Linear Algebra

Vector Part 1:
Vector and Basic Operation of Vector

Automotive Intelligence Lab.

Contents

Generating and visualizing vectors with matlab

Vector

Vector

1, 2, 3

Symbolic toolbox

- Representations of numbers or symbols in a **one-dimensional** array.

Notation for vectors

- **vectors** are typically denoted by bold lowercase Roman letters, such as $\mathbf{v}$.
- other expression : italicized (v) / with an arrow above ($\vec{\mathbf{v}}$).

Characteristics of vectors

- Dimensionality: the number of elements a vector contains.
  - Represented as $\mathbb{R}^N$
    - $\mathbb{R}$: Real Number
    - N : Dimension

[a, b, c · · · z]

- Orientation: indicates whether the vector is in column or row orientation.

Column and Row Vector

Column vector (or vector)

기본은 Column Vector

- A matrix with only one column.
- Each element of the vector is expressed as a **vertical** array.
- Column vectors are often represented as $\boldsymbol{v}$.
- Vectors are in column orientation unless otherwise specified.

Row vector

- A matrix with only one row.
- Each element of the vector is expressed as a **horizontal** array.
- Row vectors are often represented as $\boldsymbol{w}^T$.

transpose

- T represents the transpose operation.

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} .3 \\ -7 \end{bmatrix}, \boldsymbol{z} = \begin{bmatrix} 1 & 4 & 5 & 6 \end{bmatrix}$$

Example of Column Vector and Row Vector

$\boldsymbol{x}$ is a $\boxed{4}$ dimensional $\boxed{\text{Column}}$ vector
$\boldsymbol{y}$ is a $\boxed{2}$ dimensional $\boxed{\text{Column}}$ vector
$\boldsymbol{z}$ is a $\boxed{4}$ dimensional $\boxed{\text{Row}}$ vector

$\boldsymbol{x} \in \mathbb{R}^4$ can also be written.

Transpose

- **Convert row vector to column vector or vice versa, effectively flipping its orientation.**
  - Transpose of a row vector = Column vector.
  - Transpose of a column vector = Row vector.

- **Notation**
  - Transpose of $\boldsymbol{v}$ = $\boldsymbol{v}^T$.

- If we transposing **vector** twice, it returns the vector to its Original **orientation.**
  - So, $\boldsymbol{v}^{TT} = \boldsymbol{v}$.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}$$

Transpose of column vector

$$\begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Transpose of row vector

Does Vector Orientation Matter?

- **It depends on how you use vectors!**

- **In case of using vectors to store data**
  - Orientation of vector usually doesn't matter.
  - The difference is simply whether to stack information horizontally or vertically

- **In case of using vectors to perform operations**
  - Orientation of vector does matter.
    - We will study properties of vector operations which the orientation of vector is important.
  - Operation results vary depending on the orientation of vector.

[v] ← [M] [v]

input [v] → System → output [v] = [M] [Input vector]

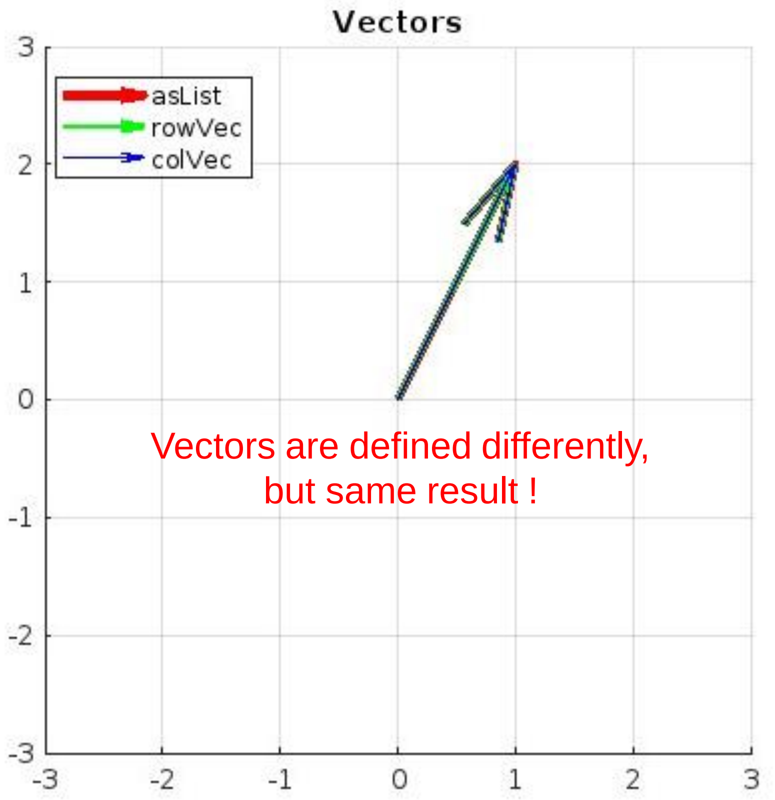
Generating and Visualizing Vectors with Matlab

Code Exercise (02_01)

- Three methods for creating vectors.

```
% Creating a vector as a MATLAB list
asList = [1, 2];

% Creating a row vector
rowVec = [1, 2]; % row

% Creating a column vector
colVec = [1; 2;]; % column

% Plotting the vectors using quiver
figure;
hold on;

% To prevent overlap, there is a 0.1 offset in the starting points of the vectors.
quiver(0, 0, asList(1), asList(2), 'r', 'LineWidth', 3, 'AutoScale', 'off', 'MaxHeadSize', 1);
quiver(0, 0, rowVec(1), rowVec(2), 'g', 'LineWidth', 2, 'AutoScale', 'off', 'MaxHeadSize', 1);
quiver(0, 0, colVec(1), colVec(2), 'b', 'LineWidth', 1, 'AutoScale', 'off', 'MaxHeadSize', 1);

% Set axes properties
axis equal;
xlim([-3, 3]);
ylim([-3, 3]);

% Show grid
grid on;

% Title for the visualization
title('Vectors');

% Legend for vectors
legend('asList', 'rowVec', 'colVec');
```

Source code



Source code result

Equivalence of Vectors

- **If and only if their corresponding entries are equal.**
  - If the corresponding components of vectors $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n)$ are equal, that is, $u_{\mathrm{i}} = v_{\mathrm{i}}$ for all i, then the two vectors are said to be equivalence or equal, denoted by $\boldsymbol{u} = \boldsymbol{v}$.
  - $\boldsymbol{u} = \boldsymbol{v}$ iff $u_1 = v_1$ and $u_2 = v_2$ in vectors in $\mathbb{R}^2$.

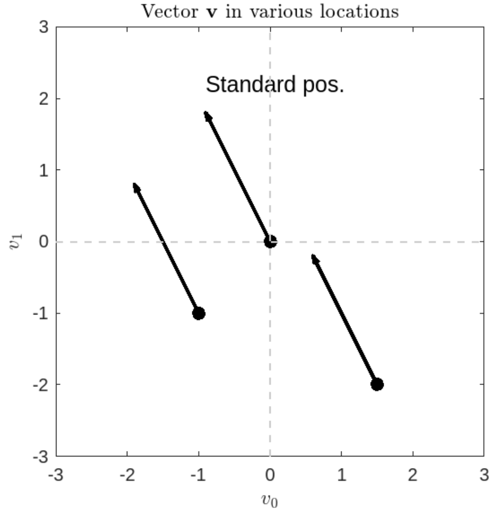
$$\boldsymbol{u} = (4,5,7,2), \boldsymbol{v} = (4,5,7,2), \boldsymbol{w} = (4,5,7,2,6)$$
$$\boldsymbol{u} = \boldsymbol{v}\,, \boldsymbol{u} \neq \boldsymbol{w}$$

Concept of equivalence between vectors

Mathematical Interpretation of Vectors

- **Algebraic interpretation of vectors**
  - A list of numbers arranged in order. ➔ useful in data science
- **Geometric interpretation of vectors**
  - A line with a specific $\boxed{\text{length (or magnitude)}}$ and $\boxed{\text{direction}}$ (or angle: measured counterclockwise from the positive x-axis). ➔ useful in physics and engineering
  - A vector representing a physical quantity with both direction and magnitude.
  - Displacement, velocity, acceleration, force, electric field, etc.

- **Standard position in Geometric interpretation.**
  - Vectors and coordinates are different!
  - All arrows represent different $\boxed{\text{Coordinates}}$ but the same $\boxed{\text{Vector}}$
  - If the vector equals the coordinate, it is a standard position.
    - A vector at the standard position has its **tail** at the **origin** and its **head points** to the **geometric coordinates**.



Various same vector $\boldsymbol{v}$

Code Exercise of Generating Different Reference Vectors using Matlab

Code Exercise (02_02)

- Generate vectors with different reference points.

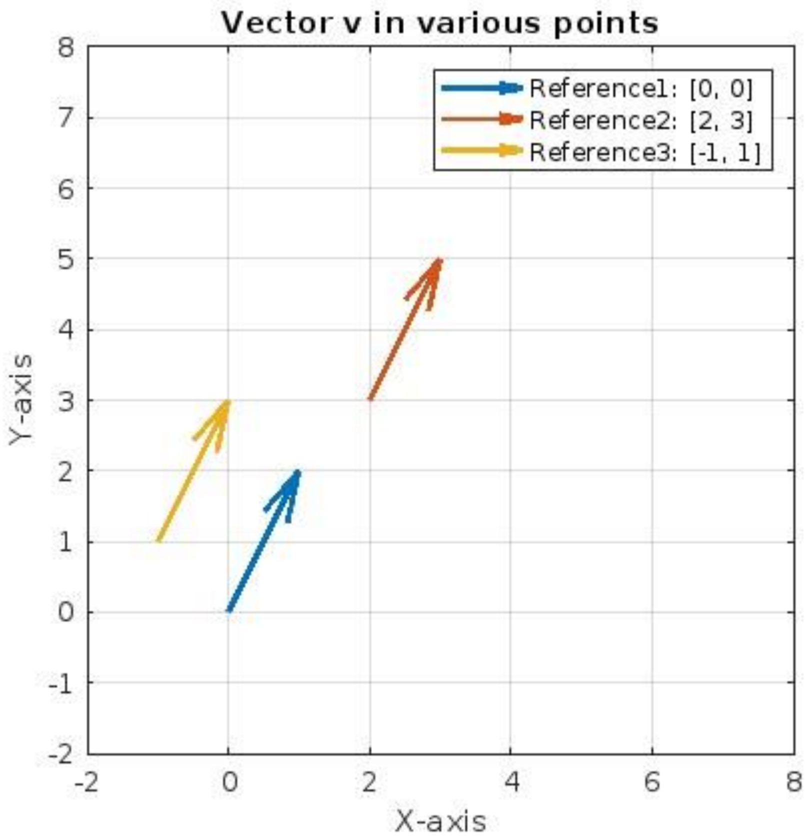
```
% Define the vector
v = [1, 2];

% Define three different reference points
reference_points = [0, 0; 2, 3; -1, 1];

% Create a figure
figure;

% Plot the vector with each reference point
for i = 1:size(reference_points, 1)
    quiver(reference_points(i, 1), reference_points(i, 2), v(1),
v(2),'LineWidth', 2, 'AutoScale', 'off', 'MaxHeadSize', 2);
    hold on;
end

% Set axes properties
axis equal;
xlim([-2, 8]);
ylim([-2, 8]);

% Show grid
grid on;

% Title for the visualization
title('Vector v in various points');

% Axes labels
xlabel('X-axis');
ylabel('Y-axis');

% Legend for vectors with different reference points
legend('Reference1: [0, 0]', 'Reference2: [2, 3]', 'Reference3: [-1,
1]');
```

