Linear Algebra

Matrices Part 2: Matrix Expansion Concept

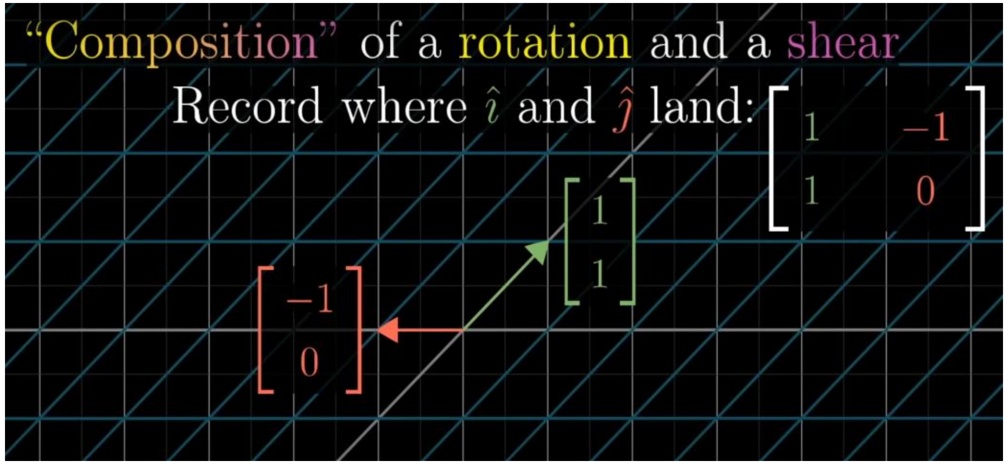
Automotive Intelligence Lab.

Contents

Matrix multiplication

Composition of the Two Separate Transformations (1)

- **Matrix multiplication**
  - Applying one ______ and then another.
  - One of linear transformation
- **This transformation can be described with a matrix its own by i-hat and j-hat like other linear transformations.**
- **Example of applying rotation and then shear matrix ➔** $\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{Shear}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation}} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$
  - $\hat{\imath}$ ends up at (1,1).
  - $\hat{\jmath}$ ends up at (-1,0).
- **So, applying rotation and then shear is a one single action, rather than two successive ones.**



Zero vector is the only vector that lands at origin (full rank)

Composition of the Two Separate Transformations (2)

Numerical example of adjusting rotation and shear.

- Apply rotation then shear matrix as shown below.
  1. Multiply the matrix $\begin{bmatrix} x \\ y \end{bmatrix}$ with the rotation matrix on the left.
  2. Multiply the matrix calculated at previous stage by the shear matrix on the left.
  3. Then, its result will be same as just applying new composition matrix.

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{\text{Shear}} \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Rotation}} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Composition}} \begin{bmatrix} x \\ y \end{bmatrix}$$

The new matrix is called "$\boxed{}$".

- Apply transformation by the matrix on the right, then apply the transformation by the matrix on the left.

Right to left

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{Product of new matrix}}$$

Another Example of Matrix Product

- **Example of multiplying matrix $\mathbf{M1} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$, and $\mathbf{M2} = \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$.**
  - Multiplication of $\mathbf{M1}$ and $\mathbf{M2}$

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix}$$

  - How to get $\hat{\imath}$?

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{2} \\ \mathbf{1} \end{bmatrix}$$

  - How to get $\hat{\jmath}$?

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{-2} \end{bmatrix}$$

  - Result of matrix product

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \mathbf{2} & \mathbf{0} \\ \mathbf{1} & \mathbf{-2} \end{bmatrix}$$

Generalization of Matrix Product

- **Write the example in general case, where matrix $\mathbf{M1}$ is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and $\mathbf{M2}$ is $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$.**
  - Multiplication of $\mathbf{M1}$ and $\mathbf{M2}$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

  - How to get $\hat{I}$?

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e \\ g \end{bmatrix} = \mathrm{e} \begin{bmatrix} a \\ c \end{bmatrix} + \mathrm{g} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ae + bg \\ ce + dg \end{bmatrix}$$

  - How to get $\hat{J}$?

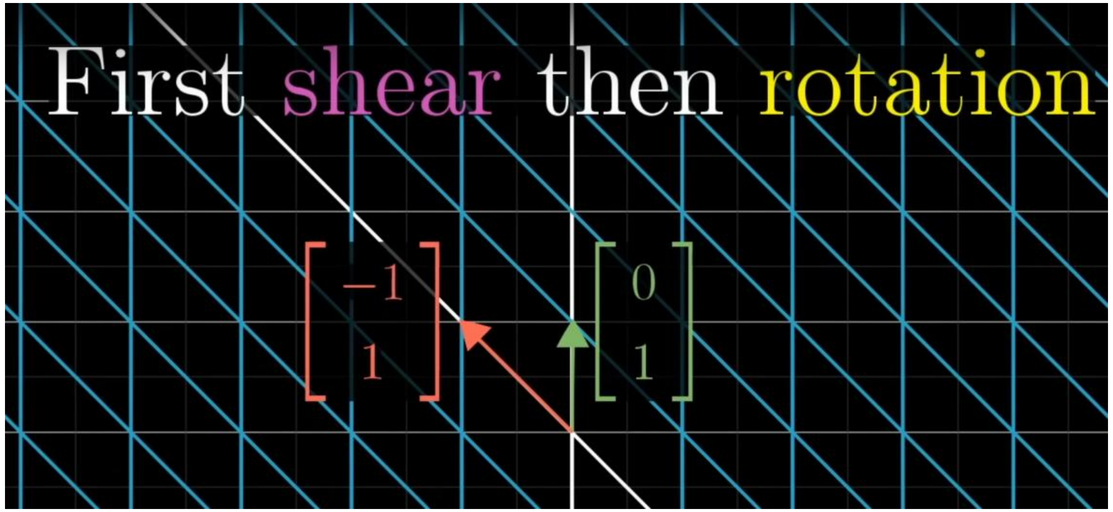
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} f \\ h \end{bmatrix} = \mathrm{f} \begin{bmatrix} a \\ c \end{bmatrix} + \mathrm{h} \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} af + bh \\ cf + dh \end{bmatrix}$$

  - Result of matrix product

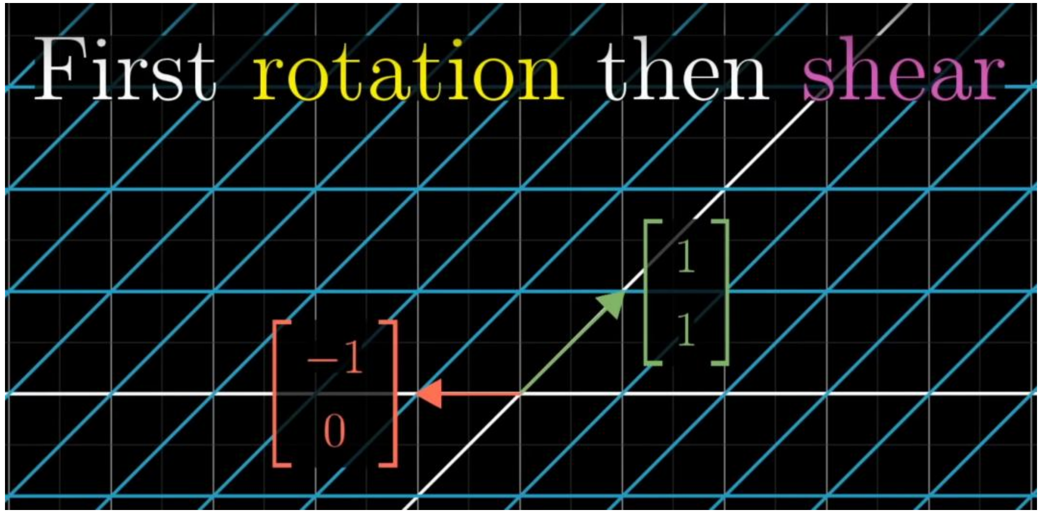
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Proof of $M_1M_2 \neq M_2M_1$

- **Proof of $M_1M_2 \neq M_2M_1$ by an example of shear and rotation.**
  - First **shear** then **rotation**.
    - I-hat end up at (0,1).
    - J-hat ends up at (-1,1).
  - First **rotation** then **shear**.
    - I-hat end up at (1,1).
    - J-hat ends up at (-1,0).
- **So, the ______ of matrix totally matters.**



First shear then rotation



First rotation then shear

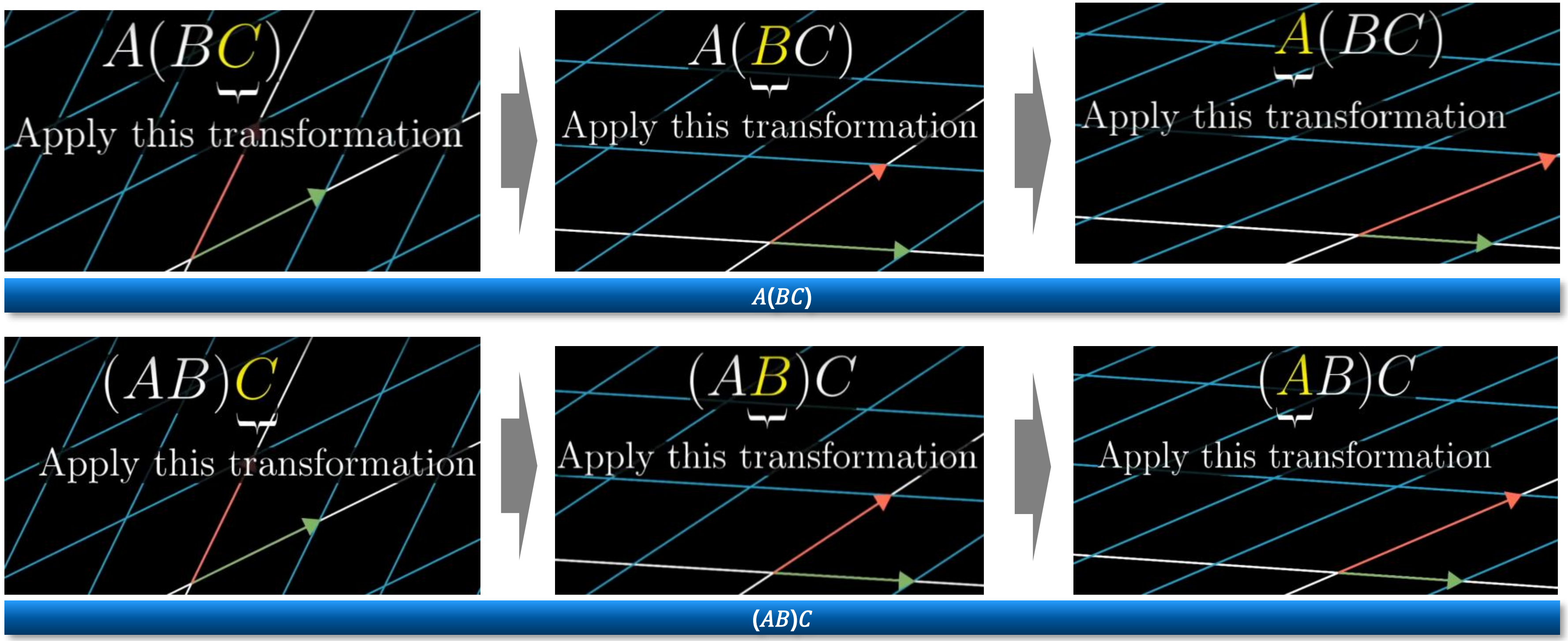
 https://youtu.be/gXjCxYuEybQ?si=nQL2sgd-0AinnIGN

Proof of Matrix Associativity

Meaning of matrix associativity

- Assume that we have three matrices A, B and C.
- If multiplying them all together, it shouldn't matter if first computing A times B, then multiply the result by C, or first multiplying B times C, then multiply that result by A on the left.
-

Same as matrix multiplication as applying one transformation after another, so



Determinant I

Definition of Determinant

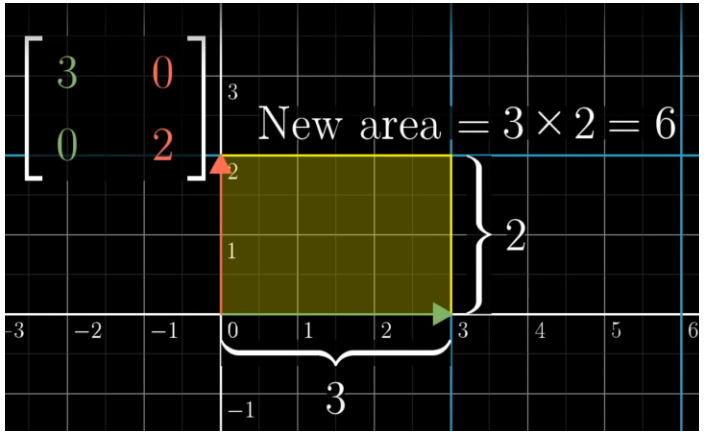
- **A number associated with a square matrix**
  - In abstract linear algebra, the determinant is a keystone quantity in several operations.
  - In practice, it can be numerically unstable for large matrices.
    - Due to underflow and overflow issues.

- **The two most important properties**
  1. Defined only for $\boxed{}$ matrices.
  2. Zero for singular (dependent vector set, reduced-rank) matrices

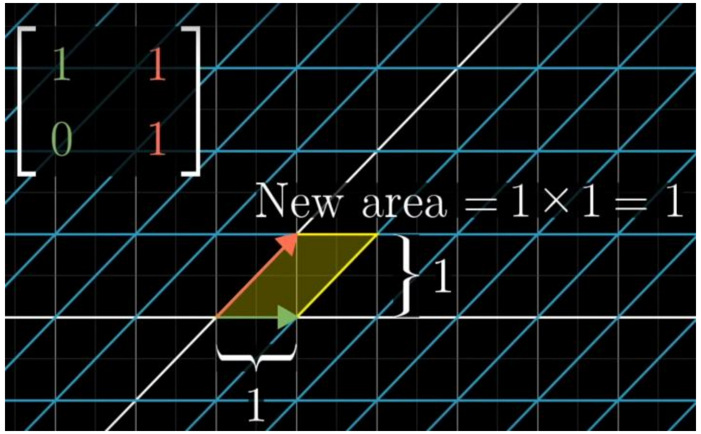
- **Notated as $det(A)$ or $|A|$**
  - $\boxed{}$ is used when it doesn't refer to a specific matrix.

Examples of Scaling through Transformation

- **Measure the amount of scaling by the example of $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$.**
  - $\hat{\imath}$ scales by the factor of 3, and $\hat{\jmath}$ scales by the factor of 2.
  - The matrix started out with 1x1 rectangle and then turns into a 2x3 rectangle.
  - So, the linear transformation has scaled its area by a factor of 6.
- **Measure the amount of scaling by the example of $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.**
  - $\hat{\imath}$ scales remains the same, and $\hat{\jmath}$ moves to (1,1).
  - The 1x1 rectangle gets slanted and turned into a parallelogram which still has the area of 1.
- **If you know how much the unit square changes, you can know how the area of the possible region in space changes.**



Example 1) Scaling of ((3,0),(0,2))



Example 2) Scaling of ((1,1),(0,1))

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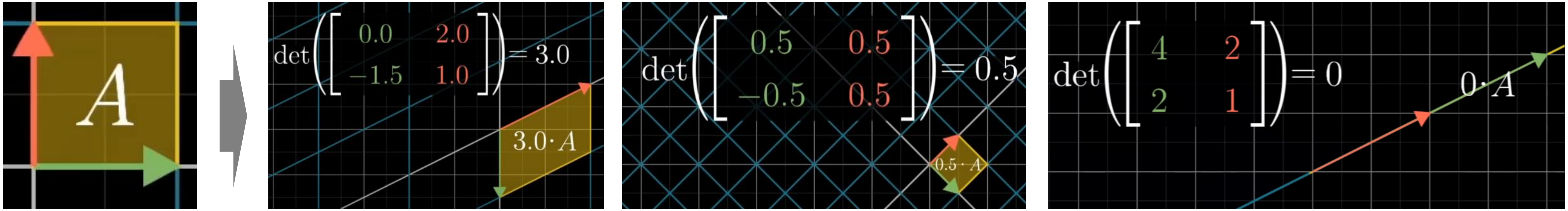
Determinant of a Transformation

- **Meaning of determinant**
  - $\boxed{\quad\quad\quad}$ **factor** by which a linear transformation changes any area.
- **Examples**
  - If transformation **increases** the area of a region by a factor of 3, the determinant of a transformation will be 3.
  - If transformation **squishes** down all areas by a factor of $^1/_2$, the determinant of a transformation will be $^1/_2$.
  - If transformation **squishes all of space onto a line**, or a **single point**, the determinant of a 2D transformation is 0.



Example of different scales of determinant

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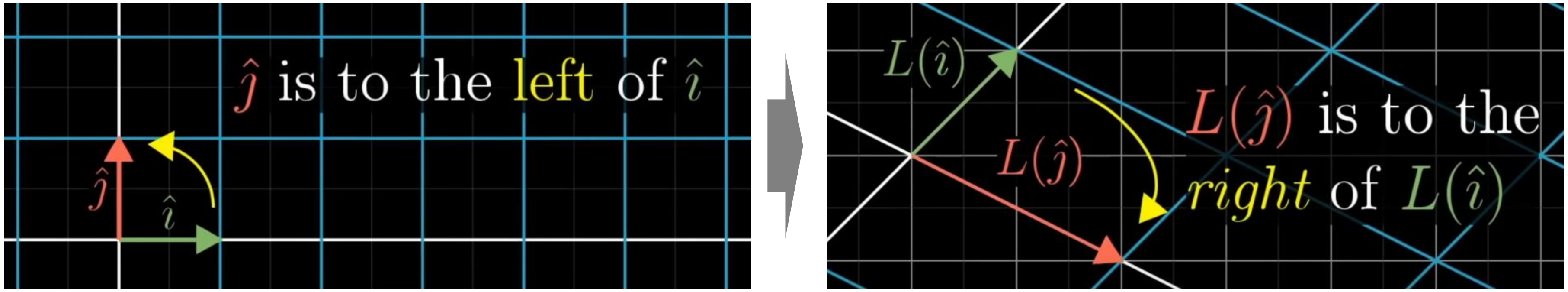
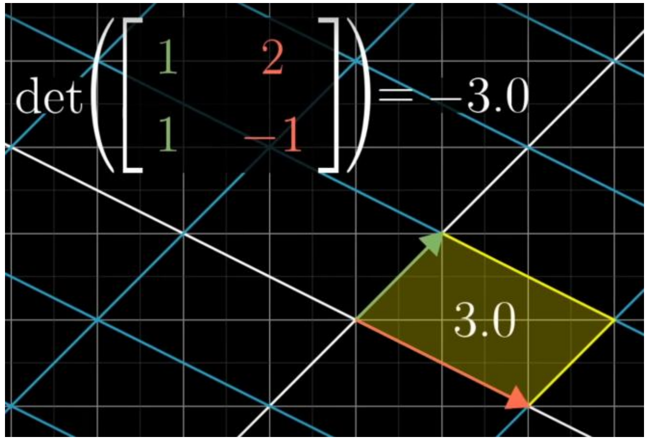
Negative Value of Determinant

- **Scaling area by a negative number means “$\boxed{\qquad\qquad}$orientation”**
- **In the aspect of $\hat{I}$ and $\hat{J}$,**
  - $\hat{J}$ is to the left of $\hat{I}$, and after a transformation, j-hat is now on the right of i-hat.
  - So, the orientation of space has been inverted.
- **Example**
  - If transformation **scales** the area of a region by a factor of 3 and its **space gets flipped over**, the determinant of a transformation will be -3.



