Proof : Rolling Around

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Proposition 1. *Locus of a point on a smaller circle (radius = R) rolling (without slipping) on the inner surface of a larger circle (radius = $2R$) is a straight line through the center of the larger circle.*

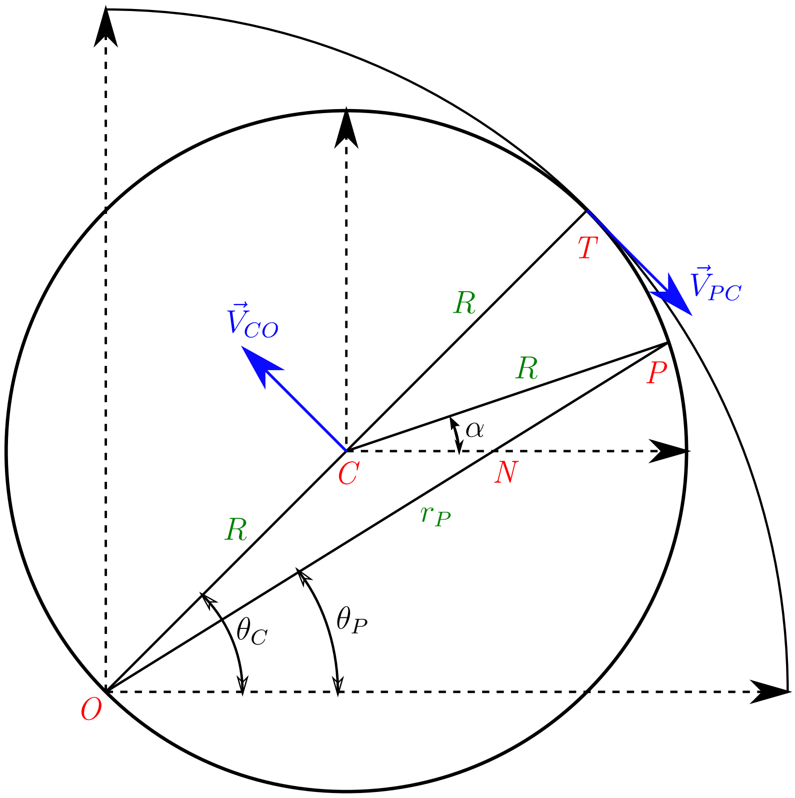

Figure 1: Schematic of motion

Proof. Let the point, center of smaller circle and the center of the larger circle (origin) (say P, C and O respectively) be represented in cylindrical co-ordinates by (r_P, θ_P), (r_C, θ_C) and (r_O, θ_O). The schematic is shown in Figure 1. By definition, $(r_O, \theta_O) = (0, 0)$

1 Locus of P

Let's consider a point P on the smaller circle whose radius makes an angle α with the x-axis. P is represented as (r_P, θ_P) and let us derive the expression for r_P during this rolling motion. From $\triangle OCP$, $\overline{OP}(r_P)$ can be calculated using cosine law.

$$\overline{OP}^2 = \overline{OC}^2 + \overline{CP}^2 - 2\,\overline{OC}\;\overline{CP}\;\cos(\angle OCP) \tag{1}$$

$\angle OCP$ can be calculated as the other two angles in $\triangle OCP$ are equal (isosceles triangle) to $(\theta_C - \theta_P)$.

$$\overline{OC} = R = \overline{PC} \implies \angle CPO = \angle POC = \theta_C - \theta_P$$
$$\therefore \angle OCP = \pi - \angle CPO - \angle POC = \pi - 2\,(\theta_C - \theta_P)$$

r_P can be calculated from Equation 1 as follows :

$$\begin{aligned} {r_P}^2 &= R^2 + R^2 - 2 \times R \times R \times \cos(\pi - 2\,(\theta_C - \theta_P)) \\ &= 2R^2\,\{1 + \cos(2\,(\theta_C - \theta_P))\} = 2R^2\,\left\{2\cos^2(\theta_C - \theta_P)\right\} \\ \Longrightarrow & \boxed{r_P = 2R\cos(\theta_C - \theta_P)} \end{aligned} \tag{2}$$

1.1 Relation between ω_{CO} and ω_{PC}

Consider the point T on the smaller circle in contact with the larger circle. The no-slip condition dictates that there will be no relative motion between the surfaces at the point of contact T.

$$\vec{V}_{TO} = 0 = \vec{V}_{TC} + \vec{V}_{CO} \tag{No-Slip Condition}$$

This condition gives us a relation between the angular velocities $\vec{\omega}_{CO}$ $(\dot{\theta}_C)$ and $\vec{\omega}_{TC}$ $(\dot{\alpha})$:

$$\begin{aligned} \vec{\omega}_{CO} &= \vec{r}_{CO} \times \vec{V}_{CO} \\ \vec{\omega}_{TC} &= \vec{r}_{TC} \times \vec{V}_{TC} = \vec{r}_{TC} \times -\vec{V}_{CO} = -\vec{\omega}_{CO} \\ \vec{\omega}_{TC} &= \boxed{\frac{d\alpha}{dt} = -\frac{d\theta_C}{dt}} = -\vec{\omega}_{CO} \end{aligned} \tag{3}$$

For deriving α, consider the following in $\triangle CNP$,

- $\angle CPN = \angle CPO = \theta_C - \theta_P$
- $\angle CNP$ and θ_P are supplementary angles $\Longrightarrow$ $\angle CNP = \pi - \theta_P$
- Thus, we can find the third angle $\angle PCN$ (α)

$$\angle PCN = \boxed{\alpha = 2\,\theta_P - \theta_C} = \pi - \angle CPN - \angle CNP \tag{4}$$

From Equations 3 and 4 :

$$2\frac{d\theta_P}{dt} - \frac{d\theta_C}{dt} = \frac{d\alpha}{dt} = -\frac{d\theta_C}{dt} \Longrightarrow \boxed{\frac{d\theta_P}{dt} = 0} \tag{5}$$

2 Conclusion

For the trajectory of P i.e., time evolution of position of P, consider a general case where initially $\theta_C(t=0) = \theta_C^0$, $\theta_P(t=0) = \theta_P^0$ and $\dot{\theta}_C = \omega$ (constant)

$$\boxed{\theta_P = \theta_P^0} \tag{6}$$

$$\begin{aligned} \theta_C &= \theta_C^0 + \omega\,t \\ r_P &= 2R\cos(\theta_C - \theta_P) \\ \Longrightarrow & \boxed{r_P = 2R\cos\left(\theta_C^0 + \omega\,t - \theta_P^0\right)} \end{aligned} \tag{7}$$

From Equation 6, we can see that the angle P makes with the x-axis (θ_P) remains constant and from Equation 7 that r_P goes to zero periodically. Therefore, P passes through origin O while maintaining a constant slope $(\tan(\theta_P))$ and thus it *slides on a straight line through the origin* during the rolling motion as proposed originally. □