Assignment 1 - Position Vectors

Saturday, February 3, 2024 3:12 PM

Notation: we let $\overrightarrow{m_i m_j}$ denote the vector starting from joint i & ending at joint j.

Fact: since links are assumed to be point masses with massless rods, we have that each r_i is just the vector from $\mathcal{I}$ to joint i.

Compute position vectors $r_i = r_i(q, x) = r_i(\bar{q})$

① Find r_1

$$r_1 = x + \overrightarrow{m_1 m_1} = x + 0 \qquad \therefore r_1 = x$$

② Find r_2

$$\overrightarrow{m_1 m_2} = \ell_1 \begin{bmatrix} \cos(q_1) \\ \sin(q_1) \end{bmatrix} \implies r_2 = x + \overrightarrow{m_1 m_2} = x + \ell_1 \begin{bmatrix} \cos(q_1) \\ \sin(q_1) \end{bmatrix}$$

③ Find r_3

$$\overrightarrow{m_2 m_3} = \ell_2 \begin{bmatrix} \cos(q_1 + q_2) \\ \sin(q_1 + q_2) \end{bmatrix}$$

$$\therefore r_3 = x + \overrightarrow{m_1 m_2} + \overrightarrow{m_2 m_3} = x + \ell_1 \begin{bmatrix} \cos(q_1) \\ \sin(q_1) \end{bmatrix} + \ell_2 \begin{bmatrix} \cos(q_1 + q_2) \\ \sin(q_1 + q_2) \end{bmatrix}$$

④ Find r_4

$$\overrightarrow{m_3 m_4} = \ell_2 \begin{bmatrix} \cos(q_1 + q_2 + q_3) \\ \sin(q_1 + q_2 + q_3) \end{bmatrix}$$

$$\therefore r_4 = x + \overrightarrow{m_1 m_2} + \overrightarrow{m_2 m_3} + \overrightarrow{m_3 m_4} = x + \ell_1 \begin{bmatrix} \cos(q_1) \\ \sin(q_1) \end{bmatrix} + \ell_2 \begin{bmatrix} \cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3) \\ \sin(q_1 + q_2) + \sin(q_1 + q_2 + q_3) \end{bmatrix}$$

⑤ Find r_5

$$\overrightarrow{m_4 m_5} = \ell_1 \begin{bmatrix} \cos(q_1 + q_2 + q_3 + q_4) \\ \sin(q_1 + q_2 + q_3 + q_4) \end{bmatrix}$$

$$\therefore r_5 = x + \overrightarrow{m_1 m_2} + \overrightarrow{m_2 m_3} + \overrightarrow{m_3 m_4} + \overrightarrow{m_4 m_5}$$

$$= x + \ell_1 \begin{bmatrix} \cos(q_1) + \cos(q_1 + q_2 + q_3 + q_4) \\ \sin(q_1) + \sin(q_1 + q_2 + q_3 + q_4) \end{bmatrix} + \ell_2 \begin{bmatrix} \cos(q_1 + q_2) + \cos(q_1 + q_2 + q_3) \\ \sin(q_1 + q_2) + \sin(q_1 + q_2 + q_3) \end{bmatrix}$$

⑥ Find r_6

$$\overrightarrow{m_3 m_6} = \ell_3 \begin{bmatrix} \cos(q_1 + q_2 + q_5) \\ \sin(q_1 + q_2 + q_5) \end{bmatrix}$$

$$\therefore r_6 = x + \overrightarrow{m_1 m_2} + \overrightarrow{m_2 m_3} + \overrightarrow{m_3 m_6}$$

$$= x + \ell_1 \begin{bmatrix} \cos(q_1) \\ \sin(q_1) \end{bmatrix} + \ell_2 \begin{bmatrix} \cos(q_1 + q_2) \\ \sin(q_1 + q_2) \end{bmatrix} + \ell_3 \begin{bmatrix} \cos(q_1 + q_2 + q_5) \\ \sin(q_1 + q_2 + q_5) \end{bmatrix}$$

Assignment 1 - Potential Function

Saturday, February 3, 2024 3:53 PM

Gravitational potential energy for each link

We know that $(r_i)_2 = [0\ 1] r_i$ is the height of link i from the ground (assumed flat).

