

Here we address the possibility of the char poly $\rho(x)$ of $\sum_{k=0}^n a_k y^{(k)} = 0$ with Date: 04.13.15

complex zeros. So, assuming that $r = a + bi$ is a zero of $\rho = \sum_{k=0}^n a_k x^k$,

then we know that $y = e^{rk} = e^{(a+bi)x} = e^{ax} e^{(bx)i}$. Recall that if $a_k \in \mathbb{R}$ for $0 \leq k \leq n$, i.e., $\rho(x) \in \mathbb{R}[x]$, then $\bar{r} = a - bi$ ($b \neq 0$) is also a zero of $\rho(x)$. Thus, in the case of $r = a \pm bi$, we are "missing" now not just one real-valued basis element, but two real valued basis elements.

Euler's identity enables us to determine these two missing basis elements:

$$(1) e^{ix} = \cos(x) + i \sin(x) \in \mathbb{C} \text{ for all } x \in \mathbb{R}$$

what led Euler to this identity are the Maclaurin series:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots = \sum_{n \geq 0} \frac{x^n}{n!};$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots = \sum_{n \geq 0} (-1)^n \frac{x^{2n}}{(2n)!};$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots = \sum_{n \geq 0} (-1)^n \frac{x^{2n+1}}{(2n+1)!};$$

These hold for all $x \in \mathbb{R}$. Notice that (1) simply gives pts on the unit circle in \mathbb{C}

$$|e^{ix}| = 1 \text{ for all } x \in \mathbb{R}$$

Note: $e^{i\pi} + 1 = 0$. ($x = \pi$)

Observe that e^{ix} is a complex-valued scalar mult of $\cos x$ & $\sin x$. So, back to the soln

$$\begin{aligned} y &= e^{ax} e^{(bx)i} \\ &= e^{ax} (\cos bx + i \sin bx) \\ &= e^{ax} \cos x + i(e^{ax} \sin bx), \end{aligned}$$

which is likewise a complex-valued linear combo of $e^{ax} \cos bx$ & $e^{ax} \sin bx$.

These are two real-valued soln to the ODE $\sum_{k=0}^n a_k y^{(k)} = 0$, i.e., there are the

two missing basis elements. therefore if $(x - r)^m$, where $r = a + bi$, is a factor of $\rho(x)$, then the corresponding basis elements are:

$$\begin{aligned} & e^{ax} \cos bx, e^{ax} \sin bx \\ & x e^{ax} \cos bx, x e^{ax} \sin bx \\ & x^2 e^{ax} \cos bx, x^2 e^{ax} \sin bx \\ & \vdots \\ & x^{m-1} e^{ax} \cos bx, x^{m-1} e^{ax} \sin bx \end{aligned}$$

Here there are the missing 2m many basis elements.

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Example.

$$y'' + y' - y = \sin^2 x$$

Here $\sum_{k=0}^n y^{(k)} = f(x)$, where $n = 2$ and $f(x) = \sin^2 x$. Notice that

$$\begin{aligned} y &= \sin^2 x \implies \\ y' &= 2 \sin x \cos x \implies \\ y'' &= 2 \cos^2 x - 2 \sin^2 x. \end{aligned}$$

$$\begin{aligned} y &= \sin^2 x; \\ y' &= 2 \sin x \cos x = \sin 2x \\ y'' &= 2 \cos^2 x - 2 \sin^2 x = 2 \cos 2x. \end{aligned}$$

the terms in there derivatives are

$$\begin{aligned} & 1, \\ & \sin^2 x, \\ & \sin x \cos x \\ & \cos^2 x. \end{aligned}$$

Another way to see these terms is as

$$\begin{aligned} & 1, \\ & \cos 2x \\ & \sin 2x. \end{aligned}$$

We consider a possible particular soln

$$y_p = a + b \cos 2x + c \sin 2x,$$

which is a linear combo of the terms above. Note that y_p is a particular soln iff

$$y_p'' + y_p' - y_p = \sin^2 x$$

Now,

$$y_p = a + b \cos 2x + c \sin 2x \implies$$

$$y_p' = -2b \sin 2x + 2c \cos 2x \implies$$

$$y_p'' = -4b \cos 2x - 4c \sin 2x \implies$$

$$y_p'' + y_p' - y_p = \sin^2 x \iff$$

$$(2c - 4b) \cos 2x - (2b + 4c) \sin 2x - b \cos 2x - c \sin 2x - a = \sin^2 x \iff$$

$$(2c - 5b) \cos 2x - (2b + 5c) \sin 2x - a = \frac{1 - \cos 2x}{2} \iff$$

$$-a - 1/2 + (2c - 5b + 1/2) \cos 2x - (2b + 5c) \sin 2x = 0.$$

This last equation is an identity if

$$a = -1/2 \text{ \& } \begin{cases} -5b + 2c + 1/2 & = 0 \\ 2b + 5c & = 0. \end{cases}$$