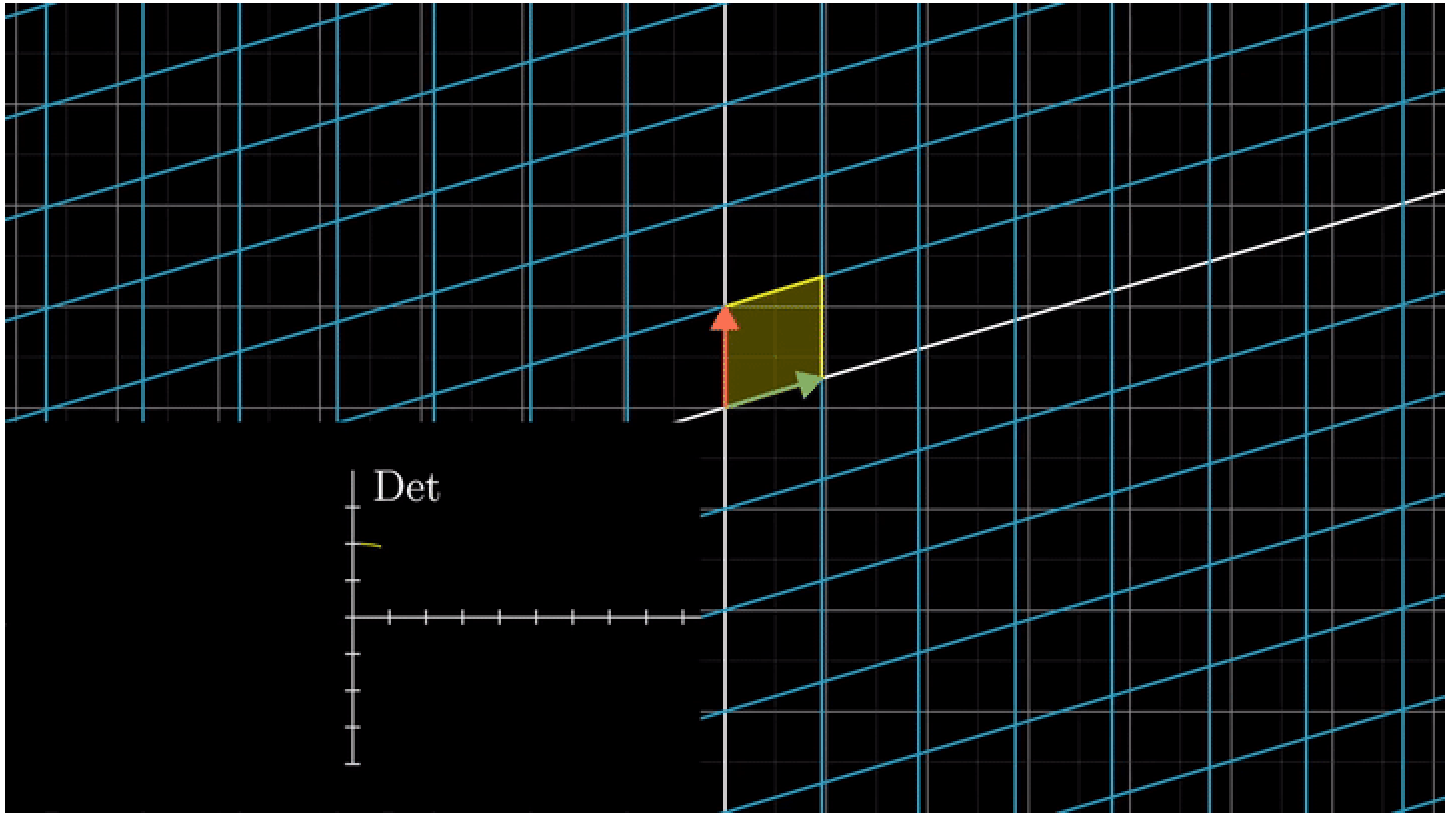
Visualization of negative value determinant



Example of determinant -3

Visualization of Determinant

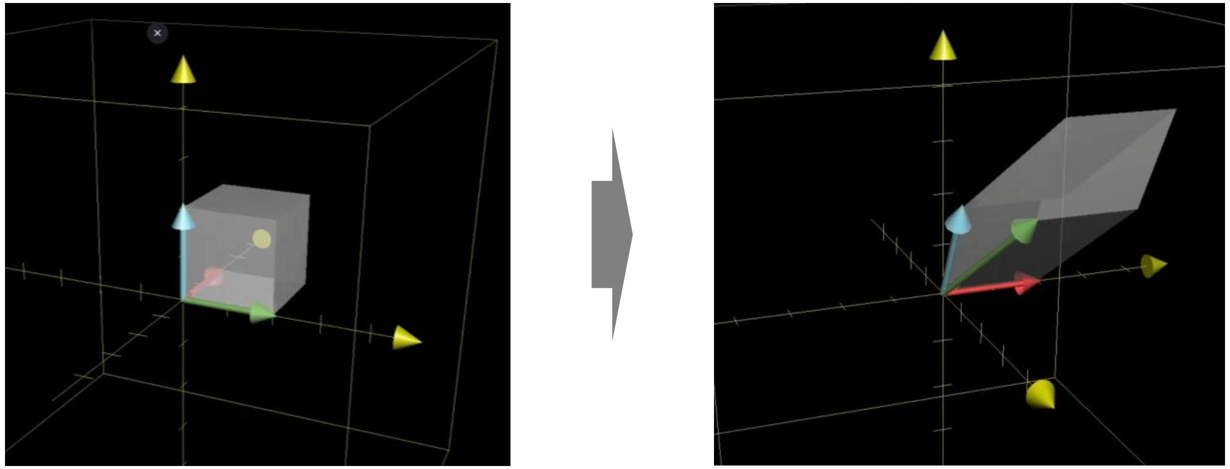
- **Why does negative area relate to orientation-flipping?**



Visualization material of determinant

Intuition Understanding of Determinant

- **Determinant in 3D tells how much a transformation scales things, and how much volumes get scaled.**
- **In the aspect of $\hat{I}$ and $\hat{J}$,**
  - 3D determinant can be described with **cube** whose edges are resting on the **basis vectors, $\hat{\boldsymbol{I}}, \hat{\boldsymbol{J}},$ and $\widehat{\boldsymbol{K}}$.**
  - After transformation, cube might get warped into slanty ways, named "$\boxed{}$".
- **So, determinant determines the volume of parallelepiped that the cube turns into.**



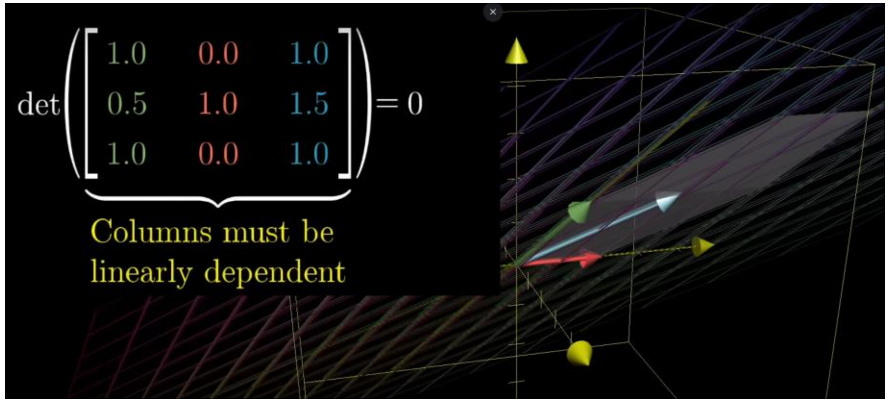
Determinant in 3D

https://youtu.be/15VOaSeHYeY?si=7OfsuKdlvlnZOP0S

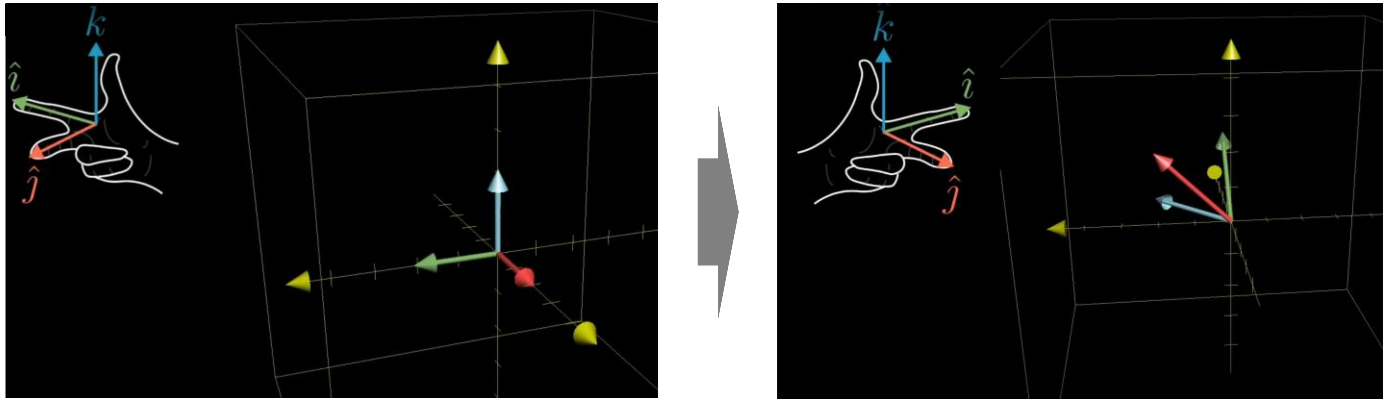
Examples of Determinant in 3D

Examples

- Determinant of **0** would mean that all of space is **squished onto something with 0 volume**, meaning either a flat plane, a line, or, in the most extreme case, onto a single point.
  - This means that the columns of the matrix are "$\boxed{}$".
- Determinant of negative would mean that "**flip of orientation**".



Determinant of 0 means linearly dependent

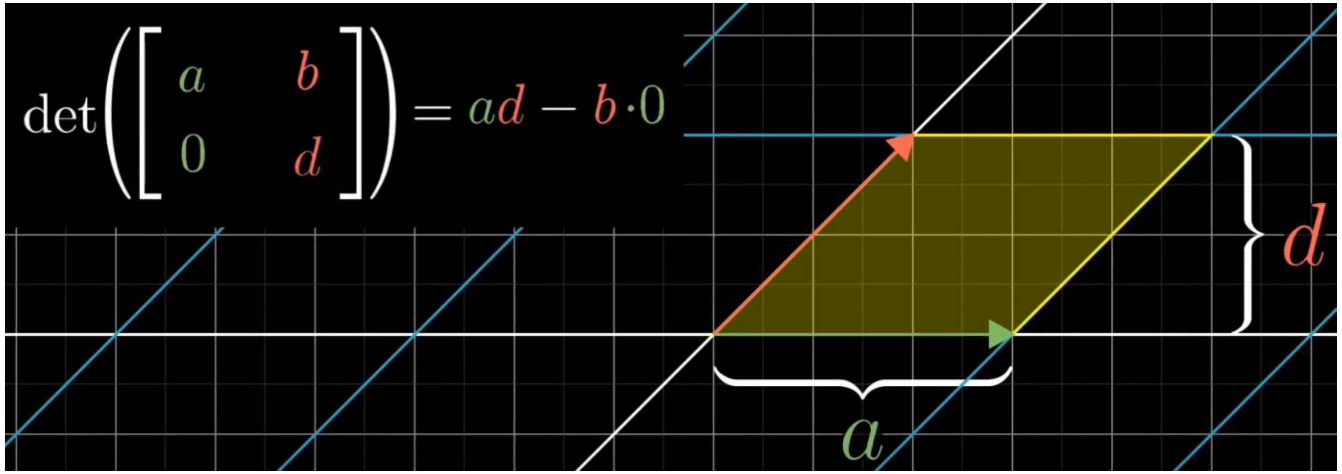


Visualization of flipped orientation (when determinant is negative)

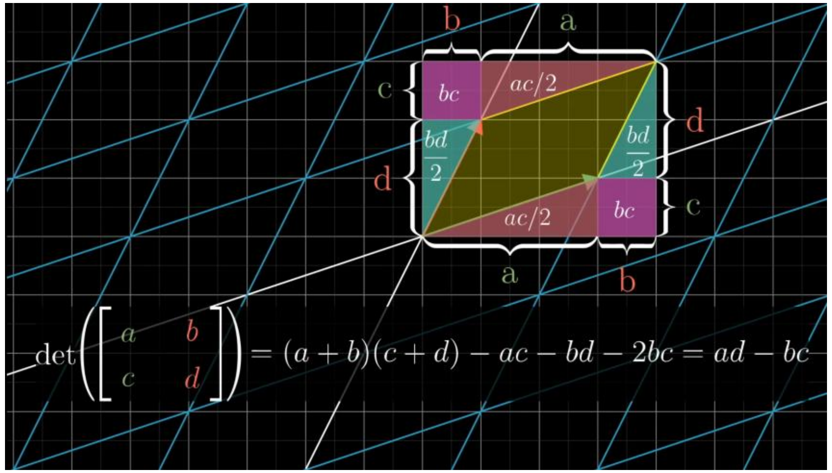
https://youtu.be/15VOaSeHYeY?si=7OfsuKdlvlnZOP0S

Computing of Determinant in 2D

- $det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$
- **Proof of the formula**
  1. Assume that b and c are both 0. Then, a will represent how much $\hat{\imath}$ is stretched in the **x direction**, and d represent how much $\hat{\jmath}$ is stretched in **y direction**. So, the output is the rectangle of $a * d$.
  2. Assume that only one of b or c are 0. Then the output will be a parallelogram with a base of a and the height of d, which still has the area of ad.
  3. Assume that both b and c are not 0. Then bc will affect how much this parallelogram is stretched or squished in the diagonal direction.



Visualization of determinant when only one of b or c are 0

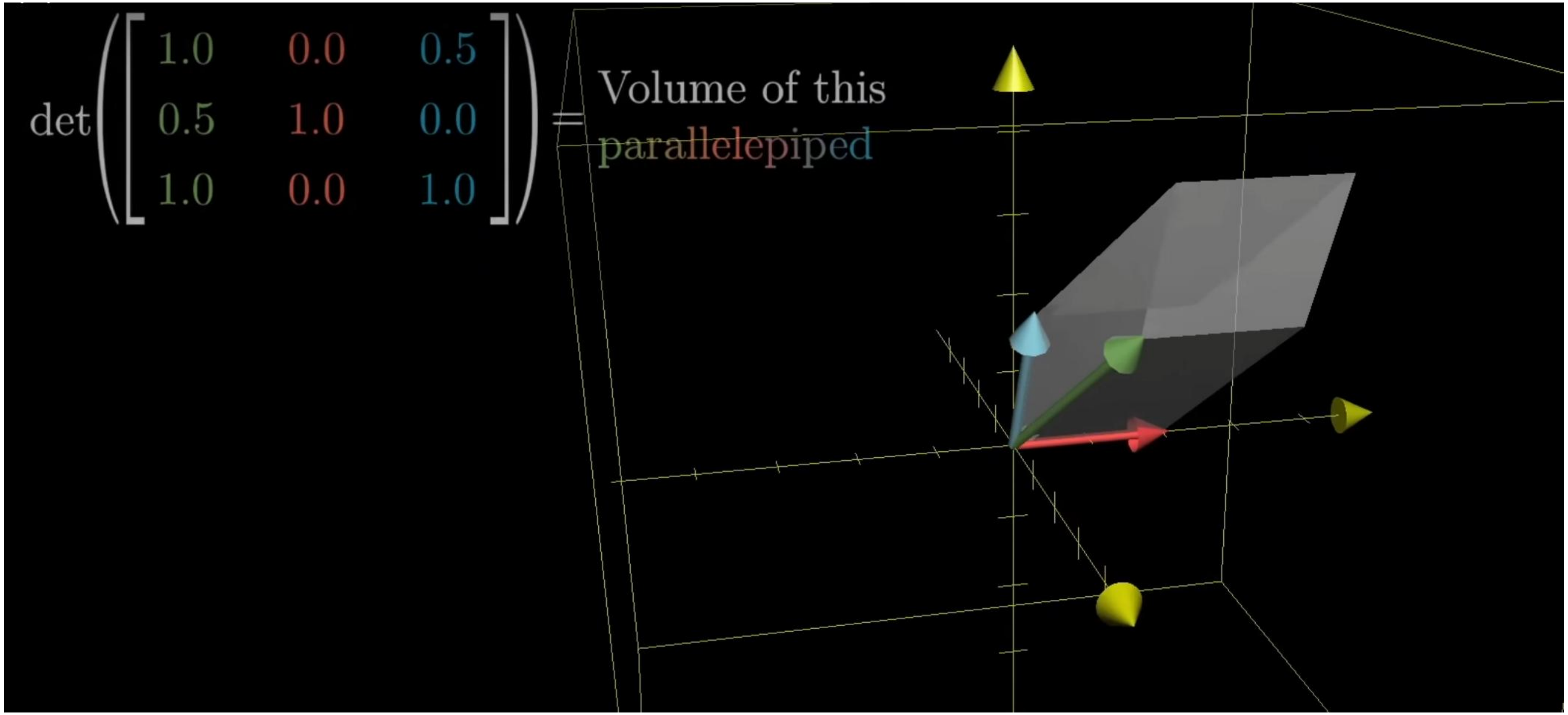


Proof of determinant formula

https://youtu.be/15VOaSeHYeY?si=7OfsuKdlvlnZOP0S

Computing of Determinant in 3D

- $det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) = a\ det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b\ det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) + c\ det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)$



Visualization of determinant in 3D

Determinant II

Applications of Determinant

- **In geometric interpretation**
  - During matrix-vector multiplication, it is related to how much the matrix stretches vectors.
  - During the transformation, a negative determinant means that one axis is rotated.

