PH558/964 Advanced Quantum Theory: Assignment 1

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The first assignment is due Noon, Monday 8th February 2021 (**PDF** via MyPlace). Marks awarded for working, explanations, clarity and conciseness. Solutions should be your own work.

I. STATES AND DENSITY OPERATORS

A. 10 marks

Let $\hat{A}$ be a linear operator in $L(\mathcal{H}^3)$, where $\mathcal{H}^3$ is a 3-dimensional Hilbert space. The non-degenerate eigenstates of $\hat{A}$ are denoted $\{|\phi_n\rangle\}_{n=1}^{3}$ with corresponding eigenvalues n. Let

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle \tag{1}$$

be an un-normalised superposition of the eigenvalues of $\hat{A}$. a) Show that the normalise state is $|\psi'\rangle = \frac{\sqrt{5}}{2}|\psi\rangle$ and, b) calculate the expectation value of $\hat{A}$ for $|\psi'\rangle$.

B. 10 marks

Consider normalised states $|\xi_1\rangle$ and $|\xi_2\rangle$ that are **not** orthogonal, $\langle\xi_1|\xi_2\rangle = c \neq 0$ ($c \in \mathbb{C}$). a) Write down a normalised state that is an equal superposition of the two states. b) Write down a density operator representing the equal statistical mixture of $|\xi_1\rangle$ and $|\xi_2\rangle$ (in terms of $|\xi_1\rangle$ and $|\xi_2\rangle$).

C. 10 marks

We prepare a 2-dimensional quantum system at random in either one of the following three states: $|\psi_1\rangle = |0\rangle$, $|\psi_2\rangle = |1\rangle$, and $|\psi_3\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$ with probabilities $p_1 = 1/8$, $p_2 = 1/8$, and $p_3 = 3/4$ respectively, without knowing which one of the states is prepared. Show that the system density operator in the basis $\{|0\rangle, |1\rangle\}$ is,

$$\hat{\rho} = \frac{1}{8}\begin{pmatrix} 4 & -3i \\ +3i & 4 \end{pmatrix}. \tag{2}$$

D. 10 marks

If Alice performs on $\hat{\rho}$ in Eq. 2 a projective measurement corresponding to observable $\hat{O} = |s\rangle\langle s|$, $|s\rangle = (|1\rangle - i|0\rangle)/\sqrt{2}$, a) what are the possible measurement outcomes, b) what are their associated probabilities.

II. DEGENERATE PROJECTIONS

A three-level system is prepared in the normalised state $|\varphi\rangle = \sqrt{\frac{1}{10}}|0\rangle + \sqrt{\frac{1}{5}}|1\rangle + \sqrt{\frac{7}{10}}|2\rangle$. Let $\hat{D} = |0\rangle\langle 0| - |1\rangle\langle 1| + |2\rangle\langle 2|$ be an observable with eigenvalues ± 1. An ideal (incomplete) projective measurement is made corresponding to $\hat{D}$ resulting in one of the eigenvalues being obtained.

A. 6 marks

a) What are the probabilities of obtaining the eigenvalues $+1$ or -1 and b) what is the expectation value $\langle\hat{D}\rangle$ in $|\varphi\rangle$?

B. 4 marks

Directly (by calculation) show that the projector $\hat{P}_{+1}$ on the $+1$ eigenspace of $\hat{D}$ satisfies $(\hat{P}_{+1})^2 = \hat{P}_{+1}$. The eigenspace is the sub-space spanned by the eigenvectors corresponding to the eigenvalue.

C. 5 marks

In the case of obtaining the result $+1$, what is the conditional (normalised) state of the system post-measurement?

III. TENSOR PRODUCTS

A. 20 marks

Let atoms A and B be *non-interacting* and have local Hamiltonians $\hat{H}_A$ and $\hat{H}_B$ respectively, with $\{E_j^A, |E_j^A\rangle\}_{j=0}^{d_A-1}$ and $\{E_k^B, |E_k^B\rangle\}_{k=0}^{d_B-1}$ their respective non-degenerate eigenenergies and eigenstates (they may be different types of atoms).