Note:

$$\begin{cases} -10b + 4c & = -1 \\ 2b + 5c & = 0 \end{cases}$$

$$\left[\begin{array}{cc|c} 10 & 25 & 0 \\ -10 & 4 & -1 \end{array} \right]$$

$$c = -1/29$$

$$b = 5/58$$

Therefore

$$y_p = -1/2 + \frac{5}{58} \cos 2x - \frac{1}{29} \sin 2x,$$

Which you have checked is an actual soln to $y'' + y' - y = \sin^2 x$.

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Example.

$$y'' + y' - y = \sec x$$

$$\begin{aligned} f(x) &= \sec x \\ f'(x) &= \sec x \tan x \\ f''(x) &= \sec x \tan^2 x + \sec^3 x \\ &= \sec x (\sec^2 x - 1) + \sec^3 x \\ &= 2 \sec^3 x - \sec x \end{aligned}$$

Put

$$B = \{1, \sec x, \sec x \tan x, \sec^3 x\}$$

Also, put

$$y_p = a + b \sec x + c \sec x \tan x + d \sec^3 x$$

$$y'_p$$

Since $\sec^3 x \tan x \notin \text{Span } B$, the method of undetermined coeffs fails.

Idea: Consider $B = \{y_1, \dots, y_n\}$, the basis of $\sum_{k=0}^n a_k y^{(k)} = 0$, and

$$(1) \quad y_p = \sum_{k=1}^n v_k(x) y_k(x).$$

We will show that a particular soln of $\sum_{k=0}^n a_k y^{(k)} = f(x)$ always has form

(1) for some $v_k(x) \in C^{(n)}(I)$. this is called "Variation of parameters."

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We want a particular soln $y = v_1(x)y_1(x) + v_2(x)y_2(x)$ to $\sum_{k=1}^2 a_k y^{(k)} =$

$f(x)$, where the hom soln space has basis $B = \{y_1, y_2\} \subset C^{(2)}(I)$. Here we require

$$y_p'' + py_p' + qy_p = f(x) \text{ (monic).}$$

$$y_p = v_1y_1 + v_2y_2 \implies$$

$$y_p' = v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2'$$

If $v_1'y_1 + v_2'y_2 = 0$ then

$$y_p' = v_1y_1' + v_2y_2'$$

Thus,

$$y_p'' = v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2''$$

$$y_p'' + py_p' + qy_p = f(x) \iff$$

$$v_1'y_1' + v_1y_1'' + v_2'y_2' + v_2y_2'' + pv_1y_1' + pv_2y_2' + qv_2y_1 + qv_2y_2 = f(x) \iff$$

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1'y_1' + v_2'y_2' = f(x) \iff$$

$$v_1'y_1' + v_2'y_2' = f(x).$$

therefore

$$\begin{cases} y_1'v_1' + y_1'v_2' = f(x) \\ y_1v_1' + y_2v_2' = 0 \end{cases}$$

iff

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore

$$v_1 = - \int \frac{y_2(x)f(x)}{|w(x)|} dx$$

and

$$v_2 = \int \frac{y_1(x)f(x)}{|w(x)|} dx$$

Hence,

$$y_p(x) =$$

Now, in general, consider

$$\sum_{k=0}^n a_k y^{(k)} = f(x)$$

with basis $B = \{y_j \in C^{(n)}(I) | 1 \leq j \leq n\}$ for the assoc hom ODE. Also, consider

$$y_p = \sum_{j=1}^n v_j y_j,$$

then

$$\begin{aligned} y_p' &= \sum_{j=1}^n (v_j' y_j + v_j y_j') \\ &= \sum_{j=1}^n v_j y_j' , \text{ if } \sum_{j=1}^n v_j' y_j = 0. \end{aligned}$$

