

Linear Algebra

Eigen value decomposition

$$A \rightarrow Q \quad Q^{-1} \quad D$$

Automotive Intelligence Lab.



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- Finding Eigenvectors
- Diagonalizing a Square Matrix
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- Special Awesomeness of Symmetric Matrices
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https://youtu.be/PFDu9oVAE-g?list=PLZHQBObOWTQDPD3MizzM2xVFitgF8hE_ab&t=82

$$\tilde{A}v = \lambda v$$

← Eigen Vector
↓
Eigenvalue

Finding Eigenvalues



Find Eigenvalues Using MATLAB

■ To eigendecompose a square matrix...,

- ▶ First, find eigenvalues first.
- ▶ Then, use each eigenvalue to find its corresponding eigenvector.
- ▶ Super easy in MATLAB.
 - Just use function **eig()**
 - Eigenvalues of the matrix below are -0.37 and 5.37 .

```
% Clear workspace, command window, and close all figures  
clc; clear; close all;  
  
% Define the matrix  
matrix = [1 2; 3 4];  
  
% Get the eigenvalues  
evals = eig(matrix);  
  
% Display the eigenvalues  
disp('Eigenvalues of the matrix:');  
disp(evals);
```

MATLAB code to find eigenvalues

■ Probably you have question...!

- ▶ How are the eigenvalues of a matrix identified?

Method to Find Eigenvalues of Matrix

■ Do some simple arithmetic!

$$\begin{aligned} & \boxed{Av = \lambda v} \\ Av - \lambda v &= 0 \\ (A - \lambda I)v &= 0 \end{aligned}$$

Reorganize eigenvalue equation

■ First equation

- ▶ Repeat of eigenvalue equation.

■ Second equation

- ▶ Simply subtracted right-hand side to set equation of right-hand side to the zeros vector.

■ Third equation

- ▶ Left-hand side of second equation has two vector terms.
 - Both of which involve v so that factor out the vector.
- ▶ After that, it leaves us with the subtraction of a matrix and a scalar $A - \lambda$.
 - Matrix-scalar subtraction is not a defined operation in linear algebra.
 - So, *shift* matrix by λ .
 - λI is sometimes called a *scalar* matrix.

Meaning of Eigenvalue Equation

- Eigenvector is in the null space of the matrix shifted by its eigenvalue.

$$\underbrace{(A - \lambda I)}_{\text{Reorganized eigenvalue equation}} v = 0$$

$v \rightarrow \boxed{A - \lambda I} \rightarrow 0$

↑
null space \Leftrightarrow $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$ $\neq 0$

- Remember...

- Ignore trivial solutions in linear algebra which means don't consider $v = 0$ to be an eigenvector.

- Matrix shifted by its eigenvalue is singular.

- Because only singular matrices have a nontrivial null space.

- What else do we know about singular matrices?

- Know that their Determinant is zero!
- Hence, we can write as below:

$$\det(A - \lambda I) = 0$$

Determinant of $A - \lambda I$

Key to Finding Eigenvalues: Determinant

■ Shift matrix by unknown eigenvalue λ .

- ▶ Set its determinant to \boxed{Zero} , and solve for λ .

■ Example of finding eigenvalues in 2×2 matrix

- ▶ You can apply quadratic formula to solve for two λ values.

$$\left| \begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0$$

$$(a - \lambda)(d - \lambda) - bc = 0$$

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

Process of finding eigenvalues



Examples to Finding Eigenvalues

■ Traditional way to find eigenvalues

- ▶ Subtract the unknown value lambda off the diagonals.
- ▶ Solve for the determinant is equal to zero.

■ Direct way to find eigenvalues

- ▶ Trace of matrix is equal to sum of the eigenvalues.
- ▶ Determinant of a matrix is equal to the product of the two eigenvalues.

Find the eigenvalues of $\begin{bmatrix} 3 & 1 \\ 4 & 1 \end{bmatrix}$

$$\begin{aligned}\det\left(\begin{bmatrix} 3 - \lambda & 1 \\ 4 & 1 - \lambda \end{bmatrix}\right) &= (3 - \lambda)(1 - \lambda) - (1)(4) \\ &= (3 - 4\lambda + \lambda^2) - 4 \\ &= \underbrace{\lambda^2 - 4\lambda - 1}_{= 0} = 0\end{aligned}$$

$$\lambda_1, \lambda_2 = \frac{4 \pm \sqrt{4^2 - 4(1)(-1)}}{2} = \frac{4 \pm \sqrt{20}}{2} = 2 \pm \sqrt{5}$$

Traditional way to find eigenvalue

$$1) \quad \frac{1}{2} \text{tr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \frac{a+d}{2} = \frac{\lambda_1 + \lambda_2}{2} = m \quad (\text{mean})$$

$$2) \quad \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc = \lambda_1 \lambda_2 = p \quad (\text{product})$$

$$3) \quad \lambda_1, \lambda_2 = m \pm \sqrt{m^2 - p}$$

$$\boxed{\begin{bmatrix} 8 & 4 \\ 2 & 6 \end{bmatrix} \begin{matrix} m = 7 \\ p = 40 \end{matrix}}$$

Direct way to find eigenvalue



Code Exercise to Find Eigenvalues

■ Code Exercise (13_01)

- ▶ Find the eigenvalues using different method.
- ▶ Use the ‘direct way’ in the previous slide.

```
% Clear workspace, command window, and close all figures  
clc; clear; close all;  
  
% 2x2 matrix : A  
A = [1 2; 3 4];  
  
% Calculate the trace of the matrix  
trA = trace(A);  
  
% Calculate the determinant of the matrix  
detA = det(A);  
  
% Calculate the eigenvalues using the direct way  
lambda1 = ;  
lambda2 = ;  
  
% Display the eigenvalues  
disp('Eigenvalues of the matrix:');  
disp([lambda1 lambda2]);
```

MATLAB code to find eigenvalues using direct way



Logical Progression of Mathematical Concepts of Eigenvalue Equation

- The matrix-vector multiplication acts like: Scalar-vector multiplication.

- Set eigenvalue equation to zeros vector, and factor out common terms.**

- ▶ Eigen vector is null space of matrix shifted by eigenvalue.
- ▶ Do not consider zeros vector to be an eigenvector.
 - Shifted matrix is singular.

- Set determinant of shifted matrix to Zero.**

- ▶ Solve for unknown eigenvalue.

- Determinant of an eigenvalue-shifted matrix set to Zero.**

- ▶ Called characteristic polynomial of the matrix.

