MA1522 Reference Notes (midterms)

github.com/reidenong/cheatsheets, AY23/24 S1

Row Echelon Form

A matrix is in row echelon form if:

(1) All zero rows are at the bottom of the matrix.
(2) The leading entries are further to the right as we move down the rows.

It is in *Reduced Row Echelon form* if:

(1) The leading entries are 1.
(2) In each pivot column, all entries except the leading entry is zero.

Elementary Row Operations

(1) Interchange two rows.
(2) Multiply a row by a nonzero constant.
(3) Add a multiple of one row to another row.

Two Linear systems have the same solution set if their augmented matrices are row equivalent.

If matrix B is obtained from matrix A by

$$A \xrightarrow{r_1} \xrightarrow{r_2} \dots \xrightarrow{r_k} B$$

Then

$$B = E_k E_{k-1} \dots E_2 E_1 A$$

where E_i is the elementary matrix corresponding to r_i.

Systems of Linear Equations

The linear system of $A\boldsymbol{x} = \boldsymbol{b}$ is homogenous if $\boldsymbol{b} = 0$. If there is a nontrivial solution, it has infinitely many solutions.

A linear system is consistent if it has at least one solution. A homogenous equation $A\boldsymbol{x} = \mathbf{0}$ is always consistent, as it has at least the trivial solution.

Types of Matrices

Scalar Matrices

A scalar matrix is a diagonal matrix where all diagonal entries are equal.

Triangular Matrices

Upper Triangular A where $a_{\{ij\}} = 0$ for $i > j$.

$$\begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & * \end{pmatrix}$$

Strictly Upper Triangular A where $a_{\{ij\}} = 0$ for $i \geq j$.

$$\begin{pmatrix} 0 & * & \dots & * \\ 0 & 0 & \dots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Lower Triangular A where $a_{\{ij\}} = 0$ for $i < j$.

$$\begin{pmatrix} * & 0 & \dots & 0 \\ * & * & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & * & * \end{pmatrix}$$

Strictly Lower Triangular A where $a_{\{ij\}} = 0$ for $i \leq j$.

$$\begin{pmatrix} 0 & 0 & \dots & 0 \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & * & 0 \end{pmatrix}$$

Scalar Multiplication and Matrix Addition

Properties:

(1) Commutative: $A + B = B + A$
(2) Associative: $(A + B) + C = A + (B + C)$
(3) Additive identity: $A + 0 = A$
(4) Additive inverse: $A + (-A) = 0$
(5) Distributive: $c(A + B) = cA + cB$
(6) Scalar addition: $(c + d)A = cA + dA$
(7) Associative: $c(dA) = (cd)A$
(8) If $aA = \mathbf{0}$, then $a = 0$ or $A = \mathbf{0}$

Matrix Multiplication

For multiplication of a $m \times n$ matrix A and a $n \times p$ matrix B,

$$AB_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Properties:

(1) Associative: $A(BC) = (AB)C$
(2) Left distributive: $A(B + C) = AB + AC$
(3) Right distributive: $(A + B)C = AC + BC$
(4) Commutes with scalar multiplication:
$$c(AB) = (cA)B = A(cB)$$
(5) Not commutative: $AB \neq BA$ in general
(6) Multiplicative Identity: $I_n A = A I_m = A$
(7) Zero divisor: There exists nonzero matrices A and B such that $AB = \mathbf{0}$
(8) Zero matrix: $A\mathbf{0} = \mathbf{0}A = \mathbf{0}$

Block Multiplication

$$AB = A(b_1 \;\; b_2 \;\; \dots \;\; b_n) = (Ab_1 \;\; Ab_2 \;\; \dots \;\; Ab_n)$$

$$AB = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_m \end{pmatrix} B = \begin{pmatrix} a_1 B \\ a_2 B \\ \dots \\ a_m B \end{pmatrix}$$

Transpose

The transpose of a $m \times n$ matrix A is a $n \times m$ matrix A^T where $A^T_{ij} = A_{ji}$.

