Here we address the possibility of the char poly $\rho(x)$ of $\sum_{k=0}^{n} a_k y^{(k)} = 0$ with complex zeros. So, assuming that $r = a + bi$ is a zero of $\rho = \sum_{k=0}^{n} a_k x^k$, then we know that $y = e^{rk} = e^{(a+bi)x} = e^{ax}e^{(bx)i}$. Recall that if $a_k \in \mathbb{R}$ for $0 \leq k \leq n$, i.e., $\rho(x) \in \mathbb{R}[x]$, then $\bar{r} = a - bi$ ($b \pm 0$ is also a zero of $\rho(x)$. Thus, in the case of $r = a \pm bi$, we are "missing" now not just one real-valued basis element, but two real valued basis elements.

Date: 04.13.15

Euler's identity enabels us to determine there two missing basis elements:

$$(1)\ e^{ix} = \cos(x) = i\sin(x) \in \mathbb{C} \text{ for all } x \in \mathbb{R}$$

what led Euler to this identity are the Maclarin series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{n\geq 0} \frac{x^n}{n!};$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^10}{10!} + \cdots = \sum_{n\geq 0} (-1)^n \frac{x^{2n}}{2n!};$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^11}{11!} + \cdots = \sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{(2n+1)!};$$

These hold for all $x \in \mathbb{R}$. Notice that (1) simply gives pts on the unit circle in $\mathbb{C}$

$$|e^{ix}| = 1 \text{ for all } x \in \mathbb{R}$$

Note: $e^{i\pi} + 1 = 0$. $(x = \pi)$
Observe that e^{ix} is a complex-valued scalar mult of $\cos x$ & $\sin x$. So, back to the soln

$$\begin{aligned} y &= e^{ax}e^{(bx)i} \\ &= e^{ax}(\cos bx + i\sin bx) \\ &= e^{ax}\cos x + i(e^{ax}\sin bx), \end{aligned}$$

which is likewise a complex-valued linear combo of $e^{ax}\cos bx$ & $e^{ax}\sin bx$. These are to real-balued soln to the ODE $\sum_{k=0}^{n} a_k y^{(k)} = 0$, i.e., there are the two missing basis elements. therefore if $(x-r)^m$, where $r = a+bi$, is a factor of $\rho(x)$, then the corespoinding basis elements are:

$$\begin{gathered} e^{ax}\cos bx \ , \ e^{ax}\sin bx \\ xe^{ax}\cos bx \ , \ xe^{ax}\sin bx \\ x^2e^{ax}\cos bx \ , \ x^2e^{ax}\sin bx \\ \vdots \vdots \\ x^{m-1}e^{ax}\cos bx \ , \ x^{m-1}e^{ax}\sin bx \end{gathered}$$

Here there are the missing 2m many basis elements. Date: 04.14.15

Example.

$$y'' + y' - y = \sin^2 x$$

Here $\sum_{k=0}^{n} y^{(k)} = f(x)$, where $n = 2$ and $f(x) = \sin^2 x$. Notice that

$$\begin{aligned} y &= \sin^2 x \implies \\ y' &= 2\sin x\cos x \implies \\ y'' &= 2\cos^2 x - 2\sin^2 x. \end{aligned}$$

$$\begin{aligned} y &= \sin^2 x; \\ y' &= 2\sin x\cos x = \sin 2x \\ y'' &= 2\cos^2 x - 2\sin^2 x = 2\cos 2x. \end{aligned}$$

the terms in there derivatives are

$$\begin{aligned} 1, \\ \sin^2 x, \\ \sin x\cos x \\ \cos^2 x. \end{aligned}$$

Another way to see these terms is as

$$\begin{aligned} 1, \\ \cos 2x \\ \sin 2x. \end{aligned}$$

We consider a posible particular soln

$$y_p = a + b\cos 2x + c\sin 2x,$$

which is a linear combo of the terms above. Note that y_p is a particular soln iff

$$y_p'' + y_p' - y_p = \sin^2 x$$

Now,

$$\begin{aligned} y_p &= a + b\cos 2x + c\sin 2x \implies \\ y_p' &= -2b\sin 2x + 2c\cos 2x \implies \\ y_p'' &= -4b\cos 2x - 4c\sin 2x \implies \end{aligned}$$