- n th order polynomial has n solutions.**

- ▶ Some of solutions might be complex-valued.
 - Called fundamental theorem of algebra.
- ▶ Characteristic polynomial of an $M \times M$ matrix will have λ^M term.
 - $M \times M$ matrix will have M eigenvalues.

Null Space

$$\begin{aligned} \mathbf{A}\mathbf{v} &= \lambda\mathbf{v} \\ \mathbf{A}\mathbf{v} - \lambda\mathbf{v} &= \mathbf{0} \\ (\mathbf{A} - \lambda\mathbf{I})\mathbf{v} &= \mathbf{0} \end{aligned}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

Finding Eigenvectors



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Find Eigenvectors Using MATLAB

■ Finding eigenvectors is super easy in MATLAB.

- ▶ Most important thing to keep in mind
 - Eigenvectors are stored in columns of the matrix.
- ▶ Columns of the matrix ~~*evecs*~~
 - Eigenvectors
 - Columns are same order as eigenvalues.
- ▶ Paired
 - Eigenvector in the first column of matrix *evecs*.
 - First eigenvalue in vector *evals*.
- ▶ People use variable names **L** & **V** or **D** & **V**.
 - **V** matrix: each column i is eigenvector v_i .
 - **L** is for Λ (capital of λ)
 - **D** is for diagonal.
 - Eigen values are often stored in a diagonal matrix.
 - Reasons will be explained later in this chapter.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% 2x2 matrix
matrix = [1 2; 3 4];

% Calculate the eigenvalues and eigenvectors
[vecs, vals] = eig(matrix);

% Display the eigenvalues and eigenvectors
disp('Eigenvalues:');
disp(diag(vals)); % Extracts and displays the eigenvalues from the diagonal matrix
disp('Eigenvectors:');
disp(vecs); % Displays the eigenvectors
```

MATLAB code to find eigenvalues

$$V = \begin{bmatrix} | & | & | & | \\ | & | & | & | \\ | & | & | & | \end{bmatrix} \dots$$

■ Important question

- ▶ Where do eigenvectors come from and how do we find them?

Important Thing to Keep In Mind About Eigenvectors When Coding

■ Eigenvectors are stored in the **columns** of the matrix.

- ▶ Not in the rows.
- ▶ Disastrous consequences in applications.
 - If accidentally using the rows instead of the columns of the eigenvectors matrix.

■ Remember common convention in linear algebra.

- ▶ Vectors are in column orientation.

Method to Find Eigenvector

■ Find vector v that is in the null space of matrix shifted by λ .

- ▶ In other words:

$$v_i \in N(A - \lambda_i I)$$

Equation of eigenvector

■ Numerical example

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \longrightarrow \lambda_1 = 3, \lambda_2 = -1$$

Example of matrix and its eigenvalues

- ▶ Focus on the first eigenvalue.

- Shift the matrix by 3 (value of first eigenvalue).
- Find a vector in its **null space**.

$$\begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Find eigenvector of the matrix

- $[1 \ 1]$: an eigenvector of the matrix **associated with** an eigenvalue of 3.
- How can we find null space vectors (eigenvectors of the matrix)?

Method to Find Null Space Vectors in Practice

■ Good way to conceptualize the solution

- ▶ Use **Gauss-Jordan** to solve a system of equations.
 - Coefficients matrix is λ shifted matrix.
 - Constants vector is zeros vector.

■ In implementation....,

- ▶ More stable numerical methods are applied for finding eigenvalues and eigenvectors.
 - Including **QR decomposition** and **Procedure** called the power method

Sign and Scale Indeterminacy of Eigenvectors

■ Return to numerical example in previous section

- ▶ Why $[1 \ 1]$ was an eigenvector of matrix?
 - $[1 \ 1]$: a basis for the null space of the matrix shifted by its eigenvalue of 3.
- ▶ Is $[1 \ 1]$ unique eigenvector of matrix?
 - No, $[4 \ 4]$ or $[-5.4 \ -5.4]$ or ...
 - Any scaled version of vector $[1 \ 1]$ is a basis for that null space.
- ▶ If v is an eigenvector of a matrix, αv can also be eigenvector.
 - αv for any real-valued α except zero.

■ Indeed, eigenvectors are important because of their direction.

- ▶ Not because of their magnitude

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \longrightarrow \lambda_1 = 3, \lambda_2 = -1 \longrightarrow \begin{bmatrix} 1-3 & 2 \\ 2 & 1-3 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Previous example about eigenvalues and eigenvector

Questions About Infinity of Possible Null Space Basis Vectors

■ Is there one “best” basis vector?

- ▶ No “best” basis vector.
- ▶ But convenient to have eigenvectors that are **unit normalized**.
 - Euclidean norm of 1.
 - Particularly useful for symmetric matrices for reasons will be explained later in this chapter.

■ What is “correct” sign of an eigenvector?

- ▶ There is none.
- ▶ Can get **different eigenvector signs** from same matrix when using different software.
 - Python, Julia, Mathematica ,
- ▶ There are principled ways for assigning a sign in applications.
 - Such as PCA.
 - But it is just common convention to facilitate interpretation.



Diagonalizing a Square Matrix

대각화



Make Equations Compact and Elegant

- Eigenvalue equation lists one eigenvalue and one eigenvector.

 - Means that an $M \times M$ matrix has M eigenvalue equations.

$M \times M$ matrix λ_i ($i = 1 \sim M$)

- Nothing wrong with that series of equations...!

 - But this equation sets are ugly.
 - Ugliness violates one of the principles of linear algebra which make equations compact and elegant.

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$$

⋮

$$A\mathbf{v}_M = \lambda_M \mathbf{v}_M$$

M eigenvalue equations of $M \times M$ matrix

- Therefore, we need to transform this series of equations into one matrix equation for compact!

Key Insight for Writing Out Matrix Eigenvalue Equation

■ Each **Column** of the eigenvectors matrix **is scaled** by exactly one eigenvalue.

- ▶ Can implement this through post multiplication by a diagonal matrix.
- ▶ Store eigenvalues in diagonal of a matrix instead of storing eigenvalues in a vector.

■ Form of diagonalization for a 3×3 matrix

- ▶ Using @ in place of numerical values in the matrix
- ▶ In the eigenvectors matrix,
 - First subscript number corresponds to eigenvector.
 - Second subscript number corresponds to eigenvector element.
- ▶ Take a moment to confirm!
 - Each eigenvalue **scales** all elements of its corresponding eigenvector and not any other eigenvectors.

$$\begin{aligned}
 Av_1 &= \lambda_1 v_1 \\
 &\vdots \\
 Av_M &= \lambda_M v_M
 \end{aligned}
 \quad \rightarrow \quad
 \begin{matrix}
 \left[\begin{matrix} @ & @ & @ \\ @ & @ & @ \\ @ & @ & @ \end{matrix} \right] \left[\begin{matrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{matrix} \right] = \left[\begin{matrix} \lambda_1 v_{11} & \lambda_2 v_{21} & \lambda_3 v_{31} \\ \lambda_1 v_{12} & \lambda_2 v_{22} & \lambda_3 v_{32} \\ \lambda_1 v_{13} & \lambda_2 v_{23} & \lambda_3 v_{33} \end{matrix} \right] \\
 &\qquad\qquad\qquad A
 \end{matrix}
 \quad = \quad
 \left[\begin{matrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{matrix} \right] \left[\begin{matrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{matrix} \right]$$

V Δ

Diagonalization for 3×3 matrix

Eigen Decomposition

$$\left[\underbrace{v_1^T}_{\downarrow}, \underbrace{v_2^T}_{\downarrow}, \underbrace{v_3^T}_{\downarrow}, \dots, \underbrace{v_m^T}_{\downarrow} \right]$$

- Consider list of equivalent declarations of matrix eigenvalue equation as shown below.

$$AV = V\Lambda$$

$$A = V\Lambda V^{-1}$$

$$\Lambda = V^{-1}AV$$

List of equivalent declarations

Code to return:

- ▶ Eigenvalues in a vector.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% 2x2 matrix
matrix = [1 2; 3 4];

% Calculate the eigenvalues and eigenvectors
[vecs, vals] = eig(matrix);

% Display the D matrix
disp('D matrix:');
disp(vals);
```