Source code

Visualization Result of Generating Vector using Matlab

Code Exercise

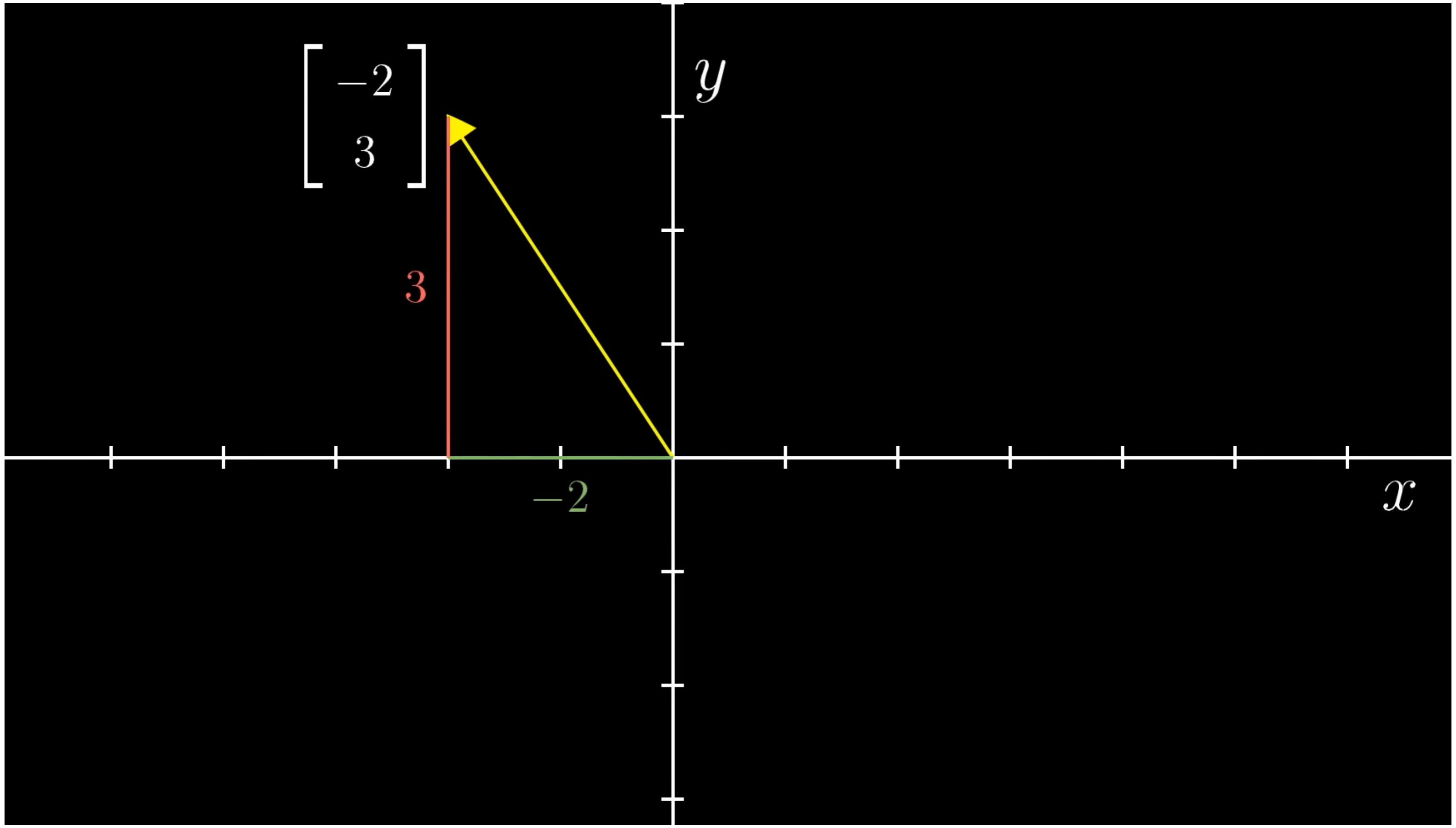
- Visualizing vectors with different reference points.



Source code result

Geometric representation of vector

- Coordinate system (0:15 ~ 4:35)
- https://youtu.be/fNk_zzaMoSs?si=HvUOkaNK1-_BCLWL&t=15



Visualization of vector

Vector operations

Vector-Vector Addition and Subtraction

Addition and subtraction of two vectors

- Vector addition, subtraction is only possible between vectors of the $\boxed{\text{same dimension}}$

$$\begin{bmatrix}4\\5\\6\end{bmatrix}+\begin{bmatrix}10\\20\\30\end{bmatrix}=\begin{bmatrix}14\\25\\36\end{bmatrix}$$

Addition between two vector

$$\begin{bmatrix}4\\5\\6\end{bmatrix}-\begin{bmatrix}10\\20\\30\end{bmatrix}=\begin{bmatrix}-6\\-15\\-24\end{bmatrix}$$

Subtraction between two vector

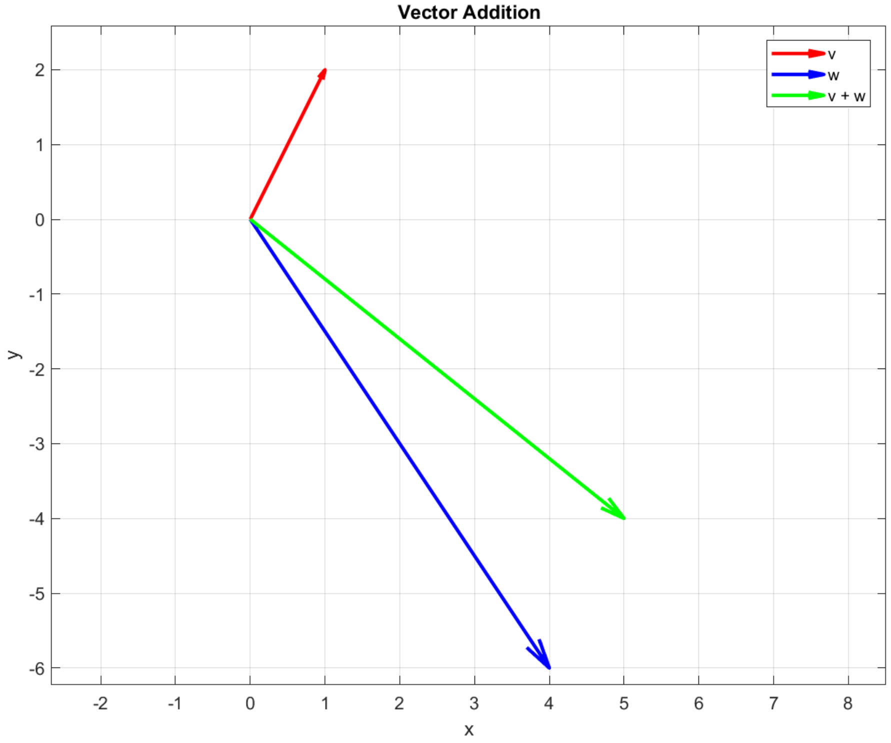
Code Exercise of Vector Addition and Subtraction using Matlab

Code Exercise (02_03)

- Addition between two vector.

```
%% Adding vectors

% Using 2D vectors here instead of 3D vectors in the book to
facilitate visualization
v = [1, 2];
w = [4, -6];
vPlusW = v + w;

% print out all three vectors
disp('v:');
disp(v);
disp('w:');
disp(w);
disp('vPlusW:');
disp(vPlusW);

% Plot vectors
quiver(0, 0, v(1), v(2), 0, 'r', 'LineWidth', 2);
hold on;
quiver(0, 0, w(1), w(2), 0, 'b', 'LineWidth', 2);
quiver(0, 0, vPlusW(1), vPlusW(2), 0, 'g', 'LineWidth', 2);
hold off;
axis equal;
xlabel('x');
ylabel('y');
title('Vector Addition');
legend('v', 'w', 'v + w');
grid on;
```

Source code



Source code result

Vector Addition and Subtraction using Broadcasting

Addition and subtraction of two vectors using Broadcasting

- Broadcasting: Mechanism that automatically aligns the sizes of arrays when performing element-wise operations.
- In MATLAB, broadcasting is possible when the dimensions of two vectors differ.

$$\begin{bmatrix}4\\5\\6\end{bmatrix} + \begin{bmatrix}10 & 20 & 30\end{bmatrix} = [\,?\,]$$

수학적으로 X

matlab

$$\begin{bmatrix}14 & 24 & 34\\15 & 25 & 35\\16 & 26 & 36\end{bmatrix}$$

Is it possible?

Code Exercise of Broadcasting using Matlab

Code Exercise (02_04)

- Broadcasting – see diagonal element.

```
% column vector and row vector
column_vector = [1; 2; 3];
row_vector = [4 5 6];

% Using 2D vectors here instead of 3D vectors in the book to
facilitate visualization
sum_result = column_vector + row_vector;
difference_result = column_vector - row_vector;

% print out all three vectors
disp('addition:');
disp(sum_result);
disp('subtraction:');
disp(difference_result);
```

Source code

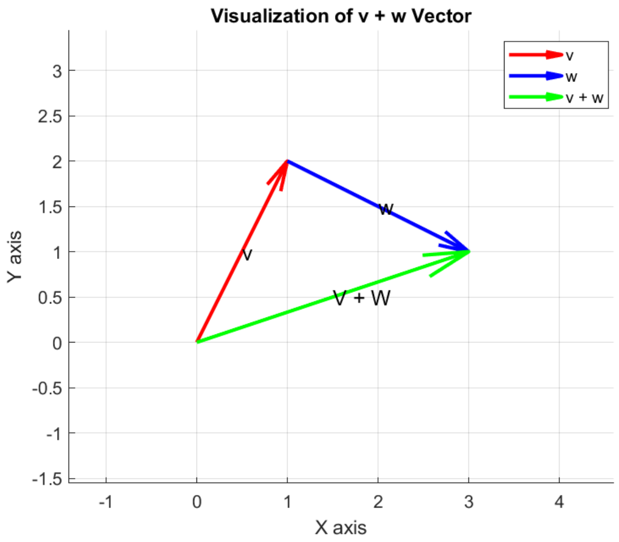
Geometric Structure of Vector Addition and Subtraction

Vector addition

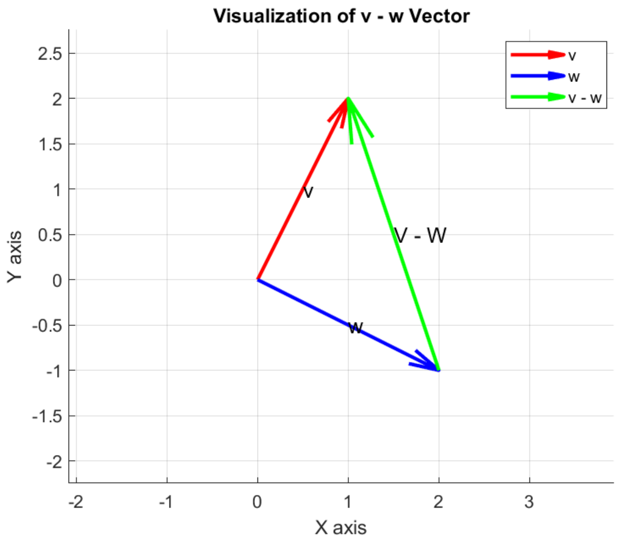
- Connecting the tail of one vector to the head of another vector.

Vector subtraction

- Positioning the tails of two vectors at the same coordinate.
- The resulting vector from subtraction is directed from the head of the second vector to the head of the first vector.



Addition between two vector



Subtraction between two vector

Scalar-Vector Multiplication

Scalar-vector multiplication

- Scalar: A quantity that is not associated with any vector or matrix, but represents $\boxed{\text{a single value}}$
  - Scalars are typically denoted by Greek lowercase letters such as α or λ.
  - example : scalar-vector multiplication can be represented as λ**w**.
    - λ : Scala
    - **w** : Vector

$$\lambda = 4, \boldsymbol{w} = \begin{bmatrix} 9 \\ 4 \\ 1 \end{bmatrix}, \lambda\boldsymbol{w} = \begin{bmatrix} 36 \\ 16 \\ 4 \end{bmatrix}$$

scalar-vector multiplication

Code Exercise of Scalar-Vector Multiplication using Matlab

Code Exercise (02_05)

- multiplication between scalar-vector.

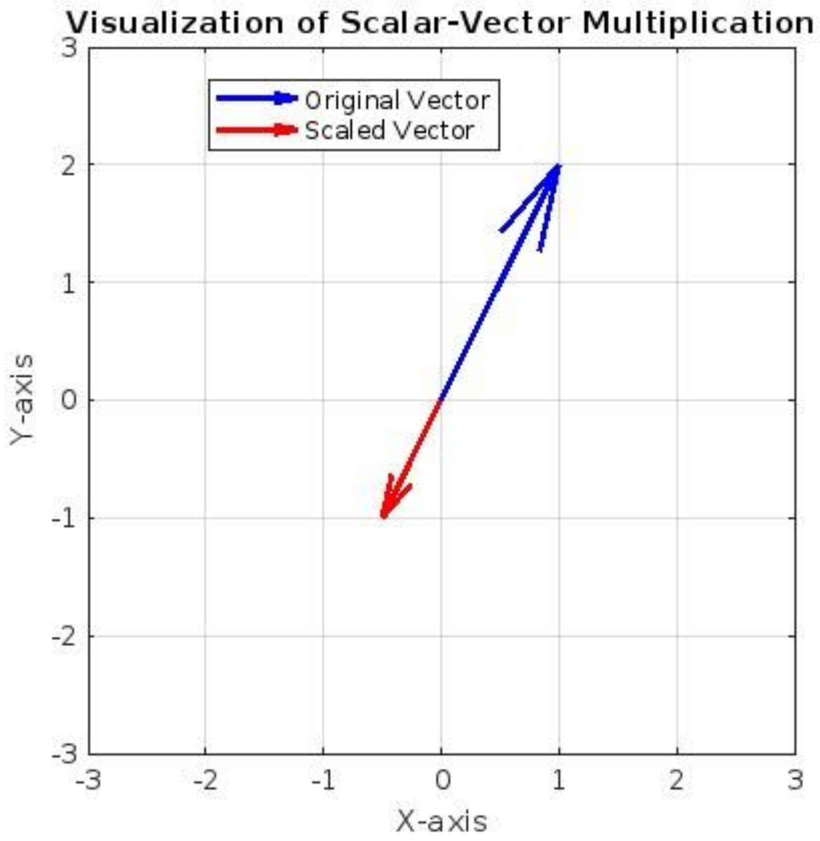
```
% Define the vector
v = [1, 2];

% Define the scalar
s = -1/2;

% Compute the scaled vector
scaled_v = s * v;

% Create a figure
figure;

% Plot the original vector
quiver(0, 0, v(1), v(2), 'b', 'LineWidth', 3, 'AutoScale',
'off','MaxHeadSize', 2);
hold on;

% Plot the scaled vector
quiver(0, 0, scaled_v(1), scaled_v(2), 'r', 'LineWidth', 2,
'AutoScale', 'off', 'MaxHeadSize', 2);

% Set axes properties
axis equal;
xlim([-3, 3]);
ylim([-3, 3]);

% Show grid
grid on;

% Title for the visualization
title('Scalar-Vector Multiplication');

% Axes labels
xlabel('X-axis');
ylabel('Y-axis');

% Legend for vectors
legend('Original Vector', 'Scaled Vector');
```

Source code

Visualization Result of Scalar-Vector Multiplication using Matlab

Code Exercise

- multiplication between scalar-vector.



