



Linear Algebra

Matrices Part 1: Matrices and Basic Operations

Automotive Intelligence Lab.



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Matrices as linear transformations

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■ Symmetric matrix

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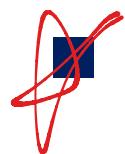


Matrices as linear transformations

Basics of Matrix-Vector Multiplication

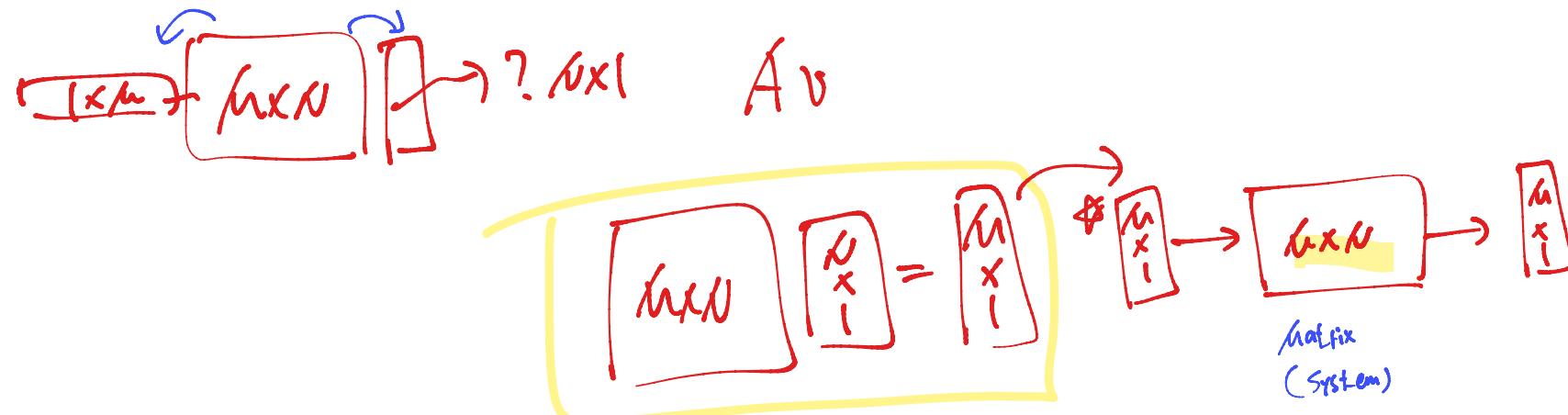
- A matrix can be right-multiplied by a column vector but not a row vector, and a matrix can be left-multiplied by a row vector but not a column vector.

 - Matrix can be If v is column vector, Av and $v^T A$ are valid, but Av^T and vA are invalid.
 - $M \times N$ matrix can be pre-multiplied by a $1 \times M$ matrix or post-multiplied by an $N \times 1$ matrix.



Result of matrix-vector multiplication is always a vector.

 - Orientation of that vector depends on the orientation of the multiplicand vector.
 - Pre-multiplying a matrix by a row vector produces another row vector.
 - Post-multiplying a matrix by a column vector produces another column vector.



What is Linear “Transformation”

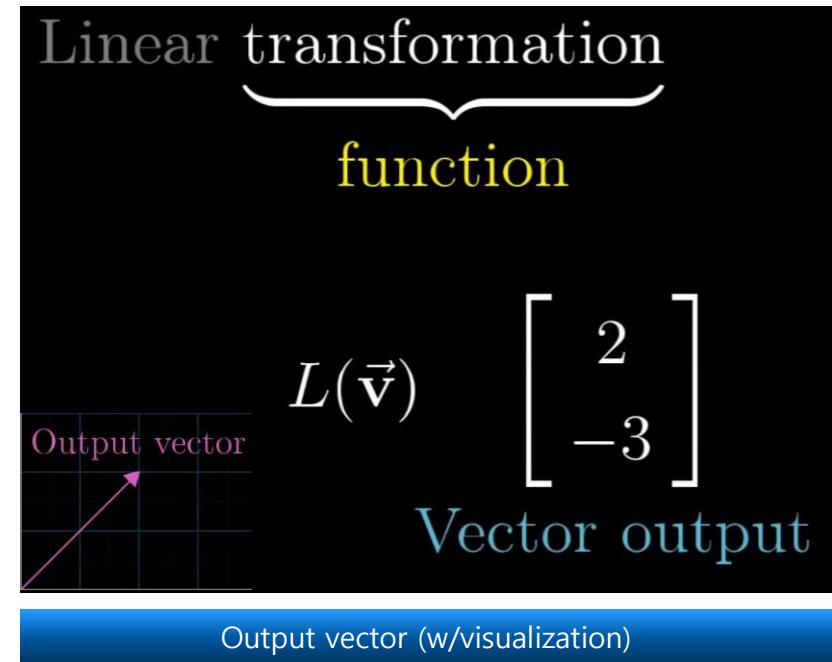
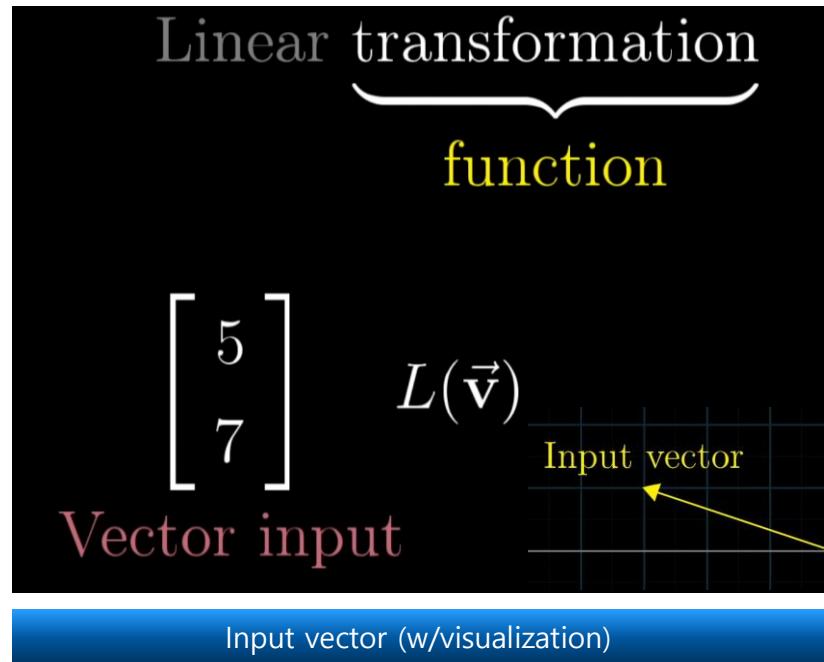
공간과 벡터의 변환

<https://youtu.be/kYB8IZa5AuE?si=kkT5-ZZpr8mFER4f> (English)
<https://youtu.be/35ESC-g49fY> (Korean)

■ The word transformation contains different meanings.

1. Function
 - It takes vector as an input, and outputs another vector.
2. Movement
 - Input vector **moves** over the corresponding output vector.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ c \end{bmatrix} + \begin{bmatrix} b \\ d \end{bmatrix}$$

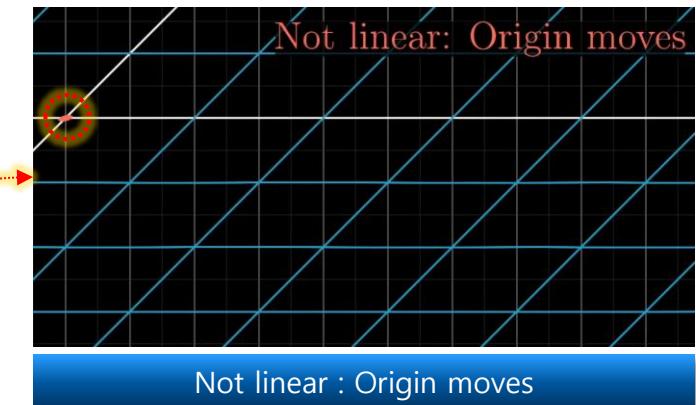
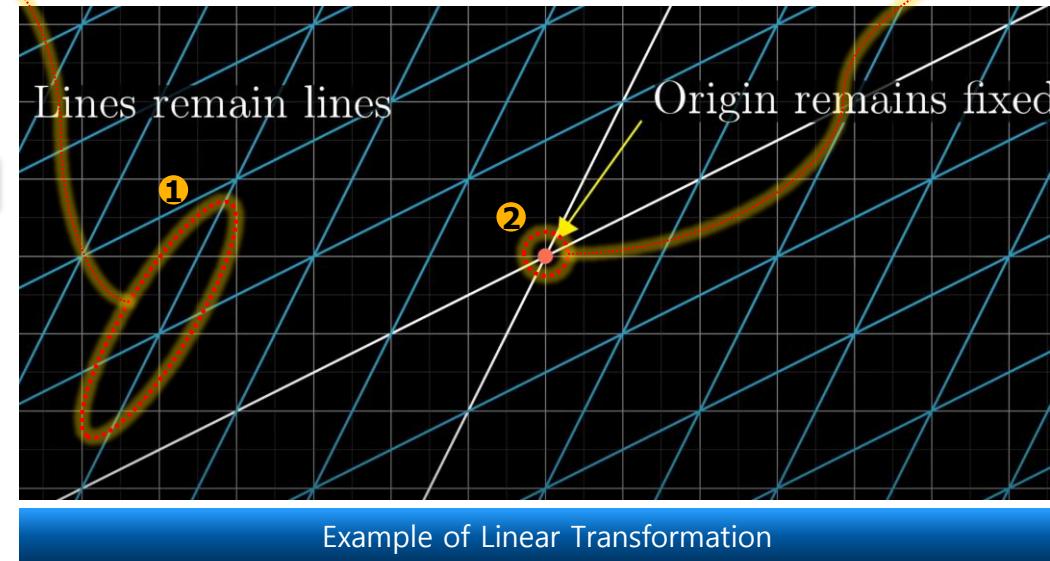
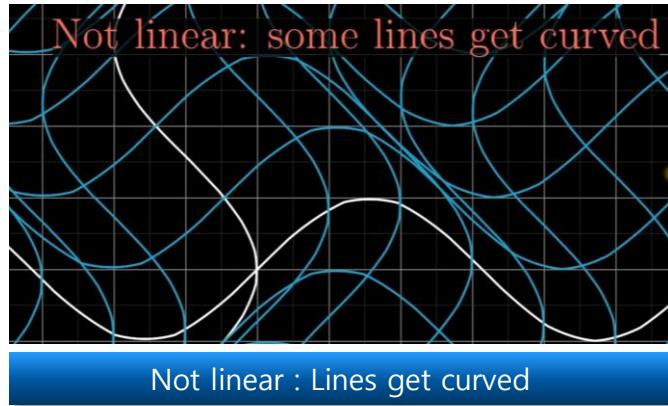


What is “Linear” Transformation

■ If a transformation has following two properties, it is linear.

1. All lines remain lines without getting *Curved*.
2. The origin must *remain fixed* in place.

■ Because of these properties, grid lines remain parallel and evenly spaced.

