PH558/964 Advanced Quantum Theory: Assignment 2

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The 2nd assignment is due 10pm Thursday 11th March 2020. **Clear working and explanations** required. Solutions should submitted as a single PDF file. LaTeX format preferred but scanned neat and legible handwritten solutions are also acceptable. Marked out of 100.

I. KRAUS OPERATORS

Let $\{\hat{K}_j\}_{j=1}^N$ be a set of Kraus operators. Let us define a new set of Kraus operators $\{\hat{K}_j^{'}\}_{j=1}^N$, where the new operators are related to the original set by

$$\hat{K}_j^{'} = \sum_k u_{jk}\hat{K}_k \tag{1}$$

where u_{jk} are (complex) elements of an $N \times N$ unitary matrix u. **Note:** u is not an operator, it is simply just a matrix of complex numbers that is unitary.

A. 10 Marks

Show that $\{\hat{K}_j^{'}\}_{j=1}^N$ is a Kraus operator set, i.e. it satisfies $\sum_j (\hat{K}_j^{'})^\dagger \hat{K}_j^{'} = \hat{\mathbb{I}}$. Hint: Use the matrix unitary condition $u^\dagger u = \mathbb{I} = uu^\dagger$. **Note:** Keep track of what are operators or just numbers in your calculation and to be clear as to summations and summation variables.

B. 5 marks

Show that $\{\hat{K}_j\}_{j=1}^N$ and $\{\hat{K}_j^{'}\}_{j=1}^N$ implement the same completely positive trace-preserving map, i.e. $\sum_j \hat{K}_j\hat{\rho}\hat{K}_j^\dagger = \sum_j \hat{K}_j^{'}\hat{\rho}(\hat{K}_j^{'})^\dagger$ for all density operators $\hat{\rho}$.

C. 5 Marks

Using the above result, show that the 2 Kraus operator sets $\{|+x\rangle\langle+x|, |-x\rangle\langle-x|\}$ and $\{\frac{\hat{\mathbb{I}}}{\sqrt{2}}, \frac{\hat{\sigma}_x}{\sqrt{2}}\}$ implement the same CP map. Draw or describe the action of the map upon the Bloch ball.

D. 10 marks

Consider the qubit CP map Λ that maps all inputs to the output $|1\rangle$, $\Lambda(\hat{\rho}) = |1\rangle\langle 1|,\ \forall\hat{\rho}$. Give a Stinespring dilation for Λ and the corresponding Kraus operators for this superoperator.

II. UNIVERSAL NOT

The (unphysical) superoperator $UNOT$ is mathematically defined by its action on pure qubit states,

$$UNOT(|\psi\rangle\langle\psi|) = |\psi^\perp\rangle\langle\psi^\perp|,\ \forall|\psi\rangle \in \mathcal{H}^2, \tag{2}$$

where $\langle\psi|\psi^\perp\rangle = 0$.

A. 6 Marks

By considering the diagonal representation of a mixed qubit density operator (i.e. $\hat{\rho} = \sum p_j|e_j\rangle\langle e_j|$, where $|e_j\rangle$ are the eigenvectors of $\hat{\rho}$ and p_j are their weights in the ensemble), show that $UNOT\left(\frac{1}{2}(\mathbb{I} + \bar{r}.\bar{\sigma})\right) = \frac{1}{2}(\mathbb{I} - \bar{r}.\bar{\sigma})$ using the definition above (Eq. 2). Here, $\bar{r}$ represents the Bloch vector of a general qubit density operator and its length satisfies $0 \leq |\bar{r}| \leq 1$.

B. 4 Marks

Hence, by using linearity of the superoperator, show that $UNOT(\hat{\sigma}_j) = -\hat{\sigma}_j,\ j = x, y, z$ by expressing the $\hat{\sigma}_j$ operators as the linear combination of suitable density operators.

C. 10 Marks

Show that $UNOT$ is trace-preserving, positive, but not a completely positive map. The latter should be shown by considering the action of $UNOT \otimes \mathbb{I}$ on a maximally entangled state of two qubits, $|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{AB}$, expressing the density operator of the input state in the Pauli ($\hat{\sigma}_j$) operator basis. I.e. determine the Choi-Jamiolkowski "state" $\hat{\rho}_{UNOT} = UNOT \otimes \mathbb{I}(|\Phi^+_{AB}\rangle\langle\Phi^+_{AB}|)$ isomorphic to $UNOT$ and show that it is not actually a state.

III. ENTANGLED STATES

A. 5 Marks

Consider the pure state of 2 qubits given by $|\Theta\rangle_{AB} = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)_{AB}$. Show that $|\Theta\rangle_{AB}$ is unentangled by giving its product form.

B. 15 Marks

A Werner state of two qubits is given by

$$\hat{\rho}^W_{AB} = p|\Psi^-_{AB}\rangle\langle\Psi^-_{AB}| + (1-p)\frac{\hat{\mathbb{I}}_{AB}}{4} \tag{3}$$

where $0 \le p \le 1$, $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{AB}$, and $\hat{\mathbb{I}}_{AB}$ is the identity operator on $\mathcal{H}^2_A \otimes \mathcal{H}^2_B$. Using the positive partial transpose criterion, determine the range of values of p over which $\hat{\rho}^W_{AB}$ is separable and when it is entangled.

C. 10 Marks

For which range of values of p does the Werner state $\hat{\rho}^W_{AB}$ violate the CHSH inequality? Consider the case where Alice and Bob have choices of measurements given by dichotomic (± 1) measurement observables $\{\hat{A}_1, \hat{A}_2\}$ and $\{\hat{B}_1, \hat{B}_2\}$ respectively and you can assume that these choices lead to the maximal CHSH violation $s = 2\sqrt{2}$ on a maximally entangled state of two qubits. There is no need to explicitly define the measurements when determining the CHSH value for the Werner state.

