# Full Class Notes

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**Theorem.** *(Existence - Uniqueness Thrm, $\exists!$ Thrm ). If $I$ is an interval, $a \in I$, and $p_u$ for all $1 \leq k \leq n-1$, and $f$ are cts on $I$, then $\forall b_i$, for $0 \leq i \leq n-1$, (then initial value problem)*

$$(1)\ y^{(n)} + \sum_{k=1}^{n} y^{(n-k)} p_u(x) = f(x)$$

& $y^{(i)}(a) = b_i$ , has a unique soln on $I$.
Note: the solns to linear ODEs are unique on the whole interval $I$.

**Theorem.** *Thrm. If $w = \{y \in \mathscr{D}^{(n)}(I)\ /\ y$ satisfies (1)$\}$ then dim $W \geq n$.*

*Proof.* put $y_j^{(i)}(a) = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases}$$

for $1 \leq i,\ j \leq n$. By the $\exists!$ Thrm, for each $j$ there is a unique soln to (1) on $I$. Now,

$$\begin{aligned} W(a) &= (y_j^{(i)}(a)) \in \text{ Mat}_n(\mathbb{R}) \\ &= I_n \end{aligned}$$

$$\therefore |W(a)| = 1 \neq 0$$

whence, $y_1, \ldots, y_n$ are linearly independent . Hence,

$$dim W \leq n$$

$\square$

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**Theorem.** *Thrm (Strong wronskian converse) If $y_1, \ldots, y_n$ are linearly independent ( in $\mathscr{D}^{(n)}(I)$ ) and $y_1, \ldots, y_n$ are solns to*

$$y^{(n)} + \sum_{k=1}^{n} y^{(n-k)} p_k(x) = 0$$

*where $p_k$ are cts on $I$ for $k = 1, \ldots, n$, then $|W(a) \neq 0$ for all $a \in I$.*

*Proof.* Assume that there is an $a \in I$ st $|W(a)| = 0$, then the linear system

$$(1)\ W(a)\vec{x} = \vec{0}$$

has a nontrivial soln, say $\vec{x} = \vec{c} \in \mathbb{R}^n$
($W(a) \in \mathrm{Mat}_n(\mathbb{R})$). Denote: $\vec{c} = (c_1, \ldots, c_n) \neq \vec{0}$.

Put $y = \sum_{j=1}^{n} c_j y_j$, then $y$ satisfies (*) and

$$y(a) = \sum_{j=1}^{n} c_j y_j(a) = 0 \implies$$

$$y'(a) = \sum_{j=1}^{n} c_j y_j'(a) = 0 \implies$$

$$y''(a) = \sum_{j=1}^{n} c_j y_j''(a) = 0 \implies$$

$$\vdots$$

$$y^{(n-1)}(a) = \sum_{j=1}^{n} c_j y_j^{(n-1)}(a) = 0 \text{ ( by(1))}$$

Notice that $y \equiv 0$ on $I$ also satisfies (*) and is such that $y^{(k)}(a) = 0$ for $1 \leq k \leq n-1$. $\therefore$ by $\exists!$ thrm,

$$\sum_{j=1}^{n} c_j y_j \equiv 0 \text{ on } I$$

Thus, since $y_1, \ldots, y_n$ are lin ind, all $c_j = 0$, which is a contradiction. □

**Theorem.** *If*

$$W = \{y \in \mathbb{R}^{(n)}(I) \ / \ y \text{ satisfies (*) }\}$$

*then dim $W \leq n$*

*Proof.* let $y_1, \dots, y_n$ be lin ind solns of (*); we show that there are scalars $c_j \in \mathbb{R}$ st $y = \sum_{j=1}^n c_j y_j$ on $I$.
Consider the linear system

$$(2)\ W(a)\vec{x} = \begin{pmatrix} y(a) \\ y'(a) \\ \vdots \\ y^{(n-1)}(a) \end{pmatrix}$$

where $W(a)$ in the Wronskian matrix of $y_1, \dots, y_n$. By the SWC, $|W(a)| \neq 0$. Thus, (2) has a unique, not-trivial soln, say $\vec{0} \pm \vec{x} = \vec{c} \neq \vec{0} \in \mathbb{R}^n$. Thus Date: 03.25.15

$$W(a)\vec{c} = \begin{pmatrix} y(a) \\ y'(a) \\ \vdots \\ y^{(n-1)}(a) \end{pmatrix}$$

has rows

$$\sum_{j=1}^n c_j y_j(a) = y(a)$$
$$\sum_{j=1}^n c_j y_j'(a) = y'(a)$$
$$\sum_{j=1}^n c_j y_j''(a) = y''(a)$$
$$\vdots$$
$$\sum_{j=1}^n c_j y_j^{(n-1)}(a) = y^{(n-1)}(a)$$

