Linear Algebra

Eigen value decomposition

Automotive Intelligence Lab.

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AI LAB Automotive Intelligence

Contents

https://youtu.be/PFDu9oVAE-g?list=PLZHQObOWTQDPD3MizzM2xVFitgF8hE_ab&t=82

Finding Eigenvalues

Find Eigenvalues Using MATLAB

- **To eigendecompose a square matrix…,**
  - First, find eigenvalues first.
  - Then, use each eigenvalue to find its corresponding eigenvector.
  - Super easy in MATLAB.
    - Just use function **eig()**
    - Eigenvalues of the matrix below are -0.37 and 5.37.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define the matrix
matrix = [1 2; 3 4];

% Get the eigenvalues
evals = eig(matrix);

% Display the eigenvalues
disp('Eigenvalues of the matrix:');
disp(evals);
```

MATLAB code to find eigenvalues

- **Probably you have question…!**
  - How are the eigenvalues of a matrix identified?

Method to Find Eigenvalues of Matrix

- **Do some simple arithmetic!**

$$\boldsymbol{Av} = \lambda\boldsymbol{v}$$
$$\boldsymbol{Av} - \lambda\boldsymbol{v} = \boldsymbol{0}$$
$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$$

Reorganize eigenvalue equation

- **First equation**
  - Repeat of eigenvalue equation.
- **Second equation**
  - Simply subtracted right-hand side to set equation of right-hand side to the zeros vector.
- **Third equation**
  - Left-hand side of second equation has two vector terms.
    - Both of which involve $\boldsymbol{v}$ so that factor out the vector.
  - After that, it leaves us with the subtraction of a matrix and a scalar $\boldsymbol{A} - \lambda$.
    - Matrix-scalar subtraction is not a defined operation in linear algebra.
    - So, $shift$ matrix by λ.
    - $\lambda\boldsymbol{I}$ is sometimes called a scalar matrix.

Meaning of Eigenvalue Equation

- **Eigenvector is in the null space of the matrix shifted by its $\boxed{}$.**

$$(A - \lambda I)v = 0$$

Reorganized eigenvalue equation

- **Remember…,**
  - Ignore trivial solutions in linear algebra which means don't consider $v = 0$ to be an eigenvector.

- **Matrix shifted by its eigenvalue is singular.**
  - Because only singular matrices have a **nontrivial null space**.

- **What else do we know about singular matrices?**
  - Know that their Determinant is **zero**!
  - Hence, we can write as below:

$$det(A - \lambda I) = 0$$

Determinant of $A - \lambda I$

Key to Finding Eigenvalues: Determinant

- **Shift matrix by unknown eigenvalue λ.**
  - Set its determinant to $\boxed{}$, and solve for λ.

- **Example of finding eigenvalues in 2×2 matrix**
  - You can apply quadratic formula to solve for two λ values.

$$\left|\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$$

$$(a - \lambda)(d - \lambda) - bc = 0$$

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

Process of finding eigenvalues

Examples to Finding Eigenvalues

- **Traditional way to find eigenvalues**
  - Subtract the unknown value lambda off the diagonals.
  - Solve for the determinant is equal to zero.

- **Direct way to find eigenvalues**
  - Trace of matrix is equal to sum of the eigenvalues.
  - Determinant of a matrix is equal to the product of the two eigenvalues.

Find the eigenvalues of $\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$

$$\det\left(\begin{bmatrix} 3-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix}\right) = (3-\lambda)(1-\lambda) - (1)(4)$$
$$= (3 - 4\lambda + \lambda^2) - 4$$
$$= \underbrace{\lambda^2 - 4\lambda - 1 = 0}$$
$$\lambda_1, \lambda_2 = \frac{4 \pm \sqrt{4^2 - 4(1)(-1)}}{2} = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5}$$

Traditional way to find eigenvalue

$$1)\quad \frac{1}{2}\mathrm{tr}\left(\begin{bmatrix} \boxed{a} & b \\ c & \boxed{d} \end{bmatrix}\right) = \frac{a+d}{2} = \frac{\lambda_1 + \lambda_2}{2} = m \quad \text{(mean)}$$

$$2)\quad \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc = \lambda_1\lambda_2 = p \quad \text{(product)}$$

$$3)\quad \lambda_1, \lambda_2 = m \pm \sqrt{m^2 - p}$$

$$\begin{bmatrix} 8 & 4 \\ 2 & 6 \end{bmatrix} \quad m = 7, \quad p = 40$$

Direct way to find eigenvalue

Code Exercise to Find Eigenvalues

Code Exercise (13_01)

- Find the eigenvalues using different method.
- Use the 'direct way' in the previous slide.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% 2x2 matrix : A
A = [1 2; 3 4];

% Calculate the trace of the matrix
trA = trace(A);

% Calculate the determinant of the matrix
detA = det(A);

% Calculate the eigenvalues using the direct way
lambda1 = ;
lambda2 = ;

% Display the eigenvalues
disp('Eigenvalues of the matrix:');
disp([lambda1 lambda2]);
```

MATLAB code to find eigenvalues using direct way

Logical Progression of Mathematical Concepts of Eigenvalue Equation

- **The matrix-vector multiplication acts like: [ ] -vector multiplication.**

- **Set eigenvalue equation to zeros vector, and factor out common terms.**
  - Eigen vector is null space of matrix shifted by eigenvalue.
  - Do not consider zeros vector to be an eigenvector.
    - Shifted matrix is singular.

$$\boldsymbol{Av} = \lambda \boldsymbol{v}$$
$$\boldsymbol{Av} - \lambda \boldsymbol{v} = \boldsymbol{0}$$
$$(\boldsymbol{A} - \lambda \boldsymbol{I})\boldsymbol{v} = \boldsymbol{0}$$

- **Set determinant of shifted matrix to [ ].**
  - Solve for unknown eigenvalue.

$$\boldsymbol{det(A - \lambda I) = 0}$$

- **Determinant of an eigenvalue-shifted matrix set to [ ].**
  - Called characteristic polynomial of the matrix.

$$\lambda^2 - (a+d)\lambda + (ad - bc) = 0$$

- **$\boldsymbol{n\ th}$ order polynomial has $\boldsymbol{n}$ solutions.**
  - Some of solutions might be complex-valued.
    - Called fundamental theorem of algebra.
  - Characteristic polynomial of an $M \times M$ matrix will have λ^M term.
    - $M \times M$ matrix will have M eigenvalues.