Now,

$$\begin{aligned} y_p'' &= \sum_{j=1}^n (v_j' y_j' + v_j y_j'') \\ &= \sum_{j=1}^n y_j y_j'' , \text{ if } \sum_{j=1}^n v_j' y_j' = 0 \end{aligned}$$

continuing,

$$y_p^{(k)} = \sum_{j=1}^n v_j y_j^{(k)} , \text{ if } \sum_{j=1}^n v_j' y_j^{(k-1)} = 0$$

for $1 \leq k \leq n-1$. Finally,

$$\begin{aligned} y_p^{(n)} &= \sum_{j=1}^n (v_j' y_j^{(n-1)} + v_j y_j^{(n)}) \\ &= \sum_{j=1}^n v_j' y_j^{(n-1)} + \sum_{j=1}^n v_j y_j^{(n)} \end{aligned}$$

Therefore

$$\begin{aligned}
& \sum_{k=1}^n a_k y_p^{(k)} = f(x) \iff \\
& a_n \sum_{j=1}^n v'_j y_j^{(n-1)} + a_n \sum_{j=1}^n v_j y_j^{(n)} + \sum_{k=1}^{n-1} a_k \sum_{j=1}^n v_j y_j^{(k)} = f(x) \iff \\
& a_n \sum_{j=1}^n v'_j y_j^{(n-1)} + \sum_{j=1}^n a_n v_j y_j^{(n)} + \sum_{j=1}^n \sum_{k=1}^{n-1} a_k v_j y_j^{(k)} = f(x) \iff \\
& a_n \sum_{j=1}^n v'_j y_j^{(n-1)} + \sum_{j=1}^n v_j \sum_{k=1}^n a_k y_j^{(k)} = f(x) \iff \\
& a_n \sum_{j=1}^n v'_n y_j^{(n-1)} = f(x)
\end{aligned}$$

since $y_j \in B$. Thus,

$$\begin{aligned}
& \sum_{j=1}^n v'_j y_j = 0 \text{ 1st eqn} \\
& \sum_{j=1}^n v'_j y'_j = 0 \text{ 2nd eqn} \\
& \vdots \\
& \sum_{j=1}^n v'_j y_j^{(k-1)} = 0 \text{ kth eqn} \\
& \vdots \\
& \sum_{j=1}^n v'_j y_j^{(n-2)} = 0 \text{ (n-1)th eqn} \\
& a_n \sum_{j=1}^n v'_j y_j^{(n-1)} = f(x) \text{ nth eqn}
\end{aligned}$$

Now,

$$\begin{aligned}
y'_p &= v'_1 y_1 + v_1 y'_1 + v'_2 y_2 + v_2 y'_2 \implies \\
y''_p &= v''_1 y_1 + v'_1 y'_1 + v_1 y''_1 + v'_2 y_2 + v_2 y'_2 + v'_2 y'_2 + v_2 y''_2 \\
&= 2(v'_1 y'_1 + v'_2 y'_2) + v_1 y''_1 + v_2 y''_2,
\end{aligned}$$

if we impose the condition $v''_1 y_1 + v''_2 y_2 = 0$. therefore

$$\begin{aligned}
f(x) &= y''_p + p y'_p + q y_p \\
&= 2(v'_1 y'_1 + v'_2 y'_2) + v_1 y''_1 + v_2 y''_2 + p v'_1 y_1 + p v_1 y'_1 + p v'_2 y_2 + p v_2 y'_2 + q v_1 y_1 + q v_2 y_2 \\
&= v_1(y''_1 + p y'_1 + q y_1) + v_2(y''_2 + p y'_2 + q y_2) + 2(v'_1 y'_1 + v'_2 y'_2) + p(v'_1 y_1 + v'_2 y_2) \\
&= 2(v'_1 y'_1 + v'_2 y'_2) + p(v'_1 y_1 + v'_2 y_2)
\end{aligned}$$

therefore,

$$f(x) = 2(v'_1 y'_1 + v'_2 y'_2) + p(v'_1 y_1 + v'_2 y_2)$$

implies

$$f(x) = v'_1(2y'_1 + p y_1) + v'_2(2y'_2 + p y_2)$$

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We want a particular soln $y = v_1(x)y_1(x) + v_2(x)y_2(x)$ to $\sum_{k=1}^2 a_k y^{(k)} = f(x)$, where the hom soln space has basis $B = \{y_1, y_2\} \subset C^{(2)}(I)$. Here we require

$$y_p'' + py_p' + qy_p = f(x) \text{ (monic).}$$

$$y_p = v_1 y_1 + v_2 y_2 \implies$$

$$y_p' = v_1' y_1 + v_1 y_1' + v_2' y_2 + v_2 y_2'$$

If $v_1' y_1 + v_2' y_2 = 0$ then

$$y_p' = v_1 y_1' + v_2 y_2'$$

Thus,

$$y_p'' = v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2''$$

$$y_p'' + py_p' + qy_p = f(x) \iff$$

$$v_1' y_1' + v_1 y_1'' + v_2' y_2' + v_2 y_2'' + pv_1 y_1' + pv_2 y_2' + qv_1 y_1 + qv_2 y_2 = f(x) \iff$$

