Probability density functions for hyperbolic and isodiachronic locations

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Animal locations are sometimes estimated with hyperbolic techniques by estimating the difference in distances of their sounds between pairs of receivers. Each pair specifies the animal's location to a hyperboloid because the speed of sound is assumed to be spatially homogeneous. Sufficient numbers of intersecting hyperboloids specify the location. A nonlinear method is developed for computing probability density functions for location. The method incorporates *a priori* probability density functions for the receiver locations, the speed of sound, winds, and the errors in the differences in travel time. The traditional linear approximation method overestimates bounds for probability density functions by one or two orders of magnitude compared with the more accurate nonlinear method. The nonlinear method incorporates a generalization of hyperbolic methods because the average speed of sound is allowed to vary between different receivers and the source. The resulting "isodiachronic" surface is the locus of points on which the difference in travel time is constant. Isodiachronic locations yield correct location errors in situations where hyperbolic methods yield incorrect results, particularly when the speed of propagation varies significantly between a source and different receivers.
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I. INTRODUCTION

Hyperbolic locations are derived by intersecting hyperboloids from estimates of the differences in distances between pairs of receivers. There may not be a published method in the bioacoustics literature for computing probability density functions for hyperbolic locations without making the linear approximation between the data and the locations. Furthermore, hyperbolic location methods for locating calling animals are restricted to the case where the speed of sound is constant throughout space, in which case the difference in distance is estimated from the difference in arrival time of the sound. A nonlinear method is developed here for estimating probability density functions for location without requiring that the speed of sound be constant. Because the average speed is different between the source and each receiver, one no longer has hyperbolic geometries that are second order polynomials in the Cartesian coordinates. Instead, a new geometrical surface, called an "isodiachron," is defined that depends on the Cartesian coordinates through a fourth order polynomial. The word isodiachron is derived from the Greek words "iso," for same, "dia," for difference, and "chron," for time. The surface is one along which the locus of points has the same difference in travel time between two points in space. This reduces to a hyperboloid only if the speed of sound is spatially homogeneous.

The approach is not as complicated nor accurate as can be obtained by doing a joint tomographic inversion for the sound speed and wind fields and for the locations of the imperfectly known locations of the receivers and the source. But the method described here may give an answer with more computational efficiency. The density functions are derived using the nonlinear relationship between the differences in travel time and the unknown quantities such as the sound speed field, wind field, and the locations of the animal and the receivers. In many practical situations, the probability density functions reveal that locations can be made with one or more orders of magnitude more accuracy than those found with the linear approximation.

The problem with linear error analysis[1–7] alone is that source location is not a linear function of the travel time differences, speed of sound, and receiver locations. One expects linear analysis of errors to yield reasonable estimates when the source is near intersecting hyperboloids that are well approximated by planes. But this is often not the case, and the curvature of the hyperboloids near a source may be significant over the region that linear error analysis prescribes. In these cases a complete nonlinear analysis of errors is warranted. As we will see, the nonlinear analysis yields much smaller errors for location than linear analysis in many cases of interest because locations will be confined to be on the curved hyperbolic surfaces.

There appears to be another technical difficulty in estimating location errors even when one accounts for the fact that the linear approximation is invalid. When one has more than the minimum number of receivers required to locate an object, the method for assessing errors does not appear to have been dealt with in the literature in a satisfying method.

Schmidt's interesting paper[8] uses Monte Carlo simulations to estimate errors in source location. Suppose for the moment that a mathematically unambiguous location can be achieved with three receivers for a two-dimensional geometry. Schmidt jiggles ideal travel time differences and receiver coordinates within their expected errors to see how much a location changes with respect to the correct location. Errors are obtained by taking the largest misfit between the jiggled and correct estimate of location. When there are more than three receivers, say $\mathcal{R}$ receivers, he suggests using this procedure for each combination of three receivers giving,

$$\binom{\mathcal{R}}{3}=\frac{\mathcal{R}!}{(\mathcal{R}-3)!3!},$$

total combinations. Each combination is referred to as a "constellation" here. Schmidt suggests using a least squares procedure to find a final estimate of error from the results of the largest misfit from each constellation. He does not prove this least-squares procedure is optimal, and indeeds states that he is not sure of any advantages in using a least-squares procedure in this situation.