For the combined (non-interacting) system of A and B, a) write down the combined Hamiltonian $\hat{H}_{AB}$ that is the observable for the total energy of the two atoms, b) work out the joint eigenenergies and eigenstates, and c) show that these obey the Time Independent Schrödinger Equation.

B. 10 marks

Consider the state of two qubits (2-level quantum systems) $|\Psi\rangle_{AB} = \frac{1}{\sqrt{30}}(|00\rangle + 2|01\rangle + 3i|10\rangle - 4|11\rangle)_{AB}$. Bob measures his qubit B in the computational basis $\{|0\rangle_B, |1\rangle_B\}$. a) What is the probability of him getting the result $|1\rangle_B$. b) What is the conditional (normalised) state of qubit A in this case? I.e. If Bob tells Alice that he obtained the result $|1\rangle_B$, then what state does Alice now know she has.

IV. POVMS

Consider a two-outcome non-projective measurement on a 2-level system that is specified by the detection operators (effects)

$$\hat{M}_0 = \begin{pmatrix} i\sqrt{q} & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}, \qquad \hat{M}_1 = \begin{pmatrix} \sqrt{1-q} & 0 \\ 0 & \sqrt{q} \end{pmatrix}, \quad (3)$$

when expressed in the computational basis $\{|0\rangle, |1\rangle\}$, where $\frac{1}{2} < q < 1$ represents the fidelity of the measurement.

A. 5 marks

a) What are the corresponding measurement operators $\{\hat{\Pi}_0, \hat{\Pi}_1\}$? b) Show that they form an allowed POVM.

B. 10 marks

The above POVM is performed on a pure state $|\nu\rangle = \cos\theta|0\rangle + \sin\theta e^{i\phi}|1\rangle$. a) If $\hat{\Pi}_0$ is the result, what is the conditional state? b) What is the unconditional state after the measurement (if you did not know the result)? Express your answers in the computational basis.

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PH 964 : Theoretical Quantum Information
ASSIGNMENT 1.

1. STATES AND DENSITY OPERATORS

A. Let $\hat{A}$ be a linear operator in $L(\mathcal{H}^3)$, where $\mathcal{H}^3$ is a 3-dimensional Hilbert space. The non-degenerate eigenstates of $\hat{A}$ are denoted $\{|\phi_n\rangle\}_{n=1}^{3}$ with corresponding eigenvalues n.
Let
$$|\psi\rangle = \frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle$$

be an un-normalized superposition of the eigenvalues of $\hat{A}$.

a) Show that the normalise state is $|\psi'\rangle = \frac{\sqrt{5}}{2}|\psi\rangle$

b) Calculate the expectation value of $\hat{A}$ for $|\psi'\rangle$

Solution a) We know for normalization
$$\langle\psi|\psi\rangle = 1.$$

Since $|\phi_1\rangle$ and $|\phi_2\rangle$ and $|\phi_3\rangle$ are orthogonal to each other
$$\langle\phi_1|\phi_2\rangle = \langle\phi_2|\phi_3\rangle = \langle\phi_3|\phi_1\rangle = 0$$
$$\& \ \langle\phi_1|\phi_1\rangle = \langle\phi_2|\phi_2\rangle = \langle\phi_3|\phi_3\rangle = 1.$$

so. let $|\psi'\rangle = x|\psi\rangle$

so $\langle\psi|\psi\rangle = 1$
$$x.\sqrt{\frac{1}{2}+\frac{1}{5}+\frac{1}{10}} = 1$$
$$x = \frac{\sqrt{5}}{2}$$
$$\boxed{|\psi'\rangle = \frac{\sqrt{5}}{2}|\psi\rangle}$$

b) $\langle \hat{A} \rangle = \langle \psi' | \hat{A} | \psi' \rangle$

$$|\psi'\rangle = \frac{\sqrt{5}}{2}\left(\frac{1}{\sqrt{2}}|\phi_1\rangle + \frac{1}{\sqrt{5}}|\phi_2\rangle + \frac{1}{\sqrt{10}}|\phi_3\rangle\right)$$

