Notes for Second Test

Joshua Bailey

March 25, 2015

Date:
02.17.15.

Ex

$$(2x \sin y \cos y)y' = 4x^2 + \sin^2 y$$
$$v = \sin y \implies$$
$$\frac{dv}{dx} = \cos y \frac{dy}{dx}$$

Thus,

$$2xvv' = 4x^2 + v^2 \text{ (hom)}, \; w = v/x$$

2.1 population models

Recall that the most basic population model,
assuming constant birth and death rates, is

$$P' = kP \text{ (separable)}$$

Now, we give birth rates and death rates the
following units:

$$\beta(t) = \text{ birth rate } \frac{\text{\# of births at t}}{\text{(unit of population at t)(unit of time)}}$$

$$\delta(t) = \text{ death rate } \frac{\text{\# of death at t}}{\text{(unit of population at t)(unit of time)}}$$

with this, for some small Δt,

$$P(t + \Delta t) \approx P(t) + \text{ (\# birth rate at t - \# death rate at t) } \Delta t \implies$$
$$= P(t) + \beta(t)P(t)\Delta t - \delta(t)P(t)\Delta t$$
$$\implies \frac{\Delta P}{\Delta t} = \frac{P(t + \Delta t) - P(t)}{\Delta t} \approx (\beta(t) - \delta(t))P(t)$$

using a diff model, the above suggest that

$$\boxed{P' = (\beta - \delta)P}$$

this is called the general population

Now, consider population with constant death rate, say δ_0, and with a birth rate given by

$$\beta = \beta_1 - \beta_0 P$$

In context, β_0 , $\beta_1 > 0$ by (1),

$$P' = (\beta_1 - \beta_0 P - \delta_0)P$$
$$= (\beta_1 - \delta_0)P - \beta_0 P^2$$
$$= \beta_0 P\Big(\frac{\beta_1 - \delta_0}{\beta_0} - P\Big)$$

which has form

$$P' = kP(M - P)$$

$k = \beta_0 > 0$ and $M = (\beta_1 - \delta_0)/\beta_0$ we see that in context, M>0 here (2) is separable; as

$$\frac{P'}{P(M-P)} = k \implies$$

$$\int \frac{1}{P(M-P)} dP = kt + C$$

Note that

$$\frac{1}{P(M-P)} = \frac{1}{M}\left(\frac{1}{M-P} + \frac{1}{P}\right)$$

Thus,

$$\frac{1}{M}(-\ln|M-P| + \ln|P|) = kt + C \implies$$

$$\ln\left|\frac{P}{M-P}\right| = Mkt + C$$

Now, $P_0 = P(0)$ yields

$$C = \ln\left|\frac{P_0}{M-P_0}\right|$$

Thus,

$$\ln\left|\frac{P}{M-P}\right| = \ln\left|\frac{P_0}{M-P_0}\right| + Mkt$$

Hence, if $P > M$ or $P > M$ then

$$\frac{P}{M-P} = \frac{P_0}{M-P_0}e^{Mkt} \implies \frac{M}{P} - 1 = \frac{M-P_0}{P_0}e^{-Mkt}$$

$$\implies \frac{M}{P} = \frac{P_0 + (M-P_0)e^{-Mkt}}{P_0}$$

$$\boxed{P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-Mkt}}}$$

Notice that

$$\lim_{t\to\infty} P(t) = \frac{MP_0}{P_0} = M$$

If $P < M$, so that $P_0 < M$, then

$$P' = kP(M-P) > 0$$

so, P is increasing to M

On the other hand, if $P > M$, so
that $P_0 > M$, then

$$P' = kP(M-P) < 0$$

so, P is decresing to M
Since $P(t) \to M$ in
either case, in context,
$M > 0$, ($M = 0$ & $P > M \implies$ *extinction*).

