# DBA3701
# Introduction to Optimization
# Lecture 8

# Non-Linear Optimization

# Objective Function and Constraints

**Optimization**

Art of finding a best solution from collection alternatives.

Everyone optimizes: application in ...

- Science: design of experiments
- Engineering: power-grid control and design
- Finance: pricing of options, optimal portfolio selection.
- Medicine: optimal radiation dose design
- Economics: optimal transition to clean energy
- Big data: machine learning ... training of neural nets

... more details tomorrow.

# Objective Function

**Ingredients of Optimization**

- **Decision Variables,** $x$, model decisions.
- **Constraints** model acceptable values of $x$.
- **Objective(s)** model our goals / performance measure.

| | |
|---|---|
| $\underset{x}{\text{minimize}}\ f(x)$ | objectve function |
| subject to $l_c \le c(x) \le u_c$ | nonlinear constraints |
| $l_A \le A^T x \le u_A$ | linear constraints |
| $l_x \le x \le u_x$ | simple bounds |
| $x \in X$ | structural constraints |

# Objective Function and Constraints

$$
\begin{array}{llll}
\underset{x}{\text{minimize}} & f(x) & & \text{objective function} \\
\text{subject to} & l_c \le c(x) & \le u_c & \text{nonlinear constraints} \\
& l_A \le A^T x & \le u_A & \text{linear constraints} \\
& l_x \le \quad x & \le u_x & \text{simple bounds} \\
& & x \in X & \text{structural constraints}
\end{array}
$$

## Basic Blanket Assumptions

We make the following blanket assumptions:

1. $x \in \mathbb{R}^n$ finite dimensional.
2. Functions, $c : \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^n \to \mathbb{R}$ are smooth.
3. Bounds, $l_c, u_c, l_A, u_A, l_x, u_x$ can be infinite.
4. Set $X \subset \mathbb{R}^n$ imposes structural restrictions $x$ (later).

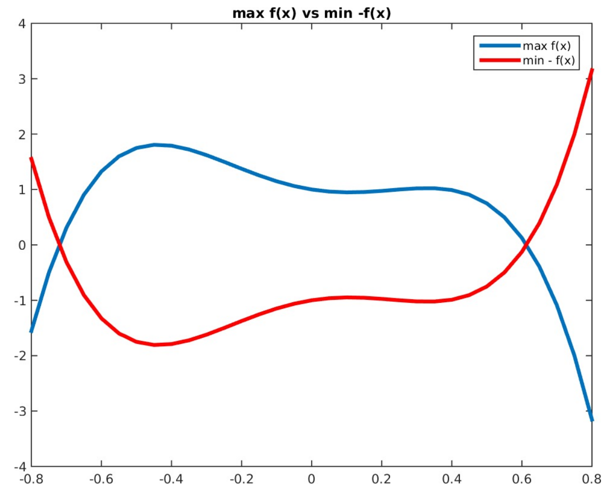
# Objective Function and Constraints

$$
\begin{array}{llll}
\underset{x}{\text{minimize}} & f(x) & & \text{objective function} \\
\text{subject to} & l_c \leq c(x) \leq u_c & & \text{nonlinear constraints} \\
& l_A \leq A^T x \leq u_A & & \text{linear constraints} \\
& l_x \leq x \leq u_x & & \text{simple bounds} \\
& x \in X & & \text{structural constraints}
\end{array}
$$

**To Minimize or To Maximize?**

$\max f(x)$ equivalent to $-\min(-f(x))$

... wlog only consider minimization



# Example: Design of Reinforced Concrete Beam

- **Variables:**
  - $x_1$ = area of re-inforcement,
  - $x_2$ = width of beam,
  - $x_3$ = depth of beam.
- **Objective:** minimizing cost of reinforced beam
- **Constraints:**
  - Support minimum amount of load.
  - Bounds on width/depth ratio and variables (positivity).

