Ordinary Differential Equations

In Class Notes

Mathematics Department
Salt Lake Community College

Contents

Chapter 1

First Test

Date: 01.12.15

1.1 Date: 1.12.15

Def: let y be a function of x and $y^{(n)}(x) = d^n y/dx^n$ for $n \in \mathbb{N}\{1, 2, 3, ...\}$ where F is a given function

$$2xy + x^2y' = 1$$
$$2xy + x^2y' - 1 = 0$$

This is an ODE, where

$$F(x, y, y') = 2xy + x^2y' - 1$$

1.2 Date: 1.13.15

Date: 01.13.15

ex: Given f on an interval I find F st $F(x)f'(x)$ for all $x \in I$ Here F is by def, an antiderivative of f on I. we denote F as $\displaystyle\int f(x)dx$. All solns F to (1) are of the form $\displaystyle\int f(x)dx + c$, where c is an arbitray constant, (1) is an ODE, i.i.,

$$y; = f(x) \iff y' - f(x) = 0$$

the latter has the form

$$f(x, y') = 0$$

A little calc III
Given $F : \mathbb{R}^n \to \mathbb{R}$, say $\vec{a} = (x_1, ..., x_1) \in R^n$, the derivative of F at $\vec{x}$ is

$$= F(x) = (F_x 1(\vec{x}), F_x 2(\vec{x}), ..., F_x n(\vec{x})) \in Mat_1 xn(R),$$

where

$$F_x i(\vec{x}) = \lim_{x \to 0} \frac{F(x_1, ..., x_i + h, ..., x_n)}{h}$$

F_{xi} is called the partial derivative of F with respect to x_i.
we call $\nabla F = F'$ the gradiant of F.
given $r : I \in R^n$, where I subset R is an integral in R, the derivative of r at t is

$$r'(t) = (x'_1(t), x'_2(t), ..., x'_n(t)),$$

$$r(t) = (x_1(t), x_2(t), ..., x_n(t)),$$

here $r'(t)$ is a tangent vector to r at $r'(t)$ in R^n.

1.3 Date: 01.14.15

Date: 01.14.15

Let $F : \mathbb{R}^n \to \mathbb{R}^m$, say $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $F(x) = (f_1)$
thrm(Chain Rule). if $\mathbb{R}^n$:

Recall: separable ODE

Date: 01.26.15.

$$y' = g(x)h(y) \text{ or } \frac{dy}{dx} = g(x)h(y)$$

"separate" the variable as

$$\frac{1}{h(y)}y' = g(x) \text{ or } \frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$\int \frac{1}{h(y)}\frac{dy}{dx}dx = \int g(x)dx$$

Now, $y = y(x)$ then by
by change of vars,

$$\int \frac{1}{h(y)}dy = \int \frac{1}{h(y)}\frac{dy}{dx}dx = \int g(x)dx$$

in short,

$$\int \frac{1}{h(y)}dy = \int g(x)dx$$

Now , once these itegrals are evaluated,
if possible, then the resolting eqn
is one of y on the left and x on the
right, which at least implicitly defines
solns y(x) to y' = h(y)g(x). this
resulting eqn may or may not be possible
to solve for y explicitly in terms of x

Example

$x' = kx$, think $x = x(t)$ this is separable; so,

$$\frac{1}{x}\frac{dx}{dt} = k$$

$$\int \frac{1}{x}\frac{dx}{dt}dt = k\int dt \implies$$

(by change of vars)

$$\int \frac{1}{x}dx = \int \frac{1}{x}\frac{dx}{dt}dt = k\int dt$$

$$\implies ln|x|kt + C,$$

$$\implies |x| = e^{kt+c} = Ce^{kt},\ C = e^c > 0$$

$$\implies x(t) = Ce^{kt},\ C \neq 0, \text{ any } k \in R$$

Example

$T' = k(A - T)$, $k > 0$
this s separable; so,

$$\frac{1}{A-T}\frac{dT}{dt} = k \implies$$

$$\int \frac{1}{A-T}dT = kt + C \implies$$

$$-ln|A-T| = kt + C \implies$$

$$\frac{1}{A-T} = Ce^{kt} \implies$$

$$A - T = Ce^{-Kt} \implies$$

$$T = A - Ce^{-kt}$$

Aside

substitution (change of vars)

$$\int f(g(x))g'(x)dx = \int f(v)dv$$

$v = g(x)$, or

$$\int f(v)\frac{v}{x}dx = \int f(v)dv$$

Date: 01.27.15.

Ex(p.43) 35

$$x(t) = ce^{kt} \text{ (form } x = kx)$$

C14 has a decay rate constant of

$$k = -0.0001216$$

Notice that

$$x(0) = C$$

so, C is called the initial value.
that notation 'x_0' is used for C
i.e., $x_0 = x(0)$. thus,

$$x(t) = x_0 e^{kt}$$

in # 35, $x(t) = x_0/6$.
so,

$$\frac{x_0}{6} = x_0 e^{kt}$$

solve for t. thus,

$$t = \frac{1}{6} ln(1/6) = \frac{1}{|k|} ln(6)$$

Torricelli's law

Think of $x = x(t)$ and $h = h(t)$
we want $x(t)$, say in particular,
we want t st $x(t) = 0$, so called
"drain time."
recall # 35, p.18, that "ground speed"
is given by $|v| =, sqrt2gx$
from "free-fall" a height x.
in contex,

$$\frac{dh}{dt} = \sqrt{2gx}$$

In "the spout" $V = ah$; so

$$\frac{dV}{dt} = a\frac{dh}{dt} a\sqrt{cgx}$$

in the tank

$$\frac{dV}{dt} = -a\sqrt{2gx}$$

let A(x) be the ? cros=sectinal
area of the tank at height x, then

$$V = \int_x^0 A(t)dt$$

by ftc(1),

$$\frac{dV}{dx} = A(x)$$

by the cain rule

$$\frac{dV}{dt} = dv/dxdx/dt = A(x)x'$$

$$a(x) * x' = -a\sqrt{2gx}$$

i.e.

$$x'A(x) = -a\sqrt{2gx}$$

which is a separable ODE. thus

$$\frac{A(x)}{\sqrt{x}}dx = -a\sqrt{2g} \implies$$

(int w.r.t t and Δ vars)

$$\int \frac{A(x)}{\sqrt{x}}dx = -at\sqrt{2g}$$

1.4 Date: 01.29.2015

Ex (p.45) # 59

revolve $x^2 = by$ about y-axis
depth is 4ft at noon, $y(0) = 4$

$$a = \pi r^2$$

depth is 1ft at 1pm same day
Recall: $\int y^{-1/2} A(y) dx = -8at$ in ft and s.
thus, by torciullis law,

$$-8\pi r^2 t = \pi b \int y^{1/2} dy \implies$$

$$-8r^2 t = \frac{2b}{3} y^{3/2} + C$$

$$y(0) = 4 \implies C = -16b/3$$

$$\therefore$$

$$\frac{2b}{3} y^{3/2} = \frac{16b}{3} - 8r^I 2t$$

Now, in 3600s (1hr), y=1, i.e., y(3600) = 1,

$$\frac{2b}{3} = \frac{16b}{3} - 8r^2(3600)$$

$$r^2 = \frac{14b}{3 * 8 * 3600} = \frac{7b}{12 * 3600} \implies$$

$$r = \frac{1}{60}\sqrt{\frac{7b}{12}}$$

Drain time t_0 is

$$0 = \frac{16b}{3} - 8r^2 t_0 \implies t_0 = \frac{2b}{3r^2}$$

Now, in particuler, if
$y = 4$, then the radius

of $A(y)$ is 2, i.e., $x = 2$.
Thus,

$$x^2 = by \implies 4 = 4b \implies b = 1 \therefore$$

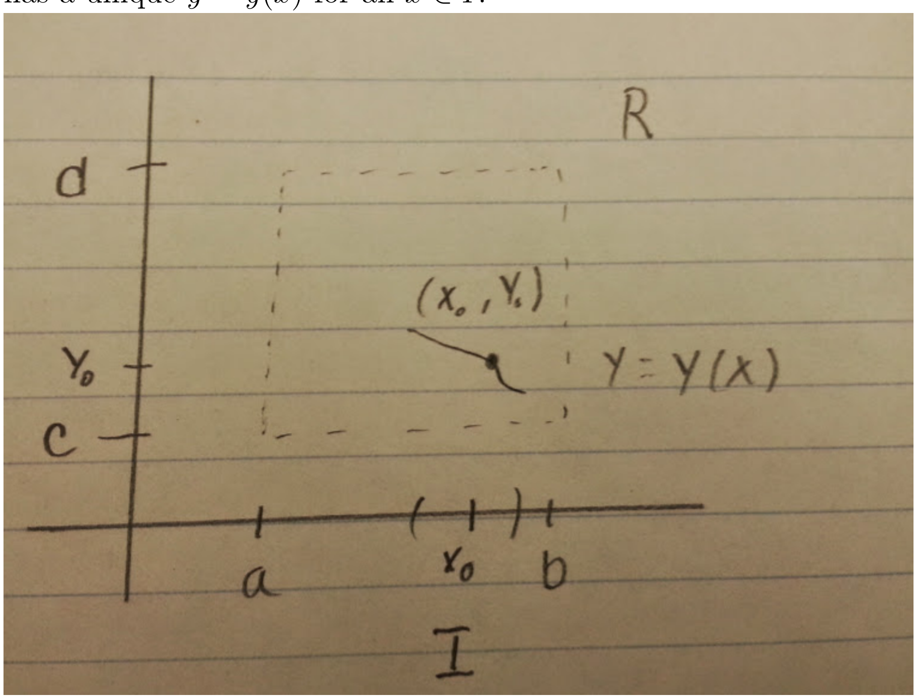
$$r = \frac{1}{60}\sqrt{\frac{17}{12}} \;\&\; t_0 = \frac{2}{3r^2}$$

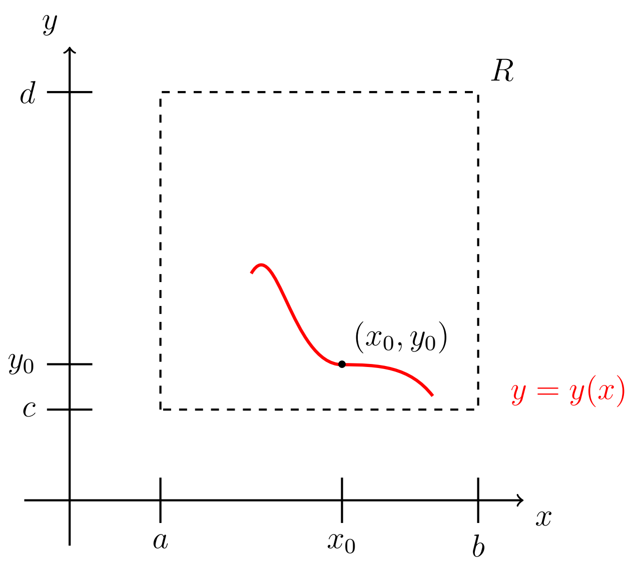
1.5 Date: 02.02.2015

Thrm existence - uniqueness thrm, $\exists!$thrm
If $f : D, \subseteq \mathbb{R}^2 \to \mathbb{R}$ and $f_y : D_2 \subseteq \mathbb{R}^2 \to \mathbb{R}$
are cts on $R = (a, b)x(c, d)$ and $x_0, y_0) \in R$ and
then these exist an interval I st $x_0 \in I$, $I \subseteq (a, b)$ and
the initial value problem

$$\frac{dy}{dx} = f(x, y) \text{ and } y_0 = y(x_0)$$

has a unique $y = y(x)$ for all $x \in I$.





$$R = (a,b)x(c,d)$$

$$= (x,y) \in R^2 | a < x < b \ \& \ c < y < d$$

The ∃! thrm is a "local" result, local
to x_0, move precisely, it just sups that there is a unique soln in I
not necessarily outside of I.

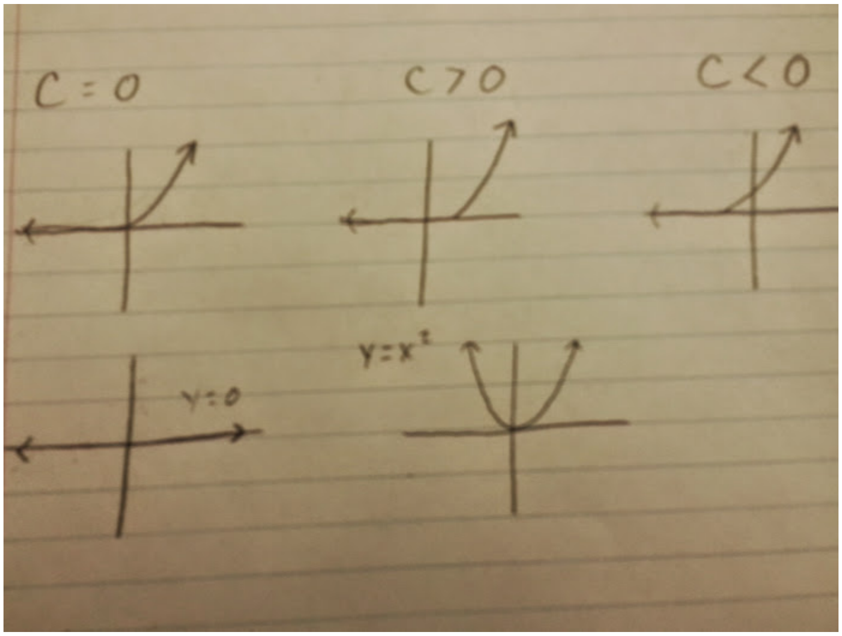
Ex(p.29) # 27

$$y' = s\sqrt{y} \ \& \ y(0) = 0$$

consider

$$y(x) = \begin{cases} 0 & x \leq c \\ (x-c)^2 & x \geq c \end{cases}$$

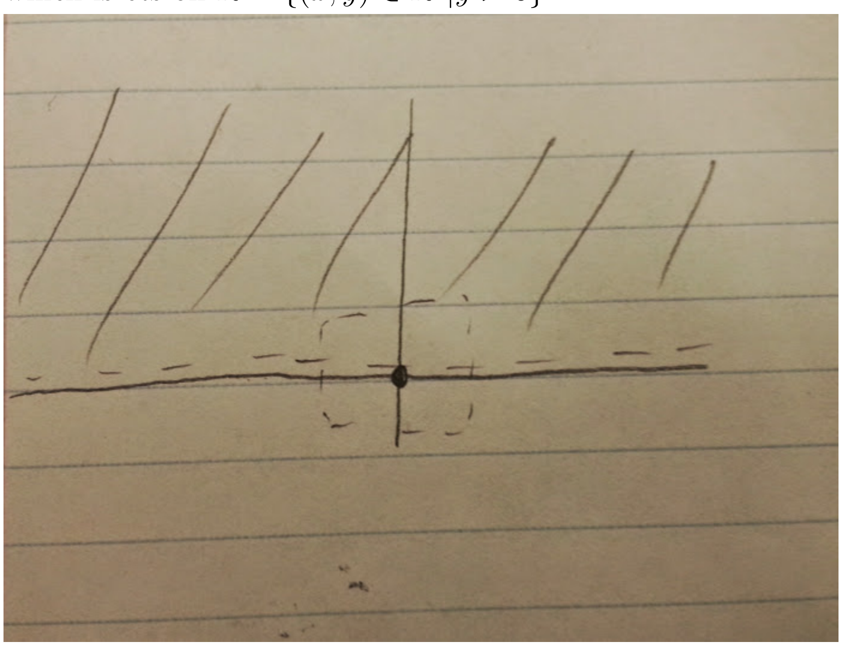
which is ctn on $\mathbb{R}$
notice that both parts
y(x) satisfy the initial value problem if
$c \geq 0$ Note:



Notice that

$$f(x,y) = 2\sqrt{y} \ \& \ f_y(x,y) = \frac{1}{\sqrt{y}}$$

which is cts on $R = \{(x,y) \in R^2 | y > 0\}$



notice that (0, 0) is not in R, not
"interior" to R. $\therefore \exists!$ thrm does not apply
Notice that if $y = f(x)$ is a soln to
an ODE on a interval I
then so is $y = f(x - c)$ a soln to
the ODE on $I - c = \{x \in R | x + c \in I\}$.