Source code result

Scalar-Vector Addition and Subtraction

- **Scalar-vector addition**
  - In linear algebra : vectors and scalars are distinct mathematical objects and **cannot be combined**.
  - In Matlab: scalars to vectors can added or subtracted. How is it possible?

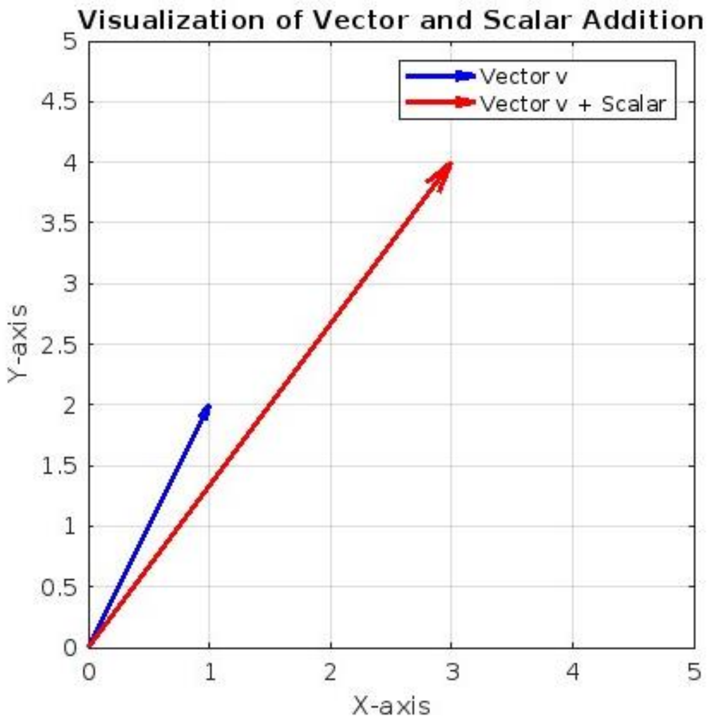
Code Exercise of Scalar-Vector Addition using Matlab

Code Exercise (02_06)

- Scalar - vector addition.

```
% Define vector
v = [1, 2];

% Define scalar
s = 2;

% Add scalar to vector
v_plus_s = v + s;

% Create figure
figure;

% Display vector v from the origin
quiver(0, 0, v(1), v(2), 'b', 'LineWidth', 2, 'AutoScale', 'off');
hold on;

% Display vector v + scalar from the origin
quiver(0, 0, v_plus_s(1), v_plus_s(2), 'r', 'LineWidth', 2, 'AutoScale', 'off');

% Set axes
axis equal;
xlim([0, 5]);
ylim([0, 5]);

% Show grid
grid on;

% Title for visualization of vector and scalar addition
title('Visualization of Vector and Scalar Addition');

% Axes labels
xlabel('X-axis');
ylabel('Y-axis');

% Legend for vectors and scalar
legend('Vector v', 'Vector v + Scalar');
```

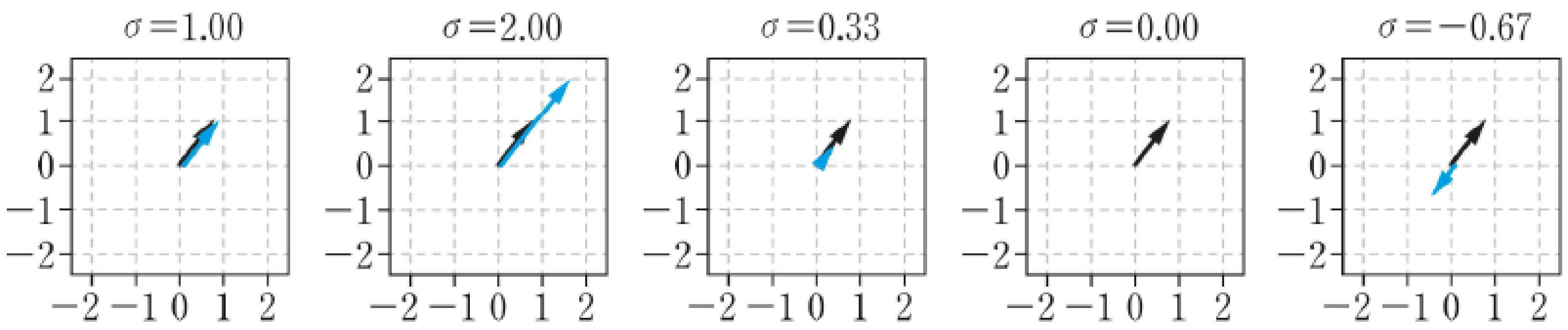
Source code



Source code result

Geometric Understanding of Scalar-Vector Multiplication

- **Geometric** understanding in scalar-vector multiplication
  - Scalars **only scale the magnitude of vectors** without changing their direction

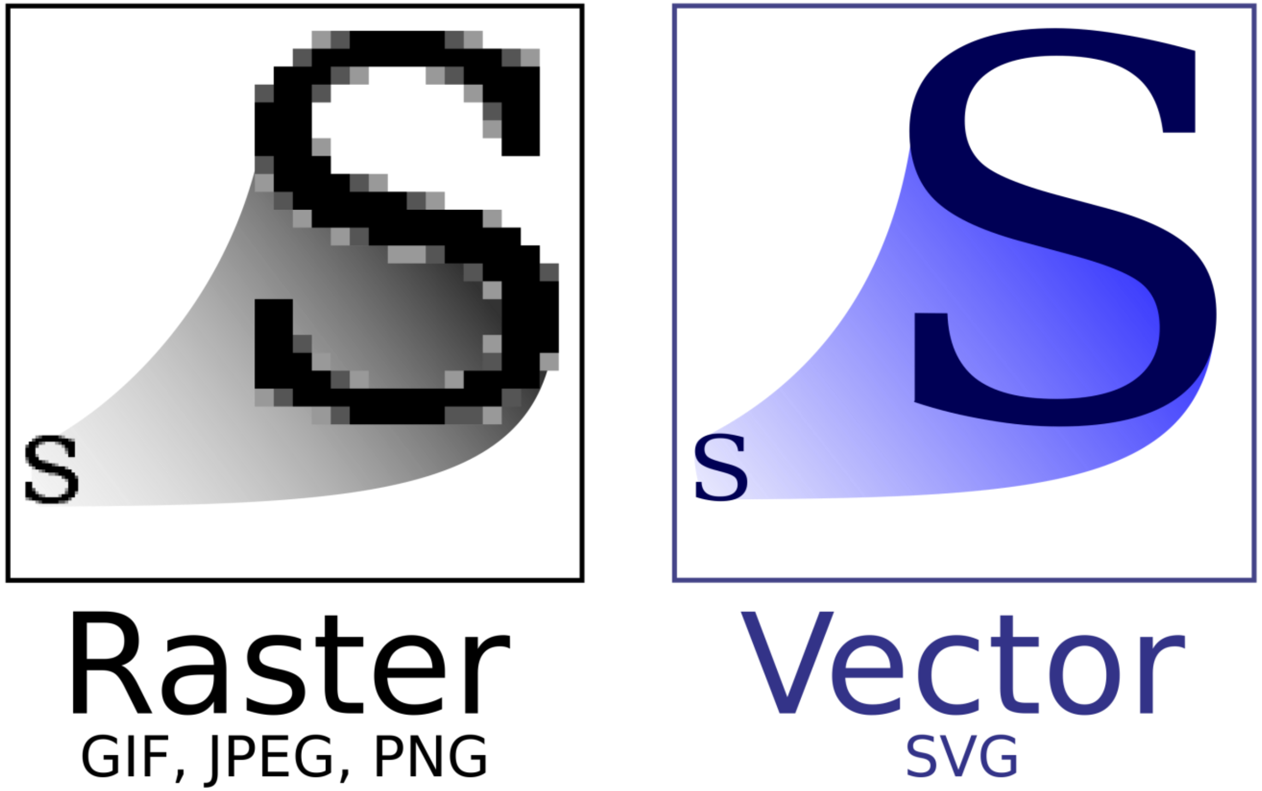


Various scalar-vector multiplication

  - In a diagram, when the scalar is negative, the vector direction is reversed (i.e., rotated 180 degrees).
  - The "rotated" vector still points along the **same infinite line**, so the **negative scalar hasn't changed its direction**.
- **Vector average**
  - Using **vector addition** and **scalar-vector multiplication.**
  - To find the average of N vectors, sum them **all together and** multiply **by the scalar 1/N**.

Example – Vector Graphics

- Vector graphics are a form of computer graphics in which visual images are created directly from geometric shapes defined on a Cartesian plane, such as points, lines, curves and polygons.



Definition of Zero Vector

Zero vector

- The zero vector (or $\boxed{\text{null Vector}}$) is a vector where all components are zero.
- Indicated using a boldfaced zero, **0**.

- In fact, using the zeros vector to solve a problem is often called the trivial solution and is excluded.
  - In linear algebra is full of statements like
    - Find a nonzeros vector that can solve…
    - Find a nontrivial solution to…

$$\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}, [0 \quad 0 \quad 0 \quad 0 \quad 0], (0 \quad 0 \quad 0 \quad 0)$$

Example of Zero vector

Properties of Vector Operations

Properties of vector operations

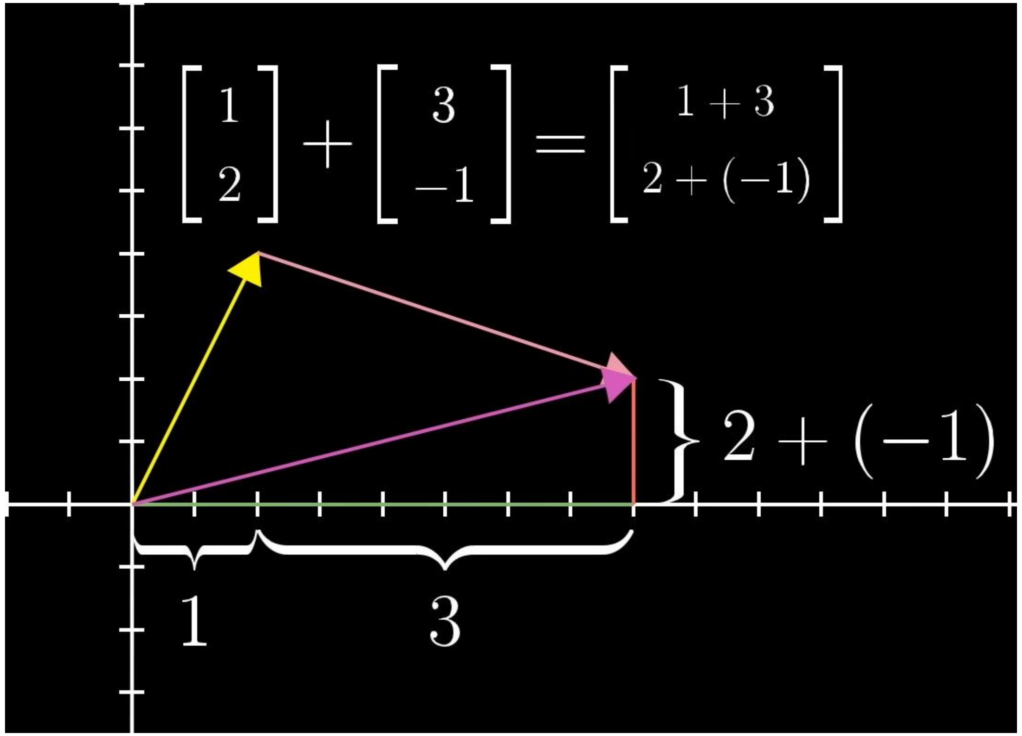
- Where α,β are scalar, $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are n-dimensional real vectors, 0 represents the zero vector.