Recall: logistics population model

Date:
02.19.15.

$$P(t) = \frac{MP_0}{P_0 + (M-P)e^{-Mkt}}$$

ex(P.88) # 22

$M = 100 \times 10^3$ total population
At $t = 0$, half the pop have heard a rumor
roughly the rumors increases
by 1000 people after 1 day

$$P_0 = 50 \times 10^3 \text{ \& } P(1) = 51 \times 10^3$$

we can solve for k

$$51 \times 10^3 = \frac{(100 \times 10^3)(50 \times 10^3)}{50 \times 10^3 + (100 \times 10^3 - 50 \times 10^3)e^{-Mk}}$$

$$\implies 51 = \frac{5000}{50 + 50e^{-Mk}} = \frac{100}{1 + e^{-Mk}}$$

$$\implies \frac{51}{100} = \frac{1}{1 + e^{-Mk}} \implies$$

$$1 + e^{-Mk} = \frac{100}{51} \implies e^{-Mk} = \frac{100}{51} - 1 \implies$$

$$-Mk = \ln\left(\frac{100}{51} - 1\right)$$

$$\implies k = -\frac{\ln(\frac{100}{51} - 1)}{10 \times 10^3} > 0$$

$\therefore$ we can now solve $P(t) = 80 \times 10^3$
for t.

Doomsday/Extinction Model:

Here we assume that

$$\beta = kP, \; k > 0$$

$$\& \; \delta = \delta_0$$

Thus, the gen pop ODE, $P' = (\beta - \delta)P$,
becomes

$$(1) \; P' = (kP - \delta_0)P = kP(P - \delta/k)$$

put $M = \delta/k > 0$, then
(1) becomes

$$(2) \; P' = kP(P - M)$$

constant (2) with the logistics
ODE: $P' = kP(M - P)$ we can solve (2); it is separable. Thus,

$$\frac{P'}{P(P-M)} = k \implies$$

$$\int \frac{1}{P(P-M)} dP = kt + C$$

Note:

$$\frac{1}{P(P-M)} = \frac{1}{M}(\frac{1}{P-M} - \frac{1}{P})$$

$\therefore$

$$\frac{1}{M}(\ln|P-M| - \ln P) = kt + C$$

$$\implies \ln\left|\frac{P-M}{P}\right| = Mk + C$$

$$\implies \frac{|P-M|}{P} = e^{Mkt+C}$$

If $P_0 = P(0)$ then

$$C = \ln\left|\frac{P_0 - M}{P_0}\right|$$

so,

$$\frac{P-M}{P} = \frac{P_0 - M}{P_0}e^{Mkt}$$

Now, in any case

$$\frac{P-M}{P} = \frac{P_0 - M}{P_0}e^{Mkt} \implies$$

$$1 - \frac{M}{P} = \frac{P_0 - M}{P_0}e^{Mkt} \implies$$

$$\frac{M}{P} = \frac{P_0 - (P_0 - M)e^{Mkt}}{P_0} \implies$$

$$\boxed{P = \frac{MP_0}{P_0 + (M - P_0)e^{Mkt}}}$$

contrast with logistic ODE
soln:

$$P = \frac{P_0 M}{P_0 + (M - P_0)e^{-Mkt}}$$

Date:
02.23.15.

With explosion/extinction modol

Notice that

$$P > M \implies P' > 0 \implies P \text{ is increasing;}$$

$$P > M \implies P' < 0 \implies P \text{ is decreasing;}$$

Now, if $P > 0$, then P has a vertical asymptote at t_0 st.

$$P_0 + (M - P_0)e^{Mkt_0} = 0 \implies$$

$$e^{Mkt_0} = \frac{P_0}{P_0 - M} > 0 \implies$$

$$Mkt_0 = \ln \frac{P_0}{P_0 - M}$$

$$\boxed{t_0 = \frac{1}{Mk} \ln \left(\frac{P_0}{P_0 - M} \right)}$$

which is the time of "doomsday," i.e., the explosion of the population

$$\lim_{t \to t_0^-} P(t) = \infty$$

On the other hand, if $P < M$, then $M - P_0 > 0$; the

$$\lim_{t \to \infty} P(t) = 0$$

This is to say, over time, extinction occurs.

equilibuim solns and stability

Def. An autonomous ODE has the form

$$(1)\ \frac{dx}{dt} = f(x)$$

Notice that the slope field in (1) is "independent" of autonomous" of t.

