Result. Limits of the form $\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials in x.

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}=\lim_{x\to\pm\infty}\frac{\overbrace{Ax^\alpha}^{\text{leading term}}+\cdots}{\underbrace{Bx^\beta}_{\text{leading term}}+\cdots}=\begin{cases}0 & \text{if } \alpha<\beta\\ \frac{A}{B} & \text{if } \alpha=\beta\\ \underbrace{\infty \text{ or } -\infty}_{\text{depends on the sign of } A \text{ and } B} & \text{if } \alpha>\beta.\end{cases}$$

There are 3 indeterminate forms of the following types:

(1) 0^0, (2) ∞^0, (3) 1^∞.

0^0: $\lim_{x\to0^+}x^x=\lim_{x\to0^+}e^{\ln(x^x)}=\lim_{x\to0^+}e^{x\ln x}=e^{\lim_{x\to0^+}x\ln x}=e^{\lim_{x\to0^+}\frac{\ln x}{x^{-1}}}$

$=e^{\lim_{x\to0^+}\frac{x^{-1}}{-x^{-2}}}=e^{\lim_{x\to0^+}-x}=e^0=1.$

∞^0: $\lim_{x\to0^+}\left(\frac{2}{x}\right)^x=\lim_{x\to0^+}\frac{2^x}{x^x}=\frac{\lim_{x\to0^+}2^x}{\lim_{x\to0^+}x^x}=\frac{2^0}{1}=1.$

1^∞: $\lim_{x\to0}(\cos x)^{\frac{1}{x}}=\lim_{x\to0}e^{\frac{\ln\cos x}{x}}=e^{\lim_{x\to0}\frac{\ln\cos x}{x}}$

$=e^{\lim_{x\to0}\frac{\frac{1}{\cos x}(-\sin x)}{1}}=e^{\lim_{x\to0}-\tan x}=e^0=1.$

f is continuous at $x=c$ if:

① $\lim_{x\to c}f(x)=\lim_{x\to c^-}f(x)=\lim_{x\to c^+}f(x)=f(c)$

Laws of limits:

1. $\lim_{x\to c}\left(f(x)+g(x)\right)=\lim_{x\to c}f(x)+\lim_{x\to c}g(x)$
2. $\lim_{x\to c}\left(kf(x)\right)=k\lim_{x\to c}f(x)$
3. $\lim_{x\to c}\left(f(x)g(x)\right)=\left(\lim_{x\to c}f(x)\right)\left(\lim_{x\to c}g(x)\right)$
4. $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)}$

If given some **indeterminate fraction** limit:

1. $\lim_{x\to6}\frac{6-x}{36-x^2}=\frac{1}{6+x}$
2. $\lim_{x\to-3}\frac{\sqrt{a}+\sqrt{b}}{9-x^2}=\frac{a-b}{(9-x^2)(\sqrt{a}-\sqrt{b})}$

If $\lim_{x\to c}g(x)=0$

1. $\lim_{x\to c}\frac{\sin g(x)}{g(x)}=\frac{g(x)}{\sin g(x)}=1$
2. $\lim_{x\to c}\frac{\tan g(x)}{g(x)}=\frac{g(x)}{\tan g(x)}=1$

Squeeze Theorem:

1. If $g(x)\le f(x)\le h(x),\forall x$ in some interval containing c, then $\lim_{x\to c}g(x)=\lim_{x\to c}h(x)=L\to\lim_{x\to c}f(x)=L$
2. $\lim_{x\to c}g(x)=0\to$

$\lim_{x\to c}g(x)\sin h(x)=\lim_{x\to c}g(x)\cos\left(h(x)\right)=0$

Intermediate Value Theorem (IVT)

If f continuous on $[a,b]$, and $f(a)\le k\le f(b)$, then $f(c)=k$ for some $c\in[a,b]$