$$y_p'' + y_p' - y_p = \sin^2 x \iff$$

$$(2c - 4b)\cos 2x - (2b + 4c)\sin 2x - b\cos 2x - c\sin 2x - a = \sin^2 x \iff$$

$$\begin{aligned} (2c - 5b)\cos 2x - (2b + 5c)\sin 2x - a = \frac{1 - \cos 2x}{2} \iff \\ -a - 1/2 + (2c - 5b + 1/2)\cos 2x - (2b + 5c)\sin 2x = 0. \end{aligned}$$

This last equation is an identity if

$$a = -1/2 \ \& \ \begin{cases} -5b + 2c + 1/2 & = 0 \\ 2b + 5c & = 0. \end{cases}$$

Note:

$$\begin{cases} -10b + 4c & = -1 \\ 2b + 5c & = 0 \end{cases}$$

$$\begin{bmatrix} 10 & 25 & | & 0 \\ -10 & 4 & | & -1 \end{bmatrix}$$

$$\begin{aligned} c &= -1/29 \\ b &= 5/58 \end{aligned}$$

Therefore

$$y_p = -1/2 + \frac{5}{58}\cos 2x - \frac{1}{29}\sin 2x,$$

Which you have cheked is an actual soln to $y'' + y' - y = \sin^2 x$.

Date: 04.15.15

Example.

$$y'' + y' - y = \sec x$$

$$\begin{aligned} f(x) &= \sec x \\ f'(x) &= \sec x \tan x \\ f''(x) &= \sec x \tan^2 x + \sec^3 x \\ &= \sec x(\sec^2 x - 1) + \sec^3 x \\ &= 2\sec^3 x - \sec x \end{aligned}$$

Put

$$B = \{1, \sec x, \sec x \tan x, \sec^3 x\}$$

Also, put

$$y_p = a + b\sec x + c\sec x\tan x + d\sec^3 x$$

$$y_p'$$

Since $\sec^3 x \tan x \notin \operatorname{Span} B$, the method of undetermined coefis fails.

Idea: Consider $B = \{y_1, \ldots, y_n\}$, the basis of $\sum_{k=0}^{n} a_k y^{(k)} = 0$, and

$$(1)\ y_p = \sum_{k=1}^{n} v_k(x) y_k(x).$$

We will show that a particular soln of $\sum_{k=0}^{n} a_k y^{(k)} = f(x)$ always has form (1) for some $v_k(x) \in C^{(n)}(I)$. this is called " Variation of parameters."

Date: 04.16.15

We want a particular soln $y = v_1(x)y_1(x) + v_2(x)y_2(x)$ to $\sum_{k=1}^{2} a_k y^{(k)} = f(x)$, where the hom soln space has basis $B = \{y_1, y_2\} \subset C^{(2)}(I)$. Here we require

$$y_p'' + py_p' + qy_p = f(x) \text{ (monic).}$$

$$y_p = v_1 y_1 + v_2 y_2 \implies$$

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$

If $v_1' y_1 + v_2' y_2 = 0$ then

$$y_p' = v_1 y_1' + v_2 y_2'$$

Thus,

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' v_2 y_2''$$

$$y_p'' + py_p' + qy_p = f(x) \iff$$

$$v_1' y_1' + v_1 y_1'' + v_2' y_2' v_2 y_2'' + pv_1 y_1' + pv_2 y_2' + qv_2 y_1 + qv_2 y_2 = f(x) \iff$$

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1' y_1' + v_2' y_2' = f(x) \iff$$

$$v_1' y_1' + v_2' y_2' = f(x).$$

therefore
$\begin{cases} y_1' v_1' + y_1' v_2' = f(x) \\ y_1 v_1' + y_2 v_2' = 0 \end{cases}$
iff

