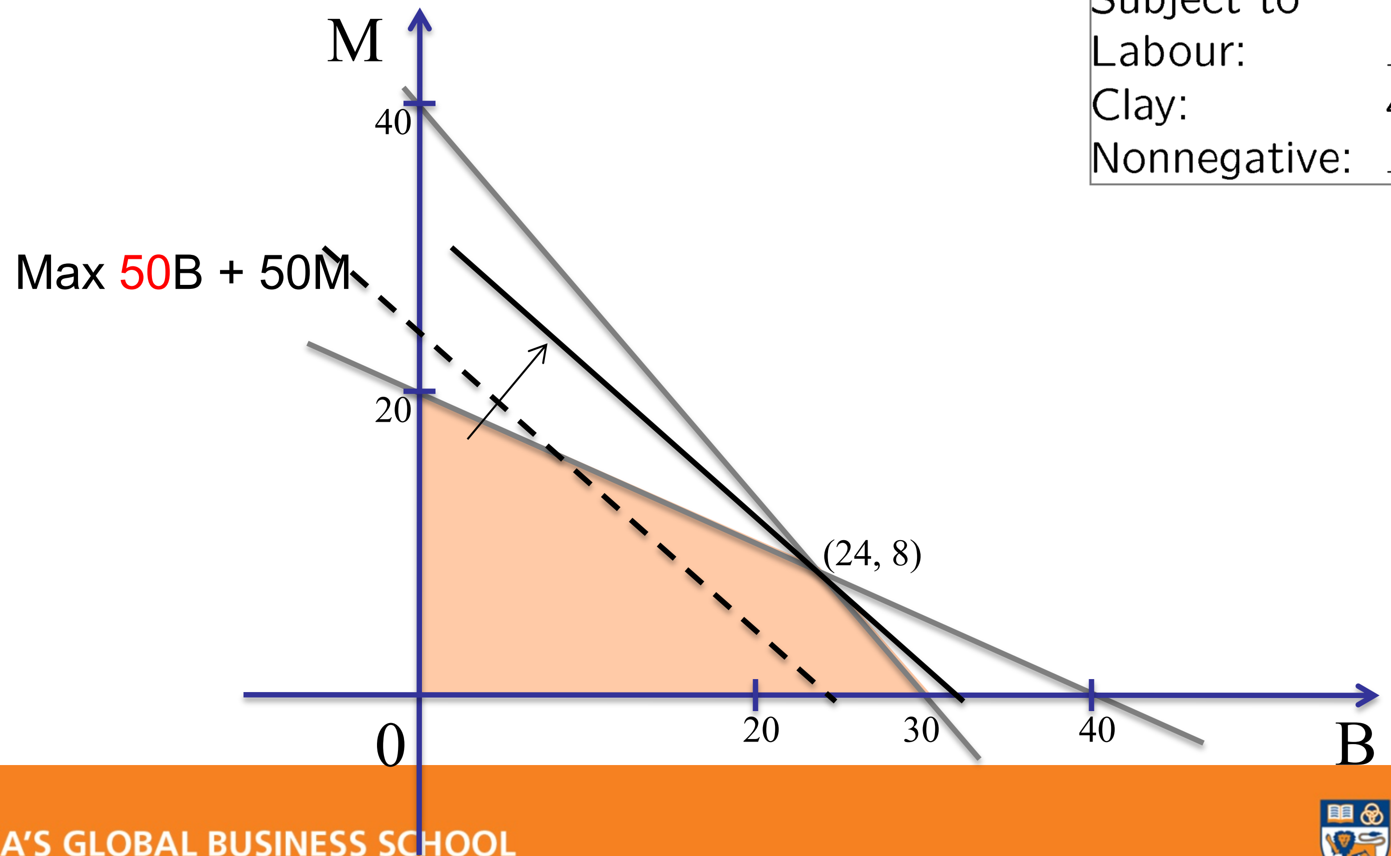
- Suppose c_1 changes, how would the optimal solution change?

Maximize	$40B + 50M$
Subject to	
Labour:	$B + 2M \leq 40$
Clay:	$4B + 3M \leq 120$
Nonnegative:	$B, M \geq 0$



Changes in Objective Function Coefficients (c_j)

- Say c_1 increases to *50*?

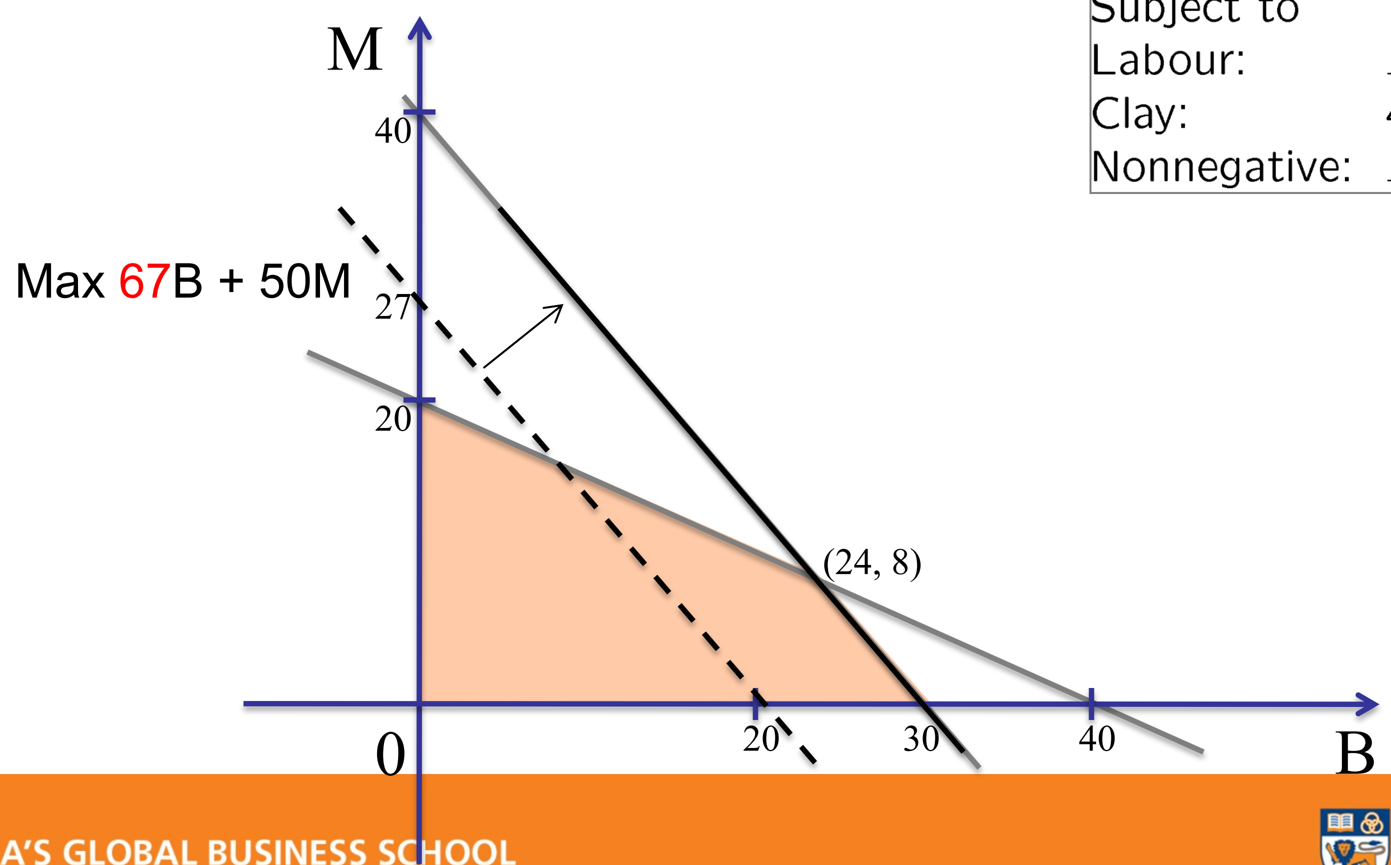


Maximize	$40B + 50M$
Subject to	
Labour:	$B + 2M \leq 40$
Clay:	$4B + 3M \leq 120$
Nonnegative:	$B, M \geq 0$

Changes in Objective Function Coefficients (c_j)

- How about c_1 increases to *67*?
- What if c_1 is larger than 67?

