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Theorem. *(Existence - Uniqueness Thrm, $\exists!$ Thrm). If I is an interval, $a \in I$, and p_u for all $1 \leq k \leq n-1$, and f are cts on I, then $\forall b_i$, for $0 \leq i \leq n-1$, (then initial value problem)*

$$(1)\ y^{(n)} + \sum_{k=1}^{n} y^{(n-k)} p_u(x) = f(x)$$

& $y^{(i)}(a) = b_i$, has a unique soln on I.
Note: the solns to linear ODEs are unique on the whole interval I.

Theorem. *Thrm. If $w = \{y \in \mathscr{D}^{(n)}(I) \ / \ y$ satisfies (1)$\}$ then dim $W \geq n$.*

Proof. put $y_j^{(i)}(a) = \delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases}$$

for $1 \leq i$, $j \leq n$. By the $\exists!$ Thrm, for each j there is a unique soln to (1) on I. Now,

$$\begin{aligned} W(a) &= (y_j^{(i)}(a)) \in \text{ Mat}_n(\mathbb{R}) \\ &= I_n \end{aligned}$$

$$\therefore |W(a)| = 1 \neq 0$$

whence, $y_1, \ldots, y_n$ are linearly independent . Hence,

$$dim W \leq n$$

□

Theorem. *Thrm (Strong wronskian converse) If $y_1, \ldots, y_n$ are linearly independent (in $\mathscr{D}^{(n)}(I)$) and $y_1, \ldots, y_n$ are solns to*

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$$y^{(n)} + \sum_{k=1}^{n} y^{(n-k)} p_k(x) = 0$$

where p_k are cts on I for $k = 1, \ldots, n$, then $|W(a) \neq 0$ for all $a \in I$.

Proof. Assume that there is an $a \in I$ st $|W(a)| = 0$, then the linear system

$$(1)\ W(a)\vec{x} = \vec{0}$$

has a nontrivial soln, say $\vec{x} = \vec{c} \in \mathbb{R}^n$
($W(a) \in \mathrm{Mat}_n(\mathbb{R})$). Denote: $\vec{c} = (c_1, \ldots, c_n) \neq \vec{0}$.

Put $y = \sum_{j=1}^{n} c_j y_j$, then y satisfies (*) and

$$y(a) = \sum_{j=1}^{n} c_j y_j(a) = 0 \implies$$

$$y'(a) = \sum_{j=1}^{n} c_j y_j'(a) = 0 \implies$$

$$y''(a) = \sum_{j=1}^{n} c_j y_j''(a) = 0 \implies$$

$$\vdots$$

$$y^{(n-1)}(a) = \sum_{j=1}^{n} c_j y_j^{(n-1)}(a) = 0 \text{ (by(1))}$$

Notice that $y \equiv 0$ on I also satisfies (*) and is such that $y^{(k)}(a) = 0$ for $1 \leq k \leq n-1$. $\therefore$ by $\exists!$ thrm,

$$\sum_{j=1}^{n} c_j y_j \equiv 0 \text{ on } I$$

Thus, since $y_1, \ldots, y_n$ are lin ind, all $c_j = 0$, which is a contradiction. □

Theorem. *If*

$$W = \{y \in \mathbb{R}^{(n)}(I)\ /\ y \textit{ satisfies (*) }\}$$

then dim $W \leq n$

Proof. let $y_1, \ldots, y_n$ be lin ind solns of (*); we show that there are scalars $c_j \in \mathbb{R}$ st $y = \sum_{j=1}^{n} c_j y_j$ on I.
Consider the linear system

$$(2)\ W(a)\vec{x} = \begin{pmatrix} y(a) \\ y'(a) \\ \vdots \\ y^{(n-1)}(a) \end{pmatrix}$$

where $W(a)$ in the Wronskian matrix of $y_1, \ldots, y_n$. By the SWC, $|W(a)| \neq 0$.
Thus, (2) has a unique, not-trivial soln, say $\vec{0} \pm \vec{x} = \vec{c} \neq \vec{0} \in \mathbb{R}^n$. Thus