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore

$$v_1 = -\int \frac{y_2(x) f(x)}{|w(x)|} dx$$

and

$$v_2 = \int \frac{y_1(x) f(x)}{|w(x)|} dx$$

Hence,

$$y_p(x) =$$

Now, in general, consider

$$\sum_{k=0}^{n} a_k y^{(k)} = f(x)$$

with basis $B = \{y_j \in C^{(n)}(I) | 1 \leq j \leq n\}$ for the assoc hom ODE. Also, consider

$$y_p = \sum_{j=1}^{n} v_j y_j,$$

then

$$\begin{aligned} y_p' &= \sum_{j=1}^{n} (v_j' y_j + v_j y_j') \\ &= \sum_{j=1}^{n} v_j y_j' \text{ , if } \sum_{j=1}^{n} v_j' y_j = 0. \end{aligned}$$

Now,

$$\begin{aligned} y_p'' &= \sum_{j=1}^{n} (v_j' y_j' + v_j y_j'') \\ &= \sum_{j=1}^{n} y_j y_j'' \text{ , if } \sum_{j=1}^{n} v_j' y_j' = 0 \end{aligned}$$

continuing,

$$y_p^{(k)} = \sum_{j=1}^{n} v_j y_j^{(k)} \text{ , if } \sum_{j=1}^{n} v_j' y_j^{(k-1)} = 0$$

for $1 \leq k \leq n-1$. Finally,

$$\begin{aligned} y_p^{(n)} &= \sum_{j=1}^{n} (v_j' y_j^{(n-1)} + v_j y_j^{(n)}) \\ &= \sum_{j=1}^{n} v_j' y_j^{(n-1)} + \sum_{j=1}^{n} v_j y_j^{(n)}) \end{aligned}$$

Therefore

$$\sum_{k=1}^{n} a_k y_p^{(k)} = f(x) \iff$$

$$a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} + a_n \sum_{j=1}^{n} v_j y_j^{(n)} + \sum_{k=1}^{n-1} a_k \sum_{j=1}^{n} v_j y_j^{(k)} = f(x) \iff$$

$$a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} + \sum_{j=1}^{n} a_n v_j y_j^{(n)} + \sum_{j=1}^{n} \sum_{k=1}^{n=1} a_k v_j y_j^{(k)} = f(x) \iff$$

$$a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} + \sum_{j=1}^{n} v_j \sum_{k=1}^{n} a_k y_j^{(k)} = f(x) \iff$$

$$a_n \sum_{j=1}^{n} v_n' y_j^{(n-1)} = f(x)$$

since $y_j \in B$. Thus,

$$\sum_{j=1}^{n} v_j' y_j = 0 \text{ 1st eqn}$$

$$\sum_{j=1}^{n} v_j' y_j' = 0 \text{ 2nd eqn}$$

$$\vdots$$

$$\sum_{j=1}^{n} v_j' y_j^{(k-1)} = 0 \text{ kth eqn}$$

$$\vdots$$

$$\sum_{j=1}^{n} v_j' y_j^{(n-2)} = 0 \text{ (n-1)th eqn}$$

$$a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} = f(x) \text{ nth eqn}$$

Now,

$$\begin{aligned} y_p' &= v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2' \implies \\ y_p'' &= v_1''y_1 + v_1'y_1' + v_1'y_1' + v_1y_1'' + v_2''y_2 + v_2'y_2' + v_2'y_2' + v_2y_2'' \\ &= 2(v_1'y_1' + v_2'y_2') + v_1y_1'' + v_2y_2'', \end{aligned}$$

if we impose the condition $v_1''y_1 + v_2''y_2 = 0$. therefore

$$\begin{aligned} f(x) &= y_p'' + py_p' + qy_p \\ &= 2(v_1'y_1' + v_2'y_2') + v_1y_1'' + v_2y_2'' + pv_1'y_1 + pv_1y_1' + pv_2'y_{1p}v_2y_2' + q_vy_1 + qv_2y_2 \\ &= v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + 2(v_1'y_1' + v_2'y_2') + p(v_1'y_1' + v_2'y_2) \\ &= 2(v_1'y_1' + v_2'y_2') + p(v_1'y_1 + v_2'y_2) \end{aligned}$$

therefore,

$$f(x) = 2(v_1'y_1' + v_2'y_2') + p(v_1'y_1 + v_2'y_2)$$

implies

$$f(x) = v_1'(2y_1' + py_1) + v_2'(2y_2' + py_2)$$

We want a particular soln $y = v_1(x)y_1(x) + v_2(x)y_2(x)$ to $\sum_{k=1}^{2} a_k y^{(k)} = f(x)$, where the hom soln space has basis $B = \{y_1, y_2\} \subset C^{(2)}(I)$. Here we require