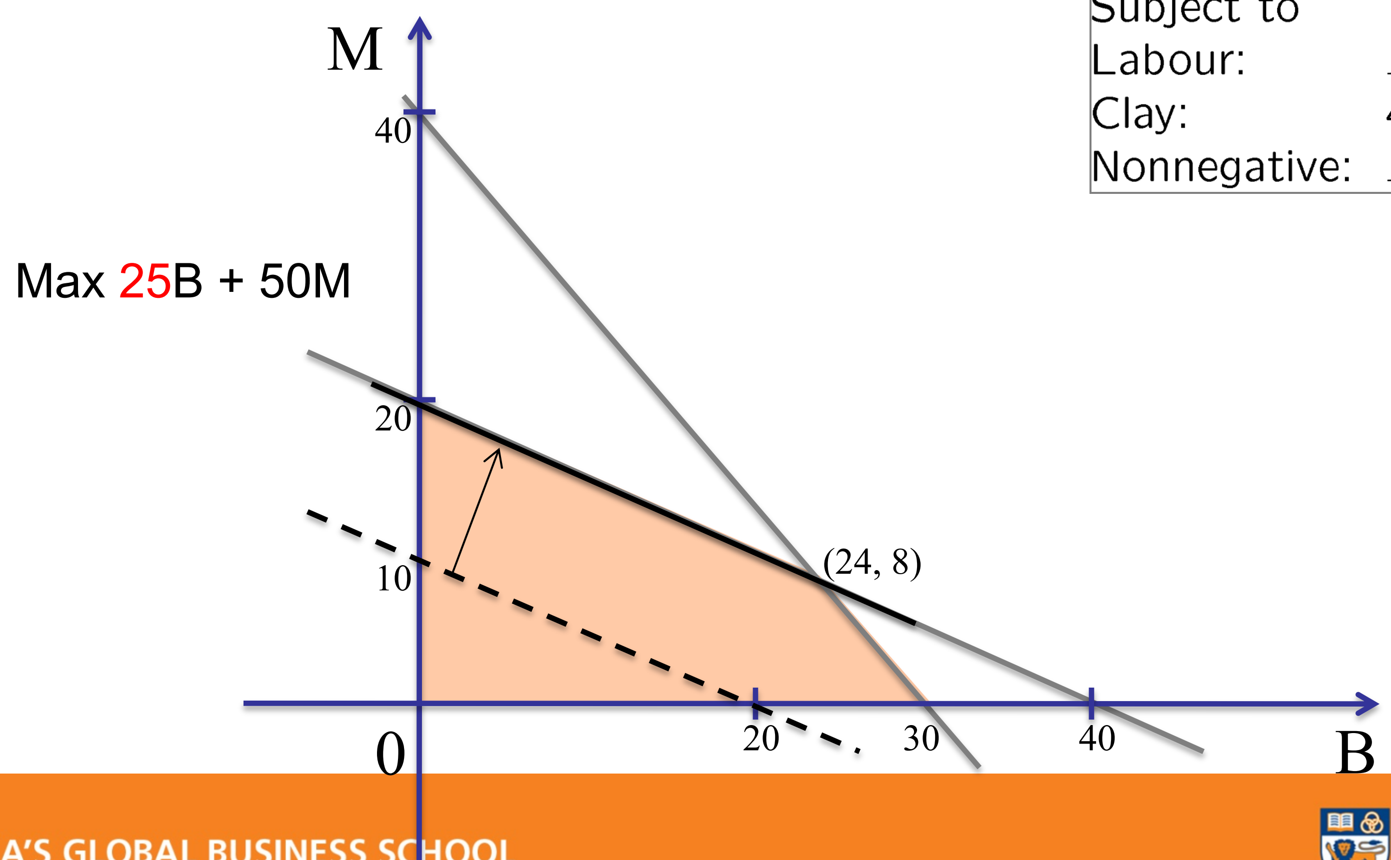
Maximize	$40B + 50M$
Subject to	
Labour:	$B + 2M \leq 40$
Clay:	$4B + 3M \leq 120$
Nonnegative:	$B, M \geq 0$



Changes in Objective Function Coefficients (c_j)

- What if c_1 decreases?
- What if c_1 is less than 25?

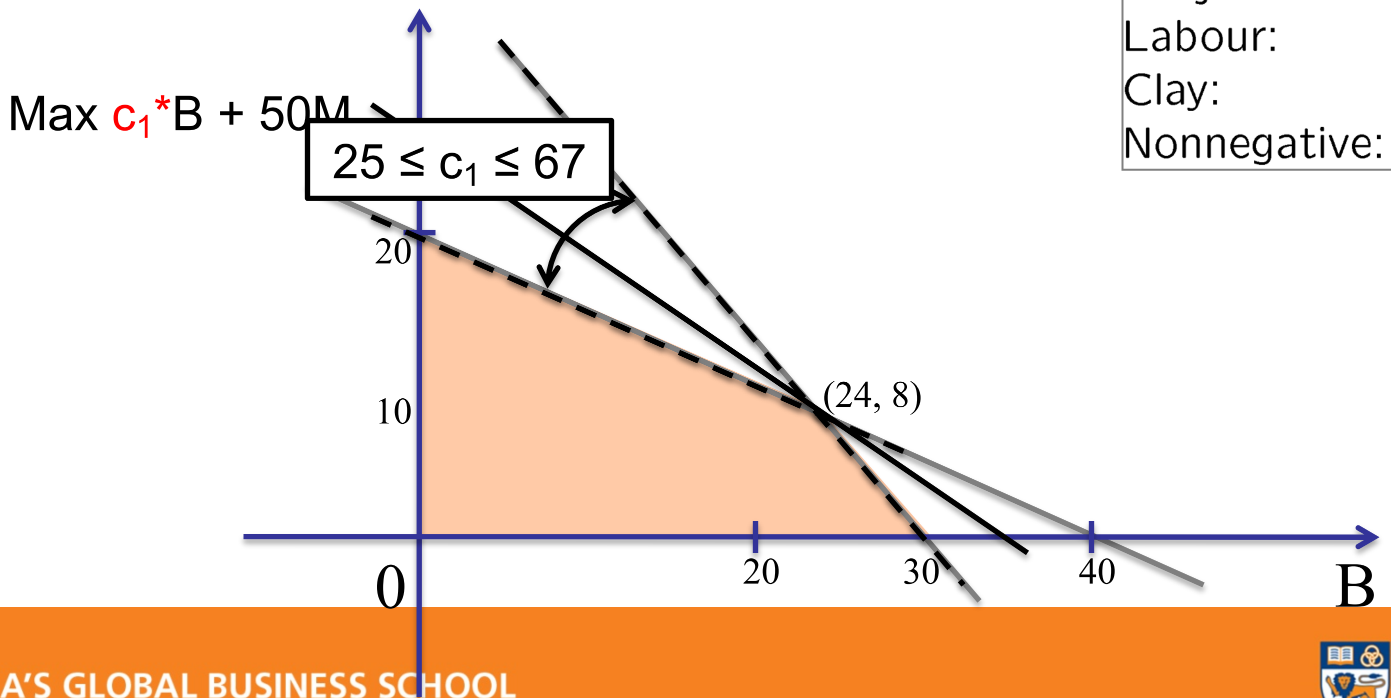
Maximize	$40B + 50M$
Subject to	
Labour:	$B + 2M \leq 40$
Clay:	$4B + 3M \leq 120$
Nonnegative:	$B, M \geq 0$



Changes in Objective Function Coefficients (c_j)