Newtons law of cooling

$$T' = k(A - T)\ ,\ k > 0$$

Recall:

$$\int \frac{1}{A - T} dt = \int k dt = kt + C \implies$$

$$-\ln |A - T| = kt + C \implies$$

$$|A - T| = e^{-kt-C}$$

where $T_0 = T(0) \implies$

$$-C = \ln |A - T_0|$$

thus,

$$|A - T| = |A - T_0| e^{-kt} \implies$$

$$A - T = (A - T_0) e^{-kt} \implies$$

$$\boxed{T(t) = A + (T_0 - A) e^{-kt}}$$

Notice that

$$\lim_{t \to \infty} = A$$

Also, notice that $T(t) \equiv A$, i.e, $T(t) = A$ for all t, in a soln to the autonomous ODE $T' = k(A - T)$. this is an example of an "equilibruim soln"

Def. $x(t) \equiv C \in \mathbb{R}$ is an equilibrium soln to $x' = f(x)$ iff $x(t) \equiv C$ is a soln to $x' = f(x)$

Def. $x = C \in \mathbb{R}$ is a critical point of $x' = f(x)$ iff $f(c) = 0$
Notice that we say that "x is a critical pt" iff $0 = f(C) = \frac{dx}{dt}$, which is similar to use in calc I of "critical pt."

Prop. $x = C$ is a critixcal pt of $x' = f(x) \iff x(t) \equiv C$ is an equilibrium soln to $x' = f(x)$
proof. easy.

Def. $C \in \mathbb{R}$ is a stable critical
point of $x' = f(x)$ iff C is a
critical pt of $x' = f(x)$ and

$$\forall \epsilon > 0 \; \exists \; \delta > 0 \forall$$

$$|x_0 - C| < \delta \implies |x(t) - C| < \epsilon$$

Ex (Logistics Modle)

$$P' = kP(M - P) \implies$$

$$P(t) = \frac{MP_0}{p_0 + (M-P)e^{-Mkt}} \implies$$

$$\lim_{t \to \infty} P(t) = M$$

Ex(Standard pop Model)

$$P' = kP \implies P(t) = P_0 e^{kt}$$

Here $P = 0$ is a critical pt; however,
$P = 0$ is not stable. Notice that $P' = kP(M - P)$ has two
critical pts, namely, $P = 0$ and $P = M$.
Here $P(t) = M$ is stable, whereas, $P(t) \equiv 0$ is not
stable.

Ex(explosion/ extinction model)

$$P' = kP(P - M) \; , \; k, M > 0$$

$$\implies P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{Mkt}}$$

Both $P = 0$ and $P = M$ are
critical pts. however, if $P_0 > M$
then ? a stable, whereas
if $P_0 < M$then only $P = 0$
is stable.

$$|x_0 - C| < \delta \implies$$

$$(x(t) - C) < \epsilon$$

Logistics Population Model with Harvesting

Date: 02.25.15.

Recall the Logistics Pop Model:

$$P' = kP(M-P),\ k,\ M > 0$$

We now consider

$$P' = kP(M-P) - h$$

where h is a constant, think: $h > 0 \implies$ harvesting;

$$h < 0 \implies \text{ stocking}$$

Notice that

$$\begin{aligned} P' &= -kP^2 + kMP - h \\ &= -k(P^2 - MP + h/k) \\ &= -k(P-N)(P-H) \end{aligned}$$

where

$$\text{H,N } = \frac{M \pm \sqrt{M^2 - 4h/k}}{2}$$

Here H and N are distinct reals if and only if $M^2 - 4h/k > 0 \iff M^2 > 4h/k \iff h < kM^2/4$. so, if $h > 0$ and H, N are distinct and real, then

$$0 < h < \frac{kM^2}{4}$$

Say, H<N. Notice that

$$\begin{gathered} h > 0 \implies \\ \sqrt{M^2 - 4h/k} < \sqrt{m^2} \implies \\ M - \sqrt{M^2 - 4h/k} > 0 \\ \therefore 0 < H \text{ ; where, } 0 < H < N \end{gathered}$$

Now, "separating," yields that

$$\frac{P'}{(P-H)(P-N)} = -k \implies$$

$$\int \frac{1}{(P-H)(P-N)} dP = -kt + C$$

Notice that

$$\frac{1}{(P-H)(P-N)} = \left(\frac{1}{P-N} - \frac{1}{P-H}\right)\frac{1}{N-H}$$