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$$y_p'' + py_p' + qy_p = f(x) \text{ (monic).}$$

$$y_p = v_1y_1 + v_2y_2 \implies$$

$$y_p' = v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2'$$

If $v_1'y_1 + v_2'y_2 = 0$ then

$$y_p' = v_1y_1' + v_2y_2'$$

Thus,

$$y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2'v_2y_2''$$

$$y_p'' + py_p' + qy_p = f(x) \iff$$

$$v_1'y_1' + v_1y_1'' + v_2'y_2'v_2y_2'' + pv_1y_1' + pv_2y_2' + qv_2y_1 + qv_2y_2 = f(x) \iff$$

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1'y_1' + v_2'y_2' = f(x) \iff$$

$$v_1'y_1' + v_2'y_2' = f(x).$$

therefore

$$\begin{cases} y_1'v_1' + y_1'v_2' = f(x) \\ y_1v_1' + y_2v_2' = 0 \end{cases}$$

iff

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore

$$v_1 = -\int \frac{y_2(x)f(x)}{|w(x)|}dx$$

and

$$v_2 = \int \frac{y_1(x)f(x)}{|w(x)|}dx$$

Hence,

$$y_p(x) =$$

Now, in general, consider

$$\sum_{k=0}^{n} a_k y^{(k)} = f(x)$$

with basis $B = \{y_j \in C^{(n)}(I) | 1 \leq j \leq n\}$ for the assoc hom ODE. Also, consider

$$y_p = \sum_{j=1}^{n} v_j y_j,$$

then

$$\begin{aligned} y_p' &= \sum_{j=1}^{n} (v_j' y_j + v_j y_j') \\ &= \sum_{j=1}^{n} v_j y_j' \text{ , if } \sum_{j=1}^{n} v_j' y_j = 0. \end{aligned}$$

Now,

$$\begin{aligned} y_p'' &= \sum_{j=1}^{n} (v_j' y_j' + v_j y_j'') \\ &= \sum_{j=1}^{n} y_j y_j'' \text{ , if } \sum_{j=1}^{n} v_j' y_j' = 0 \end{aligned}$$

continuing,

$$y_p^{(k)} = \sum_{j=1}^{n} v_j y_j^{(k)} \text{ , if } \sum_{j=1}^{n} v_j' y_j^{(k-1)} = 0$$

for $1 \leq k \leq n-1$. Finally,

$$\begin{aligned} y_p^{(n)} &= \sum_{j=1}^{n} (v_j' y_j^{(n-1)} + v_j y_j^{(n)}) \\ &= \sum_{j=1}^{n} v_j' y_j^{(n-1)} + \sum_{j=1}^{n} v_j y_j^{(n)}) \end{aligned}$$

Therefore

$$\sum_{k=1}^{n} a_k y_p^{(k)} = f(x) \iff$$

$$a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} + a_n \sum_{j=1}^{n} v_j y_j^{(n)} + \sum_{k=1}^{n-1} a_k \sum_{j=1}^{n} v_j y_j^{(k)} = f(x) \iff$$

$$a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} + \sum_{j=1}^{n} a_n v_j y_j^{(n)} + \sum_{j=1}^{n} \sum_{k=1}^{n=1} a_k v_j y_j^{(k)} = f(x) \iff$$

$$a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} + \sum_{j=1}^{n} v_j \sum_{k=1}^{n} a_k y_j^{(k)} = f(x) \iff$$