Finding Eigenvectors

Find Eigenvectors Using MATLAB

- **Finding eigenvectors is super easy in MATLAB.**
  - Most important thing to keep in mind
    - Eigenvectors are stored in columns of the matrix.
  - Columns of the matrix $evecs$
    - Eigenvectors
    - Columns are same order as eigenvalues.
  - Paired
    - Eigenvector in the first column of matrix $evecs$.
    - First eigenvalue in vector $evals$.
  - People use variable names $\boldsymbol{L}$ & $\boldsymbol{V}$ or $\boldsymbol{D}$ & $\boldsymbol{V}$.
    - $\boldsymbol{V}$ matrix: each column i is eigenvector v_i.
    - $\boldsymbol{L}$ is for $\boldsymbol{\Lambda}$ (capital of λ)
    - $\boldsymbol{D}$ is for diagonal.
      - Eigen values are often stored in a diagonal matrix.
      - Reasons will be explained later in this chapter.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% 2x2 matrix
matrix = [1 2; 3 4];

% Calculate the eigenvalues and eigenvectors
[vecs, vals] = eig(matrix);

% Display the eigenvalues and eigenvectors
disp('Eigenvalues:');
disp(diag(vals));  % Extracts and displays the eigenvalues from the diagonal matrix
disp('Eigenvectors:');
disp(vecs);  % Displays the eigenvectors
```

MATLAB code to find eigenvalues

- **Important question**
  - **Where** do eigenvectors come from and **how** do we find them?

Important Thing to Keep In Mind About Eigenvectors When Coding

- **Eigenvectors are stored in the columns of the matrix.**
  - Not in the rows.
  - Disastrous consequences in applications.
    - If accidentally using the rows instead of the columns of the eigenvectors matrix.

- **Remember common convention in linear algebra.**
  - Vectors are in column orientation.

Method to Find Eigenvector

- **Find vector v that is in the null space of matrix shifted by λ.**
  - In other words:

$$\boldsymbol{v_i} \in \boldsymbol{N}(\boldsymbol{A} - \boldsymbol{\lambda_i}\boldsymbol{I})$$

Equation of eigenvector

- **Numerical example**

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \longrightarrow \lambda_1 = 3, \lambda_2 = -1$$

Example of matrix and its eigenvalues

  - Focus on the first eigenvalue.
    - Shift the matrix by 3 (value of first eigenvalue).
    - Find a vector in its null space.

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find eigenvector of the matrix

    - $[1 \quad 1]$: an eigenvector of the matrix **associated with** an eigenvalue of 3.
    - How can we find null space vectors (eigenvectors of the matrix)?

Method to Find Null Space Vectors in Practice

- **Good way to conceptualize the solution**
  - Use **Gauss-Jordan** to solve a system of equations.
    - Coefficients matrix is λ shifted matrix.
    - Constants vector is zeros vector.

- **In implementation…,**
  - More stable numerical methods are applied for finding eigenvalues and eigenvectors.
    - Including $\boldsymbol{QR}$ **decomposition** and **Procedure** called the power method

Sign and Scale Indeterminacy of Eigenvectors

- **Return to numerical example in previous section**
  - Why $[1 \quad 1]$ was an eigenvector of matrix?
    - $[1 \quad 1]$: a basis for the null space of the matrix shifted by its eigenvalue of 3.
  - Is $[1 \quad 1]$ unique eigenvector of matrix?
    - No, $[4 \quad 4]$ or $[-5.4 \quad -5.4]$ or …
    - Any scaled version of vector $[1 \quad 1]$ is a basis for that null space.
  - If v is an eigenvector of a matrix, αv can also be eigenvector.
    - αv for any real-valued α except zero.
- **Indeed, eigenvectors are important because of their $\boxed{}$.**
  - Not because of their magnitude

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \rightarrow \lambda_1 = 3\,, \lambda_2 = -1 \rightarrow \begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Previous example about eigenvalues and eigenvector

Questions About Infinity of Possible Null Space Basis Vectors

- **Is there one “best” basis vector?**
  - No “best” basis vector.
  - But convenient to have eigenvectors that are **unit normalized**.
    - Euclidean norm of 1.
    - Particularly useful for symmetric matrices for reasons will be explained later in this chapter.

- **What is “$correct$” sign of an eigenvector?**
  - There is none.
  - Can get **different eigenvector signs** from same matrix when using different software.
    - Python, Julia, Mathematica , ….
  - There are principled ways for assigning a sign in applications.
    - Such as PCA.
    - But it is just common convention to facilitate interpretation.

Diagonalizing a Square Matrix

Make Equations Compact and Elegant

- **Eigenvalue equation lists one eigenvalue and one eigenvector.**
  - Means that an $M \times M$ matrix has M eigenvalue equations.

- **Nothing wrong with that series of equations…!**
  - But this equation sets are ugly.
  - Ugliness violates one of the principles of linear algebra which make equations compact and elegant.

$$\boldsymbol{A}\boldsymbol{v}_{\mathbf{1}} = \lambda_1 \boldsymbol{v}_{\mathbf{1}}$$
$$\vdots$$
$$\boldsymbol{A}\boldsymbol{v}_{\boldsymbol{M}} = \lambda_M \boldsymbol{v}_{\boldsymbol{M}}$$

M eigenvalue equations of $M \times M$ matrix

- **Therefore, we need to transform this series of equations into one matrix equation for compact!**

Key Insight for Writing Out Matrix Eigenvalue Equation

- **Each $\boxed{}$ of the eigenvectors matrix is scaled by exactly one eigenvalue.**
  - Can implement this through post multiplication by a diagonal matrix.
  - Store eigenvalues in diagonal of a matrix instead of storing eigenvalues in a vector.

- **Form of diagonalization for a 3×3 matrix**
  - Using @ in place of numerical values in the matrix
  - In the eigenvectors matrix,
    - First subscript number corresponds to eigenvector.
    - Second subscript number corresponds to eigenvector element.
  - Take a moment to confirm!
    - Each eigenvalue scales all elements of its corresponding eigenvector and not any other eigenvectors.

$$\begin{aligned} Av_1 &= \lambda_1 v_1 \\ &\vdots \\ Av_M &= \lambda_M v_M \end{aligned} \quad \Rightarrow \quad \underbrace{\begin{bmatrix} @ & @ & @ \\ @ & @ & @ \\ @ & @ & @ \end{bmatrix}}_{A} \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{bmatrix} = \begin{bmatrix} \lambda_1 v_{11} & \lambda_2 v_{21} & \lambda_3 v_{31} \\ \lambda_1 v_{12} & \lambda_2 v_{22} & \lambda_3 v_{32} \\ \lambda_1 v_{13} & \lambda_2 v_{23} & \lambda_3 v_{33} \end{bmatrix}$$

$$= \begin{bmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Diagonalization for 3×3 matrix

Eigen Decomposition

- **Consider list of equivalent declarations of matrix eigenvalue equation as shown below.**

$$AV = V\Lambda$$
$$A = V\Lambda V^{-1}$$
$$\Lambda = V^{-1}AV$$

List of equivalent declarations

- **Code to return:**
  - Eigenvalues in a vector.

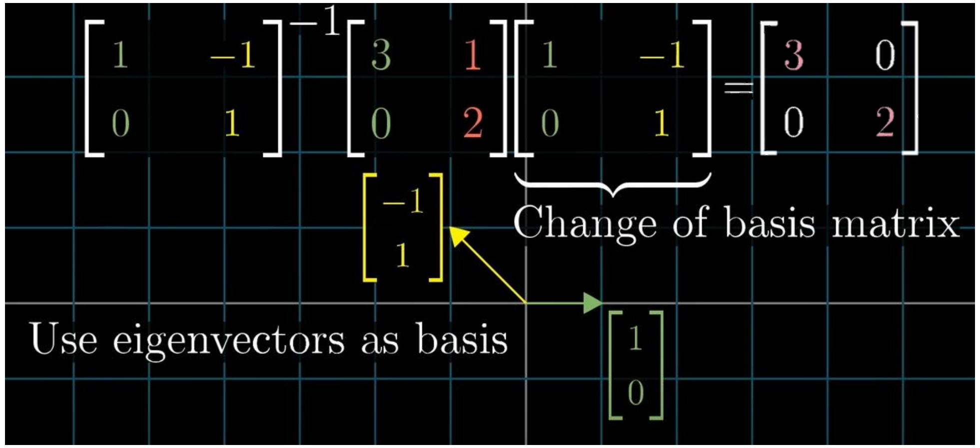
```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% 2x2 matrix
matrix = [1 2; 3 4];