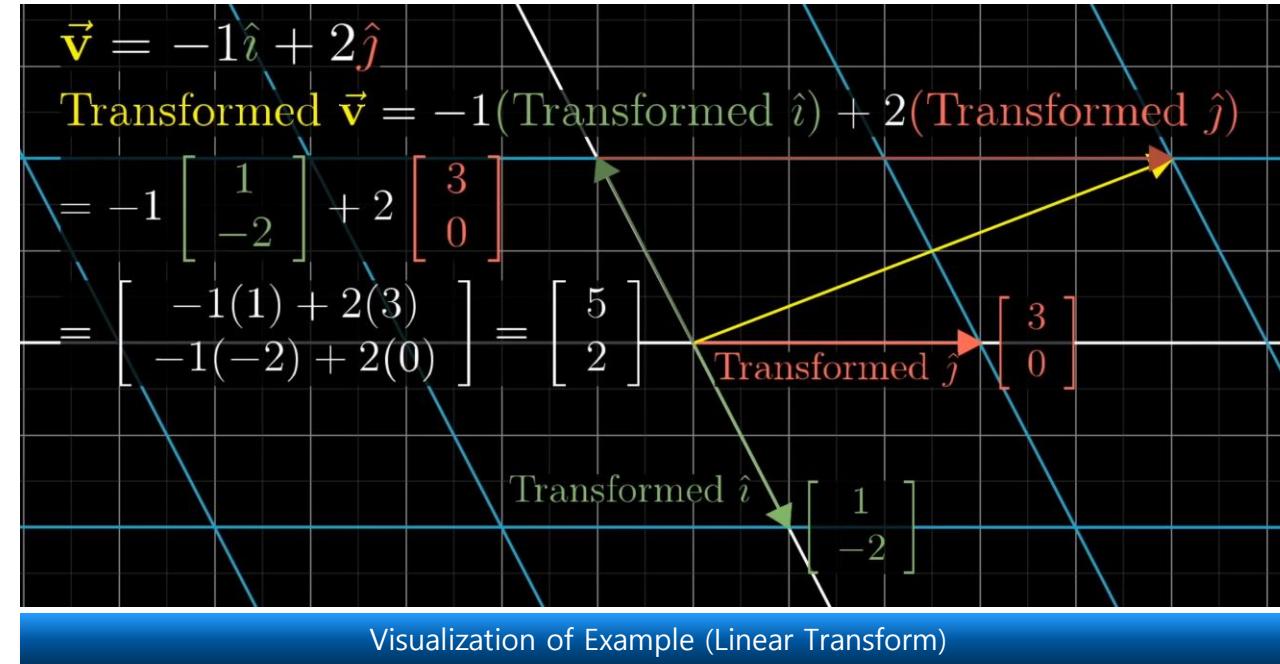
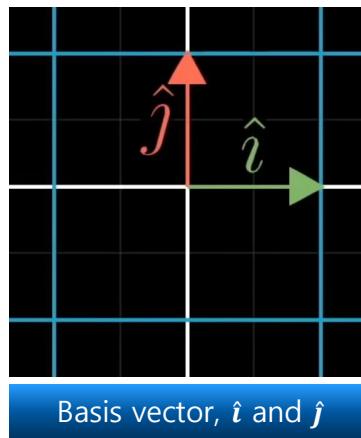


Numerical Example of Linear Transformation

■ Only consider where the two basis vectors \hat{i} , and \hat{j} each land.

■ Example

- ▶ Set vector as $\mathbf{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, meaning that \mathbf{v} equals $-1 \times \hat{i} + 2 \times \hat{j}$.
- ▶ After transformation, there is a property that grid lines remain parallel and evenly spaced.
 - \mathbf{v} lands will be $-1 \times$ the vector where \hat{i} landed $+2 \times$ the vector where \hat{j} landed.
- ▶ So, if started off as a certain linear combination of \hat{i} and \hat{j} , then ends up as that same Combination of where those two vectors landed.

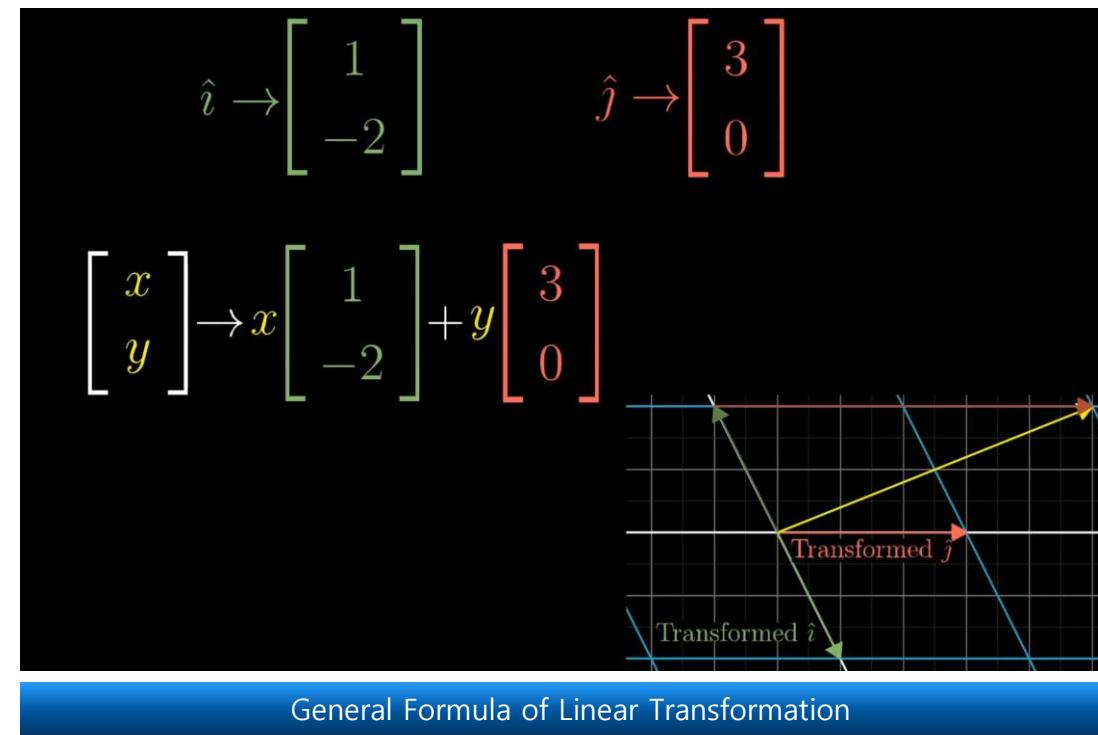


Generalization of Linear Transformation

■ Write the example with more general coordinate, x and y .

- ▶ x will land on x times the vector where \hat{i} lands, $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, plus y times the vector where \hat{j} lands, $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$.
- ▶ The summation will land at $\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow x \begin{bmatrix} 1 \\ -2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1x + 3y \\ -2x + 0y \end{bmatrix}$.

■ So, given any vector, we can calculate where that vector lands using this formula.



Numerical Example of Linear Transformation in 2×2 Matrix

■ It is common to package coordinate into a 2×2 matrix

- ▶ Column 0 is where \hat{i} lands
- ▶ Column 1 is where \hat{j} lands

■ If we expand linear transformation into 2×2 matrix form,

- ▶ Take the coordinates of the vector, multiply them by the corresponding columns of the matrix, then add together.

$$\begin{bmatrix} (3) & (2) \\ (-2) & (1) \end{bmatrix}$$

Where \hat{i} lands Where \hat{j} lands

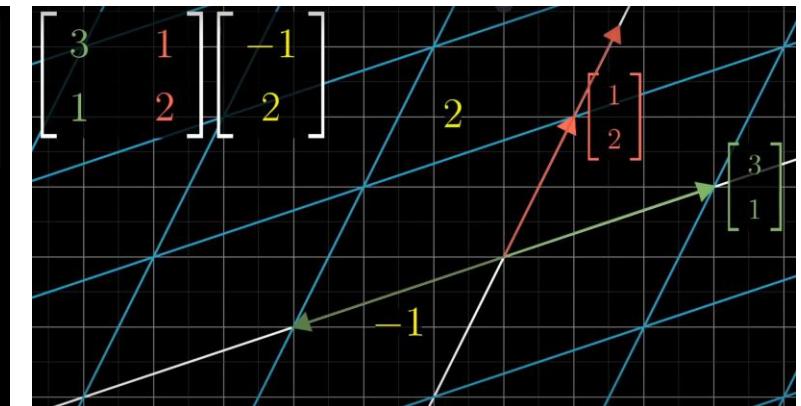
2x2 Matrix Representation of Coordinates

Numeric

$$\begin{bmatrix} 3 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix}$$

$$5 \begin{bmatrix} 3 \\ -2 \end{bmatrix} + 7 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Visual

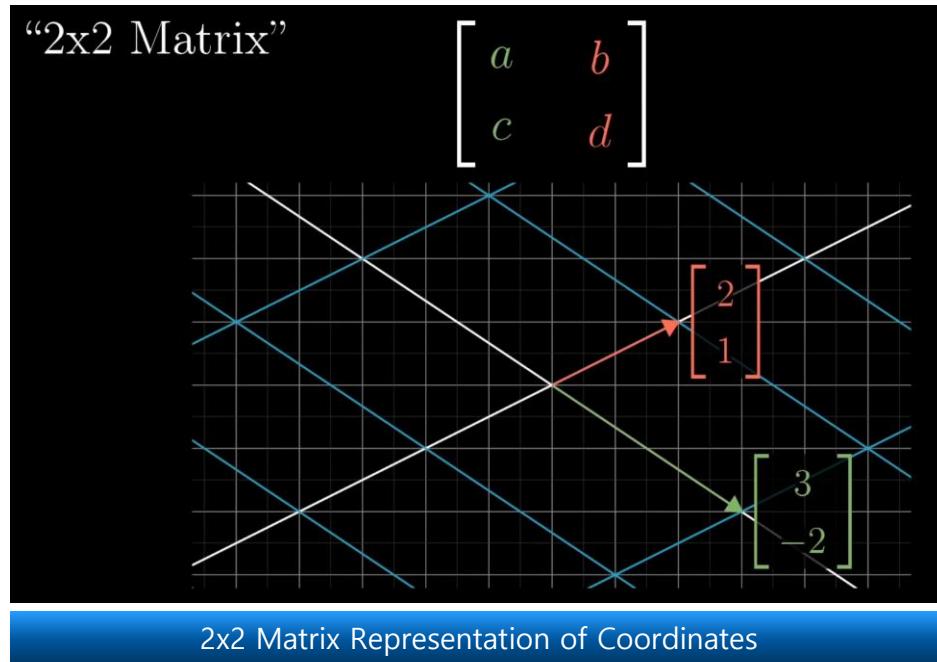


Calculation of Linear Transformation in 2x2 Matrix

Generalization of Linear Transformation in 2×2 Matrix

■ Write the example in general case, where your matrix has entries a, b, c, d .

- ▶ First column, a, c , as the place where the **first basis vector** lands.
- ▶ Second column, b, d , as the place where the **second basis vector** lands.