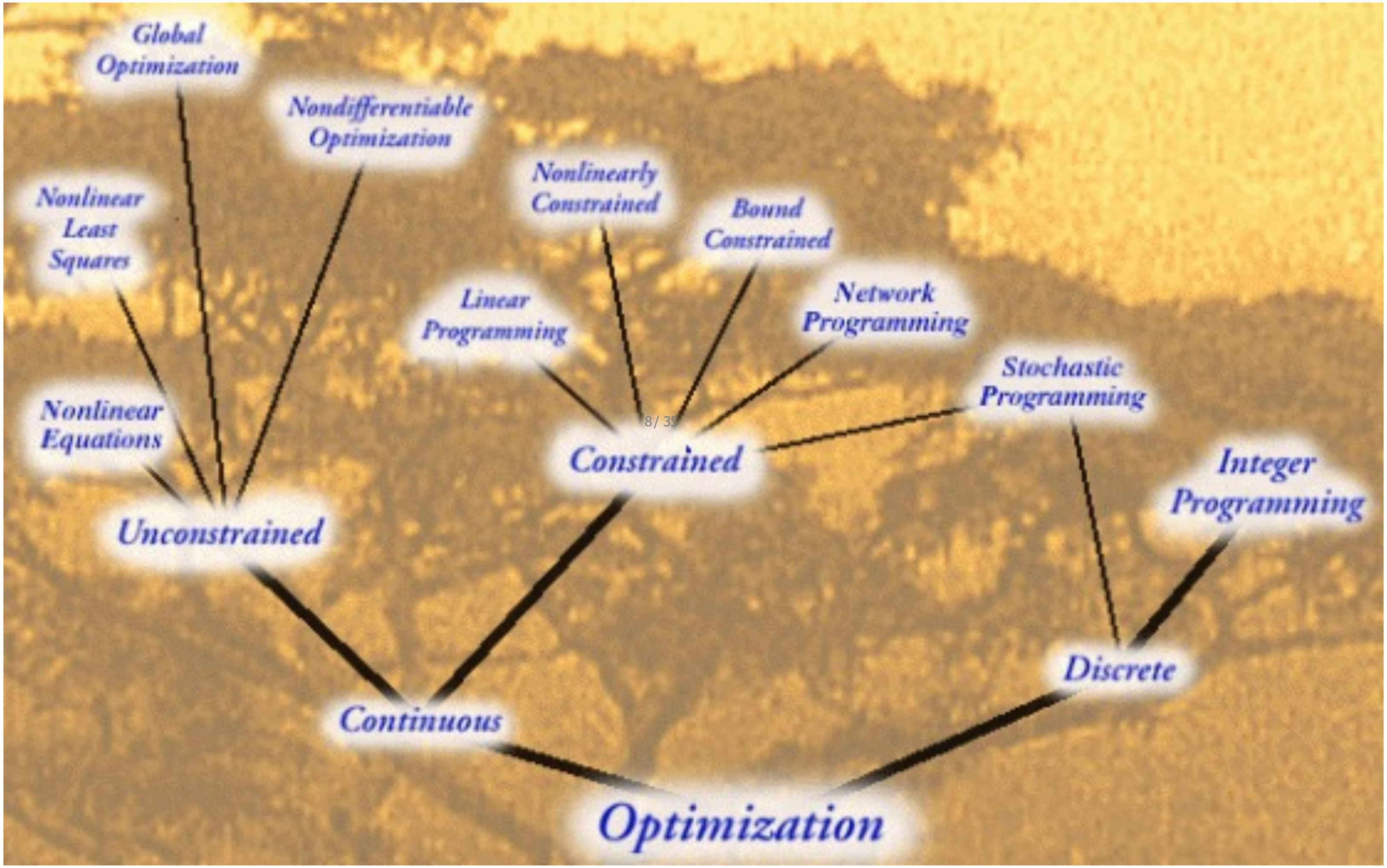
# Example: Design of Reinforced Concrete Beam

- **Variables:**
  - $x_1$ = area of re-inforcement, e.g. $x_1 \in \{40, 45, \ldots, 75\}$
  - $x_2$ = width of beam,
  - $x_3$ = depth of beam.
- **Objective:** minimizing cost of reinforced beam
- **Constraints:**
  - Support minimum amount of load.
  - Bounds on width/depth ratio and variables (positivity).