% Calculate the eigenvalues and eigenvectors
[vecs, vals] = eig(matrix);

% Display the D matrix
disp('D matrix:');
disp(vals);
```

MATLAB code to get D matrix

Example of Diagonalizing a Square Matrix

■ **Diagonalizing matrix using eigenvectors (eigenbasis).**

$$\Lambda = V^{-1}AV$$



Example of Diagonalizing matrix

Interpretations of Eigenvalues and Eigenvectors

Interpretations in Geometry

Special combination of a matrix and a vector

- Matrix *stretched* vector but did not $\boxed{}$ that vector.
  - That vector: **eigenvector** of matrix
  - Amount of stretching: **eigenvalue**
- Eigenvectors point in same direction.
  - Before and after post multiplying the matrix.
- In Fig 1.,
  - $\boldsymbol{v_1}$, $\boldsymbol{v_2}$: eigen vectors.
- In Fig 2.,
  - $\boldsymbol{w_1}$, $\boldsymbol{w_2}$: not eigen vectors.

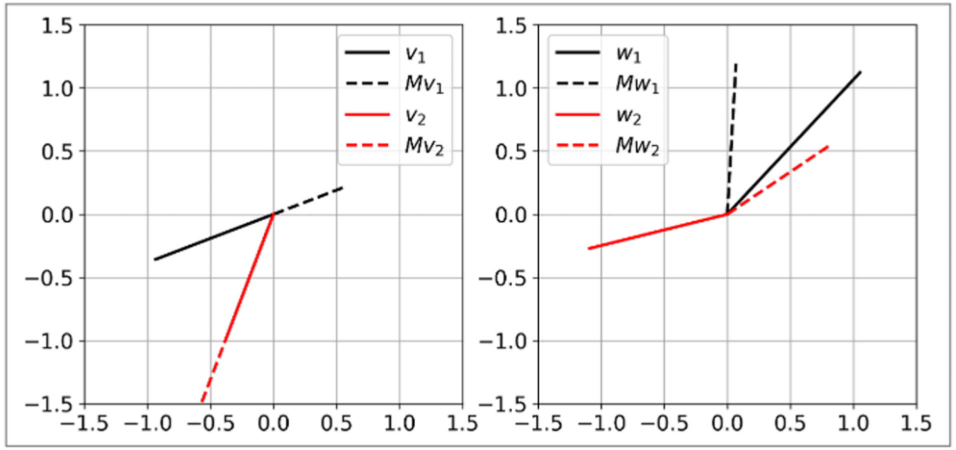


Fig 1. Fig 2.

Geometric meaning of eigenvector

- Matrix-vector multiplication acts like scalar-vector multiplication.
- Write eigenvalue equation as:

$$\boldsymbol{Av = \lambda v}$$

Eigenvalue equation

- Equation doesn't say that matrix equals the scala.
  - It says that the $effect$ of matrix on the vector is same as the $effect$ of the scalar on that same vector.

Principal Components Analysis

Implement Principal Component Analysis (PCA) on statistical data.

- To identify important patterns or structures.
  - We will practice PCA later!

Role of eigenvalue in PCA

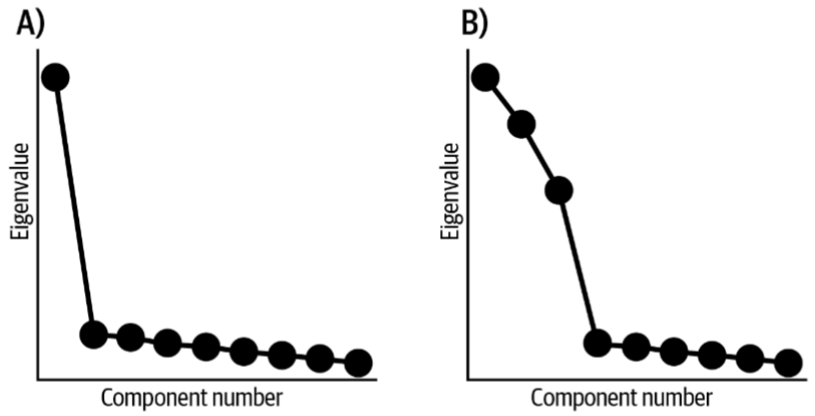
- Eigenvalue play a crucial role in PCA.
  - Represent variance of each principal component.
  - The larger the eigenvalue, the more variance (information) the principal component captures.

Scree plot

- graph of the eigenvalues of the dataset's covariance matrix

- A) 1 component accounts for most of the variance system.
  - All other components account for very little variance.

- B) 3 major subcategories.
  - Other components expect 3 major categories account for very little variance.



Example of Scree plot

Interpretations in Noise Reduction

- **Most datasets contain noise.**

- **Noise**
  - Refers to $\boxed{}$ in a dataset either unexplained or unwanted.

- **Method to reducing random noise**
  - Many ways to attenuate or eliminate noise,
    but optimal reduction strategy depends on origin of the noise or characteristics of signal
  - Method with eigenvalues and eigenvectors,
    - Identify eigenvalues and eigenvectors of a system.
    - And “project out” directions in the data space associated **with small eigen-values**.

- **Meaning of “projecting out” a data dimension**
  - Reconstruct dataset after setting some eigenvalues to zero which eigenvalues below some threshold.

Interpretations in Dimension Reduction (Data Compression)

- **It is beneficial to $compress$ data before transmitting it.**
  - Compression: Reduce the size of data while having minimal impact on the quality of the data.

- **One way to dimension-reduce a dataset**
  - Take its eigendecomposition
    - Drop eigenvalues and eigenvectors associated with small directions in data space.
    - Transmit only relatively larger eigenvector-value pairs.

- **All of the data compression idea is same!**