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$$W(a)\vec{c} = \begin{pmatrix} y(a) \\ y'(a) \\ \vdots \\ y^{(n-1)}(a) \end{pmatrix}$$

has rows

$$\sum_{j=1}^{n} c_j y_j(a) = y(a)$$
$$\sum_{j=1}^{n} c_j y_j'(a) = y'(a)$$
$$\sum_{j=1}^{n} c_j y_j''(a) = y''(a)$$
$$\vdots$$
$$\sum_{j=1}^{n} c_j y_j^{(n-1)}(a) = y^{(n-1)}(a)$$

Finally, since both y and $\sum_{j=1}^{n} c_j y_j$ are solns to the hom nth order linear ODE (*) (the ODE being home and y_j, for $1 \leq j \leq n$, being solns, so is any linear combo of y_js since the soln space to a linear home ODE is linear) and both y and $\sum_{j=1}^{n} c_j y_j$ satisfy all the same n many initial ?conbos?, by the $\exists !$ thrm, $y = \sum_{j=1}^{n} c_j y_j$ on I. $\therefore y \in$ Span $\{y_j \in W \mid 1 \leq j \leq n\}$. Thus, dim $W \leq n$. $\square$

Corollary. *If*

$$W = \{y \in D^{(n)}(I) \ / \ y \text{ satisfies (x) } \}$$

where

$$\text{(*) } y^{(n)} + \sum_{k=1}^{n} y^{(n-k)} p_k(x) = 0$$

then dim $W = n$.

Proof. We showed that $n \leq$ dim$W \leq n$. $\square$

Example. $y'' - y = 0$
Note that both $y_1 = \cosh x$ and $y_2 = \sinh x$ are solns on $\mathbb{R}$. The soln space is $2-$ dim'l . Not also that if

$$a\cosh x + b\sinh x = 0 \text{ for all } x \in \mathbb{R}$$

then in particular, $x = 0 \implies a = 0$.
$\therefore b\sinh x = 0$ for all x.
However, if $x \neq 0$ then $\sinh x \neq 0$; whence, $b = 0$. Hence, $B = \{\cosh x\ ,\ \sinh x\}$ are linearly independent solns to $y'' = y$, and the soln space has dim'n 2, B is a basis for the soln space. $\therefore$ every soln to $y'' = y$ has the form

$$y = a\cosh x + b\sinh x$$

for some $a, b \in \mathbb{R}$. Likewise, $B' = \{e^x, e^{-x}\}$ is also a basis. Note:

$$|W(x)| = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x \begin{vmatrix} 1 & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

$\therefore$ by Wronsky's Thrm, B' is lin ind.

Example. Example: $y'' + y = 0$
Here $B = \{\cos x, \sin x\}$ is a basis.

<u>nth order hom linear ODEs with constant coefficients:</u>

$$(1)\ \sum_{k=0}^{n} a_k y^{(k)} = 0$$

consider

$$y = e^{rx}$$

then

$$\begin{aligned} y' &= re^{rx} = ry \implies \\ y'' &= r^2 e^{rx} = r^2 y \implies \\ &\vdots \\ y^{(k)} &= r^k y \end{aligned}$$

thus, $y = e^{rx}$ yields

$$\begin{aligned} 0 &= \sum_{k=0}^{n} a_k y^{(k)} \\ &= \sum_{k=0}^{n} a_k r^k y \\ &= y \sum_{k=0}^{n} a_k r^{(k)} \implies \end{aligned}$$

$$\sum_{k=0}^{n} a_k r^{(k)} = 0$$

$\therefore y = e^{rx}$ is a soln to (1) if r is a root (zero) of the char poly of $\sum_{k=0}^{n} a_k y^{(k)} = 0$,

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$$\rho(x) = \sum_{k=0}^{n} a_k x^k,$$

then $y = e^{rx}$ is a soln to $\sum_{k=0}^{n} a_k y^{(k)} = 0$.