- $\boldsymbol{u} + \boldsymbol{v} = \boxed{v+u}$
- $\boldsymbol{u} + (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{v} + \boldsymbol{u}) + \boldsymbol{w}$
- $\boldsymbol{u} + \boldsymbol{0} = \boldsymbol{0} + \boldsymbol{u} = \boldsymbol{u}$
- $\boldsymbol{u} + (-\boldsymbol{u}) = (-\boldsymbol{u}) + \boldsymbol{u} = \boldsymbol{0}$
- $\alpha(\boldsymbol{u} + \boldsymbol{v}) = \alpha\boldsymbol{u} + \alpha\boldsymbol{v}$
- $(\alpha + \beta)\boldsymbol{u} = \boxed{\alpha u + \beta u}$
- $\alpha(\beta\boldsymbol{u}) = \boxed{(\alpha\beta) u}$
- $1\boldsymbol{u} = \boldsymbol{u}$

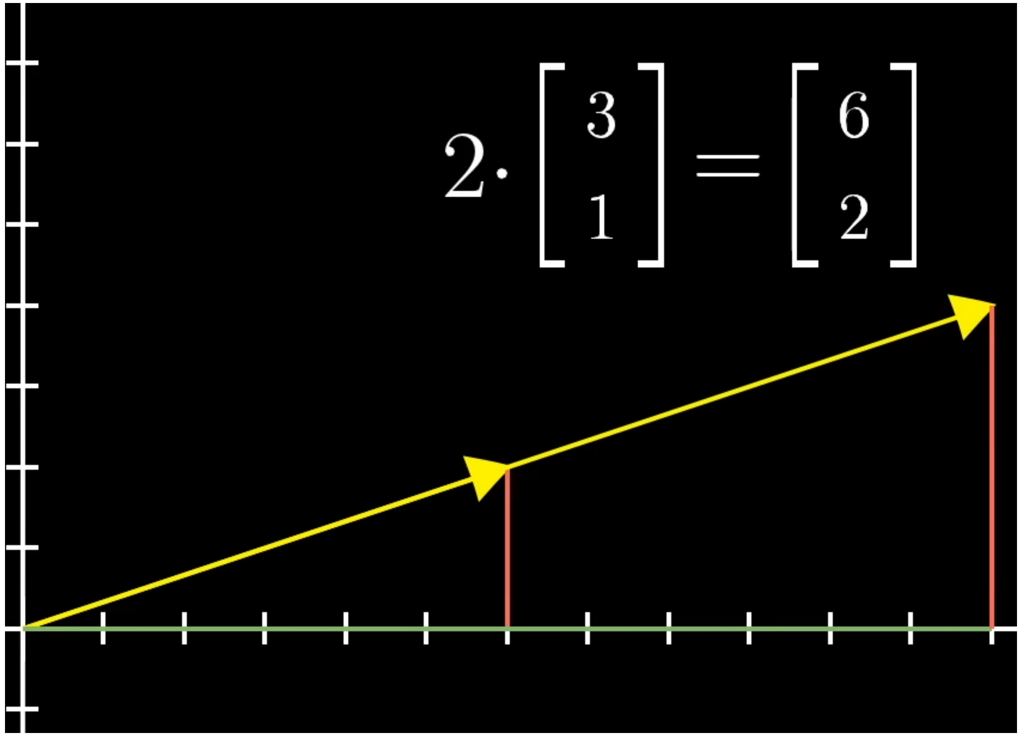
Visual Materials

Geometric representation of vector operation

- Vector addition (4:36 ~ 6:53)
  - https://youtu.be/fNk_zzaMoSs?t=276&si=iIkRwYfl8Hl1Wyo3
- Vector multiplication (6:53 ~ 8:07)
  - https://youtu.be/fNk_zzaMoSs?t=414&si=heZf3HVg9BpCFo4c



Visualization of vector addition operation



Visualization of vector multiplication operation

Vector magnitude and unit vectors

Vector Magnitude and Unit Vector

Norm

- Function that calculates the $\boxed{\text{magnitude of Vector}}$
- Vector u's norm is presented as $\boxed{\|u\|}$ and norm satisfies the following properties.
  - $\boldsymbol{u}, \boldsymbol{v}$ is vector, and α is scala.

1. $\| \boldsymbol{u} \| \geq 0$
2. $\| \alpha \boldsymbol{u} \| = |\alpha| \, \| \boldsymbol{u} \|$
3. $\| \boldsymbol{u} + \boldsymbol{v} \| \leq \| \boldsymbol{u} \| + \| \boldsymbol{v} \|$
4. $\| \boldsymbol{u} \| = 0, \text{only when } \boldsymbol{u} = 0$

$$\|v\|_p = \left[\sum_{k=1}^{N} |v_k|^p \right]^{1/p}$$

1. 2. ~~3~~

Manhattan Norm (L1 norm)

- **For a vector $v = x1, x2, \ldots . xn$, the Manhattan norm is defined as follow.**

$$\|\boldsymbol{v}\|_1 = \sum_i^n |x_i| = |x_1| + |x_2| + \cdots + |x_n|$$

- **Manhattan norm is also called** $\boxed{\text{L1 norm}}$ **and is used to define distance.**
- **Designed to express actual moving distance rather than simple straight-line distance.**

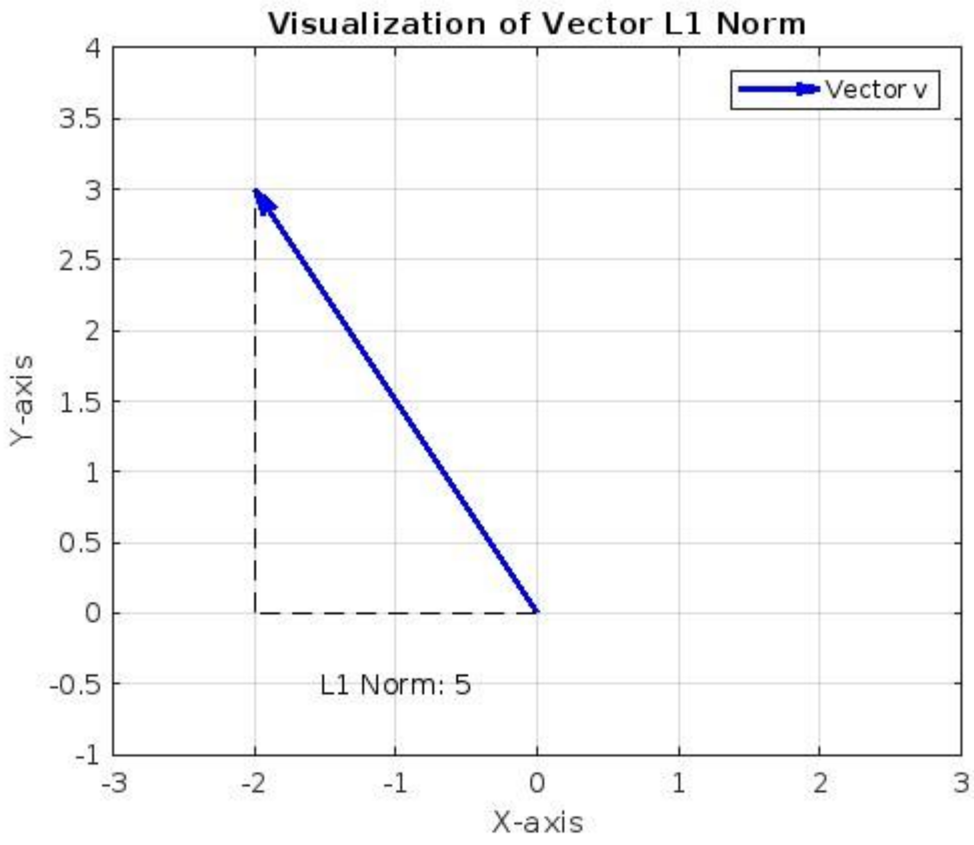
Code Exercise of Manhattan Norm Norm using Matlab

Code Exercise (02_07)

- L1 norm(Manhattan norm)

```
% Define vector
v = [-2, 3];

% Calculate L1 norm
l1_norm = norm(v, 1);

% Display vector from the origin
quiver(0, 0, v(1), v(2), 'b', 'LineWidth', 2, 'AutoScale', 'off');
hold on;

% Add lines representing movement along each axis to visualize Manhattan distance
plot([0, v(1)], [0, 0], '--k', 'LineWidth', 1); % Movement along x-axis
plot([v(1), v(1)], [0, v(2)], '--k', 'LineWidth', 1); % Movement along y-axis

% Display the value of L1 norm
text(v(1)/2, -0.5, ['L1 Norm: ', num2str(l1_norm)], 'HorizontalAlignment', 'center');

% Set axes properties
axis equal;
xlim([-3, 3]);
ylim([-1, 4]);

% Show grid
grid on;

% Title for visualization of vector L1 norm
title('Visualization of Vector L1 Norm');

% Axes labels
xlabel('X-axis');
ylabel('Y-axis');

% Legend for vectors and movement along axes
legend('Vector v');
```

Source code



Source code result

Euclidean Norm (L2 norm)

- For a vector $v = x1, x2, \dots . xn$, the Euclidean norm is defined as follow.

$$\|v\|_2 = \sqrt{\sum_i^n x_i^2} = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2 + \cdots + x_i^2}$$

- Euclidean norm is also called $\boxed{\text{L}_2 \text{ norm}}$ and is used to define distance and magnitude.
- Regardless of dimension, $\boxed{\text{L}_2 \text{ norm}}$ is obtained as the square root of the sum of the squares of the absolute values.

- When we refer to the norm of a vector, we usually mean the Euclidean norm.

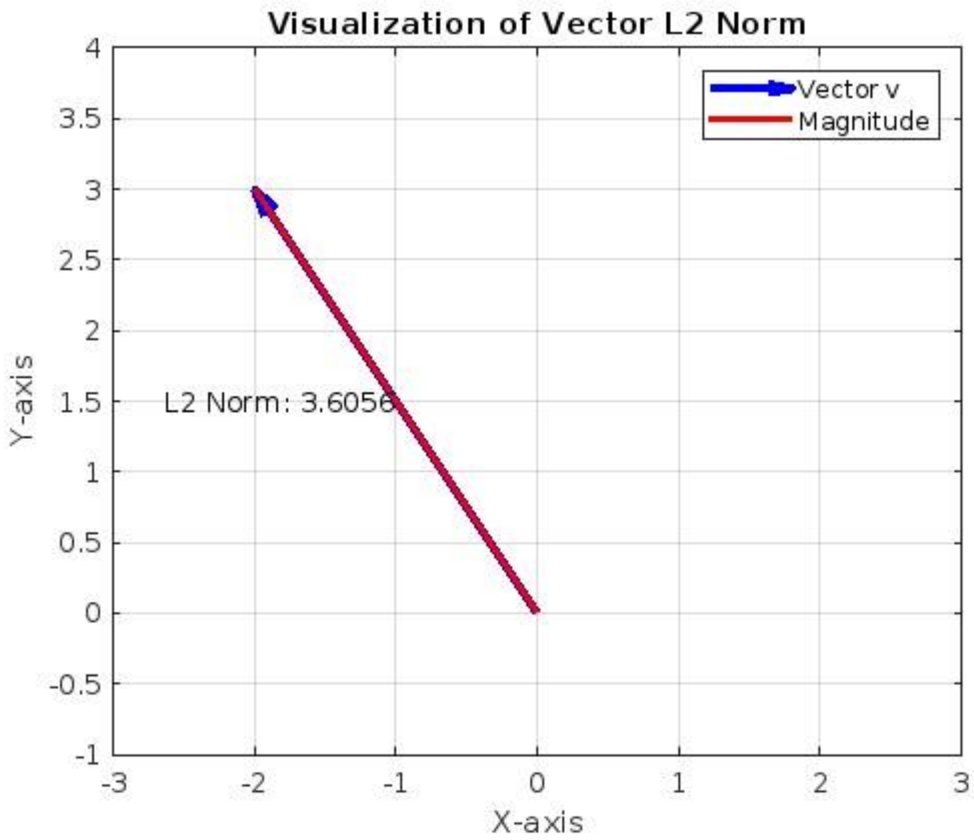
Code Exercise of Euclidean Norm Norm using Matlab

Code Exercise (02_08)

- L2 norm(Euclidean norm)

```
% Define vector
v = [-2, 3];

% Calculate L2 norm
l2_norm = norm(v, 2);

% Display vector from the origin
quiver(0, 0, v(1), v(2), 'b', 'LineWidth', 2, 'AutoScale', 'off');
hold on;

% Add line representing the vector to illustrate its magnitude
plot([0, v(1)], [0, v(2)], 'r', 'LineWidth', 2);

% Display the value of L2 norm
text(v(1)/2, v(2)/2, ['L2 Norm: ', num2str(l2_norm)], 'HorizontalAlignment', 'right');

% Set axes properties
axis equal;
xlim([-3, 3]);
ylim([-1, 4]);

% Show grid
grid on;

% Title for visualization of vector L2 norm
title('Visualization of Vector L2 Norm');

% Axes labels
xlabel('X-axis');
ylabel('Y-axis');

% Legend for vectors
legend('Vector v', 'Magnitude');
```

Source code



Source code result

Meaning of Magnitude of Vector and Code Exercise

- **Magnitude of a vector (geometric length or norm):** the distance from tail to head of a vector
  - Calculate using the standard **Euclidean distance formula** (see equation below).
  - The magnitude of a vector is indicated by double vertical bars on either side ($\|\boldsymbol{v}\|$).
    - In some cases, the squared magnitude ($\|\boldsymbol{v}\|_2$) is used, in which case the square root term on the right-hand side is removed.