  - Decompose dataset into a set of basis vectors.
    - Basis vectors that capture the most important features of data.
  - Reconstruct a high-quality version of the original data.

Special Awesomeness of Symmetric Matrices

Orthogonal Eigenvector

- **Symmetric matrices have orthogonal eigenvectors.**
  - All eigenvectors of symmetric matrix are pair-wise orthogonal.

- **Start with an example, then discuss implications of eigenvector orthogonality, finally show proof.**

Code Exercise of Orthogonal Eigenvector

Code Exercise (13_02)

- Three dot products are all zero.
  - Within computer rounding errors on order of 10^{-16}.
- Symmetric matrices were created as random matrix times its transpose.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Create a random matrix and make it symmetric
A = randi([-3, 3], 3, 3);
A = A * A';  % Symmetric matrix

% Perform eigen decomposition
[V, D] = ;

% Display the eigenvalues and eigenvectors
disp('Eigenvalues:');
disp(diag(D));
disp('Eigenvectors:');
disp(V);

% Calculate and display all pairwise dot products between eigenvectors
dot12 = ;
dot13 = ;
dot23 = ;
disp('Dot product of first and second eigenvectors:')
disp(dot12);
disp('Dot product of first and third eigenvectors:')
disp(dot13);
disp('Dot product of second and third eigenvectors:')
disp(dot23);
```

MATLAB code of orthogonal eigenvectors

Property of Orthogonal Eigenvector

- **Dot product between any pair of eigenvectors is $\boxed{}$.**
  - While dot product of eigenvector with itself is nonzero.
  - Because not consider zeros vector to be eigenvector.
  - This can be written as Eq 1..
    - $\boldsymbol{D}$: Diagonal matrix with diagonals containing norms of eigenvectors

$$\boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{D}$$

Eq 1. Property of orthogonal eigenvector

Direction VS Magnitude

- **Eigenvectors are important not magnitude but** $\boxed{}$**.**
  - Eigenvector can have any magnitude we want.
    - Except for magnitude of zero
- **Let's scale all eigenvectors so they have unit length.**
  - Question: If all eigenvectors are orthogonal and have unit length, what happens when we multiply eigenvectors matrix by its transpose?
  - Answer: As you know, it's Eq 1..
- **In other words, Eigenvectors matrix of symmetric matrix is orthogonal matrix!**

$$\boldsymbol{V}^T\boldsymbol{V} = \boldsymbol{I}$$

Eq 1. Multiply eigenvectors matrix with unit length by its transpose

Implication of Orthogonal Eigenvector

- **Eigenvectors are super easy to invert for symmetric matrices.**
  - Simply transpose them.

- **Other implications of orthogonal eigenvectors for applications.**
  - Such as principal components analysis
  - I will discuss later.

Proof of Orthogonal Eigenvector

Necessity

- Orthogonal eigenvectors of symmetric matrices is such important concept

Goal

- To show that $\boxed{}$ product between any pair of eigenvectors is zero.

Assumption

- Matrix $\boldsymbol{A}$ is symmetric.
- λ_1 and λ_2 are distinct eigenvalues of $\boldsymbol{A}$, with $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ as their corresponding eigenvectors.
  - λ_1 and λ_2 cannot equal each other.