D. 20 Marks

The singlet state of two spin-$\frac{1}{2}$ particles is given by

$$|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{AB}, \tag{4}$$

where we have identified $|\uparrow\rangle = |0\rangle, |\downarrow\rangle = |1\rangle$. Alice and Bob are given a singlet pair, each receiving one qubit, and they perform joint simultaneous projective measurements on their respective qubits. Alice and Bob decide to make projective measurements along the same axis (parallel and anti-parallel directions) of the Bloch ball, using the following procedure:

- Alice and Bob can only natively perform projective measurements in the computational basis $\{|0\rangle_A, |1\rangle_A\}$, $\{|0\rangle_B, |1\rangle_B\}$ on their own respective qubits
- They can also perform a unitary rotation $\hat{U}_A$, $\hat{U}_B$ of their respective qubit prior to measurement in the computational basis
- They always perform the same unitary rotation $\hat{U}_A = \hat{U}_B = \hat{U}$
- The unitary rotation can be expressed as $\hat{U} = |\xi\rangle\langle 0| + e^{i\gamma}|\xi^\perp\rangle\langle 1|$, $|\xi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\xi^\perp\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$ are general orthogonal normalised qubit states, and $0 \le \gamma < 2\pi$.
- After applying $(\hat{U} \otimes \hat{U})_{AB}$ on $|\Psi^-\rangle_{AB}$, Alice and Bob then measure their qubits in the computational basis. This results in the same outcome probabilities as if they had measured $|\Psi^-\rangle_{AB}$ along some other direction/basis.

Taking into considering the above situation and procedure by Alice and Bob:

1. Show that the form of $\hat{U}$ describes a general unitary operation on a qubit, i.e. that it is unitary and that all rotations of the Bloch ball can be enacted by $\hat{U}$ with suitable choices of $\{\alpha, \beta, \gamma\}$. Note: the set of rotations of the Bloch ball ($SO(3)$) is a three (real) parameter set, hence the parameters consisting of 2 complex numbers α, β and 1 real angle γ are equivalent to 5 real parameters, hence there is redundancy in this description of $\hat{U}$.
2. For a measurement on a single qubit, show that the procedure of first performing $\hat{U}$ and then measuring in the computational basis gives the same outcome probabilities as measuring along a general axis of the Bloch ball in the first place. A measurement in the computational basis is given by projecting along the Z-axis and obtaining either the $|+Z\rangle = |0\rangle$ or $|-Z\rangle = |1\rangle$ results. Other basis measurements are represented by measurement along a different axis and obtaining the results corresponding to $+1$ ($+\hat{\mathbf{n}}$) or -1 ($-\hat{\mathbf{n}}$), where $\hat{\mathbf{n}}$ is a normalised Bloch vector (not an operator).
3. Show that the outcomes of Alice and Bob are perfectly anti-correlated, i.e., whenever Alice rotates her qubit by $\hat{U}$ and measures in the $\{|0\rangle_A, |1\rangle_B\}$ basis, and Bob does the same thing on his qubit, if Alice gets the result 0, Bob gets the result 1; else if Alice gets a 1, Bob gets a 0.

IV. BONUS SECTION

Note: Bonus marks will only count if the marks obtained in the main assignment (Sections I, II, III, marked out of 100) is below 50. In that case, any marks obtained in this bonus section will be capped at the amount required to take the total score to 50, i.e. 50%. If the main assignment marks are 50 or above, the bonus marks will not count.

A. 5 Marks

A qubit density operator $\hat{\rho} = \frac{1}{2}(\hat{\mathbb{I}} + \bar{r}.\bar{\sigma})$ is diagonal in the computational basis (or Z-basis), i.e. $\hat{\rho} = \frac{1}{2}(\hat{\mathbb{I}} + n_z\hat{\sigma}_z)$, where n_z is the Z-component of the Bloch vector, $-1 \le n_z \le +1$. A measurement of the qubit is made along the Z-axis (computational basis). Express the probability of obtaining the $|+z\rangle$ and $|-z\rangle$ outcomes as a function of the value of n_z.

B. 5 Marks

The fidelity between a pure state $|\psi\rangle$ and a density operator $\hat{\rho}$ (may be mixed or pure) is defined as $F(|\psi\rangle, \hat{\rho}) = \mathrm{Tr}[|\psi\rangle\langle\psi|\hat{\rho}] = \langle\psi|\hat{\rho}|\psi\rangle$. For d-dimensional states, calculate the fidelity between a general pure state and the maximally mixed state.

C. 5 Marks

We define the Bell maximally entangled orthonormal basis states of 2 qubits as:

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B) \tag{5}$$

$$|\Phi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B) \tag{6}$$

$$|\Psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B) \tag{7}$$

$$|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B). \tag{8}$$

Express a general pure 2-qubit state, $|\Xi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB}$, in the Bell basis. I.e. in terms of $\{\alpha, \beta, \gamma, \delta\}$, find the coefficients in the expression

$$|\Xi\rangle_{AB} = (c_1|\Phi^+\rangle + c_2|\Phi^-\rangle + c_3|\Psi^-\rangle + c_4|\Psi^-\rangle)_{AB}. \tag{9}$$

D. 5 Marks

The GHZ state of 3 qubits is denoted by,

$$|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{ABC}. \tag{10}$$

Alice, Bob, and Charlie are given a GHZ state, each receiving one of the 3 qubits respectively. If they each were to measure their own qubit in the computational basis and compared their results, they would find they would either all have obtained $|0\rangle$, or all $|1\rangle$.