MATLAB code to get D matrix



Example of Diagonalizing a Square Matrix

- Diagonalizing matrix using eigenvectors (eigenbasis).

$$\Lambda = V^{-1}AV$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

Change of basis matrix

Use eigenvectors as basis

Example of Diagonalizing matrix

The diagram illustrates the process of diagonalizing a square matrix A. It shows the equation $\Lambda = V^{-1}AV$. The matrix A is $\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$. The change of basis matrix V is $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, and its inverse V^{-1} is $\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$. The resulting diagonal matrix Λ is $\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Annotations explain that the eigenvectors of A are used as the columns of V, and the inverse of V is used to change the basis of the matrix.

Interpretations of Eigenvalues and Eigenvectors



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Interpretations in Geometry

$$Av = \lambda v$$

■ Special combination of a matrix and a vector

- Matrix *stretched* vector but did not *rotate* that vector.
 - That vector: *eigenvector* of matrix
 - Amount of stretching: *eigenvalue*

- Eigenvectors point in same direction.
 - Before and after post multiplying the matrix.
- In Fig 1.,
 - v_1, v_2 : eigen vectors.
- In Fig 2.,
 - w_1, w_2 : not eigen vectors.

$$Ax=b \quad x \rightarrow [A] \rightarrow \lambda v$$

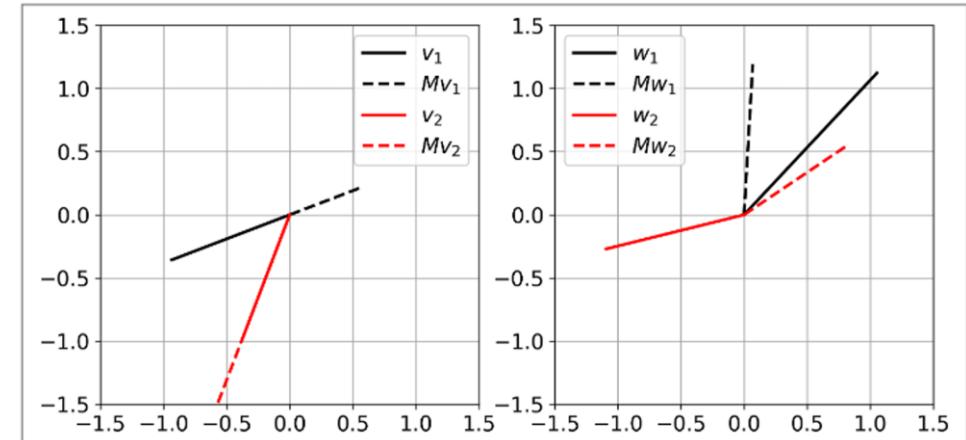


Fig 1.

Fig 2.

■ Geometric meaning of eigenvector

- Matrix-vector multiplication acts like *scalar-vector* multiplication.
- Write eigenvalue equation as:

$$Av = \lambda v$$

Eigenvalue equation

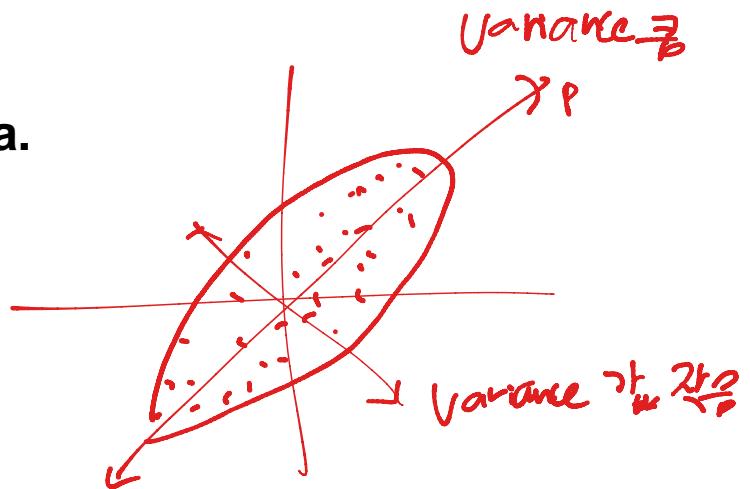
- Equation doesn't say that matrix equals the scalar.
 - It says that the *effect* of matrix on the vector is same as the *effect* of the scalar on that same vector.

Rd, pitch, yaw 3x1 (3x3)
 ↓
3x3행렬 행렬 4x1

Principal Components Analysis

■ Implement Principal Component Analysis (PCA) on statistical data.

- ▶ To identify important patterns or structures.
 - We will practice PCA later!