$$= \frac{\sqrt{5}}{2\sqrt{2}}|\phi_1\rangle + \frac{1}{2}|\phi_2\rangle + \frac{1}{2\sqrt{2}}|\phi_3\rangle$$

$$\langle \hat{A} \rangle = \frac{1}{2}\left(\left(\frac{\sqrt{5}}{\sqrt{2}}\langle\phi_1| + \langle\phi_2| + \frac{1}{\sqrt{2}}\langle\phi_3|\right)\hat{A}\left(\frac{\sqrt{5}}{\sqrt{2}}|\phi_1\rangle + |\phi_2\rangle + \frac{1}{\sqrt{2}}|\phi_3\rangle\right)\frac{1}{2}\right)$$

$=$ $\because \{|\phi_n\rangle\}_{n=1}^{3}$ with eigenvalues n.

so

$$= \frac{1}{4}\left(\frac{5}{2}.1 + 1.2 + \frac{1}{2}.3\right)$$

$$= \frac{12}{8} = \boxed{\frac{3}{2}}.$$

B. Consider normalized states $|\mathcal{E}_1\rangle$ and $|\mathcal{E}_2\rangle$ that are not orthogonal, $\langle \mathcal{E}_1|\mathcal{E}_2\rangle = c \neq 0$ $(c \in \mathbb{C})$.

a) Write down a normalised state that is an equal superposition of the two states.

b) Write down a density operator representing the equal statistical mixture of $|\mathcal{E}_1\rangle$ and $|\mathcal{E}_2\rangle$ (in terms of $|\mathcal{E}_1\rangle$ and $|\mathcal{E}_2\rangle$).

Solution a) $|\mathcal{E}\rangle = N\left(|\mathcal{E}_1\rangle + |\mathcal{E}_2\rangle\right)$

For normalisation, $\langle \mathcal{E}|\mathcal{E}\rangle = 1$.

so $1 = \langle \mathcal{E}|\mathcal{E}\rangle = |N|^2\left(\left(\langle\mathcal{E}_1| + \langle\mathcal{E}_2|\right)\left(|\mathcal{E}_1\rangle + |\mathcal{E}_2\rangle\right)\right)$

$$= |N|^2\left(1 + c^* + c + 1\right)$$

Since it has to real

$$1 = |N|^2\left(2 + 2\mathrm{Re}(c)\right)$$

$$N = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{1+Re(c)}} e^{i\phi},$$ ← this is global phase. $\phi \rightarrow$ free parameter.

so $$|\xi\rangle = \frac{1}{\sqrt{2(1+Re(c))}} \left(|\xi_1\rangle + |\xi_2\rangle \right)$$

~~[illegible]~~

b) $$\hat{\rho} = \frac{1}{2}\left(|\xi_1\rangle\langle\xi_1|\right) + \frac{1}{2}\left(|\xi_2\rangle\langle\xi_2|\right)$$

C. We prepare a 2-dimensional quantum system at random in either one of the following three states : $|\psi_1\rangle = |0\rangle, |\psi_2\rangle = |1\rangle$, $|\psi_3\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$ with probabilities $p_1 = \frac{1}{8}$, $p_2 = \frac{1}{8}$,

$p_3 = \frac{3}{4}$ respectively, without knowing which one of the states is prepared. Show that the system density operator in the basis $\{|0\rangle, \{|1\rangle\}$ is,

$$\hat{\rho} = \frac{1}{8}\begin{pmatrix} 4 & -3i \\ +3i & 4 \end{pmatrix}$$

Solution

$$|\psi_1\rangle\langle\psi_1| = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad |\psi_3\rangle\langle\psi_3| = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$$

$$|\psi_2\rangle\langle\psi_2| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

We know $\hat{\rho} = p_1(|\psi_1\rangle\langle\psi_1|) + p_2(|\psi_2\rangle\langle\psi_2|) + \dots$

so $$\hat{\rho} = \frac{1}{8}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{8}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \frac{3}{8}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

$$\boxed{\hat{\rho} = \frac{1}{8}\begin{pmatrix} 4 & -3i \\ 3i & 4 \end{pmatrix}}$$

D. If Alice performs on $\hat{\rho}$ in Eq. 2 a projective measurement corresponding to observable $\hat{O} = |s\rangle\langle s|$, $|s\rangle = ((|1\rangle - i|0\rangle)/\sqrt{2})$

a) What are the possible measurement outcomes?

b) What are their associated probabilities?