- **In data science-related applications**
  - Used algebraically.
  - Crucial step in advanced topics.
    - Matrix inverse, eigen decomposition and singular value decomposition.

Computing the Determinant

- **Computing the determinant is time-consuming and tedious.**
- **Shortcut for computing the determinant of a 2×2 matrix, which is shown in Eq 1..**
  - For a real-valued matrix, the determinant will always be a real number.

$$det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Eq 1. Computing the determinant of a 2×2 matrix

- **Code Exercise of Determinant of matrix**
  - **Code Exercise (06_01)**

```matlab
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define a 2x2 matrix
A = [3, 2; 0, 2];
B = [1, 3, 2; 3, 1, 5; 2, 4, 1];

% Calculate the determinant of matrix A,B
determinantA = det(A);
determinantB = det(B);

% Display the determinant
disp('The determinant of matrix A is:');
disp(determinantA);
disp('The determinant of matrix B is:');
disp(determinantB);
```

MATLAB code example of Determinant of matrix

Problem of Computing the Determinant

- **The shortcut method for 2×2 matrix doesn't scale up to larger matrices.**
  - Eq 1. is a "shortcut" for 3×3 matrices, but it isn't really a shortcut.
  - Eq 2. is a "shortcut" for 4×4 matrices, also not shortcut.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - ceg - bdi - afh$$

Eq 1. Computing the determinant of a 3×3 matrix

$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = afkp - aflo - agjp + \cdots - dgin + dgjm$$

Eq 2. Computing the determinant of a 4×4 matrix

Generalization of Determinant

- **Let A be an $N \times N$ matrix.**
  - $\boldsymbol{M}$ is the determinant of the $\boxed{}$.
    - $\boxed{}$: constructed by removing the ith row and jth column of $\boldsymbol{A}$.

$$det(\boldsymbol{A}) = \sum_{j=1}^{N} (-1)^{i+j} \boldsymbol{A}_{i,j} \boldsymbol{M}_{i,j}$$

Generalization of determinant

Determinant with Linear Dependencies

- **Determinants are $\boxed{}$ for any singular matrix.**

- **Any singular matrix has at least one column.**
  - Expressed as a linear combination of other columns.

- **All nonsingular (full-rank) matrices have a $\boxed{}$ determinant.**

- **In geometric meaning: $\Delta = 0$.**
  - A matrix with $\Delta = 0$ is a transformation, at least one dimension gets flattened to have surface area but no volume.

$$\begin{vmatrix} a & \lambda a \\ c & \lambda c \end{vmatrix} = ac\lambda - a\lambda c = 0$$

Eq 1. Reduced-rank matrix has a determinant of 0

Using the Determinant to Find a Missing Matrix Element

- **a, b, c, λ are the elements in the matrix and Δ is the determinant value.**
- **Assume that a, b, c and Δ are known, and λ is some unknown quantity.**
  - We can solve for $\square$ in terms of the other quantities.
- **Point**
  - If we **know the determinant of a matrix**, we can **solve for unknown variables inside the matrix**.

$$\begin{vmatrix} a & b \\ c & \lambda \end{vmatrix} \Rightarrow a\lambda - bc = \Delta$$

Eq 1. Example of finding a missing matrix element

$$\begin{vmatrix} \lambda & 1 \\ 3 & \lambda \end{vmatrix} = 1 \;\Rightarrow\; \lambda^2 - 3 = 1 \;\Rightarrow\; \lambda^2 = 4\,, \qquad \lambda = \pm 2$$

Eq 2. Numerical example to solve for unknown variable inside the matrix

The Characteristic Polynomial of Matrix

- **Combining matrix shifting with the determinant as shown in Eq 1..**
- **Why is it called a polynomial?**
  - The shifted $M \times M$ matrix has an $\lambda^{\boldsymbol{M}}$ term, therefore has M solutions.

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = \Delta$$

Eq 1. The characteristic polynomial of the matrix

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \;\Rightarrow\; \lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

Eq 2. Example of the characteristic polynomial for 2 × 2 matrices

Numerical Example of the Characteristic Polynomial

- **Let's return to the 2×2 case, this time using numbers instead of letters.**
  - Assume that it has a determinant of 0 after being shifted by some scalar λ.
  - After some algebra, the two solutions are $\lambda = \boxed{}$.

$$\det\left(\begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \boldsymbol{I}\right) = 0$$

Eq 1. The characteristic polynomial of the matrix

$$\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0 \;\Rightarrow\; (1-\lambda)^2 - 9 = 0 \qquad \lambda = \boxed{}$$

Eq 2. Example of the characteristic polynomial for 2×2 matrices

Properties of the Characteristic Polynomial

- **Let's plug them back into the shifted matrix.**
  - Both matrices have nontrivial null spaces.
    - Some non-zeros vector $\boldsymbol{y}$ such that $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{y} = 0$.
    - In this case, any scaled version of $[1 \quad -1]$ and $[1 \quad 1]$.

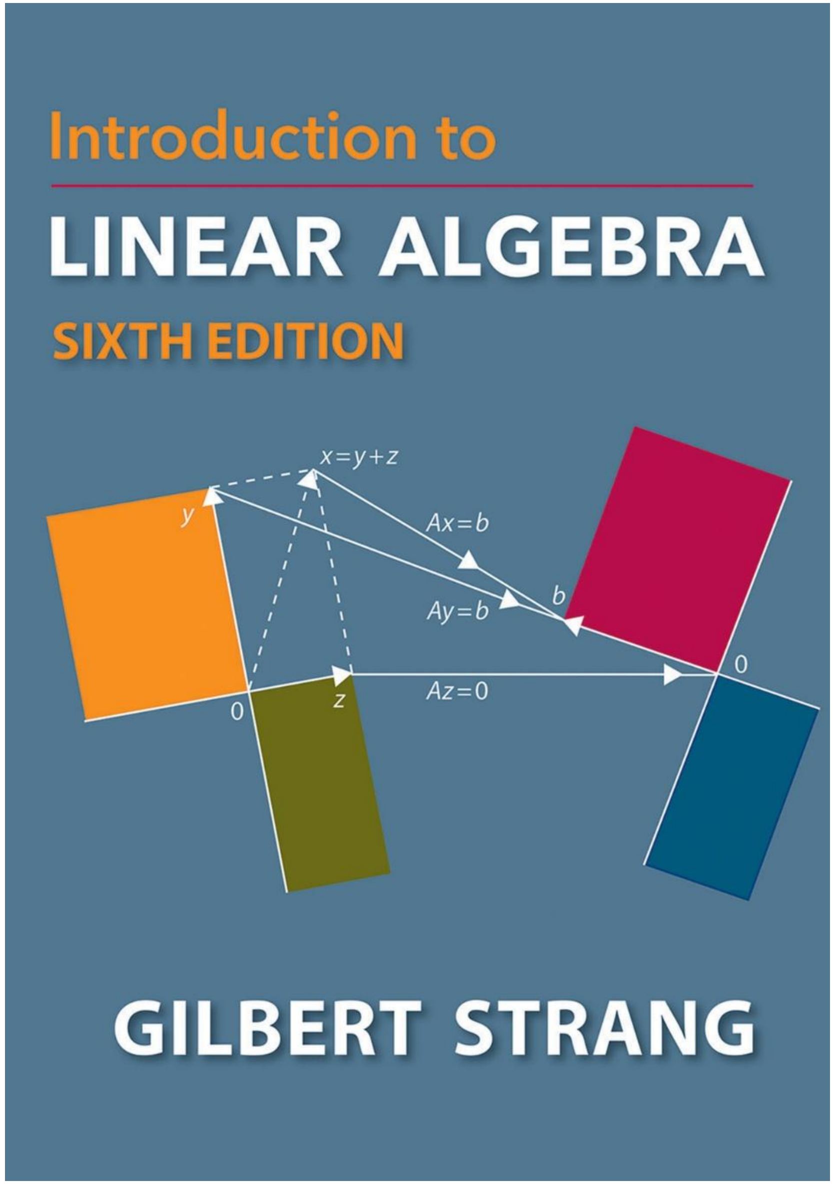
- **Every square matrix can be expressed as an equation.**
  - Directly links matrices to the fundamental theorem of algebra.
  - The solutions to the characteristic polynomial set to $\Delta = 0$ are the eigenvalues of the matrix.

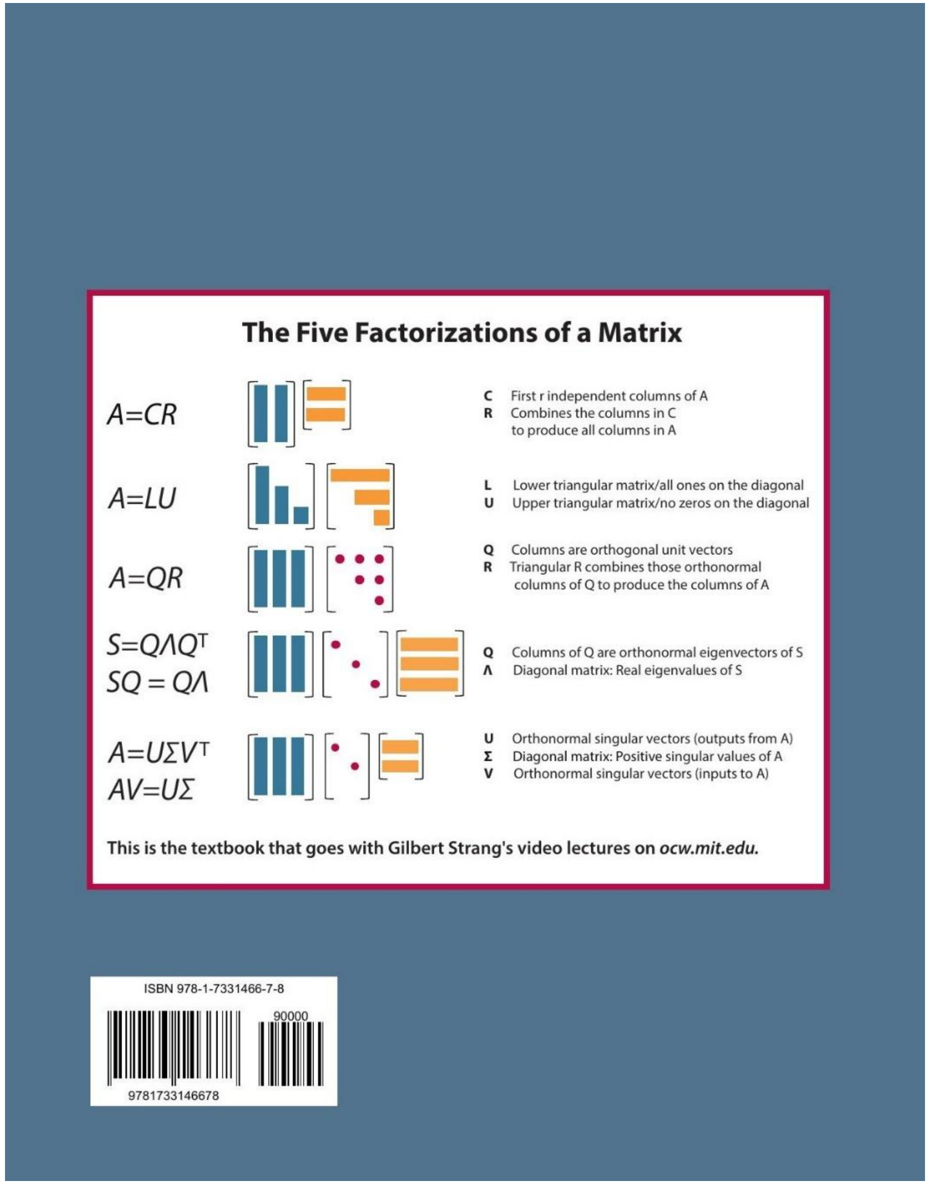
$$\lambda = -2 \Rightarrow \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \qquad \lambda = 4 \Rightarrow \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$$

Eq 1. Shifted matrix

Matrix spaces (column, row, nulls)

Gilbert Strang – Linear Algebra





Matrix Spaces

Concept of matrix spaces

- Linear transformation: linear weighted combinations of different features of a matrix.
  - Central to many topics in abstract and applied linear algebra.

Column Space of Matrix

- **Remember in vector extension part…,**
  - A linear weighted combination of vectors involves scalar multiplying and summing a set of vectors.
  - If two modifications to this concept will extend to linear weighted combination to column space of matrix.

- **Two modifications for column space of matrix**
  - Conceptualize a matrix as a set of column vectors.
  - Consider infinity of real-valued scalars instead of working with a specific set of scalars.
    - Resulting infinite set of vectors is called **column space of a matrix**.

Numerical Example of Column Space of Matrix

Expression of column space of matrix

- $C(\boldsymbol{A})$ indicates the column space of matrix $\boldsymbol{A}$.
- λ can be any possible real-valued number.

$$C\left(\begin{bmatrix}1\\3\end{bmatrix}\right) = \lambda\begin{bmatrix}1\\3\end{bmatrix}, \lambda \epsilon \mathbb{R}$$

Expression of column space

Mathematical meaning of above expression

- Column space is the set of all possible scaled versions of the column vector $[1 \quad 3]^T$.
- Is the vector $[2 \quad 6]^T$ in the column space?
  - $\boxed{}$
  - Can express that vector as the matrix times $\lambda = 2$.
- How about vector $[1 \quad 4]^T$?
  - $\boxed{}$
  - Simply no scalar that can multiply the matrix to produce that vector.

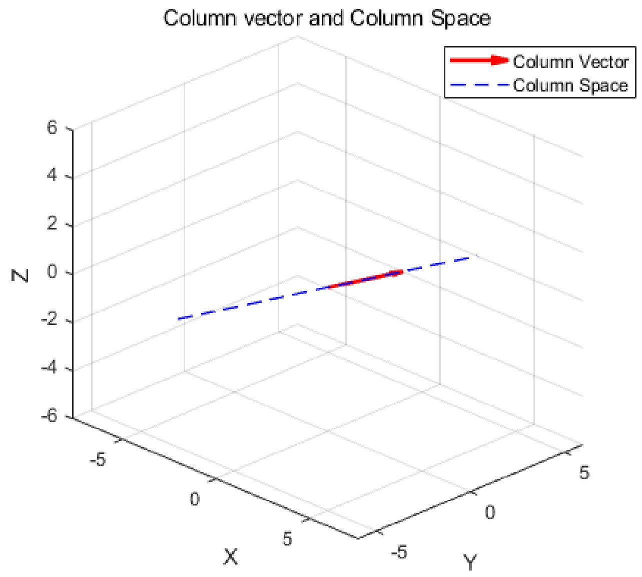
Form of Column Space of Matrix with One Column

What does the column space look like?

- Matrix with one column vector, column space is a line.
  - Passes through the origin, in the direction of the column vector.
  - Stretches out to infinity in both directions.

$$C(A) = C\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = \lambda \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \lambda \epsilon \mathbb{R}$$

Example of Column space of a matrix that has only one column



Visualization of column space of a left matrix

Example of Column Space of Matrix with Two Columns

Example of matrix with two columns

- Allow two distinct λs.
  - Both real-valued numbers but can be different from each other.

$$C\left(\begin{bmatrix}1 & 1\\3 & 2\end{bmatrix}\right) = \lambda_1\begin{bmatrix}1\\3\end{bmatrix} + \lambda_2\begin{bmatrix}1\\2\end{bmatrix}, \lambda \in \mathbb{R}$$

Example of column space of a matrix with two column

- What is the set of all vectors that can be reached by some linear combination of these two column vectors ?
  - All vectors in $\mathbb{R}^2$.
  - For example, vector $[-4 \quad 3]^T$ can be obtained by scaling the two columns.
    - In above matrix example, when λ_1, λ_2 is 11 and -15, respectively.
  - These two columns can be appropriately weighted to reach any point in $\mathbb{R}^2$.

- Always two columns of matrix cover any point in $\mathbb{R}^2$? $\boxed{}$!

Another Example of Column Space of Matrix with Two Columns

Example of another matrix with two columns

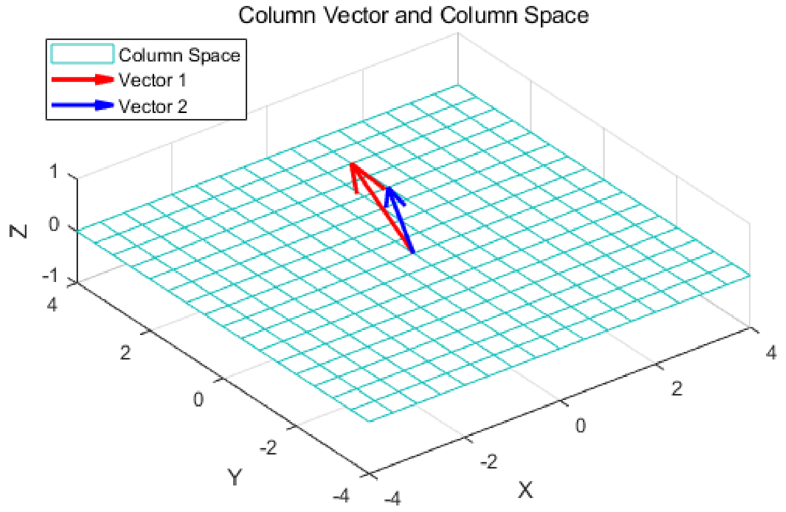
$$C\left(\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}\right) = \lambda_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \lambda_2 \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \lambda \in \mathbb{R}$$

Example of column space of another matrix with two column

- Is it possible to reach any point in $\mathbb{R}^2$?
  - $\boxed{}$
  - Can't produce another vector(ex. $[3 \quad 5]^T$) by linear combination of the two columns.
- What is dimensionality of its column space ?
  - Two columns are colinear.
    - One is already a scaled version of the other.
  - Column space of this 2×2 matrix is still just a $1D$ subspace.
- Having N columns in a matrix does not guarantee that the column space will be $N\ dimension$.
  - Dimensionality of columns space equals the number of columns only if the columns form a linearly independent set.