“2x2 Matrix”

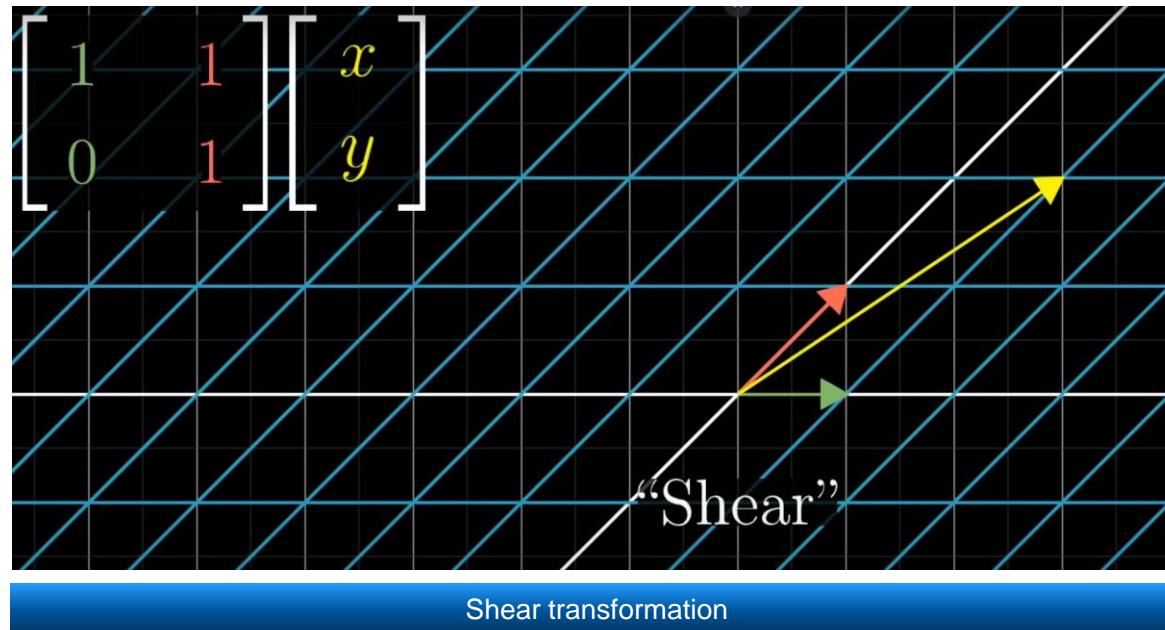
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$$

2x2 Matrix Representation of Coordinates

Example of Transformation

Shear transformation

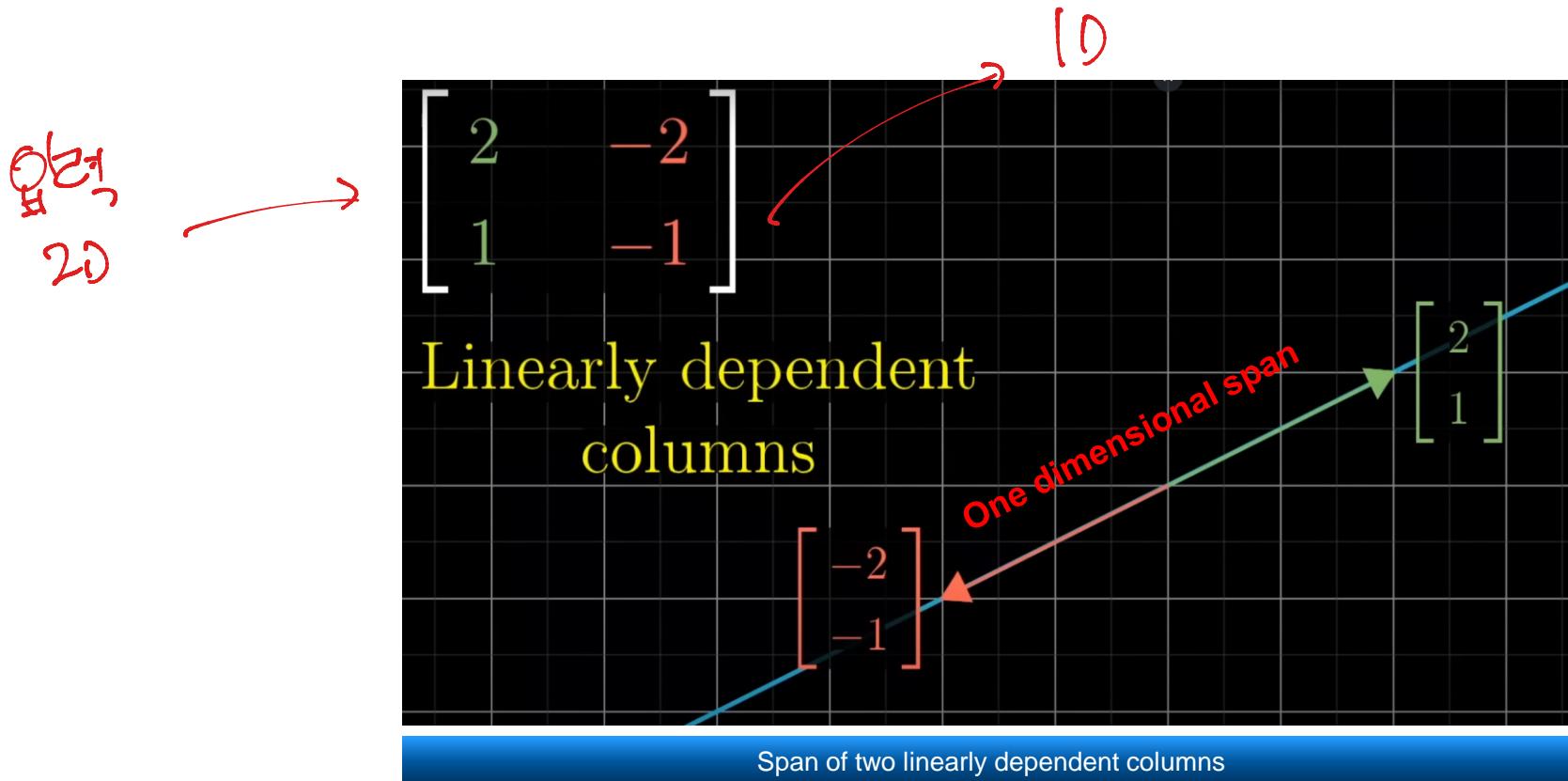
- \hat{i} remains fixed, so the first column of the matrix is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- \hat{j} moves over to the coordinates $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which become the second column of the matrix.
- Knows how a shear transforms a vector by multiplying this matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ by vector $\begin{bmatrix} x \\ y \end{bmatrix}$.



Linear Transformation in Linearly Dependent Columns

- If the vectors that i-hat and j-hat land on are **linearly dependent** one vector is scaled version of another vector.

- ▶ Linear transformation squishes all of 2D space onto the line, where those two vectors sit.
- ▶ This 2D space is called the one-dimensional span of those two linearly dependent vectors.



Interpretations of Matrix-Vector Multiplication

Linear weighted combinations (\rightarrow Linear Transformation)

- Put the individual vectors into corresponding elements of a vector, then multiply.

$$4 \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ 6 & 5 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Linear weighted combinations of column vectors

$$4[3 \ 0 \ 6] + 3[1 \ 2 \ 5] = [4 \ 3] \begin{bmatrix} 3 & 0 & 6 \\ 1 & 2 & 5 \end{bmatrix}$$

Linear weighted combinations of row vectors

Geometric transforms

- Think of a vector as a geometric line.

- Matrix-vector multiplication becomes a way of **rotating** and **scaling** that vector.

- Let's see the example of this on the next page.

$$[y] = [A][x] + [B]u$$

Common Case in Matrix-Vector Multiplication

■ Set 2×2 matrix and 2×1 vector for multiplication

- ▶ Put the individual vectors into corresponding elements of a vector, then multiply.
- ▶ Matrix M is set as $[2,3; 2,1]$, and the vector x is a vector set as $[1; 1.5]$.
- ▶ Matrix M both rotated and stretched the original vector.

■ Main point of matrix-vector multiplication

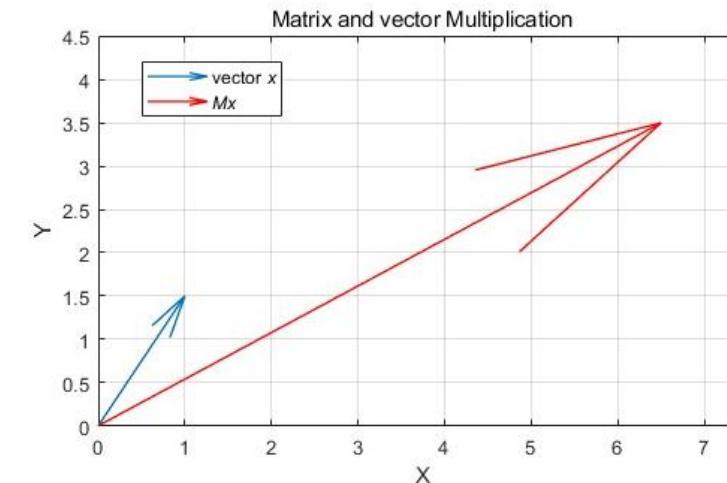
- ▶ Matrix houses a transformation that can rotate and stretch that vector.

$$M = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ 1.5 \end{bmatrix}$$

Matrix and vector

$$Mx = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1.5 \end{bmatrix} = \begin{bmatrix} 6.5 \\ 3.5 \end{bmatrix}$$

Matrix and vector multiplication



Result of the multiplication

Uncommon Case in Matrix-Vector Multiplication

■ Set another vector as $x = [1.5; 1]$.

- ▶ Matrix-vector product is no longer rotated into a different direction.
- ▶ Matrix-vector multiplication acted as if it were scalar-vector multiplication.
 - Note: Scalar-vector multiplication does not change the direction, but only changes magnitude
- ▶ It's because vector x is an Eigen Vector of matrix M and the amount by which M stretched x is its Eigenvalue
- ▶ This phenomenon will be explained later.

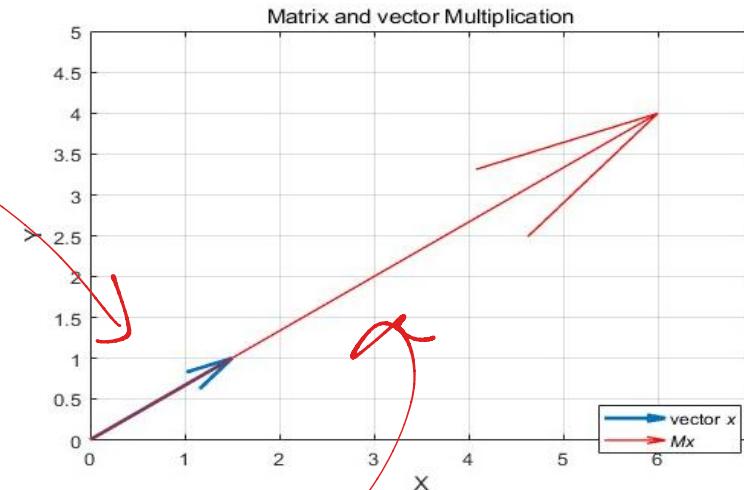
$$M = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} 1.5 \\ 1 \end{bmatrix}$$

Matrix and vector

$$Mx = \begin{bmatrix} 2 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Matrix and vector multiplication



Result of the multiplication

Code Exercise of Matrix-Vector Multiplication

Code Exercise (05_01)

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Create a 2x2 matrix with random values
matrix = [[2,3]; [2,1]];

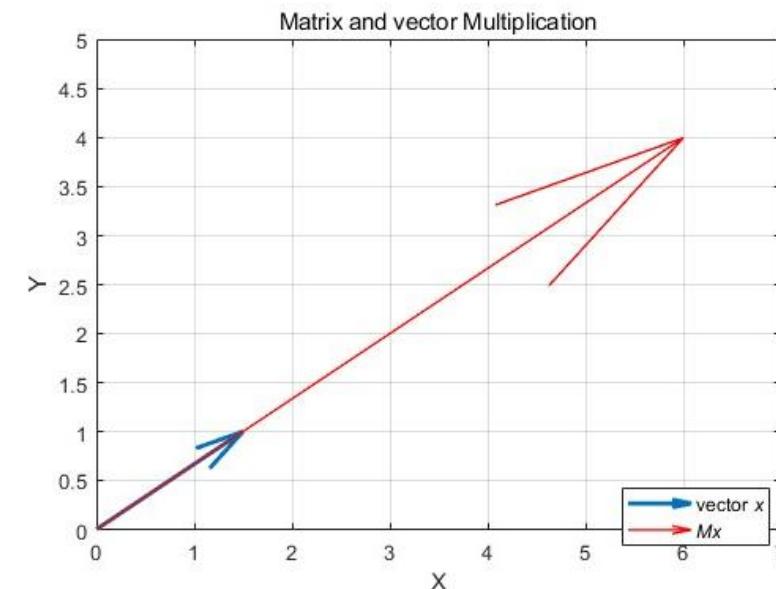
% Create a 2x1 vector with random values
vector = [1.5; 1]; % Generate a 2x1 vector with integers between 1 and 10