Date: 02.03.15.

ex p.28

15. $y' = \sqrt{x-y}$, $y(2) = 2$, no
16. $y' = \sqrt{x-y}$, $y(2) = 1$, yes
here $f(x,y) = \sqrt{x-y}$, which has
cts iff $x - y \geq 0 \iff x \geq y$
$f_y(x,y) = -\dfrac{1}{2\sqrt{x-y}}$, which is cts
on $\{(x,y) \in \mathbb{R}^2 \mid y < x\}$.

Notes

Def A first order linear ODE
has the form

$$(1)\ y' + p(x)y = q(x)$$

where p & q are cts on some interval $\mathcal{I}$.
Notice that if $q(x) = 0$ then (1) is
separable.
Key observation: the left side of
(1) resembols the product rule.
this motivates a question: Is there
a cst I(x), say on the interval $\mathcal{I}$,
st
(1') $y'I(x) + yp(x)I(x) = q(x)I(x)$,
where the left side of (1') is the dirivative
of a product?
if there is such an I(x)
then
(2) $I'(x) = p(x)I(x)$.
put $v = I(x)$, then (2) becomes

$$v' = p(x)v$$

which is separable. thus

$$\frac{v'}{v} = p(x) \implies$$

$$\int \frac{1}{v} dv = \int p(x) dx \implies$$

$$\ln |v| = \int p(x) dx + C \implies$$

$$|v| = e^{\int p(x) dx + C} = K e^{\int p(x) dx}$$

where $k > 0$ $\therefore$

$$v = K e^{\int p(x) dx} \text{ , } K \neq 0$$

Def. the intergrating factor of

$$y' + p(x) y = q(x) is$$

$$I(x) = e^{\int p(x) dx}$$

Finally, from (1')

$$(y I(x))' = q(x) I(x) \implies$$

$$y I(x) = \int q(x) I(x) dx \implies$$

$$\boxed{y = \frac{1}{I(x)} \int q(x) I(x) dx}$$

Date: 02.04.15.

Single Tanking Problem

c_i = concentration coming into the tank (constant)
r_i = rate of flow into the tank (constant)
$c_0(t)$ = concentration coming out of the tank
r_0 = rate of flow out of the tank (constant)
$x(t)$ = amount of salute in tank at time t
$V(t)$ volume of tank at time t

Units:
Concentration $= \dfrac{\text{amount of solute}}{\text{unit valume}}$
Rate $= \dfrac{\text{valume}}{\text{unit time}}$
amount = (concentration)(rate)(time)

Notice that the rate of change fo the volume is constant and is $m = r_i - r_0$; where, $V(t) = (r_i - r_0)t + V_0 = mt + V_0$, where $V_0 = V(0)$
For a small Δt
$x(t + \Delta t) = x(t) +$ amount in = amount out over time Δt

amount in over $\Delta t = c_i r_i \Delta t$;
amount out over $\Delta t \approx c_0(t) r_0 \Delta t$.
thus,

$$\Delta x = x(t + \Delta t) - x(t) \approx (c_i r_i - c_0(t) r_0)\Delta t \implies$$

$$\frac{\Delta x}{\Delta t} \approx c_i r_i - c_0(t) r_0$$

this suggest that

$$\frac{dy}{dx} = c_i r_i - c_0(t) r_0$$

Now

$$c_0(t) = \frac{x(t)}{V(t)} \implies$$

$$\frac{dx}{dt} = c_i r_i = \frac{x(t)}{V(t)} r_0$$

which is a 1st order linear ODE.
more consiely, put $x' = \dfrac{dx}{dt}$ and
$x = x(t)$, then

$$\boxed{x' + \frac{r_0}{V(t)} x = c_i r_i}$$

Recall that $V(t) = mt + v_0$, $m = r_i - r_0$; so,

$$x' + \frac{r_0}{mt + v_0} x = c_i r_i$$

Here

$$p(t) = \frac{r_0}{mt + V_0} \implies$$

$$\int p(x)dx = \frac{r_0}{m} ln(mt + V_0) + C$$

where in context, $V(t) > 0$. Choose, of ease,
$C = 0$, then

$$I(t) = e^{\int p(t)dt} = (mt + V_0)^{r_0/m}$$

So,

$$x'(mt + V_0)^{r_0/m} + r_0(mt + V_0)^{r_0/m - m} = c_i r_i (mt + V_0)^{r_0/m} \implies$$
$$(x(mt + V_0)^{r_0/m})' = c_i r_i + (mt + V_0)^{r_0/m} \implies$$
$$x(mt + V_0)^{r_0/m} = c_i r_i \int (mt + V_0)^{r_0/m} dt$$

If $r_0/m = -1$ then $r_0 = -m = -r_i + r_0$
$\implies r_i = 0$, which is not of intrest in
context ("mixing"). Thus, if $r_0/m \neq -1$
then

$$\int (mt + V_0)^{r_0/m} dt = \frac{1}{m}(mt + V_0)^{r_0/m+1} \frac{m}{r_0 + m} + C$$
$$= \frac{1}{r_i}(mt + V_0)^{r_i/m} + C$$

$$\therefore$$

$$x(mt+v_0)^{r_0/m} = c_i(mt+V_0)^{r_i/m} + C$$

at $t = 0$

$$x_0V_0^{r_0/m} = c_iV_0^{r_i/m} + C \implies$$

$$C = c_iV_0^{r_i/m} - x_0V_0^{r_0/m}$$

$$= V_0^{r_0/m}(c_iV_0 - x_0)$$

thus,

$$x(mt+V_0)^{r_0/m} = c_i(mt+V_0)^{r_0/m} + V_0^{r_0/m}(c_iV_0 - x_0$$

$$\implies x = c_i(mt+V_0) + (c_iV_0 - x_0)(\frac{V_0}{mt+V_0})^{r_0/m}$$

$$\boxed{\begin{array}{c} x = c_iV + (c_iV_0 - x_0)\left(\dfrac{V_0}{V}\right)^{r_0/(ri-r_0)} \\ \text{where } x = x(t) \ \& \ V = (r_i - r_0)t + V_0 \end{array}}$$

Date: 02.09.15.

1.6 Date: 02.09.2015

1.6 Substitutions in ODEs
Consider a slope field

$$(1)\ \frac{dy}{dx} = f(y, x)$$

i.e., a 1st order normal ODE. If

$$\alpha(x, y)$$

appers in (1), then we are compelled
to make the substitution

$$v = \alpha(x, y)$$

("alpha" for auxillary variable)
By the calc III chain rule

$$\frac{dv}{dx} = \frac{\partial\alpha}{\partial x}\frac{dx}{dx} + \frac{\partial\alpha}{\partial y}\frac{dy}{dx}$$

$$= \alpha_x + \alpha y\frac{dy}{dx}$$

If $v = \alpha(x, y) canbesoved$
for y in terms of x and v, say

$$y = \beta(x, v)$$

then from (1), we have that

$$\frac{dv}{dx} = \alpha_x + \alpha_y\frac{dy}{dx} = \alpha + \alpha_y f(x, y)$$

where,

$$\boxed{\frac{dv}{dx} = \alpha_x + \alpha f(x, \beta(x, v))}$$

which is a new ODE with dependent variable v and independent variable x.

1.6.1 ex

$$\frac{dy}{dx} = f(x, y, ax + by + c$$

put

$$v + d(x, y) = ax + by + c$$

then

$$\frac{dv}{dx} = a + b\frac{dy}{dx}$$

thus,

$$\frac{dv}{dx} = a + b\frac{dy}{dx} = a + bf(x, y, ax + by + c)$$

$$\implies \boxed{\frac{dv}{dx} = a + bf(x, (v - ax - c)/b, v)}$$

where

$$y = \beta(x, v) = \frac{v - ax - c}{b}\ , \ b \neq 0$$

1.6.2 p.74 16

$$y' = \sqrt{x + y + 1}$$

$$v = x + y + 1 \implies y = v - x - 1\ \&$$

$$\frac{dv}{dx} = 1 + \frac{dy}{dx} \implies$$

$$\frac{dv}{dx} = \sqrt{v} + 1 \text{ (separable)}$$

Def. A first order normal homogenous ODE has the form

$$\frac{dy}{dx} = f(y/x)$$

1.6.3 Ex

$$y' = \frac{xy}{x^2 + y^2}$$

In general, put

$$v = \alpha(x, y) = y/x \text{ (slope)}$$

so, $y = xv$ implies that

$$f(v) = f(y/x) = \frac{dy}{dx} = v + x\frac{dv}{dx} \implies$$

$$x\frac{dv}{dx} = f(v) - v \text{ (separable)}$$

$$\frac{1}{f(v) - v}\frac{dv}{dx} = \frac{1}{x} \implies$$

$$\int \frac{1}{f(v) - v} dv = \ln|x| + c$$

1.6.4 Ex (Revisited)

$$y' = \frac{xy}{x^2 + y^2} = \frac{y/x}{1 + (y/x)^2}, v = y/x \implies$$

$$\int \frac{1}{\frac{v}{1+v^2} - v} dv = \ln|x| + c \implies$$

$$-\int \frac{1 + v^2}{v^3} dv = \ln|x| + c \implies \dots$$

1.7 Date: 02.10.2015

Thrm. If $p(x, y) = \sum a_{i_1 i_2} x^{i_1} y^{i_2}$ and
$Q(x, y) = \sum a_{j_1 j_2} x^{j_1} y^{j_2}$ are polynomials
over $\mathbb{R}$, then if there is a $k \in \mathbb{Z}^+$ st for all i_1, i_2, j_1, j_2,

$$i_1 + i_2 = d = j_1 + j_2$$

then $p(x, y)y' = Q(x, y)$ is a 1st order linear
homogenous ODE.
Proof. notice that

$$y' \sum a_{i_1 i_2} x^{i_1} y^{i_2} = \sum a_{j_1 j_2} x^{j_1} y^{j_2} \implies$$

$$\frac{1}{x^d} y' \sum a_{i_1 i_2} x^{i_1} y^{i_2} = \frac{1}{x^d} \sum a_{j_1 j_2} x^{j_1} y^{j_2} \implies$$

$$y' \sum a_{i_1 i_2} \frac{y^{i_2}}{x^{d-i_1}} = \sum b_{j_1 j_2} \frac{y^{j_2}}{x^{d-j_1}} \implies$$

$$y' \sum a_{i_1 i_2} (\frac{y}{x})^{i_2} = \sum b_{j_1 j_2} (\frac{y}{x})^{j_2} \implies$$

$$y' = \frac{\sum a_{i_1 i_2} (\frac{y}{x})^{i_2}}{\sum b_{j_1 j_2} (\frac{y}{x})^{j_2}}$$

which is hom. endproof

Def. (i) deg $a_{i_1, i_2 \dots i_n} x_1^{i_1}, x_2^{i_2}, \dots x_n^{i_n} = \sum_{k=1}^{n} i_{ki}$

(ii) deg $((p(x_i))$
where $p(x_1, x_2, \dots, x_n = \sum a_{i_1 i_2 \dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$

1.7.1 ex p.74 2

$$2xyy' = x^2 + 2y^2 \implies$$

$$y' = (\frac{1}{2} \frac{1}{y/x} + 2(y/x)) \text{ , } v = y/x \implies$$

$$y = vx \implies y' = v + xv' \;\&\; xv' + v = \frac{1}{2v} + v \implies$$
$$v' = \frac{1}{2xv} \text{ (separable)} \implies$$
$$vv' = \frac{1}{2x} \implies$$
$$\int vdv = \frac{1}{2}\int \frac{1}{x}dx = \frac{1}{2}\ln|x| \implies$$
$$\frac{v^2}{2} = \frac{1}{2}\ln|x| + c \implies$$
$$v^2 = \ln|x| + c \implies$$
$$v = +- \sqrt{\ln|x| + c} \implies$$
$$\frac{y}{x} + -\sqrt{\ln|x| + c} \implies$$
$$y = +- x\sqrt{\ln|x| + c}$$

Def. A first order (normal)
berelli ODE has the

$$y' + yp(x) = y^n q(x)$$

side note(good book) Asmov PDE

1.8 Date: 02.11.2015

Recall: Bernulli ODE

$$y' + yp(x) = y^n q(x)$$

put $v = y^m$ then

$$\frac{dv}{dx} = my^{m-1}\frac{dy}{dx} \implies$$
$$my^{m-1}\frac{dy}{dx} + my^m p(x) = my^{m+n-1}q(x) \implies$$
$$\frac{dv}{dx}vmp(x) = my^{m+n-1}q(x)$$

want: $m + n - 1 = 0$. this requires
that $m = 1 - n$. $\therefore v = y^{1-n}$ reduses
a bernoulli ODE to a 1st order linear ODE.

1.8.1 p.74 25

$$y^2(xy' + y)(1 + x^4)^{1/2} = x$$
$$\implies (xy^2y' + y^3)\sqrt{1 + 4x} = x$$
$$\implies xy^2y'\sqrt{1 + x^4} + y^3\sqrt{1 + x^4} = x$$
$$\implies y'\sqrt{1 + x^4} + y\frac{\sqrt{1 + x^4}}{x} = y^{-2}$$
$$\implies y' + y\frac{1}{x} = y^{-2}\frac{1}{\sqrt{1 + x^4}}$$

put $v = y^3$ then

$$\frac{dv}{dx} = 3y^2\frac{dy}{dx} \implies$$
$$3y^2y' + \frac{3y^3}{x} = \frac{3}{\sqrt{1 + x^4}} \implies$$
$$v' + \frac{3v}{x} = \frac{3}{\sqrt{1 + x^4}} \text{ (linear)}$$

Here $p(x) = 3/x$; so

$$I(x) = e^{\int p(x)dx} = e^{3\ln|x|} = x^3$$

Thus,

$$v'x^{3'} + v3x^2 = \frac{3x^3}{\sqrt{1 + x^4}} \implies$$
$$(vx^3)' = \frac{3x^3}{\sqrt{1 + x^4}} \implies$$
$$vx^3 = 3\int\frac{x^3}{\sqrt{1 + x^4}}dx = \frac{3}{4}\int w^{-1/2}dw$$
$$\frac{3}{4}\frac{2}{1}w^{1/2} + c$$
$$\frac{3}{2}(\sqrt{1 + x^4} + c$$
$$w = 1 + x^4$$
$$w' = rx^3$$

$$y^3 = \frac{3}{2}(\frac{\sqrt{1+x^4}+c}{x^3}) \implies$$

$$y = (\frac{3}{2}(\frac{\sqrt{1+x^4}+c}{x^3}))^{1/3}$$

Exam 1 1. Given an ODE and a soln to it, verify it is a soln, then find a particular soln givin an initial cond.
2. Given a description of an ODE, write down the ODE.
3. Dropping ball from some h, find , ground time and speed.
4. high jump on earth given, find high jump on jupiter.
5. (a) solve ODEs
(b)
one is separable and the other is linear or bornulli
6. Torricelli problem.

