Linear Algebra

Row Reduction and LU Decomposition

Automotive Intelligence Lab.

Contents

Simultaneous equations and matrix

Solution of Simultaneous Equations

- **Think about solution of simultaneous equations as Eq 1..**
- **To solve simultaneous equations, one $\boxed{}$ must be eliminated from either upper or lower equation.**
- **Let's multiply upper equation by 2 and subtract it from lower equation in Eq 1..**
  - Upper equation: r_1, lower equation: r_2
  - $r_2 \rightarrow r_2 - 2r_1$ as Eq 2..
  - In this process, we can know that $\boldsymbol{y} = \mathbf{1}$.
  - By substituting $y = 1$ into upper equation, we can know that $\boldsymbol{x} = -\mathbf{1}$.

$$\begin{cases} 2x + 3y = 1 \\ 4x + 7y = 3 \end{cases}$$

Eq 1. Example of simultaneous equations

$$4x + 7y - 2(2x + 3y) = 3 - 2 \times 1 = 1$$

$$\Rightarrow (4x - 4x) + (7y - 6y) = y = 1$$

$$2x + 3(1) = 1 \Rightarrow x = -1$$

Eq 2. Process of solving simultaneous equations

Method to Solve Simultaneous Equations

- **We can know two methods to solve simultaneous equations.**
  - Multiplying both sides of an equation by number.
  - Combining two equations.

- **Additional technique to solve simultaneous equations.**
  - $\boxed{}$ the order of two equations.

- **In summary, there are three skills for solving simultaneous equations.**
  1. Multiplying both sides of an equation by scalar number.
  2. Combining two equations.
  3. Swapping the order of two equations.

Representation of Simultaneous Equations using Matrix

- **Simultaneous equations in Eq 1. can be expressed as matrix in Eq 2..**

- **Eq 2. can be expressed as augmented matrix in Eq 3..**
  - In augmented matrix, it can be treated like regular 2×3 matrix.
  - Long vertical bars are just auxiliary lines for visual aid.

- **In conclusion, we can solve simultaneous equations.**
  - By treating each 'row' of this augmented matrix as if it were a **single equation**.

$$\begin{cases} 2x + 3y = 1 \\ 4x + 7y = 3 \end{cases}$$

Eq 1. Example of simultaneous equations

$$\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Eq 2. Representation of simultaneous equations as matrix

$$\boldsymbol{A} = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \quad \boldsymbol{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$[\boldsymbol{A}|\boldsymbol{b}] = \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 7 & 3 \end{array}\right]$$

Eq 3. Representation of simultaneous equations as augmented matrix

Why Represent Simultaneous Equations as Matrix?

- **Isn't it more complicated?**
  - Attempting to solve simultaneous equations using $\boxed{}$.

- **To do this, calculations corresponding to "three methods" for finding solution described below must be able to be expressed on computer.**
  - Multiplying both sides of an equation by scalar number.
  - Combining two equations.
  - Swapping the order of two equations.

- **In another view,**
  - When there are two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, the operation of multiplying $\boldsymbol{A}$ and $\boldsymbol{B}$ involves $\boldsymbol{A}$ performing the operation and $\boldsymbol{B}$ functioning as the operand object that receives operation.
  - Also in case of $[\boldsymbol{A}|\boldsymbol{b}]$ as mentioned in before page, we can consider operation matrix that three skills of simultaneous equations mentioned above.
    - By multiplying operation matrix front of $[\boldsymbol{A}|\boldsymbol{b}]$ matrix, operations can be performed on the rows of the addition matrix.
    - We called this operation matrix as $\boxed{}$

Elementary Matrix

- **There are a total of three elementary row operations.**
  - 1. Row multiplication
  - 2. Row switching
  - 3. Row addition

- **Since size of augmented matrix to be multiplied later is $m \times n$,**
  - Size of elementary matrix should be $m \times m$.
  - Size of matrix may remain the same.

- **Obtain with some modification on identity matrix $\boldsymbol{I_m}$ of size $m \times m$.**
  - Change a single number in matrix $\boldsymbol{I_m}$ as below.
  - Manipulate the order of rows in matrix $\boldsymbol{I_m}$.

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example case of changing a single number for row operations

1. Row Multiplication

- **Elementary matrix that performs row multiplication**
  - Matrix that changed number of one of $\boxed{}$ elements of identity matrix as Eq 1.
- **In matrix $\boldsymbol{E}$ in Eq 1.,**
  - The second diagonal component was modified to constant s.
    - Results in operation that takes constant multiple in second row.
  - If indicated with symbol: $r_2 \rightarrow sr_2$
  - If you perform matrix operation on random 3×4 matrix $\boldsymbol{A}$, following operations are Eq 2..

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eq 1. Elementary matrix that performs row multiplication

$$\boldsymbol{EA} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ sa_{21} & sa_{22} & sa_{23} & sa_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Eq 2. Example of row multiplication

Inverse Operation of Row Multiplication

- **Inverse operation for row multiplication is to perform it ☐ times again.**
  - Inverse operation for operation that multiplies a row by s is Eq 1..

- **This represents the inverse of the elementary matrix corresponding to row multiplication.**

$$\boldsymbol{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eq 1. Example of inverse operation of row multiplication

$$\boldsymbol{E}\boldsymbol{E}^{-1} = \boldsymbol{E}^{-1}\boldsymbol{E} = I$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eq 2. Inverse operation of row multiplication

Code Exercise of Row Multiplication

Code Exercise (10_01)

- You need '**Symbolic Math Toolbox**' to run this code.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Include the symbolic toolbox
syms s;

% Define a 3x3 diagonal matrix E with diagonal entries 1, s, 1
% Use 'diag()' function
E = ;

% Calculate the inverse of matrix E
E_inv = ;

% Calculate the product of E and its inverse
product = ;