% Calculate the matrix-vector multiplication
result = matrix * vector;

% Create and visualize the original 2x1 vector
figure;
quiver(0, 0, vector(1), vector(2), 'AutoScale', 'off', 'MaxHeadSize', 1,
'LineWidth', 2);
hold on; % Keep the same figure for the next quiver plot
axis equal; % Keep the x and y scales the same
grid on; % Add a grid for better readability
title('Matrix and vector Multiplication');
xlabel('X');
ylabel('Y');
xlim([0, max([vector(1), result(1)])+1]); % Set limits based on the larger
vector
ylim([0, max([vector(2), result(2)])+1]);

% Visualize the result of matrix-vector multiplication
quiver(0, 0, result(1), result(2), 'r', 'AutoScale', 'off', 'MaxHeadSize',
1, 'LineWidth', 1);

% Add a legend for clarity
legend('vector x', '\it{M}x', 'Location', 'Best');
hold off; % Release the figure for new plots
```



MATLAB code to calculate matrix-vector multiplication and result

Matrix: introduction



Introduction of Matrix

■ Matrix: Vector taken to the next level

- ▶ Highly versatile mathematical objects.
 - Equations
 - Geometric transformations
 - Positions of particles over time
 - Financial records
 - Myriad other things
- ▶ We can use matrix in data science either.
 - Rows: Observations (e.g., customers)
 - Columns: Features (e.g., purchases)

Visualizing

■ Small matrices can simply be printed out in full.

► But matrices that you work with in practice can be large.

- Larger matrices will be visualized as [images](#)
- Numerical value of each element of the matrix maps onto a color in the image.
- Maps are pseudo-colored.
 - Mapping of numerical value onto color is arbitrary.

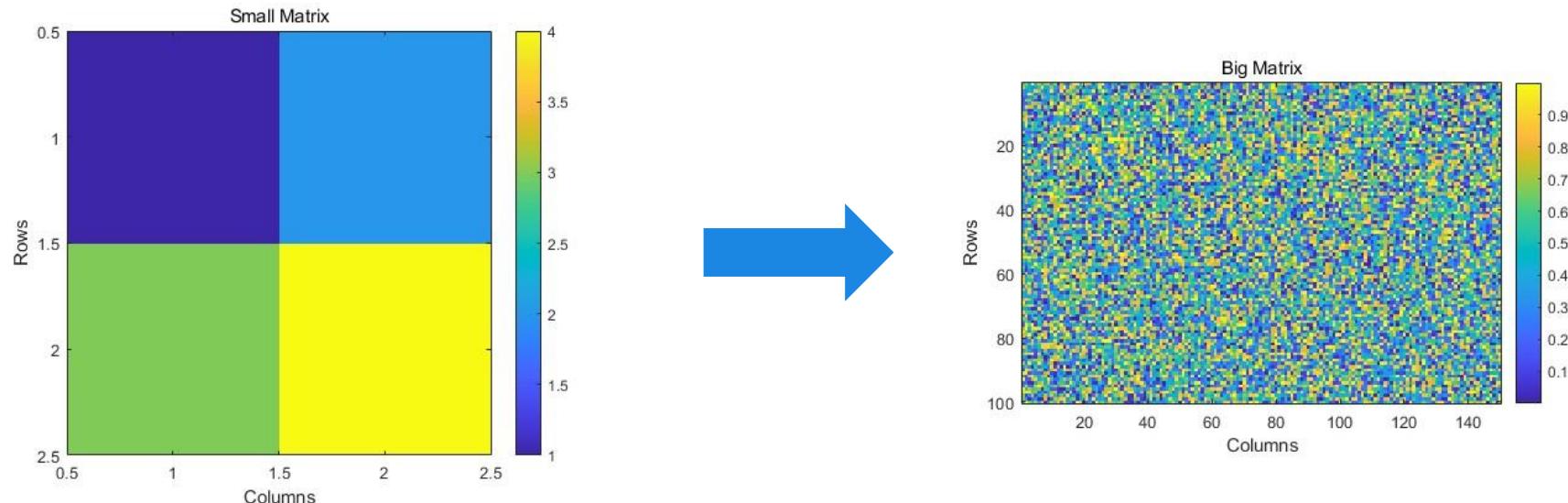


Image of small and big matrix

Indexing and Slicing

■ Matrices are indicated using bold-faced capital letters, like matrix A or M .

- ▶ Size of a matrix is indicated using (row, Column) convention.

■ Refer to specific elements of a matrix.

- ▶ Indexing the row and column position.
- ▶ Element in the 3rd row and 4th column of matrix A is indicated as $a_{3,4} = \boxed{8}$

$$A = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \\ 0 & 2 & 4 & 6 & 8 \\ 1 & 4 & 7 & 8 & 9 \end{bmatrix}$$

Example of matrix

indexing

■ Extracting a subset of rows or columns of a matrix is done through slicing.

- ▶ Following code shows an example of extracting a submatrix from rows 2-4 and columns 1-5 of a large matrix.

Code Exercise of Indexing and Slicing

■ Extracting a subset of rows or columns of a matrix is done through slicing.

- ▶ Remember that MATLAB uses index start from 1, not 0 like others.

■ Code Exercise (05_02)

```
% Clear previous figures and vars
clc; clear; close all;

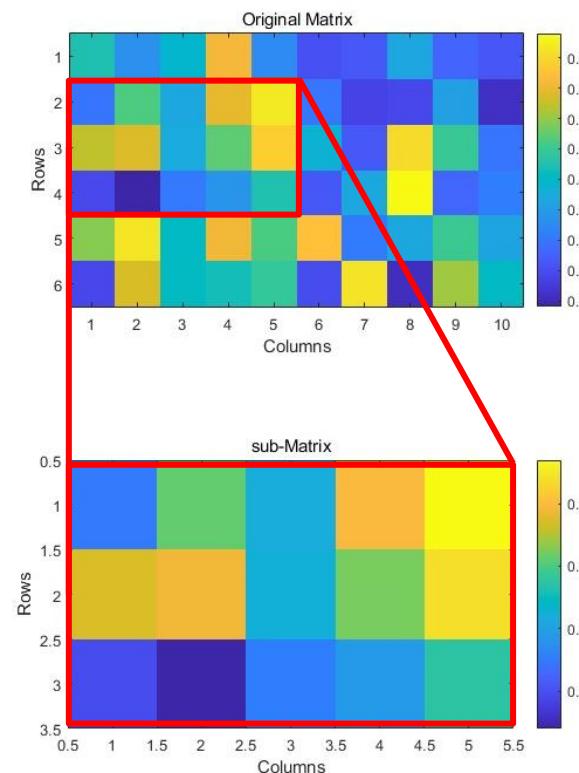
% Create a 6x10 matrix with random values
originalMatrix = rand(6, 10);

% Extract submatrix from rows 2 to 4 and columns 1 to 5
subMatrix = originalMatrix(2:4, 1:5); // Rows 2 to 4 and columns 1 to 5

% Display
disp("Original Matrix");
disp(originalMatrix);
disp("Sub-Matrix");
disp(subMatrix);

% Visualize the original matrix
figure; % Create a new figure for the original matrix
imagesc(originalMatrix); % Display the original matrix as a color image
title('Original Matrix');
xlabel('Columns');
ylabel('Rows');
colorbar; % Show a color scale
axis equal tight; % Adjust axes to fit tightly around the data

% Visualize the sub-matrix
figure; % Create a new figure for the sub-matrix
imagesc(subMatrix); % Display the sub-matrix as a color image
title('sub-Matrix');
xlabel('Columns');
ylabel('Rows');
colorbar; % Show a color scale
axis equal tight; % Adjust axes to fit tightly around the data
```



MATLAB code example of slicing matrices and results

Special Matrices

■ Number of matrices is infinite.

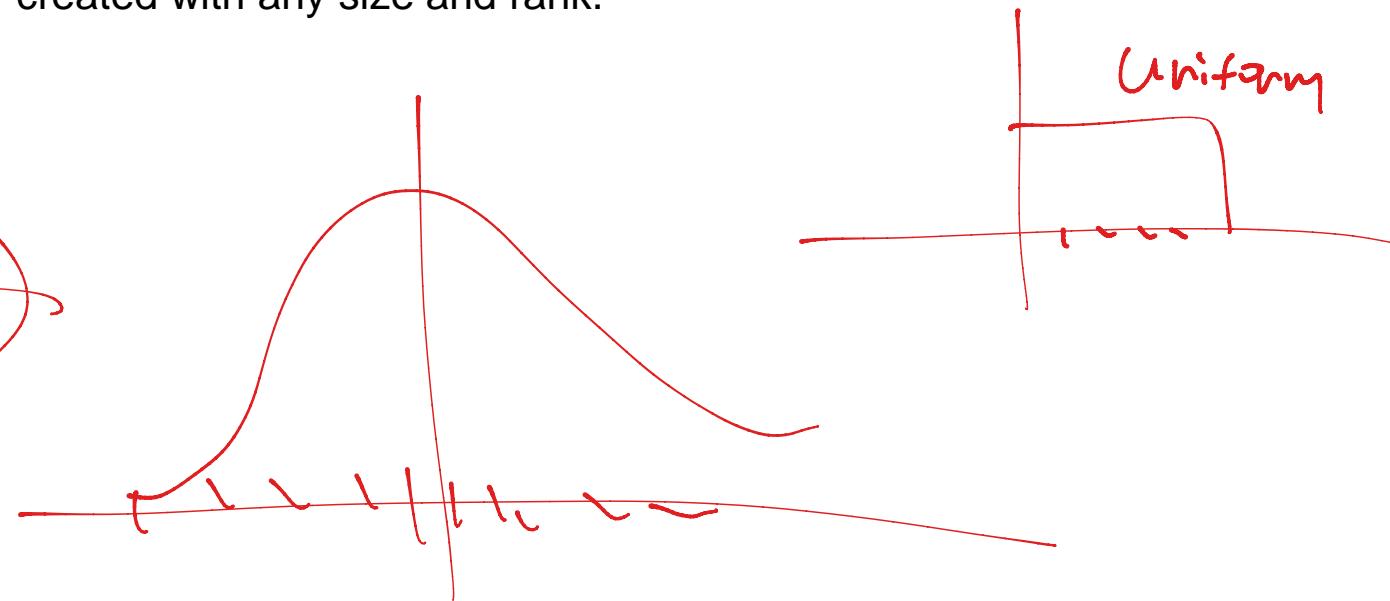
- ▶ Infinite number of ways of organizing numbers into a matrix.
- ▶ But matrices can be described using a relatively small number of characteristics.
 - It creates “families” or categories of matrices.
 - Square, diagonal, triangular, identity, zero
 - These categories appear in certain operations.
 - They have certain useful properties.
 - Some categories of matrices are used so frequently that they have dedicated MATLAB functions to create them.

Special Matrices: Random Numbers

■ Matrix can contain numbers drawn at random from some distribution, typically Gaussian (a.k.a. normal).

► Random-numbers matrices

- Can be quickly and easily created with any size and rank.



Special Matrices: Random Numbers

■ Several ways to create random matrices in MATLAB.

- ▶ Random matrices with **float** numbers using function '`rand(row, col)`'.
- ▶ Random matrices with **integer** numbers using function '`randi:(row, col)`'.

■ Code Exercise (05_03)

```
% Clear previous figures and vars
clc; clear; close all;

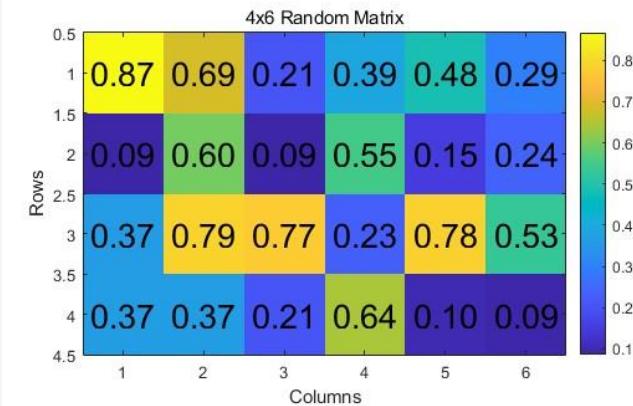
% 4x6 matrix with random values
matrix = rand(4, 6);