Solution a)

for $|s\rangle = \dfrac{|1\rangle - i|0\rangle}{\sqrt{2}}$

$$= \frac{1}{\sqrt{2}}\left(-i|0\rangle + |1\rangle\right) = \frac{-i}{\sqrt{2}}\left(|0\rangle - i|1\rangle\right)$$

$$\hat{O} = |s\rangle\langle s| = \left(\frac{1}{\sqrt{2}}\left(-i|0\rangle + |1\rangle\right) \cdot \frac{1}{\sqrt{2}}\left(+i|0\rangle + |1\rangle\right)\right)$$

$$= \frac{|1\rangle + |0\rangle}{\sqrt{2}}$$

So the possible outcomes of the projective measurement associated with $\hat{O}$ are $+1, 0$ e-values with $\frac{1}{2}$ probabilities

b) $$Pr(+1 \text{ result}) = Tr[\hat{\rho}\,|s\rangle\langle s|] = Tr\left[\left(\frac{1}{8}|0\rangle\langle 0| + \frac{1}{8}|1\rangle\langle 1| + \frac{3}{4}|s\rangle\langle s|\right)|s\rangle\langle s|\right]$$

$$= \frac{1}{8} Tr[|0\rangle\langle 0|s\rangle\langle s|] + \frac{1}{8} Tr[|1\rangle\langle 1|s\rangle\langle s|] + \frac{3}{4} \underbrace{Tr[|s\rangle\langle s|s\rangle\langle s|]}_{1}$$

$$= \frac{1}{8}\langle s|0\rangle\langle 0|s\rangle + \frac{1}{8}\langle s|1\rangle\langle 1|s\rangle + \frac{3}{4}\cdot 1$$

$$= \frac{1}{8}\cdot\frac{1}{2} + \frac{1}{8}\cdot\frac{1}{2} + \frac{3}{4}$$

$$= \frac{7}{8}.$$

$$Pr(-1 \text{ result}) = 1 - \frac{7}{8} = \frac{1}{8}.$$

II. DEGENERATE PROJECTIONS

A three-level system is prepared in the normalised state $|\psi\rangle = \sqrt{\frac{1}{10}}\,|0\rangle + \sqrt{\frac{1}{5}}\,|1\rangle + \sqrt{\frac{7}{10}}\,|2\rangle$,

Let $\hat{D} = |0\rangle\langle 0| - |1\rangle\langle 1| + |2\rangle\langle 2|$ be an observable with eigenvalues ± 1. An ideal (incomple) projective measurement is made corresponding to $\hat{D}$ resulting in one of the eigenvalues being obtained.

A. a) What are the probabilities of obtaining the eigenvalues $+1$ or -1.

b) What is the expectation value $\langle \hat{D}\rangle$ in $|\psi\rangle$?

Solution a)

$$|\psi\rangle = \sqrt{\frac{1}{10}}\,|0\rangle + \sqrt{\frac{1}{5}}\,|1\rangle + \sqrt{\frac{7}{10}}\,|2\rangle$$

and $\hat{D} = |0\rangle\langle 0| - |1\rangle\langle 1| + |2\rangle\langle 2|$

+1 eigenspace by $|0\rangle, |2\rangle$ & -1 eigenvector by $|1\rangle$

$$\hat{P}_{+1} = |0\rangle\langle 0| + |2\rangle\langle 2| \quad , \quad \hat{P}_{-1} = |1\rangle\langle 1|$$

$$Pr(+1) = \|\hat{P}_{+1}|\psi\rangle\|^2 = Tr[|\psi\rangle\langle\psi|\hat{P}_{+1}]$$

$$= \left\|\sqrt{\frac{1}{10}}\,|0\rangle + \sqrt{\frac{7}{10}}\,|2\rangle\right\|^2$$