Example. If $r_1, \ldots, r_n \in \mathbb{R}$ are paiswise distinct then $e^{r_1 x}, \ldots, e^{r_n x}$ are linearly independent (in $\mathscr{F}(\mathbb{R})$. we use the Wronskian:

$$|W(x)| = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} & \cdots & e^{r_n x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} & \cdots & r_n e^{r_n x} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} e^{r_1 x} & r_2^{n-1} e^{r_2 x} & \cdots & r_n^{n-1} e^{r_n x} \end{vmatrix}$$

$$e^{(r_1 + r_2 + \cdots + r_n)x} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{vmatrix}$$

$$e^{(r_1 + r_2 + \cdots + r_n)x} \prod_{1 \le i < j \le n} (r_j - r_i) \neq 0$$

(for all x). therefore by the wronskian thrm, $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$ ar lin ind,

Example. (Vandermonds)
consider

$$V_n(x) = \begin{pmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ 1 & r_2 & r_2^2 & \cdots & r_2^{n-1} \\ 1 & r_3 & r_3^2 & \cdots & r_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_n & r_n^2 & \cdots & r_n^{n-1} \end{pmatrix}$$

which is an $n \times n$ Vandermonde Matrix.
Put

$$f(x) = \det V_n(x) \in P_{n-1}\left[\mathbb{R}\right],$$

then $f(r_j) = \det(V_n(r_j)) = 0$ for all $2 \le j \le n$, by the alternating property of determinates. therefore by the factor Thrm,

$$f(x) = a \prod_{2 \leq j \leq n} (x - r_j),$$

where

$$a = (-1)^{n-1} \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} \\ 1 & r_2 & r_2^2 & \cdots & r_2^{n-1} \\ 1 & r_3 & r_3^2 & \cdots & r_3^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_n & r_n^2 & \cdots & r_n^{n-1} \end{vmatrix},$$

which is an $(n-1) \times (n-1)$ det. therefore by the inductive hypothesis,

$$a = (-1)^{n-1} \prod_{2 \leq i < j \leq n} (r_j - r_i)$$

therefore

$$f(x) = (-1)^{n-1} \prod_{2 \leq i < j \leq n} (r_j - r_i) \prod_{2 \leq j \leq n} (x - r_j)$$

Notice that

$$\prod_{2 \leq j \leq n} (x - r_j) = (-1)^{n-1} \prod_{2 \leq j \leq n} (r_j - x).$$

Therefore

$$\begin{aligned} f(x) &= (-1)^{n-1} \prod_{2 \leq i \leq j \leq n} (r_j - r_i)(-1)^{n-1} \prod_{2 \leq j \leq n} (r_j - x) \\ &= \prod_{2 \leq j \leq n} (r_j - x) \prod_{2 \leq i \leq j \leq n} (r_j - r_i). \end{aligned}$$

In particular,

$$\begin{aligned} \det V_n(r_1) &= f(r_1) \\ &= \prod_{2 \leq j \leq n} (r_j - r_1) \prod_{2 \leq j \leq i \leq n} (r_j - r_i) \\ &= \prod_{1 \leq i \leq j \leq n} (r_j - r_i). \end{aligned}$$

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Recall: the characteristic poly of $\sum_{k=0}^{n} a_k y^{(k)}$ is

$$\rho(x) = \sum_{k=0}^{n} a_k x^{(k)},$$

and we showed that if $r_1, \ldots, r_n$ are pairwise distinct then $B = \{e^{r_1 x}, \ldots, e^{r_n x}\}$ is linearly independent (in $\mathscr{C}^{(\infty)}(\mathbb{R})$). We also showed that if r is a zero of

$\rho(x)$ then $y = e^{rx}$ is a soln to $\sum_{k=0}^{n} a_k y^{(k)} = 0$. This all immediatly implies the following:

Theorem. *If $\rho(x) \sum_{k=0}^{n} a_k x^{(k)}$ has n many distinct real zeros then the soln space of*

$$(*)\ \sum_{k=0}^{n} a_k y^{(k)} = 0$$

*has basis $\{e^{r_1 x}, \ldots, e^{r_n x}\}$, i.e., every soln of * has the form*

$$\sum_{j=1}^{n} c_j e^{r_j x}.$$

Example. (Revisited)
Consider $y'' - y = 0$. This has char poly

$$\rho(x) = x^2 - 1 = (x-1)(x+1),$$

which has zeros ± 1.
Thus, $B = \{e^x, e^{-x}\}$ is a basis for the soln space of

$$y'' - y = 0$$

Q: What are the mising basis elements if a char poly has repeating zeros?