% Create a figure for visualization
figure;

% Display
disp("4x6 Random Matrix")
disp(matrix);

% Visualize the matrix
imagesc(matrix); % Display the matrix as a color
image
title('4x6 Random Matrix');
xlabel('Columns');
ylabel('Rows');
```

```
colorbar; % Show a color scale
axis equal tight; % Adjust axes to fit tightly
% around the data
[numRows, numCols] = size(matrix);
for row = 1:numRows
    for col = 1:numCols
        text(col, row, num2str(matrix(row, col),
        '%.2f'), ...
        'HorizontalAlignment', 'center', ...
        'VerticalAlignment', 'middle', ...
        'FontSize',20);
    end
end
```

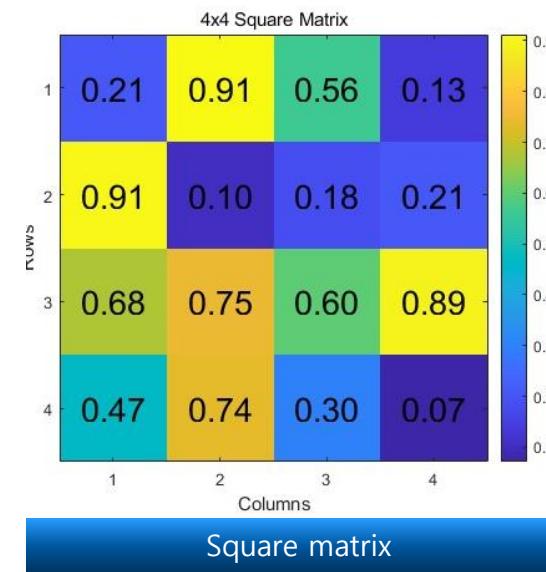


MATLAB code to make random numbers matrix and result

Special Matrices: Square VS Non-square

■ Square matrix

- ▶ Same number of rows as columns.
- ▶ Can be expressed as $R^{N \times N}$.



■ Non-square matrix

- ▶ Different number of rows and columns.
 - Sometimes called a rectangular matrix.
- ▶ Can be called **tall**.
 - Number of rows \geq number of columns
- ▶ Can be called **wide**.
 - Number of rows \leq number of columns

■ You can create square and rectangular matrices from random numbers.

- ▶ By adjusting the shape parameters in the previous code.

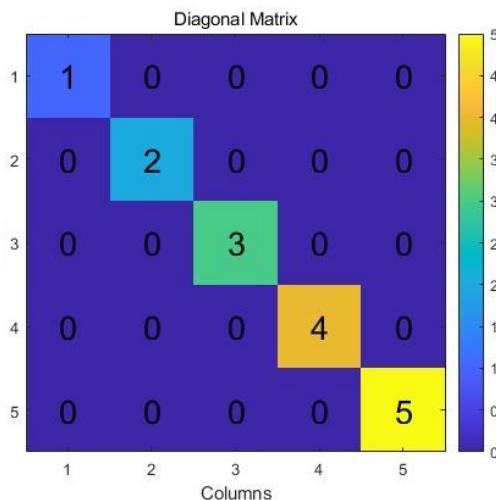
Special Matrices: Diagonal

■ Diagonal of a matrix

- ▶ Elements starting at the **top-left** and going down to the **bottom-right**

■ Diagonal matrix

- ▶ Zeros on all the off-diagonal elements.
- ▶ Diagonal elements are the only elements that may contain nonzero values.
 - Diagonal elements can contain zero too.



Diagonal Matrix

Code Exercise of Diagonal Matrix

MATLAB function `diag()`

- ▶ Input as matrix: `diag()` will return the diagonal elements of matrix as a vector.
- ▶ Input as vector: `diag()` will return a matrix with input vector elements on the diagonal.

Code Exercise (05_04)

```
% Clear previous figures and vars
clc; clear; close all;

% Create a vector for the diagonal
elements
InputMatrix = randi(10, 5);
diagonalElements = [1, 2, 3, 4, 5];

% Create a diagonal matrix using the diag
function
diagonalVector = diag(InputMatrix);
diagonalMatrix = diag(diagonalElements);

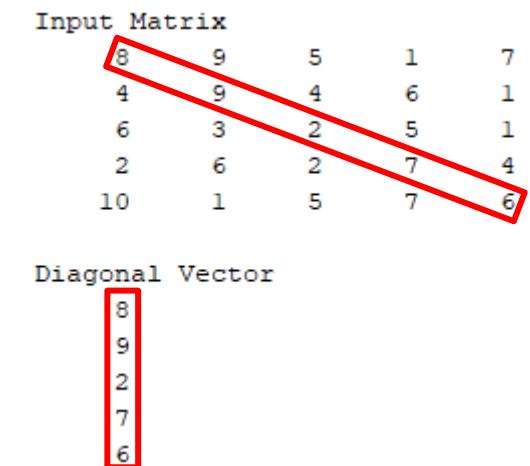
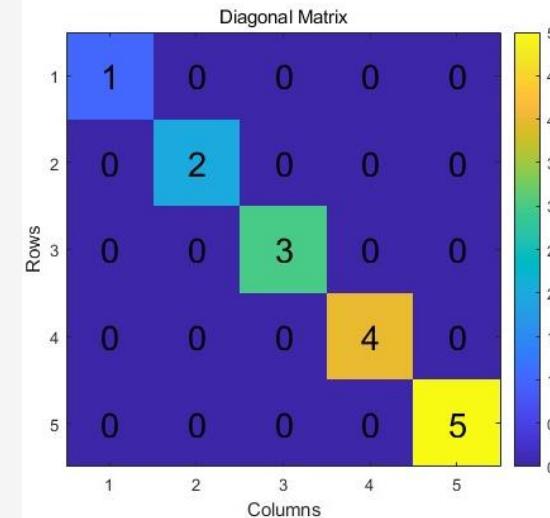
% Display
disp("Input Matrix");
disp(InputMatrix);
disp("Diagonal Vector")
disp(diagonalVector);

% Create a figure for visualization
figure;

% Visualize the diagonal matrix
imagesc(diagonalMatrix); % Display the
matrix as a color image

title('Diagonal Matrix');
xlabel('Columns');
ylabel('Rows');
colorbar; % Show a color scale
axis equal tight; % Adjust axes to fit
tightly around the data
set(gca, 'XTick',
1:length(diagonalElements), 'YTick',
1:length(diagonalElements)); % Set the
tick marks

% Add text annotations for each element
[numRows, numCols] = size(diagonalMatrix);
for row = 1:numRows
    for col = 1:numCols
        text(col, row,
num2str(diagonalMatrix(row, col),
'%d'), ...
'HorizontalAlignment',
'center', ...
'VerticalAlignment',
'middle', ...
'FontSize',20);
    end
end
```



MATLAB code of function 'diag()' and results

Special Matrices: Triangular

■ Triangular matrix

- ▶ Contains all zeros either above or below the main diagonal.
- ▶ Upper triangular
 - if the nonzero elements are above the diagonal.
- ▶ Lower triangular
 - if the nonzero elements are below the diagonal.

Code Exercise of Triangular Matrix

Code Exercise (05_05)

- ▶ Create upper triangular matrix using function ‘`triu(matrix)`’
- ▶ Create lower triangular matrix using function ‘`tril(matrix)`’.

```
% Clear workspace, command window, and
% close all figures
clc; clear; close all;

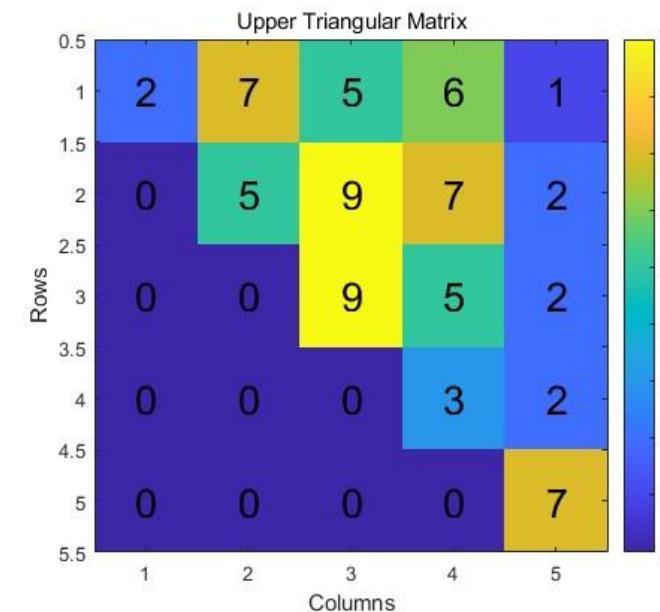
% Create a random matrix using randi
randomMatrix = randi(10, 5, 5); % Generate
% a 5x5 matrix with integers between 1 and
10

% Create upper triangular matrix from the
% random matrix
upperTriangularMatrix = triu(randomMatrix);

% You can create lower triangular matrix
% use function 'tril(matrix)'

% Create and visualize the upper
% triangular matrix
figure;
imagesc(upperTriangularMatrix); % Display
% the matrix as a color image
title('Upper Triangular Matrix');
xlabel('Columns');
ylabel('Rows');
```

```
colorbar; % Show a color scale
axis equal tight; % Adjust axes to fit
% tightly around the data
% Add text annotations for each element in
% the upper triangular matrix
[numRows, numCols] =
size(upperTriangularMatrix);
for row = 1:numRows
    for col = 1:numCols
        text(col, row,
num2str(upperTriangularMatrix(row, col),
'%d'), ...
'HorizontalAlignment',
'center', ...
'VerticalAlignment',
'middle', ...
'FontSize',20);
    end
end
```



MATLAB code to create triangular matrix and result

Special Matrices: Identity

■ Identity matrix

- ▶ Equivalent of the number 1
 - In that any matrix or vector times the identity matrix is that same matrix or vector.
- ▶ Equivalent of Square diagonal matrix
 - With all diagonal elements having a value of 1

■ Notation of Identity matrix

- ▶ Indicate using the letter I .
 - I_5 is 5×5 identity matrix.
 - If there is no subscript to indicate its size, you can infer the size from context.
 - To make the equation consistent.

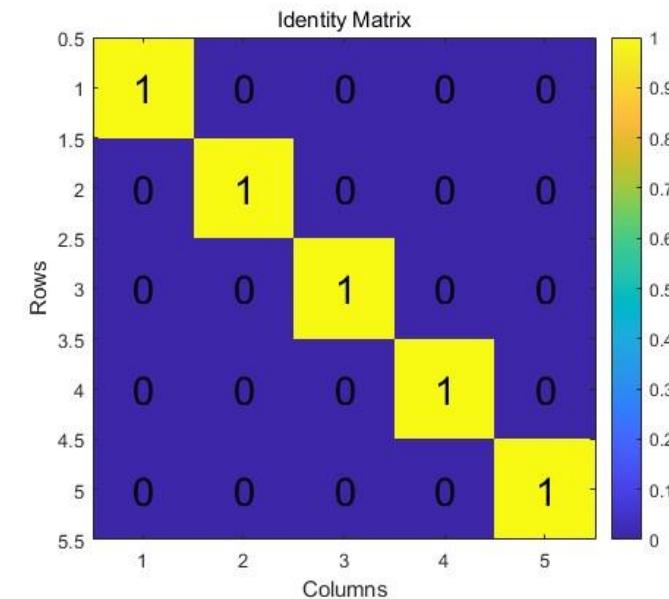
Code Exercise of Identity Matrix

- Identity matrix can be created by `eye()` in MATLAB code.
- Code Exercise (05_06)

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Create a 5x5 identity matrix
identityMatrix = eye(5, 5);

% Create and visualize the identity matrix
figure;
imagesc(identityMatrix); % Display the matrix as a color image
title('Identity Matrix');
xlabel('Columns');
ylabel('Rows');
colorbar; % Show a color scale
axis equal tight; % Adjust axes to fit tightly around the data
% Add text annotations for each element in the identity matrix
[numRows, numCols] = size(identityMatrix);
for row = 1:numRows
    for col = 1:numCols
        text(col, row, num2str(identityMatrix(row, col), '%d'), ...
            'HorizontalAlignment', 'center', ...
            'VerticalAlignment', 'middle', ...
            'FontSize', 20);
    end
end
```



MATLAB code to create identity matrix and result

Special Matrices: Zeros

■ All of zeros matrix elements are zero.

