Notes for First Test

Joshua Bailey

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1 Date: 1.12.15

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Def: let y be a function of x and $y^{(n)}(x) = d^n y/dx^n$ for $n \in \mathbb{N}\{1, 2, 3, ...\}$ where F is a given function

$$2xy + x^2y' = 1$$
$$2xy + x^2y' - 1 = 0$$

This is an ODE, where

$$F(x, y, y') = 2xy + x^2y' - 1$$

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ex: Given f on an interval I find F st $F(x)f'(x)$ for all $x \in I$ Here F is by def, an antiderivative of f on I. we denote F as $\int f(x)dx$. All solns F to (1) are of the form $\int f(x)dx + c$, where c is an arbitray constant, (1) is an ODE, i.i.,

$$y; = f(x) \iff y' - f(x) = 0$$

the latter has the form

$$f(x, y') = 0$$

A little calc III

Given $F : \mathbb{R}^n \to \mathbb{R}$, say $\vec{a} = (x_1, ..., x_1) \in R^n$, the derivative of F at $\vec{x}$ is

$$= F(x) = (F_x1(\vec{x}), F_x2(\vec{x}), ..., F_xn(\vec{x})) \in Mat_1xn(R),$$

where

$$F_xi(\vec{x}) = \lim_{x \to 0} \frac{F(x_1, ..., x_i + h, ..., x_n)}{h}$$

F_{xi} is called the partial derivative of F with respect to x_i.

we call $\nabla F = F'$ the gradiant of F.

given $r : I \in R^n$, where I subset R is an integral in R, the derivative of r at t is

$$r'(t) = (x_1'(t), x_2'(t), ..., x_n'(t)),$$

$$r(t) = (x_1(t), x_2(t), ..., x_n(t)),$$

here $r'(t)$ is a tangent vector to r at $r'(t)$ in R^n.

3 Date: 01.14.15

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Let $F : \mathbb{R}^n \to \mathbb{R}^m$, say $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $F(x) = (f_1)$

thrm(Chain Rule). if $\mathbb{R}^n$:

Recall: separable ODE

Date: 01.26.15.

$$y' = g(x)h(y) \text{ or } \frac{dy}{dx} = g(x)h(y)$$

"separate" the variable as

$$\frac{1}{h(y)}y' = g(x) \text{ or } \frac{1}{h(y)}\frac{dy}{dx} = g(x)$$

$$\int \frac{1}{h(y)}\frac{dy}{dx}dx = \int g(x)dx$$

Now, $y = y(x)$ then by
by change of vars,

$$\int \frac{1}{h(y)}dy = \int \frac{1}{h(y)}\frac{dy}{dx}dx = \int g(x)dx$$

in short,

$$\int \frac{1}{h(y)}dy = \int g(x)dx$$

Now , once these itegrals are evaluated,
if possible, then the resolting eqn
is one of y on the left and x on the
right, which at least implicitly defines
solns y(x) to y' = h(y)g(x). this
resulting eqn may or may not be possible
to solve for y explicitly in terms of x

Example

$x' = kx$, think $x = x(t)$ this is separable; so,

$$\frac{1}{x}\frac{dx}{dt} = k$$

$$\int \frac{1}{x}\frac{dx}{dt}dt = k\int dt \implies$$

(by change of vars)

$$\int \frac{1}{x}dx = \int \frac{1}{x}\frac{dx}{dt}dt = k\int dt$$

$$\implies ln|x|kt + C,$$

$$\implies |x| = e^{kt+c} = Ce^{kt},\ C = e^c > 0$$

$$\implies x(t) = Ce^{kt},\ C \neq 0, \text{ any } k \in R$$

Example

$T' = k(A - T)$, $k > 0$
this s separable; so,

$$\frac{1}{A-T}\frac{dT}{dt} = k \implies$$
$$\int \frac{1}{A-T} dT = kt + C \implies$$
$$-ln|A-T| = kt + C \implies$$
$$\frac{1}{A-T} = Ce^{kt} \implies$$
$$A - T = Ce^{-Kt} \implies$$
$$T = A - Ce^{-kt}$$