Properties:

(1) $(A^T)^T = A$
(2) $(cA)^T = cA^T$
(3) $(A + B)^T = A^T + B^T$
(4) $(AB)^T = B^T A^T$

Inverse of a Matrix

A matrix A is invertible if there exists a unique matrix B such that $AB = BA = I$.

Properties:

(1) $(A^{-1})^{-1} = A$
(2) $(cA)^{-1} = c^{-1}A^{-1}, \forall c \in \mathbb{R}$
(3) $(A^T)^{-1} = (A^{-1})^T$
(4) $(AB)^{-1} = B^{-1}A^{-1}$ if A, B are both invertible
(5) Left Cancellation Law: $AB = AC \rightarrow B = C$
(6) Right Cancellation Law: $BA = CA \rightarrow B = C$

To find an inverse, consider

$$(A \mid I) \overset{RREF}{\longrightarrow} (I \mid A^{-1})$$

Invertible Matrix Theorem

Let A be a $n \times n$ matrix. The following statements are equivalent:

(1) A is invertible.
(2) A has a left inverse
(3) A has a right inverse
(4) RREF of A is I_n
(5) A can be expressed as a product of elementary matrices
(6) Homogenous system $Ax = \mathbf{0}$ has only the trivial solution
(7) for any b, the system $Ax = b$ has a unique solution
(8) The determinant of A is nonzero
(9) The columns/rows of A are linearly independent
(10) The columns/rows of A span $\mathbb{R}^n$
(11) A^T is invertible (12) rank(A) = n (A has full rank)
(13) nullity(A) = 0

LU Decomposition

Suppose $A \overset{r_1}{\rightarrow}\overset{r_2}{\rightarrow} \dots \overset{r_k}{\rightarrow} U$, where each row operation is of the form $R_i + cR_j$ and U is a row echelon form of A. Then A can be decomposed into a *unit* lower triangular matrix and an upper triangular matrix.

(*)
(1) Not all matrices have LU
(2) nxm can have LU, $n \neq m$
(3) U does not have to be square

$$A = LU = \begin{pmatrix} 1 & 0 & \dots & 0 \\ * & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ * & * & \dots & 1 \end{pmatrix} \begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & 0 & * \end{pmatrix}$$

$$L = E_1^{-1}E_2^{-1}\dots E_k^{-1}$$

To solve $LUx = Ax = b$, solve $Ly = b$, then $Ux = y$.

Determinant

Properties:

(1) $\det(A^T) = \det(A)$
(2) $\det(AB) = \det(A)\det(B)$ for A, B of same size
(3) $\det(A^{-1}) = \frac{1}{\det(A)}$
(4) $\det(cA) = c^n \det(A)$ for $n \times n$ matrix A
(5) $\det(diag(a_1, a_2, \dots, a_n)) = a_1 \cdot a_2 \cdot \dots \cdot a_n$
(6) Determinant and Row Elementary Operations:

$A \overset{R_i + cR_j}{\longrightarrow} B$	$\det(A) = \det(B)$	$\det(B) = \det(A)$
$A \overset{cR_i}{\longrightarrow} B$	$\det(A) = \frac{1}{c}\det(B)$	$\det(B) = c\det(A)$
$A \overset{R_i \leftrightarrow R_j}{\longrightarrow} B$	$\det(A) = -\det(B)$	$\det(B) = -\det(A)$

Finding Determinants:

1. for n = 2, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = ad - bc$
2. for n = 3, $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$,
   $\det(A) = aei + bfg + cdh - ceg - bdi - afh$
3. for $n \geq 3$, use Cofactor Expansion:

$$\det(A) = \sum_{j=1}^{n} a_{ij}A_{ij} = \sum_{j=1}^{n} a_{jk}A_{jk}$$

where A_{ij} is the (i,j) cofactor of A, given by

$$\boxed{A_{ij} = (-1)^{i+j}\det(M_{ij})}$$

where M_{ij} is the (i, j) matrix minor of A, the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and jth column of A.