Eigenvector Orthogonality Proof (1)

- **Try to follow each equality step from left to right of Eq 1..**
  - Pay attention to first and last terms.
    - Terms in middle are just transformations.
- **Eq 1. are written in Eq 2..**
  - Then subtracted to set to zero.

$$\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 = (\boldsymbol{A}\boldsymbol{v}_1)^T \boldsymbol{v}_2 = \boldsymbol{v}_1^T \boldsymbol{A}^T \boldsymbol{v}_2 = \boldsymbol{v}_1^T \lambda_2 \boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2$$

Eq 1. Proof of eigenvector orthogonality for symmetric matrices

$$\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2$$

$$\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 - \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2 = 0$$

Eq 2. Continuing eigenvector orthogonality proof

Eigenvector Orthogonality Proof (2)

- **Eq 1. can be factored out as Eq 2..**
  - Both terms contain dot product $\boldsymbol{v}_1^T\boldsymbol{v}_2$.
- **Eq 2. says that two quantities multiply to produce 0.**
  - One or both of those quantities must be zero.
    - $(\lambda_1 - \lambda_2)$ cannot equal zero.
      - Because we began from assumption that they are $\boxed{}$.
  - Therefore, $\boldsymbol{v}_1^T\boldsymbol{v}_2$ must equal zero.
    - Meaning: Two eigenvectors are orthogonal.

$$\lambda_1\boldsymbol{v}_1^T\boldsymbol{v}_2 - \lambda_2\boldsymbol{v}_1^T\boldsymbol{v}_2 = 0$$

Eq 1. Continuing eigenvector orthogonality proof

$$(\lambda_1 - \lambda_2)\boldsymbol{v}_1^T\boldsymbol{v}_2 = 0$$

Eq 2. Eigenvector orthogonality proof, part 3

Eigenvector Orthogonality Proof (3)

- **Go back through Eq 1..**
  - Convince yourself that this proof fails for nonsymmetric matrices, when $\boldsymbol{A}^T \neq \boldsymbol{A}$.
  - Thus, eigenvectors of nonsymmetric matrix are not constrained to be orthogonal.
    - Linearly independent for all distinct eigenvalues.
      - But I will omit that discussion and proof.

$$\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 = (\boldsymbol{A}\boldsymbol{v}_1)^T \boldsymbol{v}_2 = \boldsymbol{v}_1^T \boldsymbol{A}^T \boldsymbol{v}_2 = \boldsymbol{v}_1^T \lambda_2 \boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2$$

$$\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 = \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2$$

$$\lambda_1 \boldsymbol{v}_1^T \boldsymbol{v}_2 - \lambda_2 \boldsymbol{v}_1^T \boldsymbol{v}_2 = 0$$

$$(\lambda_1 - \lambda_2) \boldsymbol{v}_1^T \boldsymbol{v}_2 = 0$$

Eq 1. Eigenvector orthogonality proof

Real-Valued Eigenvalues

- **Second special property of symmetric matrices**
  - Real-valued eigenvalues
  - Real-valued eigenvectors
- **Let me start by showing that matrices with all real-valued entries.**
  - Those have complex-valued eigenvalues.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define the matrix A
A = [-3 -3 0; 3 -2 3; 0 1 2];

% Perform eigen decomposition
[V, D] = eig(A);

% Extract the eigenvalues from the diagonal matrix D
eigenvalues = diag(D);

% Display the eigenvalues as a column vector
disp('Eigenvalues:');
disp(eigenvalues);
```

```
Eigenvalues:
  -2.7447 + 2.8517i
  -2.7447 - 2.8517i
   2.4895 + 0.0000i
```

MATLAB code of egiendecomposition

In Code Exercise

- **3×3 matrix A**
  - Two complex eigenvalues and one real-valued eigenvalue.
    - Eigenvectors coupled to complex-valued eigenvalues
      - Themselves be complex-valued.
  - Nothing special
    - Because matrix A comes from random integers between -3 and +3.
- **Interestingly, complex-valued solutions come in conjugate pairs.**
  - If there is $\lambda_j = a + ib$, then there is $\lambda_k = a - ib$.
  - Their corresponding eigenvectors are also complex conjugate pairs.

- **I don't go into detail about complex-valued solutions, except to show you that complex solutions to eigendecomposition are straightforward.**
  - Straightforward: Mathematically expected
    - Interpreting complex solutions in eigendecomposition is far from straightforward.

Symmetric Matrix

- **Guarantee to have [   ] eigenvalues.**
  - Also [   ] eigenvectors.
- **Let me start by modifying previous example.**
  - Make matrix symmetric.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define the matrix A
A = [-3 -3 0; -3 -2 1; 0 1 2];

% Perform eigen decomposition
[V, D] = eig(A);

% Extract the eigenvalues from the diagonal matrix D
eigenvalues = diag(D);

% Display the eigenvalues as a column vector
disp('Eigenvalues:');
disp(eigenvalues);
```

```
Eigenvalues:
  -5.5971
   0.2261
   2.3710
```

MATLAB code of eigendecomposition of symmetric matrix

Random Symmetric Matrix of Any Size

- **How to make**
  - Create random matrix.
  - Eigendecompositioning $A^T A$.
- **Where to use**
  - Confirm that eigenvalues are real-valued.

- **Guaranteed real-valued eigenvalues from symmetric matrices.**
  - It's fortunate
    - Because complex numbers are often confusing to work with.
- **In data science**
  - Lots of matrices are symmetric.
  - If you see complex eigenvalues in your data science applications,
    - It's possible that is problem with code or with data.

Eigendecomposition of Singular Matrices

Wrong Idea about Eigendecomposition of Singular Matrices

- **Students often get idea.**
  - Singular matrices cannot be
  - Eigenvectors of singular matrix must be unusual somehow.
- **That idea is completely wrong!**
  - Eigendecomposition of singular matrices is

Code Exercise of Eigendecomposition of Singular Matrix

Code Exercise (13_03)

- This rank-2 matrix has one zero-valued eigenvalue with nonzeros eigenvector.
- Explore eigendecomposition of other reduced-rank random matrices by using example code.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define the matrix
A = [1 4 7; 2 5 8; 3 6 9];

% Calculate the matrix rank
rankA = ;

% Eigen decomposition
[V, D] = ;

% Display the results
disp('Rank =')
disp(rankA);
disp('Eigenvalues:');
disp(diag(D));
disp('Eigenvectors:');
disp(V);