$$v_1(y_1'' + py_1' + qy_1) + v_2(y_2'' + py_2' + qy_2) + v_1' y_1' + v_2' y_2' = f(x) \iff$$

$$v_1' y_1' + v_2' y_2' = f(x).$$

therefore

$$\begin{cases} y_1' v_1' + y_2' v_2' = f(x) \\ y_1 v_1' + y_2 v_2' = 0 \end{cases}$$

iff

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Therefore

$$v_1 = - \int \frac{y_2(x)f(x)}{|w(x)|}dx$$

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Hence,

$$y_p(x) =$$

Now, in general, consider

$$\sum_{k=0}^n a_k y^{(k)} = f(x)$$

with basis $B = \{y_j \in C^{(n)}(I) | 1 \leq j \leq n\}$ for the assoc hom ODE. Also, consider

$$y_p = \sum_{j=1}^n v_j y_j,$$

then

$$\begin{aligned} y_p' &= \sum_{j=1}^n (v_j' y_j + v_j y_j') \\ &= \sum_{j=1}^n v_j y_j', \text{ if } \sum_{j=1}^n v_j' y_j = 0. \end{aligned}$$

Now,

$$\begin{aligned} y_p'' &= \sum_{j=1}^n (v_j' y_j' + v_j y_j'') \\ &= \sum_{j=1}^n v_j y_j'', \text{ if } \sum_{j=1}^n v_j' y_j' = 0 \end{aligned}$$

continuing,

$$y_p^{(k)} = \sum_{j=1}^n v_j y_j^{(k)}, \text{ if } \sum_{j=1}^n v_j' y_j^{(k-1)} = 0$$

for $1 \leq k \leq n-1$. Finally,

$$\begin{aligned} y_p^{(n)} &= \sum_{j=1}^n (v'_j y_j^{(n-1)} + v_j y_j^{(n)}) \\ &= \sum_{j=1}^n v'_j y_j^{(n-1)} + \sum_{j=1}^n v_j y_j^{(n)} \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{k=1}^n a_k y_p^{(k)} &= f(x) \iff \\ a_n \sum_{j=1}^n v'_j y_j^{(n-1)} + a_n \sum_{j=1}^n v_j y_j^{(n)} + \sum_{k=1}^{n-1} a_k \sum_{j=1}^n v_j y_j^{(k)} &= f(x) \iff \\ a_n \sum_{j=1}^n v'_j y_j^{(n-1)} + \sum_{j=1}^n a_n v_j y_j^{(n)} + \sum_{j=1}^n \sum_{k=1}^{n-1} a_k v_j y_j^{(k)} &= f(x) \iff \\ a_n \sum_{j=1}^n v'_j y_j^{(n-1)} + \sum_{j=1}^n v_j \sum_{k=1}^n a_k y_j^{(k)} &= f(x) \iff \\ a_n \sum_{j=1}^n v'_j y_j^{(n-1)} &= f(x) \end{aligned}$$

since $y_j \in B$. Thus,

$$\begin{aligned}
& \sum_{j=1}^n v'_j y_j = 0 \text{ (1st eqn)} \\
& \sum_{j=1}^n v'_j y'_j = 0 \text{ (2nd eqn)} \\
& \vdots \\
& \sum_{j=1}^n v'_j y_j^{(k-1)} = 0 \text{ (kth eqn)} \\
& \vdots \\
& \sum_{j=1}^n v'_j y_j^{(n-2)} = 0 \text{ ((n-1)th eqn)} \\
& a_n \sum_{j=1}^n v'_j y_j^{(n-1)} = f(x) \text{ (nth eqn)}
\end{aligned}$$

$$\implies W(y_j)_{1 \leq j \leq n} \begin{pmatrix} v'_1 \\ \vdots \\ v'_n \end{pmatrix} = f(x) \vec{e}_n$$