Recall the "ring" of polynomials

$$\mathbb{R}[x] = \{\sum_{k=0}^{n} a_k x^k \mid n \in \mathbb{N} \ \&\ a_k \in \mathbb{R}\},$$

where

$$\sum_{k=0}^{n} a_k x^{(k)} + \sum_{k=0}^{n} b_k x^{(k)} = \sum_{k=0}^{n} (a_k + b_k) x^{(k)}$$

$$\sum_{i=0}^{m} a_i x^{(i)} \sum_{j=0}^{m} b_j x^{(j)} = \sum_{k=0}^{m+n} c_k x^{(k)}, \text{ where}$$

$$c_k = \sum_{i+j=k} a_i b_j.$$

<u>Notation</u>. Put $D = \frac{d}{dx} : \mathscr{C}^{(\infty)}(I) \to \mathscr{C}^{(\infty)}(I)$
$D^0 = i$ (identity on $\mathscr{C}^{(\infty)}(I)$), and

$$D^n = DD^{n-1} \text{ for } n \in \mathbb{Z}^+.$$

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where $Dy = D(y) = \frac{dy}{dx}$
Condider $L = \sum_{k=0}^{n} a_k 0^k$, where $a_k \in \mathbb{R}$ and $D^k = \frac{d^k}{dx^k}$. Note that $L : \mathscr{C}^{(\infty)}(I) \to \mathscr{C}^{(\infty)}(I)$, where

$$L_y = L(y) = \sum_{k=0}^{n} a_k D^k y =$$

$$a_0 y + a_1 y' + a_2 y'' + \cdots + a_n y^{(n)}.$$

we show that L is a linear operation:

$$\begin{aligned} L(y_1 + y_2) &= \sum_{k=0}^{n} a_k D^k (y_1 + y_2) \\ &= \sum_{k=0}^{n} a_k (D^k y_1 + D^k y_2) \\ &= \sum_{k=0}^{n} (a_k D^k y_1 + a_k D^k y_2) \\ &= L_{y_1} + L_{y_2} \end{aligned}$$

Also,

$$\begin{aligned} L(cy) &= \sum_{k=0}^{n} a_k D^k (cy) \\ &= \sum_{k=0}^{n} a_k (c D^k y) \\ &= c \sum_{k=0}^{n} a_k D^k y \\ &= c L_y. \end{aligned}$$

Put

$$\mathbb{R}[D] = \{\sum_{k=0}^{n} a_k D^k \mid a_k \in \mathbb{R} \ \&\ n \in \mathbb{N}\}.$$

We show that $\mathbb{R}[x] = R[D]$, isomorphic as "rings," which includes multiplication. Addition is $\mathbb{R}[D]$ is defined as

$$L_1, L_2 \in \mathbb{R}[D] \ ,\ (L_1 + L_2)y = L_1 y + L_2 y;$$

Multiplication in $\mathbb{R}[D]$ is defined as

$$L_1, L_2 \in \mathbb{R}[D] \ , \ (L_1 L_2)y = (L_1 \cdot L_2)$$

Define $h : \mathbb{R}[x] \to \mathbb{R}[D]$ Such that

$$h\left(\sum_{k=0}^{n} a_k x^k\right) = \sum_{k=0}^{n} a_k D^k$$

For example, $1 \mapsto D^0 =$ Identify function in $\mathscr{C}^{(\infty)}(I)$,

$$x \mapsto D, x^2 \mapsto D^2, \ldots, x^n \mapsto D^n.$$

We show that h is a "ring" homorphism, i.e., if $f_1 = \sum_{k=0}^{m} a_k x^k$, $f_2 = \sum_{k=0}^{n} b_k x^k \in \mathbb{R}[x]$, and $L_1 = \sum_{k=0}^{m} a_k D^k$, $L_2 = \sum_{k=0}^{n} b_k D^k \in \mathbb{R}[D]$, then