% Optionally round eigenvalues and eigenvectors for display
disp('Rounded Eigenvalues:');
disp(round(diag(D).',2)); % Round and transpose for horizontal display
disp('Rounded Eigenvectors:');
disp(round(V, 2));
```

MATLAB code of eigendecomposition of singular matrix

One Special Property of Eigendecomposition of Singular Matrices

- **At least one eigenvalue is guaranteed to be zero.**
  - That doesn't mean that number of nonzero eigenvalues equals rank of matrix.
  - True for
    - Scalar values from the SVD (Singular Value Decomposition)
  - Not for
    - But if matrix is singular, then at least one eigenvalue equals zero.
- **Converse is also true.**
  - Every full-rank matrix has zero zero-valued eigenvalues.
- **Why this happens**
  - Singular matrix already has nontrivial null space.
    - $\lambda = 0$ provides nontrivial solution to $(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{v} = 0$.

- **Main take-homes of this section**
  - Eigendecomposition is valid for reduced-rank matrices.
  - Presence of at least one zero-valued eigenvalue indicates reduced-rank matrix.

Quadratic Form, Definiteness, and Eigenvalues

Quadratic Form and Definiteness

- **Quadratic form and definiteness are intimidating terms.**
  - Don’t worry.
  - They are both straightforward concepts that provide gateway to advanced linear algebra and applications.
  - advanced linear algebra technique such as
    - Principal components analysis (PCA)
    - Monte Carlo simulations
  - Integrating MATLAB code into your learning will give you huge advantage over learning about these concepts.
    - Compared to traditional linear algebra textbooks.

Quadratic Form of Matrix

- **Consider Eq 1..**
  - Pre- and postmultiply square matrix by same vector $\boldsymbol{w}$ and get scalar.
    - Notice: This multiplication is valid only for square matrices.

$$\boldsymbol{w}^T \boldsymbol{A} \boldsymbol{w} = \alpha$$

Eq 1. Quadratic form of matrix

- **This is called $\boxed{}$ on matrix $\boldsymbol{A}$.**

- **Which matrix and which vector do we use?**
  - Idea of quadratic form
    - To use one specific matrix.
    - To set of all possible vectors.
      - Appropriate size
  - Important point
    - Signs of α for all possible vectors.

Example of Quadratic Form of Matrix

For this particular matrix as Eq 1.

$$[x \quad y]\begin{bmatrix}2 & 4\\0 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 2x^2 + (0+4)xy + 3y^2$$

Eq 1. First example of quadratic form of matrix

- There is no possible combination of x and y that can give negative answer.
  - Even when x or y is negative value.
    - Because squared terms $(2x^2\ and\ 3y^2) \gg$ cross-term $(4xy)$.
- α can be nonpositive.
  - α comes from $\boldsymbol{w}^T\boldsymbol{A}\boldsymbol{w} = \alpha$.
  - Only when $x = y = 0$.
    - In remaining cases, α is always positive.

That is not trivial result of quadratic form.

- Eq 2. can have positive or negative α depending on values of x and y.
- $[x \quad y] \to [-1 \quad 1]$: $\boxed{}$ quadratic form result.
- $[x \quad y] \to [-1 \quad -1]$: $\boxed{}$ quadratic form result

$$[x \quad y]\begin{bmatrix}-9 & 4\\3 & 9\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = -9x^2 + (3+4)xy + 9y^2$$

Eq 2. Second example of quadratic form of matrix

Scalar for All Possible Vectors

- **How can you possibly know whether quadratic form will produce positive?**
  - Or negative, or zero-valued
- **Key**
  - Full-rank eigenvectors matrix spans all of $\mathbb{R}^M$.
  - Therefore, Every vector in $\mathbb{R}^M$ can be expressed.
    - As some linear weighted combination of eigenvectors.
- **Then, start from eigenvalue equation and left-multiply by eigenvector to return to quadratic form as Eq 1..**

$$\boldsymbol{Av} = \lambda \boldsymbol{v}$$
$$\boldsymbol{v}^T \boldsymbol{Av} = \lambda \boldsymbol{v}^T \boldsymbol{v}$$
$$\boldsymbol{v}^T \boldsymbol{Av} = \lambda ||\boldsymbol{v}||^2$$

Eq 1. Return to quadratic form

Key of Return to Quadratic Form

- **In Eq 1., Final equation is key.**
  - Note, $||\boldsymbol{v}^T\boldsymbol{v}||$ is strictly positive.
    - Vector magnitudes cannot be negative.
    - Ignore zeros vector.
  - Sign of right-hand side of equation is determined entirely by eigenvalue λ.
- **That equation uses only one eigenvalue and its eigenvector.**
  - But we need to know about any possible vector.

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}$$
$$\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v}^T\boldsymbol{v}$$
$$\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v} = \lambda||\boldsymbol{v}||^2$$

Eq 1. Return to quadratic form

Insight of Return to Quadratic Form

- **If equation is valid for each eigenvector-eigenvalue pair,**
  - It is valid for any combination of eigenvector-eigenvalue pairs as Eq 1..
- **In other words**
  - Set any vector $\boldsymbol{u}$ to be some linear combination of $\boxed{}$.
  - Set some scalar ζ to be that same linear combination of eigenvalues.
- **Anyway, it doesn't change principle**
  - Sign of right-hand side (quadratic form) is determined by sign of eigenvalues.

$$\boldsymbol{v}_1^T \boldsymbol{A} \boldsymbol{v}_1 = \lambda_1 \|\boldsymbol{v}_1\|^2$$
$$\boldsymbol{v}_2^T \boldsymbol{A} \boldsymbol{v}_2 = \lambda_2 \|\boldsymbol{v}_2\|^2$$
$$(\boldsymbol{v}_1 + \boldsymbol{v}_2)^T \boldsymbol{A} (\boldsymbol{v}_1 + \boldsymbol{v}_2) = (\lambda_1 + \lambda_2) \|(\boldsymbol{v}_1 + \boldsymbol{v}_2)\|^2$$
$$\boldsymbol{u}^T \boldsymbol{A} \boldsymbol{u} = \zeta \|\boldsymbol{u}\|^2$$

Eq 1. Valid for each eigenvector-eigenvalue pair

Think about Equation under Different Assumption about sign of λ

All eigenvalues are positive.

- Right-hand side of equation is always positive.
- $\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v}$ is always positive for any vector $\boldsymbol{v}$.

Eigenvalues are positive or zero.

- $\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v}$ is nonnegative
- $\boldsymbol{v}^T\boldsymbol{A}\boldsymbol{v}$ will equal zero when $\lambda = 0$.
  - $\lambda = 0$ happens when matrix is singular.

Eigenvalues are negative or zero.

- Quadratic form result will be zero or negative.

Eigenvalues are negative.

- Quadratic form result will be negative for all vectors.

Definiteness

- **Characteristic of square matrix**

- **Defined by signs of eigenvalues of matrix.**
  - Same thing as signs of quadratic form results.

- **Implication**
  - Invertibility of matrix as well as advanced data analysis methods.
    - Such as generalized eigendecomposition
      - Used in multivariate linear classifiers and signal processing.

Categories of Definiteness

- **There are 5 categories in definiteness.**
- **Five categories as shown in Table 1.**