Instead of least-squares, a Monte Carlo technique is used to estimate a probability density function of source location from each receiver constellation. For a constellation, the sound speed, winds, and receiver locations are treated as random variables with some probability density function. The probability density function can be obtained from theory, data, or a guess. When guessing, it may be advantageous to let the probability density function have the largest possible bounds with the most ignorance so as to not overestimate the accuracy of a location. For example, suppose one believes the x location of a receiver is at 10 m with an error of two centimeters. Then the associated probability density function for this parameter can be taken to be a uniformly distributed random variable in the interval [9.8, 10.2] m. With the *a priori* probability density function, the various values of the sound speed, winds, and receiver locations yield possible source locations that occupy a cloud in three-dimensional space. The actual source must lie within the intersection of the clouds from different constellations. The final probability function for the source location is estimated from the distributions within the intersection. The density function has the benefit of being able to yield many useful values such as an average location, maximum likelihood location, and any desired confidence limit.

II. LOCATING SOUNDS USING FOUR OR FIVE RECEIVERS WITH INHOMOGENEOUS SOUND SPEED FIELD

Watkins and Schevill's[9] method for locating sounds using four receivers is imitated, but instead of letting the speed of sound be constant, it is allowed to have a different average speed between the source and each receiver. Although five receivers are required in general to locate a source in three dimensions in a homogeneous sound speed field, there are spatial regions where only four are required.[8,10,11] When ambiguous solutions occur with four receivers, the ambiguity is resolved with the travel time difference between the first and fifth receiver.[11] The presentation below thus starts with a minimum constellation of four receivers.

The distance between the ith receiver at $\mathbf{r}_i$ and a source at $\mathbf{s}$ is $\|\mathbf{r}_i-\mathbf{s}\|$ so we have $\|\mathbf{r}_i-\mathbf{s}\|^2=c_i^2t_i^2=c_i^2(\tau_{i1}+t_1)^2$. The average speed and travel time of sound between the source and receiver i are c_i and t_i respectively. $\tau_{i1}\equiv t_i-t_1$ and $\mathbf{s}$ is a column vector with Cartesian coordinates $(s_x,s_y,s_z)^{\mathrm{T}}$ where T denotes transpose. Putting the first receiver at the origin of the coordinate system, one subtracts the equation for $i=1$ from $i=2,3$, and 4 to get,

$$\|\mathbf{r}_i\|^2-2\mathbf{r}_i^T\mathbf{s}=c_i^2\tau_{i1}^2+2c_i^2\tau_{i1}t_1+t_1^2(c_i^2-c_1^2).$$

This simplifies to,

$$\mathbf{R}\mathbf{s}=\tfrac{1}{2}\mathbf{b}-t_1\mathbf{f}-t_1^2\mathbf{g}, \tag{1}$$

where,

$$\mathbf{R}\equiv\begin{pmatrix} r_2(x) & r_2(y) & r_2(z)\\ r_3(x) & r_3(y) & r_3(z)\\ r_4(x) & r_4(y) & r_4(z)\end{pmatrix};$$
$$\mathbf{b}\equiv\begin{pmatrix}\|\mathbf{r}_2\|^2-c_2^2\tau_{21}^2\\ \|\mathbf{r}_3\|^2-c_3^2\tau_{31}^2\\ \|\mathbf{r}_4\|^2-c_4^2\tau_{41}^2\end{pmatrix};\quad \mathbf{f}\equiv\begin{pmatrix}c_2^2\tau_{21}\\ c_3^2\tau_{31}\\ c_4^2\tau_{41}\end{pmatrix}, \tag{2}$$

and,

$$\mathbf{g}\equiv\frac{1}{2}\begin{pmatrix}c_2^2-c_1^2\\ c_3^2-c_1^2\\ c_4^2-c_1^2\end{pmatrix}, \tag{3}$$

and where the Cartesian coordinate of $\mathbf{r}_i$ is $(r_i(x),r_i(y),r_i(z))$. Equation (1) simplifies to

$$\mathbf{s}=\mathbf{R}^{-1}\frac{\mathbf{b}}{2}-\mathbf{R}^{-1}\mathbf{f}t_1-\mathbf{R}^{-1}\mathbf{g}t_1^2, \tag{4}$$