$$a_n \sum_{j=1}^{n} v_n' y_j^{(n-1)} = f(x)$$

since $y_j \in B$. Thus,

$$\begin{aligned}
\sum_{j=1}^{n} v_j' y_j &= 0 \text{ (1st eqn)}\\
\sum_{j=1}^{n} v_j' y_j' &= 0 \text{ (2nd eqn)}\\
&\vdots\\
\sum_{j=1}^{n} v_j' y_j^{(k-1)} &= 0 \text{ (kth eqn)}\\
&\vdots\\
\sum_{j=1}^{n} v_j' y_j^{(n-2)} &= 0 \text{ ((n-1)th eqn)}\\
a_n \sum_{j=1}^{n} v_j' y_j^{(n-1)} &= f(x) \text{ (nth eqn)}
\end{aligned}$$

$$\implies W(y_j)_{1\le j\le n} \begin{pmatrix} v_1' \\ \vdots \\ v_n' \end{pmatrix} = f(x)\vec{e_n}$$

Notice that $|W(y_j)_{1\le j\le n}| \neq 0$ for all $x \in I$ since $y_j s$ are linearly independent (being that $y_j \in B$). Thus, by Cramer's Rule,

$$v_j' = \frac{D_j}{D},$$

where $D = |W(y_j)_{1\le j\le n}|$ and

$$D_j = \begin{vmatrix} y_1 & \cdots & 0 & \cdots & y_n \\ y_1' & \cdots & 0 & \cdots & y_n' \\ y_1'' & \cdots & 0 & \cdots & y_n'' \\ \vdots & & \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & f(x) & \cdots & y_n^{(n-1)} \end{vmatrix} = (-1)^{n+j} f(x) |W_{n-1}(y_k)_{\substack{1\le k\le n \\ j\neq k}}|$$

Therefore

$$v_k' = (-1)^{j+n} f(x) \frac{|W_{n-1}(y_k)_{\substack{1\le k\le n \\ j\ne k}}|}{a_n |W_n(y_k)_{1\le k\le n}|}$$

Thus,

$$(1)\ v_k(x) = \frac{(-1)^{j+n}}{a_n} \int f(x) \frac{|W_{n-1}(y_k)_{\substack{1\le k\le n \\ j\ne k}}|}{a_n |W_n(y_k)_{1\le k\le n}|} dx,$$

Where $y_p(x) = \sum_{k=1}^{n} v_k(x) y_k(x)$ is a particular soln to $\sum_{k=1}^{n} a_k y^{(k)} = f(x)$. (1) is called the variation of parameters formulas.

Ch 4 systems of ODEs

Example. Consider a curve $\vec{r} : [a, b] \to \mathbb{R}^n$, where $\vec{r}(t) = (x_1(t), \ldots, x_n(t))$ and $t \in [a, b]$. By Newton's 2nd Law, $\vec{F} = m\vec{a} = m\vec{r}''(t)$. Thus,

$$\begin{pmatrix} m(t)x_1''(t) \\ m(t)x_2''(t) \\ \vdots \\ m(t)x_n''(t) \end{pmatrix} = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix},$$

which is a system of n many 2nd order ODEs in $(x_1, \ldots, x_n)$

Date: 04.21.15

Date: 04.22.15

Example. Consider an

$$(1)\ y^{(n)} + \sum_{k=0}^{n-1} p_k(x) = f(x),$$

$$y^{(n)} + p_{n-1} y^{(n-1)} + \cdots + p_2(x) y'' + p_1(x) y' + p_0(x) y = f(x).$$

put $y_k = y^{(k-1)}$ for $1 \le k \le n+1$, then

$$\begin{aligned} y_1 &= y \implies \\ y_2 &= y' = y_1' \implies \\ y_3 &= y'' = y_1'' = y_2' \implies \\ y_4 &= y''' = y_1''' = y_2'' = y_3' \implies \\ &\vdots \\ y_{n+1} &= y^{(n)} = y_n' \end{aligned}$$

Thus, (1) becomes

$$(2)\ y_n' + \sum_{k=0}^{n-1} p_k(x) y_k' = f(x),$$

which is a first order linear ODE. puting (2) whith the alone n-1 many 1st order ODEs yields the following 1st order system: with n many ODEs:

$$\begin{cases} y_n' + \sum_{k=0}^{n-1} p(x) y_k' = f(x) \\ y_1' = y_2 \\ y_2' = y_3 \\ \vdots \\ y_{n-1}' = y_n \end{cases}$$

Again, this is a system of n many 1st linear ODEs from a single nth order linear ODE. This gives yet another example to motivate studying systems of linear ODEs.