Aside

substitution (change of vars)

$$\int f(g(x))g'(x)dx = \int f(v)dv$$

$v = g(x)$, or

$$\int f(v)\frac{v}{x}dx = \int f(v)dv$$

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Ex(p.43) 35

$$x(t) = ce^{kt} \text{ (form } x = kx)$$

C14 has a decay rate constant of

$$k = -0.0001216$$

Notice that

$$x(0) = C$$

so, C is called the initial value.
that notation 'x_0' is used for C
i.e., $x_0 = x(0)$. thus,

$$x(t) = x_0 e^{kt}$$

in # 35, $x(t) = x_0/6$.
so,

$$\frac{x_0}{6} = x_0 e^{kt}$$

solve for t. thus,

$$t = \frac{1}{6} ln(1/6) = \frac{1}{|k|} ln(6)$$

Torricelli's law

Think of $x = x(t)$ and $h = h(t)$
we want $x(t)$, say in particular,
we want t st $x(t) = 0$, so called
"drain time."
recall # 35, p.18, that "ground speed"
is given by $|v| =, sqrt2gx$
from "free-fall" a height x.
in contex,

$$\frac{dh}{dt} = \sqrt{2gx}$$

In "the spout" $V = ah$; so

$$\frac{dV}{dt} = a\frac{dh}{dt} a\sqrt{cgx}$$

in the tank

$$\frac{dV}{dt} = -a\sqrt{2gx}$$

let A(x) be the ? cros=sectinal
area of the tank at height x, then

$$V = \int_x^0 A(t)dt$$

by ftc(1),

$$\frac{dV}{dx} = A(x)$$

by the cain rule

$$\frac{dV}{dt} = dv/dx dx/dt = A(x)x'$$

$$a(x) * x' = -a\sqrt{2gx}$$

i.e.

$$x' A(x) = -a\sqrt{2gx}$$

which is a separable ODE. thus

$$\frac{A(x)}{\sqrt{x}} dx = -a\sqrt{2g} \implies$$

(int w.r.t t and Δ vars)

$$\int \frac{A(x)}{\sqrt{x}} dx = -at\sqrt{2g}$$

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Ex (p.45) # 59

revolve $x^2 = by$ about y-axis
depth is 4ft at noon, $y(0) = 4$

$$a = \pi r^2$$

depth is 1ft at 1pm same day
Recall: $\int y^{-1/2} A(y) dx = -8at$ in ft and s.
thus, by torciullis law,

$$-8\pi r^2 t = \pi b \int y^{1/2} dy \implies$$

$$-8r^2 t = \frac{2b}{3} y^{3/2} + C$$

$$y(0) = 4 \implies C = -16b/3$$

$$\therefore$$

$$\frac{2b}{3} y^{3/2} = \frac{16b}{3} - 8r^I 2t$$

Now, in 3600s (1hr), y=1, i.e., y(3600) = 1,

$$\frac{2b}{3} = \frac{16b}{3} - 8r^2(3600)$$

$$r^2 = \frac{14b}{3 * 8 * 3600} = \frac{7b}{12 * 3600} \implies$$

$$r = \frac{1}{60}\sqrt{\frac{7b}{12}}$$

Drain time t_0 is

$$0 = \frac{16b}{3} - 8r^2 t_0 \implies t_0 = \frac{2b}{3r^2}$$

Now, in particuler, if
$y = 4$, then the radius
of $A(y)$ is 2, i.e., $x = 2$.
Thus,