Instead of directly measuring the state $|GHZ\rangle_{ABC}$ in the computational basis, they instead first perform on their individual qubits the Hadamard operation $\hat{H}$ (not to be confused with the Hamiltonian operator, this should be clear from context which is meant).

The Hadamard operation is unitary and can be defined by its action on the computation basis by the following unitary transformation,

$$|0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \tag{11}$$

$$|1\rangle \to \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{12}$$

Hence, Alice, Bob, and Charlie apply the operation $\hat{H} \otimes \hat{H} \otimes \hat{H}$ to $|GHZ\rangle_{ABC}$ to obtain the new state of 3 qubits,

$$|GHZ'\rangle_{ABC} = (\hat{H} \otimes \hat{H} \otimes \hat{H})_{ABC}|GHZ\rangle_{ABC}. \tag{13}$$

Alice, Bob, and Charlie now measure $|GHZ'\rangle$ in the computational basis and compare their results. Show that in all cases, they will obtain an even number of $|1\rangle$ results (i.e. either they all get the $|0\rangle$ state, or two of them get $|1\rangle$ and the other gets $|0\rangle$).

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PH 964 : Theoretical Quantum Information

ASSIGNMENT 2

I. $\{\hat{K}_j\}_{j=1}^{N} \rightarrow$ set of Kraus operators

new set $\Rightarrow \hat{K}_j' = \sum_k u_{jk} \hat{K}_k$

A. To show $\Rightarrow \{\hat{K}_j'\}_{j=1}^{N}$ is a Kraus operator set.

Solution $\Rightarrow$

$$\sum_j (\hat{K}_j')^\dagger \hat{K}_j' = \sum_j \left(\sum_k u_{jk} \hat{K}_k\right)^\dagger \left(\sum_m u_{jm} \hat{K}_m\right) = \sum_{jkm} u_{jk}^* u_{jm} \hat{K}_k^\dagger \hat{K}_m$$

since u_{jk} element of unitary matrix, $u \Rightarrow u^\dagger u = u u^\dagger = \mathbb{1}$

Consider $AB = C$, matrix multiplications $C_{st} = \sum_j A_{sj} b_{jt}$

so $\delta_{st} = \sum_j (u^\dagger)_{sj} u_{jt} = \sum_j u_{js}^* u_{jt}$

$$\Rightarrow \sum_{km} \left(\sum_j u_{jk}^* u_{jm}\right) \hat{K}_k^\dagger \hat{K}_m = \sum_{km} \delta_{km} \hat{K}_k^\dagger \hat{K}_m$$

$$= \sum_j \hat{K}_j^\dagger K_j = \mathbb{1}$$

Hence Proved, $\{\hat{K}_j'\}$ is a Kraus Operator Set.

B. To show $\Rightarrow \sum_j \hat{K}_j \hat{\rho} K_j^\dagger = \sum_j \hat{K}_j' \hat{\rho} (\hat{K}_j')^\dagger$ for all density operators $\hat{\rho}$

Solution $\Rightarrow$

$$\Omega'(\hat\rho') = \sum_j \hat K_j' \hat\rho \hat K_j'^{\dagger} = \sum_j \left(\sum_m u_{jm} \hat K_m\right) \hat\rho \left(\sum_n u_{jn} \hat K_n\right)^{\dagger}$$

$$= \sum_{jmn} (u_{jm} u_{jn}^{*}) \hat K_m \hat\rho \hat K_n^{\dagger}$$

$$= \sum_{mn} \left(\sum_j u_{jm} u_{jn}^{*}\right) \hat K_m \hat\rho \hat K_n^{\dagger}, \text{ use } uu^{\dagger} = \mathbb{1}$$

$$= \sum_{mn} \delta_{mn} \hat K_m \hat\rho \hat K_n^{\dagger}$$

$$= \sum_n \hat K_n \hat\rho \hat K_n^{\dagger} = \Omega(\hat\rho) \quad \underline{\text{Same CP Map}}.$$

C. To show $\Rightarrow$ $\{|+x\rangle\langle +x|, |-x\rangle\langle -x|\}$ and $\left\{\frac{\hat{\mathbb{1}}}{\sqrt2}, \frac{\hat\sigma_x}{\sqrt2}\right\}$ implement same CP Map.

Solution $\Rightarrow$

$$|+x\rangle\langle +x| = \frac{1}{\sqrt2}(|0\rangle+|1\rangle)\frac{1}{\sqrt2}(\langle 0|+\langle 1|) = \frac12\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \qquad \frac{\hat{\mathbb{1}}}{\sqrt2} = \frac{1}{\sqrt2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$|-x\rangle\langle -x| = \frac{1}{\sqrt2}(|0\rangle-|1\rangle)(\langle 0|-\langle 1|) = \frac12\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} \qquad \frac{\hat\sigma_x}{\sqrt2} = \frac{1}{\sqrt2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

Considering $u = \frac{1}{\sqrt2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$, $u = u^{\dagger}$, $u^{\dagger}u = uu^{\dagger} = \frac{1}{\sqrt2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt2}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

$$u_{11}\frac{\mathbb{1}}{\sqrt2} + u_{12}\frac{\hat\sigma_x}{\sqrt2} = \frac{1}{\sqrt2}\cdot\frac{1}{\sqrt2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{1}{\sqrt2}\cdot\frac{1}{\sqrt2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac12\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} = |+x\rangle\langle +x|$$

$$u_{21}\frac{\mathbb{1}}{\sqrt2} + u_{21}\frac{\hat\sigma_x}{\sqrt2} = \frac{1}{\sqrt2}\frac{1}{\sqrt2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \frac{-1}{\sqrt2}\frac{1}{\sqrt2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \frac12\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix} = |-x\rangle\langle -x|$$

Thus the two sets of Kraus operators are related by a unitary matrix u, hence implement same CP Map.