Chapter 2

Second Test

Date: 02.17.15.

Ex

$$(2x \sin y \cos y)y' = 4x^2 + \sin^2 y$$
$$v = \sin y \implies$$
$$\frac{dv}{dx} = \cos y \frac{dy}{dx}$$

Thus,

$$2xvv' = 4x^2 + v^2 \text{ (hom)}, \; w = v/x$$

2.1 2.1 population models

Recall that the most basic population model, assuming constant birth and death rates, is

$$P' = kP \text{ (separable)}$$

Now, we give birth rates and death rates the following units:

$$\beta(t) = \text{ birth rate } \frac{\text{\# of births at t}}{\text{(unit of population at t)(unit of time)}}$$

$$\delta(t) = \text{ death rate } \frac{\text{\# of death at t}}{\text{(unit of population at t)(unit of time)}}$$

with this, for some small Δt,

$$P(t+\Delta t) \approx P(t) + \text{ (\# birth rate at t - \# death rate at t) } \Delta t \implies$$

$$= P(t) + \beta(t)P(t)\Delta t - \delta(t)P(t)\Delta t$$

$$\implies \frac{\Delta P}{\Delta t} = \frac{P(t+\Delta t) - P(t)}{\Delta t} \approx (\beta(t) - \delta(t))P(t)$$

using a diff model, the above suggest that

$$\boxed{P' = (\beta - \delta)P}$$

this is called the general population
Now, consider population with constant death rate, say δ_0, and with a birth rate given by

$$\beta = \beta_1 - \beta_0 P$$

In context, β_0 , $\beta_1 > 0$ by (1),

$$P' = (\beta_1 - \beta_0 P - \delta_0)P$$

$$= (\beta_1 - \delta_0)P - \beta_0 P^2$$

$$= \beta_0 P\Big(\frac{\beta_1 - \delta_0}{\beta_0} - P\Big)$$

which has form

$$P' = kP(M - P)$$

$k = \beta_0 > 0$ and $M = (\beta_1 - \delta_0)/\beta_0$ we see that in context, M>0 here (2) is separable; as

$$\frac{P'}{P(M-P)} = k \implies$$

$$\int \frac{1}{P(M-P)} dP = kt + C$$

Note that

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{M-P} + \frac{1}{P}\right)$$

Thus,

$$\frac{1}{M}(-\ln|M-P| + \ln|P|) = kt + C \implies$$

$$\ln\left|\frac{P}{M-P}\right| = Mkt + C$$

Now, $P_0 = P(0)$ yields

$$C = \ln\left|\frac{P_0}{M-P_0}\right|$$

Thus,

$$\ln\left|\frac{P}{M-P}\right| = \ln\left|\frac{P_0}{M-P_0}\right| + Mkt$$

Hence, if $P > M$ or $P > M$ then

$$\frac{P}{M-P} = \frac{P_0}{M-P_0}e^{Mkt} \implies \frac{M}{P} - 1 = \frac{M-P_0}{P_0}e^{-Mkt}$$

$$\implies \frac{M}{P} = \frac{P_0 + (M-P_0)e^{-Mkt}}{P_0}$$

$$\boxed{P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-Mkt}}}$$

Notice that

$$\lim_{t\to\infty} P(t) = \frac{MP_0}{P_0} = M$$

If $P < M$, so that $P_0 < M$, then

$$P' = kP(M-P) > 0$$

so, P is increasing to M

On the other hand, if $P > M$, so
that $P_0 > M$, then

$$P' = kP(M-P) < 0$$

so, P is decresing to M
Since $P(t) \to M$ in
either case, in context,
$M > 0$, ($M = 0$ & $P > M \implies$ *extinction*).

Date: 02.19.15. Recall: logistics population model

$$P(t) = \frac{MP_0}{P_0 + (M-P)e^{-Mkt}}$$

2.2 ex(P.88) # 22

$M = 100 \times 10^3$ total population
At $t = 0$, half the pop have heard a rumor
roughly the rumors increases
by 1000 people after 1 day

$$P_0 = 50 \times 10^3 \ \& \ P(1) = 51 \times 10^3$$

we can solve for k

$$51 \times 10^3 = \frac{(100 \times 10^3)(50 \times 10^3)}{50 \times 10^3 + (100 \times 10^3 - 50 \times 10^3)e^{-Mk}}$$

$$\implies 51 = \frac{5000}{50 + 50e^{-Mk}} = \frac{100}{1 + e^{-Mk}}$$

$$\implies \frac{51}{100} = \frac{1}{1 + e^{-Mk}} \implies$$

$$1 + e^{-Mk} = \frac{100}{51} \implies e^{-Mk} = \frac{100}{51} - 1 \implies$$

$$-Mk = \ln\left(\frac{100}{51} - 1\right)$$

$$\implies k = -\frac{\ln(\frac{100}{51} - 1)}{10 \times 10^3} > 0$$

$\therefore$ we can now solve $P(t) = 80 \times 10^3$
for t.

Doomsday/Extinction Model:

Here we assume that

$$\beta = kP, \ k > 0$$

$$\& \ \delta = \delta_0$$

Thus, the gen pop ODE, $P' = (\beta - \delta)P$,
becomes

$$(1) \ P' = (kP - \delta_0)P = kP(P - \delta/k)$$

put $M = \delta/k > 0$, then
(1) becomes

$$(2) \ P' = kP(P - M)$$

constant (2) with the logistics
ODE: $P' = kP(M - P)$ we can solve (2); it is separable. Thus,

$$\frac{P'}{P(P-M)} = k \implies$$

$$\int \frac{1}{P(P-M)} dP = kt + C$$

Note:

$$\frac{1}{P(P-M)} = \frac{1}{M}(\frac{1}{P-M} - \frac{1}{P})$$

$\therefore$

$$\frac{1}{M}(\ln|P-M| - \ln P) = kt + C$$

$$\implies \ln\left|\frac{P-M}{P}\right| = Mk + C$$

$$\implies \frac{|P-M|}{P} = e^{Mkt+C}$$

If $P_0 = P(0)$ then

$$C = \ln\left|\frac{P_0 - M}{P_0}\right|$$

so,

$$\frac{P-M}{P} = \frac{P_0 - M}{P_0} e^{Mkt}$$

Now, in any case

$$\frac{P-M}{P} = \frac{P_0 - M}{P_0} e^{Mkt} \implies$$

$$1 - \frac{M}{P} = \frac{P_0 - M}{P_0} e^{Mkt} \implies$$

$$\frac{M}{P} = \frac{P_0 - (P_0 - M)e^{Mkt}}{P_0} \implies$$

$$\boxed{P = \frac{MP_0}{P_0 + (M - P_0)e^{Mkt}}}$$

contrast with logistic ODE
soln:

$$P = \frac{P_0 M}{P_0 + (M - P_0)e^{-Mkt}}$$

Date: 02.23.15.

2.3 With explosion/extinction modol

Notice that

$$P > M \implies P' > 0 \implies P \text{ is increasing;}$$

$$P > M \implies P' < 0 \implies P \text{ is decreasing;}$$

Now, if $P > 0$, then P has a vertical asymptote at t_0 st.

$$P_0 + (M - P_0)e^{Mkt_0} = 0 \implies$$

$$e^{Mkt_0} = \frac{P_0}{P_0 - M} > 0 \implies$$

$$Mkt_0 = \ln \frac{P_0}{P_0 - M}$$

$$\boxed{t_0 = \frac{1}{Mk} \ln \left(\frac{P_0}{P_0 - M} \right)}$$

which is the time of "doomsday," i.e., the explosion of the population

$$\lim_{t \to t_0^-} P(t) = \infty$$

On the other hand, if $P < M$, then $M - P_0 > 0$; the

$$\lim_{t \to \infty} P(t) = 0$$

This is to say, over time, extinction occurs.

equilibuim solns and stability

Def. An autonomous ODE has the form

$$(1)\ \frac{dx}{dt} = f(x)$$

Notice that the slope field in (1) is "independent" of autonomous" of t.

Newtons law of cooling

$$T' = k(A - T)\ ,\ k > 0$$

Recall:

$$\int \frac{1}{A - T} dt = \int k dt = kt + C \implies$$

$$-\ln |A - T| = kt + C \implies$$

$$|A - T| = e^{-kt - C}$$

where $T_0 = T(0) \implies$

$$-C = \ln |A - T_0|$$

thus,

$$|A - T| = |A - T_0| e^{-kt} \implies$$

$$A - T = (A - T_0) e^{-kt} \implies$$

$$\boxed{T(t) = A + (T_0 - A) e^{-kt}}$$

Notice that

$$\lim_{t \to \infty} = A$$

Also, notice that $T(t) \equiv A$, i.e, $T(t) = A$ for all t, in a soln to the autonomous ODE $T' = k(A - T)$. this is an example of an "equilibruim soln"

Def. $x(t) \equiv C \in \mathbb{R}$ is an equilibrium soln to $x' = f(x)$ iff $x(t) \equiv C$ is a soln to $x' = f(x)$

Def. $x = C \in \mathbb{R}$ is a critical point of $x' = f(x)$ iff $f(c) = 0$

Notice that we say that "x is a critical pt" iff $0 = f(C) = \dfrac{dx}{dt}$, which is similar to use in calc I of "critical pt."

Prop. $x = C$ is a critixcal pt of $x' = f(x) \iff x(t) \equiv C$ is an equilibrium soln to $x' = f(x)$
proof. easy.

Def. $C \in \mathbb{R}$ is a stable critical point of $x' = f(x)$ iff C is a critical pt of $x' = f(x)$ and

$$\forall \epsilon > 0 \; \exists \; \delta > 0 \forall$$

$$|x_0 - C| < \delta \implies |x(t) - C| < \epsilon$$

Ex (Logistics Modle)

$$P' = kP(M - P) \implies$$

$$P(t) = \frac{MP_0}{p_0 + (M - P)e^{-Mkt}} \implies$$

$$\lim_{t\to\infty} P(t) = M$$

Ex(Standard pop Model)

$$P' = kP \implies P(t) = P_0 e^{kt}$$

Here $P = 0$ is a critical pt; however,
$P = 0$ is not stable. Notice that $P' = kP(M - P)$ has two critical pts, namely, $P = 0$ and $P = M$.
Here $P(t) = M$ is stable, whereas, $P(t) \equiv 0$ is not stable.

Ex(explosion/ extinction model)

$$P' = kP(P - M)\ ,\ k, M > 0$$

$$\implies P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{Mkt}}$$

Both $P = 0$ and $P = M$ are critical pts. however, if $P_0 > M$ then ? a stable, whereas if $P_0 < M$then only $P = 0$ is stable.

$$|x_0 - C| < \delta \implies$$

$$(x(t) - C) < \epsilon$$

Logistics Population Model with Harvesting

Date: 02.25.15.

Recall the Logistics Pop Model:

$$P' = kP(M - P),\, k,\, M > 0$$

We now consider

$$P' = kP(M - P) - h$$

where h is a constant, think: $h > 0 \implies$ harvesting;

$$h < 0 \implies \text{ stocking}$$

Notice that

$$P' = -kP^2 + kMP - h$$

$$= -k(P^2 - MP + h/k)$$

$$= -k(P - N)(P - H)$$

where

$$\text{H,N } = \frac{M \pm \sqrt{M^2 - 4h/k}}{2}$$

Here H and N are distinct reals if and only if $M^2 - 4h/k > 0 \iff M^2 > 4h/k \iff h < kM^2/4$. so, if $h > 0$ and H, N are distinct and real, then

$$0 < h < \frac{kM^2}{4}$$

Say, H<N. Notice that

$$h > 0 \implies$$

$$\sqrt{M^2 - 4h/k} < \sqrt{m^2} \implies$$

$$M - \sqrt{M^2 - 4h/k} > 0$$

$$\therefore 0 < H \text{ ; where, } 0 < H < N$$

Now, "separating," yields that

$$\frac{P'}{(P - H)(P - N)} = -k \implies$$

$$\int \frac{1}{(P-H)(P-N)} dP = -kt + C$$

Notice that

$$\frac{1}{(P-H)(P-N)} = \left(\frac{1}{P-N} - \frac{1}{P-H}\right)\frac{1}{N-H}$$

Thus,

$$\ln\left|\frac{P-N}{P-H}\right| = -(N-H)kt + C$$

If $P_0 = P(0)$ then

$$C = \ln\left|\frac{P_0 - N}{P_0 - H}\right|$$

Hence $|\frac{P-N}{P-H}| = e^c e^{-(N-H)kt} = |\frac{P_0-N}{P_0-H}|e^{-(N-H)kt}$ Now, in any case,

$$\frac{P-N}{P-H} = \frac{P_0-N}{P_0-H}e^{-(N-H)kt}$$

$$\lim_{t\to\infty}\frac{P-N}{P-H} = \lim_{t\to\infty}\frac{P_0-N}{P_0-H}e^{-(N-H)kt}$$

$$\therefore \lim_{t\to\infty}(P-N) = 0 \text{ or } \lim_{t\to\infty}(P-H) = \pm\infty$$

Q: Is $N < M$?
Q; $P(t) =$?

Recall: Date: 02.26.15

$$\frac{P-N}{P-H} = \frac{P_0-N}{P_0-H}e^{-(N-H)kt} \implies$$

$$(P-N)(P_0-H) = (P-H)(P_0-N)e^{-(N-H)kt} \implies$$

$$P(P_0-H) - N(P_0-H) = P(P_0-N)e^{-(N-H)kt} - H(P_0-N)e^{-(N-H)kt} \implies$$

$$P(P_0-H-(P_0-N)e^{-(N-H)kt}) = N(P_0-H) - H(P_0-N)e^{-(N-H)kt} \implies$$

$$\boxed{P(t) = \frac{N(P_0-H) - H(P_0-N)e^{-(N-H)kt}}{P_0-H-(P_0-N)e^{-(N-H)kt}}}$$

$\therefore$

$$\lim_{t\to\infty} P(t) = \frac{N(P_0-H)}{P_0-H} = N$$

Now, if $0 < h < kM^2/4$ then

$$0 < H < N < M$$

For recall that

$$H, N = \frac{M \pm \sqrt{M^2-4h/k}}{2}$$

so since $h > 0 \implies M^2 - 4h/k < M^2 \implies$

$$\sqrt{M^2-4h/k} < M \implies M + \sqrt{M^2-4h/k} < 2M$$

$$N = \frac{M+\sqrt{M^2-4h/k}}{2} < M$$

Here

$$N(P_0-H) - H(P_0-N)e^{-(N-H)kt_E} = 0$$

which has a soln if $P_0 < H$

Date: 03.02.15

2.4 Vertical Motion with Air Resistance

Recall Newton's 2nd Law:

$$ma = \sum F_i \text{ (net forces)}$$

Here $a = v' = dv/dt$, F_G (force due to gravity)
and F_R (force due to air resistance). Now,

$$F_G = -mg \;\&\; (F_R < 0 \iff v > 0)$$

where $g \approx 9.8m/s^2$ Empirically,

$$F_R = kv^p$$

where $1 \leq p \leq 2$ & $k > 0$
two cases, namely $p = 1$ & $p = 2$.