$$x^2 = by \implies 4 = 4b \implies b = 1 \therefore$$

$$r = \frac{1}{60}\sqrt{\frac{17}{12}} \;\&\; t_0 = \frac{2}{3r^2}$$

5 Date: 02.02.2015

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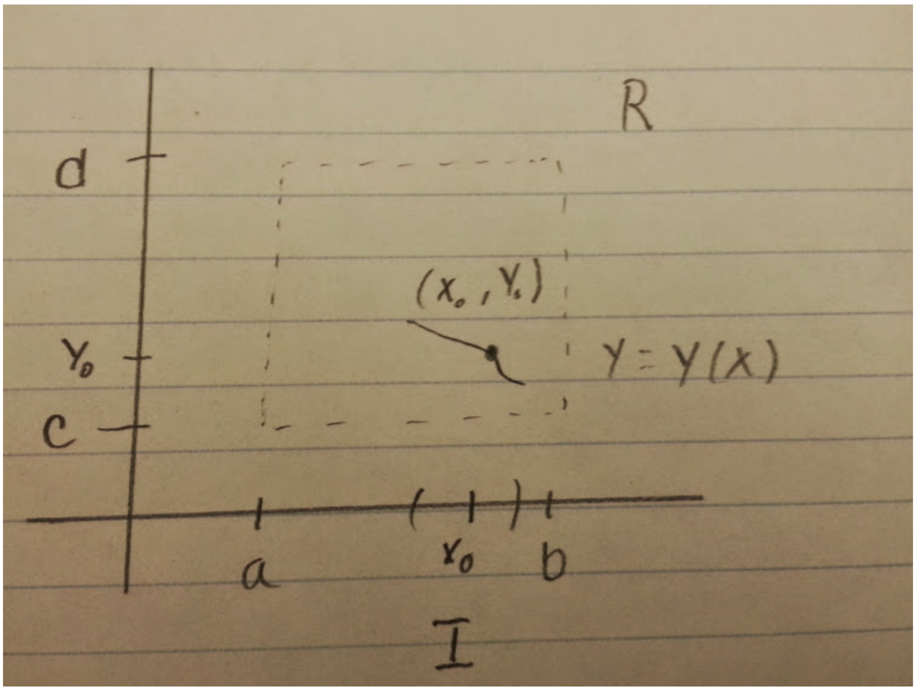
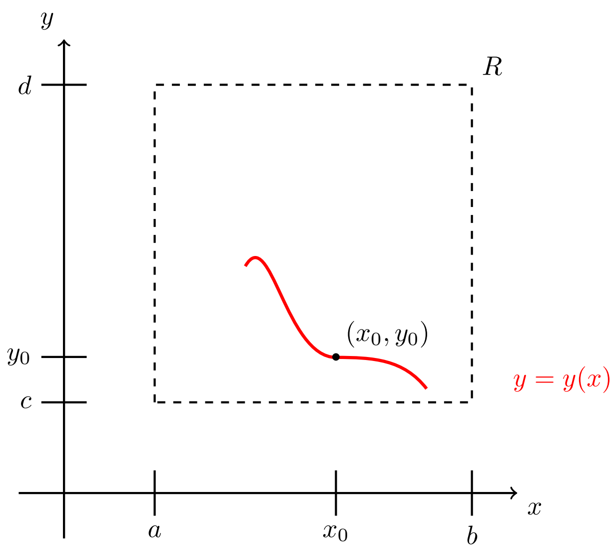
Thrm existence - uniqueness thrm, ∃!thrm
If $f : D, \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$ and $f_y : D_2 \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$
are cts on $R = (a, b)x(c, d)$ and $x_0, y_0) \in R$ and
then these exist an interval I st $x_0 \in I$, $I \subseteq (a, b)$ and
the initial value problem

$$\frac{dy}{dx} = f(x, y) \text{ and } y_0 = y(x_0)$$

has a unique $y = y(x)$ for all $x \in I$.





$$R = (a,b)x(c,d)$$
$$= (x,y) \in R^2 | a < x < b \ \& \ c < y < d$$

The $\exists!$ thrm is a "local" result, local
to x_0, move precisely, it just sups that there is a unique soln in I
not necessarily outside of I.