$$\begin{array}{lll}
\underset{x}{\text{minimize}} & f(x) = 29.4x_1 + 0.6x_2x_3 & \text{cost of beam} \\
\text{subject to} & c(x) = x_1x_2 - 7.735\frac{x_1^2}{x_2} \geq 180 & \text{load constraint} \\
& x_3 - 4x_2 \geq 0 & \text{width/depth ratio} \\
& 40 \leq x_1 \leq 77,\ x_2 \geq 0,\ x_3 \geq 0 & \text{simple bounds,}
\end{array}$$

In practice, area of reinforcement, $x_1$, is discrete ... include in X .

# NEOS Optimization Tree



# Classification by Type of Constraint

Assume $X = \mathbb{R}^n$, and $f(x)$, $c(x)$ twice continuously differentiable

- **Unconstrained Optimization** all $x \in \mathbb{R}^n$ feasible:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(x).$$

**Special Case: Least-Squares Problem**

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(x) = \sum_{j=1}^{m} (r_j(x))^2,$$

- **Bound Constrained Optimization** only bounds constraints:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \; f(x) \quad \text{subject to } l \le x \le u,$$

where $l, u \in \mathbb{R}^n$ can be infinite.

**Special case:** $f(x) = c^T x$ solved trivially.

# Classification by Constraint Type

Assume $X = \mathbb{R}^n$, and $f(x)$, $c(x)$ twice continuously differentiable

- **Linearly Constrained Optimization** nonlinear objective and linear constraints

$$
\begin{array}{lll}
\underset{x}{\text{minimize}} & f(x) & \text{objective function} \\
\text{subject to} & l_A \leq A^T x \leq u_A & \text{linear constraints} \\
 & l_x \leq \quad x \quad \leq u_x & \text{simple bounds}
\end{array}
$$



**Important Special Cases:**

- **Linear Programming** Objective function is linear:

$$f(x) = c^T x$$

- **Quadratic Programming** Objective function is quadratic:

$$f(x) = x^T Gx/2 + g^T x + a$$

wlog assume $a = 0$ Why???

# Classification by Constraint Type

Assume $X = \mathbb{R}^n$, and $f(x)$, $c(x)$ twice continuously differentiable

- **Equality Constrained Optimization** all constraints are equations:

$$\begin{aligned} &\underset{x \in \mathbb{R}^n}{\text{minimize}} \quad f(x) \\ &\text{subject to } c(x) = 0. \end{aligned}$$

**Special Case:** Only linear equality constraints: $A^T x = b$.

# Classification by Constraint Type

Assume $\mathcal{X} = \mathbb{R}^n$, and $f(x)$, $c(x)$ twice continuously differentiable

- **Nonlinearly Constrained Optimization**

$$
\begin{array}{llll}
\underset{x}{\text{minimize}} & f(x) & & \text{objective function} \\
\text{subject to} & l_c \le c(x) & \le u_c & \text{nonlinear constraints} \\
& l_A \le A^T x & \le u_A & \text{linear constraints} \\
& l_x \le \quad x & \le u_x & \text{simple bounds} \\
& & x \in \mathcal{X} & \text{structural constraints}
\end{array}
$$



**Programming vs. Optimization**

This problem is also called a Nonlinear Programming Problem.

# Classification by Type of Variables

Variable type is encoded in $x \in X$:

- **Continuous Variables** are variables with $x \in \mathbb{R}^n$
  ... leverage classical calculus.
- **Discrete Variables** $X$ is discrete subset:
  - *Binary Variables* $X = \{0, 1\}^n$ model logic.
  - *Integer Variables* $X = \mathbb{Z}^n$ model numbers of equipment.
  - *Discrete Variables* from discrete set, e.g.
    $X = \{ 1/4, 1/2, 1, 2, 4, \ldots\}$
    ... can be modeled with binary variables.

$\Rightarrow$ Integer or discrete programming problems



Often have mixture of continuous and discrete variables, called **mixed-integer programs** (MIPs).