Visualizing action through a Bloch sphere $\Longrightarrow$

consider $\{|+x\rangle\langle+x|, |-x\rangle\langle-x|\}$, projectors on $\pm\hat{\sigma}_x$ eigenvector

This is the same as the unconditional outcome of performing a measurement of $\hat{\sigma}_x$ observable and forgetting result, Bloch ball collapses along the line x-axis.

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \vec{r}\cdot\hat{\vec{\sigma}}) = \hat{P}_{+x}\hat{\rho}\hat{P}_{+x} + \hat{P}_{-x}\hat{\rho}\hat{P}_{-x}$$

$$= \hat{P}_{+x}\frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma})\hat{P}_{+x} + \hat{P}_{-x}\frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma})\hat{P}_{-x}$$

$$= \frac{1}{2}\left(\mathbb{1} + \sum_{j=x,y,z} r_j \left(\hat{P}_{+x}\hat{\sigma}_j\hat{P}_{+x} + \hat{P}_{-x}\hat{\sigma}_j\hat{P}_{-x}\right)\right)$$

Since $\hat{P}_{+x}\hat{\sigma}_z\hat{P}_{+x} = \frac{1}{2}|+x\rangle\langle+x| - \frac{1}{2}|+x\rangle\langle+x| = 0$,

$\hat{\sigma}_x = |+x\rangle\langle+x| - |-x\rangle\langle-x| \Rightarrow \hat{P}_{+x}\hat{\sigma}_x\hat{P}_{+x} + \hat{P}_{-x}\hat{\sigma}_x\hat{P}_{-x} = \hat{\sigma}_x$,

$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + r_x\hat{\sigma}_x), \text{ i.e. } \vec{r} = (r_x, r_y, r_z) \to (r_x, 0, 0)$$

D. $\Omega(\hat{\rho}) = |1\rangle\langle 1|, \ \forall \hat{\rho} \in B(\mathcal{H}^2)$

Stinespring dilation, $\hat{U}_{SE} = SWAP$, $SWAP|\psi\rangle_S|\phi\rangle_E = |\phi\rangle|\psi\rangle \ \forall |\psi\rangle, |\phi\rangle$

$$\mathrm{Tr}_{SE}\left[\hat{U}_{SE}\left(\hat{\rho}_S \otimes |1\rangle\langle 1|_E\right)\hat{U}_{SE}^\dagger\right] = \mathrm{Tr}_{SE}\left(|1\rangle_S\langle 1| \otimes \hat{\rho}_E\right)$$

$= |1\rangle\langle 1| = \Omega(\hat{\rho})$ as required.

Kraus operators

$$\hat{K}_0 = |1\rangle\langle 0| \quad , \hat{K}_1 = |1\rangle\langle 1|$$

$$\hat{K}_0^\dagger \hat{K}_0 = |0\rangle\langle 1|1\rangle\langle 0| = |0\rangle\langle 0| \;, \quad \hat{K}_1^\dagger \hat{K}_1 = |1\rangle\langle 1|$$

$$\hat{K}_0^\dagger \hat{K}_0 + \hat{K}_1^\dagger \hat{K}_1 = |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}_2$$, Hence complete

Let $\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix}$

$$\hat{K}_0 \hat{\rho} \hat{K}_0^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 1+r_z & r_x - ir_y \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1+r_z \end{pmatrix}$$

$$\hat{K}_1 \hat{\rho} \hat{K}_1^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{2}\begin{pmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1-r_z \end{pmatrix}$$

$$\Rightarrow \hat{K}_0 \rho \hat{K}_0^\dagger + \hat{K}_1 \rho \hat{K}_1^\dagger = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = |1\rangle\langle 1|$$

so this set of Kraus operators acts as required.

II. $\mathrm{UNOT}(|\psi\rangle\langle\psi|) = |\psi^\perp\rangle\langle\psi^\perp|$, $\forall |\psi\rangle \in \mathcal{H}^2$ where $\langle\psi|\psi^\perp\rangle = 0$

pure qubit state

A. $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \bar{r}.\bar{\sigma})$, let $\bar{r} = p_+ \bar{r}_+ + p_- \bar{r}_-$

where $\hat{\rho} = p_+ |\psi_+\rangle\langle\psi_+| + p_- |\psi_-\rangle\langle\psi_-|$, $|\psi_\pm\rangle = \frac{1}{2}(\mathbb{1} + \bar{r}_\pm$

$$\mathrm{UNOT}(\hat{\rho}) = p_+ \,\mathrm{UNOT}(|\psi_+\rangle\langle\psi_+|) + p_- \,\mathrm{UNOT}(|\psi_-\rangle\langle\psi_-|)$$
$$= p_+ |\psi_-\rangle\langle\psi_-| + p_- |\psi_+\rangle\langle\psi_+| = \hat{\rho}'$$
$$(\because \langle\psi_+|\psi_-\rangle = 0)$$

$$\hat{\rho}' = (1-p_-)|\psi_-\rangle\langle\psi_-| + (1-p_+)|\psi_+\rangle\langle\psi_+|$$
$$= |\psi_-\rangle\langle\psi_-| + |\psi_+\rangle\langle\psi_+| - (p_+|\psi_+\rangle\langle\psi_+| + p_-|\psi_-\rangle\langle\psi_-|)$$
$$= \mathbb{1} - \hat{\rho} = \frac{1}{2}(\mathbb{1} - \vec{r}\cdot\vec{\sigma}), \text{ i.e. } \vec{r} \xrightarrow{\mathrm{UNOT}} -\vec{r}.$$