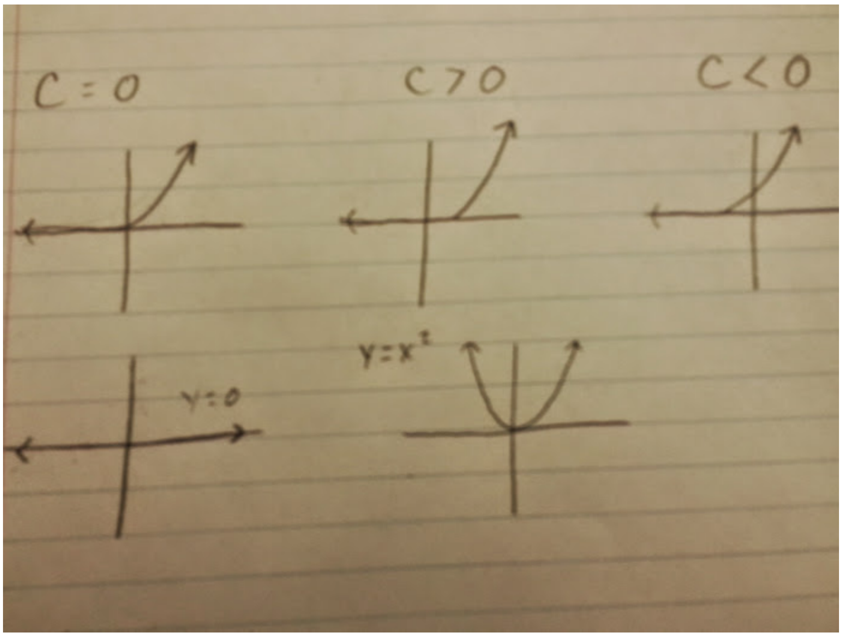
Ex(p.29) # 27

$$y' = s\sqrt{y} \ \& \ y(0) = 0$$

consider

$$y(x) = \begin{cases} 0 & x \leq c \\ (x-c)^2 & x \geq c \end{cases}$$

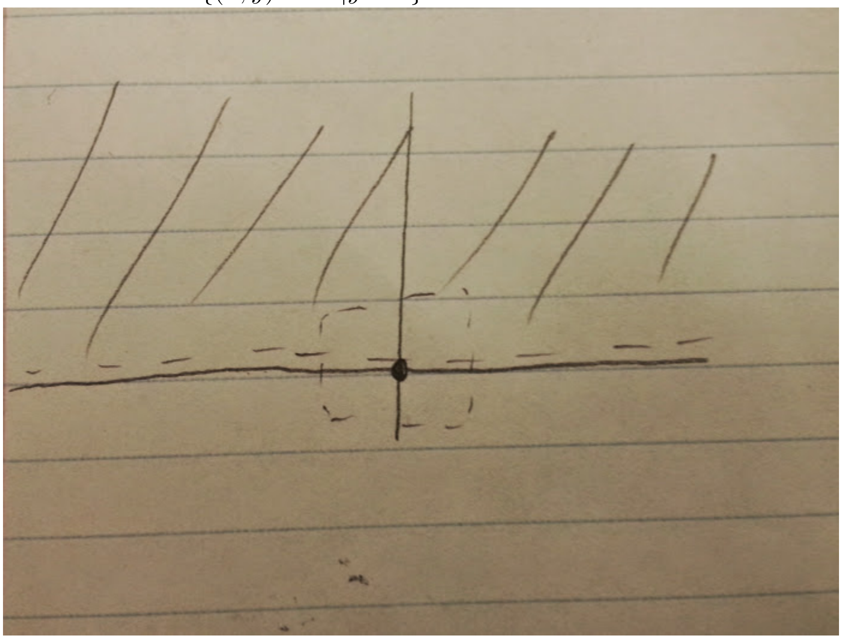
which is ctn on $\mathbb{R}$
notice that both parts
y(x) satisfy the initial value problem if
$c \geq 0$ Note:



Notice that

$$f(x, y) = 2\sqrt{y} \ \& \ f_y(x, y) = \frac{1}{\sqrt{y}}$$

which is cts on $R = \{(x, y) \in R^2 | y > 0\}$



notice that (0, 0) is not in R, not "interior" to R. $\therefore \exists!$ thrm does not apply
Notice that if $y = f(x)$ is a soln to an ODE on a interval I
then so is $y = f(x - c)$ a soln to

the ODE on $I - c = \{x \in R | x + c \in I\}$.