# A Simple Portfolio Selection Problem

**Decisions**

- $x_i$: decision variable on amount to invest in stock $i = 1, 2$

**Data**

- $\tilde{r}_i$: Reward from stock $i$ (random variable)
- $\mu_i = \mathrm{E}(\tilde{r}_i)$: expected reward from stock $i$
- $\mathrm{Var}(\tilde{r}_i)$ : variance in reward from stock $i$
- $\sigma_{ij} = \mathrm{E}((\tilde{r}_i - \mu_i)(\tilde{r}_j - \mu_j)) = \mathrm{Cov}(\tilde{r}_i, \tilde{r}_j)$. Note that $\sigma_{ii} = \mathrm{Var}(\tilde{r}_i)$ :
- Budget $B$, target $\beta$ on expected portfolio reward

# A Simple Portfolio Selection Problem

**Objective:** Minimize total portfolio variance so that

- Expected reward of total portfolio is above target $\beta$
- Total amount invested stay within our budget
- No short sales

$$
\begin{aligned}
\text{min} \quad & f(\boldsymbol{x}) = \mathrm{Var}(\tilde{r}_1 x_1 + \tilde{r}_2 x_2) = \sigma_{11} x_1^2 + \sigma_{22} x_2^2 + 2 x_1 x_2 \sigma_{12} \\
\text{s.t.} \quad & \sum_j x_j \le B \\
& \sum_j \mu_j x_j \ge \beta \\
& x_j \ge 0
\end{aligned}
$$

(Linearly constrained NLP)

# Generalization of Portfolio Variance

$$
\begin{aligned}
& \mathrm{Var}(\tilde{r}x_1 + \tilde{r}_2 x_2) \\
= \quad & \sigma_{11}x_1^2 + 2\sigma_{12}x_1x_2 + \sigma_{22}x_2^2 \\
= \quad & \sum_{i=1}^{2}\sum_{j=1}^{2}\sigma_{ij}x_ix_j \\
= \quad & \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
\end{aligned}
$$

# Generalization of Portfolio Variance

$$
\begin{aligned}
& \operatorname{Var}(\tilde{r} x_1 + \cdots + \tilde{r}_n x_n) \\
= & \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}' \begin{bmatrix} \sigma_{11} & \sigma_{12} & \ldots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \ldots & \sigma_{2n} \\ \ldots & \ddots & \ldots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_{nn} \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\
= & \boldsymbol{x}' \boldsymbol{\Sigma} \boldsymbol{x}
\end{aligned}
$$

# Generalization of Portfolio Variance

$\boldsymbol{\Sigma}$ is known as the covariance matrix, which has the following properties:

- It is a symmetric matrix.
- It is a positive definite matrix,
$$\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} > 0 \qquad \forall \boldsymbol{x}, \boldsymbol{x} \neq \boldsymbol{0}$$
- The Eigenvalues of $\boldsymbol{\Sigma}$ are real and positive.
- There exists real matrix, $\boldsymbol{A}$ such that $\boldsymbol{\Sigma} = \boldsymbol{A}'\boldsymbol{A}$.

# Covariance Matrix from Data

Let $\boldsymbol{X} \in \mathbb{R}^{T \times N}$ be a matrix that contains the stock returns of $N$ stocks over $T$ periods. In particular, $X_{tn}$ denotes the return of stock $i$ at period $t$.

Average stock return:

$$\mu_i = \frac{1}{T} \sum_{t=1}^{T} X_{ti} \qquad \boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_N \end{pmatrix}$$

# Covariance Matrix from Data

Covariance (unbiased estimation)

$$\sigma_{ij} = \frac{1}{T-1}\sum_{t=1}^{T}(X_{ti}-\mu_i)(X_{tj}-\mu_j)$$

Covariance Matrix

$$\boldsymbol{\Sigma} = \frac{1}{T-1}(\boldsymbol{X} - \mathbf{1}_T\boldsymbol{\mu}')'(\boldsymbol{X} - \mathbf{1}_T\boldsymbol{\mu}')$$

where $\mathbf{1}_T$ is a vector of ones of dimension $T$.