$$= \frac{1}{10} + \frac{7}{10} = \frac{8}{10} = \frac{4}{5}.$$

$$Pr(-1) = Tr[|\psi\rangle\langle\psi|\hat{P}_{-1}] = \left\|\sqrt{\frac{1}{5}}\,|1\rangle\right\|^2$$

$$= \frac{1}{5}.$$

b) $\langle \hat{D} \rangle = Tr[|\psi\rangle\langle\psi|\hat{D}]$

$= \sum_{\lambda} \lambda . P_r(\lambda)$

$= +1 . \frac{4}{5} - 1 . \frac{1}{5} = \boxed{\frac{3}{5}}$

B. Directly (by calculation) show that the projector $\hat{P}_{+1}$ on the +1 eigenspace of $\hat{D}$ satisfies $(\hat{P}_{+1})^2 = \hat{P}_{+1}$. The eigenspace is the sub-space spanned by the eigenvectors corresponding to the eigenvalue.

Solution $\hat{P}_{+1} = |0\rangle\langle 0| + |2\rangle\langle 2|$

$\hat{P}_{+1}^2 = (|0\rangle\langle 0| + |2\rangle\langle 2|)^2$

$= |0\rangle\langle 0|0\rangle\langle 0| + |2\rangle\langle 2|2\rangle\langle 2| + \underbrace{\cancel{|0\rangle\langle 0|2\rangle\langle 2|} + \cancel{|2\rangle\langle 2|0\rangle\langle 0|}}_{\text{over ONB}}$

$= |0\rangle\langle 0| + |2\rangle\langle 2| = \hat{P}_{+1}$.

C. In the case of obtaining the result +1, what is the conditional (normalised) state of the system post-measurement?

Solution For +1 outcome,

$|\psi^{+1}\rangle = \frac{P_{+1}|\psi\rangle}{\sqrt{P_r(+1)}} = \frac{(|0\rangle\langle 0| + |2\rangle\langle 2|)\left(\sqrt{\frac{1}{10}}|0\rangle + \sqrt{\frac{1}{5}}|1\rangle + \sqrt{\frac{7}{10}}|2\rangle\right)}{\sqrt{4/5}}$

$= \sqrt{\frac{1}{8}}|0\rangle + \sqrt{\frac{7}{8}}|2\rangle$ which is normalised.

since $\langle\psi^{+1}|\psi^{+1}\rangle = 1$.

III. TENSOR PRODUCTS

A. Let atoms A and B be non-interacting and have local Hamiltonians $\hat{H}_A$ and $\hat{H}_B$ respectively, with $\{E_j^A, |E_j^A\rangle\}_{j=0}^{d_A-1}$ and $\{E_k^B, |E_k^B\rangle\}_{k=0}^{d_B-1}$ their respective non-degenerate eigenenergies and eigenstates (they may be different types of atoms).
For the combined (non-interacting) system of A and B
a) write down the combined Hamiltonian $\hat{H}_{AB}$ that is the observable for the total energy of the two atoms.
b) work out the joint eigenergies and eigenstates.
c) show that these obey the Time Independent Schrödinger Equation.

Solution a) $\hat{H}_A$, $\hat{H}_B$ non-interacting atoms ⇒ Total energy.

$$\text{so} \quad \hat{H}_{AB} = \hat{H}_A + \hat{H}_B$$
$$= \hat{H}_A \otimes \hat{\mathbb{1}}_B + \hat{\mathbb{1}}_A \otimes \hat{H}_B$$

This is because the atoms are not interacting and their energies are independent of each other.