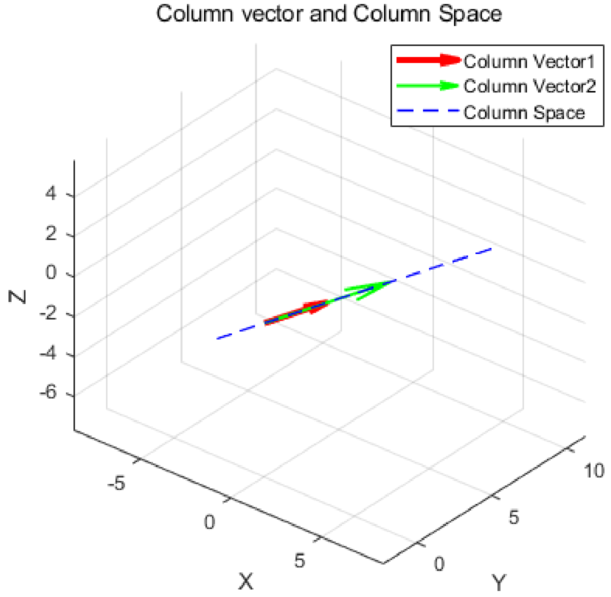
Visualization of Previous Two Examples of Column Spaces

$$C\left(\begin{bmatrix}1 & 1\\3 & 2\end{bmatrix}\right) = \lambda_1\begin{bmatrix}1\\3\end{bmatrix} + \lambda_2\begin{bmatrix}1\\2\end{bmatrix}, \lambda \in \mathbb{R}$$



Visualization of column space of first example matrix

$$C\left(\begin{bmatrix}1 & 2\\3 & 6\end{bmatrix}\right) = \lambda_1\begin{bmatrix}1\\3\end{bmatrix} + \lambda_2\begin{bmatrix}2\\6\end{bmatrix}, \lambda \in \mathbb{R}$$



Visualization of column space of second example matrix

Example of Column Space of Matrix with Two Columns in $\mathbb{R}^3$

Example of matrix with two columns in $\mathbb{R}^3$

- Two columns in below matrix are linearly independent.
  - Can't express one as a scaled version of the other.
  - Column space of this matrix is $\boxed{}$, but it is $2D$ plane embedded in $\mathbb{R}^3$.
- Column space of this matrix is an infinite $2D$ plane, but it is merely an infinitesimal slice of $3D$.
  - Like an infinitely thin piece of paper that spans universe.

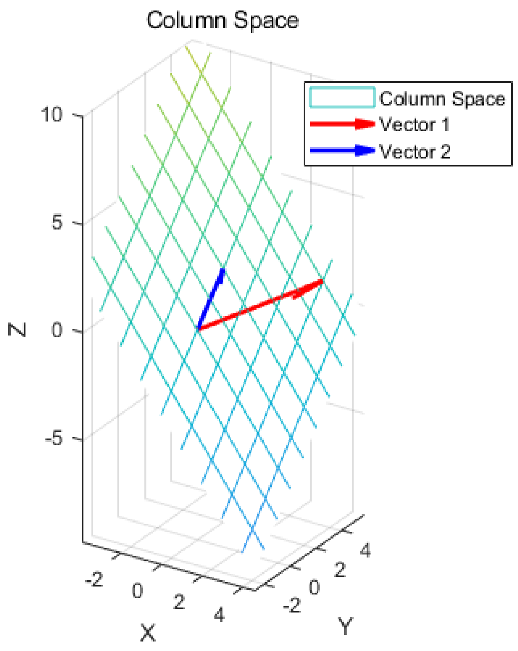
$$C\left(\begin{bmatrix} 3 & 0 \\ 5 & 2 \\ 1 & 2 \end{bmatrix}\right) = \lambda_1 \begin{bmatrix} 3 \\ 5 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \lambda \in \mathbb{R}$$

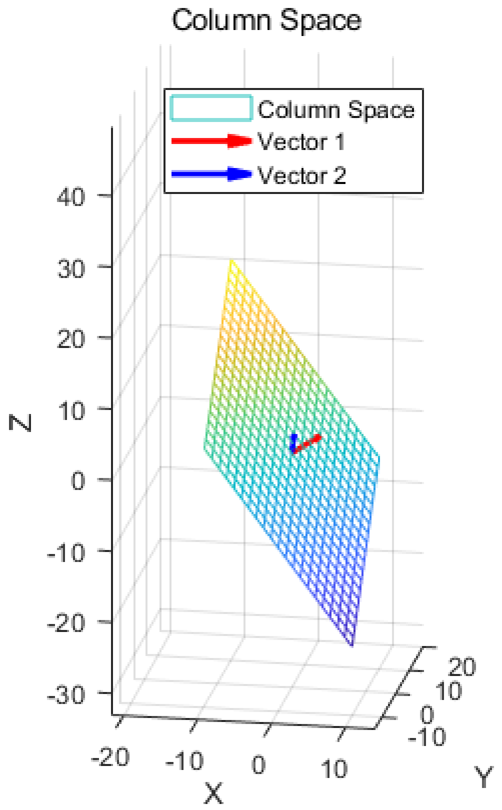
Example of column space of a matrix with two columns and three rows

Code Exercise of Column Space

Code Exercise (06_02)

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define two 3D vectors
v1 = [3; 5; 1];
v2 = [0; 2; 2];

% Calculate the normal vector (cross product of the two vectors)
normal = cross(v1, v2);

% Choose a point on the plane (for example, using v1)
point = v1;

% Equation of the column space: ax + by + cz = d
% where a, b, c are components of the normal vector, and d is the constant term of the column space equation
a = normal(1);
b = normal(2);
c = normal(3);
d = -dot(normal, point);

% Code for visualization
[x, y] = meshgrid(-10:1:10, -10:1:10); % Create a grid to represent the column space
z = (-d - a*x - b*y) / c; % Calculate z values of the column space

% Draw the column space
figure;
mesh(x, y, z);
hold on;

% Draw the two vectors
quiver3(0, 0, 0, v1(1), v1(2), v1(3), 'r', 'LineWidth', 2, 'AutoScale', 'off', 'MaxHeadSize', 1);
quiver3(0, 0, 0, v2(1), v2(2), v2(3), 'b', 'LineWidth', 2, 'AutoScale', 'off', 'MaxHeadSize', 1);

xlabel('X'); ylabel('Y'); zlabel('Z');
title('Column Space');
legend('Column Space', 'Vector 1', 'Vector 2');
axis equal;
xlim([-20 10]); ylim([-10 20]); zlim([-30 40]);
grid on;
```





MATLAB code example of Column Space and results

Row Space

- **Exact same concept of column space.**
  - Consider all possible weighted combinations of the rows instead of the columns.

- **Expression of row space**
  - Indicated as $R(\boldsymbol{A})$.

- **Properties of row space**
  - Row space of a matrix is the column space of the matrix transposed.
    - $R(\boldsymbol{A}) = \boxed{}$
  - Two matrix spaces are identical for symmetric matrices.
    - Because row space equals the column space of the matrix transpose.

Reminder of Column Space

- **Column space**
  - Summarizing column space as the following equation.

$$\boldsymbol{Ax} = \boldsymbol{b}$$

Column space equation

- **Can we find some set of coefficients in x such that the weighted combination of columns in A produce b?**
  - If yes, $\boldsymbol{b} \in C(\boldsymbol{A})$ and vector $\boldsymbol{x}$ tells how to weight the columns of $\boldsymbol{A}$ to get to $\boldsymbol{b}$.

- **How about when $\boldsymbol{b} = \boldsymbol{0}$?**

Null Space

- **Null space**
  - Summarizing null space as the following equation.

$$\boldsymbol{Ay} = \mathbf{0}$$

Null space equation

- **Can we find some set of coefficients in $\boldsymbol{y}$ such that the weighted combination of columns in $\boldsymbol{A}$ produces the zeros vector $\mathbf{0}$?**
  - Set $\boldsymbol{y} = \mathbf{0}$! (multiplying all columns by $\mathbf{0}$s will sum to the zeros vector.)
    - But it's a trivial solution and we will exclude it.
    - Therefore, question becomes different.

- **Can we find a set of weights – not all of which are $\mathbf{0}$ – that produces the zeros vector?**
  - Any vector $\boldsymbol{y}$ that can satisfy this equation is in null space of $\boldsymbol{A}$.
    - Notation: $N(\boldsymbol{A})$

Example of Null Space

Find null vector y when matrix is as below:

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

Example equation of null space

- Infinite number of vectors y that satisfy $\boldsymbol{Ay} = \mathbf{0}$ for that specific matrix $\boldsymbol{A}$.
  - Answer can be vector $[1 \quad 1]$, $[-1 \quad -1]$ or $[7.34 \quad 7.34]$.

Null space of this matrix can be expressed as:

$$N(\boldsymbol{A}) = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \lambda \in \mathbb{R}$$

Expression of null space of this matrix

Another Example of Null Space

- **Find null vector $\boldsymbol{y}$ when matrix is as below:**

$$\mathbf{A} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Second example equation of null space

  - Did you find the any vector $\boldsymbol{y}$ that satisfy $\boldsymbol{Ay} = \mathbf{0}$?
    - Matrix $\boldsymbol{A}$ has no null space.

- **Null space of this matrix is the empty set and can be expressed as:**

$$N(\boldsymbol{A}) = \boxed{\phantom{\{\mathbf{0}\}}}$$

Expression of null space of this matrix

Mathematical Property of Null Space

Difference between first example and second example

- First matrix
  - Contains columns that can be formed as scaled versions of other columns.
- Second matrix
  - Contains columns that form an independent set.

What can we find from this properties?

- Tight relationship between dimensionality of null space and linear independence of columns in a matrix
  - This relationship is given by the rank-theorem, which we will learn later.
- Key point: **Null space is empty when the columns of matrix form a linearly independent set.**

$$\boldsymbol{A_1} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$$

First Example equation of null space

$$\boldsymbol{A_2} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Second example equation of null space

Compute Null Space of a Matrix in MATLAB

Code Exercise (06_03)

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define matrices A and B
A = [1 -1; -2 2];
B = [1 -1; -2 3];

% Calculate the null spaces of A and B
nullA = null(A);
nullB = null(B);

% Display the null spaces
disp('Null space of matrix A:');
disp(nullA);
disp('Null space of matrix B:');
disp(nullB);
```

MATLAB code example of Null Space

MATLAB code and result

- Why did MATLAB choose 0.70710678 as the answer?
  - It's easier to chose 1.
  - MATLAB returned a $\boxed{}$.
- Advantages of unit vector
  - Convenient to work with.
  - Have several nice properties including numerical stability.

Shape of Null Space

Row space is orthogonal to the null space.

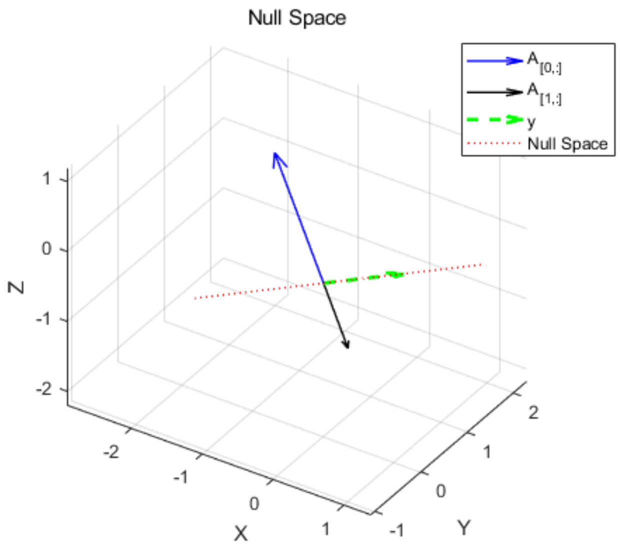
- Definition of the null space: $\boxed{}$
  - Each row of the matrix($\boldsymbol{a_i}$) leads to the expression $\boldsymbol{a_i y} = 0$.
  - Dot product between each row and null space vector is 0.

Every matrix has four associated subspaces.

- Column, row, null space.
- Fourth space is called right null space.
  - Null space of the rows.
  - Written as: $N(\boldsymbol{A^T})$

Why all the fuss about null space?

- Null space is the keystone.
  - To find eigenvectors and singular vectors.
  - You'll learn later.



Visualization of the null space of a matrix

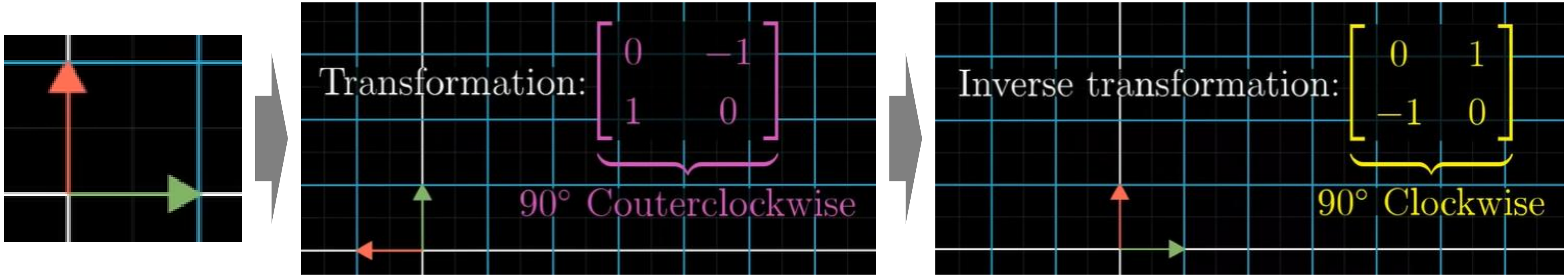
Inverse matrix, column space, and null space

Linear System of Equations

- **System of equations.**
  - List of variables
  - List of equations
- **Among system of equation, “linear system of equations” has a special form equation.**
  - Each variable is that it's scaled by some constant.
  - Scaled variables are added to each other.
- **Examples of linear system of equations.**
  - $2x + 5y + 3z = -3$
  - $4x + 0y + 8z = 0$
  - $1x + 3y + 0z = 2$
- **Linear system of equations can be packaged together into a single vector equation.**
  - $$\underbrace{\begin{bmatrix} 2 & 5 & 3 \\ 4 & 0 & 8 \\ 1 & 3 & 0 \end{bmatrix}}_{\substack{\text{Coefficients} \\ \boldsymbol{A}}} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\substack{\text{Variables} \\ \vec{\boldsymbol{x}}}} = \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}}_{\substack{\text{Constants} \\ \vec{\boldsymbol{v}}}}$$
- **So, linear equation can be represented as $A\vec{x} = \vec{v}$.**

Meaning of v

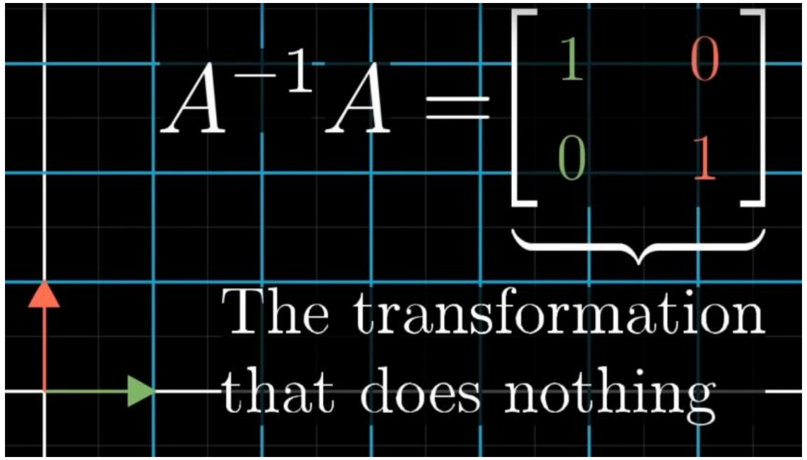
- **Matrix A corresponds with some linear transformation. So, solving $A\vec{x} = \vec{v}$ means we're looking for a vector $\vec{x}$, which, after applying the transformation, lands on $\vec{v}$.**
- **Example for solving $A\vec{x} = \vec{v}$.**
  - Assume that there are two equations $2x + 2y = -4$, and $1x + 3y = 0$. Then, linear equation can be represented as $\begin{bmatrix} \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{3} \end{bmatrix} \begin{bmatrix} \boldsymbol{x} \\ \boldsymbol{y} \end{bmatrix} = \begin{bmatrix} \mathbf{-4} \\ \mathbf{-1} \end{bmatrix}$.
  - When $Det(\boldsymbol{A}) \neq 0$ case which means, space does not get squished into a zero-area region.
    - There will always be **only one vector** that lands on $\vec{v}$, and you can find it by playing **the transformation in reverse**. Transformation in reverse corresponds to the “inverse of $\boldsymbol{A}^n$ ($\boldsymbol{A}^{-1} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}^{-1}$).
- **Example of inverse matrix.**
  1. If $\boldsymbol{A}$ is a **90-degree rotation**, then $\boldsymbol{A}^{-1}$ rotates **clockwise 90-degree**.
  2. If $\boldsymbol{A}$ is a **rightward shear** that pushes $\hat{\boldsymbol{J}}$ to the right ,then $\boldsymbol{A}^{-1}$ will be a **leftward sheer** that pushes $\hat{\boldsymbol{J}}$ to the left.