that can be squared to yield,

$$\mathbf{s}^{\mathrm{T}}\mathbf{s}=\|\mathbf{s}\|^2=\frac{a_1}{4}-a_2t_1+(a_3-a_4)t_1^2+2a_5t_1^3+a_6t_1^4, \tag{5}$$

where

$$a_1\equiv(\mathbf{R}^{-1}\mathbf{b})^{\mathrm{T}}(\mathbf{R}^{-1}\mathbf{b}),\quad a_2\equiv(\mathbf{R}^{-1}\mathbf{b})^{\mathrm{T}}(\mathbf{R}^{-1}\mathbf{f}),$$
$$a_3\equiv(\mathbf{R}^{-1}\mathbf{f})^{\mathrm{T}}(\mathbf{R}^{-1}\mathbf{f}),\quad a_4\equiv(\mathbf{R}^{-1}\mathbf{b})^{\mathrm{T}}(\mathbf{R}^{-1}\mathbf{g}), \tag{6}$$
$$a_5\equiv(\mathbf{R}^{-1}\mathbf{f})^{\mathrm{T}}(\mathbf{R}^{-1}\mathbf{g}),\quad a_6\equiv(\mathbf{R}^{-1}\mathbf{g})^{\mathrm{T}}(\mathbf{R}^{-1}\mathbf{g}),$$

and $\mathbf{R}^{-1}$ is the inverse of $\mathbf{R}$. A solution for t_1 is obtained by substituting,

$$\|\mathbf{s}\|^2=c_1^2t_1^2, \tag{7}$$

for $\|\mathbf{s}\|^2$ in Eq. (5) to yield a quartic equation in t_1,

$$a_6t_1^4+2a_5t_1^3+(a_3-a_4-c_1^2)t_1^2-a_2t_1+\frac{a_1}{4}=0, \tag{8}$$

that can be solved analytically, as discovered by Lodovico Ferrari in 1540 (Ref. 12), or numerically with a root finder. Valid roots from Eq. (8) are finally used to estimate the location of the source using Eq. (4). Note that if the sound speed is spatially homogeneous, i.e., $c_i = c_1 \ \forall \ i$, then the cubic and quartic terms vanish and the resulting quadratic equation is that found before for hyperbolic location.[9]

Ambiguous solutions occur for a spatially homogeneous sound speed field when there are two positive roots to the quadratic equation. For each ambiguous source location, one can generate a model for τ_{51} and choose the root for t_1 that yields a model for τ_{51} that is closest to that measured. In the cases investigated in this paper, the quartic equation (8) can yield four distinct positive values for t_1. This can only happen because the speed of propagation is spatially inhomogeneous. Because the values of c_i are similar in this paper, the four distinct roots yield two pairs of source locations, one pair of which is relatively close to the receivers and the other pair of which is located very far from the receivers. The distant pair is due to the fact that locations are not exactly determined by intersecting hyperboloids for a spatially inhomogeneous speed of propagation. The actual three-dimensional locus of points specified by a travel time difference is thus not quite a hyperboloid. More precisely, the hyperboloid is the locus of points $\mathbf{s}$ satisfying,

$$\|\mathbf{r}_i - \mathbf{s}\| - \|\mathbf{r}_j - \mathbf{s}\| = c\,\tau_{ij}, \tag{9}$$

where the spatially homogeneous speed of propagation is c. Letting the speed be different along each section yields the definition of the isodiachron which is the locus of points satisfying,

$$\frac{\|\mathbf{r}_i - \mathbf{s}\|}{c_i} - \frac{\|\mathbf{r}_j - \mathbf{s}\|}{c_j} = \tau_{ij}, \tag{10}$$

which turns out to depend on the Cartesian coordinates through a fourth order polynomial. The values of c_i can incorporate spatially inhomogeneous effects such as winds as well as wave speeds. The coefficients of the third and fourth powers in x, y, and z and their combinations become very small compared to the coefficients in the second, first, and zero powers as the various c_i approach the same value c (not shown).

III. PROBABILITY DENSITY FUNCTIONS FOR SOURCE LOCATION FROM RECEIVER CONSTELLATIONS

When $\mathbf{r}_i$, τ_{ij}, and c_i are random variables, then t_1 and $\mathbf{s}$ are random variables because of Eqs. (4), (8). For later convenience in comparing to linear theories, consider the situation where $(r_i(x), r_i(y), r_i(z))$, τ_{ij}, and c_i are mutually uncorrelated Gaussian random variables with given means and variances.