Example. page 255 number 12

$$\begin{cases} x' = y \\ y' = x \end{cases}$$

Note: $x = y' = x''$. So, this yields $x'' - x = 0$, which has basis $B = \{\cosh t, \sinh t\}$. Thus,

$$x = a \cosh t + b \sinh t$$

Now,

$$y = b \cosh t + a \sinh t$$

Notice that if $\vec{r}(t) = (x(t), y(t))$ then $\vec{r}'(t) = (x'(t), y'(t)) = (y(t), x(t))$. We want such an $\vec{r}(t)$. In otherwords, we may view the above system as a single. 1st order ODE of a parametric function.
Recall that

$$\cosh(\alpha + t) = \cosh \alpha \cosh t + \sinh \alpha \sinh t$$

Take $A \in \mathbb{R}$ st $a/A \geq 1$, then $a/A \in$ Range (cosh). Now, we want $\alpha \in \mathbb{R}$ st

$$a = A \cosh \alpha \ \&\ b = A \sinh \alpha.$$

So,
$\coth\alpha = \dfrac{\cosh\alpha}{\sinh\alpha} = \dfrac{a}{b}$ holds for some α st $|\alpha| < 1$ since Range(coth) $= \mathbb{R}$.
therefore for this α,

$$\begin{aligned} x &= a\cosh t + b\sinh t \\ &= A\cosh\alpha\cosh t + A\sinh\alpha\sinh t \\ &= A\cosh(\alpha + t). \end{aligned}$$

therefore $x = A\cosh(\alpha + t)$; whence, $y = x' = A\sinh(\alpha + t)$.
Hence,

$$x^2 - y^2 = A^2 \implies \frac{x^2}{A^2} - \frac{y^2}{A^2} = 1$$

These are the solns to $\vec{r}'(t) = (x'(t), y'(t))$. Note : Range(cosh) $= [1,)$. So, if $A > 0$ then $x(t) > 0$; where, $\vec{r}(t)$ coresponds to the righthand branch. Whereas, if $A < 0$ then $\vec{r}(t)$ coresponds to the left hand branch.
Notice that we soved this linear system by "substitution."

Date: 04.23.15

Example. (Recall).
let $\vec{r}(t) = (x(t), y(t))$, and consider $\vec{r}'(t) = (x'(t), y'(t))$, then

$$\begin{cases} x'(t) = y(t) \\ y'(t) = x(t) \end{cases} \iff \begin{cases} x'(t) - y(t) = 0 \\ y'(t) - x(t) = 0 \end{cases} \iff \begin{cases} L_{11}x + L_{12}y = 0 \\ L_{21}x + L_{22}y = 0 \end{cases} \iff$$

$$\begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

Where $L_{11} = D = L_{22}$ and $L_{12} = -1 = L_{21}$
In other symbols,

$$\begin{pmatrix} D & -1 \\ -1 & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

Recall that $\mathbb{R}[D] \equiv \mathbb{R}[x]$; in particular, $\mathbb{R}[D]$ is communative "ring" (multiplicative communative).
Solve by "elimination:"

$$L_{21}(L_{11}x + L_{12}y) = L_{21}0 \implies$$

$$L_{21}L_{11}x + L_{21}L_{12}y = L_{21}0$$

$$L_{11}(L_{21}x + L_{21}y) = L_{11}0 \implies$$

$$L_{11}L_{21}x + L_{11}L_{21}y = 0$$

$$L_{21}L_{12}y - L_{11}L_{22}y = 0 \implies$$

$$(L_{21}L_{12} - L_{11}L_{22})y = 0 \implies$$

$$\begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} y = 0$$

Anologosly

$$\begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} x = 0$$

Therefore

$$(|L|I) \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

where

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

Recall: If A is a sq matrix then

$$AA^a = |A|I$$

This holds for any A over a commutative ring, e.g.,

$$A \in \text{ Mat }_n(R)$$

where R is a commutative ring say $R = \mathbb{C}$
Thus if $L \in \text{ Mat}_n(\mathbb{R}[D])$ then