- ▶ Indicate using a bold-faced zero: **0**.
- ▶ It can be a bit confusing to have the same symbol indicate both a vector and a matrix.
 - But this kind of overloading is common in math and science notation.



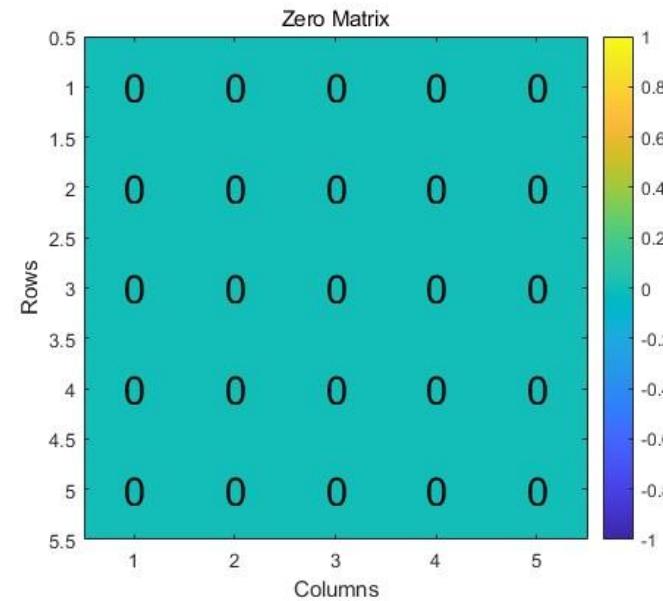
Code Exercise of Zero Matrix

- Using the `zeros()` function in MATLAB to create zeros matrix.
- Code Exercise (05_07)

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Create a 5x5 zero matrix
zeroMatrix = zeros(5, 5);

% Create and visualize the zero matrix
figure;
imagesc(zeroMatrix); % Display the matrix as a color image
title('Zero Matrix');
xlabel('Columns');
ylabel('Rows');
colorbar; % Show a color scale
axis equal tight; % Adjust axes to fit tightly around the data
% Add text annotations for each element in the zero matrix
[numRows, numCols] = size(zeroMatrix);
for row = 1:numRows
    for col = 1:numCols
        text(col, row, num2str(zeroMatrix(row, col), '%d'), ...
            'HorizontalAlignment', 'center', ...
            'VerticalAlignment', 'middle', ...
            'FontSize', 20);
    end
end
end
```



MATLAB code to create zero matrix

Matrix math: addition, scalar multiplication, Hadamard multiplication

Matrix Math: Addition and Subtraction

■ Matrix addition

- ▶ Defined only between two matrices of the **Square Size**.

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 3 & 1 \\ -1 & -4 & 2 \end{bmatrix} = \begin{bmatrix} (2+0) & (3+3) & (4+1) \\ (1-1) & (2-4) & (4+2) \end{bmatrix} = \begin{bmatrix} 2 & 6 & 5 \\ 0 & -2 & 6 \end{bmatrix}$$

Matrix addition example

Matrix Math: Shifting a Matrix

Linear-algebra is way to add a scalar to a square matrix.

- ▶ Called Shifting a matrix
- ▶ Not formally possible to add a scalar to a matrix, as in $\lambda + A$.
 - It works by adding a scalar multiplied identity matrix, as in $A + \lambda I$.

$$\begin{bmatrix} 4 & 5 & 1 \\ 0 & 1 & 11 \\ 4 & 9 & 7 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 5 & 1 \\ 0 & 7 & 11 \\ 4 & 9 & 13 \end{bmatrix}$$

Matrix addition example

- ▶ Only the diagonal elements can be changed.
 - The rest of the diagonal elements is unchanged by shifting.
- ▶ In practice, one shifts a relatively small amount.
 - To preserve as much information as possible in the matrix while benefiting from the effects of shifting, including increasing the numerical stability of the matrix.

Matrix Math: Applications of Shifting a Matrix

■ Exactly how much to Shift is a matter of on-going research in multiple areas.

- ▶ Such as machine learning, statistics, deep learning, control engineering, etc.
- ▶ For example, is shifting by $\lambda = 6$ a little or a lot ? How about $\lambda = 0.001$?
 - These numbers are “big” or “small” relative to the numerical values in the matrix.
 - Therefore, in practice, λ is usually set to be some fraction of a matrix-defined quantity such as the norm or the average of the eigenvalues.
 - You will get to explore about norm and eigenvalues in later chapters.

■ Two primary applications of shifting a matrix

- ▶ Finding the eigenvalues of a matrix.
- ▶ Regularizing matrices when fitting models to data.

Code Exercise of Matrix Shifting

Code Exercise (05_08)

```
% Clear workspace, command window, and close all
% figures
clc; clear; close all;

% Create a 3x3 square matrix with random values
squareMatrix = randi(10, 3, 3); % Generate a 3x3
matrix with integers between 1 and 10

% Define a scalar value
scalar = 5;

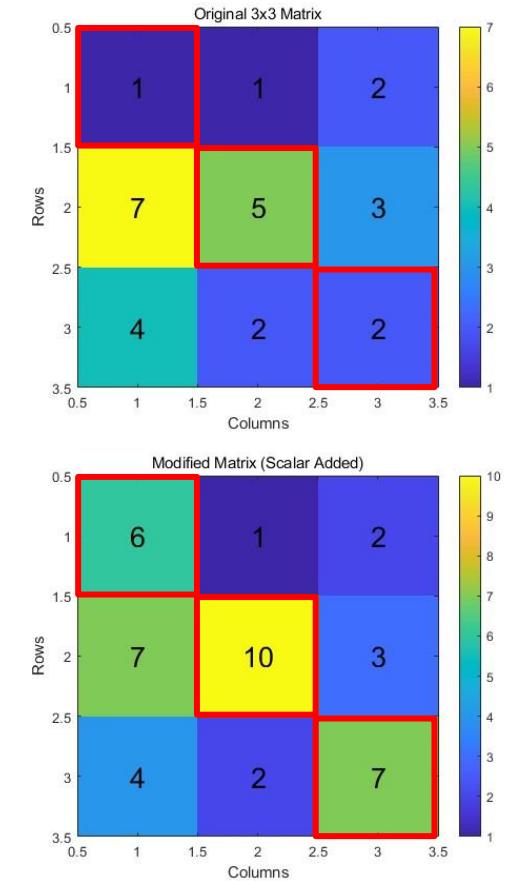
% Create a 3x3 identity matrix
identityMatrix = eye(3);

% Add the scalar multiplied by the identity matrix
% to the original matrix
modifiedMatrix = squareMatrix + scalar *
identityMatrix;

% Create and visualize the original square matrix
% in a new figure
figure;
imagesc(squareMatrix); % Display the matrix as a
color image
title('Original 3x3 Matrix');
xlabel('Columns');
ylabel('Rows');
colorbar; % Show a color scale
axis equal tight; % Adjust axes to fit tightly
around the data
% Add text annotations for each element in the
original matrix
[numRows, numCols] = size(squareMatrix);
for row = 1:numRows
```

```
    for col = 1:numCols
        text(col, row, num2str(squareMatrix(row,
col), '%d'), ...
        'HorizontalAlignment', 'center', ...
        'VerticalAlignment', 'middle', ...
        'FontSize',20);
    end
end

% Create and visualize the modified matrix in a
new figure
figure;
imagesc(modifiedMatrix); % Display the modified
matrix as a color image
title('Modified Matrix (Scalar Added)');
xlabel('Columns');
ylabel('Rows');
colorbar; % Show a color scale
axis equal tight; % Adjust axes to fit tightly
around the data
% Add text annotations for each element in the
modified matrix
for row = 1:numRows
    for col = 1:numCols
        text(col, row, num2str(modifiedMatrix(row,
col), '%d'), ...
        'HorizontalAlignment', 'center', ...
        'VerticalAlignment', 'middle', ...
        'FontSize',20);
    end
end
```



MATLAB code to calculate the matrix shifting

Matrix Math: Scalar-Matrix and Hadamard Multiplications

Scalar-matrix multiplication and Hadamard multiplication

- ▶ Work the same for matrices as they do for vectors, which is to say, **element-wise**.
- ▶ **Scalar-matrix multiplication**

- Multiply each element in the matrix by the same scalar.

$$\gamma \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \gamma a & \gamma b \\ \gamma c & \gamma d \end{bmatrix}$$

Example of scalar multiplication

- ▶ **Hadamard multiplication**
- Involving multiplying two matrices element-wise.

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \odot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 3b \\ 4c & 5d \end{bmatrix}$$

Example of Hadamard multiplication

Code Exercise of Hadamard Multiplication

Code Exercise (05_09)

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Create two 3x3 square matrices with random values
squareMatrix1 = randi(10, 3, 3); % Generate a 3x3 matrix
% with integers between 1 and 10
squareMatrix2 = randi(10, 3, 3); % Generate another 3x3
% matrix with integers between 1 and 10

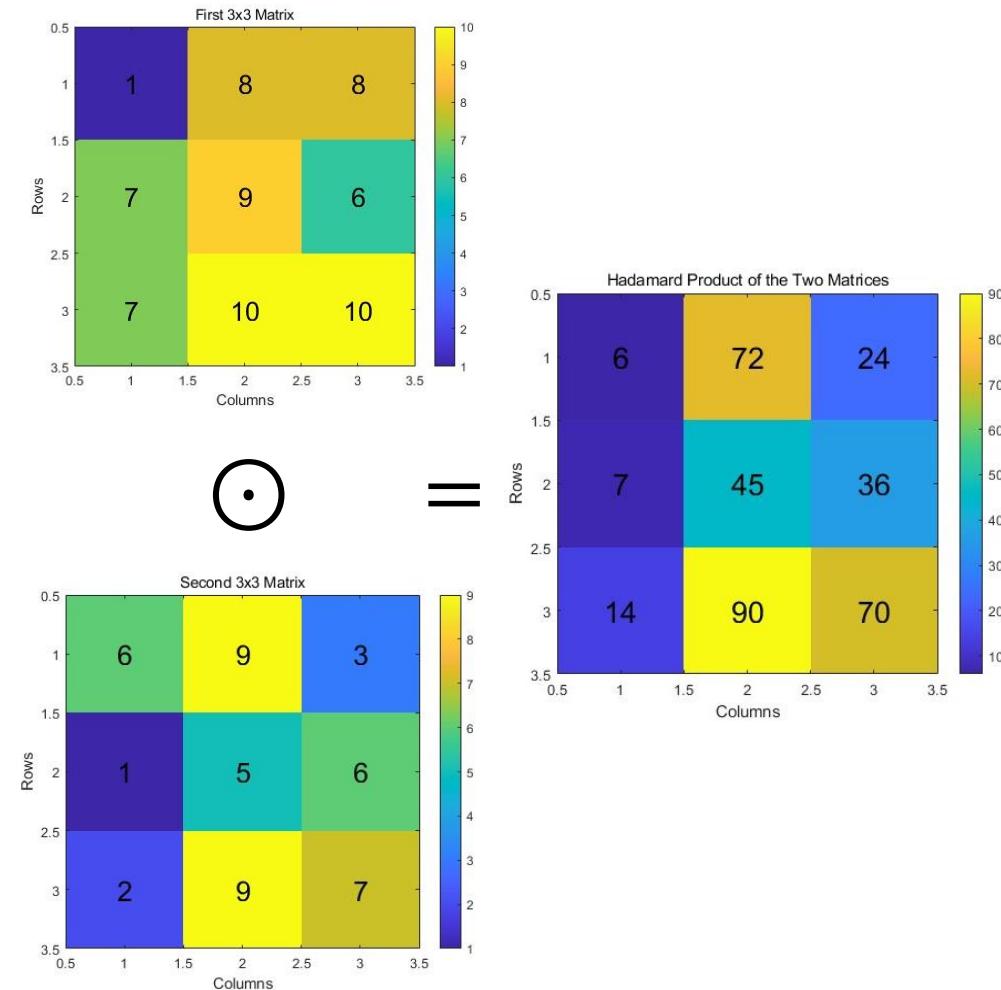
% Calculate the Hadamard product of the two matrices
hadamardProduct = squareMatrix1 * squareMatrix2;

% Create and visualize the first square matrix
figure;
imagesc(squareMatrix1); % Display the matrix as a color
image
title('First 3x3 Matrix');
xlabel('Columns');
ylabel('Rows');
colorbar; % Show a color scale
axis equal tight; % Adjust axes to fit tightly around the
data
% Add text annotations for each element in the first
matrix
[numRows, numCols] = size(squareMatrix1);
for row = 1:numRows
    for col = 1:numCols
        text(col, row, num2str(squareMatrix1(row, col),
'%d'), ...
            'HorizontalAlignment', 'center', ...
            'VerticalAlignment', 'middle', ...
            'FontSize',20);
    end
end

% Create and visualize the second square matrix
figure;
imagesc(squareMatrix2); % Display the matrix as a color
image
title('Second 3x3 Matrix');
xlabel('Columns');
ylabel('Rows');
colorbar; % Show a color scale
```

```
axis equal tight; % Adjust axes to fit tightly around the
data
% Add text annotations for each element in the second
matrix
for row = 1:numRows
    for col = 1:numCols
        text(col, row, num2str(squareMatrix2(row, col),
'%d'), ...
            'HorizontalAlignment', 'center', ...
            'VerticalAlignment', 'middle', ...
            'FontSize',20);
    end
end

% Create and visualize the Hadamard product of the
matrices
figure;
imagesc(hadamardProduct); % Display the Hadamard product
matrix as a color image
title('Hadamard Product of the Two Matrices');
xlabel('Columns');
ylabel('Rows');
colorbar; % Show a color scale
axis equal tight; % Adjust axes to fit tightly around the
data
% Add text annotations for each element in the Hadamard
product matrix
for row = 1:numRows
    for col = 1:numCols
        text(col, row, num2str(hadamardProduct(row, col),
'%d'), ...
            'HorizontalAlignment', 'center', ...
            'VerticalAlignment', 'middle', ...
            'FontSize',20);
    end
end
```