Finally, since both $y$ and $\sum_{j=1}^n c_j y_j$ are solns to the hom nth order linear ODE (*) (the ODE being home and $y_j$, for $1 \leq j \leq n$, being solns, so is any linear combo of $y_j$s since the soln space to a linear home ODE is linear) and both $y$ and $\sum_{j=1}^n c_j y_j$ satisfy all the same $n$ many initial ?conbos?, by the $\exists!$ thrm, $y = \sum_{j=1}^n c_j y_j$ on $I$. $\therefore y \in$ Span $\{y_j \in W \mid 1 \leq j \leq n\}$. Thus, dim $W \leq n$. □

**Corollary.** *If*

$$W = \{y \in D^{(n)}(I) \ / \ y \text{ satisfies (x) } \}$$

*where*

$$(*)\ y^{(n)} + \sum_{k=1}^{n} y^{(n-k)} p_k(x) = 0$$

*then dim* $W = n$.

*Proof.* We showed that $n \leq \dim W \leq n$. □

**Example.** $y'' - y = 0$
Note that both $y_1 = \cosh x$ and $y_2 = \sinh x$ are solns on $\mathbb{R}$. The soln space is $2-$ dim'l . Not also that if

$$a \cosh x + b \sinh x = 0 \text{ for all } x \in \mathbb{R}$$

then in particular, $x = 0 \implies a = 0$.
$\therefore b \sinh x = 0$ for all $x$.
However, if $x \neq 0$ then $\sinh x \neq 0$; whence, $b = 0$. Hence, $B = \{\cosh x\ ,\ \sinh x\}$ are linearly independent solns to $y'' = y$, and the soln space has dim'n 2, $B$ is a basis for the soln space. $\therefore$ every soln to $y'' = y$ has the form

$$y = a \cosh x + b \sinh x$$

for some $a, b \in \mathbb{R}$. Likewise, $B' = \{e^x, e^{-x}\}$ is also a basis. Note:

$$|W(x)| = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x \begin{vmatrix} 1 & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

$\therefore$ by Wronsky's Thrm, $B'$ is lin ind.

**Example.** Example: $y'' + y = 0$
Here $B = \{\cos x, \sin x\}$ is a basis.

<u>nth order hom linear ODEs with constant coefficients:</u>

$$(1)\ \sum_{k=0}^{n} a_k y^{(k)} = 0$$

consider

$$y = e^{rx}$$

then

$$\begin{aligned} y' &= re^{rx} = ry \implies \\ y'' &= r^2 e^{rx} = r^2 y \implies \\ &\vdots \\ y^{(k)} &= r^k y \end{aligned}$$

thus, $y = e^{rx}$ yields

$$\begin{aligned} 0 &= \sum_{k=0}^{n} a_k y^{(k)} \\ &= \sum_{k=0}^{n} a_k r^k y \\ &= y \sum_{k=0}^{n} a_k r^{(k)} \implies \end{aligned}$$

$$\sum_{k=0}^{n} a_k r^{(k)} = 0$$

$\therefore y = e^{rx}$ is a soln to (1) if $r$ is a root (zero) of the char poly of $\sum_{k=0}^{n} a_k y^{(k)} = 0$,

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$$\rho(x) = \sum_{k=0}^{n} a_k x^k,$$

then $y = e^{rx}$ is a soln to $\sum_{k=0}^{n} a_k y^{(k)} = 0$.

**Example.** If $r_1, \ldots, r_n \in \mathbb{R}$ are paiswise distinct then $e^{r_1 x}, \ldots, e^{r_n x}$ are linearly independent (in $\mathscr{F}(\mathbb{R})$. we use the Wronskian:

$$\begin{aligned} |W(x)| &= \begin{vmatrix} e^{r_1 x} & e^{r_2 x} & \cdots & e^{r_n x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} & \cdots & r_n e^{r_n x} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} e^{r_1 x} & r_2^{n-1} e^{r_2 x} & \cdots & r_n^{n-1} e^{r_n x} \end{vmatrix} \\ e^{(r_1+r_2+\cdots+r_n)x} &= \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{vmatrix} \\ e^{(r_1+r_2+\cdots+r_n)x} & \prod_{1 \le i < j \le n} (r_j - r_i) \neq 0 \end{aligned}$$

(for all $x$ ). therefore by the wronskian thrm, $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$ ar lin ind,