2 Derivatives

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

If f is differentiable at $x=x_0$, then f is continuous at $x=x_0$

$\frac{d}{dx}\left(f(x)\right)^n=nf'(x)\left(f(x)\right)^{n-1}$

$\frac{d}{dx}\cos f(x)=-f'(x)\sin f(x)$

$\frac{d}{dx}\sin f(x)=f'(x)\cos f(x)$

$\frac{d}{dx}\tan f(x)=f'(x)\sec^2 f(x)$

$\frac{d}{dx}\sec f(x)=f'(x)\sec f(x)\tan f(x)$

$\frac{d}{dx}\csc f(x)=-f'(x)\csc f(x)\cot f(x)$

$\frac{d}{dx}\cot f(x)=-f'(x)\csc^2 f(x)$

$\frac{d}{dx}e^{f(x)}=f'(x)e^{f(x)}$

$\frac{d}{dx}\ln f(x)=\frac{f'(x)}{f(x)}$

$\frac{d}{dx}\sin^{-1}f(x)=\frac{f'(x)}{\sqrt{1-f(x)^2}}$

$\frac{d}{dx}\cos^{-1}f(x)=-\frac{f'(x)}{\sqrt{1-f(x)^2}}$

$\frac{d}{dx}\tan^{-1}f(x)=\frac{f'(x)}{1+f(x)^2}$

$\frac{d}{dx}\cot^{-1}f(x)=-\frac{f'(x)}{1+f(x)^2}$

$\frac{d}{dx}\sec^{-1}f(x)=\frac{f'(x)}{|f(x)|\sqrt{f(x)^2-1}},|f(x)|>1$

$\frac{d}{dx}\sec^{-1}f(x)=-\frac{f'(x)}{|f(x)|\sqrt{f(x)^2-1}},|f(x)|>1$

$\frac{d}{dx}(uv)=\frac{du}{dx}v+u\frac{dv}{dx}$

$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{\frac{du}{dx}v-u\frac{dv}{dx}}{v^2}$

$\frac{d}{dx}\left(f\left(g(x)\right)\right)=f'\left(g(x)\right)\cdot g'(x)$

Implicit Differentiation: $\frac{dy}{dx}=-\frac{f_x}{f_y}$

Inverse Derivative:

$(f^{-1})'(a)=\frac{1}{f'(f^{-1}(a))}=\frac{1}{f'(b)}$

Parametric:

$\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, $\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy}{dx}\right)\div\frac{dx}{dt}$

Misc:

$\frac{d}{dx}f(x)^{g(x)}=f(x)^{g(x)}\left(g'(x)\ln f(x)+\frac{f'(x)}{f(x)}g(x)\right)$

$\frac{d}{dx}\log_a x=\frac{d}{dx}\frac{\ln x}{\ln a}$

3 Applications of Differentiation

$f'(x)>0$ if f is **increasing** on $[a,b]$

$f'(x)<0$ if f is **decreasing** on $[a,b]$

$f''(x)>0$ if f is **concave upwards** on $[a,b]$

$f''(x)<0$ if f is **concave downwards** on $[a,b]$

Point of Inflection: where concavity changes

Point of inflection at $c\to f''(c)=0$

Absolute max/min: max/min for all x in domain

Local max/min: max/min for all x in interval

Critical point if it is not an endpoint and $f'(c)=0$ or $f'(c)$ does not exist

Hospital Rule: if $\lim_{x\to c}f(x)=\lim_{x\to c}g(x)=0$ *or* ∞

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\lim_{x\to c}\frac{f'(x)}{g'(x)}$$

Rolle's Theorem: If f is continuous on $[a,b]$, then if $f(a)=f(b)$, there is at least one number $c\in[a,b]$, such that $f'(c)=0$

✗ **Mean Value Theorem:** : If f is continuous on $[a,b]$, there is at least one number $c\in[a,b]$, such that $f'(c)=\frac{f(b)-f(a)}{b-a}$ (gradient a to b)

4 Integrals

Factors of $Q(x)$	Partial fractions
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^2$	$\frac{A}{ax+b}+\frac{B}{(ax+b)^2}$
$ax^2+bx+c,\ b^2-4ac<0$	$\frac{Ax+B}{ax^2+bx+c}$

$\frac{P(x)}{Q(x)}$

1. $\sec^2 x-1=\tan^2 x$
2. $\csc^2 x-1=\cot^2 x$
3. $\sin A\cos A=\frac{1}{2}\sin 2A$
4. $\cos^2 A=\frac{1}{2}(1+\cos 2A)$
5. $\sin^2 A=\frac{1}{2}(1-\cos 2A)$
6. $\sin A\cos B=\frac{1}{2}(\sin(A+B)+\sin(A-B))$
7. $\cos A\sin B=\frac{1}{2}(\sin(A+B)-\sin(A-B))$
8. $\cos A\cos B=\frac{1}{2}(\cos(A+B)+\cos(A-B))$
9. $\sin A\sin B=-\frac{1}{2}(\cos(A+B)-\cos(A-B))$