MATLAB code to calculate the Hadamard multiplication

Standard matrix multiplication

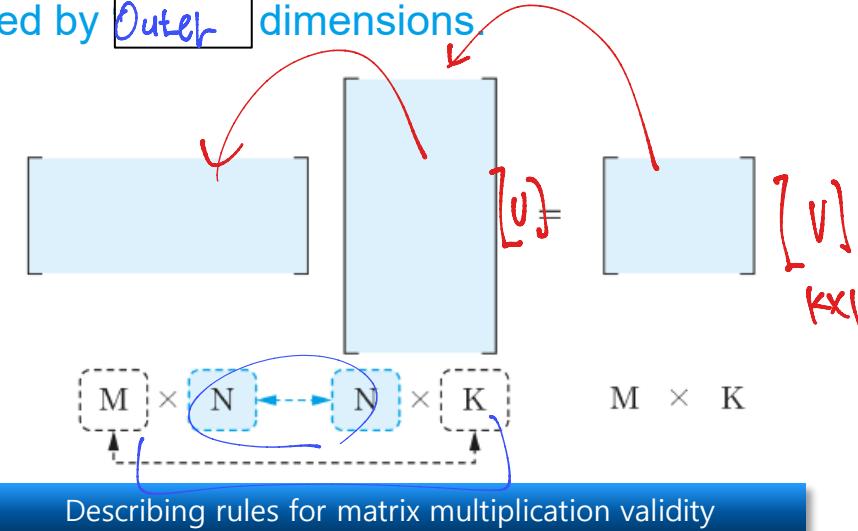
Standard Matrix Multiplication

■ Characteristics of standard matrix multiplication

- ▶ Operating row/column-wise rather than operating element-wise.
- ▶ Reduces to a systematic collection of dot products.
 - Between rows of one matrix and columns of the other matrix.
 - Formally simply called **Matrix multiplication**
 - ‘Standard’ term is added to help disambiguate from Hadamard and scalar multiplications.

Rules for Matrix Multiplication Validity

- First matrix sizes: $M \times N$
- Second matrix sizes: $N \times K$
 - ▶ Multiplying these two matrices.
 - The “Inner” dimensions: N
 - The “Outer” dimensions: M and K
 - ▶ Matrix multiplication is valid only when the **Inner** dimensions match.
 - ▶ Size of product matrix is defined by **Outer** dimensions.



- ▶ Matrix multiplication does not obey the **commutative law**.
 - If $C = AB$ and $D = BA$, then in general $C \neq D$.
 - They are equal in some special cases, but we cannot generally assume equality.

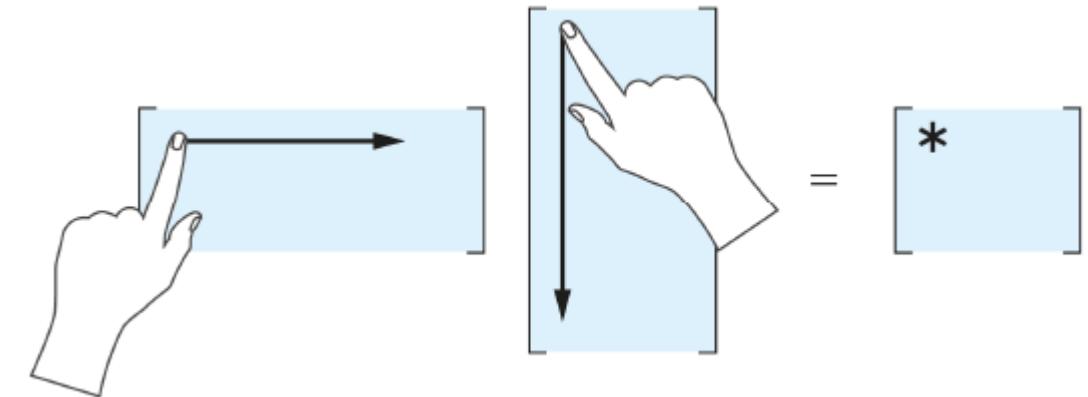
Matrix Multiplication

■ Why matrix multiplication is valid only if the number of columns in left matrix matches the number of rows in the right matrix?

- ▶ The (i, j) th element in the product matrix is the **dot product** between the i th row of the left matrix and the j th column in the right matrix.
- ▶ Dot product
 - A number that encodes the relationship between rows of the left matrix and columns of the right matrix.

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a + 3c & 2b + 3d \\ 4a + 5c & 4b + 5d \end{bmatrix}$$

Example of multiplication



Finger movements for matrix multiplication

Result of Matrix Multiplication

- Matrix that stores all the pairwise linear relationships between **rows of the left matrix** and **columns for the right matrix**.
- Can be a basis for computing covariance and correlation matrices.
 - ▶ General linear model, singular-value decomposition, and countless other applications.

Matrix operation: transpose



Transpose Operation on Matrix

■ Principle of Transpose Operation

- ▶ Simply swap the rows and Columns

■ Notation of Transpose Operation

- ▶ Indicate with a superscripted T .
- ▶ Double-transposing a matrix returns the original matrix. ($C^{TT} = C$)

$$a_{i,j}^T = a_{j,i}$$

Definition of the transpose operation

$$\begin{bmatrix} 3 & 0 & 4 \\ 9 & 8 & 3 \end{bmatrix}^T = \begin{bmatrix} 3 & 9 \\ 0 & 8 \\ 4 & 3 \end{bmatrix}$$

Example of the transpose operation



Code Exercise of Transpose Operation on Matrix

■ A few ways to transpose matrices in MATLAB

- ▶ Function < ' >
- ▶ Function <transpose(A)>

■ Code Exercise (05_10)

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Create a 3x2 matrix with random values
matrix = randi(10, 3, 2); % Generate a 3x2 matrix with integers between 1
and 10

% Transpose the matrix
transposedMatrix = matrix';
transposedMatrix2 = transpose(matrix);