Visualization of inverse matrix

Properties of Inverse Matrix

- **Properties of A^{-1}**
  - If you first **apply A** then follow it with the **transformation A^{-1}**, you end up back where you started.
  - $A^{-1}A$ corresponds to doing nothing, which is called "**identity transformation**". In this case, $\hat{I}$ and $\hat{J}$ are unmoved, so its matrix is equal as $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- **Therefore, to know the initial position of $\vec{x}$ in $A\vec{x} = \vec{v}$, you can multiply A^{-1} by both sides to create $\vec{x} = A^{-1}\vec{v}$.**
  - $A^{-1}A\vec{x} = A^{-1}\vec{v}$, then $\vec{x} = \boxed{\phantom{A^{-1}\vec{v}}}$

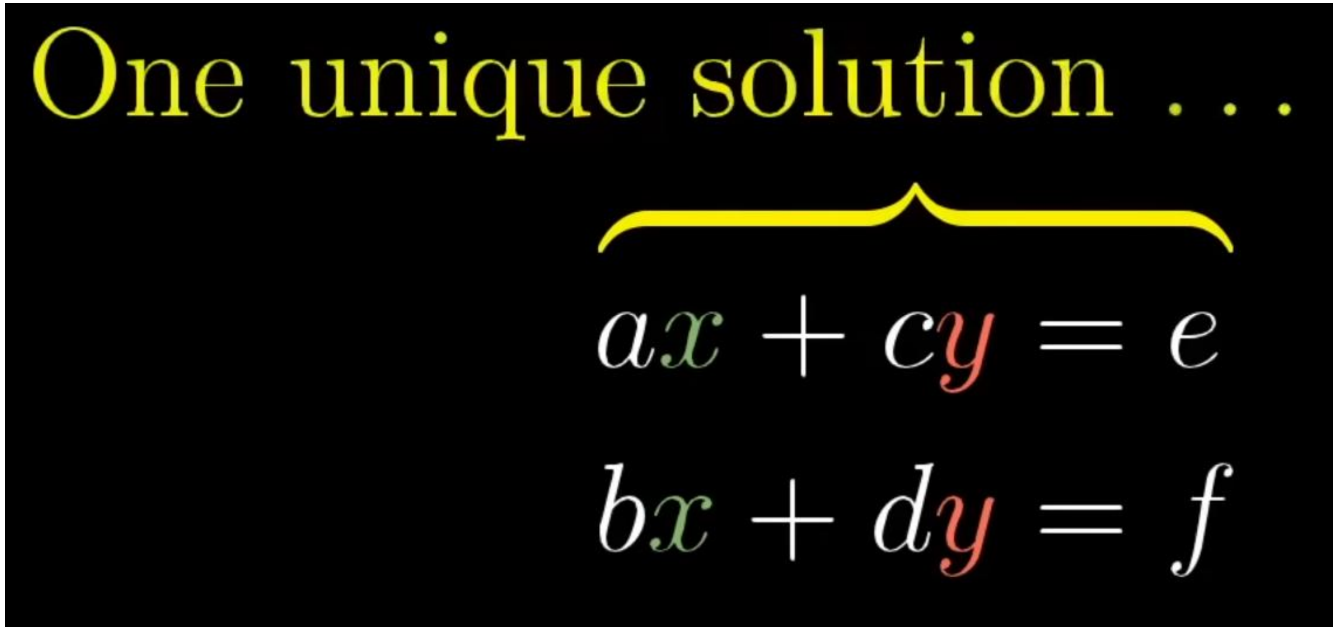


Visualization of identity transformation

https://youtu.be/bvB5uQXX7WY?si=6LtEnvFBlriJICfq

Formula of $det(A) \neq 0$

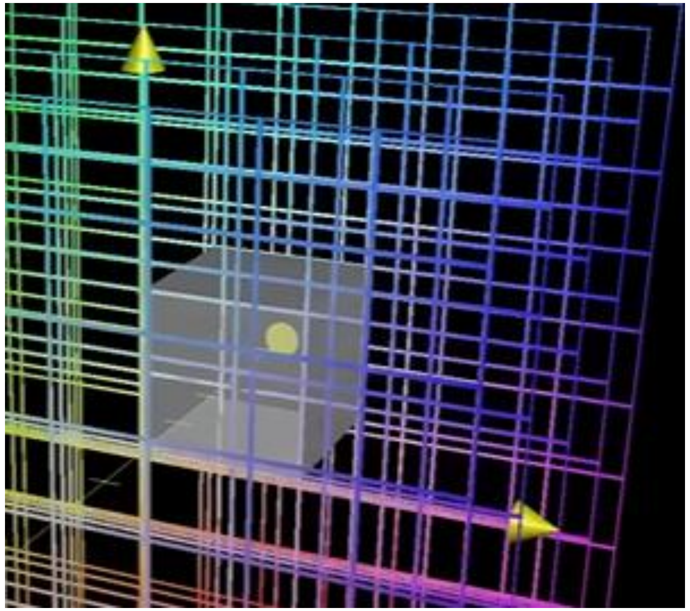
- **Two equations for two unknowns form a system of equations, and when these equations are the only ones that have a solution, then $det(A) \neq 0$.**
- **The formula is also valid for linear systems for many unknowns.**
  - If each unknown has a unique solution, the determinant is not zero.
- **If $det(A) \neq 0$, A^{-1} exists.**



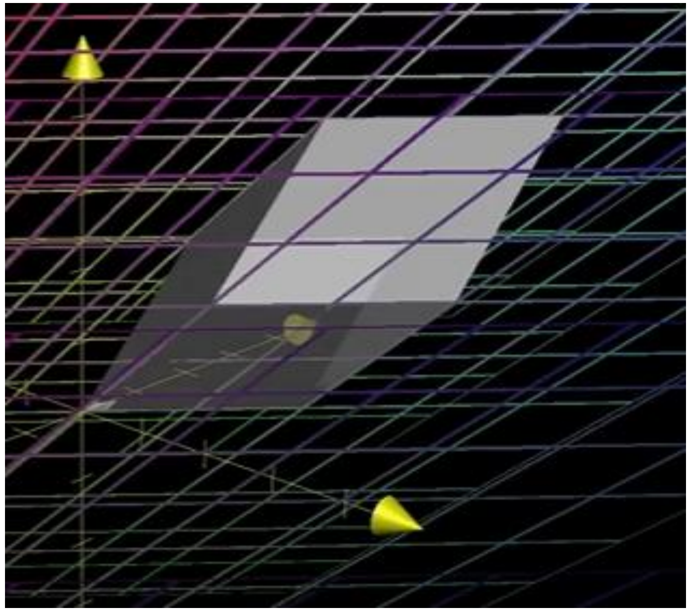
Formula of $\det(A) \neq 0$

Geometric Analysis of Inverse Matrix

- **System of equations can be interpreted geometrically.**
  - We have Transformation $\boldsymbol{A}$, vector $\vec{\boldsymbol{v}}$, vector $\vec{\boldsymbol{x}}$ that reaches $\vec{\boldsymbol{v}}$ through transformation.
  - Applying a $\boldsymbol{A}$ to a particular matrix and then applying the $\boxed{}$ can be reversed for the first time.

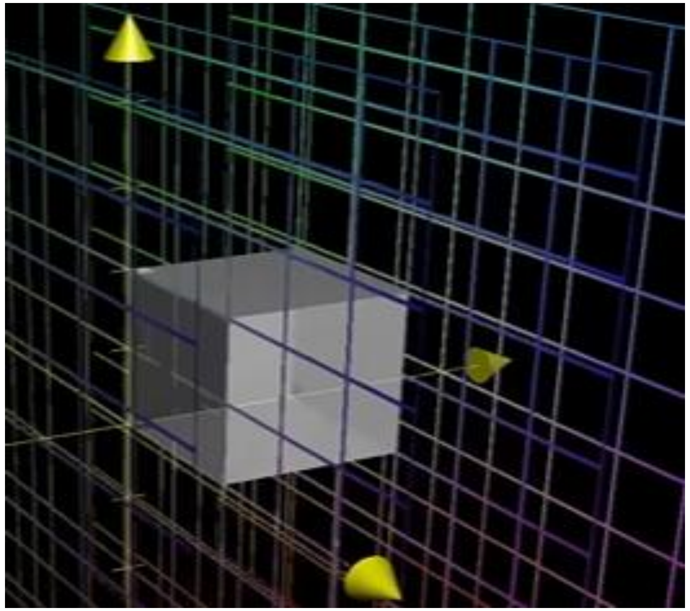


Original Space



Transformation A

$$\boldsymbol{A}\vec{\boldsymbol{x}} = \vec{\boldsymbol{v}}$$

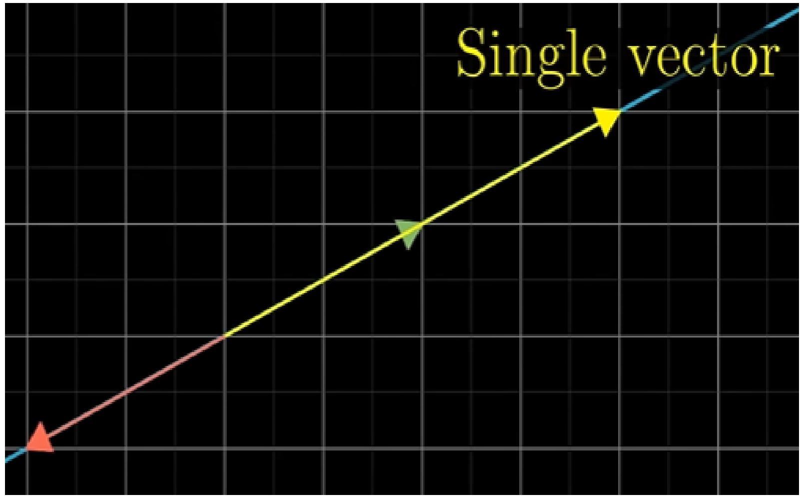


Inverse transformation A

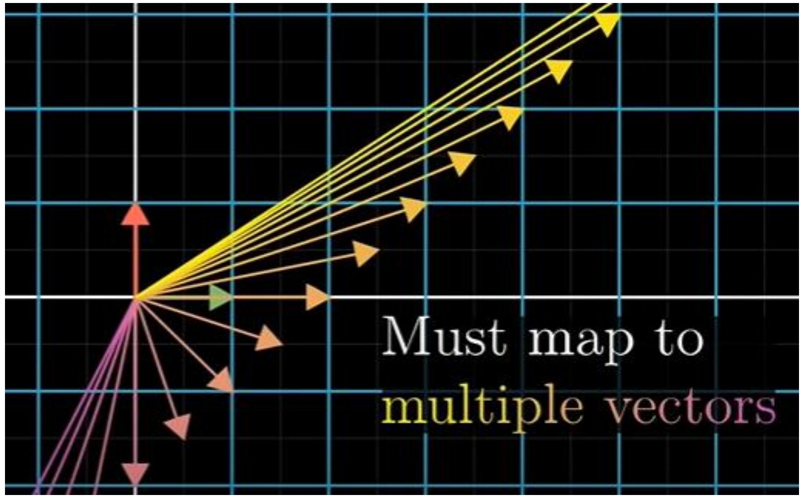
$$\boldsymbol{A}^T\boldsymbol{A}\vec{\boldsymbol{x}} = \boldsymbol{A}^T\vec{\boldsymbol{v}}$$

$A\vec{x} = \vec{v}$ when $det(A) = 0$

- **When the $det(A) = 0$**
  - Transformation squishes space into a smaller dimension.
  - There is no ______ matrix.
- **Function cannot unsquash a line to turn it into a plane.**
  - Functions can only take a single output for a single input.
  - However, if a straight line is stretched to a plane, one vector must correspond to multiple vectors.



Correspond to single vector

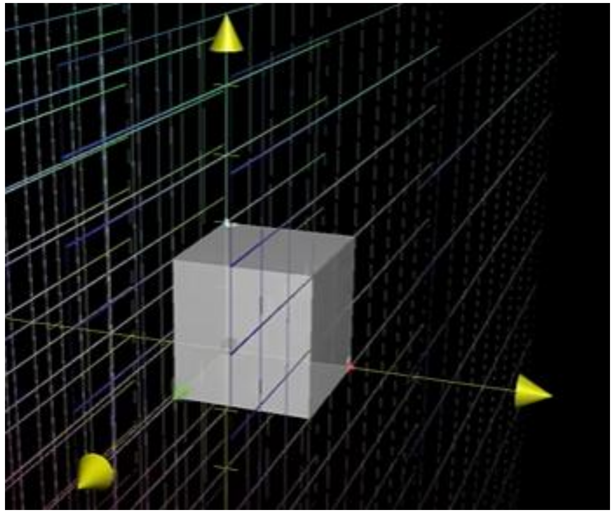


Correspond to multiple vectors

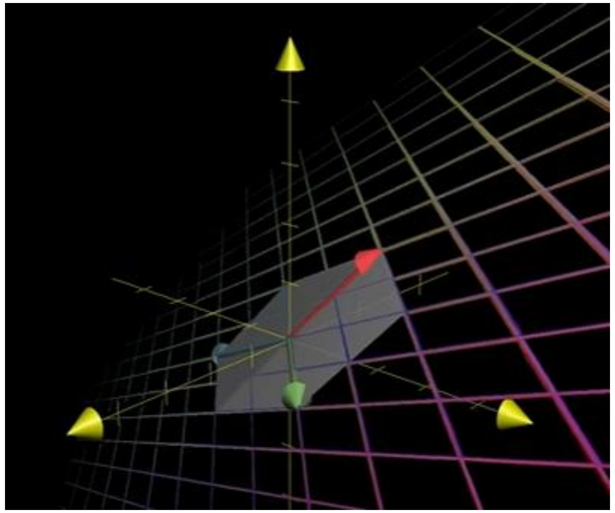
Geometric Transformation for Three Equations

Three equations for three unknown variables

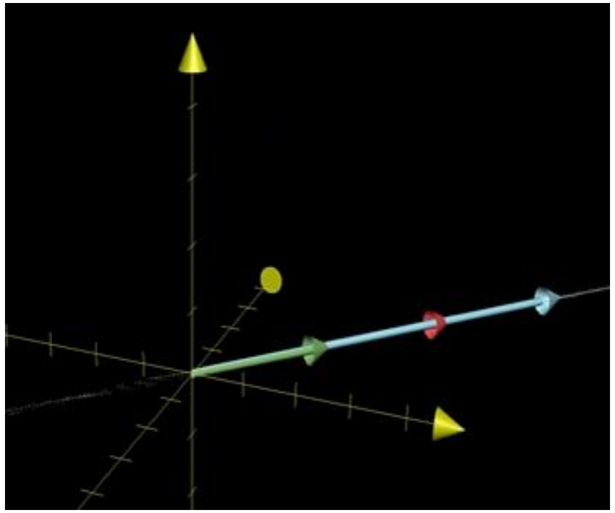
- If the transformation matrix squishes [ ] space to a plane, straight line, or point, the reverse does not exist.



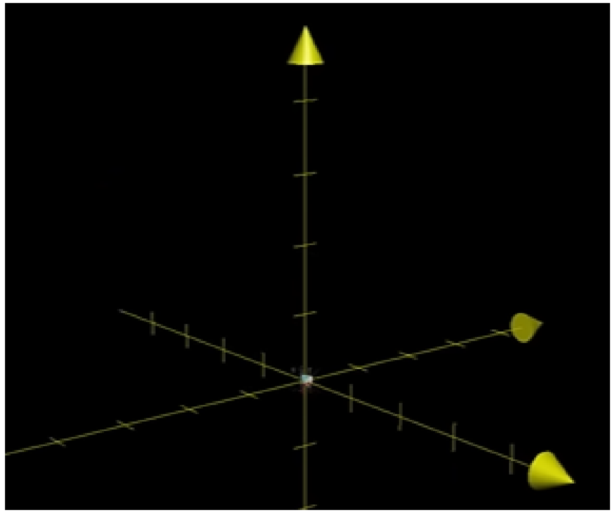
Cube in 3D



Transformation to plane in 3D



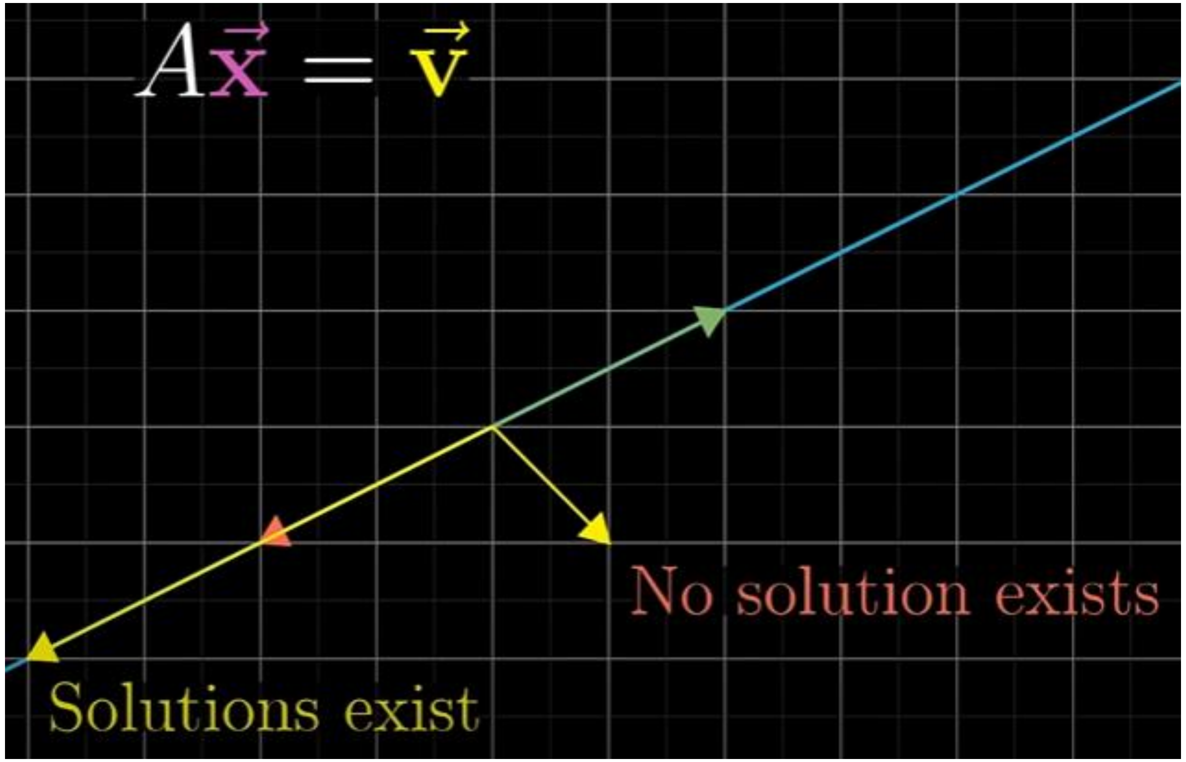
Transformation to straight line in 3D



Transformation to point in 3D

Cases Where $det(A) = 0$, but a Solution Exists

- **When the transformation matrix A converts a spatial vector $\vec{x}$ into a line vector $\vec{x}$,**
  - If the vector $\vec{v}$ exists on the transformed vector x after the transformation, there are infinitely many solutions.
  - However, if the vector $\vec{v}$ does not exist on the transformed vector $\vec{x}$, there are no solutions.