Thus,

$$\ln\left|\frac{P-N}{P-H}\right| = -(N-H)kt + C$$

If $P_0 = P(0)$ then

$$C = \ln \left| \frac{P_0 - N}{P_0 - H} \right|$$

Hence $\left|\frac{P-N}{P-H}\right| = e^c e^{-(N-H)kt} = \left|\frac{P_0-N}{P_0-H}\right| e^{-(N-H)kt}$ Now, in any case,

$$\frac{P - N}{P - H} = \frac{P_0 - N}{P_0 - H} e^{-(N-H)kt}$$

$$\lim_{t\to\infty} \frac{P - N}{P - H} = \lim_{t\to\infty} \frac{P_0 - N}{P_0 - H} e^{-(N-H)kt}$$

$$\therefore \lim_{t\to\infty} (P - N) = 0 \text{ or } \lim_{t\to\infty} (P - H) = \pm\infty$$

Q: Is $N < M$?
Q; $P(t) =$?

Recall:

Date: 02.26.15

$$\frac{P-N}{P-H} = \frac{P_0-N}{P_0-H}e^{-(N-H)kt} \implies$$

$$(P-N)(P_0-H) = (P-H)(P_0-N)e^{-(N-H)kt} \implies$$

$$P(P_0-H) - N(P_0-H) = P(P_0-N)e^{-(N-H)kt} - H(P_0-N)e^{-(N-H)kt} \implies$$

$$P(P_0-H-(P_0-N)e^{-(N-H)kt}) = N(P_0-H) - H(P_0-N)e^{-(N-H)kt} \implies$$

$$\boxed{P(t) = \frac{N(P_0-H) - H(P_0-N)e^{-(N-H)kt}}{P_0-H-(P_0-N)e^{-(N-H)kt}}}$$

$\therefore$

$$\lim_{t\to\infty} P(t) = \frac{N(P_0-H)}{P_0-H} = N$$

Now, if $0 < h < kM^2/4$ then

$$0 < H < N < M$$

For recall that

$$H, N = \frac{M \pm \sqrt{M^2-4h/k}}{2}$$

so since $h > 0 \implies M^2 - 4h/k < M^2 \implies$

$$\sqrt{M^2-4h/k} < M \implies M + \sqrt{M^2-4h/k} < 2M$$

$$N = \frac{M+\sqrt{M^2-4h/k}}{2} < M$$

Here

$$N(P_0-H) - H(P_0-N)e^{-(N-H)kt_E} = 0$$

which has a soln if $P_0 < H$

Date:
03.02.15

Vertical Motion with Air Resistance

Recall Newton's 2nd Law:

$$ma = \sum F_i \text{ (net forces)}$$

Here $a = v' = dv/dt$, F_G (force due to gravity)
and F_R (force due to air resistance). Now,

$$F_G = -mg \;\&\; (F_R < 0 \iff v > 0)$$

where $g \approx 9.8m/s^2$ Empirically,

$$F_R = kv^p$$

where $1 \leq p \leq 2$ & $k > 0$
two cases, namely $p = 1$ & $p = 2$.

$p = 1$ Here, we have that

$$mv' = F_G + F_R \implies$$

$$mv' = -mg - kv$$

since $F_R = -kv$. Notice that

$$(1)\ v' = -g - \frac{k}{m}v$$

$$= -(\frac{k}{m}v + g)$$

Also, notice that (1) is a 1st order linear ODE,

$$v' + \frac{k}{m}v = -g$$

where $\rho = k/m$, called the drag constant. Thus

$$v' = (\rho v + g) \implies$$

(sup)

$$\frac{v'}{\rho v + g} = -1 \implies \int \frac{1}{\rho v + g} dv = -t + c$$

$$\implies \frac{1}{\rho} \ln |\rho v + g| = -t + c$$

$$\implies \ln |\rho v + g| = -\rho t + c$$

$$c = \ln |\rho v_0 + g|$$

$$|\rho v + g| = |\rho v_0 + g| e^{-\rho t}$$

$$\rho v + g = |\rho v_0 + g| e^{-\rho t} \implies$$

$$\boxed{v(t) = \frac{1}{\rho}((\rho v_0 + g)e^{-\rho t} - g)}$$

Notice that $\lim_{t\to\infty} v(t) = -g/\rho$; this is
called terminal velocity. we denote this as

$$v_\tau = -g/\rho$$

Thus,

$$\boxed{v(t) = (v_0 - v_\tau)e^{-\rho t} + v_\tau}$$

Now,

$$x(t) = v_\tau t - \frac{1}{\rho}(v_0 - v_\tau)e^{-\rho t} + c \implies$$

$$c = x_0 + \frac{1}{\rho}(v_0 - v_\tau) \implies$$

$$\boxed{x(t) = x_0 + v_\tau t + \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t})}$$