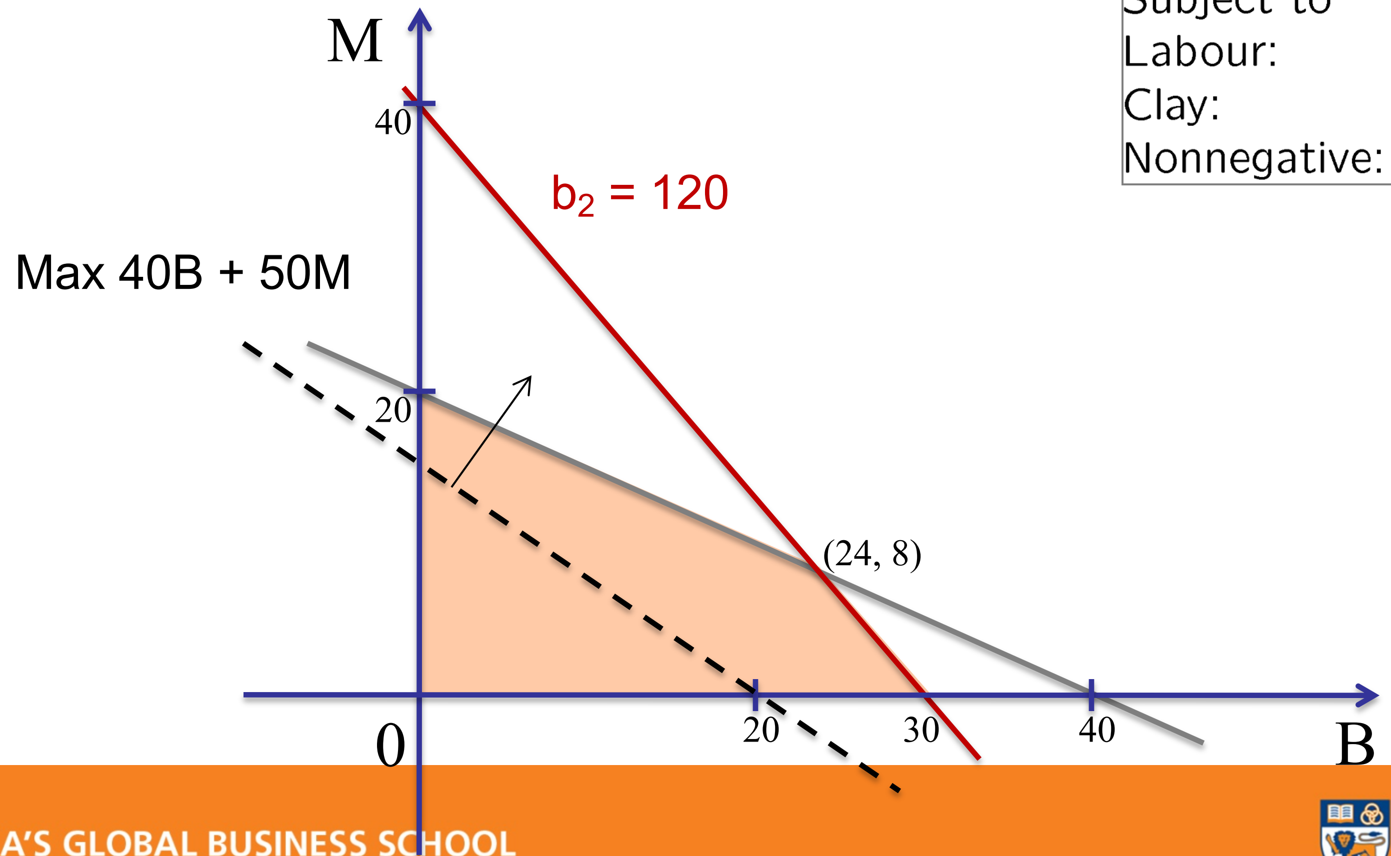
- As long as c_1 is in the range [25, 67], the ***optimal solution*** will NOT change
- The ***sensitive range*** of an ***objective coefficient*** is the range of values over which the ***current optimal solution*** will remain optimal
- How would the <u>objective value</u> change?

Maximize	$40B + 50M$
Subject to	
Labour:	$B + 2M \leq 40$
Clay:	$4B + 3M \leq 120$
Nonnegative:	$B, M \geq 0$



Changes in Constraint Quantity (b_i)

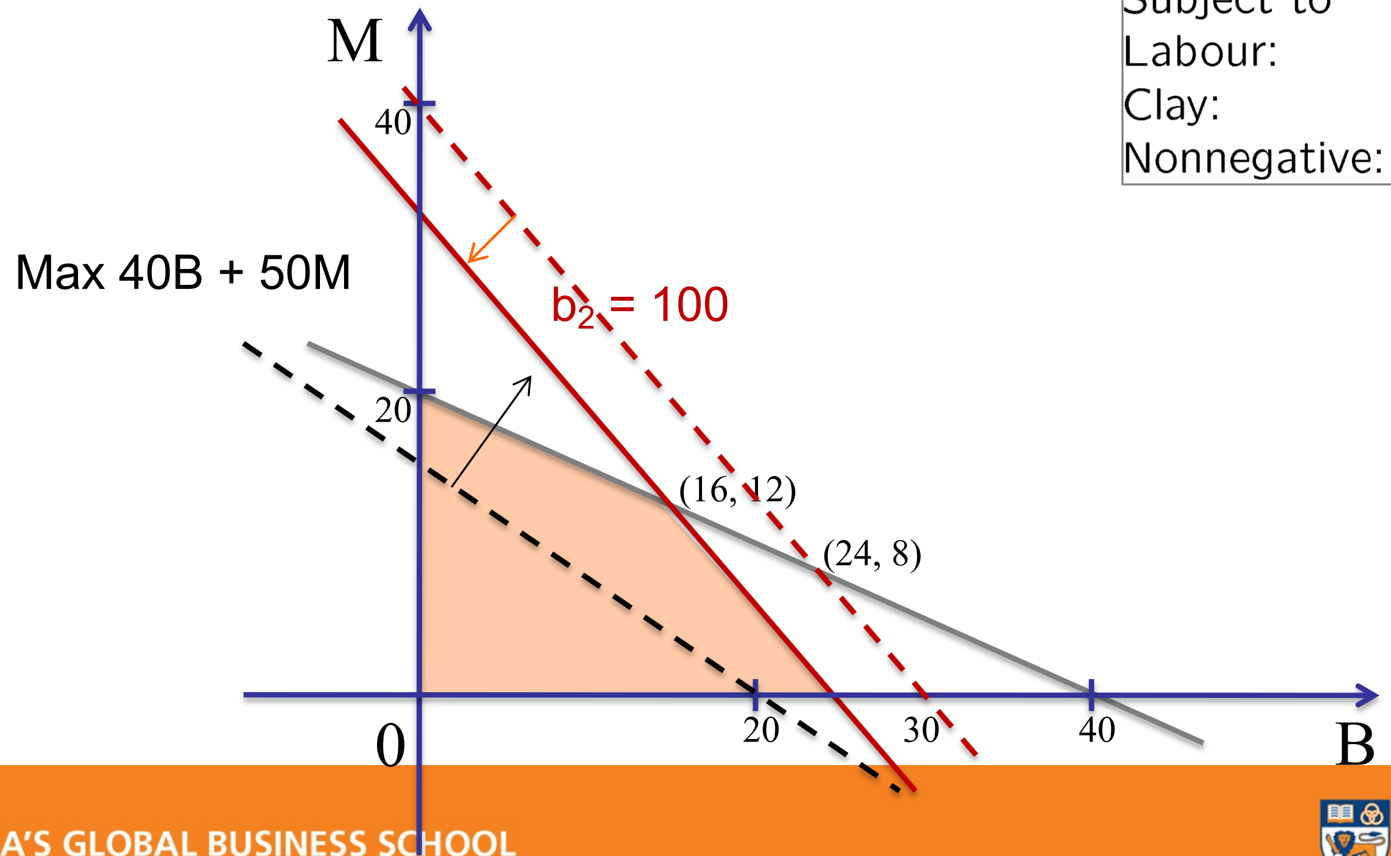
- Suppose b_2 changes, how would the optimal solution change?