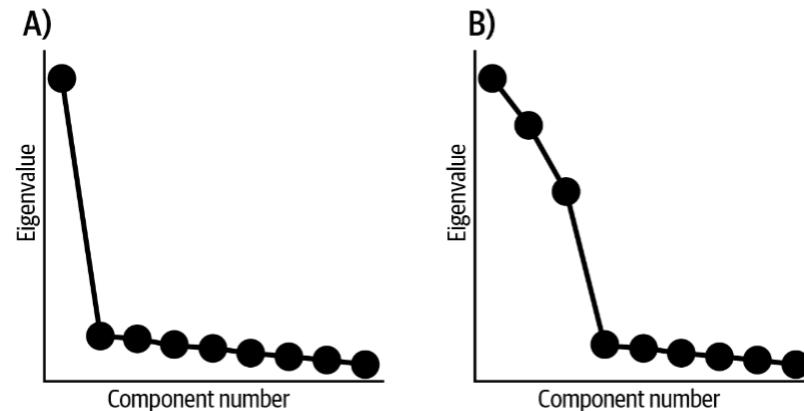


■ Role of eigenvalue in PCA

- ▶ Eigenvalue play a crucial role in PCA.
 - Represent variance of each principal component.
 - The larger the eigenvalue, the more variance (information) the principal component captures.

■ Scree plot

- ▶ graph of the eigenvalues of the dataset's covariance matrix
- ▶ A) 1 component accounts for most of the variance system.
 - All other components account for very little variance.
- ▶ B) 3 major subcategories.
 - Other components expect 3 major categories account for very little variance.

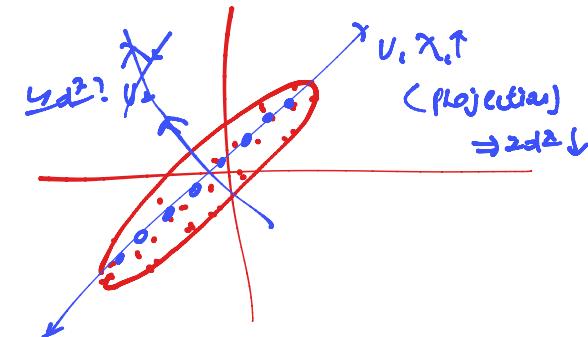


Example of Scree plot

Interpretations in Noise Reduction

- Most datasets contain noise.

- Noise
 - ▶ Refers to **Variance** in a dataset either unexplained or unwanted.



- Method to reducing random noise

- ▶ Many ways to attenuate or eliminate noise,
but optimal reduction strategy depends on origin of the noise or characteristics of signal
- ▶ Method with eigenvalues and eigenvectors,
 - Identify eigenvalues and eigenvectors of a system.
 - And “project out” directions in the data space associated **with small eigen-values**.

- Meaning of “projecting out” a data dimension

- ▶ Reconstruct dataset after setting some eigenvalues to zero which eigenvalues below some threshold.

Variance + 주요 차원은 유지한다.

Interpretations in Dimension Reduction (Data Compression)

■ It is beneficial to *compress data before transmitting it*.

- ▶ Compression: Reduce the size of data while having minimal impact on the quality of the data.

■ One way to dimension-reduce a dataset

- ▶ Take its **eigendecomposition**
 - Drop eigenvalues and eigenvectors associated with small directions in data space.
 - Transmit only relatively larger eigenvector-value pairs.

■ All of the data compression idea is same!

- ▶ Decompose dataset into a set of basis vectors.
 - Basis vectors that capture the most important features of data.
- ▶ Reconstruct a high-quality version of the original data.

Special Awesomeness of Symmetric Matrices



Orthogonal Eigenvector

- Symmetric matrices have orthogonal eigenvectors.
 - ▶ All eigenvectors of symmetric matrix are pair-wise orthogonal.
- Start with an example, then discuss implications of eigenvector orthogonality, finally show proof.

Code Exercise of Orthogonal Eigenvector

■ Code Exercise (13_02)

- ▶ Three dot products are all zero.
 - Within computer rounding errors on order of 10^{-16} .
- ▶ Symmetric matrices were created as random matrix times its transpose.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Create a random matrix and make it symmetric
A = randi([-3, 3], 3, 3);
A = A * A'; % Symmetric matrix

% Perform eigen decomposition
[V, D] = ;

% Display the eigenvalues and eigenvectors
disp('Eigenvalues:');
disp(diag(D));
disp('Eigenvectors:');
disp(V);