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ex p.28

15. $y' = \sqrt{x-y}$, $y(2) = 2$, no
16. $y' = \sqrt{x-y}$, $y(2) = 1$, yes

here $f(x, y) = \sqrt{x-y}$, which has
cts iff $x - y \geq 0 \iff x \geq y$
$f_y(x, y) = -\frac{1}{2\sqrt{x-y}}$, which is cts
on $\{(x, y) \in \mathbb{R}^2 \mid y < x\}$.

Notes

Def A first order linear ODE
has the form

$$(1)\ y' + p(x)y = q(x)$$

where p & q are cts on some interval $\mathcal{I}$.
Notice that if $q(x) = 0$ then (1) is
separable.
Key observation: the left side of
(1) resembols the product rule.
this motivates a question: Is there
a cst I(x), say on the interval $\mathcal{I}$,
st
(1') $y'I(x) + yp(x)I(x) = q(x)I(x)$,
where the left side of (1') is the dirivative
of a product?

if there is such an I(x)
then
(2) $I'(x) = p(x)I(x)$.
put $v = I(x)$, then (2) becomes

$$v' = p(x)v$$

which is separable. thus

$$\frac{v'}{v} = p(x) \implies$$

$$\int \frac{1}{v} dv = \int p(x)dx \implies$$

$$\ln|v| = \int p(x)dx + C \implies$$

$$|v| = e^{\int p(x)dx + C} = Ke^{\int p(x)dx}$$

where $k > 0$ $\therefore$

$$v = Ke^{\int p(x)dx} \text{ , } K \neq 0$$

Def. the intergrating factor of

$$y' + p(x)y = q(x)is$$

$$I(x) = e^{\int p(x)dx}$$

Finally, from (1')

$$(yI(x))' = q(x)I(x) \implies$$

$$yI(x) = \int q(x)I(x)dx \implies$$

$$\boxed{y = \frac{1}{I(x)} \int q(x)I(x)dx}$$

Date:
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Single Tanking Problem

c_i = concentration coming into the tank (constant)
r_i = rate of flow into the tank (constant)
$c_0(t)$ = concentration coming out of the tank
r_0 = rate of flow out of the tank (constant)
$x(t)$ = amount of salute in tank at time t
$V(t)$ volume of tank at time t

Units:
Concentration = $\frac{\text{amount of solute}}{\text{unit valume}}$
Rate = $\frac{\text{valume}}{\text{unit time}}$
amount = (concentration)(rate)(time)

Notice that the rate of change fo the volume is constant and
is $m = r_i - r_0$; where, $V(t) = (r_i - r_0)t + V_0 = mt + V_0$, where $V_0 = V(0)$
For a small Δt
$x(t + \Delta t) = x(t) +$ amount in = amount out over time Δt

amount in over $\Delta t = c_i r_i \Delta t$;
amount out over $\Delta t \approx c_0(t) r_0 \Delta t$.
thus,

$$\Delta x = x(t + \Delta t) - x(t) \approx (c_i r_i - c_0(t) r_0)\Delta t \implies$$

$$\frac{\Delta x}{\Delta t} \approx c_i r_i - c_0(t) r_0$$

this suggest that

$$\frac{dy}{dx} = c_i r_i - c_0(t) r_0$$

Now

$$c_0(t) = \frac{x(t)}{V(t)} \implies$$

$$\frac{dx}{dt} = c_i r_i = \frac{x(t)}{V(t)} r_0$$

which is a 1st order linear ODE.
more consiely, put $x' = \frac{dx}{dt}$ and
$x = x(t)$, then