Maximize	$40B + 50M$
Subject to	
Labour:	$B + 2M \leq 40$
Clay:	$4B + 3M \leq 120$
Nonnegative:	$B, M \geq 0$

Changes in Constraint Quantity (b_i)

- Say b_2 decreases to *100*

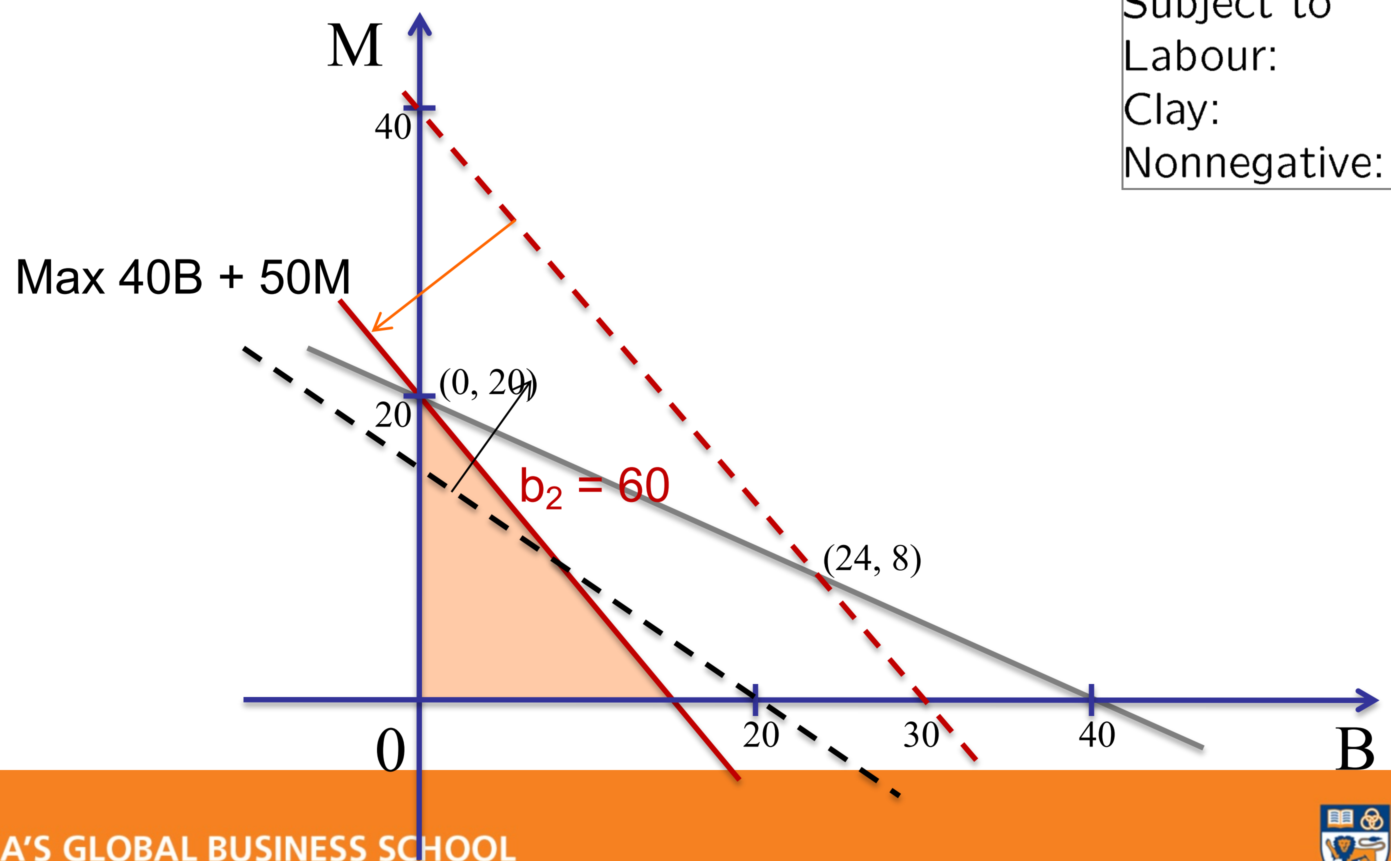


Maximize	$40B + 50M$
Subject to	
Labour:	$B + 2M \leq 40$
Clay:	$4B + 3M \leq 120$
Nonnegative:	$B, M \geq 0$

Changes in Constraint Quantity (b_i)

- How about b_2 decreases to *60*?
- What if b_2 is below *60?*

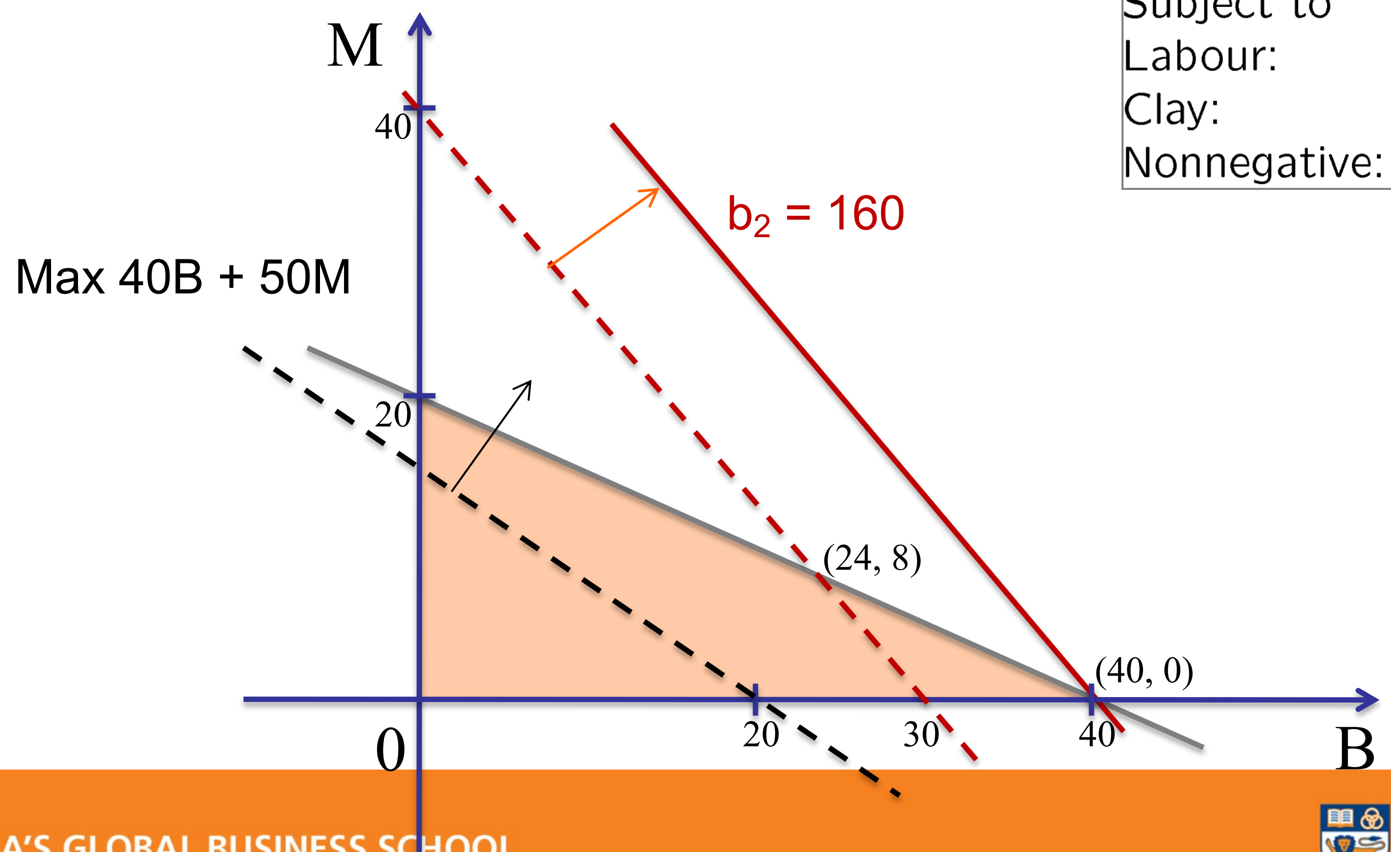
Maximize	$40B + 50M$
Subject to	
Labour:	$B + 2M \leq 40$
Clay:	$4B + 3M \leq 120$
Nonnegative:	$B, M \geq 0$