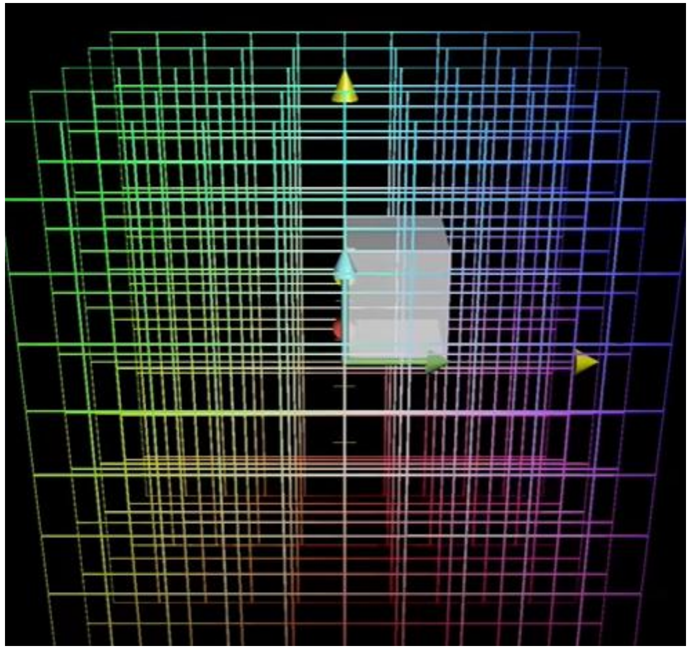


Cases of solutions exist and no exists

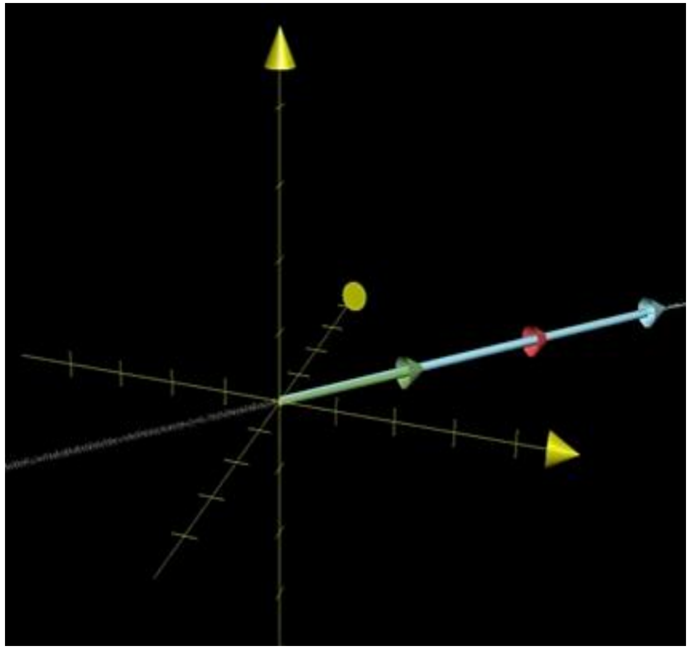
Transformation of Space Vector in Case of $det(A) = 0$

If $det(A) = 0$ in 3D

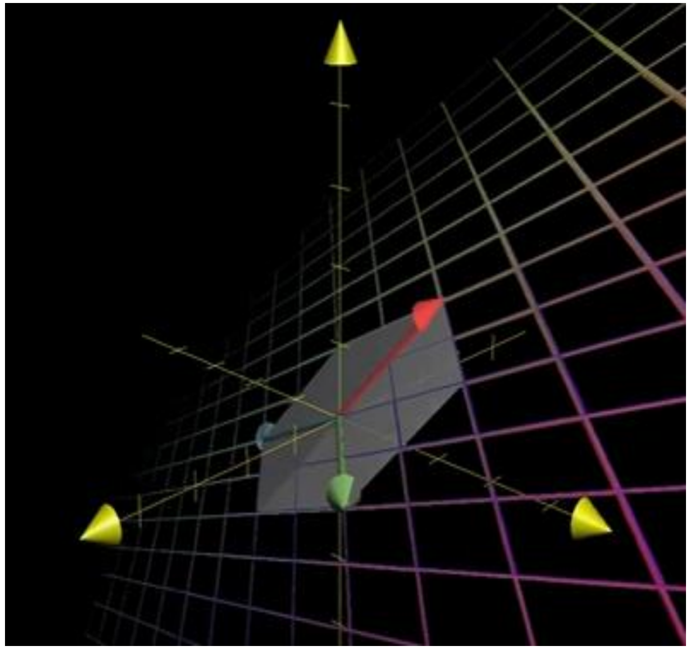
- 3D Space may be squished into a straight line.
- Or space can be squished to a plane.
- More specific expression is possible through $\boxed{}$ rather than just $det(A) = 0$.



Initial 3D space



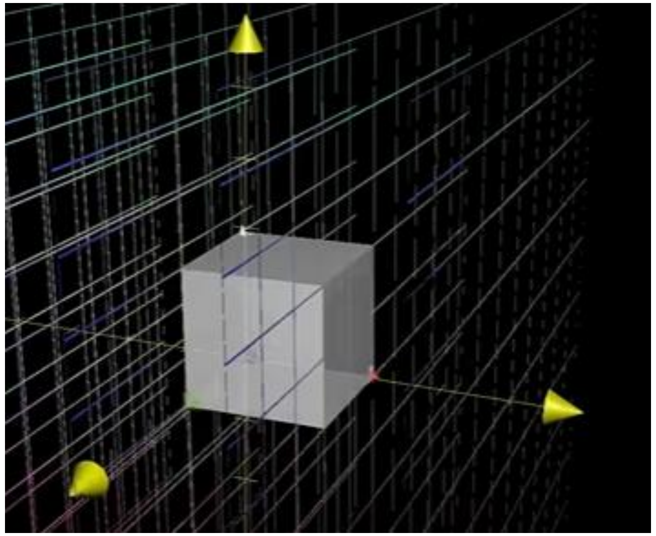
Squished to straight line



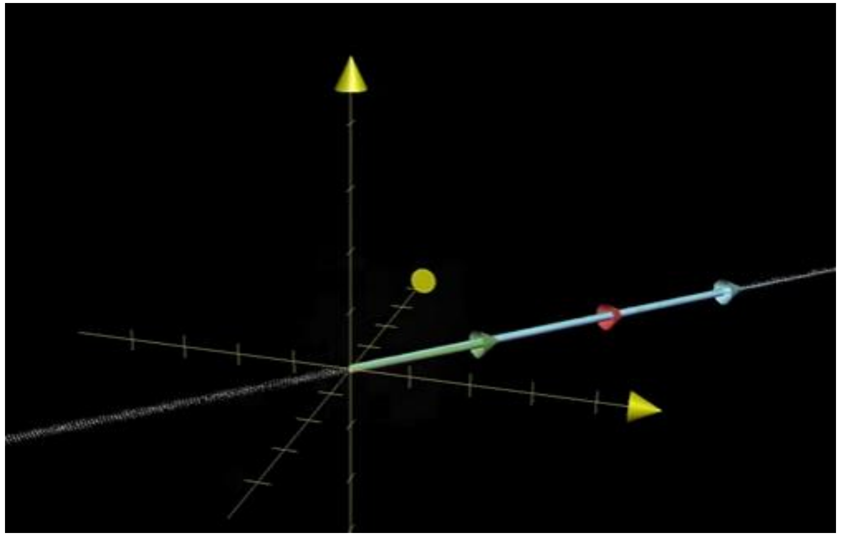
Squished to plane

Rank in Geometric Space

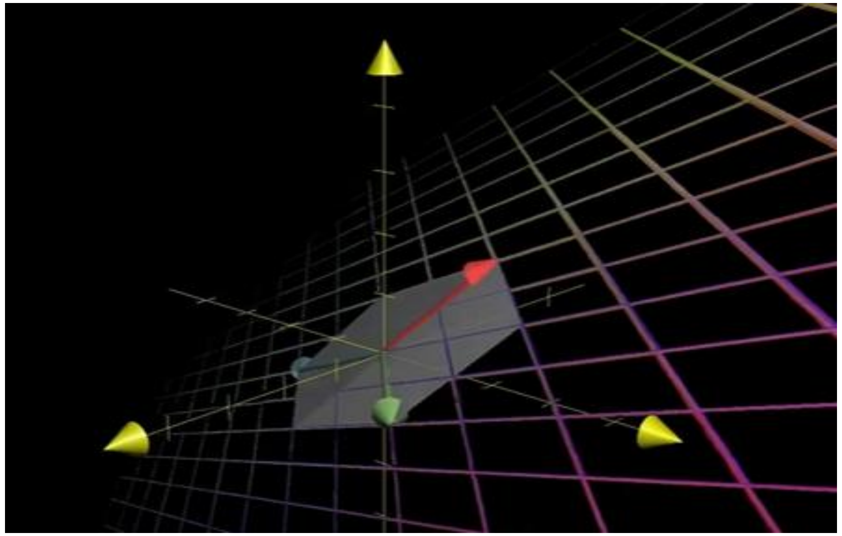
- **After applying transformation matrix A,**
  - If the output is line, that is, one-dimensional: $\boxed{}$
  - If the output is plane, that is, two dimensional: $\boxed{}$
- **In other words, "rank" refers to the number of dimensions of the output after conversion.**



Original 3D matrix



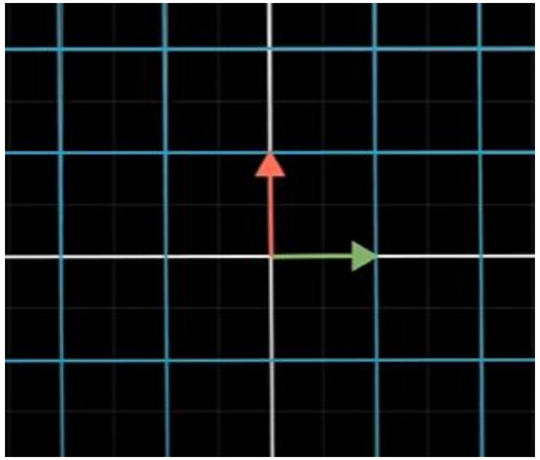
One-dimensional: rank 1



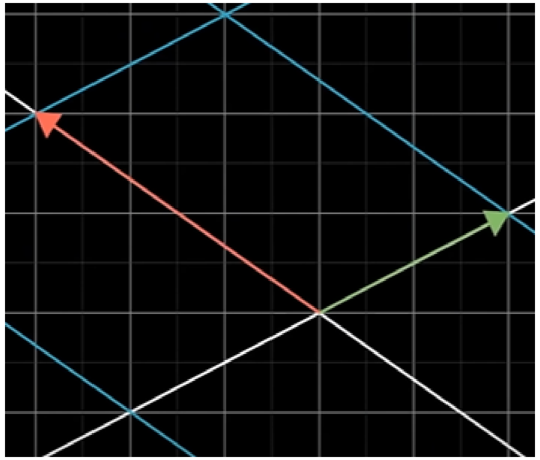
Two-dimensional: rank 2

Number of Dimensions of the Output “Rank"

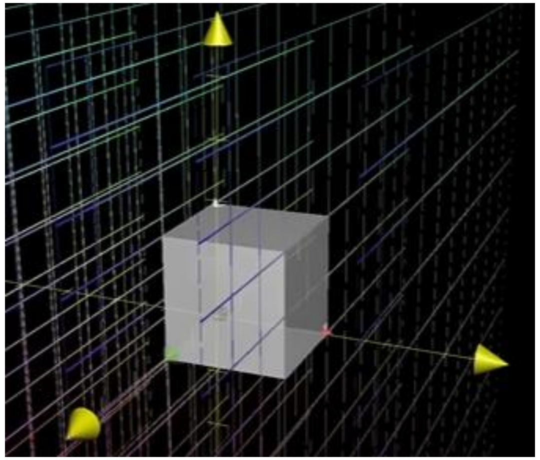
- **For a 2×2 matrix, the maximum rank is 2.**
  - In other words, a 2×2 matrix has a base vector in rank 2 that creates all two-dimensional spaces.
  - 2×2 matrix has a nonzero determinant.
- **Also, for a 3×3 matrix, the maximum rank is 3.**
  - However, if $\boldsymbol{det}(\boldsymbol{A}) = \mathbf{0}$, the rank may be reduced to 2 or 1.
- **The set of all outputs for a matrix is called the “ $\boxed{}$ of the matrix”.**



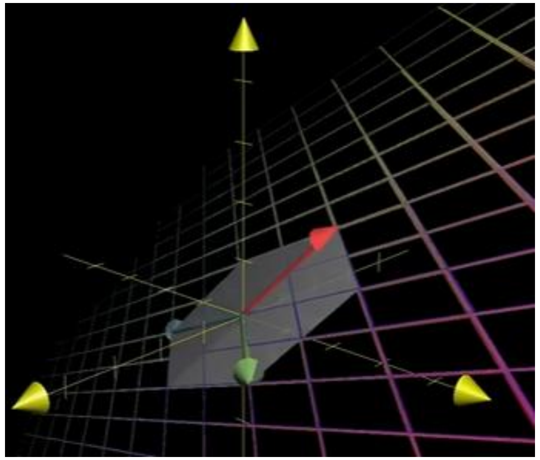
Original 2D matrix



Rank 2



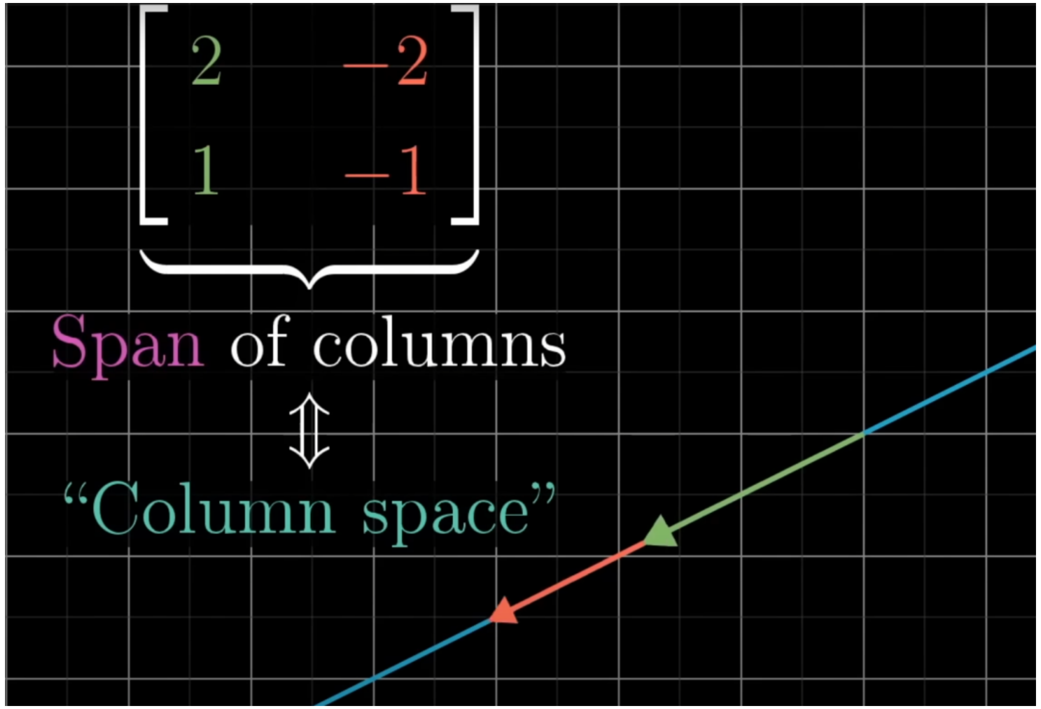
Original 3D matrix



Rank 2 when $\det = 0$

Column Space of Matrix

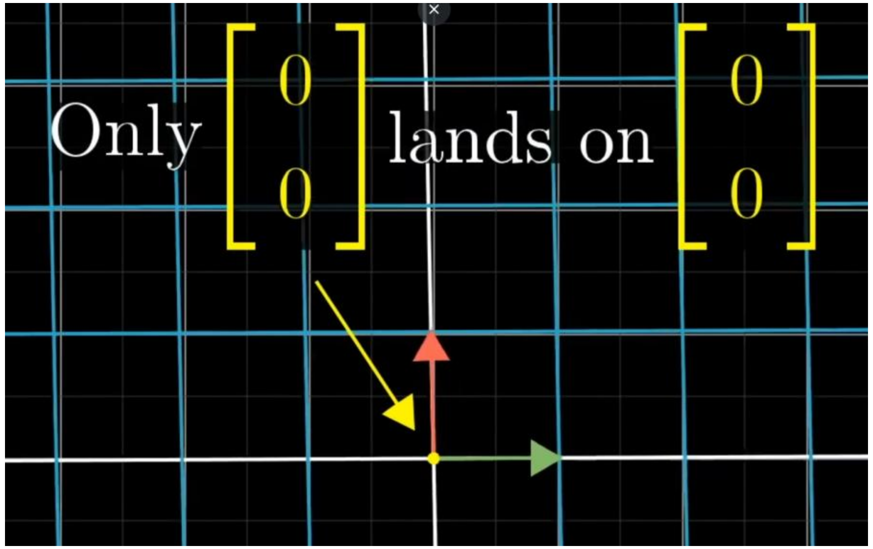
- **The columns of a matrix indicate the positions of the basis vectors they reach.**
- **The span of transformed basis vectors represents all possible outputs.**
- **In other words, the column space is the $\boxed{}$ of the columns of the matrix.**
  - Therefore, the rank is equivalent to the number of dimensions of the column space.