$\boxed{adj(A)_{ij} = (-1)^{i+j} \det(M_{ji})}$

Adjoint

With a order n square matrix A, the adjoint of A is

$$adj(A) = (A_{ij})^T = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

$$= \begin{pmatrix} M_{11} & -M_{21} & \dots & \pm M_{n1} \\ -M_{12} & M_{22} & \dots & \mp M_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ \pm M_{1n} & \mp M_{2n} & \dots & \pm M_{nn} \end{pmatrix}$$

Adjoint Formula: adj(A) is invertible then A is invertible

(1) $A \cdot adj(A) = \det(A)I$
(2) $A^{-1} = \frac{1}{\det(A)} adj(A)$
(3) $\det(adj(A)) = (\det(A))^{n-1}$
(4) $adj(A)^T = adj(A^T)$
(5) $adj(A)^{-1} = \frac{1}{\det(A)} A$
(6) $adj(A)A = A\,adj(A)$

Cramer's Rule

Let A be a invertible $n \times n$ matrix.
For any $b \in \mathbb{R}^n$, the unique solution of $Ax = b$ is given by

$$x_i = \frac{\det(A_i(b))}{\det(A)}$$

where $A_i(b)$ is the matrix obtained by replacing the ith column of A with b.

Linear Span

$$span(u_1 \ u_2 \ \dots \ u_n) = \{c_1u_1 + c_2u_2 + \dots + c_nu_n \mid c_i \in \mathbb{R}, \forall i\}$$

$$\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix} \mid x_i \in \mathbb{R}, \forall i \right\}$$

$\mathbb{R}^n$ is the set of all vectors with n-coordinates.

Theorem:

(1) $\boldsymbol{v} \in span(u_1 \ u_2 \ \dots \ u_n) \leftrightarrow$
$(u_1 \ u_2 \ \dots \ u_n)x = \boldsymbol{v}$ is consistent $\leftrightarrow$
$(u_1 \ u_2 \ \dots \ u_n \mid \boldsymbol{v})$ is consistent.

(2) $span(u_1 \ u_2 \ \dots \ u_n) = \mathbb{R}^n \leftrightarrow$
The reduced row echelon form of A has no zero rows.

Properties:

Let $S = \{u_1, u_2, ..., u_n\} \subseteq \mathbb{R}^n$.

(1) Contains Origin: $\mathbf{0} \in span(S)$

(2) Closed under addition:

$\forall u, v \in span(S), u + v \in span(S)$

(3) Closed under scalar multiplication:

$\forall u \in span(S), \forall c \in \mathbb{R}, c \cdot u \in span(S)$

(4) Contains all linear combinations:

$\forall u_1, u_2, ..., u_n \in span(S)$,

$\forall c_1, c_2, ..., c_n \in \mathbb{R}$,

$c_1 u_1 + c_2 u_2 + ... + c_n u_n \in span(S)$

Span equality:

Let $S = \{u_1, u_2, ..., u_k\}$ and $T = \{v_1, v_2, ..., v_n\}$. Then,

$$span(T) \subseteq span(S) \leftrightarrow$$
$$\forall v \in T, v \in span(S) \leftrightarrow$$
$$(S \mid T) \text{ is consistent.}$$

For equality, we need to show that $span(S) \subseteq span(T)$ and $span(T) \subseteq span(S)$.

Subspaces

A subset $V \subseteq \mathbb{R}^n$ is a subspace if:

(1) Contains Origin: $\mathbf{0} \in V$

(2) Closed under linear combination:

$\forall u, v \in V, \forall c, d \in \mathbb{R}, cu + dv \in V$

A subset $V \subseteq \mathbb{R}^n$ is a subspace if and only if it is a linear span, $V = span(S)$ for some finite set $S = \{u_1, u_2, ..., u_n\}$.

Solution Sets of linear systems

Solution sets of linear systems can be expressed implicitly or explicitly.