% Calculate and display all pairwise dot products between eigenvectors
dot12 = ;
dot13 = ;
dot23 = ;
disp('Dot product of first and second eigenvectors:')
disp(dot12);
disp('Dot product of first and third eigenvectors:')
disp(dot13);
disp('Dot product of second and third eigenvectors:')
disp(dot23);
```

MATLAB code of orthogonal eigenvectors



Property of Orthogonal Eigenvector

■ Dot product between any pair of eigenvectors is Zero.

- ▶ While dot product of eigenvector with itself is nonzero.
- ▶ Because not consider zeros vector to be eigenvector.
- ▶ This can be written as Eq 1..
 - D : Diagonal matrix with diagonals containing norms of eigenvectors

$$V^T V = D$$

Eq 1. Property of orthogonal eigenvector

Direction VS Magnitude

■ Eigenvectors are important not magnitude but direction.

- ▶ Eigenvector can have any magnitude we want.
 - Except for magnitude of zero

■ Let's scale all eigenvectors so they have unit length.

- ▶ Question: If all eigenvectors are orthogonal and have unit length, what happens when we multiply eigenvectors matrix by its transpose?
- ▶ Answer: As you know, it's Eq 1..

■ In other words, Eigenvectors matrix of symmetric matrix is orthogonal matrix!



$$V^T V = I$$

Eq 1. Multiply eigenvectors matrix with unit length by its transpose

$$V^T = V^{-1}$$

Implication of Orthogonal Eigenvector

■ Eigenvectors are super easy to invert for symmetric matrices.

- ▶ Simply transpose them.

■ Other implications of orthogonal eigenvectors for applications.

- ▶ Such as principal components analysis
- ▶ I will discuss later.

Proof of Orthogonal Eigenvector

■ Necessity

- ▶ Orthogonal eigenvectors of symmetric matrices is such important concept

■ Goal

- ▶ To show that **dot** product between any pair of eigenvectors is zero.

■ Assumption

- ▶ Matrix A is symmetric.
- ▶ λ_1 and λ_2 are distinct eigenvalues of A , with v_1 and v_2 as their corresponding eigenvectors.
 - λ_1 and λ_2 cannot equal each other.

Eigenvector Orthogonality Proof (1)

■ Try to follow each equality step from left to right of Eq 1..

- ▶ Pay attention to **first and last terms**.
 - Terms in middle are just transformations.

■ Eq 1. are written in Eq 2..

- ▶ Then subtracted to set to zero.

$$\lambda_1 \mathbf{v}_1^T \mathbf{v}_2 = (\mathbf{A} \mathbf{v}_1)^T \mathbf{v}_2 = \mathbf{v}_1^T [\cancel{\mathbf{A}^T} \mathbf{v}_2] = \mathbf{v}_1^T \lambda_2 \mathbf{v}_2 = \lambda_2 \mathbf{v}_1^T \mathbf{v}_2$$

? Symmetric

Eq 1. Proof of eigenvector orthogonality for symmetric matrices

$$\begin{aligned}\lambda_1 \mathbf{v}_1^T \mathbf{v}_2 &= \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 \\ \lambda_1 \mathbf{v}_1^T \mathbf{v}_2 - \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 &= 0\end{aligned}$$

Eq 2. Continuing eigenvector orthogonality proof

Eigenvector Orthogonality Proof (2)

■ Eq 1. can be factored out as Eq 2..

- ▶ Both terms contain dot product $\mathbf{v}_1^T \mathbf{v}_2$.

■ Eq 2. says that two quantities multiply to produce 0.

- ▶ One or both of those quantities must be zero.

- $(\lambda_1 - \lambda_2)$ cannot equal zero.
 - Because we began from assumption that they are distinct.

- ▶ Therefore, $\mathbf{v}_1^T \mathbf{v}_2$ must equal zero.

- Meaning: Two eigenvectors are orthogonal.

$$\lambda_1 \mathbf{v}_1^T \mathbf{v}_2 - \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 = 0$$

Eq 1. Continuing eigenvector orthogonality proof

$$(\lambda_1 - \lambda_2) \mathbf{v}_1^T \mathbf{v}_2 = 0$$

Eq 2. Eigenvector orthogonality proof, part 3

Zero X

36

Eigenvector Orthogonality Proof (3)

■ Go back through Eq 1..

- ▶ Convince yourself that this proof fails for nonsymmetric matrices, when $A^T \neq A$.
- ▶ Thus, eigenvectors of nonsymmetric matrix are not constrained to be orthogonal.
 - Linearly independent for all distinct eigenvalues.
 - But I will omit that discussion and proof.

$$\begin{aligned}
 \lambda_1 \mathbf{v}_1^T \mathbf{v}_2 &= (\mathbf{A} \mathbf{v}_1)^T \mathbf{v}_2 = \mathbf{v}_1^T \mathbf{A}^T \mathbf{v}_2 = \mathbf{v}_1^T \lambda_2 \mathbf{v}_2 = \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 \\
 \lambda_1 \mathbf{v}_1^T \mathbf{v}_2 &= \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 \\
 \lambda_1 \mathbf{v}_1^T \mathbf{v}_2 - \lambda_2 \mathbf{v}_1^T \mathbf{v}_2 &= 0 \\
 (\lambda_1 - \lambda_2) \mathbf{v}_1^T \mathbf{v}_2 &= 0
 \end{aligned}$$

Eq 1. Eigenvector orthogonality proof

Real-Valued Eigenvalues

■ Second special property of symmetric matrices

- ▶ Real-valued eigenvalues
- ▶ Real-valued eigenvectors

■ Let me start by showing that matrices with all real-valued entries.

- ▶ Those have complex-valued eigenvalues.

```
% Clear workspace, command window, and close all figures
clc; clear; close all;

% Define the matrix A
A = [-3 -3 0; 3 -2 3; 0 1 2];

% Perform eigen decomposition
[V, D] = eig(A);

% Extract the eigenvalues from the diagonal matrix D
eigenvalues = diag(D);

% Display the eigenvalues as a column vector
disp('Eigenvalues:');
disp(eigenvalues);
```

Eigenvalues:

-2.7447 + 2.8517i
-2.7447 - 2.8517i
2.4895 + 0.0000i

MATLAB code of egiedecomposition



In Code Exercise

■ 3×3 matrix A

- ▶ Two complex eigenvalues and one real-valued eigenvalue.
 - Eigenvectors coupled to complex-valued eigenvalues
 - Themselves be complex-valued.
- ▶ Nothing special
 - Because matrix A comes from random integers between -3 and +3.

■ Interestingly, complex-valued solutions come in conjugate pairs.

- ▶ If there is $\lambda_j = a + ib$, then there is $\lambda_k = a - ib$.
- ▶ Their corresponding eigenvectors are also complex conjugate pairs.

■ I don't go into detail about complex-valued solutions, except to show you that complex solutions to eigendecomposition are straightforward.

- ▶ Straightforward: Mathematically expected
 - Interpreting complex solutions in eigendecomposition is far from straightforward.

Symmetric Matrix

■ Guarantee to have **real-valued** eigenvalues.

▶ Also **real-valued** eigenvectors.

■ Let me start by modifying previous example.

▶ Make matrix symmetric.

```
% Clear workspace, command window, and close all figures  
clc; clear; close all;  
  
% Define the matrix A  
A = [-3 -3 0; -3 -2 1; 0 1 2];  
  
% Perform eigen decomposition  
[V, D] = eig(A);  
  
% Extract the eigenvalues from the diagonal matrix D  
eigenvalues = diag(D);  
  
% Display the eigenvalues as a column vector  
disp('Eigenvalues:');  
disp(eigenvalues);
```

Eigenvalues:
-5.5971
0.2261
2.3710

MATLAB code of eigendecomposition of symmetric matrix

Random Symmetric Matrix of Any Size

■ How to make

- ▶ Create random matrix.
- ▶ Eigendecompositioning $A^T A$.

■ Where to use

- ▶ Confirm that eigenvalues are real-valued.

■ Guaranteed **real-valued eigenvalues** from symmetric matrices.

- ▶ It's fortunate
 - Because complex numbers are often confusing to work with.

■ In data science ✓

- ▶ Lots of matrices are symmetric.
- ▶ If you see complex eigenvalues in your data science applications,
 - It's possible that is problem with code or with data.

Eigendecomposition of Singular Matrices



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Wrong Idea about Eigendecomposition of Singular Matrices

■ Students often get idea.

- ▶ Singular matrices cannot be Eigendecomposed
- ▶ Eigenvectors of singular matrix must be unusual somehow.

■ That idea is completely wrong!

- ▶ Eigendecomposition of singular matrices is perfectly fine

Code Exercise of Eigendecomposition of Singular Matrix

■ Code Exercise (13_03)

- ▶ This rank-2 matrix has one zero-valued eigenvalue with nonzeros eigenvector.
- ▶ Explore eigendecomposition of other reduced-rank random matrices by using example code.