$\therefore P_i = m_i g (r_i)_2 = m_i g [0\ 1](x + \overrightarrow{m_1 m_i})$

Using the fact that $[0\ 1] x = x_2 = 0$ because the ground is flat & our inertial frame $\mathcal{I}$ is fixed to the ground, we have:

$P_i(q) = m_i g [0\ 1] \overrightarrow{m_1 m_i}$, noting that $\overrightarrow{m_1 m_i}$ depends only on q.

Compute the potential function (of the robotic system)

$$P(q) = \sum_{i=1}^{6} P_i(q) = \sum_{i=1}^{6} m_i g [0\ 1] r_i(q,x)$$

$$= g [0\ 1] \left(\sum_{i=1}^{6} m_i r_i(q,x) \right)$$

$m_1 = m_5$
$m_2 = m_4$

$$= g [0\ 1] \Big(m_1 (r_1(q,x) + r_5(q,x)) + m_2 (r_2(q,x) + r_4(q,x)) + m_3 r_3(q,x) + m_6 r_6(q,x) \Big)$$

where the values for $r_i(q,x)$ are provided earlier & not rewritten for brevity of exposition.

Recall: $P(q)$ does not depend on x due to the multiplication with $[0\ 1]$.

Assignment 1 - Impact Map

Saturday, February 3, 2024 11:04 AM

Effect of impulsive forces $\Delta_1(\cdot)$

From the "data" structure we need:

- $\bar{D}(\cdot)$ ~ " data. Dbar(·)"
- $E(\cdot)$ ~ " data. E(·)"

Then we construct the matrix $\Delta_{\dot q}(q)$: (given to us)

$$\Delta_{\dot q}(q) = \begin{bmatrix} I_{5\times5} & O_{5\times4} \end{bmatrix} \begin{bmatrix} \bar{D}(q) & -E^T(q) \\ E(q) & O_{2\times2} \end{bmatrix}^{-1} \begin{bmatrix} \bar{D}(q) \begin{bmatrix} I_{5\times5} \\ O_{2\times5} \end{bmatrix} \\ O_{2\times5} \end{bmatrix}$$

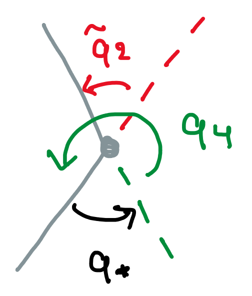
note the inverse

$\Delta_1(\cdot)$ can be written using matrix multiplication:

$$\Delta_1\left(\begin{bmatrix} q \\ \dot q \end{bmatrix}\right) = \begin{bmatrix} I_{5\times5} & O_{5\times5} \\ O_{5\times5} & \Delta_{\dot q}(q) \end{bmatrix} \begin{bmatrix} q \\ \dot q \end{bmatrix} = \begin{bmatrix} q \\ (\Delta_{\dot q}(q))\,\dot q \end{bmatrix}$$

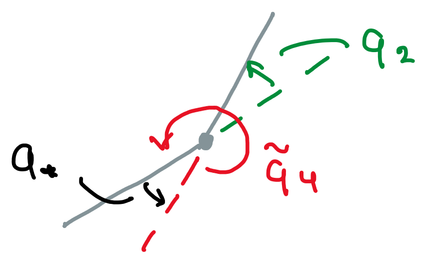
Relabelling map $T(q) = Rq + d$

① Find $\tilde q_2$



- $|\tilde q_2| = |q_*|$ because opposite angles are equal at an intersection of two lines
  ↳ in fact, $\tilde q_2 = q_*$ because they have the same orientation.
- $2\pi = q_4 + q_* \Rightarrow \tilde q_2 = q_* = 2\pi - q_4$.