(i) $h(f_1 + f_2) = L_1 + L_2$;
(ii) $h(f_1 f_2) = L_1 L_2$;

To see (i), notice that

$$\begin{aligned} f_1 + f_2 &= \sum_{k=0}^{n} a_k x^k + \sum_{k=0}^{n} b_k x^k \\ &= \sum_{k=0}^{n} (a_k + b_k) x^k \end{aligned}$$

So,

$$\begin{aligned} h(f_1 + f_2 &= \sum_{k=0}^{n} (a_k + b_k) D^k \\ &= \sum_{k=0}^{n} (a_k D^k + b_k D^k) \\ &= \sum_{k=0}^{n} a_k D^k + \sum_{k=0}^{n} b_k D^k) \\ &= L_1 + L_2 \end{aligned}$$

Also, notice that

$$f_1 f_2 = \sum_{k=0}^{n} c_k x^k \ , \ where c_k = \sum_{i+j=k} a_i b_j$$

So,

$$h(f_1 f_2) = \sum_{k=0}^{n} c_k D^k.$$

Now,

$$\begin{aligned}
c_k D^k &= \left(\sum_{i+j=k} a_i b_j \right) D^k \\
&= \sum_{i+j=k} (a_i b_j D^k) \\
&= \sum_{i+j=k} a_i D^i b_j D^j
\end{aligned}$$

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$$\begin{aligned}
L_1 L_2 y &= (L_1 \circ L_2)(y) \\
&= L_1(L_2(y)) \\
&= \sum_{i=0}^{m} a_i D^i L_2(y) \\
&= \sum_{i=0}^{m} a_i D^i \left(\sum_{j=0}^{n} b_j D^j y \right) \\
&= \sum_{i=0}^{m} \sum_{j=0}^{n} a_i b_j D^{i+j} y.
\end{aligned}$$

therefore

$$L_1 L_2 = \sum_{i=0}^{m} \sum_{j=0}^{n} a_i b_j D^{i+j}$$

Now, notice that from above

$$L_1 L_2 = \sum_{k=0}^{m+n} c_k D^k \ , \ c_k = \sum_{i+j=k} a_i b_j.$$

therefore

$$h(f_1 f_2) = \sum_{k=0}^{m+n} c_k D^k = L_1 L_2 = h(f_1) h(f_2),$$

$c_k = \sum_{i+j=k} a_i b_j$. Hence, h is a "ring" homomorphism. Clearly, h is onto, We show that h is 1-1.
Firstly, recall that

$$ker(h) = \{f \in \mathbb{R}[x] \mid h(f) = 0 \in \mathbb{R}[D]\}$$

Also, h is 1-1 $\iff$ $ker(h) = \{0\}$.

Secondly, if $n = 0$ then

$$a_0 D^0 = 0 \in \mathbb{R}[D] \implies$$
$$a_0 D^0 y = 0 \in \mathscr{C}^\infty(I) \text{ for all } y \in \mathscr{C}^\infty(I)$$

$$\implies a_0 y = 0 \text{ for all } y \in \mathscr{C}^\infty(I)$$

However, if y = 1 on I then

$$a_0 = 0$$

$$\therefore a_0 x^0 \in \mathbb{R}[x].$$

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Now, assume that

$$\sum_{k=0}^{n} a_k x^k \mapsto 0 \in \mathbb{R}[D] \implies$$

$a_k = 0$ for $0 \leq k \leq n$ (inductive hypothesis). If

$$h\left(\sum_{k=0}^{n} a_k x^k\right) = 0 \in \mathbb{R}[D],$$

then

$$\sum_{k=0}^{n} a_k D^k = 0 \in \mathbb{R}[D] \implies$$

$$(1)\ \sum_{k=0}^{n} a_k y^k = 0 \in \mathscr{C}^{(\infty)} \forall y \in \mathscr{C}^{(\infty)}(I)$$