$$\boxed{x' + \frac{r_0}{V(t)} x = c_i r_i}$$

Recall that $V(t) = mt + v_0$, $m = r_i - r_0$; so,

$$x' + \frac{r_0}{mt + v_0} x = c_i r_i$$

Here

$$p(t) = \frac{r_0}{mt + V_0} \implies$$
$$\int p(x)dx = \frac{r_0}{m} ln(mt + V_0) + C$$

where in context, $V(t) > 0$. Choose, of ease, $C = 0$, then

$$I(t) = e^{\int p(t)dt} = (mt + V_0)^{r_0/m}$$

So,

$$x'(mt + V_0)^{r_0/m} + r_0(mt + V_0)^{r_0/m - m} = c_i r_i (mt + V_0)^{r_0/m} \implies$$
$$(x(mt + V_0)^{r_0/m})' = c_i r_i + (mt + V_0)^{r_0/m} \implies$$
$$x(mt + V_0)^{r_0/m} = c_i r_i \int (mt + V_0)^{r_0/m} dt$$

If $r_0/m = -1$ then $r_0 = -m = -r_i + r_0$ $\implies r_i = 0$, which is not of intrest in context ("mixing"). Thus, if $r_0/m \neq -1$ then

$$\int (mt + V_0)^{r_0/m} dt = \frac{1}{m}(mt + V_0)^{r_0/m+1} \frac{m}{r_0 + m} + C$$
$$= \frac{1}{r_i}(mt + V_0)^{r_i/m} + C$$
$$\therefore$$
$$x(mt + v_0)^{r_0/m} = c_i (mt + V_0)^{r_i/m} + C$$

at $t = 0$

$$x_0 V_0^{r_0/m} = c_i V_0^{r_i/m} + C \implies$$
$$C = c_i V_0^{r_i/m} - x_0 V_0^{r_0/m}$$
$$= V_0^{r_0/m}(c_i V_0 - x_0)$$

thus,

$$x(mt + V_0)^{r_0/m} = c_i (mt + V_0)^{r_0/m} + V_0^{r_0/m}(c_i V_0 - x_0$$
$$\implies x = c_i(mt + V_0) + (c_i V_0 - x_0)(\frac{V_0}{mt + V_0})^{r_0/m}$$

$$\boxed{\begin{array}{c} x = c_i V + (c_i V_0 - x_0)\left(\dfrac{V_0}{V}\right)^{r_0/(ri - r_0)} \\ \text{where } x = x(t) \text{ \& } V = (r_i - r_0)t + V_0 \end{array}}$$

Date:
02.09.15.

6 Date: 02.09.2015

1.6 Substitutions in ODEs
Consider a slope field

$$(1)\ \frac{dy}{dx} = f(y,x)$$

i.e., a 1st order normal ODE. If

$$\alpha(x,y)$$

appers in (1), then we are compelled
to make the substitution

$$v = \alpha(x,y)$$

("alpha" for auxillary variable)
By the calc III chain rule

$$\frac{dv}{dx} = \frac{\partial\alpha}{\partial x}\frac{dx}{dx} + \frac{\partial\alpha}{\partial y}\frac{dy}{dx}$$

$$= \alpha_x + \alpha y\frac{dy}{dx}$$

If $v = \alpha(x,y) can be soved$
for y in terms of x and v, say

$$y = \beta(x,v)$$

then from (1), we have that

$$\frac{dv}{dx} = \alpha_x + \alpha_y\frac{dy}{dx} = \alpha + \alpha_y f(x,y)$$

where,

$$\boxed{\frac{dv}{dx} = \alpha_x + \alpha f(x,\beta(x,v))}$$

which is a new ODE with dependent variable v and independent variable x.