  - + and – signs indicate signs of eigenvalues.
  -
    - Matrix can be invertible or singular depending on numbers in matrix.
    - Not on definiteness category.

Category	Quadratic form	Eigenvalues	Invertible
Positive definite	Positive	+	Yes
Positive semidefinite	Nonnegative	+ and 0	No
Indefinite	Positive and negative	+ and -	Depends
Negative semidefinite	Nonpositive	- and 0	No
Negative definite	Negative	-	Yes

Table 1. Definiteness categories

$A^T A$ is Positive (Semi)definite

- **Specific matrix is guaranteed to be positive definite or positive semidefinite.**
  - Expressed as product of matrix and its transpose.
  - That is, $\boldsymbol{S} = \boldsymbol{A}^T\boldsymbol{A}$
  - Combination of these two categories is often written as $\boxed{}$.

- **All data covariance matrices are positive (semi)definite.**
  - Because covariance matrices defined: $\boldsymbol{A}^T\boldsymbol{A}$
    - where data matrix: $\boldsymbol{A}$
  - All covariance matrices have nonnegative eigenvalues.

- **Case1: When data matrix is full-rank,**
  - If data is stored as observations by features,
    - Full column-rank
  - Eigenvalues will be all positive.
- **Case2: If data matrix is reduced-rank,**
  - At least one zero-valued eigenvalue

Proof of $A^T A$

- **Proof that S is positive (semi)definite.**
  - Writing out its quadratic form.
  - Applying some algebra manipulations.
- **In Eq 1..**
  - Transition from first to second equation simply involves moving parentheses around.
    - Such "proof by parentheses" is common in linear algebra.

$$
\begin{aligned}
\boldsymbol{w}^T S \boldsymbol{w} &= \boldsymbol{w}^T (A^T A) \boldsymbol{w} \\
&= (\boldsymbol{w}^T A^T)(A\boldsymbol{w}) \\
&= (A\boldsymbol{w})^T (A\boldsymbol{w}) \\
&= ||A\boldsymbol{w}||^2
\end{aligned}
$$

Eq 1. Proof that S is positive (semi)definite by parentheses

Point of Proof of $A^T A$

- **Quadratic form of $A^T A$ equals $\|\boldsymbol{matrix}\|^2 * \boldsymbol{vector}$.**
- **Characteristic of magnitudes**
  - Cannot be negative.
  - Can be zero
    - Only when vector is zero.
- **If $Aw = 0$ for nontrivial w,**
  - Then $\boldsymbol{A}$ is singular.
- **Notice**
  - Although all $\boldsymbol{A}^T\boldsymbol{A}$ matrices are symmetric, not all symmetric matrices can be expressed as $\boldsymbol{A}^T\boldsymbol{A}$.
    - Matrix symmetry on its own does not guarantee positive (semi)definiteness.
      - Because not all symmetric matrices can be expressed as product of matrix and its transpose.

Importance of Quadratic Form and Definiteness

- **Importance in data science.**
  - Because some linear algebra operations are applied only to well-endowed matrices.
    - Cholesky decomposition
      - Create correlated datasets in Monte Carlo simulations.
  - Importance in optimization problems.
    -
      - Because guaranteed minimum to find
- **In your never-ending quest to improve your data science prowess,**
  - You might encounter technical papers.
    - Use abbreviation SPD (Symmetric Positive Definite).

Generalized Eigendecomposition

Eigendecomposition

- **Consider that Eq 1. is same as fundamental eigenvalue equation.**
  - This is obvious.
    - Because
    - Generalized eigendecomposition as Eq 2. involves replacing identity matrix with another matrix.
      - Not identity or zeros matrix

$$Av = \lambda I v$$

Eq 1. Assumption equal to fundamental eigenvalue equation

$$Av = \lambda B v$$

Eq 2. Generalized eigendecomposition

Generalized Eigendecomposition

- **It is also called simultaneous diagonalization of two matrices.**
- **Resulting (λ, v) pair is not eigenvalue / vector of A alone nor of B alone.**
  - Instead, two matrices share eigenvalue / vector pairs.
- **Conceptually, you can think of generalized eigendecomposition**
  - As "regular" eigendecomposition of product matrix as Eq 1..
- **Just conceptual**
  - In practice, does not require B to be invertible.
- **Not case**
  - Any two matrices can be simultaneously diagonalized.
  - If B is
    - Diagonalization is possible.

$$C = AB^{-1}$$
$$Cv = \lambda v^{-1}$$

Eq 1. "Regular" eigendecomposition of product matrix

Use Generalized Eigendecomposition in Data Science

- **Classification analysis**
- **In particular, fisher's linear discriminant analysis (LDA)**
  - Based on generalized eigendecomposition of two data covariance matrices.

Myriad Subtleties of Eigendecomposition

- **A lot of properties of eigendecomposition**
  - Sum of eigenvalues equals trace of matrix.
    - While product of eigenvalues equals determinant.
  - Not all square matrices can be diagonalized.
  - Some matrices have repeated eigenvalues.
    - Implications for their eigenvectors
  - Complex eigenvalues of real-valued matrices
    - Inside circle in complex plane.
- **Mathematical knowledge of eigenvalues runs deep.**
  - But this lecture provides essential foundational knowledge.
    - For working with eigendecomposition in applications.

Summary

Summary

- **Eigendecomposition identifies $\boldsymbol{M}$ scalar/vector pairs of an $\boldsymbol{M} \times \boldsymbol{M}$ matrix.**
  - It reflect special directions in the matrix.
  - And have myriad applications in data science.
    - As well as in geometry, physics, computational biology, and myriad other technical displines.
- **Eigenvalues can be found.**
  - Assuming that the matrix shifted by an unknown scalar λ is singular.
  - Setting its determinant to zero.
    - Called characteristic polynomial.
  - And solving for λs.
- **Eigenvectors can be found.**
  - By finding basis vector for the null space of $\boldsymbol{\lambda - shifted}$ matrix.
- **Meaning of diagonalizing a matrix.**
  - Represent matrix as $V^{-1}\Lambda V$.
    - V: matrix with eigenvectors in the columns.
    - Λ: diagonal matrix with eigenvalues in the diagonal elements.

Summary

- **Symmetric matrices have several special properties in eigendecomposition.**
  - In data science
    - All eigenvectors are pair-wise orthogonal.
      - Matrix of eigenvectors is orthogonal.
      - Inverse of eigenvectors matrix is its transpose.
- **Definiteness of matrix**
  - Signs of its eigenvalues
  - In data science
    - Positive (semi)definite
      - All eigenvalues are either nonnegative or positive.
  - Matrix times its transpose is always positive (semi)definite.
    - All covariance matrices have nonnegative eigenvalues.
- **Study of eigendecomposition**
  - Rich and detailed
  - Many fascinating subtleties, special cases, and applications

Code Exercises

A , A^{-1} Eigenvalue

- Interestingly, the eigenvectors of A^{-1} are the same as the eigenvectors of A while the eigenvalues are λ^{-1}. Prove that is the case by writing out the eigendecomposition of A and A^{-1}. Then illustrate it using a random full-rank 5×5 symmetric matrix.