```
% Clear workspace, command window, and close all figures  
clc; clear; close all;  
  
% Define the matrix  
A = [1 4 7; 2 5 8; 3 6 9];  
  
% Calculate the matrix rank  
rankA = ;  
  
% Eigen decomposition  
[V, D] = ;  
  
% Display the results  
disp('Rank =')  
disp(rankA);  
disp('Eigenvalues:');  
disp(diag(D));  
disp('Eigenvectors:');  
disp(V);  
  
% Optionally round eigenvalues and eigenvectors for display  
disp('Rounded Eigenvalues:');  
disp(round(diag(D),2)); % Round and transpose for horizontal display  
disp('Rounded Eigenvectors:');  
disp(round(V, 2));
```

MATLAB code of eigendecomposition of singular matrix



One Special Property of Eigendecomposition of Singular Matrices

■ At least one eigenvalue is guaranteed to be zero.

- ▶ That doesn't mean that number of nonzero eigenvalues equals rank of matrix. (-)
- ▶ True for Singular Values
 - Scalar values from the SVD (Singular Value Decomposition)
- ▶ Not for Eigenvalues
 - But if matrix is singular, then at least one eigenvalue equals zero.

■ Converse is also true.

- ▶ Every full-rank matrix has zero zero-valued eigenvalues.

■ Why this happens

- ▶ Singular matrix already has nontrivial null space.
 - $\lambda = 0$ provides nontrivial solution to $(A - \lambda I)v = 0$.

■ Main take-homes of this section

- ▶ Eigendecomposition is valid for reduced-rank matrices.
- ▶ Presence of at least one zero-valued eigenvalue indicates reduced-rank matrix.

Quadratic Form, Definiteness, and Eigenvalues



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Quadratic Form and Definiteness

■ Quadratic form and definiteness are intimidating terms.

- ▶ Don't worry.
- ▶ They are both straightforward concepts that provide gateway to advanced linear algebra and applications.
- ▶ advanced linear algebra technique such as
 - Principal components analysis (PCA)
 - Monte Carlo simulations
- ▶ Integrating MATLAB code into your learning will give you huge advantage over learning about these concepts.
 - Compared to traditional linear algebra textbooks.

Quadratic Form of Matrix

■ Consider Eq 1..

- ▶ Pre- and postmultiply square matrix by same vector w and get scalar.
 - Notice: This multiplication is valid only for square matrices.

$$w^T A w = \alpha$$

Eq 1. Quadratic form of matrix

■ This is called on matrix A .

■ Which matrix and which vector do we use?

- ▶ Idea of quadratic form
 - To use one specific matrix.
 - To set of all possible vectors.
 - Appropriate size
- ▶ Important point
 - Signs of α for all possible vectors.

Example of Quadratic Form of Matrix

For this particular matrix as Eq 1.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + (0 + 4)xy + 3y^2$$

Eq 1. First example of quadratic form of matrix

- ▶ There is no possible combination of x and y that can give negative answer.
 - Even when x or y is negative value.
 - Because squared terms ($2x^2$ and $3y^2$) \gg cross-term ($4xy$).
- ▶ α can be nonpositive.
 - α comes from $w^T Aw = \alpha$.
 - Only when $x = y = 0$.
 - In remaining cases, α is always positive.

That is not trivial result of quadratic form.

- ▶ Eq 2. can have positive or negative α depending on values of x and y .
- ▶ $[x \ y] \rightarrow [-1 \ 1]: \boxed{} \text{ quadratic form result.}$
- ▶ $[x \ y] \rightarrow [-1 \ -1]: \boxed{} \text{ quadratic form result}$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -9 & 4 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -9x^2 + (3 + 4)xy + 9y^2$$

Eq 2. Second example of quadratic form of matrix

Scalar for All Possible Vectors

■ How can you possibly know whether quadratic form will produce positive?

- ▶ Or negative, or zero-valued

■ Key

- ▶ Full-rank eigenvectors matrix spans all of \mathbb{R}^M .
- ▶ Therefore, Every vector in \mathbb{R}^M can be expressed.
 - As some linear weighted combination of eigenvectors.

■ Then, start from eigenvalue equation and left-multiply by eigenvector to return to quadratic form as Eq 1..

$$\begin{aligned} A\mathbf{v} &= \lambda\mathbf{v} \\ \mathbf{v}^T A\mathbf{v} &= \lambda\mathbf{v}^T \mathbf{v} \\ \mathbf{v}^T A\mathbf{v} &= \lambda\|\mathbf{v}\|^2 \end{aligned}$$

Eq 1. Return to quadratic form

Key of Return to Quadratic Form

■ In Eq 1., Final equation is key.

- ▶ Note, $\|\boldsymbol{v}^T \boldsymbol{v}\|$ is **strictly positive**.
 - Vector magnitudes cannot be negative.
 - Ignore zeros vector.
- ▶ Sign of right-hand side of equation is determined entirely by eigenvalue λ .

■ That equation uses only one eigenvalue and its eigenvector.

- ▶ But we need to know about any possible vector.

$$\begin{aligned} A\boldsymbol{v} &= \lambda\boldsymbol{v} \\ \boldsymbol{v}^T A\boldsymbol{v} &= \lambda\boldsymbol{v}^T \boldsymbol{v} \\ \boldsymbol{v}^T A\boldsymbol{v} &= \lambda\|\boldsymbol{v}\|^2 \end{aligned}$$

Eq 1. Return to quadratic form



Insight of Return to Quadratic Form

■ If equation is valid for each eigenvector-eigenvalue pair,

- ▶ It is valid for any combination of eigenvector-eigenvalue pairs as Eq 1..

■ In other words

- ▶ Set any vector u to be some linear combination of *EigenVcctors*
- ▶ Set some scalar ζ to be that same linear combination of eigenvalues.

■ Anyway, it doesn't change principle

- ▶ Sign of right-hand side (quadratic form) is determined by sign of eigenvalues.

$$\mathbf{v}_1^T \mathbf{A} \mathbf{v}_1 = \lambda_1 \|\mathbf{v}_1\|^2$$

$$\mathbf{v}_2^T \mathbf{A} \mathbf{v}_2 = \lambda_2 \|\mathbf{v}_2\|^2$$

$$(\mathbf{v}_1 + \mathbf{v}_2)^T \mathbf{A} (\mathbf{v}_1 + \mathbf{v}_2) = (\lambda_1 + \lambda_2) \|\mathbf{v}_1 + \mathbf{v}_2\|^2$$

$$\mathbf{u}^T \mathbf{A} \mathbf{u} = \zeta \|\mathbf{u}\|^2$$

Eq 1. Valid for each eigenvector-eigenvalue pair

Think about Equation under Different Assumption about sign of λ

■ All eigenvalues are positive.

- ▶ Right-hand side of equation is always positive.
- ▶ $v^T A v$ is always positive for any vector v .

■ Eigenvalues are positive or zero.

- ▶ $v^T A v$ is nonnegative
- ▶ $v^T A v$ will equal zero when $\lambda = 0$.
 - $\lambda = 0$ happens when matrix is singular.

■ Eigenvalues are negative or zero.

- ▶ Quadratic form result will be zero or negative.

■ Eigenvalues are negative.

- ▶ Quadratic form result will be negative for all vectors.

Definiteness

■ Characteristic of square matrix

■ Defined by signs of eigenvalues of matrix.

- ▶ Same thing as signs of quadratic form results.

■ Implication

- ▶ Invertibility of matrix as well as advanced data analysis methods.
 - Such as generalized eigendecomposition
 - Used in multivariate linear classifiers and signal processing.

Categories of Definiteness

- There are **5 categories in definiteness.**