Notice that $|W(y_j)_{1 \leq j \leq n}| \neq 0$ for all $x \in I$ since y_j s are linearly independent (being that $y_j \in B$). Thus, by Cramer's Rule,

$$v'_j = \frac{D_j}{D},$$

where $D = |W(y_j)_{1 \leq j \leq n}|$ and

$$D_j = \begin{vmatrix} y_1 & \cdots & 0 & \cdots & y_n \\ y'_1 & \cdots & 0 & \cdots & y'_n \\ y''_1 & \cdots & 0 & \cdots & y''_n \\ \vdots & & \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & f(x) & \cdots & y_n^{(n-1)} \end{vmatrix} = (-1)^{n+j} f(x) |W_{n-1}(y_k)_{\substack{1 \leq k \leq n \\ j \neq k}}|$$

Therefore

$$v'_k = (-1)^{j+n} f(x) \frac{|W_{n-1}(y_k)_{1 \leq k \leq n}|}{a_n |W_n(y_k)_{1 \leq k \leq n}|}$$

Thus,

$$(1) v_k(x) = \frac{(-1)^{j+n}}{a_n} \int f(x) \frac{|W_{n-1}(y_k)_{1 \leq k \leq n}|}{a_n |W_n(y_k)_{1 \leq k \leq n}|} dx,$$

Where $y_p(x) = \sum_{k=1}^n v_k(x) y_k(x)$ is a particular soln to $\sum_{k=1}^n a_k y^{(k)} = f(x)$. (1) is called the variation of parameters formulas.

Ch 4 systems of ODEs

Example. Consider a curve $\vec{r}: [a, b] \rightarrow \mathbb{R}^n$, where $\vec{r}(t) = (x_1(t), \dots, x_n(t))$ and $t \in [a, b]$. By Newton's 2nd Law, $\vec{F} = m\vec{a} = m\vec{r}''(t)$. Thus,

$$\begin{pmatrix} m(t)x_1''(t) \\ m(t)x_2''(t) \\ \vdots \\ m(t)x_n''(t) \end{pmatrix} = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix},$$

which is a system of n many 2nd order ODEs in (x_1, \dots, x_n)

Example. Consider an

$$(1) y^{(n)} + \sum_{k=0}^{n-1} p_k(x) y^{(k)} = f(x),$$

$$y^{(n)} + p_{n-1}y^{(n-1)} + \dots + p_2(x)y'' + p_1(x)y' + p_0(x)y = f(x).$$

put $y_k = y^{(k-1)}$ for $1 \leq k \leq n+1$, then

$$\begin{aligned} y_1 &= y \implies \\ y_2 &= y' = y'_1 \implies \\ y_3 &= y'' = y''_1 = y'_2 \implies \\ y_4 &= y''' = y'''_1 = y''_2 = y'_3 \implies \\ &\vdots \\ y_{n+1} &= y^{(n)} = y'_n \end{aligned}$$

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Thus, (1) becomes

$$(2) \quad y'_n + \sum_{k=0}^{n-1} p_k(x) y'_k = f(x),$$

which is a first order linear ODE. putting (2) with the alone n-1 many 1st order ODEs yields the following 1st order system: with n many ODEs:

$$\begin{cases} y'_n + \sum_{k=0}^{n-1} p_k(x) y'_k = f(x) \\ y'_1 = y_2 \\ y'_2 = y_3 \\ \vdots \\ y'_{n-1} = y_n \end{cases}$$

Again, this is a system of n many 1st linear ODEs from a single nth order linear ODE. This gives yet another example to motivate studying systems of linear ODEs.

Example. page 255 number 12

$$\begin{cases} x' = y \\ y' = x \end{cases}$$

Note: $x = y' = x''$. So, this yields $x'' - x = 0$, which has basis $B = \{\cosh t, \sinh t\}$. Thus,

$$x = a \cosh t + b \sinh t$$

Now,

$$y = b \cosh t + a \sinh t$$

Notice that if $\vec{r}(t) = (x(t), y(t))$ then $\vec{r}'(t) = (x'(t), y'(t)) = (y(t), x(t))$. We want such an $\vec{r}(t)$. In otherwords, we may view the above system as a single. 1st order ODE of a parametric function.

Recall that

$$\cosh(\alpha + t) = \cosh \alpha \cosh t + \sinh \alpha \sinh t$$

Take $A \in \mathbb{R}$ st $a/A \geq 1$, then $a/A \in \text{Range}(\cosh)$. Now, we want $\alpha \in \mathbb{R}$ st

$$a = A \cosh \alpha \text{ \& } b = A \sinh \alpha.$$

So,

$\coth \alpha = \frac{\cosh \alpha}{\sinh \alpha} = \frac{a}{b}$ holds for some α st $|\alpha| < 1$ since $\text{Range}(\coth) = \mathbb{R}$.
therefore for this α ,

$$\begin{aligned} x &= a \cosh t + b \sinh t \\ &= A \cosh \alpha \cosh t + A \sinh \alpha \sinh t \\ &= A \cosh(\alpha + t). \end{aligned}$$

therefore $x = A \cosh(\alpha + t)$; whence, $y = x' = A \sinh(\alpha + t)$.