A computer generates a single random configuration of variables for a constellation of four receivers. Each configuration consists of the set $\{r_i(x), r_i(y), r_i(z), \tau_{ij}, c_i\}$ for some $i, j \in 1,2,3,\cdots,\mathcal{R}$ and $i > j$. Then for each constellation, of which there are a total of,

$$N \equiv \binom{\mathcal{R}}{4} = \frac{\mathcal{R}!}{(\mathcal{R}-4)!\,4!}, \tag{11}$$

a source location is computed from Eq. (4) if that equation yields a unique location. If any constellation yields more than one location from this equation, a location is chosen to be that yielding the closest difference in travel time to a randomly chosen fifth receiver and receiver number one.

A valid configuration from a receiver constellation is one in which the source location lies within some predetermined spatial limits. For example, one would know that sounds from snapping shrimp occur below the surface of the water. If a receiver constellation yields a location above the surface, then that particular configuration of random variables could not have occurred in reality, and that source location is discarded.

Valid configurations from a receiver constellation define a cloud of source locations. Accurate probability density functions of location require a sufficient number of valid configurations. A sufficient number is generated for each of the N constellations.

Some constellations give better locations of the source than others. This happens for several reasons, and is usually due to the geometrical arrangement of the receivers. For example, suppose the source is near the geometric center of a pyramid and one constellation consists of four receivers on the vertices of the pyramid. That constellation would be able to locate the source rather well. Then consider another constellation of four receivers located close to a line at a great distance from the source. This constellation would not be able to locate the source as well.

The only physically possible locations for the source lie within the intersections of the clouds. The other source locations are invalid. In this paper, the upper and lower bounds of the intersected region, $(\hat{X}, \check{X})$, $(\hat{Y}, \check{Y})$, and $(\hat{Z}, \check{Z})$, are estimated along the Cartesian axes. If $(\hat{x}_c, \check{x}_c)$, $(\hat{y}_c, \check{y}_c)$, and $(\hat{z}_c, \check{z}_c)$ denote the maximum and minimum values of x, y, and z for cloud c, then the region of intersection is,

$$\begin{aligned}
\hat{X} &= \min(\hat{x}_1, \hat{x}_2, \hat{x}_3, \cdots \hat{x}_N)\\
\check{X} &= \max(\check{x}_1, \check{x}_2, \check{x}_3, \cdots \check{x}_N)\\
\hat{Y} &= \min(\hat{y}_1, \hat{y}_2, \hat{y}_3, \cdots \hat{y}_N)\\
\check{Y} &= \max(\check{y}_1, \check{y}_2, \check{y}_3, \cdots \check{y}_N)\\
\hat{Z} &= \min(\hat{z}_1, \hat{z}_2, \hat{z}_3, \cdots \hat{z}_N)\\
\check{Z} &= \max(\check{z}_1, \check{z}_2, \check{z}_3, \cdots \check{z}_N).
\end{aligned} \tag{12}$$

For each cloud c, locations outside of the bounds in Eqs. (12) are discarded. New probability density functions are formed from the remaining locations in each cloud for the x, y, and z values separately. It is not expected that the new probability density functions would asymptotically approach one another because each constellation will be able to locate the source with a different quality.

The probability density functions for the source in its x, y, and z coordinates are dependent distributions. This can be accommodated by forming the joint probability density function of location for each cloud if desired. This is not done here.

P percent confidence limits are estimated by finding the P percent confidence limits for each cloud separately using its probability density functions in x, y, and z. The Cartesian

TABLE I. The Cartesian coordinates of arrays one and two (Fig. 1) and their standard deviations. Receiver one's location is defined to be the origin of the Coordinate system, and so has zero error. The y coordinate of receiver two is defined to be at $y=0$, and thus has zero error.