Implicit form:

$$\left\{ x = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \mid Ax = b \right\}$$

Explicit form:

$$\{\boldsymbol{u} + s_1 \boldsymbol{v_1} + s_2 \boldsymbol{v_2} + ... + s_k \boldsymbol{v_k} \mid s_i \in \mathbb{R}, \forall i\}$$

where $\boldsymbol{u} + s_1 \boldsymbol{v_1} + s_2 \boldsymbol{v_2} + ... + s_k \boldsymbol{v_k}$ are the general solutions of $Ax = \boldsymbol{b}$.

The solution set $V = \{u \mid Au = \boldsymbol{b}\}$ is a subspace if and only if **b** = 0, ie. the system is homogenous.

The solution set $W = \{w \mid Aw = \boldsymbol{b}\}$ of a linear system $Ax = \boldsymbol{b}$ is given by **u** + V, where

(1) $V = \{v \mid Av = \mathbf{0}\}$ is the solution set of the homogenous

system $Ax = \mathbf{0}$ and

(2) **u** is a particular solution of $Au = \boldsymbol{b}$.

Linear Independence

A set of vectors $S = \{u_1, u_2, ..., u_n\}$ is linearly independent if the only solution to $c_1 u_1 + c_2 u_2 + ... + c_n u_n = \mathbf{0}$ is $c_1 = c_2 = ... = c_n = 0$.
A set is linearly independent iff the RREF of S has no non-pivot columns.

Special Cases:

1. $\{\mathbf{0}\}$ is always linearly dependent.
2. {v_1, v_2} is linearly dependent iff v_1 is a scalar multiple of v_2.
3. {} = ∅ is linearly independent.
4. Any subset of $\mathbb{R}^n$ containing more than n vectors must be linearly dependent.
6. Any superset of a linearly dependent set is linearly dependent.
7. Any subset of a linearly independent set is linearly independent.
8. A set S containing n vectors in $\mathbb{R}^n$ is linearly independent iff it spans $\mathbb{R}^n$
9. Basis of zero space {0} is span{}.

Basis

Let $V \subseteq \mathbb{R}^n$ be a subspace. A set $\boldsymbol{S} = \{u_1, u_2, ..., u_k\}$ is a basis of V if:

(B1)
1. $|S| = \dim(V)$
2. $S \subseteq V$
3. S is linearly independant

(B2)
1. $|S| = \dim(V)$
2. $V \subseteq span(S)$

A subset $S = \{u_1, u_2, ..., u_n\} \subseteq \mathbb{R}^n$ is a basis for $\mathbb{R}^n$ iff $|S| = n$ and $A = (u_1 \ u_2 \ \cdots \ u_n)$ is an invertible matrix.

Coordinates Relative to a basis

With basis $S = \{u_1, u_2, ..., u_n\}$, every vector $\boldsymbol{v} \in \mathbb{R}^n$ can be expressed uniquely as a linear combination of the basis vectors.

$$\boldsymbol{v} = c_1 u_1 + c_2 u_2 + ... + c_n u_n \leftrightarrow [\boldsymbol{v}]_s = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

Transition matrix are invertible

Change of Basis / Transition Matrix

Suppose there exist bases $S = \{u_1, u_2, u_3\}$ and $T = \{v_1, v_2, v_3\}$. Then, the transition matrix from T to S is

$$\mathrm{RREF}(S \mid T) = \left(\begin{array}{c|c} I_k & P_{T\to S} \\ 0 & 0 \end{array}\right)$$

Then $[w]_S = P_{T \to S}[w]_T$, and

$P_{T\to S} = ([v_1]_S \ [v_2]_S \ [v_3]_S)$

$$T = S \times P_{T \to S}$$

$P_{S\to T}$ then $(P_{S\to T})^{-1} = P_{T\to S}$

Dimension

The dimension of a subspace $V \subseteq \mathbb{R}^n$ is the number of vectors in any basis of V. The dimension of a solution space $V = \{u \mid Au = 0\}$ is the number of non-pivot columns in the RREF of A.