$$L^a L = |L|I_n,$$

where I_n $\text{Mat}_n(\mathbb{R}[D])$,

$$I_n = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \ \& \ 1_{Id} \in \mathbb{R}[D]$$

consider $L \in \text{ Mat}_n(\mathbb{R}[D])$,
$\vec{x}(t) = (x_1(t), \ldots, x_n(t))$, $\vec{F}(t) = (F_1(t), \ldots, F_n(t)) \in (C^{(d)}(I))^n$, where $d = \max\{ \text{ deg } (L_{ij}) \mid 1 \leq i\ , j \leq n\}$ and $L = (L_{ij})$, and the system of linear ODEs

$$(1)\ L\vec{x} = \vec{F} \iff \begin{pmatrix} L_{11} & L_{12} & \cdots L_{1n} \\ L_{21} & L_{22} & \cdots L_{2n} \\ \vdots & & \\ L_{n1} & L_{n2} & \cdots L_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix}$$

Here

$$L_{ij} = \sum_{k=0}^{m_{ij}} a_{i_k j_k} D^k \in \mathbb{R}[D]$$

Applying the adjoint formula to (1)

$$(2)\ (|L|I_n)\vec{x} = L^a \vec{F}.$$

Recall: $A^a = (c_{ij})^T$, $C_{ij} = (-1)^{i+j}|M_{ij}|$
In component form, (2) says that

$$(3)\ |L|x_i(t) =_i (L^a)\vec{F}(t),$$

where $_i(L^a)$ is the ith row of L^a. Notice that (3) is a linear ODE in the single unknown funct $x_i(t)$. therefore previous methods can be used to solve (3).

Now, we consider first order linear systems, at first, without constant coefficients. These have the form Date: 04.27.15

$$\vec{x}'(t) + p(t)\vec{x}(t) = \vec{q}(t),$$

where $\vec{x} : I \subseteq \mathbb{R} \to \mathbb{R}^n$ and $p(t) \in \text{ Mat}_n(C(I))$ and $\vec{q} \in (C^{(1)}(I))^n$.
In "normal form" this becomes

$$\vec{x}\,'(t) = p(t)\vec{x}(t) + \vec{q}(t)$$

Aside: $R[D]$ is a communative "ring." however, $(C(t))[D]$ this is not a communative, the point is that to use the det proputies developed last semester, we need $\text{ Mat}_n(\mathbb{R})$, where $\mathbb{R}$ is commutative ring (then thos proff genral

Example. $(((C)(I))[D]$ is not commutative).
Put

$$L_1 = -tD + 1$$
$$L_2 = t^2 D - 3,$$

then $L_1, L_2 \in (C^{(\infty)}(\mathbb{R}))[D]$. Notice that

$$\begin{aligned} L_2 x &= (t^2 D - 3)x \\ &= t^2 Dx - 3x \\ &= t^2 x' - 3x \implies \end{aligned}$$

$$\begin{aligned} L_1 L_2 x &= (-tD + 1)L_2 x \\ &= (-tD + 1)(t^2 Dx - 3x) \\ &= -tD(t^2 Dx - 3x) + t^2 Dx - 3x \\ &= -tD(t^2 Dx) + 3tDx + t^2 Dx - 3x \\ &= -t(2tDx + t^2 D^2 x) + 3 + Dx + t^2 Dx - 3x \\ &= -2t^2 Dx - t^3 D^2 x + 3tDx + t^2 Dx - 3x \\ &= -t^2 Dx - t^3 D^2 x + 3tDx - 3x. \end{aligned}$$

Also,

$$\begin{aligned} L_1 x &= (-tD + 1)x \\ &= -tDx + x \implies \\ L_2 L_1 x &= (t^2 D - 3)L_1 x \\ &= t^2 D L_1 x - 3L_1 x \\ &= t^2 D(-tDx + x) + 3tDx - 3x \\ &= t^2 D(-tDx) + t^2 Dx + 3tDx - 3x \\ &= t^2(-Dx - tD^2 x) + t^2 Dx + 3tDx - 3x \\ &= -t^2 Dx - t^2 D^2 x + t^2 dx + 3tDx - 3x \\ &= -t^3 D^2 x + 3tDx - 3x. \end{aligned}$$

therefore $L_1 L_2 \neq L_2 L_1$.