- Five categories as shown in Table 1.

- ▶ + and – signs indicate signs of eigenvalues.
- ▶ **"Depends"**
 - Matrix can be invertible or singular depending on numbers in matrix.
 - Not on definiteness category.

Category	Quadratic form	Eigenvalues	Invertible
Positive definite	Positive	+	Yes
Positive semidefinite	(Nonnegative)	+ and 0	No
Indefinite	Positive and negative	+ and -	Depends
Negative semidefinite	Nonpositive	- and 0	No
Negative definite	Negative	-	Yes

Table 1. Definiteness categories

$A^T A$ is Positive (Semi)definite

■ Specific matrix is guaranteed to be positive definite or positive semidefinite.

- ▶ Expressed as product of matrix and its transpose.
- ▶ That is, $S = A^T A$
- ▶ Combination of these two categories is often written as Positive (Semi)definite.

■ All data covariance matrices are positive (semi)definite.

- ▶ Because covariance matrices defined: $A^T A$
 - where data matrix: A ~~≠ 0~~
- ▶ All covariance matrices have nonnegative eigenvalues.

■ Case1: When data matrix is full-rank,

- ▶ If data is stored as observations by features,
 - Full column-rank
- ▶ Eigenvalues will be all positive.

■ Case2: If data matrix is reduced-rank,

- ▶ At least one zero-valued eigenvalue

Proof of $A^T A$

$$S = A^T A$$

■ Proof that S is positive (semi)definite.

- ▶ Writing out its quadratic form.
- ▶ Applying some algebra manipulations.

■ In Eq 1..

- ▶ Transition from first to second equation simply involves moving parentheses around.
 - Such “proof by parentheses” is common in linear algebra.

$$\begin{aligned} \mathbf{w}^T S \mathbf{w} &= \mathbf{w}^T (A^T A) \mathbf{w} \\ &= (\mathbf{w}^T A^T)(A \mathbf{w}) \\ &= (A \mathbf{w})^T (A \mathbf{w}) \\ &= \|A \mathbf{w}\|^2 \end{aligned}$$

Eq 1. Proof that S is positive (semi)definite by parentheses

Point of Proof of $A^T A$

■ Quadratic form of $A^T A$ equals $\|matrix\|^2 * vector$.

■ Characteristic of magnitudes

- ▶ Cannot be negative.
- ▶ Can be zero
 - Only when vector is zero.

■ If $Aw = 0$ for nontrivial w,

- ▶ Then A is singular.

■ Notice

- ▶ Although all $A^T A$ matrices are symmetric, not all symmetric matrices can be expressed as $A^T A$.
 - Matrix symmetry on its own does not guarantee positive (semi)definiteness.
 - Because not all symmetric matrices can be expressed as product of matrix and its transpose.

Importance of Quadratic Form and Definiteness

■ Importance in data science.

- ▶ Because some linear algebra operations are applied only to well-endowed matrices.
 - Cholesky decomposition
 - Create correlated datasets in Monte Carlo simulations.
- ▶ Importance in optimization problems.
 - Gradient descent
 - Because guaranteed minimum to find

■ In your never-ending quest to improve your data science prowess,

- ▶ You might encounter technical papers.
 - Use abbreviation SPD (Symmetric Positive Definite).

APC

$$J = c^T Q e + u^T R u$$

Generalized Eigendecomposition



Eigendecomposition

■ Consider that Eq 1. is same as fundamental eigenvalue equation.

▶ This is obvious.

- Because
- Generalized eigendecomposition as Eq 2. involves replacing identity matrix with another matrix.
 - Not identity or zeros matrix

$$A\mathbf{v} = \lambda I\mathbf{v}$$

Eq 1. Assumption equal to fundamental eigenvalue equation

$$A\mathbf{v} = \lambda B\mathbf{v}$$

Eq 2. Generalized eigendecomposition

Generalized Eigendecomposition

- It is also called simultaneous diagonalization of two matrices.
- Resulting (λ, v) pair is not eigenvalue / vector of A alone nor of B alone.
 - ▶ Instead, two matrices share eigenvalue / vector pairs.
- Conceptually, you can think of generalized eigendecomposition
 - ▶ As “regular” eigendecomposition of product matrix as Eq 1..
- Just conceptual
 - ▶ In practice, does not require B to be invertible.
- Not case
 - ▶ Any two matrices can be simultaneously diagonalized.
 - ▶ If B is
 - Diagonalization is possible.

$$\begin{aligned} C &= AB^{-1} \\ Cv &= \lambda v^{-1} \end{aligned}$$

Eq 1. “Regular” eigendecomposition of product matrix

Use Generalized Eigendecomposition in Data Science

- Classification analysis
- In particular, fisher's linear discriminant analysis (LDA)
 - ▶ Based on generalized eigendecomposition of two data covariance matrices.

Myriad Subtleties of Eigendecomposition

■ A lot of properties of eigendecomposition

- ▶ Sum of eigenvalues equals trace of matrix.
 - While product of eigenvalues equals determinant.
- ▶ Not all square matrices can be diagonalized.
- ▶ Some matrices have repeated eigenvalues.
 - Implications for their eigenvectors
- ▶ Complex eigenvalues of real-valued matrices
 - Inside circle in complex plane.

■ Mathematical knowledge of eigenvalues runs deep.

- ▶ But this lecture provides essential foundational knowledge.
 - For working with eigendecomposition in applications.

Summary



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Summary

■ Eigendecomposition identifies M scalar/vector pairs of an $M \times M$ matrix.

- ▶ It reflects special directions in the matrix.
- ▶ And have myriad applications in data science.
 - As well as in geometry, physics, computational biology, and myriad other technical disciplines.

■ Eigenvalues can be found.

- ▶ Assuming that the matrix shifted by an unknown scalar λ is singular.
- ▶ Setting its determinant to zero.
 - Called characteristic polynomial.
- ▶ And solving for λ s.

■ Eigenvectors can be found.

- ▶ By finding basis vector for the null space of $\lambda - \text{shifted}$ matrix.

■ Meaning of diagonalizing a matrix.

- ▶ Represent matrix as $V^{-1}\Lambda V$.
 - V : matrix with eigenvectors in the columns.
 - Λ : diagonal matrix with eigenvalues in the diagonal elements.

Summary

■ Symmetric matrices have several special properties in eigendecomposition.

- ▶ In data science
 - All eigenvectors are pair-wise orthogonal.
 - Matrix of eigenvectors is orthogonal.
 - Inverse of eigenvectors matrix is its transpose.

■ Definiteness of matrix

- ▶ Signs of its eigenvalues
- ▶ In data science
 - Positive (semi)definite
 - All eigenvalues are either nonnegative or positive.
- ▶ Matrix times its transpose is always positive (semi)definite.
 - All covariance matrices have nonnegative eigenvalues.

■ Study of eigendecomposition

- ▶ Rich and detailed
- ▶ Many fascinating subtleties, special cases, and applications

Code Exercises



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A, A^{-1} Eigenvalue

- Interestingly, the eigenvectors of A^{-1} are the same as the eigenvectors of A while the eigenvalues are λ^{-1} . Prove that is the case by writing out the eigendecomposition of A and A^{-1} . Then illustrate it using a random full-rank 5×5 symmetric matrix.