% Create and visualize the original matrix as vectors
disp("Original Matrix");
disp(matrix);
disp("Transposed Matrix using ('')");
disp(transposedMatrix);
disp("Transposed Matrix using (transpose(A))");
disp(transposedMatrix2);
```

Original Matrix

8	1
4	7
3	5

Transposed Matrix using ('')

8	4	3
1	7	5

Transposed Matrix using (transpose(A))

8	4	3
1	7	5

MATLAB code to transpose operation on matrix

Dot and Outer Product Notation

■ Dot product of vectors

- ▶ Vector a : 2×1 , vector b : 2×1 .
- ▶ The dot product is indicated as $a^T b$
 - The “inner” dimensions **match**, and the “outer” dimensions will be 1×1 as Eq 1..

■ Outer product of vectors

- ▶ A way to create matrix from a column vector and a row vector
- ▶ Vector a : 2×1 , vector b : 3×1 .
- ▶ The outer product is indicated as ab^T .
 - The “inner” dimensions **match**, and the “outer” dimensions will be 2×3 as Eq 2..
 - Two vectors in an outer product can have different dimensionality, whereas dot product must have same.

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}, \quad \mathbf{a}^T \mathbf{b} = [a_1 \quad a_2] \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = [a_1 b_1 + a_2 b_2]$$

Eq 1. Example of dot product of vectors

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \mathbf{a}\mathbf{b}^T = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} [b_1 \quad b_2 \quad b_3] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \end{bmatrix}$$

Eq 2. Example of outer product of vectors

Matrix operation: LIVE EVIL (order of operation)



The Transpose of Multiplied Matrices

■ LIVE EVIL is a palindrome.

- ▶ Palindrome is a word or phrase that is spelled the same forwards and backwards.

■ Transpose of multiplied matrices is same as the individual matrices transposed and multiplied.

- ▶ But reversed in order as shown in below equation.

■ Assume

- ▶ L, I, V and E are all matrices.
- ▶ Their sizes match to make multiplication valid.

$$(LIVE)^T = E^T V^T I^T L^T$$

Example of the LIVE EVIL rule

■ This rule applies for multiplying any number of matrices.

Symmetric matrix



Definition of Symmetric Matrix

■ The corresponding rows and columns are equal.

- ▶ When you Swap the rows and columns, nothing happens to the matrix!

■ A symmetric matrix **equals** its transpose, $A^T = A$.

■ Then, can **non-square** matrix be symmetric?

- ▶ Nope! Why can't be?
 - If matrix is of size M*N, then its transpose is of size N*M.
 - It cannot be guaranteed that M and N are always the same value.

$$A = \begin{bmatrix} a & e & f & g \\ e & b & h & i \\ f & h & c & j \\ g & i & j & d \end{bmatrix}$$

A symmetric matrix

Creating Symmetric Matrix from Nonsymmetric Matrix

■ Multiplying any matrix by its transpose, it will be a square symmetric matrix as eq 1..

■ Prove symmetry

- ▶ Recall that the definition of a symmetric matrix is one that equals its transpose as eq 2..
- ▶ The proof relies on the LIVE EVIL rule.

■ But AA^T and A^TA are square symmetric, but Not Same matrix.

- ▶ If A is non-square, then two matrix products are not even the same size.

$$\text{if } A = M \times N, \quad A^T A = (N \times M)(M \times N) = N \times N$$

Eq 1. Symmetric matrix from nonsymmetric matrix

$$(A^T A)^T = A^T A^{TT} = A^T A$$

Eq 2. Definition of a symmetric matrix



Summary



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Summary

■ Matrix

- ▶ Spreadsheet of numbers.

■ Several categories of special matrices

- ▶ Random number, square, non-square, diagonal, triangular, identity and zeros matrix.

■ Some arithmetic operations that work element-wise

- ▶ Addition, scalar multiplication and Hadamard multiplication.

■ Shifting a matrix

- ▶ Adding a constant to the diagonal elements.

■ Matrix multiplication validity

- ▶ First matrix sizes are $M \times N$, second matrix sizes are $N \times K$.

■ The transpose of multiplied matrices

- ▶ The individual matrices transposed and multiplied with their order reversed.

■ Symmetric matrix

- ▶ Each row equals its corresponding columns, $A = A^T$.
- ▶ Create from any matrix by multiplying that matrix by its transpose.



Exercise: Matrix Slicing

■ Create original matrix

- ▶ Shape: 10×10
- ▶ Value: 0-99 (Refer to the figure(Result of code))

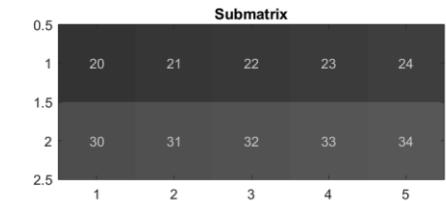
■ Create function

- ▶ Input: Original matrix, start row index, end row index, start col index, end col index
- ▶ Output: Submatrix from Original matrix
 - Output can be express various type.

Input

Original matrix										
1	0	1	2	3	4	5	6	7	8	9
2	10	11	12	13	14	15	16	17	18	19
3	20	21	22	23	24	25	26	27	28	29
4	30	31	32	33	34	35	36	37	38	39
5	40	41	42	43	44	45	46	47	48	49
6	50	51	52	53	54	55	56	57	58	59
7	60	61	62	63	64	65	66	67	68	69
8	70	71	72	73	74	75	76	77	78	79
9	80	81	82	83	84	85	86	87	88	89
10	90	91	92	93	94	95	96	97	98	99

Output



Result of code

Exercise: Matrix Addition

■ Use for loops for rows and columns to implement matrix additions for each element.

- ▶ Input: Two matrix with same size
- ▶ Output: The sum of two matrix
- ▶ Use the error() function in the MATLAB to make the error generate if the size of the two input matrix is different.

Exercise: Matrix Multiplication

■ Use for loops for rows and columns to implement dot product.

- ▶ Input: Two matrix with same size
- ▶ Output: The sum of two matrix
- ▶ Use the error() function in the MATLAB to make the error generate if dot product is not available.

Exercise: Symmetric Checker

■ Create a function that checks whether the input matrix is symmetrical or not.

- ▶ Input: a matrix
- ▶ Output: 1 (if symmetric) / 0 (if not symmetric)



Appendix



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Symbolic Math with MATLAB

- In mathematics, science, and engineering, expressions that are prone to errors when solved manually often involve symbolic variables.
- A ***symbolic variable*** refers to a variable that is represented by a symbol, such as a letter, without being assigned a specific numerical value.
- This course covers the essential concepts for [working with symbolic variables in MATLAB](#).
- Using the Symbolic Math Toolbox, you can create symbolic variables, manipulate symbolic expressions, equations, and functions, perform calculations, and simplify expressions.

$$ax^2 + bx + c = 0$$

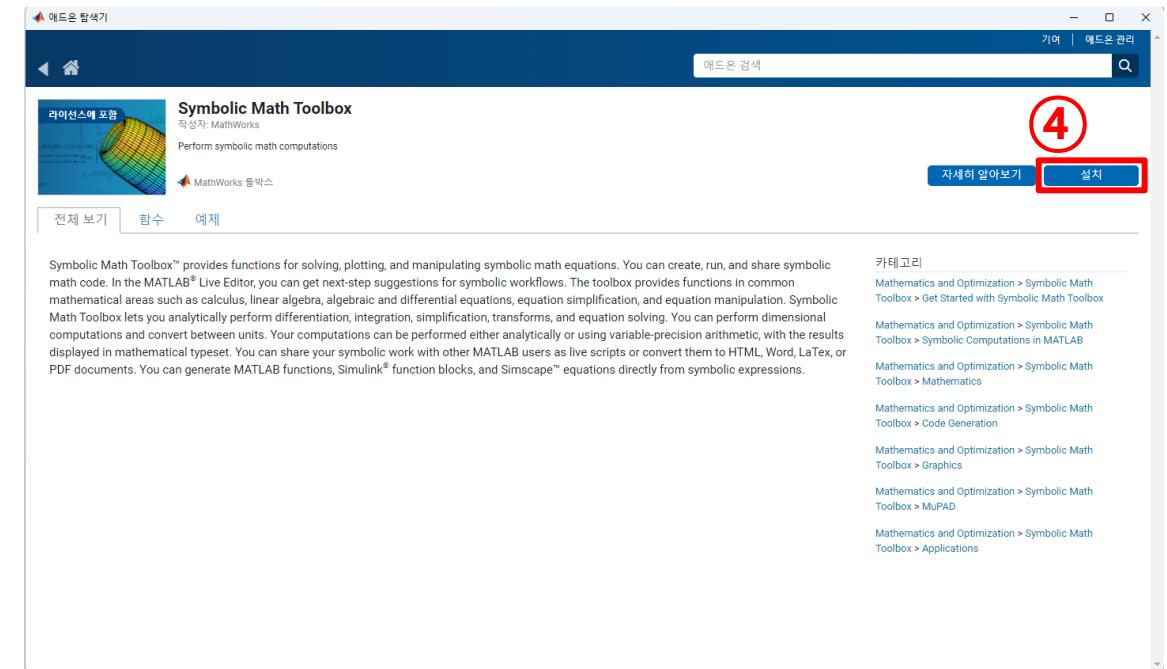
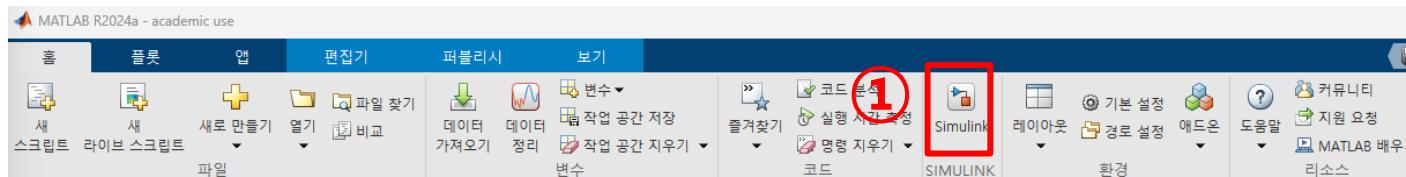
```
syms a b c x
x0 = solve(a*x^2 + b*x + c)
x0 = subs(x0, [a b c], [1 -2 -1])
```

$$\begin{aligned}x_0 &= \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a} \right) \\x_0 &= \left(\frac{1 - \frac{\sqrt{8}}{2}}{\frac{\sqrt{8}}{2} + 1} \right)\end{aligned}$$

Example of symbolic math in MATLAB

Preparation for 'Symbolic Math Toolbox'

- You need to practice using the ‘Symbolic Math Toolbox’, which appears in the latter part of the lecture.
- Follow the procedure to install the toolbox and then try the exercise yourself.



Exercise : Symbolic Variable and Function

■ Symbolic variable

- ▶ A mathematical symbol rather than a specific value.

■ Symbolic function

- ▶ An expression defined using such symbolic variables.

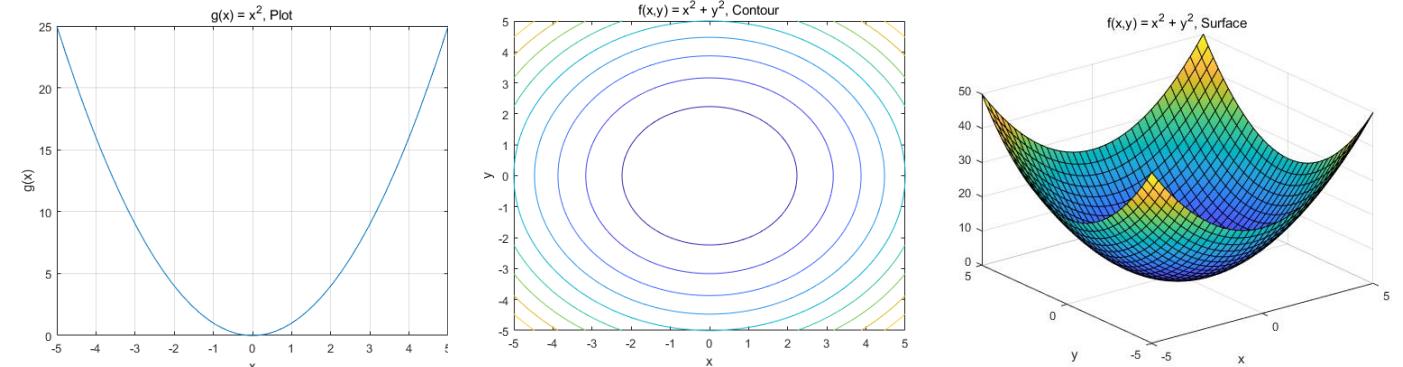
```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define symbolic variables
syms x y;

% Fig 1: Plot visualization of g(x)
g(x) = x^2; % Define symbolic function g(x)
figure;
fplot(g(x));
title('g(x) = x^2, Plot');
xlabel('x'); ylabel('g(x)');
grid on;

% Fig 2-1: Contour visualization of f(x, y)
f(x,y) = x^2 + y^2; % Define symbolic function f(x,y)
figure;
fcontour(f);
title('f(x,y) = x^2 + y^2, Contour');
xlabel('x'); ylabel('y');

% Fig 2-2: Surface visualization of f(x, y)
figure;
fsurf(f);
title('f(x,y) = x^2 + y^2, Surface');
xlabel('x'); ylabel('y');
```



MATLAB code to practice the symbolic variable and symbolic function

Supplementary material using MATLAB course

■ You can practice the symbolic math on matlab web browser independently.

- ▶ <https://matlabacademy.mathworks.com/kr/details/introduction-to-symbolic-math-with-matlab/symbolic>

■ The following chapters are recommended to be practiced.

MATLAB®의 기호 수학에 대한 기본 사항을 학습할 수 있습니다. 종이에 직접 방정식을 풀고 조작하는 대신 MATLAB을 활용할 수 있습니다.

교육과정 모듈

- ✓ > 소개 100% | 5분

- ✓ > 기호 변수 만들기 100% | 10분

- ✓ > 기호 변수를 사용한 수학 표현식 100% | 15분

- ✓ > 기호 방정식을 만들고 풀기 100% | 10분

- ✓ > 대수 조작 및 단순화 100% | 5분

- ✓ > 가정을 다루는 작업하기 100% | 10분

- ✓ > 측정 단위를 다루는 작업하기 100% | 10분

- ✓ > 기호 함수 만들기 100% | 10분

- ✓ > 기호 함수 및 기호 방정식 시각화하기 100% | 10분

Recommended chapters



**THANK YOU
FOR YOUR ATTENTION**



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