Changes in Constraint Quantity (b_i)

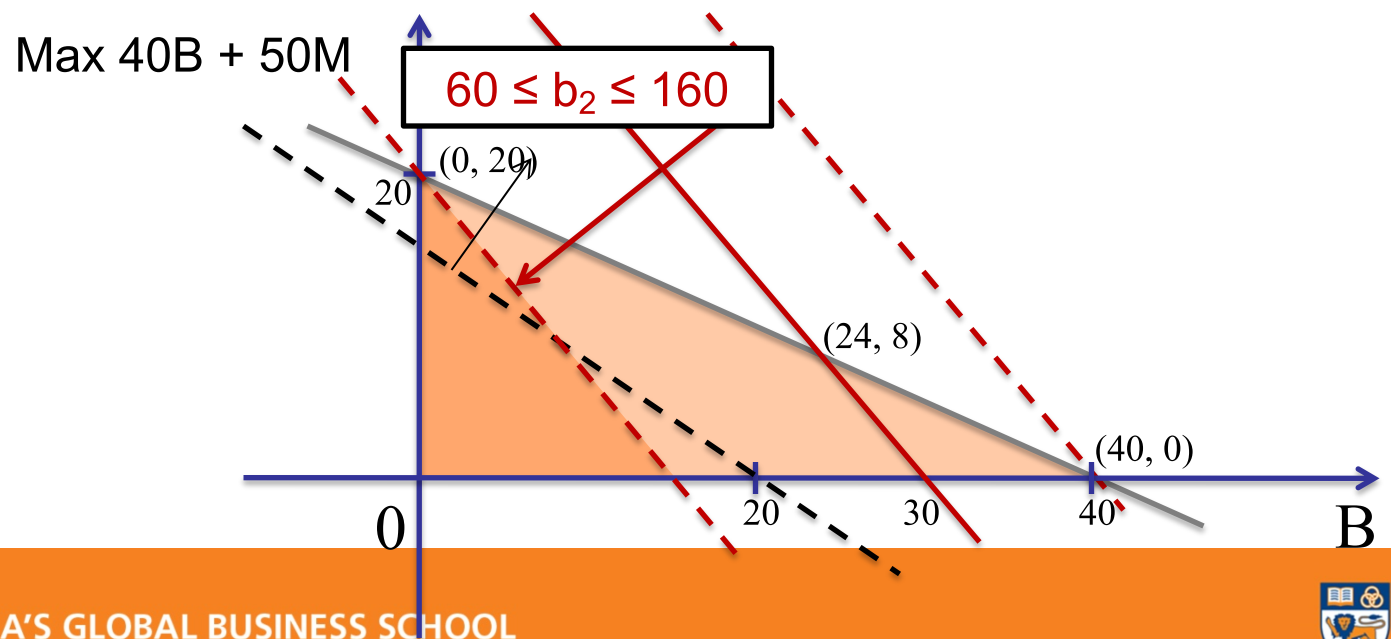
- What if b_2 increases?

Maximize	$40B + 50M$
Subject to	
Labour:	$B + 2M \leq 40$
Clay:	$4B + 3M \leq 120$
Nonnegative:	$B, M \geq 0$



Changes in Constraint Quantity (b_i)

- The optimal solution always changes if the corresponding constraint is ***binding***
- As long as b_2 is in the range [60, 160], the binding constraints will NOT change
- The ***sensitivity range*** of a ***constraint quantity***: the range of values over which the binding constraints will remain so



Excel Sensitivity Analysis Report

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
D3	Quantity-> Bowl	24	0	40	26.66666667	15
E3	Quantity-> Mug	8	0	50	30	20

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C5	Labour (Hrs) Usage	40	16	40	40	10
C6	Clay (lb) Usage	120	6	120	40	60

$$
\begin{array}{rll}
\text{max} & 40x_1 + 50x_2 & \\
\text{s.t.} & x_1 + 2x_2 \leq 40 & \text{Labour, hrs} \\
& 4x_1 + 3x_2 \leq 120 & \text{Clay, lb} \\
& x_1, x_2 \geq 0 &
\end{array}
$$

Sensitivity Analysis

- Shadow Price (or dual values): the marginal value of additional unit of resource.
  - Note that the shadow price is only valid in the sensitivity range.
- Reduced cost: the marginal value in objective if we are forced to increase a zero variable

Excel Sensitivity Analysis Report

- What happened if you increase clay by 10 units?

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
C5	Labour (Hrs) Usage	40	16	40	40	10
C6	Clay (lb) Usage	120	6	120	40	60

  - Is it within the sensitivity range?
  - If yes, use shadow price. Otherwise, resolve.

On Shadow Price

- How would objective value change when c_1 increases by 1 unit, assuming the optimal solution remains unchanged?

$$
\begin{array}{ll}
\max & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m \\
& x_1, x_2, \ldots, x_n \geq 0
\end{array}
$$

- How to compute shadow price?
  - How would objective value change when b_1 increases by 1 unit?
  - To compute shadow price, we need to solve an optimization problem call the dual problem.

Lagrange Multipliers

Consider the LO, called the **primal** with optimal solution $\mathbf{x}^*$

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

Lagrange Multipliers

Consider the LO, called the **primal** with optimal solution $\mathbf{x}^*$

$$\begin{aligned} \min \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Relax the constraint

$$\begin{aligned} g(\mathbf{p}) = \min \quad & \mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

$$g(\mathbf{p}) \leq \mathbf{c}'\mathbf{x}^* + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x}^*) = \mathbf{c}'\mathbf{x}^*$$

Lagrange Multipliers

Consider the LO, called the **primal** with optimal solution $\mathbf{x}^*$

$$\begin{array}{ll} \min & \mathbf{c}'\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

Relax the constraint

$$\begin{array}{rl} g(\mathbf{p}) = \min & \mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x}) \\ \text{s.t.} & \mathbf{x} \geq \mathbf{0} \end{array}$$

$$g(\mathbf{p}) \leq \mathbf{c}'\mathbf{x}^* + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x}^*) = \mathbf{c}'\mathbf{x}^*$$

Get the tightest lower bound. We will claim that

$$\boxed{\max_{\mathbf{p}} g(\mathbf{p}) \text{ is an LO!}}$$

Dual Function

Dual function:

$$
\begin{aligned}
g(\mathbf{p}) &= \min_{\mathbf{x}\geq\mathbf{0}} \left[\mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x})\right] \\
&= \mathbf{p}'\mathbf{b} + \min_{\mathbf{x}\geq\mathbf{0}}(\mathbf{c}' - \mathbf{p}'\mathbf{A})\mathbf{x}
\end{aligned}
$$

Dual Function

Dual function:

$$g(\mathbf{p}) = \min_{\mathbf{x}\geq\mathbf{0}} \left[\mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x})\right]$$
$$= \mathbf{p}'\mathbf{b} + \min_{\mathbf{x}\geq\mathbf{0}}(\mathbf{c}' - \mathbf{p}'\mathbf{A})\mathbf{x}$$

