Theorem. *(Existence - Uniqueness Thrm, $\exists!$ Thrm). If I is an interval, $a \in I$, and p_u for all $1 \leq k \leq n-1$, and f are cts on I, then $\forall b_i$, for $0 \leq i \leq n-1$, (then initial value problem)*

$$(1)\ y^{(n)} + \sum_{k=1}^{n} y^{(n-k)} p_u(x) = f(x)$$

& $y^{(i)}(a) = b_i$, has a unique soln on I.
Note: the solns to linear ODEs are unique on the whole interval I.

Theorem. *Thrm. If $w = \{y \in \mathscr{D}^{(n)}(I)\ /\ y$ satisfies (1)$\}$ then dim $W \geq n$.*

Proof. put $y_j^{(i)}(a) = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases}$$

for $1 \leq i,\ j \leq n$. By the $\exists!$ Thrm, for each j there is a unique soln to (1) on I. Now,

$$\begin{aligned} W(a) &= (y_j^{(i)}(a)) \in \text{ Mat}_n(\mathbb{R}) \\ &= I_n \end{aligned}$$

$$\therefore |W(a)| = 1 \neq 0$$

whence, $y_1, \ldots, y_n$ are linearly independent . Hence,

$$dim W \leq n$$

□

Theorem. *Thrm (Strong wronskian converse) If $y_1, \ldots, y_n$ are linearly independent (in $\mathscr{D}^{(n)}(I)$) and $y_1, \ldots, y_n$ are solns to*

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$$y^{(n)} + \sum_{k=1}^{n} y^{(n-k)} p_k(x) = 0$$

where p_k are cts on I for $k = 1, \ldots, n$, then $|W(a) \neq 0$ for all $a \in I$.

Proof. Assume that there is an $a \in I$ st $|W(a)| = 0$, then the linear system

$$(1)\ W(a)\vec{x} = \vec{0}$$

has a nontrivial soln, say $\vec{x} = \vec{c} \in \mathbb{R}^n$
$(W(a) \in \text{Mat}_n(\mathbb{R}))$. Denote: $\vec{c} = (c_1, \ldots, c_n) \neq \vec{0}$.

Put $y = \sum_{j=1}^{n} c_j y_j$, then y satisfies (*) and

$$y(a) = \sum_{j=1}^{n} c_j y_j(a) = 0 \implies$$

$$y'(a) = \sum_{j=1}^{n} c_j y_j'(a) = 0 \implies$$

$$y''(a) = \sum_{j=1}^{n} c_j y_j''(a) = 0 \implies$$

$$\vdots$$

$$y^{(n-1)}(a) = \sum_{j=1}^{n} c_j y_j^{(n-1)}(a) = 0 \text{ (by(1))}$$

Notice that $y \equiv 0$ on I also satisfies (*) and is such that $y^{(k)}(a) = 0$ for $1 \leq k \leq n-1$. $\therefore$ by $\exists!$ thrm,

$$\sum_{j=1}^{n} c_j y_j \equiv 0 \text{ on } I$$

Thus, since $y_1, \ldots, y_n$ are lin ind, all $c_j = 0$, which is a contradiction. $\square$

Theorem. *If*

$$W = \{y \in \mathbb{R}^{(n)}(I) \ / \ y \textit{ satisfies (*) } \}$$

then dim $W \leq n$

Proof. let $y_1, \ldots, y_n$ be lin ind solns of (*); we show that there are scalars $c_j \in \mathbb{R}$ st $y = \sum_{j=1}^{n} c_j y_j$ on I.
Consider the linear system

$$(2)\ W(a)\vec{x} = \begin{pmatrix} y(a) \\ y'(a) \\ \vdots \\ y^{(n-1)}(a) \end{pmatrix}$$

where $W(a)$ in the Wronskian matrix of $y_1, \ldots, y_n$. By the SWC, $|W(a)| \neq 0$.
Thus, (2) has a unique, not-trivial soln, say $\vec{0} \pm \vec{x} = \vec{c} \neq \vec{0} \in \mathbb{R}^n$. Thus

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$$W(a)\vec{c} = \begin{pmatrix} y(a) \\ y'(a) \\ \vdots \\ y^{(n-1)}(a) \end{pmatrix}$$

has rows

$$\sum_{j=1}^{n} c_j y_j(a) = y(a)$$
$$\sum_{j=1}^{n} c_j y_j'(a) = y'(a)$$
$$\sum_{j=1}^{n} c_j y_j''(a) = y''(a)$$
$$\vdots$$
$$\sum_{j=1}^{n} c_j y_j^{(n-1)}(a) = y^{(n-1)}(a)$$