- **Code Exercise (02_09)**
  - Vector norm & length

```
%% Vector Norm
v = [-2, 3];

% Norm of vector
v_L1_norm = norm(v, 1);
v_L2_norm = norm(v, 2);

% Display norm of vector

disp(['Vector L1 norm: ', num2str(v_L1_norm)]);
disp(['Vector L2 norm: ', num2str(v_L2_norm)]);
```

Source code

$$\|\boldsymbol{v}\| = \sqrt{\sum_{i=1}^{n} \boldsymbol{v}_i^2}$$

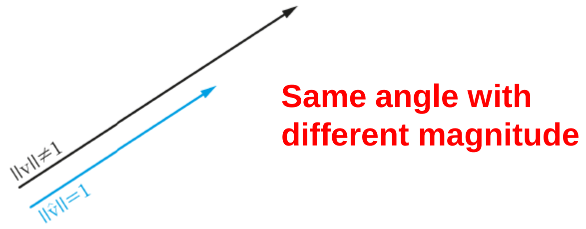
Euclidean distance formula

Unit Vector

- **A vector with a geometric length of 1.**
  - Examples) Orthogonal matrices and rotation matrices, eigenvectors, singular vectors, etc.
  - The unit vector is defined as $\boxed{\|v\| = 1}$
- **How to create the associated unit vector?**
  - By scalar multiplication of the reciprocal of the vector norm.

$$\hat{\boldsymbol{v}} = \frac{1}{\|\boldsymbol{v}\|}\boldsymbol{v}$$

- **The general convention to denote a unit vector($\hat{v}$) in the same direction as the parent vector (v).**



$$\begin{bmatrix} a & b & c \end{bmatrix}^T \begin{bmatrix} e & f & g \end{bmatrix}^T \rightarrow \underline{ae+bf+cg}$$

Vector dot product

Definition of Vector Dot Product

- **The dot product (also known as the** Scalar **product or** inner **product) is one of the most important operations in the entirety of linear algebra.**
  - It forms the basis of many operations and algorithms such as convolution, correlation, Fourier transform, matrix multiplication, linear feature extraction, signal filtering, etc.
  - The ways to denote the dot product between two vectors include:
    - The general notation $\boldsymbol{a}^T\boldsymbol{b}$.
    - $\boldsymbol{a} \cdot \boldsymbol{b}$ or $< \boldsymbol{a}, \boldsymbol{b} >$

Vector → Column

a^t

b

⇒ scalar

  - To calculate the dot product:
    - Multiply corresponding elements from the two vectors and then sum all the results.
    - The dot product is only defined between two vectors of the same dimension

$$\delta = \sum_{i=1}^{n} a_i b_i$$

Dot product formula

Calculation of Vector Dot Product

Dot product is defined by following equation

- Let $\boldsymbol{u}$, and $\boldsymbol{v}$ vectors such that :

$\boxed{v} \cdot \boxed{v} \rightarrow$ Scalar

$$\boldsymbol{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- Dot product of $\boldsymbol{u}$ and $\boldsymbol{v}$ is defined as $\boxed{\text{Scalar}}$, and represented as $< \boldsymbol{u}, \boldsymbol{v} >$ or $\boldsymbol{u} \cdot \boldsymbol{v}$.

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T \boldsymbol{v} = [u_1 \quad u_2 \quad \cdots \quad u_n] \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n$$

$$\begin{aligned} [1\ 2\ 3\ 4] \cdot [5\ 6\ 7\ 8] &= 1 \times 5 + 2 \times 6 + 3 \times 7 + 4 \times 8 \\ &= 5 + 12 + 21 + 32 \\ &= 70 \end{aligned}$$

Example of dot product calculation

Properties of Vector Dot Product

Scala multiplication

- When positive scalar is multiplied by a vector, its dot product $\boxed{\text{increases}}$ by that factor.
  - $(\alpha \boldsymbol{u})^T \boldsymbol{v} = \alpha(\underline{\boldsymbol{u}^T \boldsymbol{v}})$
  - If dot product of $\boldsymbol{v}$ and $\boldsymbol{w}$ is 70, and value of scala s is 10, then the dot product of $s\boldsymbol{v}$ and $\boldsymbol{w}$ will be $\boxed{700}$
- If you try multiplying negative scalar the magnitude of the dot product remains the same, but the sign is opposite.
- Scala of value 0
  - If s = 0, then the dot product is also $\boxed{0}$

$$\begin{cases} a \cdot a = a^T a = \|a\|^2 & \text{norm} \\ a \cdot b = b \cdot a \Rightarrow a^T b = b^T a \end{cases}$$

The dot product is a measure of $\boxed{\text{Similarity}}$ or $\boxed{\text{mapping}}$ between two vectors.

- Pearson correlation coefficient: the normalized dot product between two variables.

Vector 방향이 비슷하면 dot 값 ↑

→ Similarity

고차원의 벡터는 그려볼 수 X ⇒ dot로 similarity 비교

word2vec 단어의 유사성 비교

Code Exercise of Vector Dot Product using Matlab

Code Exercise (02_10)

- dot() function

```
%% Dot product
v = [0, 1, 2];
u = [13, 21, 34];

s = 10;

% scala multiplcate dot product
dot_product = dot(v, u);
scala_multiplicated = dot(s*v, u);

% show the result
disp('Dot Product:');
disp(dot_product);
disp('Scala multiplicated:');
disp(scala_multiplicated);
```

Source code

Property and Code Exercise of Dot Product Distributive Law

Distributive law of the dot product

- The dot product of the sum of vectors is equal to $\boxed{\text{the sum of the dot products}}$

$$\boldsymbol{a}^T(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{a}^T\boldsymbol{b} + \boldsymbol{a}^T\boldsymbol{c}$$

Distributive law of dot product

Code Exercise (02_11)

- Distributive law of dot product.

```
%% The dot product is distributive

% some random vectors
v = [0, 1, 2];
w = [3, 5, 8];
u = [13, 21, 34];

% two ways to compute
res1 = dot(v, w + u);
res2 = dot(v, w) + dot(v, u);

% show that they are equivalent
disp('res1:');
disp(res1);
disp('res2:');
disp(res2);
```

Source code

The two results, res1 and res2, are the same (the answer is 110). This indicates that the distributive property of the dot product holds.

Geometric Definition of Dot Product

Geometric interpretation of dot product

- Multiplication the magnitudes of two vectors and increasing the size by the $\boxed{\text{Cosine Value}}$ of the angle between the two vectors.
- Eq 1. and Eq 2. are mathematically equivalent but expressed differently.

$a \cdot b = \|a\| \|b\| \times \cos(\theta)$ — 0 max, 90° min

$$\delta = \sum_{i=1}^{n} a_i b_i$$

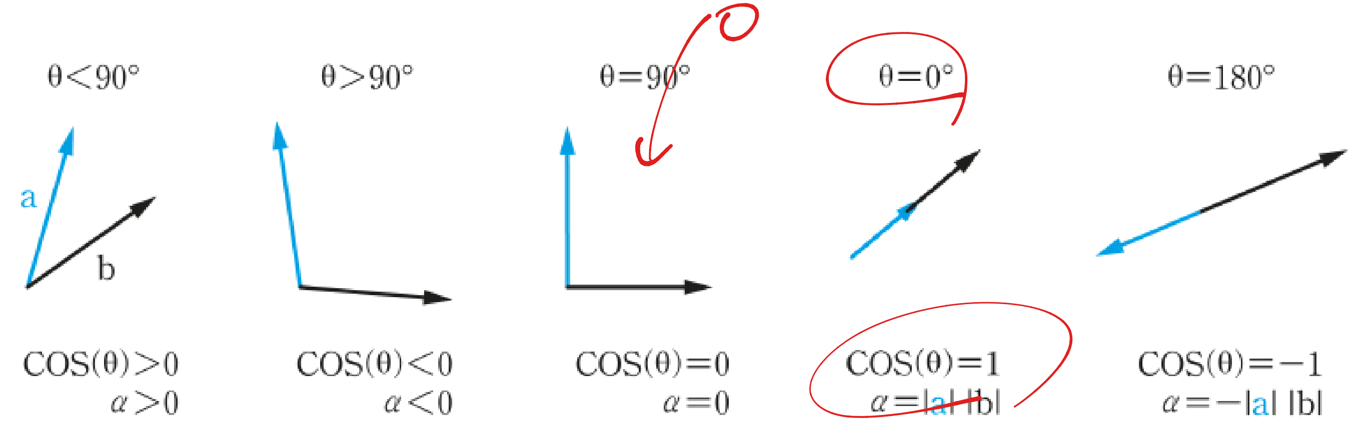
Eq 1. Dot product formula

magnitude magnitude

$$\alpha = \cos(\theta_{\mathrm{v,w}}) \|\mathbf{v}\| \|\mathbf{w}\|$$

Eq 2. Geometric definition of vector dot product

- Five cases of dot product sign depending on the angle between two vectors.



Dot product sign of two vectors present geometric relationship between vectors

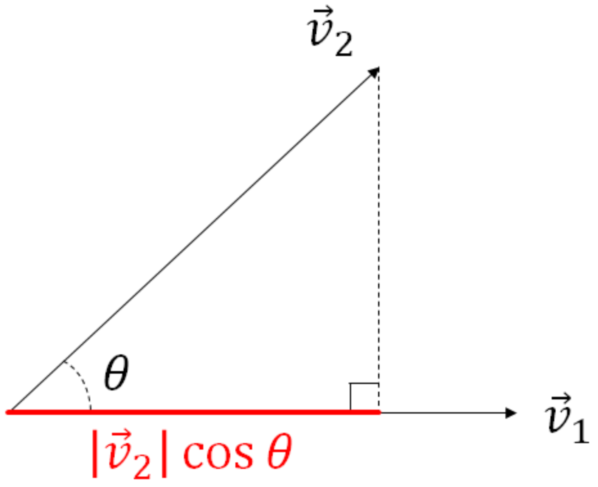
Reference Materials of Vector Dot Product

- **Geometric meaning of vector dot product**
  - https://angeloyeo.github.io/2020/09/09/row_vector_and_inner_product.html#%ED%96%89%EB%B2%A1%ED%84%B0%EC%9D%98-%EC%8B%9C%EA%B0%81%ED%99%94

$$\vec{v}_1 \cdot \vec{v}_2 = \begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax + by$$

why?