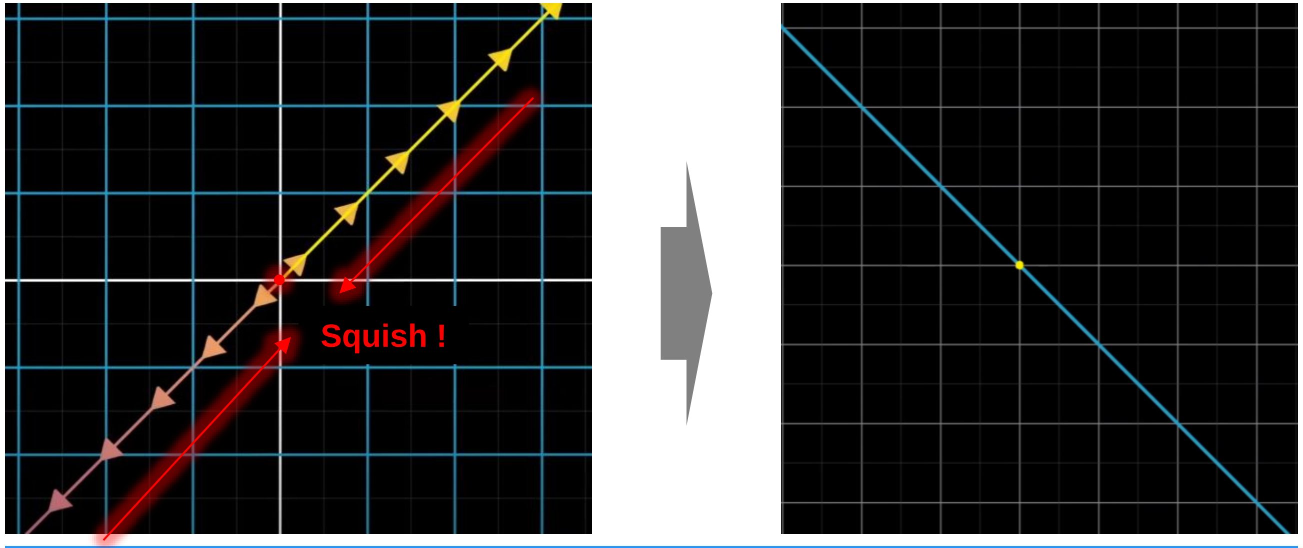
Meaning of columns space

Full Rank

- **A matrix is said to be of "full rank" when its rank is maximal.**
  - Rank equals the number of columns in the matrix.
- **Since linear transformations must always keep the origin fixed, all column spaces necessarily include the zero vector.**
  - In a **full rank transformation**, the only vector that reaches the origin is the **zero vector itself**.
  - For matrices **that aren't full rank**, which squish to a smaller dimension, there are a whole bunch of vectors that land on zero.



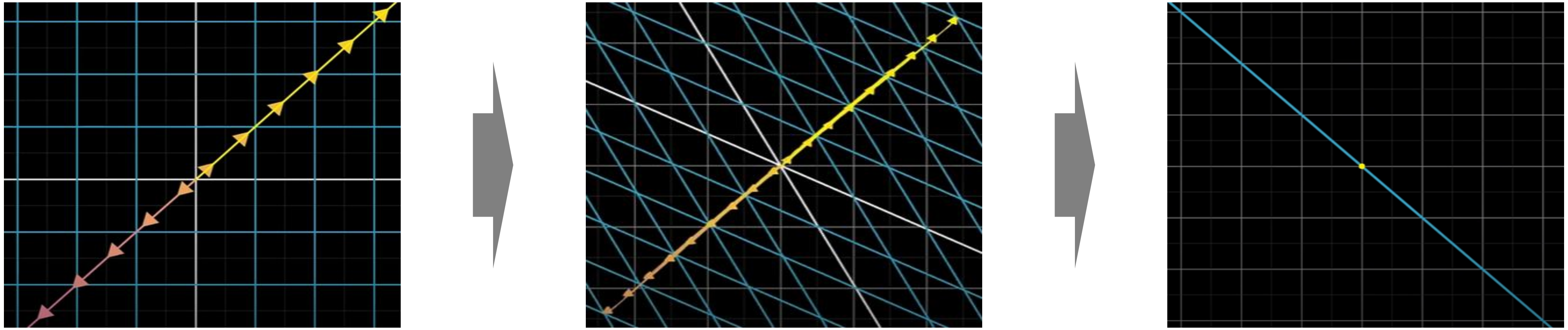
Zero vector is the only vector that lands at origin (full rank)



When squishing the vectors to a smaller dimension (not full rank)

Null Space of Matrix: Squishes 2D Space to Line

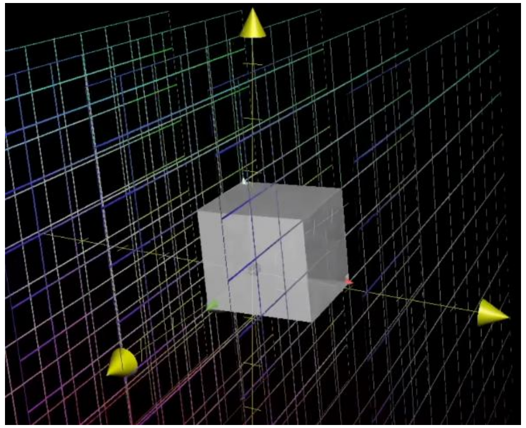
- **For full rank, the only vector that reaches the origin by transformation is the zero vector itself.**
  - In cases where the matrix is not of full rank, the space squishes to a lower dimension, resulting in a full line of vectors that can reach the origin.
  - Case 1: 2D transformation squishes **2D space** onto a **line**.
    - Separated line in a different direction full of vectors that get squished onto the origin.
  - Case 2: 3D transformation squishes 3D space squishes to a plane.
  - Case 3: 3D transformation squishes 3D space to a line.

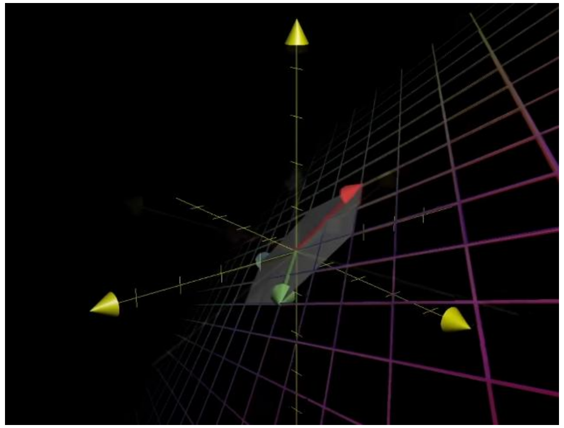


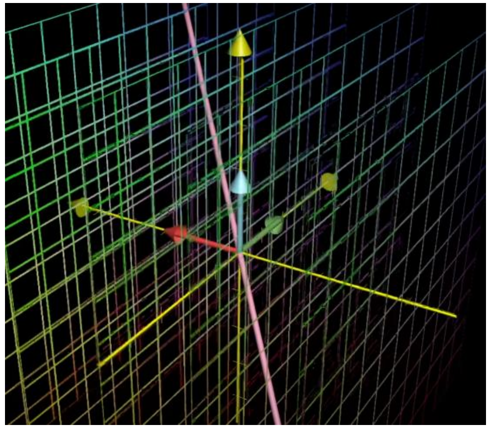
If 2D transformation squishes space to a line, a line in a different direction full of vectors that get squished onto the origin

Null Space of Matrix: Squishes 3D Space to Plane

- **For full rank, the only vector that reaches the origin by transformation is the zero vector itself.**
  - In cases where the matrix is not of full rank, the space squishes to a lower dimension, resulting in a full line of vectors that can reach the origin.
  - Case 1: 2D transformation squishes 2D space onto a line.
  - Case 2: 3D transformation squishes **3D space** squishes to a **plane**.
    - There is also a full line of vectors that land on the origin.
  - Case 3: 3D transformation squishes 3D space to a line.



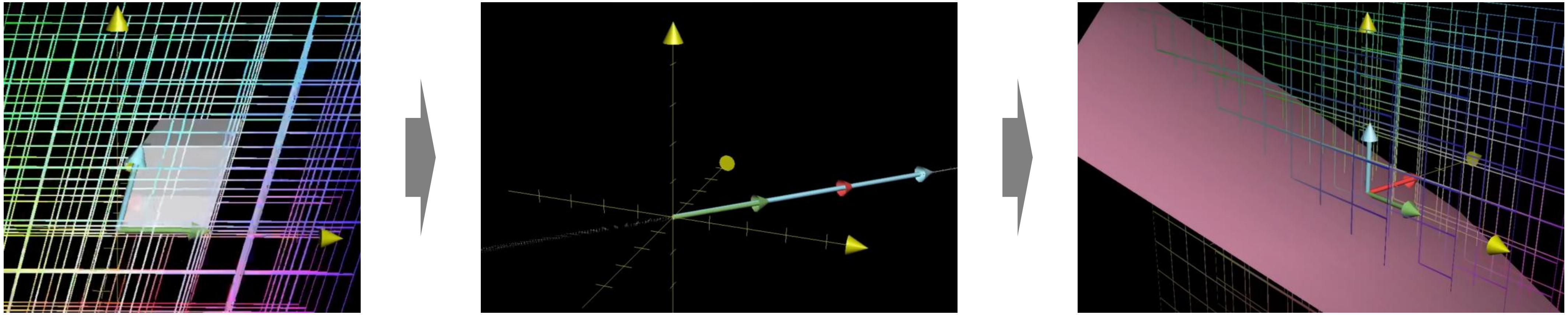




If 3D transformation squishes space to a plane, there is a full line of vectors that land on the origin

Null Space of Matrix: Squishes 3D Space to Line

- **For full rank, the only vector that reaches the origin by transformation is the zero vector itself.**
  - In cases where the matrix is not of full rank, the space squishes to a lower dimension, resulting in a full line of vectors that can reach the origin.
  - Case 1: 2D transformation squishes 2D space onto a line.
  - Case 2: 3D transformation squishes 3D space squishes to a plane.
  - Case 3: 3D transformation squishes **3D space** to a **line**.
    - There exists a plane filled with vectors that reach the origin.



If 3D transformation squishes space to a line, there exists a plane filled with vectors that reach the origin

Null Space of Matrix: Summary

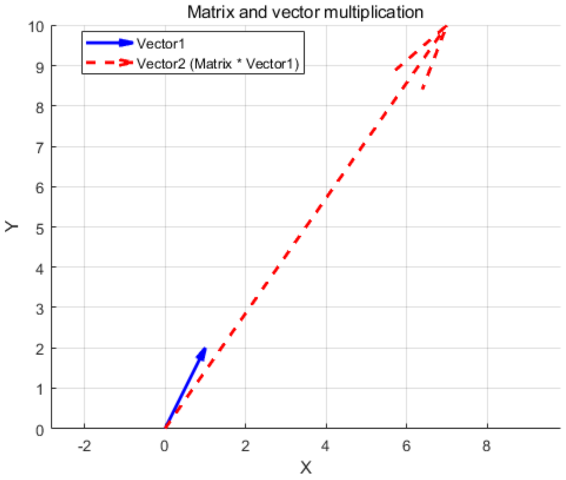
- **For full rank, the only vector that reaches the origin by transformation is the zero vector itself.**
  - In cases where the matrix is not of full rank, the space squishes to a lower dimension, resulting in a full line of vectors that can reach the origin.
  - Case 1: 2D transformation squishes 2D space onto a line.
  - Case 2: 3D transformation squishes 3D space squishes to a plane.
  - Case 3: 3D transformation squishes 3D space to a line.

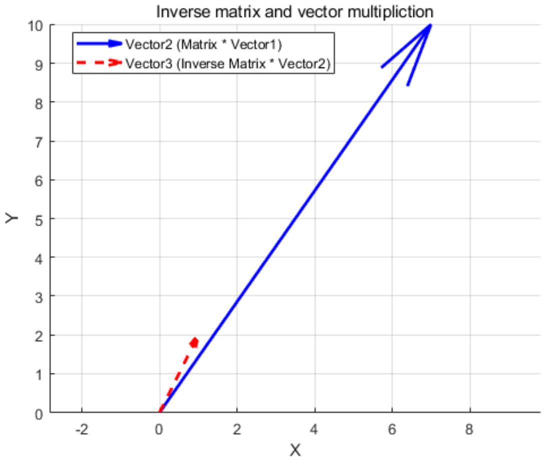
- **This set of vectors reaching the origin is called the "null space of the matrix".**
  - Null space is the space of all vectors that reach a null, i.e. zero vector.
  - In terms of simultaneous equations, this is the case where $\vec{v}$ is zero vector, where zero space is all possible solutions of the equation.

Code Exercise of Inverse Matrix

Code Exercise (06_04)

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define a square matrix
matrix = [1, 3; 2, 4];

% Define a 2D column vector
vector1 = [1; 2];

% Check if the matrix is singular (det(matrix) == 0 means
it's singular)
if det(matrix) == 0
    disp('The matrix is singular and does not have an
inverse.');
else
    % Calculate the inverse of the matrix
    inverseMatrix = inv(matrix);

    % Multiply the matrix by vector1 to get vector2
    vector2 = matrix * vector1;

    % Multiply vector2 by the inverse matrix to get vector3
    vector3 = inverseMatrix * vector2;

    % Print the matrices
    disp('Original Matrix:');
    disp(matrix);
    disp('Inverse Matrix:');
    disp(inverseMatrix);

    % Create the first figure for visualization of vector1
and vector2
    figure;
    hold on;
    grid on;
    axis equal;

    % Visualize vector1
    quiver(0, 0, vector1(1), vector1(2), 'b', 'LineWidth', 2,
'AutoScale', 'off', 'MaxHeadSize', 0.5);

    % Visualize vector2
    quiver(0, 0, vector2(1), vector2(2), 'r--', 'LineWidth',
2, 'AutoScale', 'off', 'MaxHeadSize', 0.5);

    % Labels and title for the first figure
    xlabel('X');
    ylabel('Y');
    title('Matrix and vector multiplication');
    legend({'Vector1', 'Vector2 (Matrix * Vector1)'},
'Location', 'best');

    % Create the second figure for visualization of vector2
and vector3
    figure;
    hold on;
    grid on;
    axis equal;

    % Visualize vector2 again for comparison
    quiver(0, 0, vector2(1), vector2(2), 'b', 'LineWidth', 2,
'AutoScale', 'off', 'MaxHeadSize', 0.5);

    % Visualize vector3
    quiver(0, 0, vector3(1), vector3(2), 'r--', 'LineWidth',
2, 'AutoScale', 'off', 'MaxHeadSize', 0.5);