Finally, since both y and $\sum_{j=1}^{n} c_j y_j$ are solns to the hom nth order linear ODE (*) (the ODE being home and y_j, for $1 \leq j \leq n$, being solns, so is any linear combo of y_js since the soln space to a linear home ODE is linear) and both y and $\sum_{j=1}^{n} c_j y_j$ satisfy all the same n many initial ?conbos?, by the $\exists!$ thrm, $y = \sum_{j=1}^{n} c_j y_j$ on I. $\therefore y \in \text{Span}\ \{y_j \in W \mid 1 \leq j \leq n\}$. Thus, dim $W \leq n$. $\square$

Corollary. *If*

$$W = \{y \in D^{(n)}(I) \ / \ y \textit{ satisfies (x) } \}$$

where

$$\textit{(*)}\ y^{(n)} + \sum_{k=1}^{n} y^{(n-k)} p_k(x) = 0$$

then dim $W = n$.

Proof. We showed that $n \leq$ dim$W \leq n$. $\square$

Example. $y'' - y = 0$
Note that both $y_1 = \cosh x$ and $y_2 = \sinh x$ are solns on $\mathbb{R}$. The soln space is $2-$ dim'l . Not also that if

$$a \cosh x + b \sinh x = 0 \text{ for all } x \in \mathbb{R}$$

then in particular, $x = 0 \implies a = 0$.
$\therefore b \sinh x = 0$ for all x.
However, if $x \neq 0$ then $\sinh x \neq 0$; whence, $b = 0$. Hence, $B = \{\cosh x\ ,\ \sinh x\}$ are linearly independent solns to $y'' = y$, and the soln space has dim'n 2, B is a basis for the soln space. $\therefore$ every soln to $y'' = y$ has the form

$$y = a\cosh x + b\sinh x$$

for some $a, b \in \mathbb{R}$. Likewise, $B' = \{e^x, e^{-x}\}$ is also a basis. Note:

$$|W(x)| = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x \begin{vmatrix} 1 & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

$\therefore$ by Wronsky's Thrm, B' is lin ind.

Example. Example: $y'' + y = 0$
Here $B = \{\cos x, \sin x\}$ is a basis.

<u>nth order hom linear ODEs with constant coefficients:</u>

$$(1)\ \sum_{k=0}^{n} a_k y^{(k)} = 0$$

consider

$$y = e^{rx}$$

then

$$\begin{aligned} y' &= re^{rx} = ry \implies \\ y'' &= r^2 e^{rx} = r^2 y \implies \\ &\vdots \\ y^{(k)} &= r^k y \end{aligned}$$

thus, $y = e^{rx}$ yields

$$\begin{aligned} 0 &= \sum_{k=0}^{n} a_k y^{(k)} \\ &= \sum_{k=0}^{n} a_k r^k y \\ &= y \sum_{k=0}^{n} a_k r^{(k)} \implies \end{aligned}$$

$$\sum_{k=0}^{n} a_k r^{(k)} = 0$$

$\therefore y = e^{rx}$ is a soln to (1) if r is a root (zero) of the char poly of $\sum_{k=0}^{n} a_k y^{(k)} = 0$,

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$$\rho(x)=\sum_{k=0}^{n} a_k x^k,$$

then $y=e^{rx}$ is a soln to $\sum_{k=0}^{n} a_k y^{(k)}=0$.

Example. If $r_1,\ldots,r_n\in\mathbb{R}$ are paiswise distinct then $e^{r_1x},\ldots,e^{r_nx}$ are linearly independent (in $\mathscr{F}(\mathbb{R})$. we use the Wronskian:

$$|W(x)|=\begin{vmatrix} e^{r_1x} & e^{r_2x} & \cdots & e^{r_nx}\\ r_1e^{r_1x} & r_2e^{r_2x} & \cdots & r_ne^{r_nx}\\ \vdots & \vdots & \ddots & \vdots\\ r_1^{n-1}e^{r_1x} & r_2^{n-1}e^{r_2x} & \cdots & r_n^{n-1}e^{r_nx}\end{vmatrix}$$

$$e^{(r_1+r_2+\cdots+r_n)x}=\begin{vmatrix} 1 & 1 & \cdots & 1\\ r_1 & r_2 & \cdots & r_n\\ \vdots & \vdots & \ddots & \vdots\\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1}\end{vmatrix}$$

$$e^{(r_1+r_2+\cdots+r_n)x}\prod_{1\le i<j\le n}(r_j-r_i)\neq 0$$

(for all x). therefore by the wronskian thrm, $e^{r_1x},e^{r_2x},\ldots,e^{r_nx}$ ar lin ind,