```
% create the matrix
A = randn(5,5);
A = A' * A;

% compute its inverse
Ai = ;

% eigenvalues of A and Ai
eigvals_A = ;
eigvals_Ai = ;

% compare them (hint: sorting helps!)
disp('Eigenvalues of A:')
disp(sort(eigvals_A))

disp(' ')
disp('Eigenvalues of inv(A):')
disp(sort(eigvals_Ai))

disp(' ')
disp('Reciprocal of evals of inv(A):')
disp(sort(1./eigvals_Ai))
```

Sample code

Interesting property of random matrices

- One interesting property of random matrices is that their complex-valued eigenvalues are distributed in a circle with a radius proportional to the size of the matrix. To demonstrate this, compute 123 random 42×42 matrices, extract their eigenvalues, divide by the square root of the matrix size (42), and plot the eigenvalues on the complex plane, as in **Figure below**.

```

nIter = 123;
matsize = 42;
% fill this evals variable
evals = zeros(nIter, matsize);

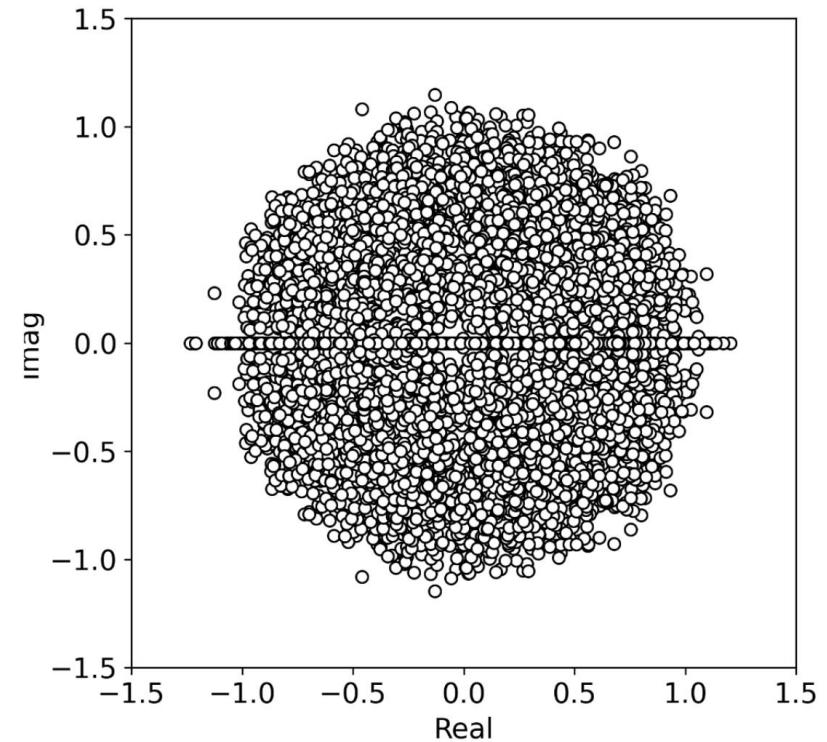
% create the matrices and get their scaled eigenvalues
for i = 1:nIter
    % declear (matsize, matsize) sized matrix A in every iteration
    A = ;
    evals(i, :) = ;
end

% visualization
% and show in a plot
figure('Position', [100, 100, 600, 600]);

plot(real(evals(:)), imag(evals(:)), 'ko', 'MarkerFaceColor', 'w');
xlim([-1.5, 1.5]);
ylim([-1.5, 1.5]);
xlabel('Real');
ylabel('Imag');

```

Sample code



Method to Create Random Symmetric Matrices

- Start by creating a 4×4 diagonal matrix with positive numbers on the diagonals (they can be, for example, the numbers 1,2,3,4). Then create a 4×4 Q matrix from the QR decomposition of a random-numbers matrix. Use these matrices as the eigenvalues and eigenvectors, and multiply them appropriately to assemble a matrix. Confirm that the assembled matrix is symmetric, and that its eigenvalues equal the eigenvalues you specified.

```
% Create the Lambda matrix with positive values
Lambda = diag(rand(4,1) * 5);
randnMat = randn(4,4);

% create Q
;

% reconstruct to a matrix
A = ;

% the matrix minus its transpose should be zeros (within precision error)
result = ;

disp(result);

% sort(diag(Lambda)) and sort(eig(A)) disp same result
% print sorted diagonal of Lambda
disp('Sorted diagonal of Lambda:')
disp(sort(diag(Lambda)))

% print sorted eigenvalues of A
disp('Sorted eigenvalues of A:')
disp(sort(eig(A)))
```

Sample code





**THANK YOU
FOR YOUR ATTENTION**



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