B. Consider $|+x\rangle\langle+x| = \frac{1}{2}(\mathbb{1} + \sigma_x)$, $r_x = 1$

$$|-x\rangle\langle-x| = \frac{1}{2}(\mathbb{1} - \sigma_x), \; r_x = -1$$
$$\Rightarrow \hat{\sigma}_x = |+x\rangle\langle+x| - |-x\rangle\langle-x|$$
$$\mathrm{UNOT}(\hat{\sigma}_x) = \mathrm{UNOT}(|+x\rangle\langle+x| - |-x\rangle\langle-x|) = -\hat{\sigma}_x$$

Similarly for $\hat{\sigma}_y, \hat{\sigma}_z$, thus

$$\mathrm{UNOT}(\hat{\sigma}_j) = -\hat{\sigma}_j \text{ for } j = x, y, z.$$

C. $\mathrm{Tr}[\mathrm{UNOT}(\hat{\rho})] = \mathrm{Tr}[\mathrm{UNOT}(\frac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}))]$

$$= \mathrm{Tr}\left[\frac{1}{2}(\mathbb{1} - \vec{r}\cdot\vec{\sigma})\right]$$
$$= \frac{1}{2}\mathrm{Tr}[\mathbb{1}] - \mathrm{Tr}\left[\sum_{j=x,y,z} r_j \hat{\sigma}_j\right]$$
$$= \frac{1}{2}\cdot 2 - 0 \quad (\because \sigma_{x,y,z} \text{ is traceless})$$
$$= 1 = \mathrm{Tr}[\hat{\rho}]$$

Hence trace preserving.

Let $\hat{A} = \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j|$, $\hat{A} \in B(\mathcal{H}^2)$, diagonal representation, $\lambda_j \geq 0$.

$\hat{A} \geq 0$, positive semi-definite operator.

$\hat{A}' = UNOT(\hat{A}) = \sum_j \lambda_j |\lambda_j^\perp\rangle\langle\lambda_j^\perp|$, still in diagonal form.

$\Rightarrow \hat{A}' \geq 0$ as eigenvalues still all non-negative.

Hence UNOT is a positive map.

Now consider $UNOT \otimes \mathbb{1} \left(|\Psi^+\rangle\langle\Psi^+|_{AB}\right) = \hat{\rho}_{UNOT}$, $|\Psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$$|\Psi^+\rangle\langle\Psi^+|_{AB} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$$ in $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis

$$\mathbb{1} \otimes \mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\sigma_x \otimes \sigma_x = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_y \otimes \sigma_y = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \sigma_z \otimes \sigma_z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow |\Psi^+\rangle\langle\Psi^+|_{AB} = \frac{1}{4}\left(\mathbb{1}\otimes\mathbb{1} + \sigma_x \otimes \sigma_x - \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z\right)_{AB}$$

$$\Rightarrow NOT \otimes \mathbb{1}\left(|\Psi^+\rangle\langle\Psi^+|_{AB}\right) = \frac{1}{4}\left(UNOT(\mathbb{1})\otimes\mathbb{1} + UNOT(\hat{\sigma}_x)\otimes\sigma_x + UNOT(\hat{\sigma}_y)\otimes\sigma_y + UNOT(\hat{\sigma}_z)\otimes\sigma_z\right)$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Subspace spanned by $|00\rangle, |11\rangle$, operator $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -\hat{\sigma}_x$

which has a −ve eigenvalue

$\Rightarrow$ UNOT $\otimes \mathbb{1}$ not a positive map.
$\Rightarrow$ UNOT not completely positive.

III. A. $|\ominus\rangle_{AB} = \frac{1}{2}(|00\rangle - |01\rangle - |10\rangle + |11\rangle)_{AB}$.

$$|\ominus\rangle_{AB} = \frac{1}{2}\left(|0\rangle(|0\rangle - |1\rangle) - |1\rangle(|0\rangle - |1\rangle)\right)_{AB}$$
$$= \frac{1}{2}(|0\rangle - |1\rangle)_A(|0\rangle - |1\rangle)_B.$$
$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)_A \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)_B = |-x\rangle_A|-x\rangle_B$$

product form
non-entangled

B. $\hat{\rho}^{w}_{AB} = p|\Psi^-_{AB}\rangle\langle\Psi^-_{AB}| + (1-p)\frac{\mathbb{1}_{AB}}{4}$, $0 \leq p \leq 1$

$$|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{AB}$$

PPT Criterion : $\hat{\rho}_{AB} \in B(\mathcal{H}^2 \otimes \mathcal{H}^2)$

$\hat{\rho}_{AB}$ separable $\Leftrightarrow$ $T \otimes \mathbb{1}(\hat{\rho}_{AB}) \geq 0$

Use the fact that $\frac{\mathbb{1}}{4}_{AB}$ shares eigenvectors with $\hat{\rho}^{w}_{AB}$

$\Rightarrow$ can easily work out e-values of $T \otimes \mathbb{1}(\hat{\rho}^{w}_{AB})$

e-vals of $T \otimes \mathbb{1}(\hat{\rho}^{w}_{AB}) = p\,\underset{\text{spectrum}}{\sigma}\left(T \otimes \mathbb{1}(|\Psi^-\rangle\langle\Psi^-|_{AB})\right)$

$+ (1-p) \cdot \frac{1}{4}$ ← e-vals of $\frac{\mathbb{1}}{4}$,

$T \otimes \mathbb{1}(\mathbb{1}/4) = \frac{\mathbb{1}}{4}$

Proof Let $\hat{A} = \sum_j \lambda_j^A |\lambda_j\rangle\langle\lambda_j|$, diagonal representation