$p = 1$ Here, we have that

$$mv' = F_G + F_R \implies$$

$$mv' = -mg - kv$$

since $F_R = -kv$. Notice that

$$(1)\ v' = -g - \frac{k}{m}v$$

$$= -(\frac{k}{m}v + g)$$

Also, notice that (1) is a 1st order linear ODE,

$$v' + \frac{k}{m}v = -g$$

where $\rho = k/m$, called the drag constant. Thus

$$v' = (\rho v + g) \implies$$

(sup)

$$\frac{v'}{\rho v + g} = -1 \implies \int \frac{1}{\rho v + g} dv = -t + c$$

$$\implies \frac{1}{\rho} \ln|\rho v + g| = -t + c$$

$$\implies \ln|\rho v + g| = -\rho t + c$$

$$c = \ln|\rho v_0 + g|$$

$$|\rho v + g| = |\rho v_0 + g| e^{-\rho t}$$

$$\rho v + g = |\rho v_0 + g| e^{-\rho t} \implies$$

$$\boxed{v(t) = \frac{1}{\rho}((\rho v_0 + g)e^{-\rho t} - g)}$$

Notice that $\lim_{t \to \infty} v(t) = -g/\rho$; this is called terminal velocity. we denote this as

$$v_\tau = -g/\rho$$

Thus,

$$\boxed{v(t) = (v_0 - v_\tau)e^{-\rho t} + v_\tau}$$

Now,

$$x(t) = v_\tau t - \frac{1}{\rho}(v_0 - v_\tau)e^{-\rho t} + c \implies$$

$$c = x_0 + \frac{1}{\rho}(v_0 - v_\tau) \implies$$

$$\boxed{x(t) = x_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t})}$$

Date: 03.03.15 Recall that $F_R = \pm kv^p$, $F_G = -mg$, and

$$ma = \sum F = F_G + F_R \implies$$

$$(1)\ v' = -g \pm \frac{k}{m} v^p = -g \pm \rho v^p$$

where drag $\rho = k/m$.
p=2 there are 2 cases:
(i) upward motion, $F_R = -kv^2$;
(ii) downward motion, $F_R = kv^2$.

(i) Upward motion Here (1) becomes

$$v' = -g - \rho v^2 = -g(\frac{\rho}{g}v^2 + 1) = -g((v\sqrt{\frac{\rho}{g}})^2 + 1)$$

$$\implies \int \frac{1}{(v\sqrt{\rho/g})^2 + 1} dv = -gt + c \implies$$

$$\frac{1}{\sqrt{\rho/g}} \arctan(v\sqrt{\rho/g}) = -gt + c \implies \arctan(v\sqrt{\rho/g}) = -t\sqrt{\rho g} + c$$

$$\implies c = \arctan(v_0\sqrt{\rho/g}) \implies$$

$$v\sqrt{\rho/g} = \tan(\arctan(v_0\sqrt{\rho/g}) - t\sqrt{\rho g}) \implies$$

$$\boxed{v(t) = \sqrt{g/\rho}\tan(\arctan(v_0\sqrt{\rho/g}) - t\sqrt{\rho g})}$$

Thus,

$$x(t) = \sqrt{g/\rho}\frac{1}{\sqrt{\rho g}} \ln|\cos(\arctan(v_0\sqrt{\rho/g}) - t\sqrt{\rho g})| + c$$

$$\implies = \frac{1}{\rho} \ln|\cos(\arctan(v_0\sqrt{\rho/g} - t\sqrt{\rho g}| + c$$

$$\implies c = x_0 - \frac{1}{\rho} \ln|\cos(\arctan(v_0\sqrt{\rho/g}))| \implies$$

$$\boxed{x(t) = x_0 + \frac{1}{\rho} \ln\left|\frac{\cos(\arctan(v_0\sqrt{\rho/g}) - t\sqrt{\rho g}}{\cos(\arctan(v_0\sqrt{\rho/g}))}\right|}$$

Here $v(t) = 0$ allows us to find
time of max height, say t_m

$$\boxed{t_m = \frac{1}{\sqrt{\rho g}} \arctan(v_0\sqrt{\rho/g})}$$

(ii) Downward Motion

$$v' = -g + \rho v^2 = -g(1 - (v\sqrt{\rho/g})^2) \implies$$

$$\int \frac{1}{1 - (v\sqrt{\rho/g})^2} dv = -gt + c$$

Date: 03.04.15

$$\frac{1}{\sqrt{\rho/g}} \operatorname{arctanh}(v\sqrt{\rho/g} = -gt + c \implies$$

$$\operatorname{arctanh}(v\sqrt{\rho/g} = -\sqrt{\rho g}t + c \implies$$

$$c = \operatorname{arctanh}(v_0\sqrt{\rho/g}) \implies$$

$$v\sqrt{\rho/g} = \tanh(\operatorname{arctanh}(v_0\sqrt{\rho/g} - t\sqrt{\rho g} \implies$$

$$\boxed{v(t) = \sqrt{g/\rho}\tanh(\operatorname{arctanh}(v_0\sqrt{\rho/g}) - t\sqrt{\rho g})}$$

$\therefore$

$$x(t) = \sqrt{g/\rho}(-\frac{1}{\sqrt{\rho g}} \ln|\cosh(\operatorname{arctanh}(v_)\sqrt{\rho/g} - t\sqrt{\rho g})| + c$$

$$= -\frac{1}{\rho} \ln|\cosh(\operatorname{arctanh}(v_0\sqrt{\rho/g}) - t\sqrt{\rho g}| + c$$

$$\implies c = x_0 + \frac{1}{\rho} \ln|\cosh(\operatorname{arctanh}(v_0\sqrt{\rho/g}))|$$

$$\therefore \boxed{x(t) = x_0 - \frac{1}{\rho} \ln|\frac{\cosh(\operatorname{arctanh}(x_0\sqrt{\rho/g} - t\sqrt{\rho g}}{\cosh(\operatorname{arctanh}(v_0\sqrt{\rho/g}))}}$$

Thrm(Inverse Function trm). if $f'(x) \neq 0$ then f^{-1} is
diff at $y = f(x)$ and

$$\frac{df^{-1}}{dy}(y) = \frac{1}{\frac{df}{dx}(x)}$$

$$(\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}})$$

where $y = f(x) \iff x = f^{-1}(y)$

Aside:

$$\text{Ex } y = f(x) = \tanh x \implies$$

$$\frac{df^{-1}}{dy}(y) = \frac{1}{\frac{df}{dx}(x)} = \frac{1}{\operatorname{sech}^2 x} = \frac{1}{1 - \tanh^2 x} = \frac{1}{1 - y^2}$$

$$\text{Ex } \int \tanh x dx = \int \frac{\sinh x}{\cosh x} dx \int \frac{1}{u} du = \ln|\cosh x| + c$$

$u = \cosh x$
$u' = \sinh x$

Aside:

$$\cosh x = \frac{e^x - e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh' x = \sinh x$$

$$\sinh' x = \cosh x$$

$$\frac{f(x) \pm f(-x)}{2}$$

Ex

Date: 03.05.15

2.5 Escape Velocity

Recall Newton's Gravitational Law:

$$F = \frac{GmM}{r^2}$$

where $G \approx 6.67 \times 10^{-11}$. Let m be the mass of a projectile from a planet's surface of mass M of radius R. By Newton's 2nd Law,

$$ma = -\frac{GmM}{r^2} \implies$$

$$v' = -\frac{GM}{r^2}$$

By the chain rule,

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$$

Thus,

$$v\frac{dv}{dr} = -\frac{GM}{r^2} \text{ (separable)}$$

$$\int v dv = -GM \int r^{-2} dr \implies$$

$$\frac{v^2}{2} = \frac{GM}{r} + c \implies$$

$$c = \frac{v_0^2}{2} - \frac{GM}{r_0}$$

where $r_0 = R \therefore$

$$\frac{v^2}{2} = \frac{v_0^2}{2} - \frac{GM}{R} + \frac{GM}{r} \implies$$

$$v^2 = v_0^2 - \frac{2GM}{R} + \frac{2GM}{r}$$

$$> v_0^2 - \frac{2GM}{R}$$

To "escape" the gravitational force of the planet, we must have that $v > 0$ for all r. This happens if

$$v^2 > v_0^2 - \frac{2GM}{R} > 0 \iff$$

$$v_0^2 > \frac{2GM}{R} \implies$$

Ex (p.109) # 30

Newton's 2nd Law:

$$ma = \text{ net forces } = F_e + F_m$$

By Newton's Gravitational law,

$$F_e = -\frac{GmM_e}{r^2} \;\&$$

$$F_m = \frac{GmM_m}{(s-r)^2}$$

$\therefore$

$$\frac{dv}{dt} = \frac{GMm}{(s-r)^2} - \frac{GM_e}{r^2}$$

As before,

$$v\frac{dv}{dr} = G(\frac{M_m}{(s-r)^2} - \frac{M_e}{r^2})$$

which is separable. Thus,

$$\frac{v^2}{2} = G(M_m \int \frac{1}{(s-r)^2} dr - M_e \int \frac{1}{r^2} dr)$$

$$= G(\frac{M_m}{s-r} + \frac{M_e}{r} + c \implies$$

$$\frac{v_0^2}{2} = G(\frac{M_m}{s-R} + \frac{M_e}{R}) + c$$

$\implies$

$$\frac{v^2}{2} = G(\frac{M_m}{s-r} + \frac{M_e}{r}) + \frac{v_0^2}{2} - G(\frac{M_m}{s-R} + \frac{M_e}{R})$$

Recall: Date: 03.09.15

$$ma = \sum F_i \ , \ (r_0 = R)$$

$$(r_0 = R) \ , \ R \leq r \leq s$$

where

$$F_e = -\frac{GmM_e}{r^2} \ \& \ F_m = \frac{GmM_m}{(s-r)^2}$$

thus,

$$a = \frac{d^2 r}{dt^2} = F_m + F_e = \frac{GM_m}{(s-r)^2} - \frac{GM_e}{r^2}$$

By chain rule,

$$\frac{d^2 r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$$

so,

$$v\frac{dv}{dr} = \frac{GM_m}{(s-r)^2} - \frac{GM_e}{r^2} \text{ (sep)}$$

$$\frac{v^2}{2} = \frac{GM_m}{s-r} + \frac{GM_e}{r} + c \implies$$

$$c = \frac{v_0^2}{2} - \frac{GM_m}{s-R} - \frac{GM_e}{R}$$

Hence,

$$\frac{v^2}{2}\frac{GM_m}{s-r} + \frac{GM_e}{r} + \frac{v_0^2}{2} - \frac{GM_m}{s-R} - \frac{GM_e}{R}$$

we want $v > 0$

$$a = 0 \implies$$

$$\frac{GM_m}{(s-r)^2} = \frac{GM_e}{r^2} \implies$$

$$(\frac{s-r}{r}) = \frac{M_m}{M_e} \implies$$

$$\frac{s}{r} - 1 = \sqrt{M_m/M_e} \implies$$

$$\frac{s}{r} = 1 + \sqrt{M_m/M_e} = \frac{\sqrt{M_e}\sqrt{M_e}}{\sqrt{M_e}}$$

$$r = \frac{s\sqrt{M_e}}{\sqrt{M_m} + \sqrt{M_e}}$$

Also, notice that

$$v = 0 \implies$$

$$\frac{v_0^2}{2} = \frac{GM_m}{s-r} + \frac{GM_e}{R} - \frac{GM_m}{s-R} - \frac{GM_e}{r}$$

$\therefore$

$$v_0 = \sqrt{2G(\frac{M_m}{s-R} + \frac{M_e}{R} - \frac{M_m + M_e + 2\sqrt{M_m M_e}}{s})} \implies$$

$$\boxed{v_0 = \sqrt{2G(\frac{M_m}{s-R} + \frac{M_e}{R} - \frac{1}{s}(\sqrt{M_m} + \sqrt{M_e})^2)}}$$

Aside:

$$\frac{1}{r} = \frac{\sqrt{M_m} + \sqrt{M_e}}{s\sqrt{M_e}} \implies$$

$$\frac{M_m}{s-r} = \frac{M_m + \sqrt{M_m M_e}}{s}$$

$$\frac{M_e}{r} = \frac{M_e + \sqrt{M_m M_e}}{s}$$

Aside:

$$r\sqrt{M_m} + r\sqrt{M_e} = s\sqrt{M_e} \implies$$

$$r\sqrt{M_m} = \sqrt{M_e}(s-r) \implies$$

$$\frac{1}{s-r} = \frac{1}{r}\sqrt{\frac{M_e}{M_m}} \implies$$

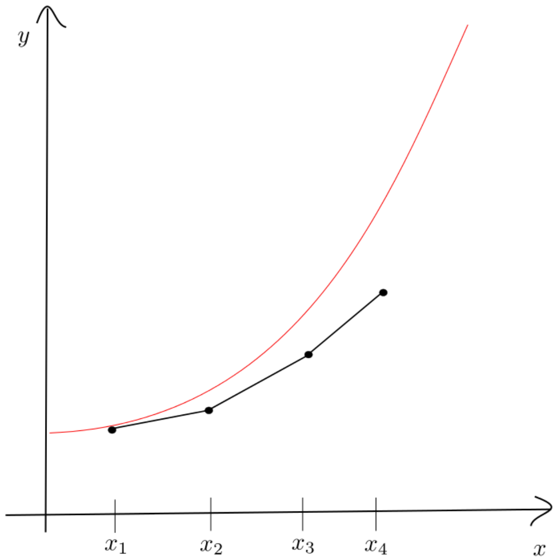
$$\frac{M_m}{s-r} = \frac{\sqrt{M_m M_e}}{r}$$

Euler's Method

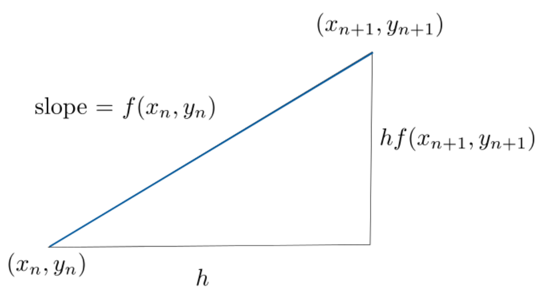
Given a slope field, $y' = f(x, y)$,
& a specific soln to the initial
value problem

$$\frac{dy}{dx} = f(x, y) \;\&\; (x_0, y_0)$$

say $y = y(x)$, then $y(x_0) = y_0$, &
Euler's method gives an algorithem
for estimating the exact soln $y = y(x)$