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$$\vec{x}\,'(t) = p(t)\vec{x}(t) + \vec{q}(t)$$

where $p(t) \in \operatorname{Mat}_n(C(I))$. Now, we consider the case where $p(t) \in \operatorname{Mat}_n(\mathbb{R})$, i.e., the entries of the matrix p are costant functions. In the hom case, this becomes

$$\vec{x}\,'(t) = A\vec{x}(t)$$

or

$$(1)\ \vec{x}\,' = A\vec{x}$$

where $A \in \operatorname{Mat}_n(\mathbb{R})$.
Notice that if $n = 1$ then (1) becomes

$$x' = ax,$$

Which has solns $x(t) = ke^{at}$. This leads us to a conjecture in the more general case,

$$x_i(t) = k_i e^{a_i t} \text{ for } 1 \leq i \leq n.$$

In other words,

$$\vec{x}(t) = \begin{pmatrix} k_1 e^{a_1 t} \\ k_2 e^{a_2 t} \\ \vdots \\ k_n e^{a_n t} \end{pmatrix}$$

Since $\vec{x}' = (x_i')$,

$$\vec{x}'(t) = \begin{pmatrix} a_1 k_1 e^{a_1 t} \\ a_2 k_2 e^{a_2 t} \\ \vdots \\ a_n k_n e^{a_n t} \end{pmatrix}$$

therefore this $\vec{x}(t)$ is a soln to (1) $\iff$

$$\begin{pmatrix} a_1 k_1 e^{a_1 t} \\ a_2 k_2 e^{a_2 t} \\ \vdots \\ a_n k_n e^{a_n t} \end{pmatrix} = A \begin{pmatrix} k_1 e^{a_1 t} \\ k_2 e^{a_2 t} \\ \vdots \\ k_n e^{a_n t} \end{pmatrix}$$

Here we see that if $a_i = a_j$ for all i and j , say $a = a_i$, then $\vec{x}'(t) = a\vec{x}(t)$ and so, (1) becomes

$$a\vec{x}(t) = A\vec{x}(t)$$

Now, notice that

$$\vec{x}(t) = e^{at}\vec{k},$$

where $\vec{k} = (k_1, k_2, \ldots, k_n)$. therefore (1) now becomes

$$ae^{at}\vec{k} = A(e^{at}\vec{k}) \implies$$

$$A\vec{k} = a\vec{k}.$$

Recall: an eigenvalue of $A \in \text{ Mat}_n(\mathbb{R})$ is $\lambda \in \mathbb{R}$ iff there is $\vec{0} \neq \vec{v} \in \mathbb{R}^n$ st

$$A\vec{v} = \lambda\vec{v}.$$

Here $\vec{v}$ is called an eigevector assoc with λ. therefore from above, we see that $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a soln to $\vec{x}' = A\vec{x}$ iff λ is an eiginvalue of A and $\vec{v}$ is an assoc eigenvector of λ.
Recall: to find eigenvalues of $A \in \text{ Mat}_n(\mathbb{R})$, notice that for $\vec{v} \neq \vec{0}$,

$$\begin{aligned} A\vec{v}\lambda\vec{v} &\iff \lambda\vec{v} - A\vec{v} = \vec{0} \\ &\iff (\lambda I_n - A)\vec{v} = \vec{0}, \end{aligned}$$

which is a hom system with a nontivial soln $\vec{v}$. So, therefore

$$\det(\lambda I_n - A) = 0.$$

Also, recall that the char poly of $A \in \text{ Mat}_n(\mathbb{R})$,

$$p_A(x) = \det(xI_n - A) \in \mathbb{R}[x],$$

which is an nth degree poly over $\mathbb{R}$. So, therefore to find eigenvalues of A, we must find the zeros of $p_A(x)$. Once we have the eigenvalue λ, to find an assoc $\vec{v} \pm \vec{0}$, we solve for $\vec{v}$ in $(A - \lambda I_n)\vec{v} = \vec{0}$.