$\hat{B} = \sum_j \lambda_j^B |\lambda_j\rangle\langle\lambda_j|$, $|\lambda_j\rangle$ common e-vectors

$$\hat{C} = p\hat{A} + (1-p)\hat{B},\ \hat{C}|\lambda_j\rangle = (\underbrace{p\lambda_j^A + (1-p)\lambda_j^B}_{\text{e-value of } |\lambda_j\rangle \text{ in } \hat{C}})|\lambda_j\rangle$$

e-values of $T\otimes\mathbb{1}(|\Psi^-\rangle\langle\Psi^-|_{AB})$

$$T\otimes\mathbb{1}(|\Psi^-\rangle\langle\Psi^-|) = T\otimes\mathbb{1}\left(\frac{|01\rangle\langle01| - |01\rangle\langle10| - |10\rangle\langle01| + |10\rangle\langle10|}{2}\right)$$

$$= \frac{|01\rangle\langle01| - |00\rangle\langle11| - |11\rangle\langle00| + |10\rangle\langle10|}{2}$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \text{ in } \{|00\rangle, |01\rangle, |10\rangle, |11\rangle\} \text{ basis}$$

$$\downarrow$$

e-vals are $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right\}$

Hence e-values of $\hat{\rho}_{AB}^W = p\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right\} + (1-p)\cdot\frac{1}{4}$

Min. e-val is $(1-p)\frac{1}{4} - \frac{p}{2} = \frac{1-3p}{4}$

$\frac{1-3p}{4} < 0 \Rightarrow p > \frac{1}{3}$ for $\hat{\rho}_{AB}^W$ entangled.

i.e. $\frac{1}{3} < p \leq 1$ necessary & sufficient for $\hat{\rho}_{AB}^W$ entangled

C. Consider a Bell experiment on $|\Psi^-\rangle_{AB}$ resulting in $S = 2\sqrt{2}$.

The expectation values of the measurements on $\frac{\mathbb{1}_{AB}}{4}$ will give
$\langle \hat{A}^j \hat{B}^k \rangle = 0$ for $j = 1, 2$,
$\hat{A}^j, \hat{B}^k$ being the choice of observables to measure.
Hence the value of S on $\hat{\rho}^{w}_{AB}$ is $p \cdot 2\sqrt{2} + (1-p) \cdot 0$

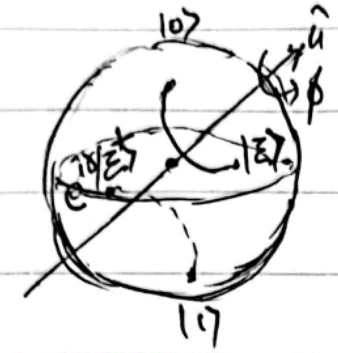
For a violation of the Bell CHSH inequality.
$S > 2 \Rightarrow p > \frac{1}{\sqrt{2}}$, i.e. $\frac{1}{\sqrt{2}} < p \leq 1$

D. $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

① $\hat{u} = |\xi\rangle\langle 0| + e^{i\gamma}|\xi^\perp\rangle\langle 1|$, $|\xi\rangle = \alpha|0\rangle + \beta|1\rangle$
$|\xi^\perp\rangle = \beta^*|0\rangle - \alpha^*|1\rangle$
$\langle \xi^\perp | \xi \rangle = (\beta\langle 0| - \alpha\langle 1|)(\alpha|0\rangle + \beta|1\rangle)$
$= \beta\alpha - \alpha\beta = 0$, hence orthonormal

$\hat{u}^\dagger \hat{u} = (|0\rangle\langle\xi| + e^{-i\gamma}|1\rangle\langle\xi^\perp|)(|\xi\rangle\langle 0| + e^{i\gamma}|\xi^\perp\rangle\langle 1|)$
$= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbb{1}$
Similarly $\hat{u}\hat{u}^\dagger = \mathbb{1}$, resolution into ONB.
Hence $\hat{u}$ is unitary.

Now, to show $\Rightarrow \hat{U}$ is general.



so $|0\rangle$ rotated in $|\xi\rangle$

$|1\rangle$ rotated in $e^{i\gamma}|\xi^{\perp}\rangle$

(not observable in this representation, matters when acting on superposition of $|0\rangle,|1\rangle$)

We can observe that all rotations of the form $(\hat{u}, \phi)$ can be described by specifying the mapping $\begin{array}{l} |0\rangle \to |\xi\rangle \\ |1\rangle \to e^{i\gamma}|\xi^{\perp}\rangle \end{array}$

We can do parameter count, for convenience, we express.

$$\left.\begin{array}{l} |\xi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2} e^{i\delta}|1\rangle \\ |\xi^{\perp}\rangle = \cos\frac{\pi-\theta}{2}|0\rangle + \sin\left(\frac{\pi-\theta}{2}\right)e^{-i\delta}|1\rangle \end{array}\right] \begin{array}{l} \theta, \delta \\ \frac{1}{2} \text{ parameters} \end{array}$$

together with γ gives 3-parameters, same as $(\hat{u}, \phi)$

$$\hat{U} = \begin{pmatrix} \alpha & e^{i\gamma}\beta^* \\ \beta & -e^{i\gamma}\alpha^* \end{pmatrix}$$ [columns are orthogonal]

$$\begin{array}{cc} \uparrow & \\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} & e^{i\gamma}\begin{pmatrix} \beta^* \\ -\alpha^* \end{pmatrix} \\ |\xi\rangle & |\xi^{\perp}\rangle \end{array}$$

$$= \underset{\substack{\uparrow \\ \text{global phase}}}{e^{i\nu}} \underbrace{\begin{pmatrix} \cos\frac{\theta}{2} & e^{i\gamma}\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2}e^{i\delta} & -e^{i\gamma}\cos\frac{\theta}{2} \end{pmatrix}}_{\substack{\uparrow \\ SU(2) \text{ matrix}}}$$

(2) Let a single qubit pure state be $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$

$$\hat{u}|\psi\rangle = \left(|\xi\rangle\langle 0| + e^{i\gamma}|\xi^\perp\rangle\langle 0|\right)\left(c_0|0\rangle + c_1|1\rangle\right)$$

$$= c_0|\xi\rangle + e^{i\gamma}c_1|\xi^\perp\rangle = |\psi'\rangle$$

$|\langle 0|\psi'\rangle|^2 = |c_0\alpha + e^{i\gamma}c_1\beta^*|^2 = P_r(0)$ outcome prob of $|0\rangle$.