Note that

$$\min_{\mathbf{x}\geq\mathbf{0}} (\mathbf{c}' - \mathbf{p}'\mathbf{A})\mathbf{x} = \begin{cases} 0, & \text{if } \mathbf{c}' - \mathbf{p}'\mathbf{A} \geq \mathbf{0}' \\ -\infty, & \text{otherwise.} \end{cases}$$

Dual Function

Dual function:

$$
\begin{aligned}
g(\mathbf{p}) &= \min_{\mathbf{x}\geq\mathbf{0}} \left[\mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x})\right] \\
&= \mathbf{p}'\mathbf{b} + \min_{\mathbf{x}\geq\mathbf{0}}(\mathbf{c}' - \mathbf{p}'\mathbf{A})\mathbf{x}
\end{aligned}
$$

Note that

$$
\min_{\mathbf{x}\geq\mathbf{0}} \ (\mathbf{c}' - \mathbf{p}'\mathbf{A})\mathbf{x} = \begin{cases} 0, & \text{if } \mathbf{c}' - \mathbf{p}'\mathbf{A} \geq \mathbf{0}' \\ -\infty, & \text{otherwise.} \end{cases}
$$

Dual:

$$
\begin{aligned}
\max\nolimits_{\mathbf{p}} \ g(\mathbf{p}) \quad = \quad & \max \quad \mathbf{p}'\mathbf{b} \\
& \text{s.t.} \quad \mathbf{p}'\mathbf{A} \leq \mathbf{c}'
\end{aligned}
$$

General Form of The Dual

Primal			**Dual**		
min	$\mathbf{c}'\mathbf{x}$		max	$\mathbf{p}'\mathbf{b}$	
s.t.	$\mathbf{a}_i'\mathbf{x} \geq b_i$	$i \in M_1$	s.t.	$p_i \geq 0$	$i \in M_1$
	$\mathbf{a}_i'\mathbf{x} \leq b_i$	$i \in M_2$		$p_i \leq 0$	$i \in M_1$
	$\mathbf{a}_i'\mathbf{x} = b_i$	$i \in M_3$		p_i free	$i \in M_3$
	$x_j \geq 0$	$j \in N_1$		$\mathbf{p}'\mathbf{A}_j \leq c_j$	$j \in N_1$
	$x_j \leq 0$	$j \in N_2$		$\mathbf{p}'\mathbf{A}_j \geq c_j$	$j \in N_2$
	x_j free	$j \in N_3$		$\mathbf{p}'\mathbf{A}_j = c_j$	$j \in N_3$

- The number of dual variables equals to the number of primal constraints.
- The number of dual constraints equals to the number of primal variables.

Example

$$
\begin{array}{llllll}
\min & x_1 & + & 2x_2 & + & 3x_3 & \\
\text{s.t.} & -x_1 & + & 3x_2 & & & = 5 \\
& 2x_1 & - & x_2 & + & 3x_3 & \geq 6 \\
& & & & & x_3 & \leq 4 \\
& x_1 \geq 0 \\
& x_2 \leq 0 \\
& x_3 \text{ free}
\end{array}
$$

Example

$$
\begin{array}{llllllll}
\min & x_1 & + & 2x_2 & + & 3x_3 & \\
\text{s.t.} & -x_1 & + & 3x_2 & & & = 5 \\
 & 2x_1 & - & x_2 & + & 3x_3 & \geq 6 \\
 & & & & & x_3 & \leq 4 \\
 & x_1 \geq 0 & & & & & \\
 & x_2 \leq 0 & & & & & \\
 & x_3 \text{ free} & & & & &
\end{array}
\quad\Bigg|\quad
\begin{array}{llllllll}
\max & 5p_1 & + & 6p_2 & + & 4p_3 & \\
\text{s.t.} & p_1 \text{ free} & & & & & \\
 & p_2 \geq 0 & & & & & \\
 & p_3 \leq 0 & & & & & \\
 & -p_1 & + & 2p_2 & & & \leq 1 \\
 & 3p_1 & - & p_2 & & & \geq 2 \\
 & & & 3p_2 & + & p_3 & = 3.
\end{array}
$$

The Dual Problem

Primal	min	max	dual
constraints	$\geq b_i$ $\leq b_i$ $= b_i$	≥ 0 ≤ 0 free	variables
variables	≥ 0 ≤ 0 free	$\leq c_j$ $\geq c_j$ $= c_j$	constraints

Theorem: The dual of the dual is the primal.

Theorem: Equivalent primal problems have equivalent dual problems.

Primal and Dual Problems

Primal

$$
\begin{array}{lll}
\max & c_1x_1 + c_2x_2 + \cdots + c_nx_n & \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 & : p_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 & : p_2 \\
& \vdots & \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m & : p_m \\
& x_1, x_2, \ldots, x_n \ge 0 &
\end{array}
$$

Dual

$$
\begin{array}{lll}
\min & b_1p_1 + b_2p_2 + \cdots + b_mp_m & \\
\text{s.t.} & a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \ge c_1 & : x_1 \\
& a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \ge c_2 & : x_2 \\
& \vdots & \\
& a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \ge c_n & : x_n \\
& p_1, p_2, \ldots, p_m \ge 0 &
\end{array}
$$

Matrix view

Primal

$$\begin{aligned} \max \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\ & \boldsymbol{x} \ge \boldsymbol{0} \end{aligned}$$

$$\begin{aligned} \max \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\ & x_1, x_2, \ldots, x_n \ge 0 \end{aligned}$$

Dual

$$\begin{aligned} \min \quad & \boldsymbol{b}'\boldsymbol{p} \\ \text{s.t.} \quad & \boldsymbol{A}'\boldsymbol{p} \ge \boldsymbol{c} \\ & \boldsymbol{p} \ge \boldsymbol{0} \end{aligned}$$

$$\begin{aligned} \min \quad & b_1p_1 + b_2p_2 + \cdots + b_mp_m \\ \text{s.t.} \quad & a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \ge c_1 \\ & a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \ge c_2 \\ & \vdots \\ & a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \ge c_n \\ & p_1, p_2, \ldots, p_m \ge 0 \end{aligned}$$

Weak Duality

Theorem: If $\boldsymbol{x}$ is primal feasible, and $\boldsymbol{p}$ is dual feasible, then $\boldsymbol{b}'\boldsymbol{p} \geq \boldsymbol{c}'\boldsymbol{x}$.

Hence, if $\boldsymbol{c}'\boldsymbol{x} = \boldsymbol{b}'\boldsymbol{p}$, then $\boldsymbol{x}$ is optimal in the primal and $\boldsymbol{p}$ is optimal in the dual.