② Find $\tilde q_4$

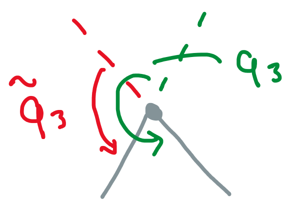


By the same reasoning as above, we conclude

$q_2 = q_* = 2\pi - \tilde q_4$

Rearranging, we have: $\tilde q_4 = 2\pi - q_2$

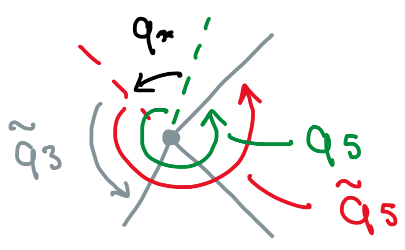
③ Find $\tilde q_3$



Similarly: $2\pi = q_3 + \tilde q_3$

$\Rightarrow \tilde q_3 = 2\pi - q_3$

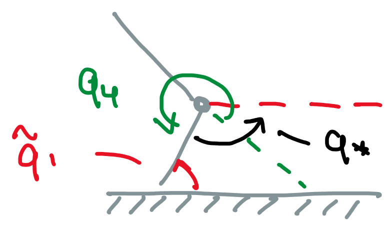
④ Find $\tilde q_5$



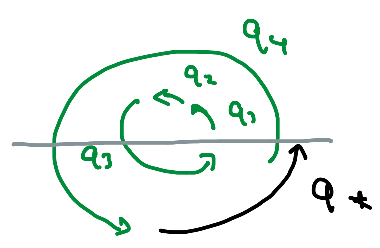
- By construction, $q_* + \tilde q_5 = q_5 \Rightarrow \tilde q_5 = q_5 - q_*$
- By supplementary angles, $\pi = q_* + \tilde q_3 = q_* + 2\pi - q_3$
  $\Rightarrow q_* = \pi - 2\pi + q_3 = q_3 - \pi$

$\therefore \tilde q_5 = \pi + q_5 - q_3$

⑤ Find $\tilde q_1$



- By following the arcs traced out by:
  $q_1 \to q_2 \to q_3 \to q_4 \to q_*$
  we notice that 2 circles are drawn.
  the sum of q_1 to q_* forms 2 loops: That is,
  $2\cdot 2\pi = q_1 + q_2 + q_3 + q_4 + q_*$



- Next, we notice that $\tilde q_1$ & q_* are supplementary angles. Hence
  $\pi = \tilde q_1 + q_* = \tilde q_1 + 4\pi - q_1 - q_2 - q_3 - q_4$

$\therefore \tilde q_1 = q_1 + q_2 + q_3 + q_4 - 3\pi$

Stacking all quantities into a matrix:

$$\tilde q = \begin{bmatrix} \tilde q_1 \\ \tilde q_2 \\ \tilde q_3 \\ \tilde q_4 \\ \tilde q_5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{bmatrix} + \begin{bmatrix} -3\pi \\ 2\pi \\ 2\pi \\ 2\pi \\ \pi \end{bmatrix} = Rq + d = T(q)$$

Effect of relabelling $\Delta_2(\cdot)$

Create the matrix R & vector d as presented above.

The we construct $\Delta_2(\cdot)$:

$$\Delta_2\left(\begin{bmatrix} q \\ \dot q \end{bmatrix}\right) = \begin{bmatrix} Rq + d \\ R\dot q \end{bmatrix} = \begin{bmatrix} R & O_{5\times5} \\ O_{5\times5} & R \end{bmatrix} \begin{bmatrix} q \\ \dot q \end{bmatrix} + \begin{bmatrix} d \\ O_{5\times1} \end{bmatrix}$$

Impact Map

From the "data" structure we need

- $\bar{D}(\cdot)$ ~ " data. Dbar(·)"
- $E(\cdot)$ ~ " data. E(·)"

Then the impact map is computed as (using our quantities above)

$$\Delta\left(\begin{bmatrix} q \\ \dot q \end{bmatrix}\right) = (\Delta_2 \circ \Delta_1)\left(\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} R & O_{5\times5} \\ O_{5\times5} & R \end{bmatrix} \begin{bmatrix} I_{5\times5} & O_{5\times5} \\ O_{5\times5} & \Delta_{\dot q}(q) \end{bmatrix} \begin{bmatrix} q \\ \dot q \end{bmatrix} + \begin{bmatrix} d \\ O_{5\times1} \end{bmatrix}$$

$$= \begin{bmatrix} R & O_{5\times5} \\ O_{5\times5} & R\Delta_{\dot q}(q) \end{bmatrix} \begin{bmatrix} q \\ \dot q \end{bmatrix} + \begin{bmatrix} d \\ O_{5\times1} \end{bmatrix}$$