Recall that $F_R = \pm kv^p$, $F_G = -mg$, and

Date: 03.03.15

$$ma = \sum F = F_G + F_R \implies$$

$$(1)\ v' = -g \pm \frac{k}{m} v^p = -g \pm \rho v^p$$

where drag $\rho = k/m$.
p=2 there are 2 cases:
(i) upward motion, $F_R = -kv^2$;
(ii) downward motion, $F_R = kv^2$.

(i) Upward motion Here (1) becomes

$$v' = -g - \rho v^2 = -g(\frac{\rho}{g} v^2 + 1) = -g((v\sqrt{\frac{\rho}{g}})^2 + 1)$$

$$\implies \int \frac{1}{(v\sqrt{\rho/g})^2 + 1} dv = -gt + c \implies$$

$$\frac{1}{\sqrt{\rho/g}} \arctan(v\sqrt{\rho/g}) = -gt + c \implies \arctan(v\sqrt{\rho/g}) = -t\sqrt{\rho g} + c$$

$$\implies c = \arctan(v_0\sqrt{\rho/g}) \implies$$

$$v\sqrt{\rho/g} = \tan(\arctan(v_0\sqrt{\rho/g}) - t\sqrt{\rho g}) \implies$$

$$\boxed{v(t) = \sqrt{g/\rho}\tan(\arctan(v_0\sqrt{\rho/g}) - t\sqrt{\rho g})}$$

Thus,

$$x(t) = \sqrt{g/\rho}\frac{1}{\sqrt{\rho g}} \ln|\cos(\arctan(v_0\sqrt{\rho/g}) - t\sqrt{\rho g})| + c$$

$$\implies = \frac{1}{\rho} \ln|\cos(\arctan(v_0\sqrt{\rho/g} - t\sqrt{\rho g}| + c$$

$$\implies c = x_0 - \frac{1}{\rho} \ln|\cos(\arctan(v_0\sqrt{\rho/g}))| \implies$$

$$\boxed{x(t) = x_0 + \frac{1}{\rho} \ln\left|\frac{\cos(\arctan(v_0\sqrt{\rho/g}) - t\sqrt{\rho g}}{\cos(\arctan(v_0\sqrt{\rho/g}))}\right|}$$

Here $v(t) = 0$ allows us to find
time of max height, say t_m

$$\boxed{t_m = \frac{1}{\sqrt{\rho g}} \arctan(v_0\sqrt{\rho/g})}$$

(ii) Downward Motion

$$v' = -g + \rho v^2 = -g(1 - (v\sqrt{\rho/g})^2) \implies$$

$$\int \frac{1}{1 - (v\sqrt{\rho/g})^2} dv = -gt + c$$

Date:
03.04.15

$$\frac{1}{\sqrt{\rho/g}}\operatorname{arctanh}(v\sqrt{\rho/g} = -gt + c \implies$$

$$\operatorname{arctanh}(v\sqrt{\rho/g} = -\sqrt{\rho g}t + c \implies$$

$$c = \operatorname{arctanh}(v_0\sqrt{\rho/g}) \implies$$

$$v\sqrt{\rho/g} = \tanh(\operatorname{arctanh}(v_0\sqrt{\rho/g} - t\sqrt{\rho g} \implies$$

$$\boxed{v(t) = \sqrt{g/\rho}\tanh(\operatorname{arctanh}(v_0\sqrt{\rho/g}) - t\sqrt{\rho g})}$$

$\therefore$

$$x(t) = \sqrt{g/\rho}(-\frac{1}{\sqrt{\rho g}}\ln|\cosh(\operatorname{arctanh}(v_)\sqrt{\rho/g} - t\sqrt{\rho g})| + c$$

$$= -\frac{1}{\rho}\ln|\cosh(\operatorname{arctanh}(v_0\sqrt{\rho/g}) - t\sqrt{\rho g}| + c$$

$$\implies c = x_0 + \frac{1}{\rho}\ln|\cosh(\operatorname{arctanh}(v_0\sqrt{\rho/g}))|$$

$$\therefore \boxed{x(t) = x_0 - \frac{1}{\rho}\ln|\frac{\cosh(\operatorname{arctanh}(x_0\sqrt{\rho/g} - t\sqrt{\rho g}}{\cosh(\operatorname{arctanh}(v_0\sqrt{\rho/g}))}}$$