b) Let atom A be in $|E_j^A\rangle$ and atom B in $|E_k^B\rangle$

since independent, ⇒ $|E_{jk}^{AB}\rangle = |E_j^A\rangle \otimes |E_k^B\rangle$

Total energy is the sum of each e-value
$$E_{tot}^{AB} = E_j^A + E_k^B$$

c) We know that
$$\hat{H}|\psi\rangle = E|\psi\rangle \text{ for } |\psi\rangle \text{ an e-state of } \hat{H}$$

$$\hat{H}_{AB}(|E_j^A\rangle \otimes E_k^B\rangle) = (\hat{H}_A \otimes \hat{\mathbb{1}}_B)(|E_j^A\rangle \otimes |E_k^B\rangle) + (\hat{\mathbb{1}}_A \otimes \hat{H}_B)(|E_j^A\rangle \otimes |E_k^B\rangle)$$

$$= E_j^A\left(|E_j^A\rangle \otimes |E_k^B\rangle\right) + E_k^B\left(|E_j^A\rangle \otimes |E_k^B\rangle\right)$$

$$= \underbrace{\left(E_j^A + E_k^B\right)}_{\text{e-values}}\underbrace{\left(|E_j^A\rangle \otimes |E_k^B\rangle\right)}_{\text{e-vectors}}$$

B. Consider the state of two qubits (2-level quantum systems)
$|\Psi\rangle_{AB} = \frac{1}{\sqrt{30}}\left(|00\rangle + 2|01\rangle + 3i|10\rangle - 4|11\rangle\right)_{AB}$. Bob

measures his qubit B in the computational basis $\{|0\rangle_B, |1\rangle_B\}$

a) What is the probability of him getting the result $|1\rangle_B$?
b) What is the conditional (normalised) state of qubit A in this case?
i.e. if Bob tells Alice that he obtained the result $|1\rangle_B$, then what state does Alice now know she has.

Solution a) $|\Psi\rangle_{AB} = \frac{1}{\sqrt{30}}\left(\left(|0\rangle + 3i|1\rangle\right)_A|0\rangle_B + \left(2|0\rangle - 4|1\rangle\right)_A|1\rangle_B\right)_{AB}$

$$P_r(|1\rangle_B) = Tr\left[\left(|\Psi\rangle\langle\Psi|_{AB}\right)\left(\mathbb{1}_A \otimes |1\rangle\langle 1|_B\right)\right]$$

$$= \left\|\frac{1}{\sqrt{30}}\left(2|0\rangle - 4|1\rangle\right)_A\right\|^2$$

$$= \frac{1}{30}(4+16) = \frac{20}{30} = \boxed{\frac{2}{3}}$$

b) State of Alice conditional on Bob obtaining result $|1\rangle_B$

$$|\Psi_A\rangle = \frac{1}{\sqrt{30}}\left(2|0\rangle - 4|1\rangle\right)_A \cdot \sqrt{\frac{3}{2}}$$

$\rightarrow$ normalization

$\sqrt{\frac{30}{16+4}}$

$$= \frac{1}{\sqrt{5}}\left(|0\rangle - 2|1\rangle\right)$$

~~Full state given by~~

Full state $|\psi\rangle_{AB} = \left(\frac{1}{\sqrt{5}}\left(|0\rangle - 2|1\rangle\right)\right)_A \otimes |1\rangle_B$.

IV. POVMs.

Consider a two-outcome non-projective measurement on a 2-level system that is specified by the detection operators (effects)

$$\hat{M}_0 = \begin{pmatrix} i\sqrt{q} & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}, \quad \hat{M}_1 = \begin{pmatrix} \sqrt{1-q} & 0 \\ 0 & \sqrt{q} \end{pmatrix}$$

when expressed in the computational basis $\{|0\rangle, |1\rangle\}$, where $\frac{1}{2} < q < 1$ represents the fidelity of the measurement.

A. a) What are the corresponding measurement operators $\{\hat{\Pi}_0, \hat{\Pi}_1\}$?

b) Show that they form an allowed POVM.

Solution a) We know that

$$\hat{\Pi}_0 = \hat{M}_0^{+}\hat{M}_0 = \begin{pmatrix} -i\sqrt{q} & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}\begin{pmatrix} i\sqrt{q} & 0 \\ 0 & \sqrt{1-q} \end{pmatrix} = \begin{pmatrix} q & 0 \\ 0 & 1-q \end{pmatrix}$$

$$\hat{\Pi}_1 = \hat{M}_1^{+}\hat{M}_1 = \begin{pmatrix} \sqrt{1-q} & 0 \\ 0 & \sqrt{q} \end{pmatrix}\begin{pmatrix} \sqrt{1-q} & 0 \\ 0 & \sqrt{q} \end{pmatrix} = \begin{pmatrix} 1-q & 0 \\ 0 & q \end{pmatrix}$$

b) $\hat{\Pi} = \sum \hat{\Pi}_j = \hat{\Pi}_0 + \hat{\Pi}_1$

$$= \begin{pmatrix} q & 0 \\ 0 & 1-q \end{pmatrix} + \begin{pmatrix} 1-q & 0 \\ 0 & q \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

Hence it forms resolution of identity.