    % Labels and title for the second figure
    xlabel('X');
    ylabel('Y');
    title('Inverse matrix and vector multipliction');
    legend({'Vector2 (Matrix * Vector1)', 'Vector3 (Inverse
Matrix * Vector2)'}, 'Location', 'best');
end
```





MATLAB code example of Inverse matrix and results

Rank

Concept of Rank

Rank

- A number associated with a matrix.
- Related to dimensionalities of matrix subspaces.
- Has important implications for matrix operations.
  - Inverting matrices.
  - Determining the number of solutions to a system of equations.

Properties of Rank

- **Rank is a non-negative integer.**
  - A matrix can have a rank of $0,1,2,\ldots,$ but not $-2\ or\ 3.14$.
- **Every matrix has one unique rank.**
  - Matrix cannot simultaneously have multiple distinct ranks.
    - Also means that rank is a feature of the matrix, not of the rows or the columns.
- **Rank of a matrix is indicated: $r(\boldsymbol{A})$ or $rank(\boldsymbol{A})$.**
- **Maximum possible rank of a matrix**
  - Smaller of its row or column count.
  - Maximum possible rank is $min\{M, N\}$.
  - Maximum possible rank is called "Full-rank"
    - Rank $r < min\{M, N\}$ is variously called "reduced-rank," "rank-deficient," or "singular."
- **Scalar multiplication does not affect the matrix rank.**
  - Exception of $\boxed{}$
  - $\mathbf{0}$ transforms the matrix into the zero matrix with a rank of $\mathbf{0}$.

Several Equivalent Interpretations and Definitions of Matrix Rank

- **Largest number of columns (or rows) that form a linearly independent set.**

- **Dimensionality of the column space.**
  - Same as the dimensionality of the row space.

- **Number of dimensions containing information in the matrix.**
  - Not the same as the total number of columns or rows in the matrix.
  - Because of possible linear dependencies, number of nonzero singular values of the matrix.

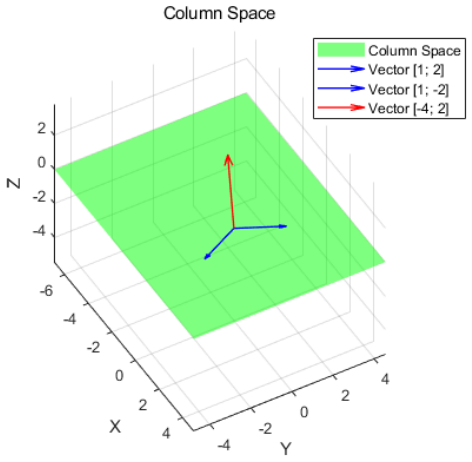
- **Surprising that definition of rank is same for columns and rows.**
  - Even for non-square matrix ?
    -

Example of Rank of Non-square Matrix

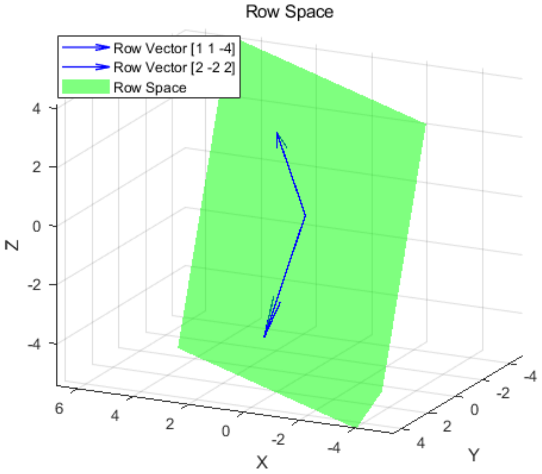
- **Column and row space of the matrix of right matrix.**
  - Column space : $\mathbb{R}^2$
  - Row space : $\mathbb{R}^3$
  - Three columns do not form a linearly independent set.
    - Described as a linear combination of the other two.
  - But they do span all $\mathbb{R}^2$.
    - The column space of the matrix is $2D$.
    - Two rows do form a linearly independent set.
      - Subspace they span is a $2D$ plane in $\mathbb{R}^3$.
  - Column space and row space of the matrix are different.
    - But ______ of those matrix spaces is the same.
      - Dimensionality is rank of the matrix.
    - Matrix has a rank of 2.

$$\begin{bmatrix} 1 & 1 & -4 \\ 2 & -2 & 2 \end{bmatrix}$$

Non-square matrix



Column space



Row space

Guess Rank

- **Guess rank of matrix based on previous descriptions.**
  - We didn't learn yet, but let's try.

$$\boldsymbol{A} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 4 & 12 \end{bmatrix} \quad \boldsymbol{C} = \begin{bmatrix} 1 & 3.1 \\ 2 & 6 \\ 4 & 12 \end{bmatrix}$$

$r(A) = \square$ $r(B) = \square$ $r(C) = \square$

$$\boldsymbol{D} = \begin{bmatrix} 1 & 3 & 2 \\ 6 & 6 & 1 \\ 4 & 2 & 0 \end{bmatrix} \quad \boldsymbol{E} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \boldsymbol{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$r(D) = \square$ $r(E) = \square$ $r(F) = \square$

- **Focus on rank corresponds to the largest number of columns that can form a linearly independent set.**
  - Which also corresponds to the dimensionality of the column space of the matrix.

Ranks of Special Matrices (I)

- **Vectors**
  - All of vectors have rank of $\boxed{}$
  - Only exception is the zeros vector.
- **Zeros matrices**
  - Zeros matrix of any size (including zeros vector) has a rank of $\mathbf{0}$.
- **Identity matrices**
  - Rank of identity matrix equals the number of rows.
    - Equal the number of columns.
  - $r(\boldsymbol{I_N}) = N$.
  - A special case of a diagonal matrix.
- **Diagonal matrices**
  - Rank of diagonal matrix equals the number of nonzero diagonal elements.
  - Each row contains maximum one nonzero element.
    - Impossible to create a nonzero number through weighted combinations of zeros.

Ranks of Special Matrices (II)

Triangular matrices

- Full rank only if there are nonzero values in all diagonal elements.
- With at least one zero in the diagonal will be reduced rank.

Random matrices

- Impossible to know a priori.
  - Depends on **distribution** of numbers from which elements in the matrix were drawn.
  - Depends on **probability** of drawing each number.
  - Example of 2×2 matrix populated with either $0s$ or $1s$
    - In some case, it can have a rank of 0 if the individual elements all equal 0.
    - Another case, it can have a rank of 2 if identity matrix is randomly selected.
- Way to create random matrices with guaranteed maximum possible rank.
  - In MATLAB, you can create random matrix using ‘randn()’ function

Ranks of Special Matrices (III)

$\boldsymbol{Rank}\ -1$ matrices

- Definition: Matrix that has a **rank of 1.**
- Only one column's (or row's) worth of information in the matrix.
- All other columns (or rows) are simply linear multiples.

$$\begin{bmatrix} -2 & -4 & -4 \\ -1 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 2 & 1 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 12 & 4 & 4 & 12 & 4 \\ 6 & 2 & 2 & 6 & 2 \\ 9 & 3 & 3 & 9 & 3 \end{bmatrix}$$

Example of $rank\ -1$ matrices

- Regardless of the size, **each column (or row) is scaled copy of the first column (or row)**.
- Method to create $rank\ -1$ matrix
  - Taking the outer product between two nonzero vectors.
    - Third matrix above is the outer product of $[4 \quad 2 \quad 3]^T$ and $[3 \quad 1 \quad 1 \quad 3 \quad 1]$.
- $Rank-1$ matrices are important in **eigen decomposition** and **singular value decomposition**.

Rank of Added and Multiplied Matrices

- **If you know the ranks of matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, do you automatically know the $\boldsymbol{rank}$ of $\boldsymbol{A}+\boldsymbol{B}$ or $\boldsymbol{AB}$?**
  - Answer: $\boxed{}$
  - But here are the rules of $rank$.
    - $rank(\boldsymbol{A}+\boldsymbol{B}) \leq rank(\boldsymbol{A}) + rank(\boldsymbol{B})$
    - $rank(\boldsymbol{AB}) \leq \min\{rank(\boldsymbol{A}), rank(\boldsymbol{B})\}$
  - Recommend memorizing the following rules:
    - Cannot know exact rank of a summed or product matrix.
      - Based on knowing ranks of individual matrices (except zeros matrix).
      - Instead, individual matrices provide upper bounds for the rank of the summed or product matrix.
    - Rank of a summed matrix **could be greater** than ranks of individual matrices.
    - Rank of a product matrix **cannot be greater** than the largest rank of the multiplying matrices.

Rank of Shifted Matrices

- **Shifted matrices have full rank.**
  - Goals of shifting a square matrix
    - **To increase its rank** from $r < M$ to $r = \boxed{}$
  - Obvious example: shifting zeros matrix by the identity matrix.
    - Rank of resulting sum $\mathbf{0} + \boldsymbol{I}$ is a full rank matrix.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\mathbf{0} + \boldsymbol{I}$ is a full rank matrix

Example of Shifting Matrices

- **Leftmost matrix has a rank of 2.**
  - Before add $0.1 * \boldsymbol{I}$ in Leftmost matrix.
    - Third column equals the second minus the first.
  - After add $0.1 * \boldsymbol{I}$ in Leftmost matrix.
    - Impossible to produce third column.
      - By some linear combination of the first two.
  - Information in the matrix has **hardly changed**.
    - Pearson correlation between the elements in the original and shifted matrix: $\rho = 0.99999997$.
    - **It's significant implications**.
      - Ex) $rank - 2$ matrix cannot be inverted whereas shifted matrix can.

$$\begin{bmatrix} 1 & 3 & 2 \\ 5 & 7 & 2 \\ 2 & 2 & 0 \end{bmatrix} + .01 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1.01 & 3 & 2 \\ 5 & 7.01 & 2 \\ 2 & 2 & .01 \end{bmatrix}$$

Example of shifting matrix

Code Exercise of Rank

Code Exercise (06_05)

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define the 1x3 vector
vector = [1 2 4];

% Define the 2x3 matrices
matrix1 = [1, 2, 4; 3, 6, 12];
matrix2 = [1, 2, 4; 3.1, 6, 12];

% Define the 3x3 identical (I think you meant identity) matrix and
3x3 zero matrix
identityMatrix = eye(3);
zeroMatrix = zeros(3, 3);

% Calculate the ranks
vector_rank = rank(vector);
matrix1_rank = rank(matrix1);
matrix2_rank = rank(matrix2);
identity_rank = rank(identityMatrix);
zero_rank = rank(zeroMatrix);

% Calculate and display the rank of each
disp('Rank of the 1x3 vector: ');
disp(vector_rank);
disp('Rank of the first 2x3 matrix: ');
disp(matrix1_rank);
disp('Rank of the second 2x3 matrix: ');
disp(matrix2_rank);
disp('Rank of the 3x3 identity matrix: ');
disp(identity_rank);
disp('Rank of the 3x3 zero matrix: ');
disp(zero_rank);
```

MATLAB code example of Null Space

Rank Applications

Augmenting Matrices

- **Meaning: to add extra columns to the right-hand side of the matrix.**
  - "Base" $M \times N$ matrix and "extra" $M \times K$ matrix.
    - Augmented matrix size: $M \times (N + K)$.
    - **Valid only two matrices have same number of rows.**
      - They can have different number of

$$\begin{bmatrix} 4 & 5 & 6 \\ 0 & 1 & 2 \\ 9 & 9 & 4 \end{bmatrix} \sqcup \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 & 1 \\ 0 & 1 & 2 & 2 \\ 9 & 9 & 4 & 3 \end{bmatrix}$$

Example of augmenting matrices

- **Problem: whether a vector is in the column space of a matrix .**
  - Mathematically written as $\boldsymbol{v} \in C(\boldsymbol{A})$.
  - Use augmenting matrices to solve.

Solve $v \in C(A)$

- **Algorithm for determining whether a vector is in the column space of a matrix.**
  1. Augment the matrix with the vector.
     - Call original matrix $\boldsymbol{A}$ and the augmented matrix $\widetilde{\boldsymbol{A}}$.
  2. **Compute the ranks of** the two matrices.
  3. Compare the two ranks.
     - One of two possible outcomes.
     - $rank(\boldsymbol{A}) = rank(\widetilde{\boldsymbol{A}})$
       - Vector $\boldsymbol{v}$ is **in the column space of matrix** $\boldsymbol{A}$.
     - $rank(\boldsymbol{A}) < rank(\widetilde{\boldsymbol{A}})$
       - Vector $\boldsymbol{v}$ is **not in the column space of matrix** $\boldsymbol{A}$.

Reason of Solving $\boldsymbol{v} \in \boldsymbol{C}(\boldsymbol{A})$

- **Repeat the result of solving $\boldsymbol{v} \in C(\boldsymbol{A})$.**
  - $rank(\boldsymbol{A}) = rank(\widetilde{\boldsymbol{A}})$
    - $\boldsymbol{v} \in C(\boldsymbol{A})$
  - $rank(\boldsymbol{A}) < rank(\widetilde{\boldsymbol{A}})$
    - $\boldsymbol{v} \notin C(\boldsymbol{A})$

- **Reason of the result.**
  - If $\boldsymbol{v} \in C(\boldsymbol{A})$
    - **$\boldsymbol{v}$ can be expressed as some linear weighted combination of the columns of A.**
    - Columns of augmented matrix $\widetilde{\boldsymbol{A}}$ form a linearly.
    - In terms of span, vector $\boldsymbol{v}$ is redundant in $\widetilde{\boldsymbol{A}}$.
      - **Rank stays the same.**
  - If $\boldsymbol{v} \notin C(\boldsymbol{A})$
    - **$\boldsymbol{v}$ cannot be expressed as some linear weighted combination of the columns of A.**
    - $\boldsymbol{v}$ has added new information into $\widetilde{\boldsymbol{A}}$.
      - Rank will be $\boxed{}$

Matrix norms

Matrix Norms

- **No ‘$\boldsymbol{the}$ matrix norm’.**
  - Multiple distinct norms that can be computed from a matrix.
  - Somewhat similar to vector norms.
    - Each matrix norm provides one number that characterizes a matrix as vector norm characterizes a vector.
    - Indicate using double-vertical lines as shown in below.

$$||\boldsymbol{A}||$$

Norm of A matrix

Different Meaning of Different Matrix Norms

- **Different matrix norms have different meanings.**
  - Myriad of matrix norms can be broadly divided into 2 families.
    - Element-wise (also sometimes called entry-wise)
      - Computed based on the individual elements of the matrix.
      - Can be interpreted to reflect the magnitudes of the elements in the matrix.
    - Induced
      - A measure of how much that transformation scales.
        - How much stretches or shrinks that vector.

- **We will study element-wise norms!**

Element-wise Norm

Euclidean norm

- A direct extension of the vector norm to matrices.

- Also called Frobenius norm.
  - Computed as the square root of the sum of all matrix elements squared as below Eq 1..

- In Eq 1.,
  - i and j correspond to the M rows and N columns.
  - Sub-scripted A_F indicating the Frobenius norm

$$||\boldsymbol{A}||_F = \sqrt{\sum_{i=1}^{M}\sum_{j=1}^{N} {a^2}_{ij}}$$

Eq 1. Frobenius norm

Another Notation of Frobenius Norm

Frobenius norm

- Also called $l2$ norm.

- $l2$ norm gets its name from the general formula.
  - Element-wise p-norms

$$\|\boldsymbol{A}\|_p = (\sum_{i=1}^{M}\sum_{j=1}^{N}|a_{ij}|^p)^{1/p}$$

General formula for element-wise p- norms

  - Get Frobenius norm when $p = \boxed{}$

Applications of Matrix Norm: Matrix Distance

- **Computing measure of matrix distance.**
  - Distance between a matrix and itself is 0.
  - How about distance between two distinct matrices?
    - Distance increases as the numerical values in those matrices **become increasingly dissimilar**.

- **Computation of Frobenius matrix**
  - Simply by replacing matrix $\boldsymbol{A}$ with matrix $\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$.
    - This equation will be come out later.

Applications of Matrix Norm: Regularization

- **Matrix norms have several application in machine learning and statistical analysis.**

- **Regularization in machine learning**
  - Aims to improve model fitting and to increase generalization of models to unseen data

- **Basic idea of regularization**
  - Add a matrix norm as a cost function to minimization algorithm.
  - Prevent model parameters from becoming too large.
    - $l2$ regularization also called "ridge regression".
  - Prevent encouraging sparse solutions.
    - $l1$ regularization also called "lasso regression".

- **Modern deep learning architectures rely on matrix norms.**
  - To achieve impressive performance at solving computer vision problems.

Code Exercise of Matrix Norm

Code Exercise (06_06)

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define a 2x2 matrix
matrix = [1, 2; 3, 4];

% Calculate the norm of the matrix
frobeniusNorm = norm(matrix, 'fro');
L2Norm = norm(matrix(:), 2);

% Display the Frobenius norm
disp('Frobenius norm of the matrix: ');
disp(frobeniusNorm);
disp('L2 norm of the matrix');
disp(L2Norm);
```

MATLAB code example of Matrix Norm

Matrix Trace

Definition of $trace$ in matrix

- Sum of its diagonal elements.

Notation of $trace$

- Indicated as $tr(\boldsymbol{A})$.

Exists only for square matrices.

- Both of the below matrices have same trace

$$\begin{bmatrix} 4 & 5 & 6 \\ 0 & 1 & 4 \\ 9 & 9 & 9 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 8 & 0 \\ 1 & 2 & 6 \end{bmatrix}$$

Both of matrices have the same trace

Matrix Trace and Frobenius Norm

Properties of matrix trace

- Trace of a matrix equals the sum of its eigenvalues.
  - A measure of the “volume” of its eigenspace.

- Frobenius norm can **be calculated with trace of matrix**
  - Square root of the trace times its transpose as shown in below Eq 1..

- Why Eq 1. works?
  - Each diagonal element of the matrix $A^T A$ is defined by dot product of each row with itself.

$$||\boldsymbol{A}||_{\boldsymbol{F}} = \sqrt{\sum_{i=1}^{M} \sum_{j=1}^{N} {a_{ij}}^2} = \sqrt{tr(\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{A})}$$

Eq 1. Calculation of Frobenius norm

Code Exercise of Matrix Trace

Code Exercise (06_07)

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define 3x3 matrices
matrix1 = [1, 2, 3; 4, 5, 6; 7, 8, 9];
matrix2 = [2, 3, 0; -2, 7, 4; -4, -1, 6];

% Calculate the trace of the matrix
tra1 = trace(matrix1);
tra2 = trace(matrix2);

% Display the trace of matrices
disp('Matrix1');
disp(matrix1);
disp('Matrix2');
disp(matrix2);
disp('Trace of the matrix1: ');
disp(tra1);
disp('Trace of the matrix1: ');
disp(tra2);
```

MATLAB code example of Matrix Trace

Summary

Summary

- **Two kinds of matrix norms**
  - Element-wise: reflects the magnitudes of the elements in the matrix.
    - Called the Frobenius norm (Euclidean norm or the $l2$ norm)
  - Included: reflects the geometric-transformative effect of the matrix on vectors.
- **The trace of a matrix is the sum of the diagonal elements.**
- **Four matrix spaces: column, row, null, left-null**
  - The set of linear weighted combinations of different features of the matrix.
- **The column space of the matrix**
  - All linear weighted combinations of the columns in the matrix, written as $C(\boldsymbol{A})$.
- **If some vector b is in the column space of a matrix**
  - Some vector $\boldsymbol{x}$ such that $\boldsymbol{Ax} = \boldsymbol{b}$.
- **The row space of the matrix**
  - The set of linear weighted combinations of the rows of the matrix, written as $R(\boldsymbol{A})$ or $C(\boldsymbol{A}^T)$.
- **The null space of the matrix**
  - The set of vectors that linearly combines the columns to produce the zeros vector.

Summary

Rank

- Nonnegative integer associated with a matrix
- The largest number of columns (or rows) that can form a linearly independent set.
- Reduced-rank or singular
  - Matrices with a rank smaller than maximum possible
- Full-rank
  - Shifting a square matrix by adding a constant to the diagonal
- One application: determine whether a vector is in the column space of a matrix
  - Comparing the rank of the matrix to the rank of the vector-augmented matrix.

Determinant

- A number associated with a square matrix.
- Zero for all reduced-rank matrices
- Nonzero for all full-rank matrices

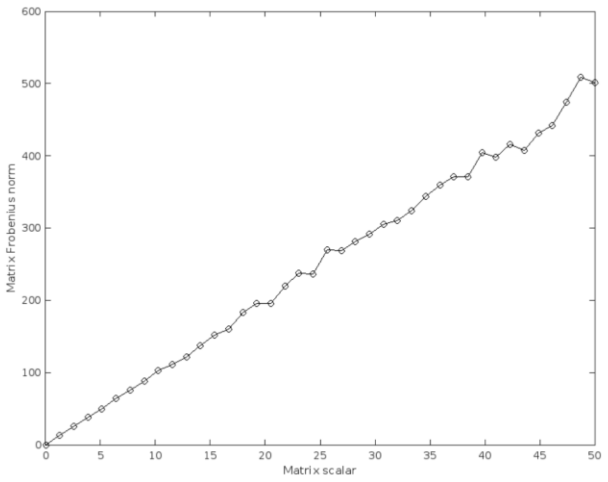
Characteristic polynomial

- Transform a square matrix, shifted by λ, into an equation that equals the determinant.

Code exercises

Frobenius Norm Exercise

- **The norm of a matrix is related to the scale of the numerical values in the matrix. In this exercise, you will create an experiment to demonstrate this. In each of 10 experiment iterations, create a 10×10 random numbers matrix and compute its Frobenius norm. Then repeat this experiment 40 times, each time scalar multiplying the matrix by a different scalar that ranges between 0 and 50. The result of the experiment will be a 40×10 matrix of norms. Figure shows the resulting norms, averaged over the 10 experiment iterations. This experiment also illustrates two additional properties of matrix norms: they are strictly nonnegative and can equal 0 only for the zeros matrix.**



Result of figure

Matrix size and ranks

- I will now show you how to create random matrices with arbitrary rank (subject to the constraints about matrix sizes, etc.). To create an $M \times N$ matrix with rank r, multiply a random $M \times r$ matrix with an $r \times N$ matrix.

Matrix, Transpose Matrix Rank and Size

- **Interestingly, the matrices $A, A^T, A^T A,$ and AA^T all have the same rank. Write code to demonstrate this, using random matrices of various sizes, shapes (square, tall, wide), and ranks.**

THANK YOU
FOR YOUR ATTENTION