	x (m)	y (m)	z (m)
		Array 1	
R1	0 ± 0	0 ± 0	0 ± 0
R2	1000 ± 2	0 ± 0	0 ± 1
R3	1000 ± 2	1000 ± 2	0 ± 1
R4	0 ± 2	1000 ± 2	0 ± 1
R5	0 ± 5	0 ± 5	-100 ± 5
SOURCE	551	451	-100
		Array 2	
R1	0 ± 0	0 ± 0	0 ± 0
R2	1414 ± 2	0 ± 0	0 ± 1
R3	534 ± 2	400 ± 2	0 ± 1
R4	1459 ± 5	-1052 ± 5	-25 ± 5
R5	1459 ± 20	-1052 ± 20	-95 ∓ 20
R6	0 ± 20	0 ± 20	-100 ± 20
R7	1414 ± 20	0 ± 20	-100 ± 20
SOURCE	860	47	-5

upper and lower bounds for a specified confidence limit for constellation c are denoted $[\hat{P}_c(x),\check{P}_c(x)]$, $[\hat{P}_c(y),\check{P}_c(y)]$, and $[\hat{P}_c(z),\check{P}_c(z)]$. The final bounds for the source are chosen from the smallest bound for x, y, and z from each constellation. For example, suppose constellation p has the smallest value of $\hat{P}(x)-\check{P}(x)$ for all values of c, constellation q has the smallest value of $\hat{P}(y)-\check{P}(y)$ for all values of c, and constellation r has the smallest value of $\hat{P}(z)-\check{P}(z)$ for all values of c. Then the final confidence limits for the source are $[\hat{P}_p(x),\check{P}_p(x)]$, $[\hat{P}_q(y),\check{P}_q(y)]$, and $[\hat{P}_r(z),\check{P}_r(z)]$.

IV. EXAMPLES

Examples below utilize 2000 valid configurations of random variables to estimate probability density functions for each cloud.

A. Hyperbolic location

Because error bars derived from linearized hyperbolic location techniques[1–7] assume the speed of sound or light is spatially homogeneous, comparison with a linear theory is done using a spatially homogeneous value of 1475 m/s. The travel time differences are computed for this speed. In the error analysis, the sound speed is assumed to be a Gaussian random variable with mean 1475 m/s and standard deviation of 10 m/s. This standard deviation is realistic if one considers paths emitted from a shallow source to receivers at perhaps 3 m and 100 m depth because the surface region can be very warm compared with temperatures below. Travel time differences are assumed to be mutually uncorrelated Gaussian random variables with means given by true values and standard deviations of 0.000 141 4 s. Receiver locations are assumed to be mutually uncorrelated Gaussian random variables with means given by their true values and standard deviations as shown in Table I.

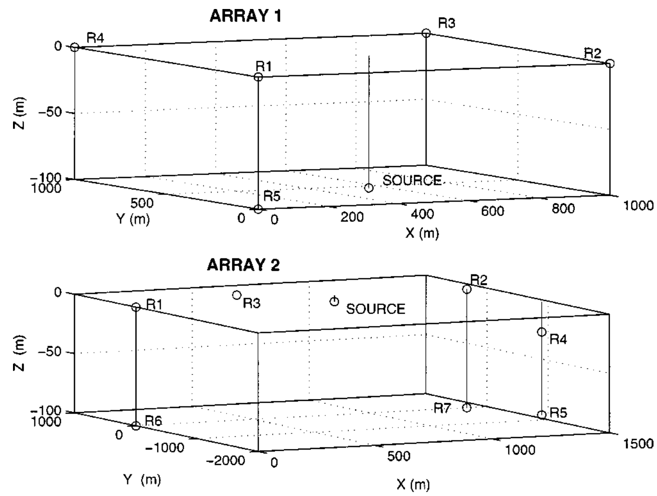

FIG. 1. The receiver and source locations for arrays one and two.

The errors in the travel time differences and receiver locations may not be mutually uncorrelated as assumed above. For example, errors in τ_{i1} may be correlated with τ_{j1} because both involve t_1. Also, for example, the errors in the location of receiver 3 are not necessarily uncorrelated between two constellations that both contain receiver 3. Linear error analysis can accommodate correlations between random variables,[7] as can the nonlinear analysis in this paper. However, incorporating these correlations leads to larger computational times in the nonlinear analysis, so they are not implemented. Error models that follow are done assuming that random variables are mutually uncorrelated for both the linear and nonlinear analysis. We found that the results were identical for the linear analysis[7] when the random variables were correlated.