Thrm(Inverse Function trm). if $f'(x) \neq 0$ then f^{-1} is diff at $y = f(x)$ and

$$\frac{df^{-1}}{dy}(y) = \frac{1}{\frac{df}{dx}(x)}$$

$$(\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}})$$

where $y = f(x) \iff x = f^{-1}(y)$

Aside:

$$\text{Ex } y = f(x) = \tanh x \implies$$

$$\frac{df^{-1}}{dy}(y) = \frac{1}{\frac{df}{dx}(x)} = \frac{1}{\operatorname{sech}^2 x} = \frac{1}{1-\tanh^2 x} = \frac{1}{1-y^2}$$

$$\text{Ex } \int \tanh x dx = \int \frac{\sinh x}{\cosh x}dx \int \frac{1}{u}du = \ln|\cosh x| + c$$

$u = \cosh x$
$u' = \sinh x$

Aside:

$$\cosh x = \frac{e^x - e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh' x = \sinh x$$

$$\sinh' x = \cosh x$$

$$\frac{f(x) \pm f(-x)}{2}$$

Ex

Date:
03.05.15

Escape Velocity

Recall Newton's Gravitational Law:

$$F = \frac{GmM}{r^2}$$

where $G \approx 6.67 \times 10^{-11}$. Let m be the mass of a projectile from a planet's surface of mass M of radius R. By Newton's 2nd Law,

$$ma = -\frac{GmM}{r^2} \implies$$

$$v' = -\frac{GM}{r^2}$$

By the chain rule,

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$$

Thus,

$$v\frac{dv}{dr} = -\frac{GM}{r^2} \text{ (separable)}$$

$$\int v dv = -GM \int r^{-2} dr \implies$$

$$\frac{v^2}{2} = \frac{GM}{r} + c \implies$$

$$c = \frac{v_0^2}{2} - \frac{GM}{r_0}$$

where $r_0 = R \therefore$

$$\frac{v^2}{2} = \frac{v_0^2}{2} - \frac{GM}{R} + \frac{GM}{r} \implies$$

$$v^2 = v_0^2 - \frac{2GM}{R} + \frac{2GM}{r}$$

$$> v_0^2 - \frac{2GM}{R}$$

To "escape" the gravitational force of the planet, we must have that $v > 0$ for all r. This happens if

$$v^2 > v_0^2 - \frac{2GM}{R} > 0 \iff$$

$$v_0^2 > \frac{2GM}{R} \implies$$

Ex (p.109) # 30

Newton's 2nd Law:

$$ma = \text{ net forces } = F_e + F_m$$

By Newton's Gravitational law,

$$F_e = -\frac{GmM_e}{r^2} \ \&$$

$$F_m = \frac{GmM_m}{(s-r)^2}$$

$\therefore$

$$\frac{dv}{dt} = \frac{GMm}{(s-r)^2} - \frac{GM_e}{r^2}$$

As before,

$$v\frac{dv}{dr} = G(\frac{M_m}{(s-r)^2} - \frac{M_e}{r^2})$$

which is separable. Thus,

$$\frac{v^2}{2} = G(M_m \int \frac{1}{(s-r)^2} dr - M_e \int \frac{1}{r^2} dr)$$

$$= G(\frac{M_m}{s-r} + \frac{M_e}{r} + c \implies$$

$$\frac{v_0^2}{2} = G(\frac{M_m}{s-R} + \frac{M_e}{R}) + c$$

$\implies$

$$\frac{v^2}{2} = G(\frac{M_m}{s-r} + \frac{M_e}{r}) + \frac{v_0^2}{2} - G(\frac{M_m}{s-R} + \frac{M_e}{R})$$