# Compact form of Portfolio Selection Problem

$$\begin{aligned} \text{min} \quad & \boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} \\ \text{s.t.} \quad & \mathbf{1}'\boldsymbol{x} \leq B \\ & \boldsymbol{\mu}'\boldsymbol{x} \geq \beta \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned}$$

# Quadratic Optimization Problems

$$\begin{array}{ll} \min & \boldsymbol{x}'\boldsymbol{M}\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x}'\boldsymbol{M}_i\boldsymbol{x} \leq \boldsymbol{a}_i\boldsymbol{x} + b_i \quad \forall i = \{1, \ldots, m\} \end{array}$$

where

$$\boldsymbol{M}, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_m$$

are **symmetric positive semidefinite (PSD)** matrices, i.e.

$$\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} \geq 0 \qquad \forall \boldsymbol{x}$$

# On Positive Semidefinite Matrix

- If leading principal minors of matrix are positive, then matrix is positive definite.

- However, if one of them is zero, we cannot conclude that the matrix is positive semidefinite. If this is the case, add the diagonal elements of the matrix by a very small number, $\epsilon$ and then check the leading principal minors again. If they are positive then, the matrix is positive semidefinite.

# Quadratic Optimization Problems

- Quadratic optimization problems (QOP) can be solved efficiently, though not as great as LOP.

- Some solver can incorporate integer constraints in their quadratic optimization package.

# Are these QOP?

$$\begin{array}{ll} \min & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{array}$$

$$\begin{array}{ll} \max & -\boldsymbol{x}'\boldsymbol{M}\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & -\boldsymbol{x}'\boldsymbol{M}_i\boldsymbol{x} + \boldsymbol{a}_i'\boldsymbol{x} \geq \boldsymbol{b}_i \quad \forall i = \{1, \ldots, m\} \end{array}$$

where

$$\boldsymbol{M}, \boldsymbol{M}_1, \ldots, \boldsymbol{M}_m$$

are PSD matrices.

# Are these QOP?

$$\begin{array}{ll} \min & 5x_1^2 + 6x_2^2 + 10x_3^2 + 4x_1x_2 + 3x_1x_3 - 2x_2x_3 - 6x_1 + 4x_2 \\ \text{s.t.} & -5x_1^2 - 4x_2^2 - 6x_3^2 + 4x_1x_2 - 2x_1x_3 \geq -3x_1 + x_3 + 9 \\ & \ldots \end{array}$$

$$\begin{array}{ll} \min & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' \begin{bmatrix} 5 & 2 & 1.5 \\ 2 & 6 & 1 \\ 1.5 & 1 & 10 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - 6x_1 + 4x_2 \\ \text{s.t.} & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}' \begin{bmatrix} 5 & -2 & 1 \\ -2 & 4 & 0 \\ 1 & 0 & 6 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - 3x_1 + x_3 \leq -9 \\ & \ldots \end{array}$$

$$\begin{bmatrix} 5 & 2 & 1.5 \\ 2 & 6 & 1 \\ 1.5 & 1 & 10 \end{bmatrix} \text{ and } \begin{bmatrix} 5 & -2 & 1 \\ -2 & 4 & 0 \\ 1 & 0 & 6 \end{bmatrix} \text{ are positive definite matrices.}$$

# Are these QOP?

$$\begin{array}{ll} \max & \boldsymbol{x}'\boldsymbol{M}\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \ldots \end{array}$$

where $\boldsymbol{M}$ is a PSD matrix.

$$\begin{array}{ll} \min & \boldsymbol{x}'\boldsymbol{M}\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \ldots \end{array}$$

where $\boldsymbol{M}$ is any symmetric matrix.

$$\begin{array}{ll} \min & \boldsymbol{x}'\boldsymbol{M}\boldsymbol{x} + \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} & \boldsymbol{x}'\boldsymbol{M}_1\boldsymbol{x} = \boldsymbol{a}'\boldsymbol{x} + b \\ & \ldots \end{array}$$

where $\boldsymbol{M}, \boldsymbol{M}_1$ are PSD matrices.