Column and Row Space

The column space of a matrix A is the subspace of $\mathbb{R}^m$ spanned by its column vectors.

$$v \in \mathrm{colspace}(A) \leftrightarrow Ax = v \text{ is consistent.}$$

ERO $(A \to B)$ ie A and B row equivalent.

row
1. Row(A) = Row(B)
2. linearity destroyed

col
1. Col(A) ≠ Col(B)
2. linearity preserved

Null(A) = Null(B)

Let **A** be a $m \times n$ matrix.

$A_{m \times n}$

Subspace	Subspace of	Basis	Dimension
Col(**A**)	$\mathbb{R}^m$	Columns of **A** corresponding to pivot columns in RREF	rank(**A**) = no. of pivot columns in RREF
Row(**A**)	$\mathbb{R}^n$	Nonzero rows of RREF	rank(**A**) = no. of nonzero rows in RREF
Null(**A**)	$\mathbb{R}^n$	Vectors in general solution to $\mathbf{Ax} = \mathbf{0}$	nullity(**A**) = no. of nonpivot columns in RREF

Rank

rank(A) = dimension of colspace(A)
= dimension of rowspace(A)
= number of pivot columns in RREF
= number of nonzero rows in RREF

Geometry is ONLY preserved for vectors expressed in orthonormal basis.

1. rank(A^T) = rank(A) = rank(A^TA) = rank(AA^T).
2. rank($A_{m,n}$) $\leq$ min(m, n), with equality when *full rank*.
3. rank(AB) $\leq$ min(rank(A), rank(B)) | S. Col(AB) $\subseteq$ Col(A)
4. rank(AB) $\leq$ min(rank(A), rank(B))

Nullspace

Nullspace of A is the solution space to $Ax = \mathbf{0}$.

$$\begin{aligned} \text{null}(A) &= \{x \in \mathbb{R}^n \mid Ax = \mathbf{0}\} \\ &= \text{null}(A^TA) \end{aligned}$$

$$\begin{aligned} \text{nullity}(A) &= \dim(\text{null}(A)) \\ &= \text{number of non-pivot columns in RREF} \end{aligned}$$

Then by the *Rank-Nullity Theorem*,

$$\boxed{\text{rank}(A) + \text{nullity}(A) = \text{number of columns in A}}$$

Dot Product, norm

$$\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u}^T v$$
$$\|u\| = \sqrt{\boldsymbol{u} \cdot \boldsymbol{u}}$$
$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|u\|\|v\|}$$

Where θ is the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

Orthogonality

Two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$. A set of vectors are orthogonal if every pair of vectors in the set are orthogonal. It is linearly independent if it does not have the zero vector.

0 can be in Orthogonal Set but NOT Orthonormal Set

A set of vectors is orthonormal if it is orthogonal and every vector in the set has norm 1. It is then guaranteed to be linearly independent.

$Col(A)^\perp = Null(A^T)$; $Row(A)^\perp = Null(A)$

Orthogonal Basis

$w \in Null(A^T)$ is $\perp$ to col(A).

A basis is orthogonal if it is a orthogonal set.
Let $S = \{u_1, u_2, ..., u_n\}$ be an orthogonal basis. Then for any $v \in V$, To express a $v \in \mathbb{R}^n$ as linear combi of orthogonal set

$$\boxed{v = \left(\frac{v \cdot u_1}{\|u_1\|^2}\right)u_1 + \left(\frac{v \cdot u_2}{\|u_2\|^2}\right)u_2 + ... + \left(\frac{v \cdot u_n}{\|u_n\|^2}\right)u_n}$$

Orthogonal Projection

same formula to find wp !!!

vector $n \in \mathbb{R}^n$ is orthogonal to subspace V if $\forall v \in V, n \cdot v = 0$. proj of u in direction of v = $\frac{u \cdot v}{\|v\|^2} v$

Consider subspace $V \subseteq \mathbb{R}^n$. Every vector $w \in \mathbb{R}^n$ can be decomposed uniquely as $w = v + n$, where n is orthogonal to V and $v \in V$ is the orthogonal projection of w onto V. $w = w_p + w_n$