% Display the matrix E, its inverse, and their product
disp('Matrix E:');
disp(E);
disp('Inverse of Matrix E:');
disp(E_inv);
disp('Product of E and E_inv (should be the identity matrix):');
disp(product);
```

MATLAB code of row multiplication

2. Row Switching

- **Matrix that changed $\boxed{}$ of identity matrix**

- **If you want to perform operation that switch row 3 and 2, operate as Eq 1..**

- **Permutation matrix**
  - Among elementary matrices, matrix that performs row switching.
  - If indicated with symbol,
    - $\boldsymbol{P}$: Permutation
    - Number of two rows to be replaced are conventionally written, such as $\boldsymbol{P}_{ij}$.
      - To specify which two rows you want to change the order of.

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \boldsymbol{P}_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Eq 1. Elementary matrix that performs row switching

Example of Row Switching

- **If you perform matrix operation on random 3×4 matrix A,**
  - Following operations are performed as Eq 1..

$$\boldsymbol{P}_{32}\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix}$$

Eq 1. Example of row switching

Inverse Operation of Row Switching

- **Inverse matrix of elementary matrix (or permutation matrix) is $\boxed{}$**
  - This may seem pretty obvious.
  - All you have to do is swap lines 1 and 3 again.
    - To reverse the operation that swaps lines 1 and 3.

$$\boldsymbol{P}_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{P}_{31}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Example of inverse operation of row switching

Code Exercise of Row Switching

Code Exercise (10_02)

- You need '**Symbolic Math Toolbox**' to run this code.
- You can change the permutation matrix.
  - How about changing permutation matrix to P_{31}?

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Include the symbolic toolbox
syms a11 a12 a13 a14 a21 a22 a23 a24 a31 a32 a33 a34;

% Define a 3x4 symobolic matrix A
A = [a11 a12 a13 a14; a21 a22 a23 a24; a31 a32 a33 a34];

% Define the permutation matrix P32
P32 = ;

% Calculate the product of P32 and A
product = ;

% Display the matrix A, permutation matrix P32, and their product
disp('Matrix A:');
disp(A);
disp('Matrix P32');
disp(P32);
disp('Product of P32 and A');
disp(product);
```

MATLAB code of row switching

3. Row-Addition Matrix

■ Operation that $\boxed{}$ different rows

▶ You must also perform process of replacing added result in certain row as Eq 1..

- Convert row 2 to row 2 plus s times row 1.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} + sa_{11} & a_{22} + sa_{12} & a_{23} + sa_{13} & a_{24} + sa_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Eq 1. Example of row-addition matrix

Example of Row-Addition Matrix

- **Let's consider matrix E that performs the operation below.**
  - If matrix before the operation is called A,
    - It can be thought as Eq 1..

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow E = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} + sa_{11} & a_{22} + sa_{12} & a_{23} + sa_{13} & a_{24} + sa_{14} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

Eq 1. Example of row-addition matrix

Code Exercise of Row-Addition Matrix

Code Exercise (10_03)

- You need '**Symbolic Math Toolbox**' to run this code.
- You can change the matrix E.
  - For example, change the matrix E to $\begin{bmatrix} 1 & s & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Include the symbolic toolbox
syms s a11 a12 a13 a14 a21 a22 a23 a24 a31 a32 a33 a34;

% Define a 3x4 symobolic matrix A
A = [a11 a12 a13 a14; a21 a22 a23 a24; a31 a32 a33 a34];

% Define the permutation matrix E
E = ;

% Calculate the product of E and A
product = ;

% Display the matrix A, E, and their product
disp('Matrix A:');
disp(A);
disp('Matrix E');
disp(E);
disp('Product of E and A');
disp(product);
```

MATLAB code of row-addition

Meaning of Row-Addition Matrix

- **Let's consider how matrix $\boldsymbol{E}$ performs row-addition operations.**
- **First, operation affects each row of output matrix.**
  - Operation performed using each row of matrix $\boldsymbol{E}$.
- **In Eq 1.,**
  - Each row of matrix multiplied on left affects each row of output matrix.
  - Also indicates how much $\boxed{}$ to give to each row of operated matrix.

weighting $= 1$ **for row 2**

weighting $= 0$ **for row 3**

weighting $= s$ **for row 1**

$$\begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{bmatrix}$$

Eq 1. Affect of each row of matrix

Result of Row-Addition Matrix

- **Therefore, when row addition operation is performed on output matrix, Eq 1. occurs.**

$$\boldsymbol{s} \times (\boldsymbol{a_{11}}\ \ \boldsymbol{a_{12}}\ \ \boldsymbol{a_{13}}\ \ \boldsymbol{a_{14}})$$
$$+\boldsymbol{1} \times (\boldsymbol{a_{21}}\ \ \boldsymbol{a_{22}}\ \ \boldsymbol{a_{23}}\ \ \boldsymbol{a_{24}})$$
$$\rightarrow \boldsymbol{r_2} \text{ of output matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} - & r_1 & - \\ - & r_2 & - \\ - & r_3 & - \end{bmatrix}$$

Eq 1. Affect of each row of matrix

Result of Row-Addition Matrix

- **Using row-addition operation,**
  - You can erase specific element to $\mathbf{0}$ as Eq 1..

- **You can use elementary matrix E to substitute $\boldsymbol{row\,2 \;=\; row\,2 \;-\; row\,1}$.**
  - To make the first element in row 2 of matrix $\boldsymbol{A}$ to $\mathbf{0}$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & -2 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

$$\boldsymbol{EA} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & -2 & 2 \\ 0 & 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -5 & -2 \\ 0 & 3 & 2 & 1 \end{bmatrix}$$

Eq 1. Example of row-addition matrix

Inverse Operation of Row-Addition Matrix

- **Multiply minus to s and add again.**

$$\boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boldsymbol{E}^{-1}\boldsymbol{E} = \boldsymbol{I}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example of inverse operation of row-addition matrix

Code Exercise of Inverse of Row-Addition Matrix

Code Exercise (10_04)

- You need '**Symbolic Math Toolbox**' to run this code.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Include the symbolic toolbox
syms s;

% Define the matrix E
E = ;

% Define the inverse matrix of E
E_inv = ;

% Calculate the product of E_inv and E
product = ;

% Display the matrix E ,its inverse and the product of them
disp('Matrix E:');
disp(E);
disp('Matrix E_inv');
disp(E_inv);
disp('Product of E_inv and E');
disp(product);
```

MATLAB code of row-addition

Solving Simultaneous Equations using Elementary Matrix

- **Let's solve simultaneous equations.**
  - Use elementary matrix and check results by implementing it directly in MATLAB.
- **Eq 1. can be represented in the form of a matrix, like Eq 2..**
- **To remove term related to $\square$ in the second equation, perform Eq 3..**

$$\begin{cases} 2x + 3y = 1 \\ 4x + 7y = 3 \end{cases}$$

Eq 1. Example of simultaneous equations

$$[\boldsymbol{A}|\boldsymbol{b}] = \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 7 & 3 \end{array}\right]$$

Eq 2. Representation of simultaneous equations as augmented matrix

$$r_2 \rightarrow r_2 - 2r_1$$

Eq 3. Process of solving simultaneous equations

Solving Simultaneous Equations using Augmented Matrix

- **Let's multiply augmented matrix.**

$$E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$\Rightarrow E_1[A|b] = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 7 & 3 \end{array}\right]$$

$$= \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

Process of solving simultaneous equations using augmented matrix

Solving Simultaneous Equations using Elementary Matrix

- Let's perform following operation to remove second element 3 of first row as Eq 1..
- To do this, let's multiply elementary matrix as Eq 2..