# Another Portfolio Selection Problem

**Objective:** Maximize probability that portfolio returns exceeds target $\beta$.

- Expect reward of portfolio is above $\beta$.
- Total Investment stays within budget $B$
- No short sales
- Assume returns are normally distributed.

$$
\begin{array}{rl}
\max & \mathbb{P}(\tilde{\boldsymbol{r}}'\boldsymbol{x} \geq \beta) \\
\text{s.t.} & \mathbf{1}'\boldsymbol{x} \leq B \\
& \boldsymbol{\mu}'\boldsymbol{x} \geq \beta \\
& \boldsymbol{x} \geq \mathbf{0}
\end{array}
$$

# Another Portfolio Selection Problem

Under normal distribution assumption,

$$
\begin{aligned}
& \mathbb{P}(\tilde{\boldsymbol{r}}'\boldsymbol{x} \geq \beta) \\
= \;\; & \mathbb{P}\left( \underbrace{\frac{\boldsymbol{\mu}'\boldsymbol{x} - \tilde{\boldsymbol{r}}'\boldsymbol{x}}{\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}}}_{=Z} \leq \frac{\boldsymbol{\mu}'\boldsymbol{x} - \beta}{\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}} \right) \\
= \;\; & \Phi\left( \frac{\boldsymbol{\mu}'\boldsymbol{x} - \beta}{\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}} \right)
\end{aligned}
$$

# Another Portfolio Selection Problem

$$
\begin{array}{ll}
\max & \mathbb{P}(\tilde{\boldsymbol{r}}'\boldsymbol{x} \geq \beta) \\
\text{s.t.} & \mathbf{1}'\boldsymbol{x} \leq B \\
& \boldsymbol{\mu}'\boldsymbol{x} \geq \beta \\
& \boldsymbol{x} \geq \mathbf{0}
\end{array}
\Leftrightarrow
\begin{array}{ll}
\max & \Phi\left(\frac{\boldsymbol{\mu}'\boldsymbol{x}-\beta}{\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}}\right) \\
\text{s.t.} & \mathbf{1}'\boldsymbol{x} \leq B \\
& \boldsymbol{\mu}'\boldsymbol{x} \geq \beta \\
& \boldsymbol{x} \geq \mathbf{0}
\end{array}
\Leftrightarrow
\begin{array}{ll}
\max & \dfrac{\boldsymbol{\mu}'\boldsymbol{x}-\beta}{\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}} \\
\text{s.t.} & \mathbf{1}'\boldsymbol{x} \leq B \\
& \boldsymbol{\mu}'\boldsymbol{x} \geq \beta \\
& \boldsymbol{x} \geq \mathbf{0}
\end{array}
$$

Recall Sharpe ratio: $\frac{\boldsymbol{\mu}'\boldsymbol{x}-\beta}{\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}}$

How do we solve the problem?

# Another Portfolio Selection Problem

Observe:

$$\begin{array}{ll} \max & \dfrac{\boldsymbol{\mu}'\boldsymbol{x} - \beta}{\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}} \\ \text{s.t.} & \mathbf{1}'\boldsymbol{x} \le B \\ & \boldsymbol{\mu}'\boldsymbol{x} \ge \beta \\ & \boldsymbol{x} \ge \mathbf{0} \end{array} \Leftrightarrow \begin{array}{ll} \min & \dfrac{\sqrt{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}}{\boldsymbol{\mu}'\boldsymbol{x} - \beta} \\ \text{s.t.} & \mathbf{1}'\boldsymbol{x} \le B \\ & \boldsymbol{\mu}'\boldsymbol{x} \ge \beta \\ & \boldsymbol{x} \ge \mathbf{0} \end{array} \Leftrightarrow \begin{array}{ll} \min & \dfrac{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}{(\boldsymbol{\mu}'\boldsymbol{x} - \beta)^2} \\ \text{s.t.} & \mathbf{1}'\boldsymbol{x} \le B \\ & \boldsymbol{\mu}'\boldsymbol{x} \ge \beta \\ & \boldsymbol{x} \ge \mathbf{0} \end{array}$$