Hence,

$$x^2 - y^2 = A^2 \implies \frac{x^2}{A^2} - \frac{y^2}{A^2} = 1$$

These are the solns to $\vec{r}'(t) = (x'(t), y'(t))$. Note : $\text{Range}(\cosh) = [1, \infty)$. So, if $A > 0$ then $x(t) > 0$; where, $\vec{r}(t)$ corresponds to the righthand branch.

Whereas, if $A < 0$ then $\vec{r}(t)$ corresponds to the left hand branch.

Notice that we solved this linear system by "substitution."

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Example. (Recall).

let $\vec{r}(t) = (x(t), y(t))$, and consider $\vec{r}'(t) = (x'(t), y'(t))$, then

$$\begin{aligned} \begin{cases} x'(t) = y(t) \\ y'(t) = x(t) \end{cases} &\iff \begin{cases} x'(t) - y(t) = 0 \\ y'(t) - x(t) = 0 \end{cases} &\iff \begin{cases} L_{11}x + L_{12}y = 0 \\ L_{21}x + L_{22}y = 0 \end{cases} &\iff \\ &\iff \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0} \end{aligned}$$

Where $L_{11} = D = L_{22}$ and $L_{12} = -1 = L_{21}$

In other symbols,

$$\begin{pmatrix} D & -1 \\ -1 & D \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

Recall that $\mathbb{R}[D] \equiv \mathbb{R}[x]$; in particular, $\mathbb{R}[D]$ is commutative "ring" (multiplicative commutative).

Solve by "elimination:"

$$L_{21}(L_{11}x + L_{12}y) = L_{21}0 \implies$$

$$L_{21}L_{11}x + L_{21}L_{12}y = L_{21}0$$

$$L_{11}(L_{21}x + L_{21}y) = L_{11}0 \implies$$

$$L_{11}L_{21}x + L_{11}L_{21}y = 0$$

$$L_{21}L_{12}y - L_{11}L_{22}y = 0 \implies$$

$$(L_{21}L_{12} - L_{11}L_{22})y = 0 \implies$$

$$\begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} y = 0$$

Analogously

$$\begin{vmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{vmatrix} x = 0$$

Therefore

$$(|L|I) \begin{pmatrix} x \\ y \end{pmatrix} = \vec{0}$$

where

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

Recall: If A is a sq matrix then

$$AA^a = |A|I$$

This holds for any A over a commutative ring, e.g.,

$$A \in \text{Mat}_n(R)$$

where R is a commutative ring say $R = \mathbb{C}$

Thus if $L \in \text{Mat}_n(\mathbb{R}[D])$ then

$$L^a L = |L|I_n,$$

where $I_n \in \text{Mat}_n(\mathbb{R}[D])$,

$$I_n = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{pmatrix}, \text{ \& } 1_{Id} \in \mathbb{R}[D]$$

consider $L \in \text{Mat}_n(\mathbb{R}[D])$,
 $\vec{x}(t) = (x_1(t), \dots, x_n(t))$, $\vec{F}(t) = (F_1(t), \dots, F_n(t)) \in (C^{(d)}(I))^n$, where
 $d = \max\{\deg(L_{ij}) \mid 1 \leq i, j \leq n\}$ and $L = (L_{ij})$, and the system of
linear ODEs

$$(1) \quad L\vec{x} = \vec{F} \iff \begin{pmatrix} L_{11} & L_{12} & \cdots & L_{1n} \\ L_{21} & L_{22} & \cdots & L_{2n} \\ \vdots & & & \\ L_{n1} & L_{n2} & \cdots & L_{nn} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} F_1(t) \\ F_2(t) \\ \vdots \\ F_n(t) \end{pmatrix}$$

Here

$$L_{ij} = \sum_{k=0}^{m_{ij}} a_{ijk} D^k \in \mathbb{R}[D]$$

Applying the adjoint formula to (1)

$$(2) \quad (|L|I_n)\vec{x} = L^a \vec{F}.$$

Recall: $A^a = (c_{ij})^T$, $C_{ij} = (-1)^{i+j}|M_{ij}|$

In component form, (2) says that

$$(3) \quad |L|x_i(t) = {}_i(L^a)\vec{F}(t),$$

where ${}_i(L^a)$ is the i th row of L^a . Notice that (3) is a linear ODE in the single unknown function $x_i(t)$. therefore previous methods can be used to solve (3).