Let subspace V have basis $S = \{v_1, ..., v_n\}$. Let $A = (v_1 ... v_n)$. Then the orthogonal projection of w onto V is

$$\boxed{\text{proj} = A(A^TA)^{-1}A^Tw = A\hat{w}}$$

Where $\hat{w}$ is a least Square solution of $Ax = w$. Alternatively, one can use the method above of expressing v as a linear combination of the (orthogonal) basis vectors to show the projection of w onto v

Gram-Schmidt Process

Linearly Indp set → Orthogonal set

Let $S = \{u_1, u_2, ..., u_n\}$ be a linearly independent set.

$$v_1 = u_1$$
$$v_2 = u_2 - \left(\frac{u_2 \cdot v_1}{\|v_1\|^2}\right)v_1$$
$$v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{\|v_1\|^2}\right)v_1 - \left(\frac{u_3 \cdot v_2}{\|v_2\|^2}\right)v_2$$
$$\vdots$$
$$v_n = u_n - \left(\frac{u_n \cdot v_1}{\|v_1\|^2}\right)v_1 - \left(\frac{u_n \cdot v_2}{\|v_2\|^2}\right)v_2 - ... - \left(\frac{u_n \cdot v_{n-1}}{\|v_{n-1}\|^2}\right)v_{n-1}$$

Then $S' = \{v_1, v_2, ..., v_n\}$ is an orthogonal basis for $span(S)$.

Best approx. Theorem → $\|w - w_p\| \leq \|w - v\|$ for all $v \in V \subseteq \mathbb{R}^n$
∴ w_p is closest to w.

Orthogonal Matrices

$Q_{n \times m} \rightarrow Q^TQ$ — diagonal (orthogonal) / Identity (orthonormal)

A square matrix A is orthogonal if
$\boxed{A^T = A^{-1}} \leftrightarrow A^TA = I. = AA^T$
Then the columns and rows of A form an orthonormal basis for $\mathbb{R}^n$. → Product of orthogonal matrix is an orthogonal matrix

QR Factorization

If matrix A has linearly independent columns, A can be uniquely written as $A = QR$, where Q is an orthonormal set and R is an invertible upper triangular matrix with +ve diagonals.

L.I.

$Q^TQ = I$

Algorithm for QR Factorization:
(1) Gram-Schmidt on A to obtain orthonormal set Q.
(2) $R = Q^TA$, ensuring R has +ve diagonals by multiplying the column of Q by −1 as needed.

unique sol if A is invertible ⟷ A^TA invertible

Least Square Approximation

A vector $u \in \mathbb{R}^n$ is a least square solution to $Ax = b$ if for every vector $v \in \mathbb{R}^n$, $\|Au - b\| \leq \|Av - b\|$.
The least square solution is given by the solution set of $A^TAx = A^Tb$.

If col of A are linearly independent then orthogonal projection of $w \in \mathbb{R}^n$ onto subspace : $w_p = A(A^TA)^{-1}A^Tw$

Eigenvalues and Eigenvectors

With square matrix A, λ /v are eigenvalue/eigenvector if $v \neq 0$ and $Av = \lambda v$.

$\det(\lambda I - A) = \det(\lambda I - A^T)$ ∴ Hence A and A^T same eigenvalues.

The nontrivial solutions to $(\lambda I - A)x = 0$ are the eigenvectors of A with eigenvalue λ. If λ is a eigenvalue of A, it is a eigenvalue of A^T as they share the same characteristic polynomial.

If $\lambda = 0$, A is singular.

The algebraic multiplicity of λ is the number of times λ appears as a root of the characteristic polynomial of A.

The geometric multiplicity of λ is the dimension of the eigenspace of λ, null($A - \lambda I$).

$$\boxed{1 \leq \text{geometric multiplicity} \leq \text{algebraic multiplicity}, \forall \lambda_i}$$

eigenvalues of triangular matrixes are the diagonal entries. The algebraic multiplicity is the number of times it appears on diagonal entries.