To mimic marine examples, array one has five receivers separated by O(1000) m horizontally and up to 100 m vertically (Fig. 1, Table I). The five receiver constellations [Eq. (11)], one through five, are {1,2,3,4}, {1,2,3,5}, {1,2,4,5}, {1,3,4,5}, and {2,3,4,5}, respectively. The 100% confidence limits for the source are computed from Eq. (12) for increasing numbers of constellations where N is set to 1, 2, 3, 4, and 5 respectively in this equation for Fig. 2. As more clouds are intersected, the limits for the source decrease monotonically, with the biggest improvement occurring with the addition of constellation two with one. Constellation one only uses the receivers at z equal zero. Constellation two is the first one that includes the receiver at $z=-100$ m. This deeper receiver not only helps in locating the source's vertical coordinate, but significantly helps locate the horizontal coordinates as well. The probability density functions for the source location come from constellations two and four (Fig. 3). These density functions appear to be approximately Gaussian. The 68% confidence limits span only a few meters in the horizontal coordinates and are about 50 m in the vertical coordinate (Table II). The confidence limits from the nonlinear analysis are about a factor of ten less than those from the standard linear analysis of errors in the horizontal coordinates. The nonlinear analysis has smaller limits because the hyperboloids are not well approximated by planes in the horizontal directions as required by the linear analysis. Non-

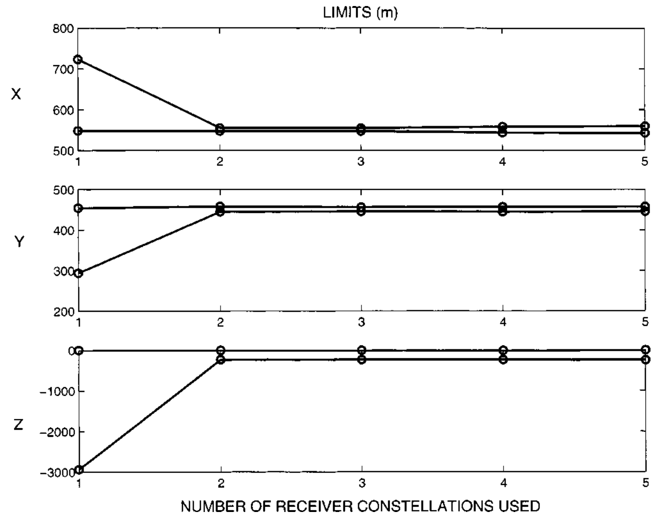

FIG. 2. 100% confidence limits for source location as a function of the number of receiver constellations used from array one (Fig. 1). There are five ways of choosing four receivers from five total without replacement. A receiver constellation consists of one of the choices of four receivers. As more constellations are used to locate the source, the bounds for the source's location decrease monotonically. The lines join results from different numbers of constellations.

linear analysis yields somewhat larger limits in z than linear analysis (Table II). Similarity of results in z indicates that the hyperboloids are fairly well approximated as planes in the vertical coordinate in the vicinity of the source.

Array two has seven receivers with the largest horizontal and vertical separations being about 1400 and 100 m, respectively (Fig. 1). There are 35 receiver constellations [Eq. (11)] of which the first four provide most of the accuracy for locating the source at the 100% confidence limits (Fig. 4). These constellations are the first of the 35 that include all the deeper receivers. There are modest increases in accuracy from other constellations, most notably 29 and 33. The probability density functions in the x-y-z coordinates come from constellations 7, 4, and 26, respectively (Fig. 5). The distributions in x and y do not look very Gaussian, while the

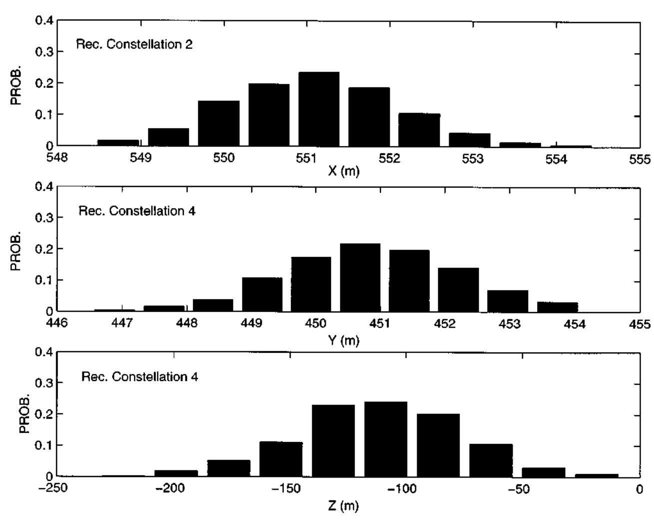

FIG. 3. Probability density functions for source location from array one (Fig. 1, top), calculated from the nonlinear method in this paper. Receiver constellation 2 is receivers 1, 2, 3, and 5 (Fig. 1, top). Receiver constellation 4 is receivers 1, 3, 4, and 5.