Now, we consider first order linear systems, at first, without constant coefficients. These have the form

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$$\vec{x}'(t) + p(t)\vec{x}(t) = \vec{q}(t),$$

where $\vec{x}: I \subseteq \mathbb{R} \rightarrow \mathbb{R}^n$ and $p(t) \in \text{Mat}_n(C(I))$ and $\vec{q} \in (C^{(1)}(I))^n$.

In "normal form" this becomes

$$\vec{x}'(t) = p(t)\vec{x}(t) + \vec{q}(t)$$

Aside: $R[D]$ is a commutative "ring." however, $(C(t))[D]$ this is not a commutative, the point is that to use the det properties developed last semester, we need $\text{Mat}_n(\mathbb{R})$, where \mathbb{R} is commutative ring (then thos proff genral

Example. $((C(I))[D])$ is not commutative).

Put

$$L_1 = -tD + 1$$

$$L_2 = t^2D - 3,$$

then $L_1, L_2 \in (C^{(\infty)}(\mathbb{R}))[D]$. Notice that

$$\begin{aligned} L_2x &= (t^2D - 3)x \\ &= t^2Dx - 3x \\ &= t^2x' - 3x \implies \end{aligned}$$

$$\begin{aligned} L_1L_2x &= (-tD + 1)L_2x \\ &= (-tD + 1)(t^2Dx - 3x) \\ &= -tD(t^2Dx - 3x) + t^2Dx - 3x \\ &= -tD(t^2Dx) + 3tDx + t^2Dx - 3x \\ &= -t(2tDx + t^2D^2x) + 3tDx + t^2Dx - 3x \\ &= -2t^2Dx - t^3D^2x + 3tDx + t^2Dx - 3x \\ &= -t^2Dx - t^3D^2x + 3tDx - 3x. \end{aligned}$$

Also,

$$\begin{aligned} L_1x &= (-tD + 1)x \\ &= -tDx + x \implies \\ L_2L_1x &= (t^2D - 3)L_1x \\ &= t^2DL_1x - 3L_1x \\ &= t^2D(-tDx + x) + 3tDx - 3x \\ &= t^2D(-tDx) + t^2Dx + 3tDx - 3x \\ &= t^2(-Dx - tD^2x) + t^2Dx + 3tDx - 3x \\ &= -t^2Dx - t^3D^2x + t^2dx + 3tDx - 3x \\ &= -t^3D^2x + 3tDx - 3x. \end{aligned}$$

therefore $L_1L_2 \neq L_2L_1$.

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$$\vec{x}'(t) = p(t)\vec{x}(t) + \vec{q}(t)$$

where $p(t) \in \text{Mat}_n(C(I))$. Now, we consider the case where $p(t) \in \text{Mat}_n(\mathbb{R})$, i.e., the entries of the matrix p are constant functions. In the hom case, this becomes

$$\vec{x}'(t) = A\vec{x}(t)$$

or

$$(1) \quad \vec{x}' = A\vec{x}$$

where $A \in \text{Mat}_n(\mathbb{R})$.

Notice that if $n = 1$ then (1) becomes

$$x' = ax,$$

Which has solns $x(t) = ke^{at}$. This leads us to a conjecture in the more general case,

$$x_i(t) = k_i e^{a_i t} \text{ for } 1 \leq i \leq n.$$

In other words,

$$\vec{x}(t) = \begin{pmatrix} k_1 e^{a_1 t} \\ k_2 e^{a_2 t} \\ \vdots \\ k_n e^{a_n t} \end{pmatrix}$$

Since $\vec{x}' = (x'_i)$,

$$\vec{x}'(t) = \begin{pmatrix} a_1 k_1 e^{a_1 t} \\ a_2 k_2 e^{a_2 t} \\ \vdots \\ a_n k_n e^{a_n t} \end{pmatrix}$$

therefore this $\vec{x}(t)$ is a soln to (1) \iff

$$\begin{pmatrix} a_1 k_1 e^{a_1 t} \\ a_2 k_2 e^{a_2 t} \\ \vdots \\ a_n k_n e^{a_n t} \end{pmatrix} = A \begin{pmatrix} k_1 e^{a_1 t} \\ k_2 e^{a_2 t} \\ \vdots \\ k_n e^{a_n t} \end{pmatrix}$$