Primal

$$\begin{array}{ll} \max & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

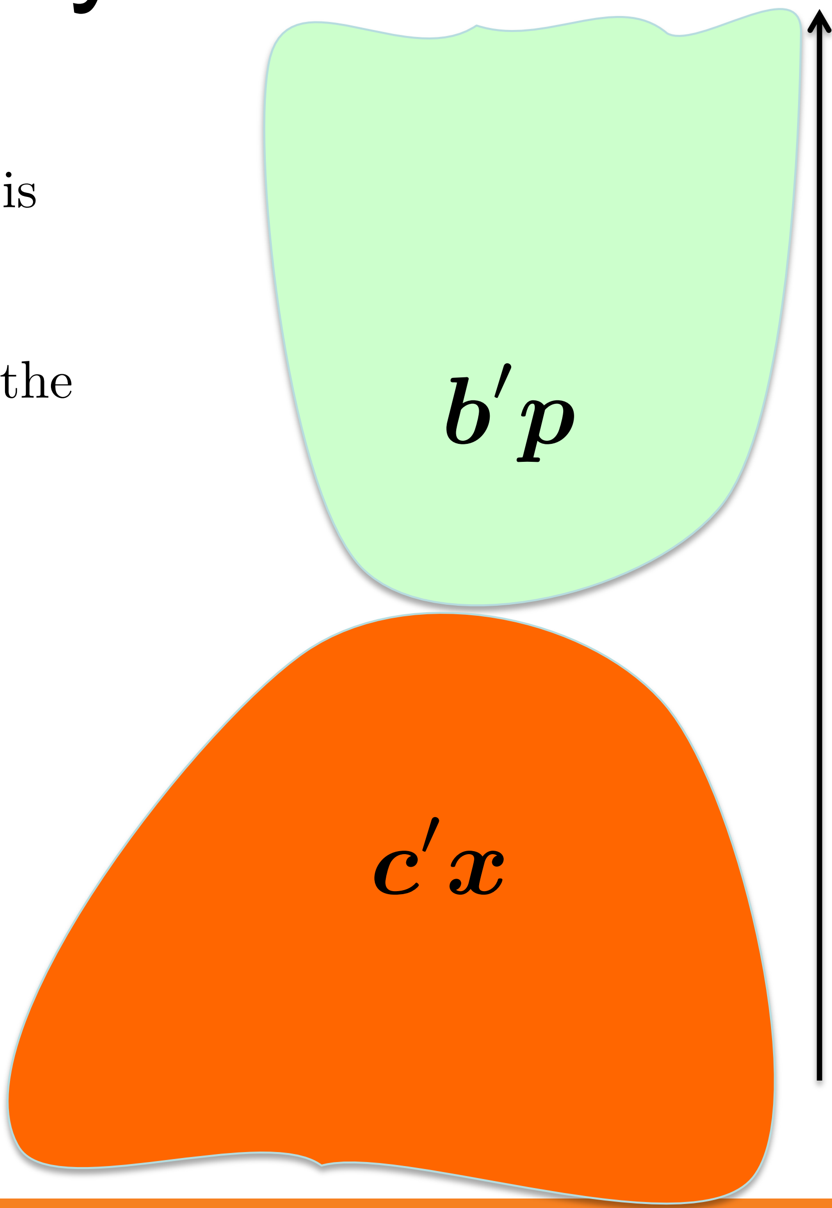
Dual

$$\begin{array}{ll} \min & \boldsymbol{b}'\boldsymbol{p} \\ \text{s.t.} & \boldsymbol{A}'\boldsymbol{p} \geq \boldsymbol{c} \\ & \boldsymbol{p} \geq \boldsymbol{0} \end{array}$$

Weak Duality

Theorem: If $\boldsymbol{x}$ is primal feasible, and $\boldsymbol{p}$ is dual feasible, then $\boldsymbol{b}'\boldsymbol{p} \geq \boldsymbol{c}'\boldsymbol{x}$.

Hence, if $\boldsymbol{c}'\boldsymbol{x} = \boldsymbol{b}'\boldsymbol{p}$, then $\boldsymbol{x}$ is optimal in the primal and $\boldsymbol{p}$ is optimal in the dual.



Weak Duality

Proof

$$\begin{array}{ll} \max & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

$$\begin{array}{ll} \min & \boldsymbol{b}'\boldsymbol{p} \\ \text{s.t.} & \boldsymbol{A}'\boldsymbol{p} \geq \boldsymbol{c} \\ & \boldsymbol{p} \geq \boldsymbol{0} \end{array}$$

- $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \Leftrightarrow \boldsymbol{x}'\boldsymbol{A}' \leq \boldsymbol{b}'$
- $\boldsymbol{x}'\boldsymbol{A}'\boldsymbol{p} \leq \boldsymbol{b}'\boldsymbol{p}$
- Observe that $(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p}) \leq \boldsymbol{0}$
- Since $\boldsymbol{x} \geq \boldsymbol{0}$, we have $\boldsymbol{x}'(\boldsymbol{c} - \boldsymbol{A}'\boldsymbol{p}) \leq 0$
- Hence, $\boldsymbol{c}'\boldsymbol{x} \leq \boldsymbol{x}'\boldsymbol{A}'\boldsymbol{p} \leq \boldsymbol{b}'\boldsymbol{p}$

Strong Duality

Theorem: If the primal LOP has optimal solution, so does the dual, and the optimal cost are equal.

$$Z_P = \quad \max \quad \boldsymbol{c}'\boldsymbol{x} \quad \text{s.t.} \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \quad \boldsymbol{x} \geq \boldsymbol{0}$$

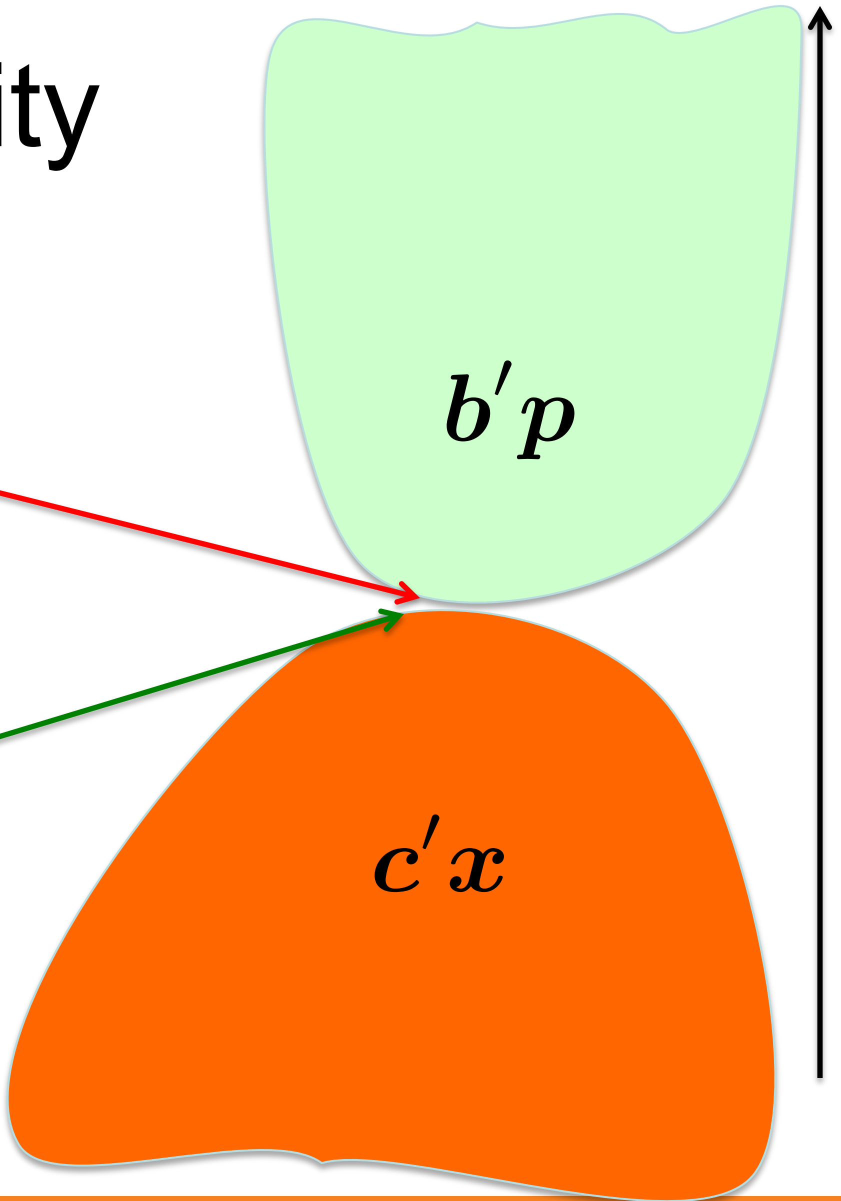
$$Z_D = \quad \min \quad \boldsymbol{b}'\boldsymbol{p} \quad \text{s.t.} \quad \boldsymbol{A}'\boldsymbol{p} \geq \boldsymbol{c}, \quad \boldsymbol{p} \geq \boldsymbol{0}$$

$$Z_D = Z_P$$

Strong Duality

$$\begin{array}{ll} \min & \boldsymbol{b}'\boldsymbol{p} \\ \text{s.t.} & \boldsymbol{A}'\boldsymbol{p} \geq \boldsymbol{c} \\ & \boldsymbol{p} \geq \boldsymbol{0} \end{array}$$

$$\begin{array}{ll} \max & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$



Complementary Slackness

- The optimal solutions of the Primal and Dual has the following complementary slackness conditions:
  - If a Primal constraint is not binding (i.e. slack > 0), then the corresponding Dual variable must be zero.
  - If a Primal variable is strictly greater than zero, then the corresponding Dual constraint must be binding.
  - If a Dual constraint is not binding (i.e. slack > 0), then the corresponding Primal variable must be zero.
  - If a Dual variable is strictly greater than zero, then the corresponding Primal constraint must be binding.