6.1 ex

$$\frac{dy}{dx} = f(x,y,ax+by+c$$

put

$$v + d(x,y) = ax + by + c$$

then

$$\frac{dv}{dx} = a + b\frac{dy}{dx}$$

thus,

$$\frac{dv}{dx} = a + b\frac{dy}{dx} = a + bf(x, y, ax + by + c)$$

$$\implies \boxed{\frac{dv}{dx} = a + bf(x, (v - ax - c)/b, v)}$$

where

$$y = \beta(x, v) = \frac{v - ax - c}{b} \text{ , } b \neq 0$$

6.2 p.74 16

$$y' = \sqrt{x + y + 1}$$

$$v = x + y + 1 \implies y = v - x - 1 \text{ \&}$$

$$\frac{dv}{dx} = 1 + \frac{dy}{dx} \implies$$

$$\frac{dv}{dx} = \sqrt{v} + 1 \text{ (separable)}$$

Def. A first order normal homogenous ODE has the form

$$\frac{dy}{dx} = f(y/x)$$

6.3 Ex

$$y' = \frac{xy}{x^2 + y^2}$$

In general, put

$$v = \alpha(x, y) = y/x \text{ (slope)}$$

so, $y = xv$ implies that

$$f(v) = f(y/x) = \frac{dy}{dx} = v + x\frac{dv}{dx} \implies$$

$$x\frac{dv}{dx} = f(v) - v \text{ (separable)}$$

$$\frac{1}{f(v) - v}\frac{dv}{dx} = \frac{1}{x} \implies$$

$$\int \frac{1}{f(v) - v} dv = \ln|x| + c$$

6.4 Ex (Revisited)

$$y' = \frac{xy}{x^2 + y^2} = \frac{y/x}{1 + (y/x)^2}, v = y/x \implies$$

$$\int \frac{1}{\frac{v}{1+v^2} - v} dv = \ln|x| + c \implies$$

$$-\int \frac{1 + v^2}{v^3} dv = \ln|x| + c \implies \ldots$$

7 Date: 02.10.2015

Date: 02.10.15

Thrm. If $p(x,y) = \sum a_{i_1 i_2} x^{i_1} y^{i_2}$ and
$Q(x,y) = \sum a_{j_1 j_2} x^{j_1} y^{j_2}$ are polynomials
over $\mathbb{R}$, then if there is a $k \in \mathbb{Z}^+$ st for all i_1, i_2, j_1, j_2,

$$i_1 + i_2 = d = j_1 + j_2$$

then $p(x,y)y' = Q(x,y)$ is a 1st order linear
homogenous ODE.
Proof. notice that

$$y' \sum a_{i_1 i_2} x^{i_1} y^{i_2} = \sum a_{j_1 j_2} x^{j_1} y^{j_2} \implies$$

$$\frac{1}{x^d} y' \sum a_{i_1 i_2} x^{i_1} y^{i_2} = \frac{1}{x^d} \sum a_{j_1 j_2} x^{j_1} y^{j_2} \implies$$

$$y' \sum a_{i_1 i_2} \frac{y^{i_2}}{x^{d-i_1}} = \sum b_{j_1 j_2} \frac{y^{j_2}}{x^{d-j_1}} \implies$$

$$y' \sum a_{i_1 i_2} (\frac{y}{x})^{i_2} = \sum b_{j_1 j_2} (\frac{y}{x})^{j_2} \implies$$

$$y' = \frac{\sum a_{i_1 i_2} (\frac{y}{x})^{i_2}}{\sum b_{j_1 j_2} (\frac{y}{x})^{j_2}}$$

which is hom. endproof

Def. (i) deg $a_{i_1, i_2 \dots i_n} x_1^{i_1}, x_2^{i_2}, \dots x_n^{i_n} = \sum_{k=1}^{n} i_{ki}$
(ii) deg $((p(x_i))$
where $p(x_1, x_2, \dots, x_n = \sum a_{i_1 i_2 \dots i_n} x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$