Also $q > 1/2 \Rightarrow$ ~~...~~ $0 < 1-q < \frac{1}{2}$

Thus $\hat{\Pi}_0 > 0$ & $\hat{\Pi}_1 > 0$

$\boxed{\text{Therefore allowed POVM}}$

B. The above POVM is performed on a pure state $|\psi\rangle = \cos\theta|0\rangle + \sin\theta e^{i\phi}|1\rangle$

a) If $\hat{\Pi}_0$ is the result, what is the conditional state?

b) What is the unconditional state after the measurement (if you did not know the result)? Express your answers in the computational basis.

Solution a) $|\psi\rangle = \cos\theta|0\rangle + \sin\theta e^{i\phi}|1\rangle$

For $\hat{\Pi}_0$ result,

$$|\psi'\rangle = \frac{\hat{M}_0|\psi\rangle}{\sqrt{Pr(\Pi_0)}} = \frac{i\sqrt{q}\cos\theta|0\rangle + \sqrt{1-q}\sin\theta e^{i\phi}|1\rangle}{\sqrt{q\cos^2\theta + (1-q)\sin^2\theta}}$$

$$= \frac{i\sqrt{q}\cos\theta|0\rangle + \sqrt{1-q}\sin\theta e^{i\phi}|1\rangle}{\sqrt{2q\cos 2\theta + \sin^2\theta}}$$

b) Unconditional state is given by

$$\hat{\rho}' = \hat{M}_0\hat{\rho}\hat{M}_0^\dagger + \hat{M}_1\hat{\rho}\hat{M}_1^\dagger$$

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

$$= \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta e^{-i\phi} \\ \cos\theta\sin\theta e^{i\phi} & \sin^2\theta \end{pmatrix}$$

$$\hat{\rho}' = \begin{pmatrix} i\sqrt{q} & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta e^{-i\phi} \\ \cos\theta\sin\theta e^{i\phi} & \sin^2\theta \end{pmatrix}\begin{pmatrix} -i\sqrt{q} & 0 \\ 0 & \sqrt{1-q} \end{pmatrix}$$

$$+ \begin{pmatrix} \sqrt{1-q} & 0 \\ 0 & \sqrt{q} \end{pmatrix}\begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta e^{-i\phi} \\ \cos\theta\sin\theta e^{i\phi} & \sin^2\theta \end{pmatrix}\begin{pmatrix} \sqrt{1-q} & 0 \\ 0 & \sqrt{q} \end{pmatrix}$$

$$\hat{\rho}' = \begin{pmatrix} q\cos^2\theta & i\sqrt{q(1-q)}\cos\theta\sin\theta e^{-i\phi} \\ -i\sqrt{q(1-q)}\cos\theta\sin\theta e^{i\phi} & (1-q)\sin^2\theta \end{pmatrix}$$

$$+ \begin{pmatrix} (1-q)\cos^2\theta & \sqrt{q(1-q)}\cos\theta\sin\theta e^{-i\phi} \\ \sqrt{q(1-q)}\cos\theta\sin\theta e^{i\phi} & q\sin^2\theta \end{pmatrix}$$

$$\hat{\rho}' = \begin{pmatrix} \cos^2\theta & (1+i)\sqrt{q(1-q)}\cos\theta\sin\theta e^{-i\phi} \\ (1-i)\sqrt{q(1-q)}\cos\theta\sin\theta e^{i\phi} & \sin^2\theta \end{pmatrix}$$

and the modulus of $\hat{\rho}'$ is $\cos^2\theta + \sin^2\theta = 1$.

so pure state.