=



$$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1||\vec{v}_2| \cos\theta$$

Geometrical proof of vector dot product

Geometric Proofs of Vector Dot Product

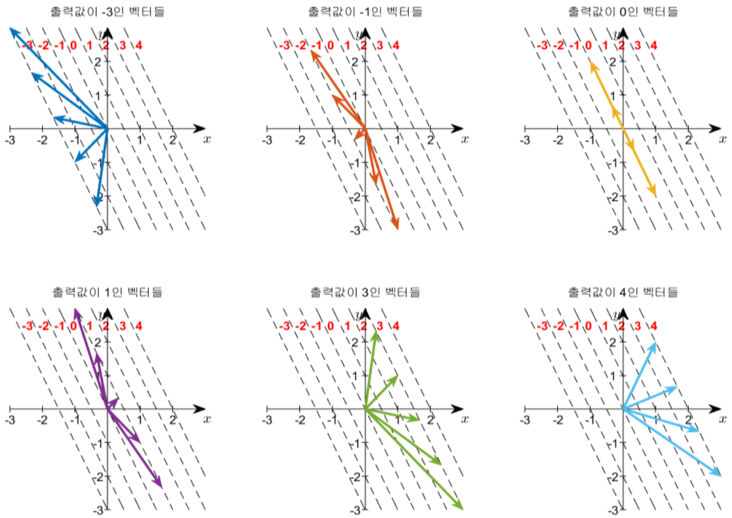
$\begin{bmatrix}2\\1\end{bmatrix}\cdot\begin{bmatrix}x\\y\end{bmatrix} \to 2x+y=4$

$y=-2x+4$

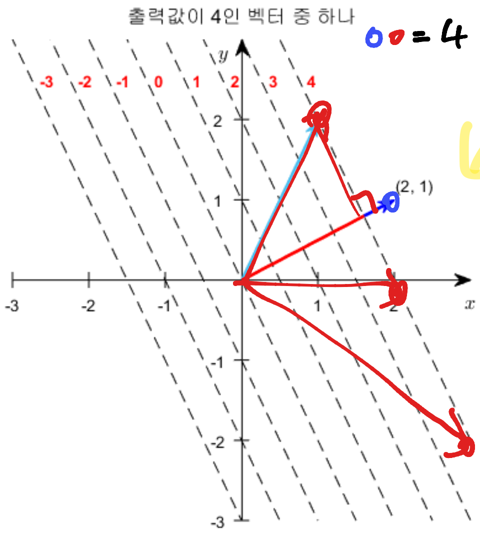
Geometric meaning of vector dot product

1. Represent a map where locations of equal height are connected by a single line.
2. Consider the case where the output scalar value is 4.
3. Since the dashed lines corresponding to 2x+y=4 are all perpendicular to the row vector [2,1],
   - $4\times 2 = d \times \sqrt{20}$, $d = \frac{4}{\sqrt{5}}$
   - Length of row vector [2,1] is $\sqrt{5}$, and multiplication of d and row vector is, $d \times \sqrt{5} = \frac{4}{\sqrt{5}} \times \sqrt{5} = 4$

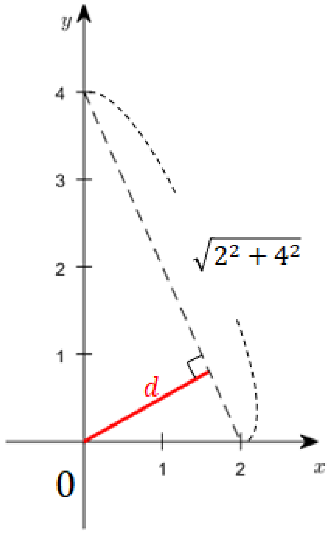
▶ So, product of the projection length of a column vector and the $\boxed{\text{length of a row vector}}$ = dot product value.



1) Visualization of different output of 2x+y (-3 ~ 4)



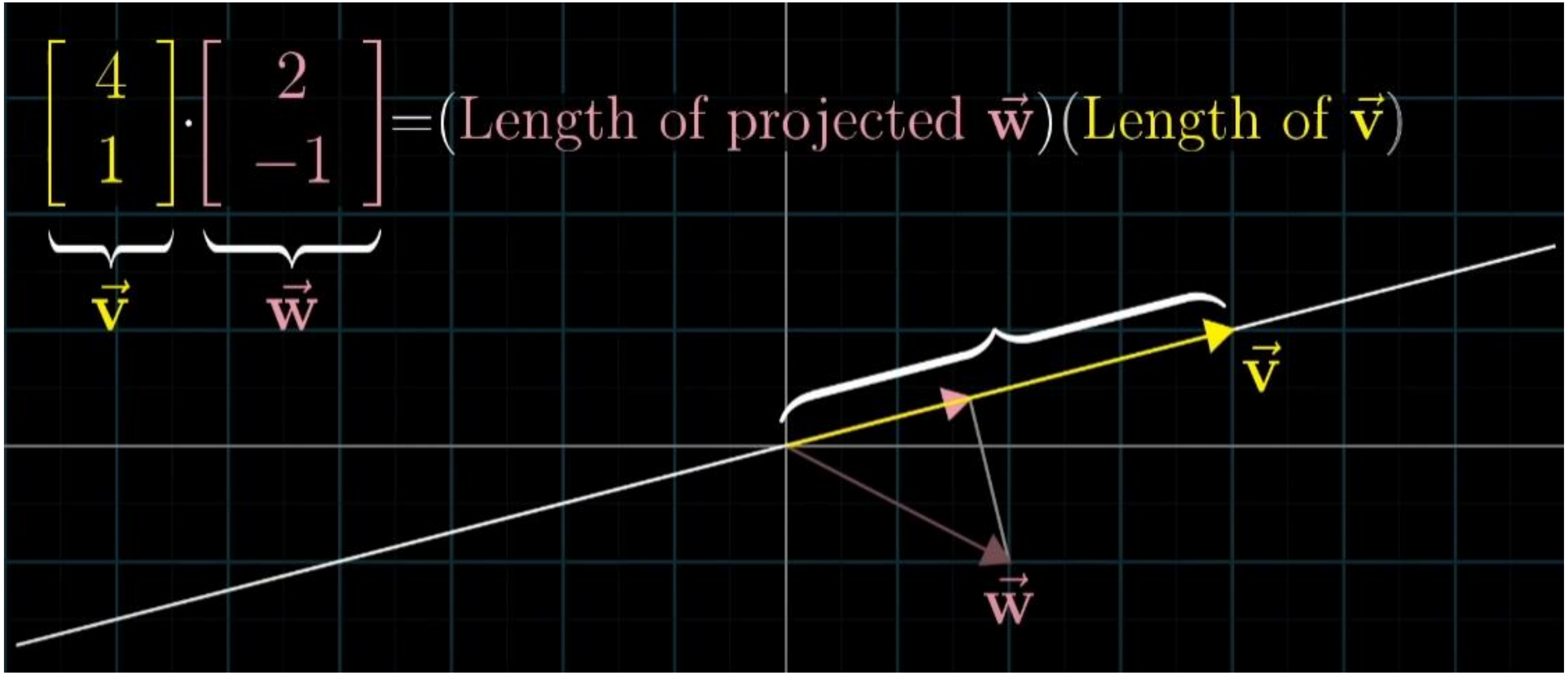
2) 2x + y = 4



3) Distance d

Visual Materials (1)

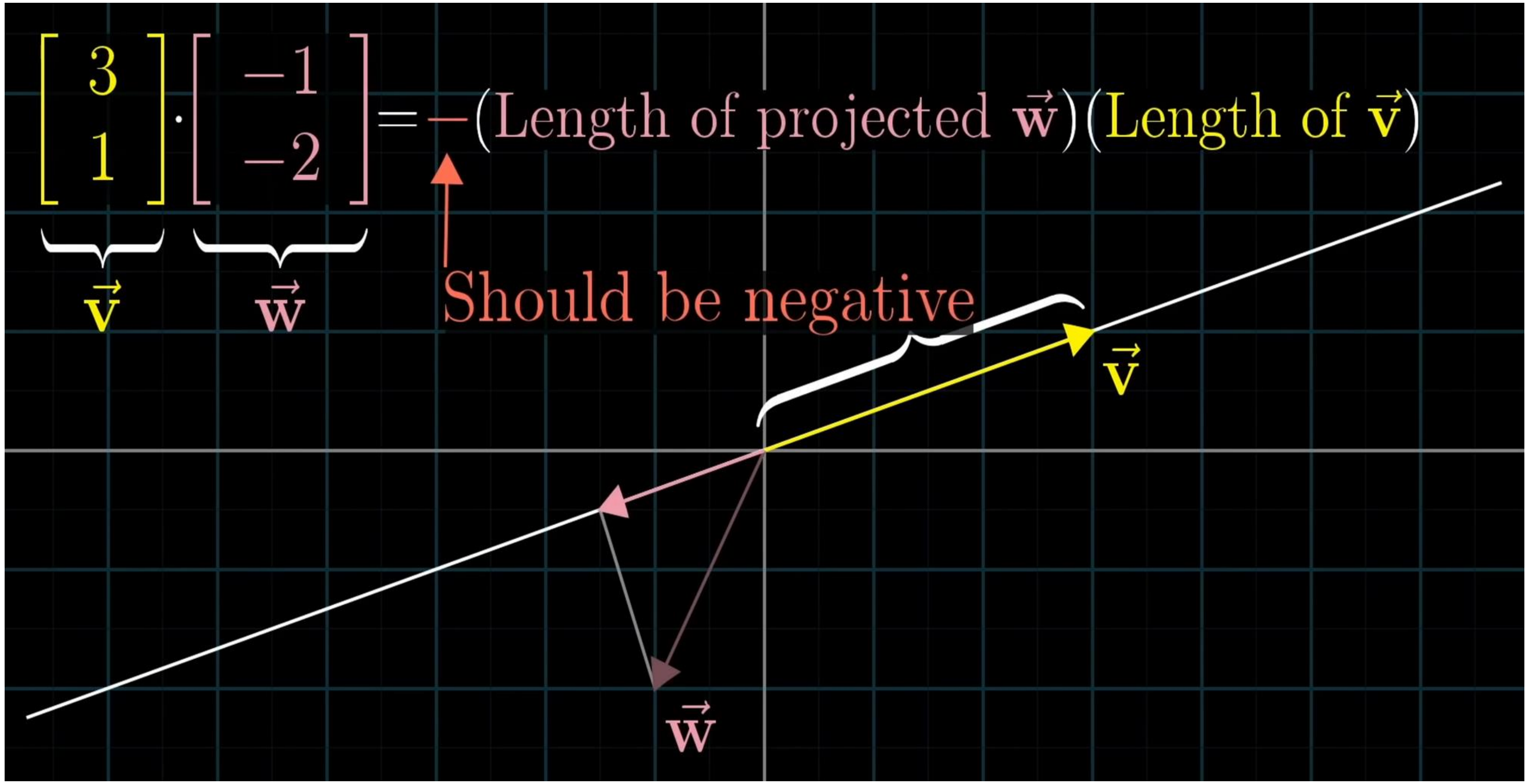
- **Geometric representation of vector dot product with different angles**
  - Dot products, geometric interpretation (0:51 ~ 3:55)
  - https://youtu.be/LyGKycYT2v0?si=kSluHVZr478QXAkE&t=51



Visualization of dot production when angle between v,w is acute

Visual Materials (2)

- **Geometric representation of vector dot product with different angles**
  - Dot products, geometric interpretation (0:51 ~ 3:55)
  - https://youtu.be/LyGKycYT2v0?si=kSluHVZr478QXAkE&t=51



Visualization of dot production when angle between v,w is acute

Other vector multiplications

Definition and Properties of Vector Cross Product

Cross product

- The cross product($\boldsymbol{x} \times \boldsymbol{y}$) of vectors $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$ in the $\mathbb{R}^3$ space is defined as follows.
- $\boldsymbol{x} \times \boldsymbol{y}$ is called ' $\boldsymbol{x}$ cross $\boldsymbol{y}$ '.

$$\boldsymbol{x} \times \boldsymbol{y} = (x_2y_3 - x_3y_2,\ x_3y_1 - x_1y_3,\ x_1y_2 - x_2y_1)$$

Characteristics of cross product

- Characteristics of cross product of $\mathbb{R}^3$ space vector.
- The following properties hold for vector $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$ in $\mathbb{R}^3$ space and scalar c.

(1) $\boldsymbol{x} \times \boldsymbol{y} = -\boldsymbol{y} \times \boldsymbol{x}$

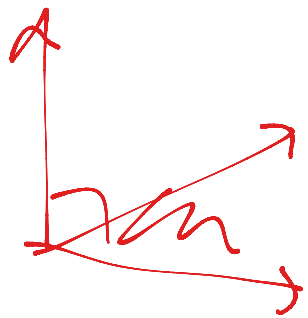
(2) $\boldsymbol{x} \times (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} \times \boldsymbol{y}) + (\boldsymbol{x} \times \boldsymbol{z})$

(3) $(\boldsymbol{x} + \boldsymbol{y}) \times \boldsymbol{z} = (\boldsymbol{x} \times \boldsymbol{z}) + (\boldsymbol{y} \times \boldsymbol{z})$

(4) $c(\boldsymbol{x} \times \boldsymbol{y}) = (c\boldsymbol{x}) \times \boldsymbol{y} = \boldsymbol{x} \times (c\boldsymbol{y})$

(5) $\boldsymbol{x} \times 0 = 0 \times \boldsymbol{x} = 0$

(6) $\boldsymbol{x} \times \boldsymbol{x} = 0$



Geometric Definition of Vector Cross Product

- **Geometric definition of vector cross product**
  - Normal vector of a plane and cross product.
    - Normal vector of a plane can be calculated through the cross product of the vectors corresponding to line segments forming the plane.