7.1 ex p.74 2

$$2xyy' = x^2 + 2y^2 \implies$$

$$y' = (\frac{1}{2} \frac{1}{y/x} + 2(y/x)) \text{ , } v = y/x \implies$$

$$y = vx \implies y' = v + xv' \text{ \& } xv' + v = \frac{1}{2v} + v \implies$$

$$v' = \frac{1}{2xv} \text{ (separable) } \implies$$

$$vv' = \frac{1}{2x} \implies$$

$$\int v dv = \frac{1}{2} \int \frac{1}{x} dx = \frac{1}{2} \ln |x| \implies$$

$$\frac{v^2}{2} = \frac{1}{2} \ln |x| + c \implies$$

$$v^2 = \ln |x| + c \implies$$

$$v = +-\sqrt{\ln|x| + c} \implies$$
$$\frac{y}{x} +-\sqrt{\ln|x| + c} \implies$$
$$y = +-x\sqrt{\ln|x| + c}$$

Def. A first order (normal)
berelli ODE has the

$$y' + yp(x) = y^n q(x)$$

side note(good book) Asmov PDE

8 Date: 02.11.2015

Recall: Bernulli ODE

$$y' + yp(x) = y^n q(x)$$

put $v = y^m$ then

$$\frac{dv}{dx} = my^{m-1}\frac{dy}{dx} \implies$$
$$my^{m-1}\frac{dy}{dx} + my^m p(x) = my^{m+n-1}q(x) \implies$$
$$\frac{dv}{dx}vmp(x) = my^{m+n-1}q(x)$$

want: $m + n - 1 = 0$. this requires
that $m = 1 - n$. $\therefore v = y^{1-n}$ reduses
a bernoulli ODE to a 1st order linear ODE.

8.1 p.74 25

$$y^2(xy' + y)(1 + x^4)^{1/2} = x$$
$$\implies (xy^2y' + y^3)\sqrt{1 + 4x} = x$$
$$\implies xy^2y'\sqrt{1 + x^4} + y^3\sqrt{1 + x^4} = x$$
$$\implies y'\sqrt{1 + x^4} + y\frac{\sqrt{1 + x^4}}{x} = y^{-2}$$
$$\implies y' + y\frac{1}{x} = y^{-2}\frac{1}{\sqrt{1 + x^4}}$$

put $v = y^3$ then

$$\frac{dv}{dx} = 3y^2\frac{dy}{dx} \implies$$
$$3y^2y' + \frac{3y^3}{x} = \frac{3}{\sqrt{1 + x^4}} \implies$$
$$v' + \frac{3v}{x} = \frac{3}{\sqrt{1 + x^4}} \text{ (linear)}$$

Here $p(x) = 3/x$; so

$$I(x) = e^{\int p(x)dx} = e^{3\ln|x|} = x^3$$

Thus,

$$v'x^{3\prime} + v3x^2 = \frac{3x^3}{\sqrt{1+x^4}} \implies$$

$$(vx^3)' = \frac{3x^3}{\sqrt{1+x^4}} \implies$$

$$vx^3 = 3\int \frac{x^3}{\sqrt{1+x^4}}dx = \frac{3}{4}\int w^{-1/2}dw$$

$$\frac{3}{4}\frac{2}{1}w^{1/2} + c$$

$$\frac{3}{2}(\sqrt{1+x^4} + c$$

$$w = 1 + x^4$$

$$w' = rx^3$$

$$y^3 = \frac{3}{2}(\frac{\sqrt{1+x^4}+c}{x^3}) \implies$$

$$y = (\frac{3}{2}(\frac{\sqrt{1+x^4}+c}{x^3}))^{1/3}$$

Exam 1 1. Given an ODE and a soln to it, verify it is a soln, then find a particular soln givin an initial cond.

2. Given a description of an ODE, write down the ODE.

3. Dropping ball from some h, find , ground time and speed.

4. high jump on earth given, find high jump on jupiter.

5. (a) solve ODEs

(b)

one is separable and the other is linear or bornulli

6. Torricelli problem.