(*) Product of eigenvalues each raised to its algebraic multiplicity is the determinant of A.

Diagonalization

A is diagonalizable if there exists invertible P such that $P^{-1}AP$ is a diagonal matrix.

$D = P^{-1}AP$ or $A = PDP^{-1}$

A is diagonalizable iff

(1) the characteristic polynomial of A splits into linear factors

and

(2) the algebraic multiplicity of each eigenvalue equals its geometric multiplicity

(3) There exists a basis of $\mathbb{R}^n$ of eigenvectors of A.

If $A_{n\times n}$ diagonalizable, then $\bigcup^k S_{\lambda_k} = \mathbb{R}^n$, where S_{λ_k} is the basis for each eigenspace E_{λ_k}. The eigenvectors for A form a basis for $\mathbb{R}^n$, ie $\forall u \in \mathbb{R}^n, v = c_1v_1 + ... + c_nv_n$ where v is the eigenvectors that form the basis for each eigenspace.

If λ_1, λ_2 are distinct eigenvalues of A, then the eigenspaces are linearly independent

Algorithm for diagonalization:

(*)

(1) Find Eigenvalues of A.

(2) For each Eigenvalue, find a basis for its Eigenspace.

(3) The bases of the Eigenvalues form the columns of P.

(4) D is a diagonal matrix of Eigenvalues, where each align to their corresponding bases in P.

Powers of Diagonalizable matrices

$k>0 \rightarrow A$ non-invertible

$k \in \mathbb{Z} \rightarrow A$ invertible

$$\boxed{A^k = P\cdot D^k \cdot P^{-1}} = P\begin{pmatrix} \lambda_1^k & 0 & ... & 0\\ 0 & \lambda_2^k & ... & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & ... & \lambda_n^k\end{pmatrix}P^{-1}$$

might need to perform GS on vectors in some E_λ to make them orthonormal

Symmetric matrixes

Orthogonally Diagonalizable / Symmetric

A is orthogonally diagonalizable if $A = PDP^T$ for some orthogonal matrix P. Algorithm is the same as diagonalization, except the basis of each eigenspace is change to an orthonormal basis. Then $\boldsymbol{A} = \boldsymbol{A^T}$.

eigenspaces of distinct eigenvalues are orthogonal to each other.

Stochastic Matrices

(1) A is square.

(2) Sum of the columns of A is 1.

(3) All entries of A are nonnegative. $|\lambda| \leq 1$

(4) 1 is an eigenvalue of A, as $A^T(1; ...; 1) = (1; ...; 1)$

$A^k, k \in \mathbb{Z}^+$, all a_{ij} entries are +ve.

Regular Stochastic Matrix will always converge to equil vector.

A Markov chain is a sequence of probability vectors $x_0, x_1, x_2, ...$ such that $x_{k+1} = Ax_k$ for some stochastic matrix A.

Singular Value Decomposition

Every $m \times n$ matrix can be written as $A = \boldsymbol{U\Sigma V^T}$, where

(1) U is a order m orthogonal matrix

(2) V is a order n orthogonal matrix

(3) Σ is of the form

$$\Sigma_{m,n} = \begin{pmatrix} D & 0\\ 0 & 0\end{pmatrix}$$

where D has the roots of the eigenvalues of A^TA.

non-negative

symmetric matrix of order n for any $A_{m\times n}$

SVD Algorithm (autoSVD) :

(1) find the eigenvalues of A^TA, arranging the nonzero ones in descending order with duplicates. Find Σ by using the roots of these eigenvalues.

(2) Find an orthogonal basis for each eigenspace, then set $V = (v_1v_2...v_n)$ where v_i is the unit vector associated to the ith eigenvalue.

non-zero val of σ_i

(3) Let $u_i = \frac{1}{\sigma_i}Av_i$ for i = 1,2,...,r. Extend $\{u_1, ..., u_r\}$ to an orthonormal basis $\{u_1, ..., u_r, ..., u_m\}$ of $\mathbb{R}^m$ to get U.