$$r_1 \rightarrow r_1 - 3r_2$$

Eq 1. Process of solving simultaneous equations

$$\boldsymbol{E}_2 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix} \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 2 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right]$$

Eq 2. Multiply elementary matrix

Result of Simultaneous Equations

- Lastly, let's multiply the first row by $1/2$.
- To do this, let's multiply elementary matrix as Eq 1..
- Therefore, it can be confirmed through final augmented matrix.
  - ▶ $x = \boxed{}, y = \boxed{}$

$$\boldsymbol{E}_3 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \left[\begin{array}{cc|c} 2 & 0 & -2 \\ 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

Eq 1. Multiply elementary matrix

Summary of using Elementary Matrix

- **If you think about this process carefully,**
  - You can see that result can be obtained.
    - By using elementary matrices $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, and $\boldsymbol{E}_3$ in order as Eq 1..
- **With computer,**
  - Represent them into operations and equations.
  - Obtain solutions with simple coding as Fig 1..

$$\boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1[\boldsymbol{A}|\boldsymbol{b}] = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 7 & 3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 1 \end{array}\right]$$

Eq 1. Result via elementary matrix

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix E1, E2, E3 and augmented A
E3 = [0.5 0; 0 1];
E2 = [1 -3; 0 1];
E1 = [1 0; -2 1];
augmented_A = [2 3 1; 4 7 3];

% Calculate the multiplication
result = E3*E2*E1*augmented_A;

% Display the result
disp("result")
disp(result);
```

```
result
     1     0    -1
     0     1     1
```

Fig 1. Solving simultaneous equations using elementary matrix

LU Decomposition

Introduction of Triangular Matrix

- **Matrix in which values of terms above or below diagonal elements are all 0.**

- **Lower triangular matrix**
  - Matrix whose $\boxed{}$ diagonal terms are all *0*.
- **Upper triangular matrix**
  - Matrix whose $\boxed{}$ diagonal terms are all *0*.

$$\boldsymbol{L} = \begin{bmatrix} l_{1,1} & 0 & 0 & 0 & 0 \\ l_{2,1} & l_{2,2} & 0 & 0 & 0 \\ l_{3,1} & l_{3,2} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ l_{n,1} & l_{n,2} & \cdots & l_{n,n-1} & l_{n,n} \end{bmatrix}$$

Lower triangular matrix

$$\boldsymbol{U} = \begin{bmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \cdots & u_{1,n} \\ 0 & u_{2,2} & u_{2,3} & \cdots & u_{2,n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & u_{n-1,n} \\ 0 & 0 & 0 & 0 & u_{n,n} \end{bmatrix}$$

Upper triangular matrix

Remind Solving Simultaneous Equations

- **Elementary matrix**
  - Row multiplication
  - Row switching
  - Row addition

- **If we think again about process of obtaining solution through elementary row operations,**
  - Equation at bottom leaves only expression for last unknown.
  - Equation above it leaves only last two unknowns, thereby eliminating the unknowns.
  - Obtain value of last unknown from bottom equation.
  - Substitute into equation above it to obtain value of next unknown.
  - You can see that it is possible to obtain values of unknowns one by one in this order.

- **This process is called back substitution.**
  - Because it calculates from last unknown to first unknown.

$$\begin{cases} x+y+z=6 \\ 2x+3y-z=5 \\ 2x+3y+3z=17 \end{cases} \Rightarrow \begin{array}{l} r_2 \to r_2-2r_1 \\ r_3 \to r_3-2r_1 \\ r_3 \to r_3-r_2 \end{array} \Rightarrow \begin{cases} x+y+z=6 \\ 0x+y-3z=-7 \\ 0x+0y+4z=12 \end{cases} \Rightarrow \begin{array}{l} z=3 \\ \Rightarrow r_2 \to y=2 \\ \Rightarrow r_1 \to x=1 \end{array}$$

Back substitution

Simultaneous Equations represented as Matrix

- **Let's find solution using matrix.**
  - Perform elementary row operations.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & -1 & 5 \\ 2 & 3 & 3 & 17 \end{array}\right]$$

$$r_2 \to r_2 - 2r_1$$
$$r_3 \to r_3 - 2r_1$$
$$r_3 \to r_3 - r_2$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 4 & 12 \end{array}\right]$$

Representation of simultaneous equations as augmented matrix

Back Substitution represented as Matrix

- **If elementary row operations are expressed using elementary matrices,**
  - They can be summarized as Eq 1..

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{r_3 \rightarrow (r_3 - r_2)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}_{r_3 \rightarrow (r_3 - 2r_1)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{r_2 \rightarrow (r_2 - 2r_1)} \underbrace{\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & 3 & -1 & 5 \\ 2 & 3 & 3 & 17 \end{array}\right]}_{[\boldsymbol{A}|\boldsymbol{b}]} = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 4 & 12 \end{array}\right]$$

Eq 1. Perform back substitution with final result obtained through elementary matrix operations

Convert Coefficient Matrix A into Upper Triangular Matrix

- **Let's try something a little different by applying this idea.**
  - If we multiply elementary matrix in the same way for matrix $\boldsymbol{A}$,
    - Only has equation coefficients instead of $[\boldsymbol{A}|\boldsymbol{b}]$.
  - We can obtain form without augmenting matrix on right side.
    - The result will be in form of $\boxed{}$ triangular matrix introduced earlier.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}}_{r_3 \to (r_3 - r_2)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}_{r_3 \to (r_3 - 2r_1)} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{r_2 \to (r_2 - 2r_1)} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix}}_{\boldsymbol{A}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

Convert coefficient matrix $\boldsymbol{A}$ into upper triangular matrix through elementary matrix operations

Multiply by Inverse of Elementary Matrix

- **If you multiply inverse matrices of elementary matrices that were multiplied in front of coefficient matrix A in order,**
  - You can rewrite matrix A as follows.

$$r_3 \rightarrow (r_3 - r_2) \qquad r_2 \rightarrow (r_2 - 2r_1)$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$r_3 \rightarrow (r_3 - 2r_1)$$

Multiplying inverse matrix of elementary matrix

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

Multiplying inverse matrix of elementary matrix

Inverse of Elementary Matrix

- **Inverse matrices of elementary matrices have very simple form.**
  - Row multiplication as Eq 1.
  - Row addition as Eq 2.
  - Elementary matrix that changes order of rows as Eq 3.

$$\boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \boldsymbol{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/s & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eq 1. Relationship between row multiplication matrix and its inverse matrix

$$\boldsymbol{E} = \begin{bmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \boldsymbol{E}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eq 2. Relationship between elementary matrix that performs row addition and its inverse matrix

$$\boldsymbol{E} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \boldsymbol{E}^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Eq 3. Relationship between elementary matrix and its inverse matrix that performs function of changing order of rows

LU Decomposition

- **If calculate inverse matrices and combine them into one matrix through matrix multiplication,**
  - They can be combined into $\boxed{}$ triangular matrix as shown in equation below.

$$\overset{\boldsymbol{A}}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \underset{\boldsymbol{L}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}} \underset{\boldsymbol{U}}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}}$$

Matrix A represented as product of lower triangular matrix and upper triangular matrix

Example of LU Decomposition

Code Exercise (10_05)

- Function for LU decomposition is in MATLAB.

$$\begin{bmatrix} 2 & 2 & 4 \\ 1 & 0 & 3 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ {}^{1}/_{2} & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Numerical example of LU decomposition

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix A
A = [2 2 4; 1 0 3; 2 1 2];

% LU decomposition
[L,U] = ;

% Display the results
disp("L")
disp(L);
disp("U")
disp(U);
```