```
% create the matrix
A = randn(5,5);
A = A' * A;

% compute its inverse
Ai = ;

% eigenvalues of A and Ai
eigvals_A = ;
eigvals_Ai = ;

% compare them (hint: sorting helps!)
disp('Eigenvalues of A:')
disp(sort(eigvals_A))

disp(' ')
disp('Eigenvalues of inv(A):')
disp(sort(eigvals_Ai))

disp(' ')
disp('Reciprocal of evals of inv(A):')
disp(sort(1./eigvals_Ai))
```

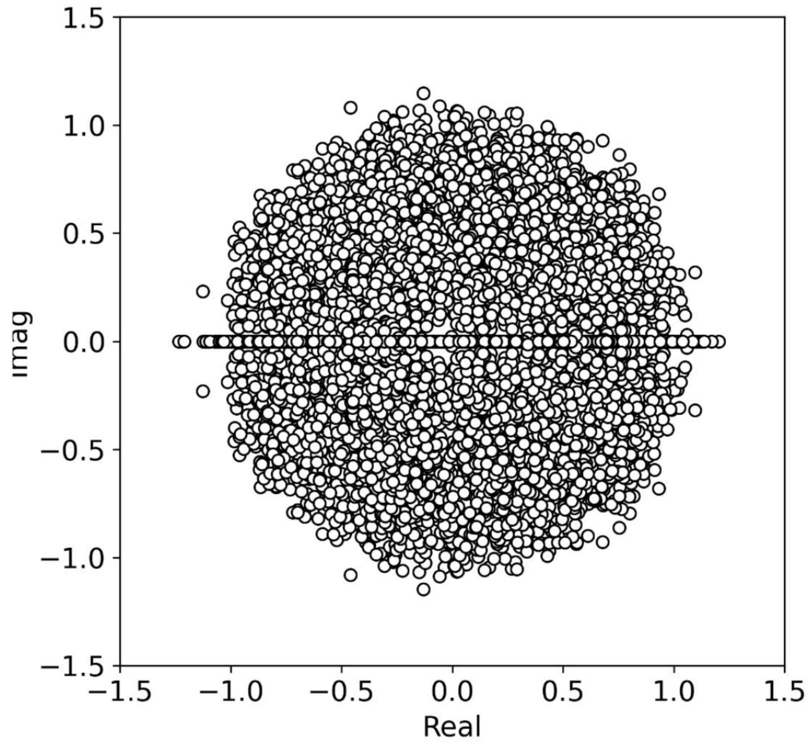
Sample code

Interesting property of random matrices

- **One interesting property of random matrices is that their complex-valued eigenvalues are distributed in a circle with a radius proportional to the size of the matrix. To demonstrate this, compute 123 random 42×42 matrices, extract their eigenvalues, divide by the square root of the matrix size (42), and plot the eigenvalues on the complex plane, as in Figure below.**

```
nIter = 123;
matsize = 42;
% fill this evals variable
evals = zeros(nIter, matsize);

% create the matrices and get their scaled eigenvalues
for i = 1:nIter
    % declear (matsize, matsize) sized matrix A in every iteration
    A = ;
    evals(i, :) = ;
end

% visualization
% and show in a plot
figure('Position', [100, 100, 600, 600]);

plot(real(evals(:)), imag(evals(:)), 'ko', 'MarkerFaceColor', 'w');
xlim([-1.5, 1.5]);
ylim([-1.5, 1.5]);
xlabel('Real');
ylabel('Imag');
```

Sample code



Method to Create Random Symmetric Matrices

- **Start by creating a 4×4 diagonal matrix with positive numbers on the diagonals(they can be, for example, the numbers 1,2,3,4). Then create a 4×4 Q matrix from the QR decomposition of a random-numbers matrix. Use these matrices as the eigenvalues and eigenvectors, and multiply them appropriately to assemble a matrix. Confirm that the assembled matrix is symmetric, and that its eigenvalues equal the eigenvalues you specified.**

```
% Create the Lambda matrix with positive values
Lambda = diag(rand(4,1) * 5);
randnMat = randn(4,4);

% create Q
;

% reconstruct to a matrix
A = ;

% the matrix minus its transpose should be zeros (within precision error)
result = ;

disp(result);

% sort(diag(Lambda)) and sort(eig(A)) disp same result
% print sorted diagonal of Lambda
disp('Sorted diagonal of Lambda:')
disp(sort(diag(Lambda)))

% print sorted eigenvalues of A
disp('Sorted eigenvalues of A:')
disp(sort(eig(A)))
```

Sample code

THANK YOU
FOR YOUR ATTENTION