$|\langle 1|\psi'\rangle|^2 = |c_0\beta - e^{i\gamma}c_1\alpha^*|^2 = P_r(1)$ outcome prob of $|1\rangle$

probability check for sum to be 1.

$$p(0) = (c_0^*\alpha^* + e^{-i\gamma}c_1^*\beta)(c_0\alpha + e^{i\gamma}c_1\beta^*)$$

$$= |c_0|^2|\alpha|^2 + c_0^*c_1\alpha^*\beta^*e^{i\gamma} + e^{-i\gamma}c_1^*\beta c_0\alpha + |c_1|^2|\beta|^2$$

$$= |c_0|^2|\alpha|^2 + 2\mathrm{Re}(c_1^*c_0\alpha\beta) + |c_1|^2|\beta|^2$$

$$p(1) = (c_0^*\beta^* - e^{-i\gamma}c_1^*\alpha)(c_0\beta - e^{i\gamma}c_1\alpha^*)$$

$$= |c_0|^2|\beta|^2 - e^{-i\gamma}c_1^*\alpha c_0\beta - e^{+i\gamma}c_0^*\beta^*c_1\alpha^* + |c_1|^2|\alpha|^2$$

$$= |c_0|^2|\beta|^2 - 2\mathrm{Re}(c_1^*c_0\alpha\beta) + |c_1|^2|\alpha|^2.$$

$$p(0) + p(1) = |c_0|^2|\alpha|^2 + |c_1|^2|\beta|^2 + |c_0|^2|\beta|^2 + |c_1|^2|\alpha|^2$$

$$= (|c_0|^2 + |c_1|^2)(|\alpha|^2 + |\beta|^2)$$

$$= 1.1 = 1.$$

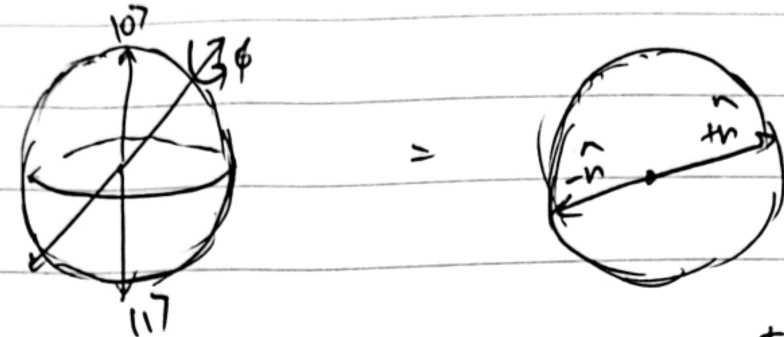
Let $|+\hat{n}\rangle = \alpha^*|0\rangle + e^{-i\gamma}\beta|1\rangle$

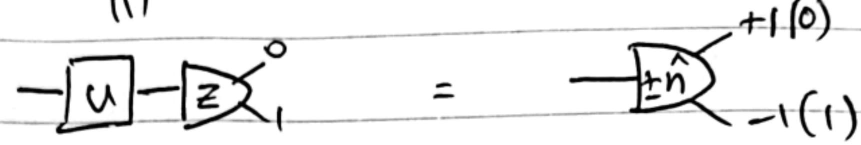
$|-\hat{n}\rangle = \beta^*|0\rangle - e^{+i\gamma}\alpha|1\rangle$

then $|\langle +\hat{n}|\psi\rangle|^2 = |\alpha c_0 + e^{i\gamma}\beta^*c_1|^2 = P_r(0)$

$|\langle -\hat{n}|\psi\rangle|^2 = |\beta c_0 - e^{-i\gamma}\alpha^*c_1|^2 = P_r(1)$

Hence measuring $|\psi'\rangle = \hat{u}|\psi\rangle$ in $\{|0\rangle, |1\rangle\}$ basis gives same probabilities as measuring $|\psi\rangle$ in $\{|+\hat{n}\rangle, |-\hat{n}\rangle\}$ basis





(3) $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right) \xrightarrow{U_A \otimes U_B} \frac{1}{\sqrt{2}}\left(U|0\rangle \otimes U|1\rangle - U|1\rangle \otimes U|0\rangle\right)$

$$= \frac{1}{\sqrt{2}}\Big((\alpha|0\rangle + \beta|1\rangle) e^{i\gamma}(\beta^*|0\rangle - \alpha^*|1\rangle) - e^{i\gamma}(\beta^*|0\rangle - \alpha^*|1\rangle)(\alpha|0\rangle + \beta|1\rangle) \Big)$$

$$= \frac{1}{\sqrt{2}}\Big(\overbrace{(\alpha\beta^* e^{i\gamma} - e^{i\gamma}\beta^*\alpha)}^{0}|00\rangle - \overbrace{(|\alpha|^2 + |\beta|^2)}^{1} e^{i\gamma}|01\rangle + \overbrace{(|\beta|^2 + |\alpha|^2)}^{1} e^{i\gamma}|10\rangle - \overbrace{(e^{i\gamma}\beta\alpha^* - e^{i\gamma}\alpha^*\beta)}^{0}|11\rangle \Big)$$

$$= -\frac{e^{i\gamma}}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$$

→ same as before upto global phase

$$p(|01\rangle_{AB}) = \left|\langle 01|\left(-\frac{e^{i\gamma}}{\sqrt{2}}(|01\rangle - |10\rangle)\right)\right|^2 = \frac{1}{2}$$

$$p(|10\rangle_{AB}) = \left|\langle 10|\left(-\frac{e^{i\gamma}}{\sqrt{2}}(|01\rangle - |10\rangle)\right)\right|^2 = \frac{1}{2}$$

$$p(|00\rangle) = p(|11\rangle) = 0$$

only anti-correlated results

⇒ Measurements in $\{|0\rangle, |1\rangle\}$ basis remain perfectly anti-correlated