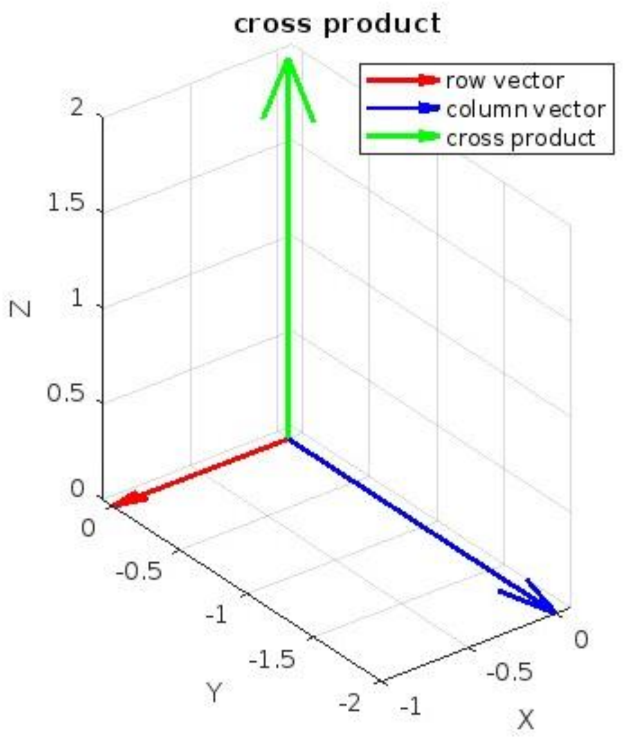
Code Exercise of Vector Cross Product

Code Exercise (02_12)

- Operation cross product between two vectors, one along the column direction and the other along the row direction.

```
% two vectors
row_vector = [-1 0 0];
column_vector = [0; -2; 0];

% cross product
cross_product = cross(row_vector, column_vector);

% result
disp('Cross Product:');
disp(cross_product);

% visualization
figure;
quiver3(0, 0, 0, row_vector(1), row_vector(2), row_vector(3), 'r', 'LineWidth', 2, 'AutoScale', 'off',
'MaxHeadSize', 0.5);
hold on;
quiver3(0, 0, 0, column_vector(1), column_vector(2), column_vector(3), 'b', 'LineWidth', 2, 'AutoScale', 'off',
'MaxHeadSize', 0.5);
quiver3(0, 0, 0, cross_product(1), cross_product(2), cross_product(3), 'g', 'LineWidth', 2, 'AutoScale', 'off',
'MaxHeadSize', 0.5);
legend('row vector', 'column vector', 'cross product');
xlabel('X');
ylabel('Y');
zlabel('Z');
title('cross product');
axis equal;
grid on;
```

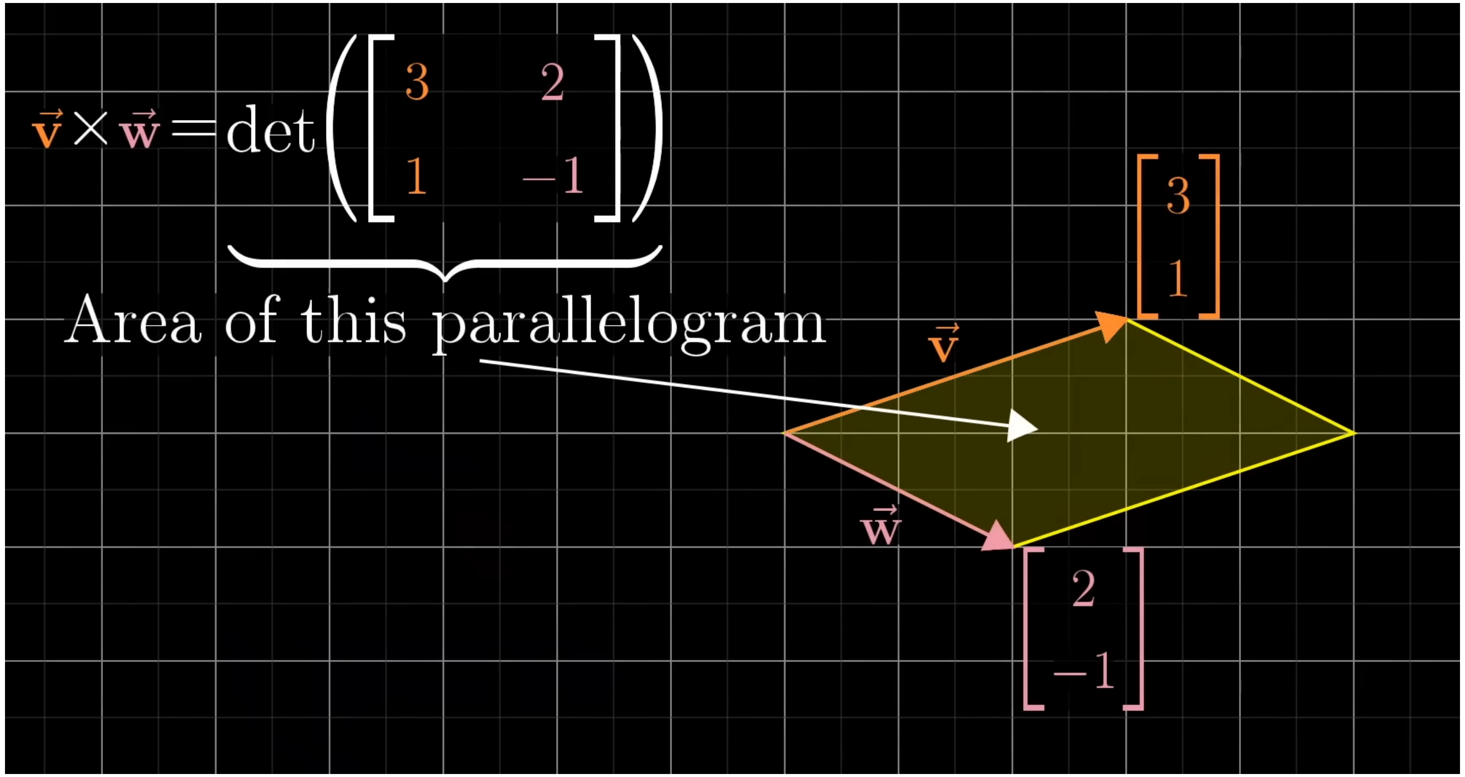
Source code



Source code result

Visual materials

- **Geometric representation of vector cross product**
  - Cross product (0:40 ~)
  - https://youtu.be/eu6i7WJeinw?si=POJURAxWpOe_oQNa&t=40



Visualization of vector cross product in 2-dimensions

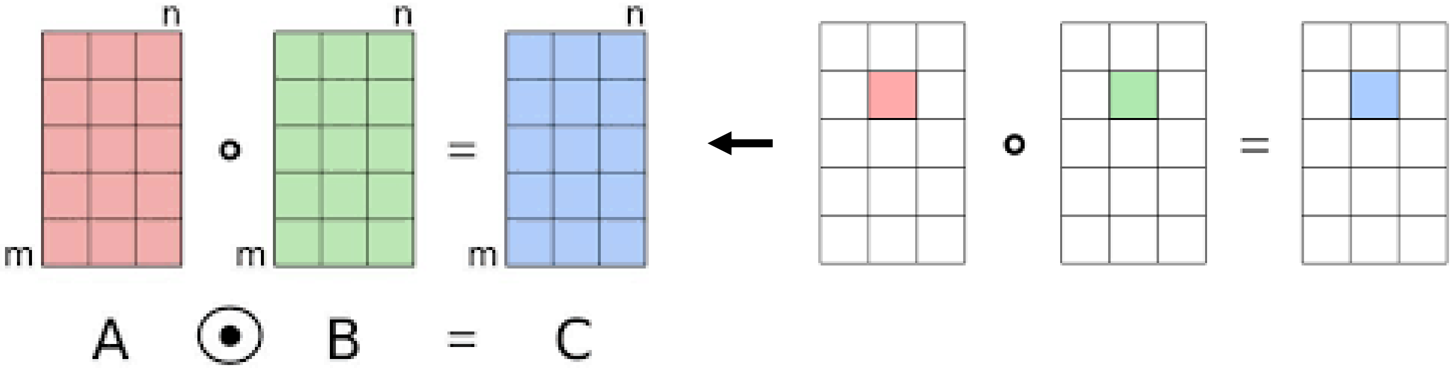
Definition of Hadamard Product

Hadamard product

- Implementation of Hadamard product
  - Operation that multiplies corresponding elements of two vectors of the same size.
    - The result of multiplication is vector of Same dimension with two vectors.
  - The symbol used to denote the Hadamard product is $\odot$.

$$\begin{bmatrix} 5 \\ 4 \\ 8 \\ 2 \end{bmatrix} \odot \begin{bmatrix} 1 \\ 0 \\ .5 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 4 \\ -2 \end{bmatrix}$$

Representation of the Hadamard product (Vector)



Representation of the Hadamard product (Matrix)

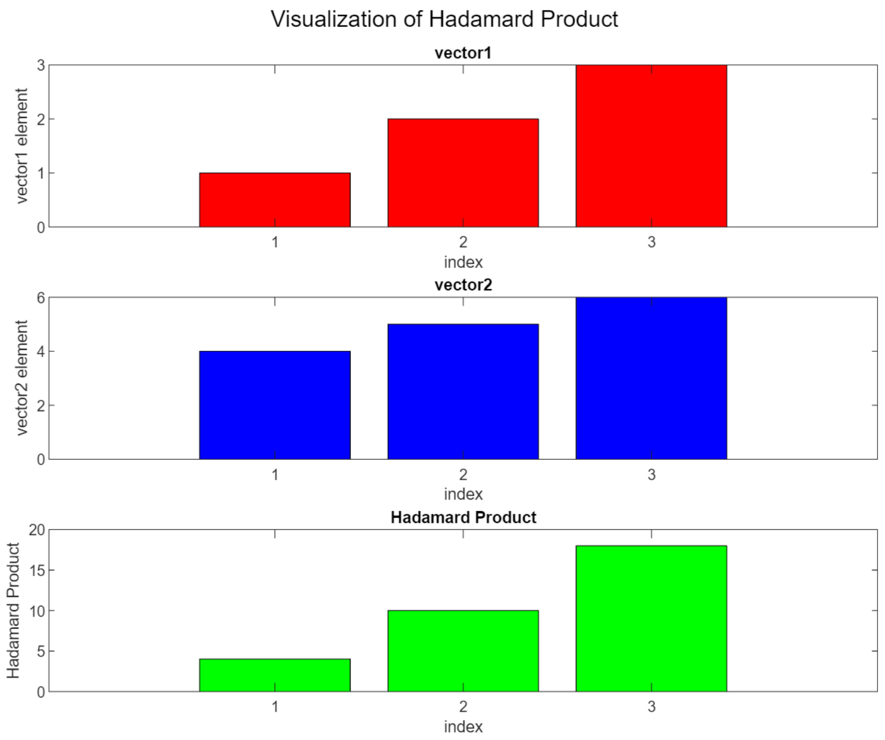
Code Exercise of Hadamard Product using Matlab

Code Exercise (02_13)

- Multiplication between two vectors or matrices.

```
% two vectors
vector1 = [1 2 3];
vector2 = [4 5 6];

% Hadamard product - operator: .*
hadamard_product = vector1 .* vector2;

% plot
subplot(3, 1, 1);
bar(vector1, 'r');
xlabel('index');
ylabel('vector1 element');
title('vector1');

subplot(3, 1, 2);
bar(vector2, 'b');
xlabel('index');
ylabel('vector2 element');
title('vector2');

subplot(3, 1, 3);
bar(hadamard_product, 'g');
xlabel('index');
ylabel('Hardamard Product');
title('Hardamard Product');

sgtitle('visualization of Hardamard Product');
```

Source code



Source code result

Orthogonal vector decomposition

분리

Definition of Orthogonality and Decomposition

Concept of orthogonality

- In mathematics, orthogonality is the generalization of the geometric notion of **perpendicularity**.
- If dot product of two vector is zero, they are Orthogonal.

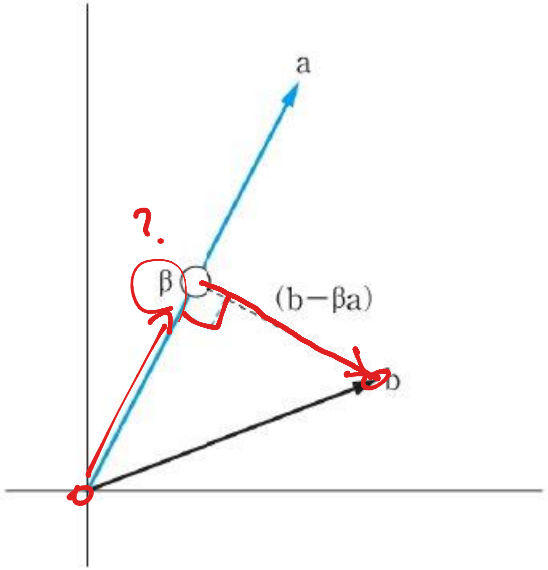
Concept of decomposition

- Scalar decomposition
  - The number 42.01 = 42 + 0.01
  - Prime factorization : decompose the number 42 into the product of the prime number 2, 3 and 7.