MATLAB code of LU decomposition

Use of Permutation Matrix

- **For some matrices, LU decomposition may not be possible without row swap.**
- **Consider LU decomposition, which also includes row swap operations.**
  - Consider matrix $\boldsymbol{A}$ as shown below.
  - By using only row addition or row scaling among elementary lower triangular matrices, the final output of this type of matrix cannot be upper triangular matrix because first and second elements in first row are already set to 0.
  - Therefore, rows of $\boldsymbol{A}$ must be changed and started to be able to use only elementary matrix corresponding to row addition and row scaling of lower triangular matrix.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & -1 \end{bmatrix}$$

Example of matrix A

LU Decomposition including Row Operations

- **First, let's replace rows 1 and 3 and then consider LU decomposition.**
  - To achieve this, multiply matrix $\boldsymbol{P}_{13}$ to matrix $\boldsymbol{A}$.

$$\boldsymbol{P}_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{P}_{13}\boldsymbol{A} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Result of $P_{13}A$

Result of LU Decomposition including Row Operations

- **Perform $\boldsymbol{r_2} \rightarrow \boldsymbol{r_2} - (^1/_2)\boldsymbol{r_1}$.**
  - Result is an upper triangular matrix.
- **Thus, consider that you can take inverse matrix of elementary row operations and write it as follows.**

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1/2 & 3/2 \\ 0 & 0 & 3 \end{bmatrix}$$

Result is an upper triangular matrix

$$\boldsymbol{P}_{13}\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -1/2 & 3/2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{L}} \underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 0 & -1/2 & 3/2 \\ 0 & 0 & 3 \end{bmatrix}}_{\boldsymbol{U}} = LU$$

Representation of LU decomposition

PLU Decomposition

- **When performing LU decomposition by changing $\boxed{}$ order of matrix A to be decomposed in advance.**
- **Since inverse matrix of row permutation matrix is $\boxed{}$**
  - Original coefficient matrix A can be decomposed as follows.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 1 & 1 \\ 2 & 3 & -1 \end{bmatrix}$$

$$\boldsymbol{P}_{13}\boldsymbol{L}\boldsymbol{U} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{\boldsymbol{P}} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{L}} \underbrace{\begin{bmatrix} 2 & 3 & -1 \\ 0 & -1/2 & 3/2 \\ 0 & 0 & 3 \end{bmatrix}}_{\boldsymbol{U}}$$

Result of decomposition of coefficient matrix A

Code Exercise of PLU decomposition

Code Exercise (10_06)

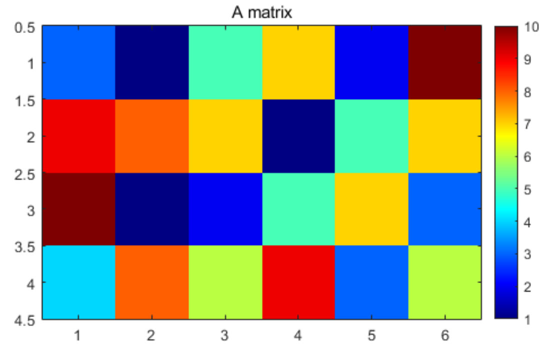
- Use MATLAB function $lu()$.

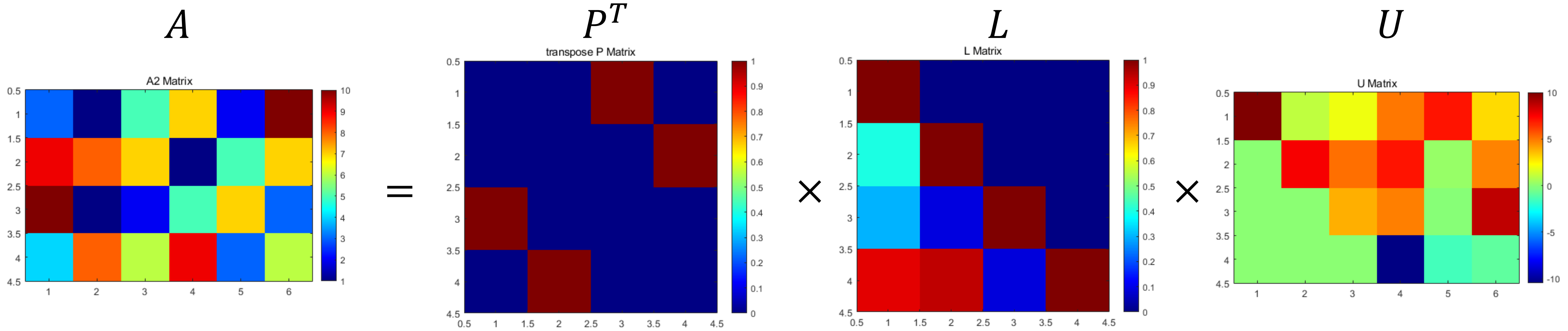
```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix A
A = [3 1 5 7 2 10;
     9 8 7 1 5 7;
     10 1 2 5 7 3;
     4 8 6 9 3 6];

% LU decomposition
[L,U,P] = ;

% Verify the equality of A and transpose(P)*L*U
A2 = ;

% Visualize the results
figure;
imagesc(A); % Display the matrix as a color image
title('A matrix');
colorbar; % Show a color scale
colormap jet; % Use the jet color map
axis equal tight;  % Adjust axes to fit the data

figure;
imagesc(P');
title('transpose P Matrix');
colorbar;
colormap jet;
axis equal tight;

figure;
imagesc(L);
title('L Matrix');
colorbar;
colormap jet;
axis equal tight;

figure;
imagesc(U);
title('U Matrix');
colorbar;
colormap jet;
axis equal tight;

figure;
imagesc(A2);
title('A2 Matrix');
colorbar;
colormap jet;
axis equal tight;
```