**Example.** (Vandermonds)
consider

$$V_n(x) = \begin{pmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ 1 & r_2 & r_2^2 & \cdots & r_2^{n-1} \\ 1 & r_3 & r_3^2 & \cdots & r_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_n & r_n^2 & \cdots & r_n^{n-1} \end{pmatrix}$$

which is an $n \times n$ Vandermonde Matrix.
Put

$$f(x) = \det V_n(x) \in P_{n-1}\left[\mathbb{R}\right],$$

then $f(r_j) = \det(V_n(r_j)) = 0$ for all $2 \leq j \leq n$, by the alternating property of determinates. therefore by the factor Thrm,

$$f(x) = a \prod_{2 \leq j \leq n} (x - r_j),$$

where

$$a = (-1)^{n-1} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ 1 & r_2 & r_2^2 & \cdots & r_2^{n-1} \\ 1 & r_3 & r_3^2 & \cdots & r_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_n & r_n^2 & \cdots & r_n^{n-1} \end{vmatrix},$$

which is an $(n-1) \times (n-1)$ det. therefore by the inductive hypothesis,

$$a = (-1)^{n-1} \prod_{2 \leq i < j \leq n} (r_j - r_i)$$

therefore

$$f(x) = (-1)^{n-1} \prod_{2 \leq i < j \leq n} (r_j - r_i) \prod_{2 \leq j \leq n} (x - r_j)$$

Notice that

$$\prod_{2 \leq j \leq n} (x - r_j) = (-1)^{n-1} \prod_{2 \leq j \leq n} (r_j - x).$$

Therefore

$$\begin{aligned} f(x) &= (-1)^{n-1} \prod_{2 \leq i \leq j \leq n} (r_j - r_i)(-1)^{n-1} \prod_{2 \leq j \leq n} (r_j - x) \\ &= \prod_{2 \leq j \leq n} (r_j - x) \prod_{2 \leq i \leq j \leq n} (r_j - r_i). \end{aligned}$$

In particular,

$$\begin{aligned} \det V_n(r_1) &= f(r_1) \\ &= \prod_{2 \leq j \leq n} (r_j - r_1) \prod_{2 \leq j \leq i \leq n} (r_j - r_i) \\ &= \prod_{1 \leq i \leq j \leq n} (r_j - r_i). \end{aligned}$$

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Recall: the characteristic poly of $\sum_{k=0}^{n} a_k y^{(k)}$ is

$$\rho(x) = \sum_{k=0}^{n} a_k x^{(k)},$$

and we showed that if $r_1, \ldots, r_n$ are pairwise distinct then $B = \{e^{r_1 x}, \ldots, e^{r_n x}\}$ is linearly independent (in $\mathscr{C}^{(\infty)}(\mathbb{R})$ ). We also showed that if $r$ is a zero of $\rho(x)$ then $y = e^{rx}$ is a soln to $\sum_{k=0}^{n} a_k y^{(k)} = 0$. This all immediatly implies the following:

**Theorem.** *If $\rho(x) \sum_{k=0}^{n} a_k x^{(k)}$ has $n$ many distinct real zeros then the soln space of*

$$(*) \ \sum_{k=0}^{n} a_k y^{(k)} = 0$$

*has basis $\{e^{r_1 x}, \ldots, e^{r_n x}\}$, i.e., every soln of * has the form*

$$\sum_{j=1}^{n} c_j e^{r_j x}.$$

**Example.** (Revisited)
Consider $y'' - y = 0$. This has char poly

$$\rho(x) = x^2 - 1 = (x-1)(x+1),$$

which has zeros $\pm 1$.
Thus, $B = \{e^x, e^{-x}\}$ is a basis for the soln space of

$$y'' - y = 0$$

Q: What are the mising basis elements if a char poly has repeating zeros?

Recall the "ring" of polynomials

$$\mathbb{R}[x] = \{\sum_{k=0}^{n} a_k x^k \mid n \in \mathbb{N} \ \& \ a_k \in \mathbb{R}\},$$

where

$$\sum_{k=0}^{n} a_k x^{(k)} + \sum_{k=0}^{n} b_k x^{(k)} = \sum_{k=0}^{n} (a_k + b_k) x^{(k)}$$

$$\sum_{i=0}^{m} a_i x^{(i)} \sum_{j=0}^{m} b_j x^{(j)} = \sum_{k=0}^{m+n} c_k x^{(k)}, \text{ where}$$

$$c_k = \sum_{i+j=k} a_i b_j.$$

Notation. Put $D = \frac{d}{dx} : \mathscr{C}^{(\infty)}(I)$
$D^0 = i$ (identity on $\mathscr{C}^{(\infty)}(I)$ ), and

$$D^n = DD^{n-1} \text{ for } n \in \mathbb{Z}^+.$$

## Books

Hausen and sullivan "Real Analysis"
Reference: I.N. Herstein's "Topics in Abstract Algebra" (2nd ed), "Abstract Algebra" (3rd ed)
Artin's "Algebra"
(Formal poly : See hungerfords springer GTM.)