⇒ Measuring $|\Psi^-\rangle_{AB}$ in $\{|0\rangle, |1\rangle\}$ basis gives same correlations as measuring $|\Psi^-\rangle_{AB}$ in $\{|+\hat{n}\rangle, |-\hat{n}\rangle\}$ basis.

IV. A. $\hat{\rho} = \frac{1}{2}(\mathbb{1} + \bar{r} \cdot \bar{\sigma})$, $\hat{\rho} = \frac{1}{2}(\mathbb{1} + n_z \hat{\sigma}_z)$

$$\hat{\rho} = \frac{1}{2}\begin{pmatrix} 1+n_z & 0 \\ 0 & 1-n_z \end{pmatrix}, \quad -1 \leq n_z \leq 1$$

$$Pr(0) = Tr[|0\rangle\langle 0|\hat{\rho}] = Tr\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \frac{1+n_z}{2} & 0 \\ 0 & \frac{1-n_z}{2} \end{pmatrix}\right]$$

$$= Tr\begin{bmatrix} \frac{1+n_z}{2} & 0 \\ 0 & 0 \end{bmatrix} = \frac{1+n_z}{2} = Pr(+z)$$

$$Pr(1) = Tr[|1\rangle\langle 1|\hat{\rho}] = \frac{1-n_z}{2} = Pr(-z)$$

$$Pr(0) + Pr(1) = \frac{1+n_z}{2} + \frac{1-n_z}{2} = 1$$

B. $F(|\psi\rangle, \hat{\rho}) = Tr[|\psi\rangle\langle\psi|\hat{\rho}] = \langle\psi|\hat{\rho}|\psi\rangle$

Let $\hat{\rho} = \frac{\mathbb{1}}{d}$ $\Rightarrow$ d-dimensional maximally mixed state

Let $|\psi\rangle \in \mathcal{H}^d$ be a pure state

$$F\left(|\psi\rangle, \frac{\mathbb{1}}{d}\right) = \langle\psi|\frac{\mathbb{1}}{d}|\psi\rangle = \frac{\langle\psi|\psi\rangle}{d} = \frac{1}{d}$$

C. $\mathbb{1} = \sum_{j=1}^{4} |\Psi_j\rangle\langle\Psi_j|$, $\begin{Bmatrix} |\Psi_1\rangle = |\Phi^+\rangle, |\Psi_2\rangle = |\Phi^-\rangle \\ |\Psi_3\rangle = |\Psi^+\rangle, |\Psi_4\rangle = |\Psi^-\rangle \end{Bmatrix}$

ONB

$$|\Xi\rangle = \left(\sum_j |\Psi_j\rangle\langle\Psi_j|\right)|\Xi\rangle$$

$c_j = \langle \Psi_j | \Xi \rangle = \langle \Psi_j | (\alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle) \rangle$

$c_1 = \frac{1}{\sqrt{2}} (\langle 00| + \langle 11|)(\alpha|00\rangle + \underbrace{\beta|01\rangle + \gamma|10\rangle}_{\text{no overlap with } |\Psi_1\rangle} + \delta|11\rangle)$

so, $c_1 = \frac{\alpha + \delta}{\sqrt{2}}$, $c_2 = \frac{\alpha - \delta}{\sqrt{2}}$, $c_3 = \frac{\beta + \gamma}{\sqrt{2}}$, $c_4 = \frac{\beta - \gamma}{\sqrt{2}}$

D. $|GHZ\rangle_{ABC} = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)_{ABC}$

$Pr(|000\rangle) = |\langle 000 | GHZ \rangle|^2 = \frac{1}{2} = Pr(|111\rangle)$

only perfectly correlated results in $\hat{\sigma}_z$ basis

$|GHZ'\rangle = H^{\otimes 3} |GHZ\rangle$

$= \frac{1}{\sqrt{2}} (H|0\rangle_A \otimes H|0\rangle_B \otimes H|0\rangle_C + H|1\rangle_A \otimes H|1\rangle_B \otimes H|1\rangle_C)$

$= \frac{1}{\sqrt{2}} \left(\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) + \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \frac{1}{\sqrt{2}} (|0\rangle - |1\rangle) \right)_{ABC}$

$= \frac{1}{4} \Big(|000\rangle + |001\rangle + |010\rangle + |100\rangle + |101\rangle + |110\rangle + |011\rangle + |111\rangle$
$+ |000\rangle - |001\rangle - |010\rangle + |011\rangle - |100\rangle + |101\rangle + |110\rangle - |111\rangle \Big)_{ABC}$

($\because \pm 1$ sign depends whether even or odd # of $|1\rangle$ term)

$$= \frac{1}{2}\left(|000\rangle + |011\rangle + |101\rangle + |110\rangle\right)_{ABC}$$

When each qubit measured in $\{|0\rangle\}$ basis, only get

$A=0$,	$B=0$	, $C=0$	, $p=\frac{1}{4}$
0	1	1	$p=\frac{1}{4}$
1	0	1	$p=\frac{1}{4}$
1	1	0	$p=\frac{1}{4}$

Either 0 "1"s or 2 "1"s.