MATLAB code to verify LU decomposition with permutation matrix

Visualization of results of PLU decomposition

A





Visualization results of LU decomposition with permutation matrix

Use of LU Decomposition

- **Find solution to $\boldsymbol{Ax} = \boldsymbol{b}$.**
  - If A is square matrix and can be decomposed as $\boldsymbol{A} = \boldsymbol{LU}$,
    - You can think about it as follows.
      - $\boldsymbol{Ux}$ can also be thought of as a kind of column vector.
      - Therefore, replace it with column vector named $\boldsymbol{Ux} = \boldsymbol{c}$.
    - It becomes the same problem as $\boldsymbol{Lc} = \boldsymbol{b}$.

$$
\begin{aligned}
&\boldsymbol{Ax} = \boldsymbol{b} \\
\Rightarrow\ &(\boldsymbol{LU})\boldsymbol{x} = \boldsymbol{b} \\
\Rightarrow\ &\boldsymbol{L}(\boldsymbol{Ux}) = \boldsymbol{b} \\
\Rightarrow\ &\boldsymbol{Lc} = \boldsymbol{b}
\end{aligned}
$$

Using LU decomposition to solve $\boldsymbol{Ax} = \boldsymbol{b}$

Characteristic of LU Decomposition

- **However, if you think about it carefully,**
  - $\boldsymbol{L}$ is lower triangular matrix.
  - Solution for lower triangular matrix can be easily obtained.
    - By using $\boxed{}$ substitution.
- **Then we solve problem as $\boldsymbol{Ux = c}$, we will get answer to $\boldsymbol{x}$.**
  - In this case, solution can be easily obtained.
    - By using $\boxed{}$ substitution.

Example of LU Decomposition

- **LU decomposition for matrix $\boldsymbol{A}$ is as Eq 1..**
- **In $\boldsymbol{Ax} = \boldsymbol{b}$,**
  - If $\boldsymbol{b}$ is $[\mathbf{6}, \mathbf{5}, \mathbf{17}]^T$, $\boldsymbol{LUx} = \boldsymbol{b}$ is Eq 2..

$$\boldsymbol{A} = \boldsymbol{LU}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

Eq 1. LU decomposition for matrix $\boldsymbol{A}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 17 \end{bmatrix}$$

Eq 2. Substitute LU decomposition

Example of Shaping LU Decomposition

- **If $\boldsymbol{LUx} = \boldsymbol{b}$ is changed to $\boldsymbol{Lc} = \boldsymbol{b}$,**
  - It becomes Eq 1..
  - Then, we can easily know that $c_1 = 6, c_2 = -7, c_3 = 12$.
  - Consider that additional problem we need to solve is $\boldsymbol{Ux} = \boldsymbol{c}$.
    - Using Eq 2. and back-substitution, we can see that $z = 3, y = 2$, and $x = 1$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 17 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 17 \end{bmatrix}$$

Result of $\boldsymbol{LUx} = \boldsymbol{b}$

$$\boldsymbol{Ux} = \boldsymbol{c} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 12 \end{bmatrix}$$

Result of $\boldsymbol{Ux} = \boldsymbol{c}$

Properties of Determinant

- **In same principle, if matrix A can be LU decomposed,**
  - You can consider Eq 1..
- **Eq 2. is established.**
  - Due to properties of determinant.

$$\boldsymbol{A} = \boldsymbol{LU}$$

Eq 1. LU decomposition

$$det(\boldsymbol{A}) = det(\boldsymbol{L})det(\boldsymbol{U})$$

Eq 2. Property of determinant

Easy Way to Obtain Determinant

- **Determinant of A can be easily obtained.**
  - Since both $\boldsymbol{L}$ and $\boldsymbol{U}$ are triangular matrices, consider that determinant is calculated only by multiplying $\boxed{\qquad}$ components.
- **In other words, if L and U decomposed from A are the same as lower triangular matrix and upper triangular matrix,**
  - Determinant of $\boldsymbol{A}$ is the same as Eq 1..
  - It can be calculated simply.

$$det(\boldsymbol{A}) = \prod_{i=1}^{n} l_{i,i} \prod_{j=1}^{n} l_{j,j} = \prod_{i=1}^{n} l_{i,i} u_{i,i}$$

Eq 1. Determinant of $\boldsymbol{A}$

Gauss-Jordan Elimination

Introduction

In LU decomposition, row operation is not only possible on square matrix.

- How to create something similar to an upper triangular matrix?
  - When number of expressions and variables are different ?
- What if all the numbers on the diagonal elements are eliminated?
  - By taking a row operation.

$$\boldsymbol{Ax} = \boldsymbol{b}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\boldsymbol{Ux} = \boldsymbol{c}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\boldsymbol{E_3 E_2 E_1}\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{E_3 E_2 E_1}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Process to get upper triangular matrix by basic row operations

LU Decomposition and REF, RREF

- **REF: Row-Echelon Form**

- **RREF: $\boxed{\qquad\qquad}$ Row-Echelon Form**

- **Performing a row operation on a rectangular matrix.**
  - Same as obtaining upper triangular matrix through LU decomposition.
  - Matrix in below figure: Row-Echelon matrix
    - Or called Row-Echelon matrix of given matrix.
    - $\blacktriangle$, -: non-zero elements.

$$
\begin{bmatrix}
\blacktriangle & - & - & - & - & - & - & - \\
0 & \blacktriangle & - & - & - & - & - & - \\
0 & 0 & \blacktriangle & - & - & - & - & - \\
0 & 0 & 0 & \blacktriangle & - & - & - & - \\
0 & 0 & 0 & 0 & 0 & 0 & \blacktriangle & - \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

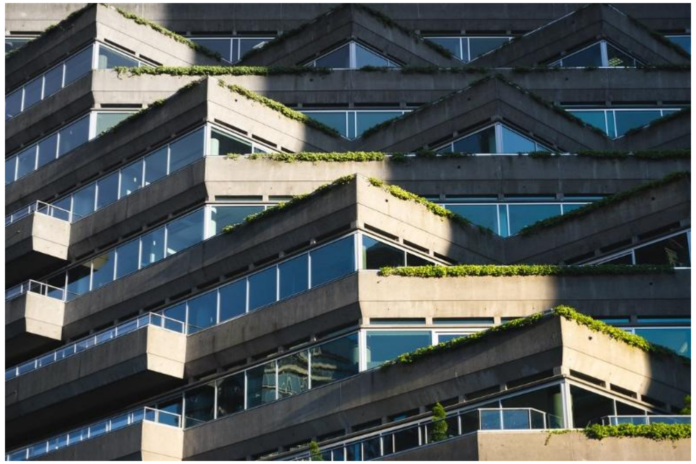
Form of Row-Echelon matrix

What is Echelon?