TABLE II. 68% confidence limits for source location corresponding to arrays one and two for the nonlinear and linear analyses.

Cartesian coordinate	68% Confidence limits (m)	
	Nonlinear	Linear
	Array 1	
x	550 to 552	530 to 573
y	450 to 452	432 to 471
z	−145 to −77	−121 to −80
	Array 2	
x	860 to 865	236 to 1486
y	46 to 62	−1890 to 1988
z	−138 to 65	−163 to 153

distribution in z looks more Gaussian-like. These departures from Gaussian distributions are quite different than the Gaussian distributions usually assumed from linear analyses. This time, the 68% confidence limits from the nonlinear analyses are two orders of magnitude smaller than those from standard linear analysis in x and y (Table II). The linear and nonlinear confidence limits are similar for the vertical coordinate.

B. Isodiachronic location

It appears there are two extreme situations in which isodiachronic locations are useful.

The first is one where the speed of sound is similar, but not exactly the same between each source and receiver. Consider an atmospheric example for locating a sound at Cartesian coordinate (20,100,7) m from five receivers at (0,0,0), (25,0,3), (50,3,5), (30,40,9), and (1,30,6) m, respectively. The speed of sound is a typical 330 m/s. The speed of propagation is made to be inhomogeneous by introducing a wind of 10 m/s in the positive y direction. Next, simulated values of the travel time differences are computed using these values. The source is located using hyperbolic and isodiachronic location. It will be seen that only isodiachronic location yields a correct solution.

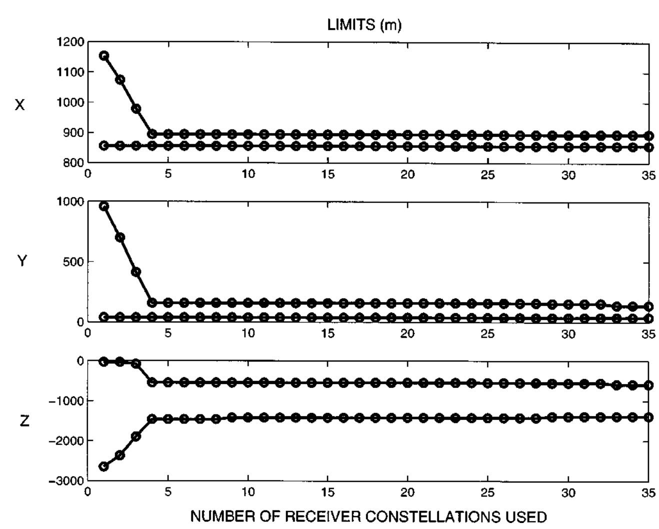

FIG. 4. Same as Fig. 2 except for array two in Fig. 1 and there are 35 ways of choosing 4 receivers from a total of 7 without replacement.

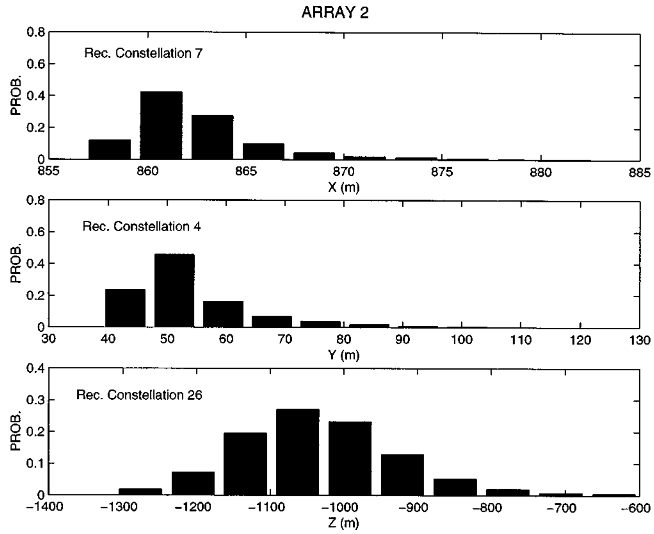

FIG. 5. Probability density functions for source location from array two (Fig. 1, bottom) using the nonlinear method of this paper. Receiver constellations 7, 4, and 26 are composed from receivers {1,2,4,7}, {1,2,3,7}, and {2,3,6,7}, respectively.