# Another Portfolio Selection Problem

Observe:

$$\begin{aligned} \min \quad & \frac{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}{(\boldsymbol{\mu}'\boldsymbol{x}-\beta)^2} \\ \text{s.t.} \quad & \mathbf{1}'\boldsymbol{x} \leq B \\ & \boldsymbol{\mu}'\boldsymbol{x} \geq \beta \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \min \quad & \frac{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}{y^2} \\ \text{s.t.} \quad & \mathbf{1}'\boldsymbol{x} \leq B \\ & \boldsymbol{\mu}'\boldsymbol{x} - \beta = y \\ & y \geq 0 \\ & \boldsymbol{x} \geq \mathbf{0} \end{aligned}$$

# Another Portfolio Selection Problem

Assume there exists an optimum solution where $\boldsymbol{\mu}'\boldsymbol{x} > \beta$ or equivalently $y > 0$. Otherwise, finite meaningful solution may not exist.

$$
\begin{aligned}
\min \quad & \frac{\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x}}{y^2} \\
\text{s.t.} \quad & \mathbf{1}'\boldsymbol{x} \le B \\
& \boldsymbol{\mu}'\boldsymbol{x} - \beta = y \\
& y \ge 0 \\
& \boldsymbol{x} \ge \mathbf{0}
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
\min \quad & \frac{\boldsymbol{x}'}{y}\boldsymbol{\Sigma}\frac{\boldsymbol{x}}{y} \\
\text{s.t.} \quad & \mathbf{1}'\frac{\boldsymbol{x}}{y} \le B\frac{1}{y} \\
& \boldsymbol{\mu}'\frac{\boldsymbol{x}}{y} - \beta\frac{1}{y} = 1 \\
& \frac{1}{y} \ge 0 \\
& \frac{\boldsymbol{x}}{y} \ge \mathbf{0}
\end{aligned}
$$

# Another Portfolio Selection Problem

Change of variables: Let $\bar{\boldsymbol{x}} = \boldsymbol{x}/y$, $z = 1/y$.

$$
\begin{aligned}
\min \quad & \frac{\boldsymbol{x}'}{y}\boldsymbol{\Sigma}\frac{\boldsymbol{x}}{y} \\
\text{s.t.} \quad & \mathbf{1}'\frac{\boldsymbol{x}}{y} \le B\frac{1}{y} \\
& \boldsymbol{\mu}'\frac{\boldsymbol{x}}{y} - \beta\frac{1}{y} = 1 \\
& \frac{1}{y} \ge 0 \\
& \frac{\boldsymbol{x}}{y} \ge \mathbf{0}
\end{aligned}
\quad \Leftrightarrow \quad
\begin{aligned}
\min \quad & \bar{\boldsymbol{x}}'\boldsymbol{\Sigma}\bar{\boldsymbol{x}} \\
\text{s.t.} \quad & \mathbf{1}'\bar{\boldsymbol{x}} \le Bz \\
& \boldsymbol{\mu}'\bar{\boldsymbol{x}} - \beta z = 1 \\
& z \ge 0 \\
& \bar{\boldsymbol{x}} \ge \mathbf{0}
\end{aligned}
$$

# Another Portfolio Selection Problem

Solve the following QOP:

$$
\begin{array}{ll}
\min & \bar{\boldsymbol{x}}'\boldsymbol{\Sigma}\bar{\boldsymbol{x}} \\
\text{s.t.} & \mathbf{1}'\bar{\boldsymbol{x}} \leq Bz \\
 & \boldsymbol{\mu}'\bar{\boldsymbol{x}} - \beta z = 1 \\
 & z \geq 0 \\
 & \bar{\boldsymbol{x}} \geq \mathbf{0}
\end{array}
$$

Actual portfolio weights: $\boldsymbol{x} = \bar{\boldsymbol{x}}y = \bar{\boldsymbol{x}}/z$.