Orthonormal Basis of Col(A), rank(A) = r

By solving for $\text{Null}(u_1 ... u_r)^T$

Linear Transformations

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation if $\forall u, v \in \mathbb{R}^n$

(1) $T(u + v) = T(u) + T(v)$

(2) $T(cu) = cT(u), \forall c \in \mathbb{R}$

(3) $T(\mathbf{0}) = \mathbf{0}$

Range of T:

Col(A)

$$R(T) = \{T(u) \mid u \in \mathbb{R}^n\}$$

$$\text{rank}(T) = \dim(R(T)) = \dim(\text{colspace}(A)) = \text{rank}(A)$$

Kernel of T:

Null(A)

$$\ker(T) = \{u \in \mathbb{R}^n \mid T(u) = \mathbf{0}\}$$

$$\text{nullity}(T) = \dim(\ker(T)) = \text{nullity}(A)$$

Rank-Nullity Theorem:

Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then

$$\boxed{\text{rank}(T) + \text{nullity}(T) = n}$$

One-to-One (Injective)

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is *one-to-one* if $T(u) = T(v) \rightarrow u = v$.

T is One-to-One $\leftrightarrow \ker(T) = \{\mathbf{0}\} \leftrightarrow \text{nullity}(T) = 0$

Onto (Surjective)

$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is *onto* if $R(T) = \mathbb{R}^m$.

T is Onto $\leftrightarrow R(T) = \mathbb{R}^m \leftrightarrow \text{rank}(T) = m$

Invertible Matrix Theorem (extended)

if *invertible* A describes a linear transformation T,

(xi) rank(A) = n, ie. A has full rank

(xii) nullity(A) = 0

(xiii) 0 is not an eigenvalue of A

(xiv) T is one-to-one (injective)

(xv) T is onto (surjective)

Finding Standard Matrix A

we need $\{T(u_1), ..., T(u_n)\}$ for a *basis* $\{u_1, ..., u_n\}$ of $\mathbb{R}^n$.

Then

$$\boxed{A = (T(u_1)\ \dots\ T(u_n))(u_1 ... u_n)^{-1}}$$

($A_{m\times n}$) Full Rank

Col

1. rank(A) = n
2. Row(A) = $\mathbb{R}^n$
3. Col of A are linearly independent
4. Homogenous sol^n has only trivial sol^n, ie Null(A) = {0}
5. A^TA invertible matrix of order n.
6. A has left inverse
7. left inverse: $(A^TA)^{-1}A^T$
8. $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by A is injective

Row

1. rank(A) = m
2. Col(A) = $\mathbb{R}^m$
3. Rows are linearly independent
4. $Ax = b$ is consistent for all $b \in \mathbb{R}^m$
5. AA^T is invertible matrix of order m
6. A has right inverse
7. Right inverse: $A^T(AA^T)^{-1}$
8. $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by A is surjective

Algorithm for eqm vector

① Find an eigenvector u in eigenspace E_1, by finding Null$(I-P)$, P is stochastic matrix.

②

$$\text{eqm vector} = \frac{1}{\text{sum of entries}} \cdot u$$

Linear transformation wrt basis

If $T: \mathbb{R}^n \to \mathbb{R}^m$ and $S = \{u_1, \dots u_n\}$ is basis of $\mathbb{R}^n$, then

$$[T]_S = (T(u_1) \dots T(u_n)).$$

Futhermore, $T(v) = [T]_S [v]_S$

Standard matrix of T,

$$A = [T]_E = [T]_S P_{E \to S}$$

$$P_{E \to S} = (P_{S \to E})^{-1} = (u_1 \dots u_n)^{-1}$$

To show that a transformation is unique

$Au_i = Bu_i$ for all i, for another linear transformation B, then

$$A = B.$$

so A is unique.

Nilpotent

- 0 is its only eigenvalue