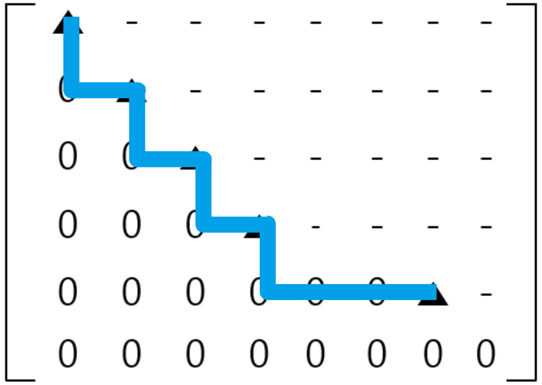
- **If translated into Korean, “사다리꼴” (trapezoid)**
  - **Mistranslation**…! Then, how to translate Echelon?

- **Echelon**
  - Means “**ladder**” shape, not “trapezoid” shape
  - “**ladder**” shape means “ $\boxed{}$ **architecture**”
    - **0** is concentrated at the bottom of the matrix, their shape looks like a staircase.

Trapezoid architecture



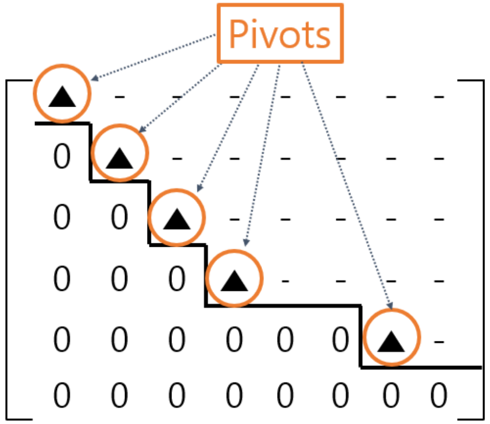
Step-like architecture



Form of Row-Echelon matrix

Three Characteristics of Row-Echelon Matrix

- **All non-zero rows are above row 0.**
  - Rows where all elements are **0** are at the ______ of the matrix.
- **Leading coefficient in a non-zero row**
  - Always exists to the ______ of the first non-zero entry in the row above.
- **All column entries under pivot are 0.**
  - Pivot: part where you step on the foot at the end of each step.

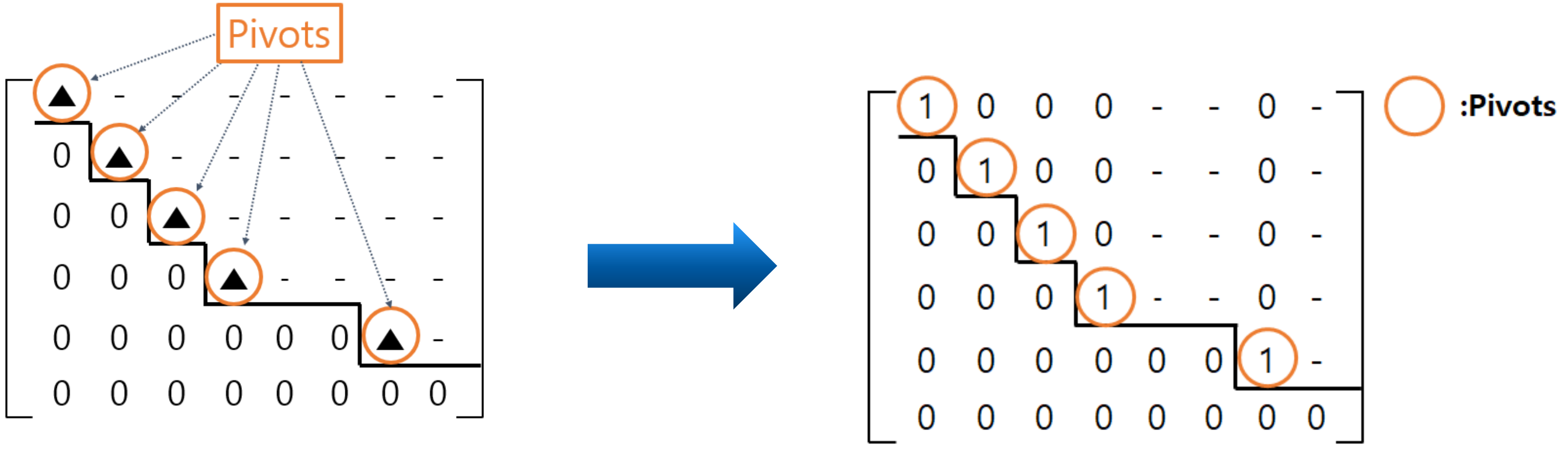


Form of Row-Echelon matrix

Characteristics of Reduced Row-Echelon Matrix

Reduced Row-Echelon Form (RREF)

- Make all pivots as □.
- Make the numbers above the pivots **0** as well.



Reduced Row-Echelon Form

Example of form of REF

- **Distinguish REF with 5 example matrices!**
  - Hint: Consider three characteristics of Row-Echelon matrix.
  - –: non zero number

- **Is the first matrix in row-echelon form?**
  - ☐, then why is it?
- **Is the second matrix in row-echelon form?**
  - ☐, then why is it?
- **Is the third matrix in row-echelon form?**
  - ☐, then why isn't it?
- **Is the fourth matrix in row-echelon form?**
  - ☐, then why isn't it?
- **Is the fifth matrix in row-echelon form?**
  - ☐, then why isn't it?

$$\begin{bmatrix} 1 & - & - & - \\ 0 & 1 & - & - \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix 1

$$\begin{bmatrix} 0 & 3 & - & - & - & - & - \\ 0 & 0 & 2 & - & - & - & - \\ 0 & 0 & 0 & 0 & 0 & 5 & - \\ 0 & 0 & 0 & 0 & 0 & 0 & - \end{bmatrix}$$

Matrix 2

$$\begin{bmatrix} 1 & - & - & - \\ 0 & 2 & - & - \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Matrix 3

$$\begin{bmatrix} 0 & 1 & - & - \\ 1 & - & - & - \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix 4

$$\begin{bmatrix} 3 & - & - & - \\ 0 & 4 & - & - \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix 5

Example of form of RREF

- **Distinguish RREF with 3 example matrices!**
  - Hint: Consider definition of RREF.