All Gaussian random variables in this simulation are truncated to have a maximum of two standard deviations for the following two reasons. First, many experimental situations are inaccurately represented by assuming that random variations differ from an estimate by say ten standard deviations. Instead, it is more realistic to truncate the variations. Second, it is important to note that a realistic truncation is easy to impose with the models developed here but is difficult to implement with analytical and linear approximations for error.

The standard deviation for receiver locations is 0.02 m. The variations are zero for the x, y, and z coordinates of receiver 1, the y and z coordinates of receiver 2, and the z coordinate of receiver 3. The coordinates with zero variations merely define the origin and orientation of the coordinate system. The x and z components of the winds are modeled to have a value of 0 m/s. For hyperbolic location, the speed of acoustic propagation must be spatially homogeneous. The mean and standard deviation for sound speed are 330 m/s and 10 m/s, respectively. For isodiachronic location, the speed of propagation is inhomogeneous. The speed of sound is taken to be 330 m/s. The *a priori* value of the wind in the y direction has mean 0 and standard deviation 10 m/s. Values of the speed of acoustic propagation, c_i, between receiver i and the source are unknown because the location of the source is initially unknown. Therefore, it is impossible to precompute the component of the wind vector along the direction from the source to each receiver. Instead, the value for each c_i is computed using a direction chosen at random through the simulated field of sound speed and wind. The error in travel time due to the straight path approximation is typically less than a microsecond at these ranges.[6] The travel time differences are derived with ideal values for means and from a standard deviation of 16 μs. The 16 μs value is derived from Eq. 41 in Ref. 6 using an rms bandwidth of 1000 Hz and a peak signal-to-noise ratio of 20 dB in the cross-correlation function of the signals between receivers.

Incorrect locations are obtained using the hyperbolic method. For example, the source's 100% confidence limits for x are 19.02 to 19.05 m, but its actual x location is 20 m. Similarly, the 100% confidence limits for y are 103 to 105 m, but the actual value is 100 m. So given *a priori* variations of receiver locations, travel time differences, and environmental variations, the hyperbolic method always yields incorrect answers.

With isodiachronic location, 95% confidence limits for the source are x: 19.5 to 23.7 m, y: 92.6 to 106 m, z: -26 to 13 m. These are correct. Other confidence limits could be given but they are not shown because the point is that isodiachronic location yields a correct answer at a stringent confidence of 95%.

The second case where isodiachronic location would be useful is one where the speed of sound is quite different between the source and each receiver. In this case, hyperbolic locations would be inappropriate to use because the speed of sound is not nearly constant in space. For example suppose low frequency sources such as Finback whales are located. Suppose some receivers close to the source pick up only the first acoustic path through the sea, while other distant receivers pick up only the acoustic path that propagates below the sea-floor[13] because the paths through the water are blocked by seamounts. The speed of propagation along the water and solid-Earth paths can differ by more than a factor of 2.[13] In other scientific fields, sounds can propagate to receivers along paths with different speeds of sound, such as from vehicles where paths propagate through the air and ground.

V. CONCLUSION

A method is developed for computing probability density functions for hyperbolic locations without relying on any linear approximation between travel time differences at pairs of receivers and the location of the source. In cases of practical interest, the confidence limits for location can be one or two orders of magnitude smaller with the nonlinear analysis than the linear one. The method for computing probability density functions includes *a priori* information about the probability density functions of the receiver locations, the speed of sound, and the errors in the differences in travel time.

It appears to be useful to relax the traditional assumption that the speed of acoustic propagation be spatially homogeneous for hyperbolic locations. Instead, one can allow the average speed to be different between each receiver and the source. This leads to a new geometrical surface, called an isodiachron, that approaches a hyperboloid when the speed of propagation is spatially homogeneous.

The ideas in this paper need to be tried with data.

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