Find y_1 & y_{n+1} in general
slope $= f(x_0, y_0)$



$h = x_1 - x_0$

$$y = f(x_0, y_0)(x - x_0) + y_0 \implies$$

$$y_1 = f(x_0, y_0)h + y_0$$

In general,

$$\boxed{y_{n+1} = hf(x_n, y_n) + y_n}$$

Here

$$\boxed{y_n \approx y(x_n)}$$

Recall that an nth order linear ODE has the form Date: 03.11.15

$$(1)\ y^{(n)} + \sum_{k=1}^{n} Pk(x)y^{(n-k)} = f(x)$$

where Pk and f are cts for $1 \leq k \leq n$.
The associated homogeneous nth order linear ODE to (1) is
(2)

$$y^{(n)} + \sum_{k=1}^{n} Pk(x)y^{(n-k)} = 0$$

i.e., (1) with $f(x) \equiv 0$
Notation. put

$$V = y : I \to \mathbb{R} \mid \text{y has nth order derivative on I}$$

then V is an $\mathbb{R}$- linear space. put

$$W = y \in V \mid \text{y is a soln to (2)}$$

then the following holds

thrm. $W \leq V$, i.e., the set of all solns to (2) is a linear space.
Proof. If $y \in W$ & $c \in \mathbb{R}$ then

$$(cy)^{(n)} + \sum_{k=1}^{n} Pk(x)(cy)^{(n-k)} =$$

$$(cy)^{(n)} + c\sum_{k=1}^{n} Pk(x)y^{(n-k)}$$

$$c(y^{(n)} + \sum_{k=1}^{n} Pk(x)y^{(n-k)})$$

$$c \cdot 0 = 0$$

$\therefore cy \in W$
If $y_1, y_2 \in W$ then

$$(y_1 + y_2)^{(n)} + \sum_{k=1}^{n} Pk(x)(y_1 + y_2)^{(n-k)} =$$

$$y_1^{(n)} + \sum_{k=1}^{n} Pk(x)y_1^{(n-k)} + y_2^{(n)} + \sum_{k=1}^{n} Pk(x)y_2^{(n-k)} =$$
$$0 + 0 = 0$$

$\therefore y_1 + y_2 \in W$. Hence, $W \in \mathbb{R}$-linear subspace of V.

Recall: Thrm (wronskian thrm). if $f_1, ..., f_n$ are linearly independent in $c^{(n-1)}(I) = \{f : I \to \mathbb{R} \mid f^{(n-1)}$ is cts on I$\}$, then the wronskian of $f_1, ..., f_n$ is identically 0 for all $x \in I$, i.e., for all $x \in I$,

$$|W(f_1, ..., f_n)(x)| = det \begin{pmatrix} f_1(x) & \cdots & f_n(x) \\ f_1'(x) & \cdots & f_n'(x) \\ f_1''(x) & \cdots & f_n''(x) \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix} = 0$$

Aside

$$A\vec{x} = \vec{b}$$

the soln space of

$$A\vec{x} = \vec{0}$$

is a linear space. the soln space of

$$A\vec{x} = \vec{b}$$

is a affine linear space, with solns

$$\vec{x} = \vec{x_0} + \vec{x_1}$$

where $\vec{x_0}$ is any hom soln

Exam 2

1. hom ODE
2. ODE needing a subst to reduse to a 1st order linear/sep
3. exact ODE
4. population Model (logistic pop)
5. population Model (havesting a logistic pop)

Chapter 3

Third Test

Date: 03.23.15

Theorem. *(Existence - Uniqueness Thrm, $\exists!$ Thrm). If I is an interval, $a \in I$, and p_u for all $1 \leq k \leq n-1$, and f are cts on I, then $\forall b_i$, for $0 \leq i \leq n-1$, (then initial value problem)*

$$(1)\ y^{(n)} + \sum_{k=1}^{n} y^{(n-k)} p_u(x) = f(x)$$

& $y^{(i)}(a) = b_i$, has a unique soln on I.
Note: the solns to linear ODEs are unique on the whole interval I.

Theorem. *Thrm. If $w = \{y \in \mathscr{D}^{(n)}(I) \ / \ y$ satisfies (1)$\}$ then dim $W \geq n$.*

Proof. put $y_j^{(i)}(a) = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases}$$

for $1 \leq i, j \leq n$. By the $\exists!$ Thrm, for each j there is a unique soln to (1) on I. Now,

$$\begin{aligned} W(a) &= (y_j^{(i)}(a)) \in \text{ Mat}_n(\mathbb{R}) \\ &= I_n \end{aligned}$$

$$\therefore |W(a)| = 1 \neq 0$$

whence, $y_1, \ldots, y_n$ are linearly independent . Hence,

$$dimW \leq n$$

□

Date: 03.24.15

Theorem. *Thrm (Strong wronskian converse) If $y_1, \ldots, y_n$ are linearly independent (in $\mathscr{D}^{(n)}(I)$) and $y_1, \ldots, y_n$ are solns to*

$$y^{(n)} + \sum_{k=1}^{n} y^{(n-k)} p_k(x) = 0$$

where p_k are cts on I for $k = 1, \ldots, n$, then $|W(a) \neq 0$ for all $a \in I$.

Proof. Assume that there is an $a \in I$ st $|W(a)| = 0$, then the linear system

$$(1)\ W(a)\vec{x} = \vec{0}$$

has a nontrivial soln, say $\vec{x} = \vec{c} \in \mathbb{R}^n$
$(W(a) \in \mathrm{Mat}_n(\mathbb{R}))$. Denote: $\vec{c} = (c_1, \ldots, c_n) \neq \vec{0}$.

Put $y = \displaystyle\sum_{j=1}^{n} c_j y_j$, then y satisfies (*) and

$$y(a) = \sum_{j=1}^{n} c_j y_j(a) = 0 \implies$$

$$y'(a) = \sum_{j=1}^{n} c_j y_j'(a) = 0 \implies$$

$$y''(a) = \sum_{j=1}^{n} c_j y_j''(a) = 0 \implies$$

$$\vdots$$

$$y^{(n-1)}(a) = \sum_{j=1}^{n} c_j y_j^{(n-1)}(a) = 0 \text{ (by(1))}$$

Notice that $y \equiv 0$ on I also satisfies (*) and is such that $y^{(k)}(a) = 0$ for $1 \leq k \leq n-1$. $\therefore$ by $\exists!$ thrm,

$$\sum_{j=1}^{n} c_j y_j \equiv 0 \text{ on } I$$

Thus, since $y_1, \dots, y_n$ are lin ind, all $c_j = 0$, which is a contradiction. $\square$

Theorem. *If*

$$W = \{y \in \mathbb{R}^{(n)}(I) \ / \ y \textit{ satisfies (*) } \}$$

then dim $W \leq n$

Proof. let $y_1, \dots, y_n$ be lin ind solns of (*); we show that there are scalars $c_j \in \mathbb{R}$ st $y = \displaystyle\sum_{j=1}^{n} c_j y_j$ on I.
Consider the linear system

$$(2)\ W(a)\vec{x} = \begin{pmatrix} y(a) \\ y'(a) \\ \vdots \\ y^{(n-1)}(a) \end{pmatrix}$$

where $W(a)$ in the Wronskian matrix of $y_1, \dots, y_n$. By the SWC, $|W(a)| \neq 0$. Thus, (2) has a unique, not-trivial soln, say $\vec{0} \pm \vec{x} = \vec{c} \neq \vec{0} \in \mathbb{R}^n$. Thus Date: 03.25.15

$$W(a)\vec{c} = \begin{pmatrix} y(a) \\ y'(a) \\ \vdots \\ y^{(n-1)}(a) \end{pmatrix}$$

has rows

$$\sum_{j=1}^{n} c_j y_j(a) = y(a)$$
$$\sum_{j=1}^{n} c_j y_j'(a) = y'(a)$$
$$\sum_{j=1}^{n} c_j y_j''(a) = y''(a)$$
$$\vdots$$
$$\sum_{j=1}^{n} c_j y_j^{(n-1)}(a) = y^{(n-1)}(a)$$

Finally, since both y and $\sum_{j=1}^{n} c_j y_j$ are solns to the hom nth order linear ODE (*) (the ODE being home and y_j, for $1 \leq j \leq n$, being solns, so is any linear combo of y_js since the soln space to a linear home ODE is linear) and both y and $\sum_{j=1}^{n} c_j y_j$ satisfy all the same n many initial ?conbos?, by the $\exists!$ thrm, $y = \sum_{j=1}^{n} c_j y_j$ on I. $\therefore y \in \text{Span}\,\{y_j \in W \mid 1 \leq j \leq n\}$. Thus, dim $W \leq n$. $\square$

Corollary. *If*

$$W = \{y \in D^{(n)}(I) \ / \ y \textit{ satisfies (x) } \}$$

where

$$(*)\ y^{(n)} + \sum_{k=1}^{n} y^{(n-k)} p_k(x) = 0$$

then dim $W = n$.

Proof. We showed that $n \leq \ \dim W \leq n$. $\square$

Example. $y'' - y = 0$
Note that both $y_1 = \cosh x$ and $y_2 = \sinh x$ are solns on $\mathbb{R}$. The soln space is $2-$ dim'l . Not also that if

$$a \cosh x + b \sinh x = 0 \text{ for all } x \in \mathbb{R}$$

then in particular, $x = 0 \implies a = 0$.
$\therefore b \sinh x = 0$ for all x.
However, if $x \neq 0$ then $\sinh x \neq 0$; whence, $b = 0$. Hence, $B = \{\cosh x\ ,\ \sinh x\}$ are linearly independent solns to $y'' = y$, and the soln space has dim'n 2, B is a basis for the soln space. $\therefore$ every soln to $y'' = y$ has the form

$$y = a \cosh x + b \sinh x$$

for some $a, b \in \mathbb{R}$. Likewise, $B' = \{e^x, e^{-x}\}$ is also a basis. Note:

$$|W(x)| = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x \begin{vmatrix} 1 & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

$\therefore$ by Wronsky's Thrm, B' is lin ind.

Example. Example: $y'' + y = 0$
Here $B = \{\cos x, \sin x\}$ is a basis.

<u>nth order hom linear ODEs with constant coefficients:</u>

$$(1)\ \sum_{k=0}^{n} a_k y^{(k)} = 0$$

consider

$$y = e^{rx}$$

then

$$\begin{aligned} y' &= re^{rx} = ry \implies \\ y'' &= r^2 e^{rx} = r^2 y \implies \\ &\vdots \\ y^{(k)} &= r^k y \end{aligned}$$

thus, $y = e^{rx}$ yields

$$\begin{aligned} 0 &= \sum_{k=0}^{n} a_k y^{(k)} \\ &= \sum_{k=0}^{n} a_k r^k y \\ &= y \sum_{k=0}^{n} a_k r^{(k)} \implies \end{aligned}$$

$$\sum_{k=0}^{n} a_k r^{(k)} = 0$$

$\therefore y = e^{rx}$ is a soln to (1) if r is a root (zero) of the char poly of $\sum_{k=0}^{n} a_k y^{(k)} = 0$,

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$$\rho(x) = \sum_{k=0}^{n} a_k x^k,$$

then $y = e^{rx}$ is a soln to $\sum_{k=0}^{n} a_k y^{(k)} = 0$.

Example. If $r_1, \ldots, r_n \in \mathbb{R}$ are paiswise distinct then $e^{r_1 x}, \ldots, e^{r_n x}$ are linearly independent (in $\mathscr{F}(\mathbb{R})$. we use the Wronskian:

$$|W(x)| = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} & \cdots & e^{r_n x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} & \cdots & r_n e^{r_n x} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} e^{r_1 x} & r_2^{n-1} e^{r_2 x} & \cdots & r_n^{n-1} e^{r_n x} \end{vmatrix}$$

$$e^{(r_1 + r_2 + \cdots + r_n)x} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{vmatrix}$$

$$e^{(r_1+r_2+\cdots+r_n)x}\prod_{1\le i<j\le n}(r_j-r_i)\neq 0$$

(for all x). therefore by the wronskian thrm, $e^{r_1x}, e^{r_2x}, \ldots, e^{r_nx}$ ar lin ind,

Example. (Vandermonds)
consider

$$V_n(x)=\begin{pmatrix}1 & x & x^2 & \cdots & x^{n-1}\\ 1 & r_2 & r_2^2 & \cdots & r_2^{n-1}\\ 1 & r_3 & r_3^2 & \cdots & r_3^{n-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & r_n & r_n^2 & \cdots & r_n^{n-1}\end{pmatrix}$$

which is an $n\times n$ Vandermonde Matrix.
Put

$$f(x)=\det V_n(x)\in P_{n-1}\left[\mathbb{R}\right],$$

then $f(r_j)=\det(V_n(r_j))=0$ for all $2\le j\le n$, by the alternating property of determinates. therefore by the factor Thrm,

$$f(x)=a\prod_{2\le j\le n}(x-r_j),$$

where

$$a=(-1)^{n-1}\begin{vmatrix}1 & x & x^2 & \cdots & x^{n-1}\\ 1 & r_2 & r_2^2 & \cdots & r_2^{n-1}\\ 1 & r_3 & r_3^2 & \cdots & r_3^{n-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & r_n & r_n^2 & \cdots & r_n^{n-1}\end{vmatrix},$$

which is an $(n-1)\times(n-1)$ det. therefore by the inductive hypothesis,

$$a=(-1)^{n-1}\prod_{2\le i<j\le n}(r_j-r_i)$$

therefore

$$f(x)=(-1)^{n-1}\prod_{2\le i<j\le n}(r_j-r_i)\prod_{2\le j\le n}(x-r_j)$$

Notice that

$$\prod_{2\le j\le n}(x-r_j) = (-1)^{n-1}\prod_{2\le j\le n}(r_j-x).$$

Therefore

$$\begin{aligned} f(x) &= (-1)^{n-1}\prod_{2\le i\le j\le n}(r_j-r_i)(-1)^{n-1}\prod_{2\le j\le n}(r_j-x)\\ &= \prod_{2\le j\le n}(r_j-x)\prod_{2\le i\le j\le n}(r_j-r_i). \end{aligned}$$

In particular,

$$\begin{aligned} \det V_n(r_1) &= f(r_1)\\ &= \prod_{2\le j\le n}(r_j-r_1)\prod_{2\le j\le i\le n}(r_j-r_i)\\ &= \prod_{1\le i\le j\le n}(r_j-r_i). \end{aligned}$$

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Recall: the characteristic poly of $\sum_{k=0}^{n} a_k y^{(k)}$ is

$$\rho(x) = \sum_{k=0}^{n} a_k x^{(k)},$$

and we showed that if $r_1,\ldots,r_n$ are pairwise distinct then $B = \{e^{r_1x},\ldots,e^{r_nx}\}$ is linearly independent (in $\mathscr{C}^{(\infty)}(\mathbb{R})$). We also showed that if r is a zero of $\rho(x)$ then $y = e^{rx}$ is a soln to $\sum_{k=0}^{n} a_k y^{(k)} = 0$. This all immediatly implies the following:

Theorem. *If $\rho(x)\sum_{k=0}^{n} a_k x^{(k)}$ has n many distinct real zeros then the soln space of*

$$(*)\ \sum_{k=0}^{n} a_k y^{(k)} = 0$$

has basis $\{e^{r_1x}, \ldots, e^{r_nx}\}$, *i.e., every soln of* $*$ *has the form*

$$\sum_{j=1}^{n} c_j e^{r_jx}.$$

Example. (Revisited)
Consider $y'' - y = 0$. This has char poly

$$\rho(x) = x^2 - 1 = (x-1)(x+1),$$

which has zeros ± 1.
Thus, $B = \{e^x, e^{-x}\}$ is a basis for the soln space of

$$y'' - y = 0$$

Q: What are the mising basis elements if a char poly has repeating zeros?