- **Is the first matrix in row-echelon form?**
  - ☐, then why is it?
- **Is the second matrix in row-echelon form?**
  - ☐, then why isn't it?
- **Is the third matrix in row-echelon form?**
  - ☐, then why is it?

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix 1

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 4 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix 2

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1/4 & 5/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Matrix 3

Get REF and RREF by Hand

- **Let's find REF and RREF with matrix A with elementary row operation**

- **Step 1.**
  - $r_2 \rightarrow r_2 - 2r_1$
    $r_3 \rightarrow r_3 - 3r_1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 4 & 0 & 0 & -6 \\ 0 & 0 & 2 & 3 & 5 \end{bmatrix} \longrightarrow \text{REF}$$

Result of Step 1.

- **Step 2.**
  - $r_2 \rightarrow \frac{1}{4} r_2$
    $r_2 \rightarrow \frac{1}{2} r_3$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 3/2 & 5/2 \end{bmatrix}$$

Result of Step 2.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 6 & 4 & 6 & 2 \\ 3 & 3 & 8 & 12 & 17 \end{bmatrix}$$

Matrix A

- **Step 3.**
  - $r_1 \rightarrow r_1 - r_2$
    $r_1 \rightarrow r_1 - 2r_3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 3/2 & 5/2 \end{bmatrix} \longrightarrow \text{RREF}$$

Result of Step 3.

Get REF Using MATLAB

- **REF is not unique.**
  - In MATLAB, result of REF can be different from the answer obtained by hand.
  - Even if the pivot value is not abbreviated when calculating the REF of a certain matrix, it is still treated as REF.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 6 & 4 & 6 & 2 \\ 3 & 3 & 8 & 12 & 17 \end{bmatrix}$$

Matrix A

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 4 & 0 & 0 & -6 \\ 0 & 0 & 2 & 3 & 5 \end{bmatrix}$$

$REF(A)_1$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 & -3 \\ 0 & 0 & 2 & 3 & 5 \end{bmatrix}$$

$REF(A)_2$

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix A
A = [1 1 2 3 4;2 6 4 6 2; 3 3 8 12 17];

% Calculate REF of A
[~,ref_A] = lu(A);

% Display the result
disp("A")
disp(A);
disp("ref_A")
disp(ref_A);
```

MATLAB code

```
A
     1     1     2     3     4
     2     6     4     6     2
     3     3     8    12    17

ref_A
    3.0000    3.0000    8.0000   12.0000   17.0000
         0    4.0000   -1.3333   -2.0000   -9.3333
         0         0   -0.6667   -1.0000   -1.6667
```

Answer of MATLAB

Get RREF Using MATLAB

- **RREF is unique.**
  - In MATLAB, result of RREF is always same with the answer obtained by hand.
  - In MATLAB, we can obtain RREF using a function called $\boxed{}$.
  - RREF is unique because it decomposes the pivot and eliminates all elements above the pivot.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 4 \\ 2 & 6 & 4 & 6 & 2 \\ 3 & 3 & 8 & 12 & 17 \end{bmatrix}$$

Matrix A

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & -3/2 \\ 0 & 0 & 1 & 3/2 & 5/2 \end{bmatrix}$$

$RREF(A)$

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix A
A = [1 1 2 3 4;2 6 4 6 2; 3 3 8 12 17];

% Calculate RREF of A
rref_A = rref(A);

% Display the result
disp("A")
disp(A);
disp("rref_A")
disp(rref_A);
```

MATLAB code

```
A
     1     1     2     3     4
     2     6     4     6     2
     3     3     8    12    17

rref_A
    1.0000         0         0         0    0.5000
         0    1.0000         0         0   -1.5000
         0         0    1.0000    1.5000    2.5000
```

Answer of MATLAB

Application of RREF

- **When multiple types of $\boldsymbol{b}$ vectors in $\boldsymbol{Ax} = \boldsymbol{b}$,**
  - x can be obtained all at once by RREF.
- **Using augmented matrix and performing Gauss-Jordan elimination, it is possible to solve three equations at once!**

$$\begin{cases} 3x - z = 1 \\ x + 2y + 3z = 1 \\ 2x - y + z = 1 \end{cases}$$

$$\begin{cases} 3x - z = 2 \\ x + 2y + 3z = 2 \\ 2x - y + z = 2 \end{cases}$$

$$\begin{cases} 3x - z = 3 \\ x + 2y + 3z = 3 \\ 2x - y + z = 3 \end{cases}$$

3 equations

$$\begin{bmatrix} 3 & 0 & -1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\Big\downarrow [\boldsymbol{A}|\boldsymbol{B}]$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 1 & 2 & 3 \\ 2 & -1 & 1 & 1 & 2 & 3 \end{array}\right]$$

Augmented matrix

Application of RREF: Inverse Matrix

How to get inverse matrix by Gauss-Jordan elimination?

- Apply the fact that we can use an augmented matrix.
- Matrix $\boldsymbol{B}$ must be multiplied by matrix $\boldsymbol{A}$, and matrix $\boldsymbol{I}$ should be obtained as a result.
  - It means matrix $\boldsymbol{B}$ is the inverse of the matrix $\boldsymbol{A}$.
    - $\boldsymbol{B} = \boxed{}$

$$\boldsymbol{Set}\ [\boldsymbol{A}|\boldsymbol{I}]$$

$$\left[\begin{array}{ccc|ccc} 3 & 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 2 & -1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\boldsymbol{Gauss - Jordan\ Elimination} \qquad \boldsymbol{Make}\ [\boldsymbol{I}|\boldsymbol{A}^{-1}]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/4 & 1/20 & 1/10 \\ 0 & 1 & 0 & 1/4 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/4 & 3/20 & 3/10 \end{array}\right] \boldsymbol{A}^{-1}$$

Augmented matrix

Row Equivalent

Solution is still same.

- Even if row operation is applied.

- Original matrix: A
  - REF of original matrix: U
  - RREF of original matrix: R

- All of solution x is same.
  - $Ax = b\ , Ux = c, Rx = d$
  - c, d, and vector b on the right side transformed
    - By changing the original matrices A and U.