Recall: Date: 03.09.15

$$ma = \sum F_i \ , \ (r_0 = R)$$

$$(r_0 = R) \ , \ R \leq r \leq s$$

where

$$F_e = -\frac{GmM_e}{r^2} \ \& \ F_m = \frac{GmM_m}{(s-r)^2}$$

thus,

$$a = \frac{d^2r}{dt^2} = F_m + F_e = \frac{GM_m}{(s-r)^2} - \frac{GM_e}{r^2}$$

By chain rule,

$$\frac{d^2r}{dt^2} = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$$

so,

$$v\frac{dv}{dr} = \frac{GM_m}{(s-r)^2} - \frac{GM_e}{r^2} \text{ (sep)}$$

$$\frac{v^2}{2} = \frac{GM_m}{s-r} + \frac{GM_e}{r} + c \implies$$

$$c = \frac{v_0^2}{2} - \frac{GM_m}{s-R} - \frac{GM_e}{R}$$

Hence,

$$\frac{v^2}{2}\frac{GM_m}{s-r} + \frac{GM_e}{r} + \frac{v_0^2}{2} - \frac{GM_m}{s-R} - \frac{GM_e}{R}$$

we want $v > 0$

$$a = 0 \implies$$

$$\frac{GM_m}{(s-r)^2} = \frac{GM_e}{r^2} \implies$$

$$(\frac{s-r}{r}) = \frac{M_m}{M_e} \implies$$

$$\frac{s}{r} - 1 = \sqrt{M_m/M_e} \implies$$

$$\frac{s}{r} = 1 + \sqrt{M_m/M_e} = \frac{\sqrt{M_e}\sqrt{M_e}}{\sqrt{M_e}}$$

$$r = \frac{s\sqrt{M_e}}{\sqrt{M_m} + \sqrt{M_e}}$$

Also, notice that

$$v = 0 \implies$$

$$\frac{v_0^2}{2} = \frac{GM_m}{s-r} + \frac{GM_e}{R} - \frac{GM_m}{s-R} - \frac{GM_e}{r}$$

$\therefore$

$$v_0 = \sqrt{2G(\frac{M_m}{s-R} + \frac{M_e}{R} - \frac{M_m + M_e + 2\sqrt{M_m M_e}}{s})} \implies$$

$$\boxed{v_0 = \sqrt{2G(\frac{M_m}{s-R} + \frac{M_e}{R} - \frac{1}{s}(\sqrt{M_m} + \sqrt{M_e})^2)}}$$

Aside:

$$\frac{1}{r} = \frac{\sqrt{M_m} + \sqrt{M_e}}{s\sqrt{M_e}} \implies$$

$$\frac{M_m}{s-r} = \frac{M_m + \sqrt{M_m M_e}}{s}$$

$$\frac{M_e}{r} = \frac{M_e + \sqrt{M_m M_e}}{s}$$

Aside:

$$r\sqrt{M_m} + r\sqrt{M_e} = s\sqrt{M_e} \implies$$

$$r\sqrt{M_m} = \sqrt{M_e}(s-r) \implies$$

$$\frac{1}{s-r} = \frac{1}{r}\sqrt{\frac{M_e}{M_m}} \implies$$

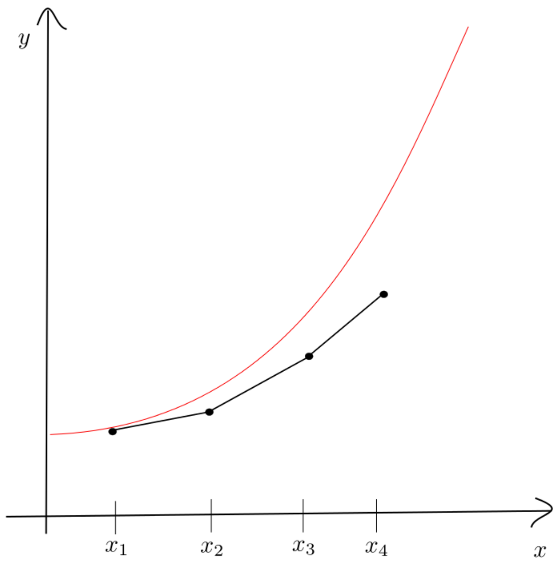
$$\frac{M_m}{s-r} = \frac{\sqrt{M_m M_e}}{r}$$