Thus, if $a_{n+1} \pm 0$ then
Take $y \equiv 1 \in \mathscr{C}^{(\infty)}(I)$, then (1) becomes $a_0 y \equiv 0 \in \mathscr{C}^{(\infty)}(I)$; thus, $a_0 = 0$.
Thus,

$$(2)\ \sum_{k=1}^{n+1} a_k y^{(k)} = 0 \in \mathscr{C}^{(\infty)}(I)$$

take $y \equiv x \in \mathscr{C}^{(\infty)}(I)$, then (2) becomes $a_1 y' \equiv 0 \in \mathscr{C}^{(\infty)}(I)$; thus, $0 = a_1$, $y'(x) = a_1$ for all $x \in I$. therefore $a_1 = 0$. Thus,

$$(3)\ \sum_{k=2}^{n+1} a_k y^{(k)} = 0 \in \mathscr{C}^{(\infty)}(I)$$

Take $y = x^2, etc, then\ (n+1)\ a_{n+1} y^{n+1} = 0 \in \mathscr{C}^{(\infty)}(I)$

Take $y \equiv x^{n+1}$, then as above, $a_{n+1} = 0$, a contradiction. $\therefore$ $a_{n+1} = 0$. As such, (1) becomes

$$\sum_{k=0}^{n} a_k y^k = 0 \in \mathscr{C}^{(\infty)}(I)$$

Hence, by the inductive hypo, $a_k = 0$ for $0 \leq k \leq n$. therefore $ker(h) = (0)$; whence, h is an isomorphis. therefore $\mathbb{R}[x] \equiv \mathbb{R}[D]$ as "rings" Date: 04.07.15
Was not at school this day Date: 04.08.15
Recall: $B = \{x^k e^{rx} \in {}^{(\infty)}(\mathbb{R}) \mid 0 \leq k \leq m-1$
where $y \in B$ are solns to $(D-r)^m y = 0$.
B is linearly independent since

$$\sum_{k=0}^{m-1} c_k x^k e^{rk} = 0 \implies \sum_{k=0}^{m-1} c_k x^{k-1} = 0 \implies$$

$c_k = 0$ for all $0 \leq k \leq m-1$ this is because $\left\{1, x, x^2, \ldots, x^{m-1}\right\} \subset \mathbb{R}[x]$ is linearly independence. therefore we have found the missing basis elements. In summary, if

$$\sum_{k=0}^{n} a_k y^{(k)} = 0 \;\&\; \rho p(x) = a_n \prod_{k=0}^{m} (x - r_k)^{m_k}$$

where $r_k \in \mathbb{R}$ for $1 \leq k \leq m$, then the soln space of $\sum_{k=0}^{n} a_k y^{(k)} = 0$ has basis

$$B = \{x^j e^{r_k x} \in {}^{(\infty)}(\mathbb{R}) \mid 0 \leq j \leq m-1 \;\&\; 1 \leq k \leq m\}$$

Note: since $\sum_{k=1}^{m} m_k = n$, $cardB = n$

Example. say $\sum_{k=0}^{n} a_k y^{(k)} = 0$ has the factored form

$$(D-2)^3 (D+1)(D-4)^2 y = 0$$

then basis elements are :

$$e^{2x}, xe^{2x}, x^2 e^{2x}, e^{-x}, e^{4x}, xe^{4x}.$$

Exam 3
hw up through 3.3, excluding problems involving $\mathbb{C}$ - valued zeros
1a Given functions, prove they are lin independent
1b Verify these are solns to a given lin ODE
1c Solve the linear ODE initial value prob
2 given a 4th order lin hom ODE find a basis for its soln space, find the gen soln
3 Find a basis for a 2nd order hom ODE; find the gen soln 3cont Given a particular soln to the linear non-hom ODE, find the gen soln to the non-hom lin ODE

$$Ly = 0$$

$$\sum a_k y^{(k)} = 0$$

$$Ly = f$$

Books

Hausen and sullivan "Real Analysis"
Reference: I.N. Herstein's "Topics in Abstract Algebra" (2nd ed), "Abstract Algebra" (3rd ed)
Artin's "Algebra"
(Formal poly : See hungerfords springer GTM.)