Recall the "ring" of polynomials

$$\mathbb{R}[x] = \{\sum_{k=0}^{n} a_k x^k \mid n \in \mathbb{N} \ \& \ a_k \in \mathbb{R}\},$$

where

$$\sum_{k=0}^{n} a_k x^{(k)} + \sum_{k=0}^{n} b_k x^{(k)} = \sum_{k=0}^{n} (a_k + b_k) x^{(k)}$$

$$\sum_{i=0}^{m} a_i x^{(i)} \sum_{j=0}^{m} b_j x^{(j)} = \sum_{k=0}^{m+n} c_k x^{(k)}, \text{ where}$$

$$c_k = \sum_{i+j=k} a_i b_j.$$

Notation. Put $D = \dfrac{d}{dx} : \mathscr{C}^{(\infty)}(I) \to \mathscr{C}^{(\infty)}(I)$
$D^0 = i$ (identity on $\mathscr{C}^{(\infty)}(I)$), and

$$D^n = DD^{n-1} \text{ for } n \in \mathbb{Z}^+.$$

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where $Dy = D(y) = \dfrac{dy}{dx}$

Condider $L = \sum_{k=0}^{n} a_k 0^k$, where $a_k \in \mathbb{R}$ and $D^k = \dfrac{d^k}{dx^k}$. Note that $L : \mathscr{C}^{(\infty)}(I) \to \mathscr{C}^{(\infty)}(I)$, where

$$L_y = L(y) = \sum_{k=0}^{n} a_k D^k y =$$

$$a_0 y + a_1 y' + a_2 y'' + \cdots + a_n y^{(n)}.$$

we show that L is a linear operation:

$$\begin{aligned} L(y_1 + y_2) &= \sum_{k=0}^{n} a_k D^k (y_1 + y_2) \\ &= \sum_{k=0}^{n} a_k (D^k y_1 + D^k y_2) \\ &= \sum_{k=0}^{n} (a_k D^k y_1 + a_k D^k y_2) \\ &= L_{y_1} + L_{y_2} \end{aligned}$$

Also,

$$\begin{aligned} L(cy) &= \sum_{k=0}^{n} a_k D^k (cy) \\ &= \sum_{k=0}^{n} a_k (c D^k y) \\ &= c \sum_{k=0}^{n} a_k D^k y \\ &= c L_y. \end{aligned}$$

Put

$$\mathbb{R}[D] = \{\sum_{k=0}^{n} a_k D^k \mid a_k \in \mathbb{R} \ \& \ n \in \mathbb{N}\}.$$

We show that $\mathbb{R}[x] = R[D]$, isomorphic as "rings," which includes multiplication. Addition is $\mathbb{R}[D]$ is defined as

$$L_1, L_2 \in \mathbb{R}[D] \ , \ (L_1 + L_2)y = L_1 y + L_2 y;$$

Multiplication in $\mathbb{R}[D]$ is defined as

$$L_1, L_2 \in \mathbb{R}[D] \ , \ (L_1 L_2)y = (L_1 \cdot L_2)$$

Define $h : \mathbb{R}[x] \to \mathbb{R}[D]$ Such that

$$h\left(\sum_{k=0}^{n} a_k x^k\right) = \sum_{k=0}^{n} a_k D^k$$

For example, $1 \mapsto D^0 =$ Identify function in $\mathscr{C}^{(\infty)}(I)$,

$$x \mapsto D, x^2 \mapsto D^2, \ldots, x^n \mapsto D^n.$$

We show that h is a "ring" homorphism, i.e., if $f_1 = \sum_{k=0}^{m} a_k x^k$, $f_2 = \sum_{k=0}^{n} b_k x^k \in \mathbb{R}[x]$, and $L_1 = \sum_{k=0}^{m} a_k D^k$, $L_2 = \sum_{k=0}^{n} b_k D^k \in \mathbb{R}[D]$, then

(i) $h(f_1 + f_2) = L_1 + L_2$;
(ii) $h(f_1 f_2) = L_1 L_2$;

To see (i), notice that

$$\begin{aligned} f_1 + f_2 &= \sum_{k=0}^{n} a_k x^k + \sum_{k=0}^{n} b_k x^k \\ &= \sum_{k=0}^{n} (a_k + b_k) x^k \end{aligned}$$

So,

$$
\begin{aligned}
h(f_1 + f_2 &= \sum_{k=0}^{n}(a_k + b_k)D^k \\
&= \sum_{k=0}^{n}(a_k D^k + b_k D^k) \\
&= \sum_{k=0}^{n} a_k D^k + \sum_{k=0}^{n} b_k D^k) \\
&= L_1 + L_2
\end{aligned}
$$

Also, notice that

$$
f_1 f_2 = \sum_{k=0}^{n} c_k x^k \ , \ where c_k = \sum_{i+j=k} a_i b_j
$$

So,

$$
h(f_1 f_2) = \sum_{k=0}^{n} c_k D^k.
$$

Now,

$$
\begin{aligned}
c_k D^k &= \left(\sum_{i+j=k} a_i b_j \right) D^k \\
&= \sum_{i+j=k} (a_i b_j D^k) \\
&= \sum_{i+j=k} a_i D^i b_j D^j
\end{aligned}
$$

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$$
\begin{aligned}
L_1 L_2 y &= (L_1 \circ L_2)(y) \\
&= L_1(L_2(y)) \\
&= \sum_{i=0}^{m} a_i D^i L_2(y) \\
&= \sum_{i=0}^{m} a_i D^i \left(\sum_{j=0}^{n} b_j D^j y \right) \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} a_i b_j D^{i+j} y.
\end{aligned}
$$

therefore

$$L_1 L_2 = \sum_{i=0}^{m} \sum_{j=0}^{n} a_i b_j D^{i+j}$$

Now, notice that from above

$$L_1 L_2 = \sum_{k=0}^{m+n} c_k D^k \ , \ c_k = \sum_{i+j=k} a_i b_j.$$

therefore

$$h(f_1 f_2) = \sum_{k=0}^{m+n} c_k D^k = L_1 L_2 = h(f_1) h(f_2),$$

$c_k = \sum_{i+j=k} a_i b_j$. Hence, h is a "ring" homomorphism. Clearly, h is onto, We show that h is 1-1.
Firstly, recall that

$$ker(h) = \{ f \in \mathbb{R}[x] \mid h(f) = 0 \in \mathbb{R}[D] \}$$

Also, h is 1-1 $\iff ker(h) = \{0\}$.

Secondly, if $n = 0$ then

$$a_0 D^0 = 0 \in \mathbb{R}[D] \implies$$

$$a_0 D^0 y = 0 \in \mathscr{C}^\infty(I) \text{ for all } y \in \mathscr{C}^\infty(I)$$

$$\implies a_0 y = 0 \text{ for all } y \in \mathscr{C}^\infty(I)$$

However, if y = 1 on I then

$$a_0 = 0$$

Date: 04.06.15

$$\therefore a_0 x^0 \in \mathbb{R}[x].$$

Now, assume that

$$\sum_{k=0}^{n} a_k x^k \mapsto 0 \in \mathbb{R}[D] \implies$$

$a_k = 0$ for $0 \leq k \leq n$ (inductive hypothesis). If

$$h\left(\sum_{k=0}^{n} a_k x^k\right) = 0 \in \mathbb{R}[D],$$

then

$$\sum_{k=0}^{n} a_k D^k = 0 \in \mathbb{R}[D] \implies$$

$$(1)\ \sum_{k=0}^{n} a_k y^k = 0 \in \mathscr{C}^{(\infty)} \forall y \in \mathscr{C}^{(\infty)}(I)$$

Thus, if $a_{n+1} \pm 0$ then
Take $y \equiv 1 \in \mathscr{C}^{(\infty)}(I)$, then (1) becomes $a_0 y \equiv 0 \in \mathscr{C}^{(\infty)}(I)$; thus, $a_0 = 0$. Thus,

$$(2)\ \sum_{k=1}^{n+1} a_k y^{(k)} = 0 \in \mathscr{C}^{(\infty)}(I)$$

take $y \equiv x \in \mathscr{C}^{(\infty)}(I)$, then (2) becomes $a_1 y' \equiv 0 \in \mathscr{C}^{(\infty)}(I)$; thus, $0 = a_1$, $y'(x) = a_1$ for all $x \in I$. therefore $a_1 = 0$. Thus,

$$(3)\ \sum_{k=2}^{n+1} a_k y^{(k)} = 0 \in \mathscr{C}^{(\infty)}(I)$$

Take $y = x^2, etc, then\ (n+1)\ a_{n+1}y^{n+1} = 0 \in \mathscr{C}^{(\infty)}(I)$
Take $y \equiv x^{n+1}$, then as above, $a_{n+1} = 0$, a contradiction. $\therefore a_{n+1} = 0$. As such, (1) becomes

$$\sum_{k=0}^{n} a_k y^k = 0 \in \mathscr{C}^{(\infty)}(I)$$

Hence, by the inductive hypo, $a_k = 0$ for $0 \leq k \leq n$. therefore $ker(h) = (0)$; whence, h is an isomorphis. therefore $\mathbb{R}[x] \equiv \mathbb{R}[D]$ as "rings" Date: 04.07.15
Was not at school this day Date: 04.08.15
Recall: $B = \{x^k e^{rx} \in {}^{(\infty)}(\mathbb{R}) \mid 0 \leq k \leq m-1$
where $y \in B$ are solns to $(D-r)^m y = 0$.
B is linearly independent since

$$\sum_{k=0}^{m-1} c_k x^k e^{rk} = 0 \implies \sum_{k=0}^{m-1} c_k x^{k-1} = 0 \implies$$

$c_k = 0$ for all $0 \leq k \leq m-1$ this is because $\left\{1, x, x^2, \ldots, x^{m-1}\right\} \subset \mathbb{R}[x]$ is linearly independence. therefore we have found the missing basis elements.
In summary, if

$$\sum_{k=0}^{n} a_k y^{(k)} = 0 \ \&\ \rho p(x) = a_n \prod_{k=0}^{m} (x - r_k)^{m_k}$$

where $r_k \in \mathbb{R}$ for $1 \leq k \leq m$, then the soln space of $\displaystyle\sum_{k=0}^{n} a_k y^{(k)} = 0$ has basis

$$B = \{x^j e^{r_k x} \in {}^{(\infty)}(\mathbb{R}) \mid 0 \leq j \leq m-1 \ \&\ 1 \leq k \leq m\}$$

Note: since $\displaystyle\sum_{k=1}^{m} m_k = n$, $cardB = n$

Example. say $\displaystyle\sum_{k=0}^{n} a_k y^{(k)} = 0$ has the factored form

$$(D-2)^3(D+1)(D-4)^2 y = 0$$

then basis elements are :

$$e^{2x}, xe^{2x}, x^2e^{2x}, e^{-x}, e^{4x}, xe^{4x}.$$

Exam 3
hw up through 3.3, excluding problems involving $\mathbb{C}$ - valued zeros
1a Given functions, prove they are lin independent
1b Verify these are solns to a given lin ODE
1c Solve the linear ODE initial value prob
2 given a 4th order lin hom ODE find a basis for its soln space, find the gen soln
3 Find a basis for a 2nd order hom ODE; find the gen soln 3cont Given a particular soln to the linear non-hom ODE, find the gen soln to the non-hom lin ODE

$$Ly = 0$$

$$\sum a_k y^{(k)} = 0$$

$$Ly = f$$

Books

Hausen and sullivan "Real Analysis"
Reference: I.N. Herstein's "Topics in Abstract Algebra" (2nd ed), "Abstract Algebra" (3rd ed)
Artin's "Algebra"
(Formal poly : See hungerfords springer GTM.)

Chapter 4

Fourth Test

Date: 04.13.15

Here we address the possibility of the char poly $\rho(x)$ of $\sum_{k=0}^{n} a_k y^{(k)} = 0$ with complex zeros. So, assuming that $r = a + bi$ is a zero of $\rho = \sum_{k=0}^{n} a_k x^k$, then we know that $y = e^{rk} = e^{(a+bi)x} = e^{ax}e^{(bx)i}$. Recall that if $a_k \in \mathbb{R}$ for $0 \leq k \leq n$, i.e., $\rho(x) \in \mathbb{R}[x]$, then $\bar{r} = a - bi$ ($b \pm 0$ is also a zero of $\rho(x)$. Thus, in the case of $r = a \pm bi$, we are "missing" now not just one real-valued basis element, but two real valued basis elements.
Euler's identity enabels us to determine there two missing basis elements:

$$(1)\ e^{ix} = \cos(x) = i\sin(x) \in \mathbb{C} \text{ for all } x \in \mathbb{R}$$

what led Euler to this identity are the Maclarin series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{n\geq 0} \frac{x^n}{n!};$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^10}{10!} + \cdots = \sum_{n\geq 0} (-1)^n \frac{x^{2n}}{2n!};$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^11}{11!} + \cdots = \sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{(2n+1)!};$$

These hold for all $x \in \mathbb{R}$. Notice that (1) simply gives pts on the unit circle in $\mathbb{C}$

$$|e^{ix}| = 1 \text{ for all } x \in \mathbb{R}$$

Note: $e^{i\pi} + 1 = 0$. $(x = \pi)$
Observe that e^{ix} is a complex-valued scalar mult of $\cos x$ & $\sin x$. So, back to the soln

$$\begin{aligned} y &= e^{ax} e^{(bx)i} \\ &= e^{ax}(\cos bx + i \sin bx) \\ &= e^{ax} \cos x + i(e^{ax} \sin bx), \end{aligned}$$

which is likewise a complex-valued linear combo of $e^{ax} \cos bx$ & $e^{ax} \sin bx$. These are to real-balued soln to the ODE $\displaystyle\sum_{k=0}^{n} a_k y^{(k)} = 0$, i.e., there are the two missing basis elements. therefore if $(x - r)^m$, where $r = a + bi$, is a factor of $\rho(x)$, then the corespoinding basis elements are:

$$\begin{gathered} e^{ax} \cos bx \ , \ e^{ax} \sin bx \\ x e^{ax} \cos bx \ , \ x e^{ax} \sin bx \\ x^2 e^{ax} \cos bx \ , \ x^2 e^{ax} \sin bx \\ \vdots \vdots \\ x^{m-1} e^{ax} \cos bx \ , \ x^{m-1} e^{ax} \sin bx \end{gathered}$$