Euler's Method

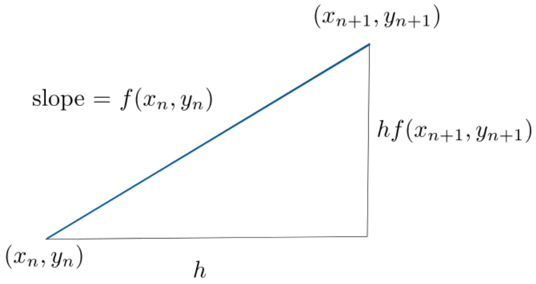
Given a slope field, $y' = f(x, y)$,
& a specific soln to the initial
value problem

$$\frac{dy}{dx} = f(x, y) \text{ \& } (x_0, y_0)$$

say $y = y(x)$, then $y(x_0) = y_0$, &
Euler's method gives an algorithem
for estimating the exact soln $y = y(x)$



Find y_1 & y_{n+1} in general
slope $= f(x_0, y_0)$



$h = x_1 - x_0$

$$y = f(x_0, y_0)(x - x_0) + y_0 \implies$$
$$y_1 = f(x_0, y_0)h + y_0$$

In general,

$$\boxed{y_{n+1} = hf(x_n, y_n) + y_n}$$

Here

$$\boxed{y_n \approx y(x_n)}$$

Recall that an nth order linear ODE has the form

Date: 03.11.15

$$(1)\ y^{(n)} + \sum_{k=1}^{n} Pk(x)y^{(n-k)} = f(x)$$

where Pk and f are cts for $1 \leq k \leq n$.
The associated homogeneous nth order linear ODE to (1) is
(2)

$$y^{(n)} + \sum_{k=1}^{n} Pk(x)y^{(n-k)} = 0$$

i.e., (1) with $f(x) \equiv 0$
Notation. put

$$V = y : I \to \mathbb{R} \mid \text{y has nth order derivative on I}$$

then V is an $\mathbb{R}$- linear space. put

$$W = y \in V \mid \text{y is a soln to (2)}$$

then the following holds

thrm. $W \leq V$, i.e., the set of all solns to (2) is a linear space.
Proof. If $y \in W$ & $c \in \mathbb{R}$ then

$$(cy)^{(n)} + \sum_{k=1}^{n} Pk(x)(cy)^{(n-k)} =$$

$$(cy)^{(n)} + c\sum_{k=1}^{n} Pk(x)y^{(n-k)}$$

$$c(y^{(n)} + \sum_{k=1}^{n} Pk(x)y^{(n-k)})$$

$$c \cdot 0 = 0$$

$\therefore cy \in W$
If $y_1, y_2 \in W$ then

$$(y_1 + y_2)^{(n)} + \sum_{k=1}^{n} Pk(x)(y_1 + y_2)^{(n-k)} =$$

$$y_1^{(n)} + \sum_{k=1}^{n} Pk(x)y_1^{(n-k)} + y_2^{(n)} + \sum_{k=1}^{n} Pk(x)y_2^{(n-k)} =$$

$$0 + 0 = 0$$

$\therefore y_1 + y_2 \in W$. Hence, $W \in \mathbb{R}$-linear subspace of V.

Recall: Thrm (wronskian thrm). if $f_1, ..., f_n$ are linearly independent in $c^{(n-1)}(I) =$

$\{f : I \to \mathbb{R} \mid f^{(n-1)}$ is cts on I$\}$, then the wronskian of $f_1, ..., f_n$ is identically 0 for all $x \in I$, i.e., for all $x \in I$,

$$|W(f_1, ..., f_n)(x)| = det \begin{pmatrix} f_1(x) & \cdots & f_n(x) \\ f_1'(x) & \cdots & f_n'(x) \\ f_1''(x) & \cdots & f_n''(x) \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{pmatrix} = 0$$

Aside

$$A\vec{x} = \vec{b}$$

the soln space of

$$A\vec{x} = \vec{0}$$

is a linear space. the soln space of

$$A\vec{x} = \vec{b}$$

is a affine linear space, with solns

$$\vec{x} = \vec{x_0} + \vec{x_1}$$

where $\vec{x_0}$ is any hom soln

Exam 2

1. hom ODE
2. ODE needing a subst to reduse to a 1st order linear/sep
3. exact ODE
4. population Model (logistic pop)
5. population Model (havesting a logistic pop)