- A, U, R: row equivalent
  - No change in the row space.
    - Even if a row operation is performed.

$$\begin{cases} 3x + 3y + z = 3 \\ 4x + 5y + 2z = 1 \\ 2x + 5y + z = 3 \end{cases}$$

$$\begin{bmatrix} 3 & 3 & 1 \\ 4 & 5 & 2 \\ 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \quad : Ax = b$$

$$\begin{bmatrix} 4 & 5 & 2 \\ 0 & 5/2 & 0 \\ 0 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 5/2 \\ 3 \end{bmatrix} \quad : Ux = c$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 6 \end{bmatrix} \quad : Rx = \mathrm{d}$$

$$All\ of\ solution: x = 2, y = 1, z = -6$$

Row equivalent

Application of REF: Determination of Linear Dependence of rows

Review of linear dependent or independent

- If Eq 1. can be established with $\boldsymbol{c_1}$ and $\boldsymbol{c_2}$ other than $\mathbf{0}$, then two vectors $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ are linear ______.
  - If can't established, then linear independent.

$$\boldsymbol{c_1 v_1 + c_2 v_2 = 0}$$

Eq 1.

Obtaining REF or RREF

- Performed through a ______
- If some row elements become all zero,
  - That row could be obtained by a linear combination of other rows.
  - That row is **linearly dependent** with other rows.

$$A = \begin{bmatrix} 5 & 3 & 3 \\ 5 & 1 & 3 \\ 5 & 2 & 3 \end{bmatrix} \xrightarrow[\boldsymbol{r_3 \to r_3 - \frac{1}{2}r_1 - \frac{1}{2}r_2}]{\textbf{To make REF}} A = \begin{bmatrix} 5 & 3 & 3 \\ 5 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Third row eliminated through row operation

Summary

Summary

Simultaneous equations and matrix

- Expression of simultaneous equations with matrix
  - Augmented matrix
- Skill to solve
  - Row operation
    - Row multiplication
    - Row switching
    - Row addition

LU decomposition

- To decompose matrix $\boldsymbol{A} = \boldsymbol{LU}$
  - $\boldsymbol{L}$: lower triangular matrix
  - $\boldsymbol{U}$: upper triangular matrix

Summary

Gauss-Jordan Elimination

- Row operation to make form of matrix:
  - Row echelon form(REF)
  - Reduced echelon form(RREF)

- Application of REF and RREF
  - Can get solution from multiple equations.
  - Calculate matrix inverse.
  - Can determine linear dependence of rows.

Code Exercises

Solving Simultaneous Equations

- Implement simultaneous equation in this lecture.
- In Eq 1., you can get matrix $\boldsymbol{B}$ after solving simultaneous equation.
  - $r_2 \rightarrow (r_2 - 2r_1)$
  - $r_3 \rightarrow (r_3 - 2r_1)$
  - $r_3 \rightarrow (r_3 - r_2)$
- Make 3 elementary matrices and multiply with matrix $\boldsymbol{A}$ to get matrix $\boldsymbol{B}$.
- Refer to page 9,10, and 17.

$$\text{3 elementary matrices} \times \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix}}_{\boldsymbol{A}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}}_{\boldsymbol{B}}$$

Eq 1.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix A
A = [1 1 1;2 3 -1; 2 3 3];

% Matrix B
B = [1 1 1;0 1 -3;0 0 4];

% Implement three elementary Matrix
% like elem = [1 0 0;0 1 0;0 0 1];
% r_2 -> r_2 - 2*r_1
elem_1 = ;
% r_3 -> r_3 - 2*r_1
elem_2 = ;
% r_3 -> r_3 - r_2
elem_3 = ;

% Multiply with matrix A
result = ;

% Check that result and B are same
disp(B);
disp(result);
```

Sample code

Multiply by Inverse of Elementary Matrix

- **If you multiply inverse matrices of elementary matrices that were multiplied in front of coefficient matrix A in order,**
  - You can rewrite matrix A as follows.

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}}_{r_3 \rightarrow (r_3 - r_2)}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}}_{r_3 \rightarrow (r_3 - 2r_1)}\underbrace{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}_{r_2 \rightarrow (r_2 - 2r_1)}\underbrace{\begin{bmatrix}1&1&1\\2&3&-1\\2&3&3\end{bmatrix}}_{A}=\begin{bmatrix}1&1&1\\0&1&-3\\0&0&4\end{bmatrix}$$

Multiplying inverse matrix of elementary matrix

$$\underbrace{\begin{bmatrix}1&1&1\\2&3&-1\\2&3&3\end{bmatrix}}_{A}=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}^{-1}\begin{bmatrix}1&1&1\\0&1&-3\\0&0&4\end{bmatrix}$$

Multiplying inverse matrix of elementary matrix

Multiply by Inverse of Elementary Matrix

- **If you multiply inverse matrices of elementary matrices that were multiplied in front of coefficient matrix A in order,**
  - You can rewrite matrix $\boldsymbol{A}$ as follows.

$$\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}}_{r_3 \to (r_3 - r_2)}\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}}_{r_3 \to (r_3 - 2r_1)}\underbrace{\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}}_{r_2 \to (r_2 - 2r_1)}\underbrace{\begin{bmatrix}1&1&1\\2&3&-1\\2&3&3\end{bmatrix}}_{\boldsymbol{A}}=\begin{bmatrix}1&1&1\\0&1&-3\\0&0&4\end{bmatrix}$$

Multiplying inverse matrix of elementary matrix

$$\underbrace{\begin{bmatrix}1&1&1\\2&3&-1\\2&3&3\end{bmatrix}}_{\boldsymbol{A}}=\begin{bmatrix}1&0&0\\-2&1&0\\0&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\0&1&0\\-2&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}^{-1}\begin{bmatrix}1&1&1\\0&1&-3\\0&0&4\end{bmatrix}$$

Multiplying inverse matrix of elementary matrix

LU Implementation

- **Implement LU decomposition with multiplying inverse of elementary matrix.**
  - Do not use lu() function in MATLAB.
  - https://kr.mathworks.com/help/matlab/ref/lu.html
- **You can use three elementary matrices in previous exercise.**

$$\underset{\boldsymbol{A}}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 3 & 3 \end{bmatrix}} = \mathbf{L} * \underset{\boldsymbol{U}}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}}$$

Eq 1.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Matrix A
A = [1 1 1;2 3 -1; 2 3 3];

% Matrix B
U = [1 1 1;0 1 -3;0 0 4];

% Implement three elementary Matrix
% like elem = [1 0 0;0 1 0;0 0 1];
% r_2 -> r_2 - 2*r_1
elem_1 = ;
% r_3 -> r_3 - 2*r_1
elem_2 = ;
% r_3 -> r_3 - r_2
elem_3 = ;
% Get inverse matrix
elem_1_inv = ;
elem_2_inv = ;
elem_3_inv = ;
% Get L matrix
L = ;
% display L & U
disp(L);
disp(U);
% Check that L * U is same with A
disp('--');
disp(L * U);
disp(A);
```

Sample code

THANK YOU FOR YOUR ATTENTION