- Vector decomposition
  - To decompose a single vector into two vectors, one orthogonal to the reference vector and the other parallel to the reference vector.
    - The orthogonal vector decomposition has direct relevance to statistics in the Gram-Schmidt process and QR decomposition.

Example of Vector Decomposition

- **Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ exist in the standard position.**
- **Search the nearest point from $\boldsymbol{a}$ to the head of $\boldsymbol{b}$.**
  - It can be expressed as an optimization problem, where vector $\boldsymbol{b}$ is projected onto vector a such that the projection distance is Minimalized
  - The point is $\beta\boldsymbol{a}$ that reduces the magnitude of $\boldsymbol{a}$.
  - Find Scalar β.



Vector decomposition visualization

Definition of Orthogonal Projection

Orthogonal projection

- It can be inferred that $\boldsymbol{b} - \beta\boldsymbol{a}$ is orthogonal to $\beta\boldsymbol{a}$.
  - Hence, these vectors are vertical. Therefore, dot product between two vectors should be $\boxed{0}$

$$\boldsymbol{a}^T(\boldsymbol{b} - \beta\boldsymbol{a}) = 0$$

  - Finding β.

$$\boldsymbol{a}^T\boldsymbol{b} - \beta\boldsymbol{a}^T\boldsymbol{a} = 0$$

$$\beta\boldsymbol{a}^T\boldsymbol{a} = \boldsymbol{a}^T\boldsymbol{b}$$

$$\beta = \frac{\boldsymbol{a}^T\boldsymbol{b}}{\boldsymbol{a}^T\boldsymbol{a}}$$

Scalar

Orthogonal projection

Decompose Target Vector and Terminology

'Target vector' and 'Reference vector'

- The goal is to decompose the target vector into two different vectors.
  - Sum of the two vector is the target vector.
  - One **orthogonal** to the reference vector but the other **parallel** to the reference vector.

Terminology clarification

- Target vector is $\boldsymbol{t}$, reference vector is $\boldsymbol{r}$.
- $\boldsymbol{t}_{\perp \boldsymbol{r}}$ is vertical component created from target vector, $\boldsymbol{t}_{\parallel \boldsymbol{r}}$ is parallel component created from target vector.

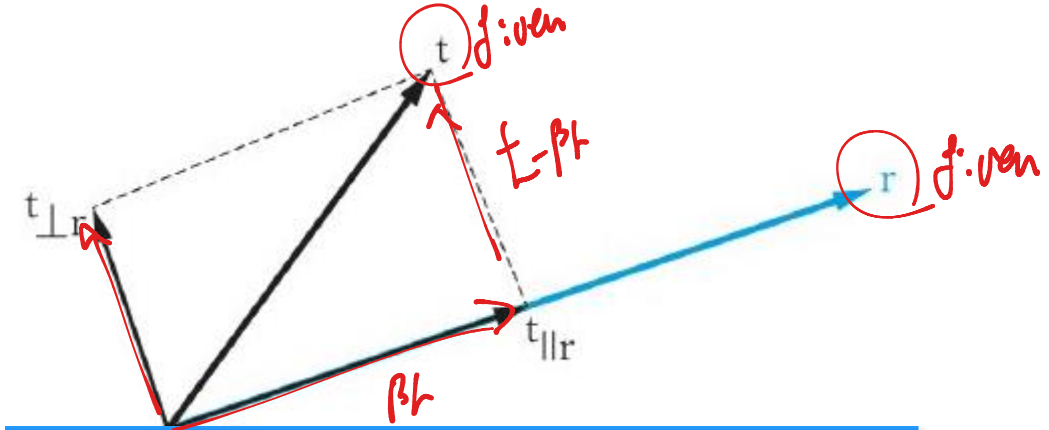


Figure of orthogonal vector decomposition

Parallel Component Generated from Target Vector

Parallel component

- Vector that resizing the size of r is $\boxed{\text{Parallel}}$ to r.
- In Eq 1., only scalar β is calculated. Here, the resized vector β is calculated.
- $\boxed{\text{Sum}}$ of the two vector components is the target vector.

$t_{\parallel r} = \beta \vec{r}$

$t_{\perp r} = t - \beta \vec{r}$

$$\beta = \frac{\boldsymbol{a}^T \boldsymbol{b}}{\boldsymbol{a}^T \boldsymbol{a}}$$

Eq 1. Orthogonal projection

$$\boldsymbol{t} = \boldsymbol{t}_{\perp \boldsymbol{r}} + \boldsymbol{t}_{\parallel \boldsymbol{r}}$$
$$\boldsymbol{t}_{\perp \boldsymbol{r}} = \boldsymbol{t} - \boldsymbol{t}_{\parallel \boldsymbol{r}}$$

Eq 2. Parallel component of target vector

Vertical Component Generated from Target Vector

Vertical component

- Is vertical component really orthogonal to the reference vector?

- Calculate if the dot product between $\boxed{\text{vertical component}}$ and the $\boxed{\text{reference vector}}$ is 0.
  - Prove it!

matlab 해보기.

$$(\boldsymbol{t}_{\perp \boldsymbol{r}})^T \boldsymbol{r} = \boldsymbol{0}$$

$$\left(\boldsymbol{t} - \boldsymbol{r}\frac{\boldsymbol{t}^T\boldsymbol{r}}{\boldsymbol{r}^T\boldsymbol{r}}\right)^T \boldsymbol{r} = \boldsymbol{0}$$

dot product of perpendicular component and reference vector

Summary

Summary

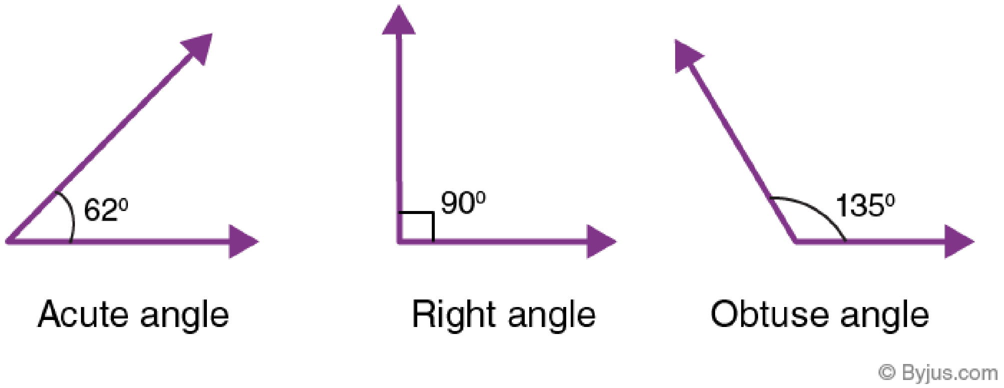
- **Vector is a list of numbers arranged in a** $\boxed{\text{Column}}$ **or** $\boxed{\text{row}}$
  - The number of elements in a vector is called its $\boxed{\text{dimension}}$, and vector can be represented as a single line in a geometric space with the same number of axes as its dimension.

- **Vector arithmetic operations such as addition, minus and Hadamard product are calculated** $\boxed{\text{element wise}}$**.**

- **The dot product is calculated by multiplying corresponding elements of two vectors of the same** $\boxed{\text{dimension}}$ **and summing them up, resulting in a single number encoding the relationship between the two vectors.**

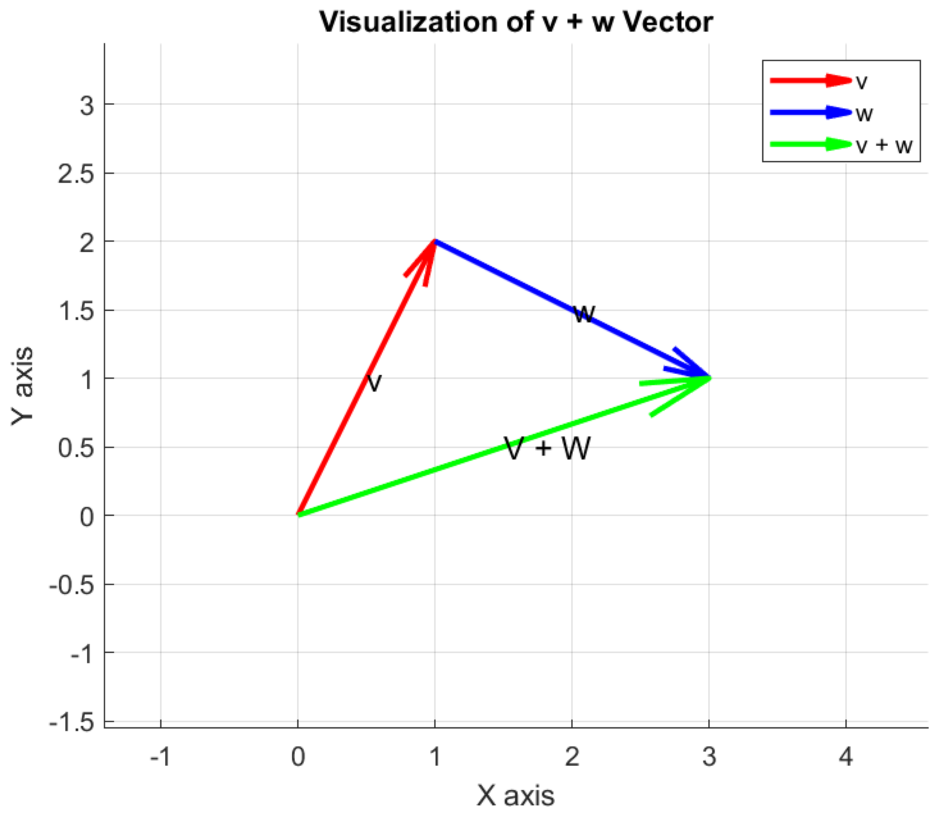
Summary

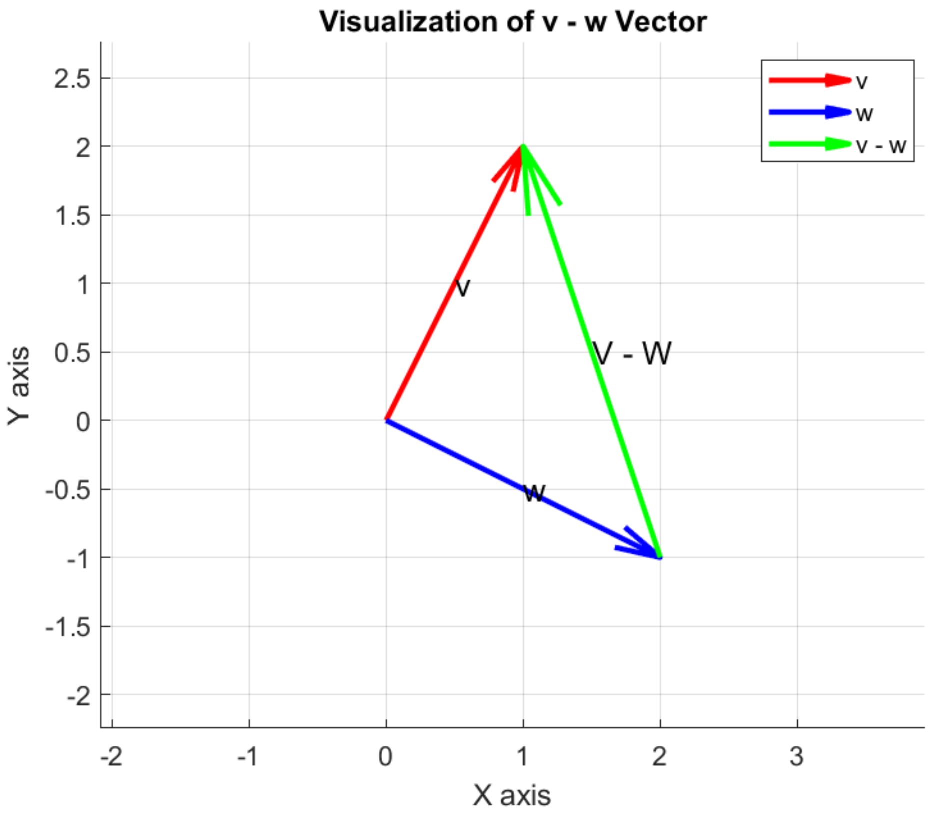
- If the two vectors orthogonal, the result of dot product is 0 and that means geometrically that the vectors meet at right angles
- Orthogonal vector decomposition is dividing one vector to reference vector, Vertical vector and Parallel vector.
- Decomposition equation can be derived geometrically, but one must remember the phrase 'mapping to size', a concept implied by the equation.



Exercise

1. Write the code that creates figure.





Exercise

2. Implement a function that takes a vector as input and outputs a unit vector in the same direction.

Exercise

3. Write the for loop that transposes row vector to column vector without using built-in functions (e.g., A.T).

THANK YOU
FOR YOUR ATTENTION