Here we see that if $a_i = a_j$ for all i and j , say $a = a_i$, then $\vec{x}'(t) = a\vec{x}(t)$ and so, (1) becomes

$$a\vec{x}(t) = A\vec{x}(t)$$

Now, notice that

$$\vec{x}(t) = e^{at}\vec{k},$$

where $\vec{k} = (k_1, k_2, \dots, k_n)$. therefore (1) now becomes

$$ae^{at}\vec{k} = A(e^{at}\vec{k}) \implies$$

$$A\vec{k} = a\vec{k}.$$

Recall: an eigenvalue of $A \in \text{Mat}_n(\mathbb{R})$ is $\lambda \in \mathbb{R}$ iff there is $\vec{0} \neq \vec{v} \in \mathbb{R}^n$ st

$$A\vec{v} = \lambda\vec{v}.$$

Here \vec{v} is called an eigenvector assoc with λ . therefore from above, we see that $\vec{x}(t) = e^{\lambda t}\vec{v}$ is a soln to $\vec{x}' = A\vec{x}$ iff λ is an eigenvalue of A and \vec{v} is an assoc eigenvector of λ .

Recall: to find eigenvalues of $A \in \text{Mat}_n(\mathbb{R})$, notice that for $\vec{v} \neq \vec{0}$,

$$\begin{aligned} A\vec{v} = \lambda\vec{v} &\iff \lambda\vec{v} - A\vec{v} = \vec{0} \\ &\iff (\lambda I_n - A)\vec{v} = \vec{0}, \end{aligned}$$

which is a hom system with a nontrivial soln \vec{v} . So, therefore

$$\det(\lambda I_n - A) = 0.$$

Also, recall that the char poly of $A \in \text{Mat}_n(\mathbb{R})$,

$$p_A(x) = \det(xI_n - A) \in \mathbb{R}[x],$$

which is an n th degree poly over \mathbb{R} . So, therefore to find eigenvalues of A , we must find the zeros of $p_A(x)$. Once we have the eigenvalue λ , to find an assoc $\vec{v} \neq \vec{0}$, we solve for \vec{v} in $(A - \lambda I_n)\vec{v} = \vec{0}$.

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Example. $\begin{cases} x' &= 2x + 3y \\ y' &= 2x + y \end{cases} \iff \vec{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \vec{x}$
 put

$$A = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix},$$

then

$$\rho_A(x) = \begin{vmatrix} x-2 & -3 \\ -2 & x-1 \end{vmatrix} \implies$$

$$\begin{aligned} &= (x-2)(x-1) - 6 \\ &= x^2 - 3x - 4 \\ &= (x-4)(x+1). \end{aligned}$$

therefore the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$. Now, let us find assoc eigenvectors:

$$\lambda_1 = 4 \implies \begin{pmatrix} -2 & 3 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \vec{0}.$$

Take $(u, v) = (3, 2)$, say, then $\vec{v}_1 = (3, 2)$ is an assoc eigenvector of $\lambda_1 = 4$. Therefore $\vec{x} = e^{4t}(3, 2)$ is a soln. Note:

$$\begin{aligned} x &= 3e^{4t} \\ y &= 2e^{4t} \end{aligned}$$

$$\implies y = \frac{2}{3}x$$

where where $x, y > 0$.

$$\lambda_2 = -1 \implies \begin{pmatrix} 3 & 3 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \vec{0}.$$

Take $(u, v) = (1, -1)$, say, then $\vec{v}_2 = (1, -1)$ is an assoc eigenvector of $\lambda_2 = -1$. therefore $\vec{x}_2 = e^{-t}(1, -1)$

Final exam

Homework up through 5.2

1.a solve a 1st order initial value problem

2.a Find a basis for the soln space of a 2nd order hom ODE (complex)

2.b Find a general soln

3 reduce the given 2nd order system to a first order system

4.a Find a particular soln to a 3rd order nonhom ODE

4.b Now, find a general form of all hom solns

4.c Finally, find a gen form of all non-hom solns

5 Solve a 1st order linear system with subst, elim, cramers, adjnts i.e., not with eigenvalues.

6 Show a given system has no soln.

7a Given a linear system, write it in diff operator 'D' notation.

7b Find the char poly of the system

7c solve the system

8a prove given vector-valued functions are linearly independent

8b determine the dims of the soln space

8c if the given vector-valued functions are solns to a linear system of ODEs, then write down a general soln, 6 Show a given system has no soln.

9 use the eigenvalue method to solve a 1st order linear system.

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