Here there are the missing 2m many basis elements. Date: 04.14.15

Example.

$$y'' + y' - y = \sin^2 x$$

Here $\displaystyle\sum_{k=0}^{n} y^{(k)} = f(x)$, where $n = 2$ and $f(x) = \sin^2 x$. Notice that

$$\begin{aligned} y &= \sin^2 x \implies \\ y' &= 2 \sin x \cos x \implies \\ y'' &= 2 \cos^2 x - 2 \sin^2 x. \end{aligned}$$

$$
\begin{aligned}
y &= \sin^2 x; \\
y' &= 2\sin x \cos x = \sin 2x \\
y'' &= 2\cos^2 x - 2\sin^2 x = 2\cos 2x.
\end{aligned}
$$

the terms in there derivatives are

$$
\begin{aligned}
1, \\
\sin^2 x, \\
\sin x \cos x \\
\cos^2 x.
\end{aligned}
$$

Another way to see these terms is as

$$
\begin{aligned}
1, \\
\cos 2x \\
\sin 2x.
\end{aligned}
$$

We consider a posible particular soln

$$y_p = a + b\cos 2x + c\sin 2x,$$

which is a linear combo of the terms above. Note that y_p is a particular soln iff

$$y_p'' + y_p' - y_p = \sin^2 x$$

Now,

$$
\begin{aligned}
y_p &= a + b\cos 2x + c\sin 2x \implies \\
y_p' &= -2b\sin 2x + 2c\cos 2x \implies \\
y_p'' &= -4b\cos 2x - 4c\sin 2x \implies
\end{aligned}
$$

$$y_p'' + y_p' - y_p = \sin^2 x \iff$$

$$
\begin{gathered}
(2c-4b)\cos 2x-(2b+4c)\sin 2x-b\cos 2x-c\sin 2x-a=\sin^2 x \iff \\
(2c-5b)\cos 2x-(2b+5c)\sin 2x-a=\frac{1-\cos 2x}{2} \iff \\
-a-1/2+(2c-5b+1/2)\cos 2x-(2b+5c)\sin 2x=0.
\end{gathered}
$$

This last equation is an identity if

$$
a=-1/2 \ \& \ \begin{cases} -5b+2c+1/2 & =0 \\ 2b+5c & =0. \end{cases}
$$

Note:

$$
\begin{cases} -10b+4c & =-1 \\ 2b+5c & =0 \end{cases}
$$

$$
\begin{bmatrix} 10 & 25 & | & 0 \\ -10 & 4 & | & -1 \end{bmatrix}
$$

$$
\begin{aligned}
c &= -1/29 \\
b &= 5/58
\end{aligned}
$$

Therefore

$$
y_p=-1/2+\frac{5}{58}\cos 2x-\frac{1}{29}\sin 2x,
$$

Which you have cheked is an actual soln to $y''+y'-y=\sin^2 x$.

Date: 04.15.15

Example.

$$
y''+y'-y=\sec x
$$

$$
\begin{aligned}
f(x) &= \sec x \\
f'(x) &= \sec x\tan x \\
f''(x) &= \sec x\tan^2 x+\sec^3 x \\
&= \sec x(\sec^2 x-1)+\sec^3 x \\
&= 2\sec^3 x-\sec x
\end{aligned}
$$

Put

$$B = \{1, \sec x, \sec x \tan x, \sec^3 x\}$$

Also, put

$$y_p = a + b \sec x + c \sec x \tan x + d \sec^3 x$$

$$y_p'$$

Since $\sec^3 x \tan x \notin$ Span B, the method of undetermined coefis fails.

Idea: Consider $B = \{y_1, \ldots, y_n\}$, the basis of $\sum_{k=0}^{n} a_k y^{(k)} = 0$, and

$$(1)\ y_p = \sum_{k=1}^{n} v_k(x) y_k(x).$$

We will show that a particular soln of $\sum_{k=0}^{n} a_k y^{(k)} = f(x)$ always has form (1) for some $v_k(x) \in C^{(n)}(I)$. this is called " Variation of parameters."

Date: 04.16.15

We want a particular soln $y = v_1(x)y_1(x) + v_2(x)y_2(x)$ to $\sum_{k=1}^{2} a_k y^{(k)} = f(x)$, where the hom soln space has basis $B = \{y_1, y_2\} \subset C^{(2)}(I)$. Here we require

$$y_p'' + p y_p' + q y_p = f(x) \text{ (monic)}.$$

$$y_p = v_1 y_1 + v_2 y_2 \implies$$

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$

If $v_1' y_1 + v_2' y_2 = 0$ then

$$y_p' = v_1 y_1' + v_2 y_2'$$

Thus,

$$y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2'v_2y_2''$$

$$y_p'' + py_p' + qy_p = f(x) \iff$$

$$v_1'y_1' + v_1y_1'' + v_2'y_2'v_2y_2'' + pv_1y_1' + pv_2y_2' + qv_2y_1 + qv_2y_2 = f(x) \iff$$

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1'y_1' + v_2'y_2' = f(x) \iff$$

$$v_1'y_1' + v_2'y_2' = f(x).$$

therefore
$\begin{cases} y_1'v_1' + y_1'v_2' = f(x) \\ y_1v_1' + y_2v_2' = 0 \end{cases}$
iff

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore

$$v_1 = -\int \frac{y_2(x)f(x)}{|w(x)|}dx$$

and

$$v_2 = \int \frac{y_1(x)f(x)}{|w(x)|}dx$$

Hence,

$$y_p(x) =$$

Now, in general, consider

$$\sum_{k=0}^{n} a_k y^{(k)} = f(x)$$

with basis $B = \{y_j \in C^{(n)}(I) | 1 \leq j \leq n\}$ for the assoc hom ODE. Also, consider

$$y_p = \sum_{j=1}^{n} v_j y_j,$$

then

$$\begin{aligned} y_p' &= \sum_{j=1}^{n} (v_j' y_j + v_j y_j') \\ &= \sum_{j=1}^{n} v_j y_j' \text{ , if } \sum_{j=1}^{n} v_j' y_j = 0. \end{aligned}$$

Now,

$$\begin{aligned} y_p'' &= \sum_{j=1}^{n} (v_j' y_j' + v_j y_j'') \\ &= \sum_{j=1}^{n} y_j y_j'' \text{ , if } \sum_{j=1}^{n} v_j' y_j' = 0 \end{aligned}$$

continuing,

$$y_p^{(k)} = \sum_{j=1}^{n} v_j y_j^{(k)} \text{ , if } \sum_{j=1}^{n} v_j' y_j^{(k-1)} = 0$$

for $1 \leq k \leq n-1$. Finally,

$$\begin{aligned} y_p^{(n)} &= \sum_{j=1}^{n} (v_j' y_j^{(n-1)} + v_j y_j^{(n)}) \\ &= \sum_{j=1}^{n} v_j' y_j^{(n-1)} + \sum_{j=1}^{n} v_j y_j^{(n)}) \end{aligned}$$

Therefore

$$\sum_{k=1}^{n} a_k y_p^{(k)} = f(x) \iff$$

$$a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} + a_n \sum_{j=1}^{n} v_j y_j^{(n)} + \sum_{k=1}^{n-1} a_k \sum_{j=1}^{n} v_j y_j^{(k)} = f(x) \iff$$

$$a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} + \sum_{j=1}^{n} a_n v_j y_j^{(n)} + \sum_{j=1}^{n} \sum_{k=1}^{n=1} a_k v_j y_j^{(k)} = f(x) \iff$$

$$a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} + \sum_{j=1}^{n} v_j \sum_{k=1}^{n} a_k y_j^{(k)} = f(x) \iff$$

$$a_n \sum_{j=1}^{n} v_n' y_j^{(n-1)} = f(x)$$

since $y_j \in B$. Thus,

$$\begin{aligned}
\sum_{j=1}^{n} v_j' y_j &= 0 \text{ 1st eqn} \\
\sum_{j=1}^{n} v_j' y_j' &= 0 \text{ 2nd eqn} \\
&\vdots \\
\sum_{j=1}^{n} v_j' y_j^{(k-1)} &= 0 \text{ kth eqn} \\
&\vdots \\
\sum_{j=1}^{n} v_j' y_j^{(n-2)} &= 0 \text{ (n-1)th eqn} \\
a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} &= f(x) \text{ nth eqn}
\end{aligned}$$

Now,

$$\begin{aligned}
y_p' &= v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2' \implies \\
y_p'' &= v_1'' y_1 + v_1' y_1' + v_1' y_1' + v_1 y_1'' + v_2'' y_2 + v_2' y_2' + v_2' y_2' + v_2 y_2'' \\
&= 2(v_1' y_1' + v_2' y_2') + v_1 y_1'' + v_2 y_2'',
\end{aligned}$$

if we impose the condition $v_1'' y_1 + v_2'' y_2 = 0$. therefore

$$\begin{aligned}
f(x) &= y_p'' + p y_p' + q y_p \\
&= 2(v_1' y_1' + v_2' y_2') + v_1 y_1'' + v_2 y_2'' + p v_1' y_1 + p v_1 y_1' + p v_2' y_{1p} v_2 y_2' + q_v y_1 + q v_2 y_2 \\
&= v_1(y_1'' + p y_1' + q y_1) + v_2(y_2'' + p y_2' + q y_2) + 2(v_1' y_1' + v_2' y_2') + p(v_1' y_1' + v_2' y_2) \\
&= 2(v_1' y_1' + v_2' y_2') + p(v_1' y_1 + v_2' y_2)
\end{aligned}$$

therefore,

$$f(x) = 2(v_1' y_1' + v_2' y_2') + p(v_1' y_1 + v_2' y_2)$$

implies

$$f(x) = v_1'(2y_1' + p y_1) + v_2'(2y_2' + p y_2)$$

Date: 04.20.15

We want a particular soln $y = v_1(x)y_1(x) + v_2(x)y_2(x)$ to $\sum_{k=1}^{2} a_k y^{(k)} = f(x)$, where the hom soln space has basis $B = \{y_1, y_2\} \subset C^{(2)}(I)$. Here we require

$$y_p'' + py_p' + qy_p = f(x) \text{ (monic).}$$

$$y_p = v_1y_1 + v_2y_2 \implies$$

$$y_p' = v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2'$$

If $v_1'y_1 + v_2'y_2 = 0$ then

$$y_p' = v_1y_1' + v_2y_2'$$

Thus,

$$y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2'v_2y_2''$$

$$y_p'' + py_p' + qy_p = f(x) \iff$$

$$v_1'y_1' + v_1y_1'' + v_2'y_2'v_2y_2'' + pv_1y_1' + pv_2y_2' + qv_2y_1 + qv_2y_2 = f(x) \iff$$

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1'y_1' + v_2'y_2' = f(x) \iff$$

$$v_1'y_1' + v_2'y_2' = f(x).$$

therefore

$$\begin{cases} y_1'v_1' + y_1'v_2' = f(x) \\ y_1v_1' + y_2v_2' = 0 \end{cases}$$

iff

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore

$$v_1 = -\int \frac{y_2(x) f(x)}{|w(x)|} dx$$

and

$$v_2 = \int \frac{y_1(x) f(x)}{|w(x)|} dx$$

Hence,

$$y_p(x) =$$

Now, in general, consider

$$\sum_{k=0}^{n} a_k y^{(k)} = f(x)$$

with basis $B = \{y_j \in C^{(n)}(I) | 1 \leq j \leq n\}$ for the assoc hom ODE. Also, consider

$$y_p = \sum_{j=1}^{n} v_j y_j,$$

then

$$\begin{aligned} y_p' &= \sum_{j=1}^{n} (v_j' y_j + v_j y_j') \\ &= \sum_{j=1}^{n} v_j y_j' \text{ , if } \sum_{j=1}^{n} v_j' y_j = 0. \end{aligned}$$

Now,

$$
\begin{aligned}
y_p'' &= \sum_{j=1}^{n}(v_j'y_j' + v_jy_j'') \\
&= \sum_{j=1}^{n} y_jy_j'' \text{ , if } \sum_{j=1}^{n} v_j'y_j' = 0
\end{aligned}
$$

continuing,

$$
y_p^{(k)} = \sum_{j=1}^{n} v_jy_j^{(k)} \text{ , if } \sum_{j=1}^{n} v_j'y_j^{(k-1)} = 0
$$

for $1 \leq k \leq n-1$. Finally,

$$
\begin{aligned}
y_p^{(n)} &= \sum_{j=1}^{n}(v_j'y_j^{(n-1)} + v_jy_j^{(n)}) \\
&= \sum_{j=1}^{n} v_j'y_j^{(n-1)} + \sum_{j=1}^{n} v_jy_j^{(n)})
\end{aligned}
$$

Therefore

$$
\sum_{k=1}^{n} a_ky_p^{(k)} = f(x) \iff
$$

$$
a_n\sum_{j=1}^{n} v_j'y_j^{(n-1)} + a_n\sum_{j=1}^{n} v_jy_j^{(n)} + \sum_{k=1}^{n-1} a_k\sum_{j=1}^{n} v_jy_j^{(k)} = f(x) \iff
$$

$$
a_n\sum_{j=1}^{n} v_j'y_j^{(n-1)} + \sum_{j=1}^{n} a_nv_jy_j^{(n)} + \sum_{j=1}^{n}\sum_{k=1}^{n=1} a_kv_jy_j^{(k)} = f(x) \iff
$$

$$
a_n\sum_{j=1}^{n} v_j'y_j^{(n-1)} + \sum_{j=1}^{n} v_j\sum_{k=1}^{n} a_ky_j^{(k)} = f(x) \iff
$$

$$
a_n\sum_{j=1}^{n} v_n'y_j^{(n-1)} = f(x)
$$

since $y_j \in B$. Thus,

$$
\begin{aligned}
\sum_{j=1}^{n} v_j' y_j &= 0 \text{ (1st eqn)} \\
\sum_{j=1}^{n} v_j' y_j' &= 0 \text{ (2nd eqn)} \\
&\vdots \\
\sum_{j=1}^{n} v_j' y_j^{(k-1)} &= 0 \text{ (kth eqn)} \\
&\vdots \\
\sum_{j=1}^{n} v_j' y_j^{(n-2)} &= 0 \text{ ((n-1)th eqn)} \\
a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} &= f(x) \text{ (nth eqn)}
\end{aligned}
$$

$$
\implies W(y_j)_{1 \le j \le n} \begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} = f(x)\vec{e_n}
$$

Notice that $|W(y_j)_{1 \le j \le n}| \neq 0$ for all $x \in I$ since $y_j s$ are linearly independent (being that $y_j \in B$). Thus, by Cramer's Rule,

$$
v_j' = \frac{D_j}{D},
$$

where $D = |W(y_j)_{1 \le j \le n}|$ and

$$
D_j = \begin{vmatrix} y_1 & \cdots & 0 & \cdots & y_n \\ y_1' & \cdots & 0 & \cdots & y_n' \\ y_1'' & \cdots & 0 & \cdots & y_n'' \\ \vdots & & \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & f(x) & \cdots & y_n^{(n-1)} \end{vmatrix} = (-1)^{n+j} f(x) |W_{n-1}(y_k)_{\substack{1 \le k \le n \\ j \neq k}}|
$$

Therefore

$$v_k' = (-1)^{j+n} f(x) \frac{|W_{n-1}(y_k)_{\substack{1 \le k \le n \\ j \ne k}}|}{a_n |W_n(y_k)_{1 \le k \le n}|}$$

Thus,

$$(1)\ v_k(x) = \frac{(-1)^{j+n}}{a_n} \int f(x) \frac{|W_{n-1}(y_k)_{\substack{1 \le k \le n \\ j \ne k}}|}{a_n |W_n(y_k)_{1 \le k \le n}|} dx,$$

Where $y_p(x) = \displaystyle\sum_{k=1}^{n} v_k(x) y_k(x)$ is a particular soln to $\displaystyle\sum_{k=1}^{n} a_k y^{(k)} = f(x)$. (1) is called the variation of parameters formulas.