Implication of Strong Duality

- How to compute shadow price?
  - How would objective value change when b_1 increases by 1 unit, assuming the dual problem remains optimal?

$$
\begin{array}{ll}
\max & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le b_m \\
& x_1, x_2, \ldots, x_n \ge 0
\end{array}
$$

$$
\begin{array}{ll}
\min & b_1p_1 + b_2p_2 + \cdots + b_mp_m \\
\text{s.t.} & a_{11}p_1 + a_{21}p_2 + \cdots + a_{m1}p_m \ge c_1 \\
& a_{12}p_1 + a_{22}p_2 + \cdots + a_{m2}p_m \ge c_2 \\
& \vdots \\
& a_{1n}p_1 + a_{2n}p_2 + \cdots + a_{mn}p_m \ge c_n \\
& p_1, p_2, \ldots, p_m \ge 0
\end{array}
$$

  - What is the Shadow Price of a constraint that is not binding?
    - Implications of Complementary Slackness.

Primal/Dual Generalization

- The Primal/Dual problems pairs are interchangeable, i.e., Dual of the Dual is the Primal.

Primal with equality

Primal

$$
\begin{array}{lll}
\max & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 & \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 = b_1 & : p_1 \\
& a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \leq b_2 & : p_2 \\
& a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \leq b_3 & : p_3 \\
& a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 \leq b_4 & : p_4 \\
& x_1, x_2, x_3, x_4 \geq 0 &
\end{array}
$$

Dual

$$
\begin{array}{lll}
\min & b_1p_1 + b_2p_2 + b_3p_3 + b_4p_4 & \\
\text{s.t.} & a_{11}p_1 + a_{21}p_2 + a_{31}p_3 + a_{41}p_4 \geq c_1 & : x_1 \\
& a_{12}p_1 + a_{22}p_2 + a_{32}p_3 + a_{42}p_4 \geq c_2 & : x_2 \\
& a_{13}p_1 + a_{23}p_2 + a_{33}p_3 + a_{43}p_4 \geq c_3 & : x_3 \\
& a_{14}p_1 + a_{24}p_2 + a_{34}p_3 + a_{44}p_4 \geq c_4 & : x_4 \\
& p_2, p_3, p_3 \geq 0, p_1 \text{free} &
\end{array}
$$

- Dual variables associated with primal equality constraints are unconstrained or free.

Dual with equality

Primal

$$\begin{array}{lll} \max & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 & \\ \text{s.t.} & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \le b_1 & : p_1 \\ & a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \le b_2 & : p_2 \\ & a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \le b_3 & : p_3 \\ & a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 \le b_4 & : p_4 \\ & x_1, x_2, x_4 \ge 0, x_3 \text{ free} & \end{array}$$

Dual

$$\begin{array}{lll} \min & b_1p_1 + b_2p_2 + b_3p_3 + b_4p_4 & \\ \text{s.t.} & a_{11}p_1 + a_{21}p_2 + a_{31}p_3 + a_{41}p_4 \ge c_1 & : x_1 \\ & a_{12}p_1 + a_{22}p_2 + a_{32}p_3 + a_{42}p_4 \ge c_2 & : x_2 \\ & a_{13}p_1 + a_{23}p_2 + a_{33}p_3 + a_{43}p_4 = c_3 & : x_3 \\ & a_{14}p_1 + a_{24}p_2 + a_{34}p_3 + a_{44}p_4 \ge c_4 & : x_4 \\ & p_2, p_3, p_3, p_4 \ge 0 & \end{array}$$

•Primal variable associated with dual equality constraints are unconstrained or free.

Primal inequality change

Primal

$$
\begin{aligned}
\max \quad & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 \\
\text{s.t.} \quad & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \geq b_1 && : p_1 \\
& a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \leq b_2 && : p_2 \\
& a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \leq b_3 && : p_3 \\
& a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 \leq b_4 && : p_4 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

Dual

$$
\begin{aligned}
\min \quad & b_1p_1 + b_2p_2 + b_3p_3 + b_4p_4 \\
\text{s.t.} \quad & a_{11}p_1 + a_{21}p_2 + a_{31}p_3 + a_{41}p_4 \geq c_1 && : x_1 \\
& a_{12}p_1 + a_{22}p_2 + a_{32}p_3 + a_{42}p_4 \geq c_2 && : x_2 \\
& a_{13}p_1 + a_{23}p_2 + a_{33}p_3 + a_{43}p_4 \geq c_3 && : x_3 \\
& a_{14}p_1 + a_{24}p_2 + a_{34}p_3 + a_{44}p_4 \geq c_4 && : x_4 \\
& p_2, p_3, p_3 \geq 0, p_1 \leq 0
\end{aligned}
$$

Dual inequality change

Primal

$$
\begin{array}{lll}
\max & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 & \\
\text{s.t.} & a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 \leq b_1 & : p_1 \\
& a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 \leq b_2 & : p_2 \\
& a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4 \leq b_3 & : p_3 \\
& a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 \leq b_4 & : p_4 \\
& x_1, x_2, x_4 \geq 0, x_3 \leq 0 &
\end{array}
$$

Dual

$$
\begin{array}{lll}
\min & b_1p_1 + b_2p_2 + b_3p_3 + b_4p_4 & \\
\text{s.t.} & a_{11}p_1 + a_{21}p_2 + a_{31}p_3 + a_{41}p_4 \geq c_1 & : x_1 \\
& a_{12}p_1 + a_{22}p_2 + a_{32}p_3 + a_{42}p_4 \geq c_2 & : x_2 \\
& a_{13}p_1 + a_{23}p_2 + a_{33}p_3 + a_{43}p_4 \leq c_3 & : x_3 \\
& a_{14}p_1 + a_{24}p_2 + a_{34}p_3 + a_{44}p_4 \geq c_4 & : x_4 \\
& p_2, p_3, p_3, p_4 \geq 0 &
\end{array}
$$

Primal and Dual Problem

Primal

$$\begin{array}{lll} \min & -10x_1 + 12x_2 - 12x_3 & \\ \text{s.t.} & x_1 + 2x_2 + 6x_3 \leq 20 & : p_1 \\ & -2x_1 + x_2 - 2x_3 \geq -20 & : p_2 \\ & 2x_1 + 4x_2 + x_3 = -20 & : p_3 \\ & x_1 \geq 0, x_2 \leq 0, x_3 \text{ free} & \end{array}$$

Dual

Assessing Dual Solution in Python

- Manufacturing problem

x_j = amount of product j produced

$$
\begin{aligned}
\max \quad & \sum_{j=1}^{n} c_j x_j \\
\text{s.t.} \quad & a_{11}x_1 + \ldots + a_{1n}x_n \leq b_1 \\
& \vdots \\
& a_{m1}x_1 + \ldots + a_{mn}x_n \leq b_m \\
& x_1, \ldots, x_n \geq 0
\end{aligned}
$$

```python
# print optimal solutions
print("Primal Solutions:")
for v in mod.getVars():
    print('%s %g' % (v.varName, v.x))

# print optimal value
print('Obj: %g' % mod.objVal)

print("Dual Solutions:")
for d in mod.getConstrs():
    print('%s %g' % (d.ConstrName, d.Pi))
```