Example. (Vandermonds)
consider

$$V_n(x)=\begin{pmatrix} 1 & x & x^2 & \cdots & x^{n-1}\\ 1 & r_2 & r_2^2 & \cdots & r_2^{n-1}\\ 1 & r_3 & r_3^2 & \cdots & r_3^{n-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & r_n & r_n^2 & \cdots & r_n^{n-1}\end{pmatrix}$$

which is an $n\times n$ Vandermonde Matrix.
Put

$$f(x)=\det V_n(x)\in P_{n-1}[\mathbb{R}],$$

then $f(r_j)=\det(V_n(r_j))=0$ for all $2\le j\le n$, by the alternating property of determinates. therefore by the factor Thrm,

$$f(x)=a\prod_{2\le j\le n}(x-r_j),$$

where

$$a=(-1)^{n-1}\begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1}\\ 1 & r_2 & r_2^2 & \cdots & r_2^{n-1}\\ 1 & r_3 & r_3^2 & \cdots & r_3^{n-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & r_n & r_n^2 & \cdots & r_n^{n-1}\end{vmatrix},$$

which is an $(n-1)\times(n-1)$ det. therefore by the inductive hypothesis,

$$a = (-1)^{n-1} \prod_{2\le i<j\le n} (r_j - r_i)$$

therefore

$$f(x) = (-1)^{n-1} \prod_{2\le i<j\le n} (r_j - r_i) \prod_{2\le j\le n} (x - r_j)$$

Notice that

$$\prod_{2\le j\le n} (x - r_j) = (-1)^{n-1} \prod_{2\le j\le n} (r_j - x).$$

Therefore

$$\begin{aligned} f(x) &= (-1)^{n-1} \prod_{2\le i\le j\le n} (r_j - r_i)(-1)^{n-1} \prod_{2\le j\le n} (r_j - x) \\ &= \prod_{2\le j\le n} (r_j - x) \prod_{2\le i\le j\le n} (r_j - r_i). \end{aligned}$$

In particular,

$$\begin{aligned} \det V_n(r_1) &= f(r_1) \\ &= \prod_{2\le j\le n} (r_j - r_1) \prod_{2\le j\le i\le n} (r_j - r_i) \\ &= \prod_{1\le i\le j\le n} (r_j - r_i). \end{aligned}$$

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Recall: the characteristic poly of $\sum_{k=0}^{n} a_k y^{(k)}$ is

$$\rho(x) = \sum_{k=0}^{n} a_k x^{(k)},$$

and we showed that if $r_1, \dots, r_n$ are pairwise distinct then $B = \{e^{r_1 x}, \dots, e^{r_n x}\}$ is linearly independent (in $\mathscr{C}^{(\infty)}(\mathbb{R})$). We also showed that if r is a zero of $\rho(x)$ then $y = e^{rx}$ is a soln to $\sum_{k=0}^{n} a_k y^{(k)} = 0$. This all immediatly implies the following:

Theorem. *If $\rho(x) \sum_{k=0}^{n} a_k x^{(k)}$ has n many distinct real zeros then the soln space of*

$$(*) \ \sum_{k=0}^{n} a_k y^{(k)} = 0$$

*has basis $\{e^{r_1 x}, \dots, e^{r_n x}\}$, i.e., every soln of * has the form*

$$\sum_{j=1}^{n} c_j e^{r_j x}.$$

Example. (Revisited)
Consider $y'' - y = 0$. This has char poly

$$\rho(x) = x^2 - 1 = (x-1)(x+1),$$

which has zeros ± 1.
Thus, $B = \{e^x, e^{-x}\}$ is a basis for the soln space of

$$y'' - y = 0$$

Q: What are the mising basis elements if a char poly has repeating zeros?

Recall the "ring" of polynomials

$$\mathbb{R}[x] = \{\sum_{k=0}^{n} a_k x^k \mid n \in \mathbb{N} \ \& \ a_k \in \mathbb{R}\},$$

where

$$\sum_{k=0}^{n} a_k x^{(k)} + \sum_{k=0}^{n} b_k x^{(k)} = \sum_{k=0}^{n} (a_k + b_k) x^{(k)}$$

$$\sum_{i=0}^{m} a_i x^{(i)} \sum_{j=0}^{m} b_j x^{(j)} = \sum_{k=0}^{m+n} c_k x^{(k)}, \text{ where}$$

$$c_k = \sum_{i+j=k} a_i b_j.$$

<u>Notation</u>. Put $D = \frac{d}{dx} : \mathscr{C}^{(\infty)}(I)$
$D^0 = i$ (identity on $\mathscr{C}^{(\infty)}(I)$), and

$$D^n = DD^{n-1} \text{ for } n \in \mathbb{Z}^+.$$

Books

Hausen and sullivan "Real Analysis"
Reference: I.N. Herstein's "Topics in Abstract Algebra" (2nd ed), "Abstract Algebra" (3rd ed)
Artin's "Algebra"
(Formal poly : See hungerfords springer GTM.)