Ch 4 systems of ODEs

Example. Consider a curve $\vec{r} : [a, b] \to \mathbb{R}^n$, where $\vec{r}(t) = (x_1(t), \ldots, x_n(t))$ and $t \in [a, b]$. By Newton's 2nd Law, $\vec{F} = m\vec{a} = m\vec{r}''(t)$. Thus,

$$\begin{pmatrix} m(t)x_1''(t) \\ m(t)x_2''(t) \\ \vdots \\ m(t)x_n''(t) \end{pmatrix} = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix},$$

which is a system of n many 2nd order ODEs in $(x_1, \ldots, x_n)$

Date: 04.21.15
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Example. Consider an

$$(1)\ y^{(n)} + \sum_{k=0}^{n-1} p_k(x) = f(x),$$

$$y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_2(x) y'' + p_1(x) y' + p_0(x) y = f(x).$$

put $y_k = y^{(k-1)}$ for $1 \le k \le n+1$, then

$$
\begin{aligned}
y_1 &= y \implies \\
y_2 &= y' = y_1' \implies \\
y_3 &= y'' = y_1'' = y_2' \implies \\
y_4 &= y''' = y_1''' = y_2'' = y_3' \implies \\
&\vdots \\
y_{n+1} &= y^{(n)} = y_n'
\end{aligned}
$$

Thus, (1) becomes

$$(2)\ y_n' + \sum_{k=0}^{n-1} p_k(x) y_k' = f(x),$$

which is a first order linear ODE. puting (2) whith the alone n-1 many 1st order ODEs yields the following 1st order system: with n many ODEs:

$$
\begin{cases}
y_n' + \sum\limits_{k=0}^{n-1} p(x) y_k' = f(x) \\
y_1' = y_2 \\
y_2' = y_3 \\
\vdots \\
y_{n-1}' = y_n
\end{cases}
$$

Again, this is a system of n many 1st linear ODEs from a single nth order linear ODE. This gives yet another example to motivate studying systems of linear ODEs.

Example. page 255 number 12

$$
\begin{cases}
x' = y \\
y' = x
\end{cases}
$$

Note: $x = y' = x''$. So, this yields $x'' - x = 0$, which has basis $B = \{\cosh t, \sinh t\}$. Thus,

$$x = a\cosh t + b\sinh t$$

Now,

$$y = b\cosh t + a\sinh t$$

Notice that if $\vec{r}(t) = (x(t), y(t))$ then $\vec{r}'(t) = (x'(t), y'(t)) = (y(t), x(t))$. We want such an $\vec{r}(t)$. In otherwords, we may view the above system as a single. 1st order ODE of a parametric function.
Recall that

$$\cosh(\alpha + t) = \cosh\alpha\cosh t + \sinh\alpha\sinh t$$

Take $A \in \mathbb{R}$ st $a/A \geq 1$, then $a/A \in$ Range (cosh). Now, we want $\alpha \in \mathbb{R}$ st

$$a = A\cosh\alpha \ \&\ b = A\sinh\alpha.$$

So,
$\coth\alpha = \dfrac{\cosh\alpha}{\sinh\alpha} = \dfrac{a}{b}$ holds for some α st $|\alpha| < 1$ since Range(coth) $= \mathbb{R}$.
therefore for this α,

$$\begin{aligned} x &= a\cosh t + b\sinh t \\ &= A\cosh\alpha\cosh t + A\sinh\alpha\sinh t \\ &= A\cosh(\alpha + t). \end{aligned}$$

therefore $x = A\cosh(\alpha + t)$; whence, $y = x' = A\sinh(\alpha + t)$.
Hence,

$$x^2 - y^2 = A^2 \implies \frac{x^2}{A^2} - \frac{y^2}{A^2} = 1$$

These are the solns to $\vec{r}'(t) = (x'(t), y'(t))$. Note : Range(cosh) $= [1,)$. So, if $A > 0$ then $x(t) > 0$; where, $\vec{r}(t)$ coresponds to the righthand branch. Whereas, if $A < 0$ then $\vec{r}(t)$ coresponds to the left hand branch.
Notice that we soved this linear system by "substitution."

Date: 04.23.15

Example. (Recall).
let $\vec{r}(t) = (x(t), y(t))$, and consider $\vec{r}'(t) = (x'(t), y'(t))$, then

$$\begin{cases} x'(t) = y(t) \\ y'(t) = x(t) \end{cases} \iff \begin{cases} x'(t) - y(t) = 0 \\ y'(t) - x(t) = 0 \end{cases} \iff \begin{cases} L_{11}x + L_{12}y = 0 \\ L_{21}x + L_{22}y = 0 \end{cases} \iff$$

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

Where $L_{11} = D = L_{22}$ and $L_{12} = -1 = L_{21}$
In other symbols,

$$\begin{pmatrix} D & -1 \\ -1 & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

Recall that $\mathbb{R}[D] \equiv \mathbb{R}[x]$; in particular, $\mathbb{R}[D]$ is communative "ring" (multiplicative communative).
Solve by "elimination:"

$$L_{21}(L_{11}x + L_{12}y) = L_{21}0 \implies$$

$$L_{21}L_{11}x + L_{21}L_{12}y = L_{21}0$$

$$L_{11}(L_{21}x + L_{21}y) = L_{11}0 \implies$$

$$L_{11}L_{21}x + L_{11}L_{21}y = 0$$

$$L_{21}L_{12}y - L_{11}L_{22}y = 0 \implies$$

$$(L_{21}L_{12} - L_{11}L_{22})y = 0 \implies$$

$$\begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} y = 0$$

Anologosly

$$\begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} x = 0$$

Therefore

$$(|L|I) \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

where

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

Recall: If A is a sq matrix then

$$AA^a = |A|I$$

This holds for any A over a commutative ring, e.g.,

$$A \in \text{ Mat }_n(R)$$

where R is a commutative ring say $R = \mathbb{C}$
Thus if $L \in \text{ Mat}_n(\mathbb{R}[D])$ then

$$L^a L = |L| I_n,$$

where I_n $\text{Mat}_n(\mathbb{R}[D])$,

$$I_n = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \text{ , \& } 1_{Id} \in \mathbb{R}[D]$$

consider $L \in \text{ Mat}_n(\mathbb{R}[D])$,
$\vec{x}(t) = (x_1(t), \dots, x_n(t))$, $\vec{F}(t) = (F_1(t), \dots, F_n(t)) \in (C^{(d)}(I))^n$, where $d = \max\{ \deg (L_{ij}) \mid 1 \leq i , j \leq n\}$ and $L = (L_{ij})$, and the system of linear ODEs

$$(1)\ L\vec{x} = \vec{F} \iff \begin{pmatrix} L_{11} & L_{12} & \cdots L_{1n} \\ L_{21} & L_{22} & \cdots L_{2n} \\ \vdots & & \\ L_{n1} & L_{n2} & \cdots L_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix}$$

Here

$$L_{ij} = \sum_{k=0}^{m_{ij}} a_{i_k j_k} D^k \in \mathbb{R}[D]$$

Applying the adjoint formula to (1)

$$(2)\ (|L| I_n)\vec{x} = L^a \vec{F}.$$

Recall: $A^a = (c_{ij})^T$, $C_{ij} = (-1)^{i+j}|M_{ij}|$
In component form, (2) says that

$$(3)\ |L|x_i(t) =_i (L^a)\vec{F}(t),$$

where ${}_i(L^a)$ is the ith row of L^a. Notice that (3) is a linear ODE in the single unknown funct $x_i(t)$. therefore previous methods can be used to solve (3).

Date: 04.27.15

Now, we consider first order linear systems, at first, without constant coefficients. These have the form

$$\vec{x}'(t) + p(t)\vec{x}(t) = \vec{q}(t),$$

where $\vec{x} : I \subseteq \mathbb{R} \to \mathbb{R}^n$ and $p(t) \in \operatorname{Mat}_n(C(I))$ and $\vec{q} \in (C^{(1)}(I))^n$.
In "normal form" this becomes

$$\vec{x}\,'(t) = p(t)\vec{x}(t) + \vec{q}(t)$$

Aside: $R[D]$ is a communative "ring." however, $(C(t))[D]$ this is not a communative, the point is that to use the det proputies developed last semester, we need $\operatorname{Mat}_n(\mathbb{R})$, where $\mathbb{R}$ is commutative ring (then thos proff genral

Example. $(((C)(I))[D]$ is not commutative).
Put

$$L_1 = -tD + 1$$

$$L_2 = t^2D - 3,$$

then $L_1, L_2 \in (C^{(\infty)}(\mathbb{R}))[D]$. Notice that

$$\begin{aligned} L_2x &= (t^2D - 3)x \\ &= t^2Dx - 3x \\ &= t^2x' - 3x \implies \end{aligned}$$

$$
\begin{aligned}
L_1 L_2 x &= (-tD+1)L_2 x \\
&= (-tD+1)(t^2 Dx - 3x) \\
&= -tD(t^2 Dx - 3x) + t^2 Dx - 3x \\
&= -tD(t^2 Dx) + 3tDx + t^2 Dx - 3x \\
&= -t(2tDx + t^2 D^2 x) + 3 + Dx + t^2 Dx - 3x \\
&= -2t^2 Dx - t^3 D^2 x + 3tDx + t^2 Dx - 3x \\
&= -t^2 Dx - t^3 D^2 x + 3tDx - 3x.
\end{aligned}
$$

Also,

$$
\begin{aligned}
L_1 x &= (-tD+1)x \\
&= -tDx + x \implies \\
L_2 L_1 x &= (t^2 D - 3)L_1 x \\
&= t^2 D L_1 x - 3L_1 x \\
&= t^2 D(-tDx + x) + 3tDx - 3x \\
&= t^2 D(-tDx) + t^2 Dx + 3tDx - 3x \\
&= t^2(-Dx - tD^2 x) + t^2 Dx + 3tDx - 3x \\
&= -t^2 Dx - t^2 D^2 x + t^2 dx + 3tDx - 3x \\
&= -t^3 D^2 x + 3tDx - 3x.
\end{aligned}
$$

therefore $L_1 L_2 \neq L_2 L_1$.

Date: 04.29.15

$$\vec{x}\,'(t) = p(t)\vec{x}(t) + \vec{q}(t)$$

where $p(t) \in \text{ Mat}_n(C(I))$. Now, we consider the case where $p(t) \in \text{ Mat}_n(\mathbb{R})$, i.e., the entries of the matrix p are costant functions. In the hom case, this becomes

$$\vec{x}\,'(t) = A\vec{x}(t)$$

or

$$(1)\ \vec{x}\,' = A\vec{x}$$

where $A \in \text{Mat}_n(\mathbb{R})$.
Notice that if $n = 1$ then (1) becomes

$$x' = ax,$$

Which has solns $x(t) = ke^{at}$. This leads us to a conjecture in the more general case,

$$x_i(t) = k_i e^{a_i t} \text{ for } 1 \leq i \leq n.$$

In other words,

$$\vec{x}(t) = \begin{pmatrix} k_1 e^{a_1 t} \\ k_2 e^{a_2 t} \\ \vdots \\ k_n e^{a_n t} \end{pmatrix}$$

Since $\vec{x}' = (x_i')$,

$$\vec{x}'(t) = \begin{pmatrix} a_1 k_1 e^{a_1 t} \\ a_2 k_2 e^{a_2 t} \\ \vdots \\ a_n k_n e^{a_n t} \end{pmatrix}$$

therefore this $\vec{x}(t)$ is a soln to (1) $\iff$

$$\begin{pmatrix} a_1 k_1 e^{a_1 t} \\ a_2 k_2 e^{a_2 t} \\ \vdots \\ a_n k_n e^{a_n t} \end{pmatrix} = A \begin{pmatrix} k_1 e^{a_1 t} \\ k_2 e^{a_2 t} \\ \vdots \\ k_n e^{a_n t} \end{pmatrix}$$

Here we see that if $a_i = a_j$ for all i and j , say $a = a_i$, then $\vec{x}'(t) = a\vec{x}(t)$ and so, (1) becomes

$$a\vec{x}(t) = A\vec{x}(t)$$

Now, notice that

$$\vec{x}(t) = e^{at}\vec{k},$$

where $\vec{k} = (k_1, k_2, \ldots, k_n)$. therefore (1) now becomes

$$ae^{at}\vec{k} = A(e^{at}\vec{k}) \implies$$

$$A\vec{k} = a\vec{k}.$$

Recall: an eigenvalue of $A \in \text{ Mat}_n(\mathbb{R})$ is $\lambda \in \mathbb{R}$ iff there is $\vec{0} \neq \vec{v} \in \mathbb{R}^n$ st

$$A\vec{v} = \lambda\vec{v}.$$

Here $\vec{v}$ is called an eigevector assoc with λ. therefore from above, we see that $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a soln to $\vec{x}' = A\vec{x}$ iff λ is an eiginvalue of A and $\vec{v}$ is an assoc eigenvector of λ.
Recall: to find eigenvalues of $A \in \text{ Mat}_n(\mathbb{R})$, notice that for $\vec{v} \neq \vec{0}$,

$$\begin{aligned} A\vec{v}\lambda\vec{v} &\iff \lambda\vec{v} - A\vec{v} = \vec{0} \\ &\iff (\lambda I_n - A)\vec{v} = \vec{0}, \end{aligned}$$

which is a hom system with a nontivial soln $\vec{v}$. So, therefore

$$\det(\lambda I_n - A) = 0.$$

Also, recall that the char poly of $A \in \text{ Mat}_n(\mathbb{R})$,

$$p_A(x) = \text{ det}(xI_n - A) \in \mathbb{R}[x],$$

which is an nth degree poly over $\mathbb{R}$. So, therefore to find eigenvalues of A, we must find the zeros of $p_A(x)$. Once we have the eigenvalue λ, to find an assoc $\vec{v} \pm \vec{0}$, we solve for $\vec{v}$ in $(A - \lambda I_n)\vec{v} = \vec{0}$.