# A Real Portfolio Optimization Problem

**Problem**

- We currently own $z_i$ shares from stock $i, i \in S$
- $P_i$ current price of stock $i$
- We consider buying and selling stocks in $S$, and consider buying new stocks from the set $B (B \cap S = \emptyset)$
- Set of stocks $B \cup S = \{1, \ldots, n\}$

**Data**

- Forecasted prices next period (say next month) and their correlations.

$$\mathrm{E}(\tilde{p}_i) = \mu_i, \sigma_{ij} = \mathrm{Cov}(\tilde{p}_i, \tilde{p}_j)$$

$$\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)', \qquad \boldsymbol{\Sigma} = \sigma_{ij}$$

# A Real Portfolio Optimization Problem

**Issues and Objectives**

- Mutual fund regulation: we cannot sell a stock if we do not own it
- Transaction costs
- Turnover
- Liquidity
- Volatility
- **Objective:** Maximize expected wealth next period minus transaction costs

**Decision variables**

$$x_i = \begin{cases} \text{\# shares bought or sold if } i \in S \\ \text{\# shares bought if } i \in B \end{cases}$$

By convention:

$$\begin{array}{ll} x_i \geq 0 & \text{buy} \\ x_i < 0 & \text{sell} \end{array}$$

# A Real Portfolio Optimization Problem

**Transaction costs**

- Small investors only pay commission costs: $a_i$ \$ per share traded.
- Transaction costs: $a_i|x_i|$
- Large investors (like portfolio managers of large funds) may affect price: price become $p_i + b_i x_i$.
- Price impact costs:$(p_i + b_i x_i)x_i - p_i x_i = b_i x_i^2$
- Total transaction costs:

$$c_i(x_i) = a_i|x_i| + b_i x_i^2$$

# A Real Portfolio Optimization Problem

**Liquility**

- Suppose you own 50% of the outstanding stock of a company
- How difficult is it to sell it?
- Reasonable to bound the percentage of ownership on a particular stock
- Thus, for **liquidity** reasons $\frac{z_i + x_i}{z_i^{total}} \leq \gamma_i$
- $z_i^{total} =\#$ outstanding shares of stock $i$
- $\gamma_i$ maximum allowable percentage of ownership

# A Real Portfolio Optimization Problem

**Turnover**

- Because of transaction costs: $|x_i|$ should be small:

$$|x_i| \leq \delta_i.$$

Alternatively,

$$\sum_{i=1}^{N} p_i |x_i| \leq t$$

**Balanced portfolio**

- Need the value of stocks we buy and sell to balance out:

$$\left| \sum_{i=1}^{n} p_i x_i \right| \leq L$$

- No short sales:

$$z_i + x_i \geq 0$$

# A Real Portfolio Optimization Problem

**Expected value and volatality**

- Expected value of portfolio

$$\mathrm{E}\left(\sum_{i=1}^{n} \tilde{p}_i(z_i + x_i)\right) = \sum_{i=1}^{n} \mu_i(z_i + x_i)$$

- Variance of portfolio

$$\mathrm{Var}\left(\sum_{i=1}^{n} \tilde{p}_i(z_i + x_i)\right) = (\boldsymbol{z} + \boldsymbol{x})'\boldsymbol{\Sigma}(\boldsymbol{z} + \boldsymbol{x})$$

# A Real Portfolio Optimization Problem

$$
\begin{array}{lll}
\max & \displaystyle\sum_{i=1}^{n} \left(\mu_i(z_i + x_i) - a_i|x_i| - b_i x_i^2\right) & \\
\text{s.t.} & (\boldsymbol{z} + \boldsymbol{x})' \boldsymbol{\Sigma} (\boldsymbol{z} + \boldsymbol{x}) \leq s & \\
& z_i + x_i \leq \gamma_i z_i^{total} & i = 1, \ldots, n \\
& -\delta_i \leq x_i \leq \delta_i & i = 1, \ldots, n \\
& -L \leq \displaystyle\sum_{i=1}^{n} p_i x_i \leq L & \\
& \displaystyle\sum_{i=1}^{n} p_i |x_i| \leq t & \\
& z_i + x_i \geq 0 & i = 1, \ldots, n
\end{array}
$$

Is this a QOP?