$\sin^2 x+\cos^2 x=1$

$\boxed{\text{Trigo substitute for Integrals}}$

① $\sqrt{a^2-(x+b)^2}\to x+b=a\sin\theta$

② $\sqrt{a^2+(x+b)^2}\to x+b=a\tan\theta$

③ $\sqrt{(x+b)^2-a^2}$

$\hookrightarrow x+b=a\sec\theta$

Integration by Substitution:

$\int f\left(g(x)\right)g'(x)dx=\int f(u)du\,,u=g(x)$

Integration by Parts

$\boxed{\int u.\frac{dv}{dx}=uv-\int v.\frac{du}{dx}}$

Logarithmic Function	$\ln(ax+b)$ or its higher powers	differentiate it
Inverse Trigonometric Functions	$\sin^{-1}(ax+b)$, $\cos^{-1}(ax+b)$, $\tan^{-1}(ax+b)$	differentiate it
Algebraic Functions	Power functions x^a, polynomials	differentiate it
Trigonometric Functions	$\sin(ax+b)$, $\cos(ax+b)$, $\tan(ax+b)$, $\csc(ax+b)$, $\sec(ax+b)$, $\cot(ax+b)$, or a combinations of these	differentiate it integrate it
Exponential Functions	e^{ax+b}	integrate it

Notice: MUST be linear

1. $\int(ax+b)^n\,dx=\frac{(ax+b)^{n+1}}{(n+1)a}+C\quad(n\neq-1)$
2. $\int\frac{1}{ax+b}\,dx=\frac{1}{a}\ln|ax+b|+C$
3. $\int e^{ax+b}\,dx=\frac{1}{a}e^{ax+b}+C$
4. $\int\sin(ax+b)\,dx=-\frac{1}{a}\cos(ax+b)+C$
5. $\int\cos(ax+b)\,dx=\frac{1}{a}\sin(ax+b)+C$
6. $\int\tan(ax+b)\,dx=\frac{1}{a}\ln|\sec(ax+b)|+C$
7. $\int\sec(ax+b)\,dx=\frac{1}{a}\ln|\sec(ax+b)+\tan(ax+b)|+C$
8. $\int\csc(ax+b)\,dx=-\frac{1}{a}\ln|\csc(ax+b)+\cot(ax+b)|+C$
9. $\int\cot(ax+b)\,dx=-\frac{1}{a}\ln|\csc(ax+b)|+C$
10. $\int\sec^2(ax+b)\,dx=\frac{1}{a}\tan(ax+b)+C$
11. $\int\csc^2(ax+b)\,dx=-\frac{1}{a}\cot(ax+b)+C$
12. $\int\sec(ax+b)\tan(ax+b)\,dx=\frac{1}{a}\sec(ax+b)+C$
13. $\int\csc(ax+b)\cot(ax+b)\,dx=-\frac{1}{a}\csc(ax+b)+C$
14. $\int\frac{1}{a^2+(x+b)^2}\,dx=\frac{1}{a}\tan^{-1}\left(\frac{x+b}{a}\right)+C$
15. $\int\frac{1}{\sqrt{a^2-(x+b)^2}}\,dx=\sin^{-1}\left(\frac{x+b}{a}\right)+C$
16. $\int\frac{-1}{\sqrt{a^2-(x+b)^2}}\,dx=\cos^{-1}\left(\frac{x+b}{a}\right)+C$
17. $\int\frac{1}{a^2-(x+b)^2}\,dx=\frac{1}{2a}\ln\left|\frac{x+b+a}{x+b-a}\right|+C$
18. $\int\frac{1}{(x+b)^2-a^2}\,dx=\frac{1}{2a}\ln\left|\frac{x+b-a}{x+b+a}\right|+C$
19. $\int\frac{1}{\sqrt{(x+b)^2+a^2}}\,dx=\ln\left|(x+b)+\sqrt{(x+b)^2+a^2}\right|+C$
20. $\int\frac{1}{\sqrt{(x+b)^2-a^2}}\,dx=\ln\left|(x+b)+\sqrt{(x+b)^2-a^2}\right|+C$
21. $\int\sqrt{a^2-x^2}\,dx=\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}+C$
22. $\int\sqrt{x^2-a^2}\,dx=\frac{x}{2}\sqrt{x^2-a^2}-\frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right|+C$

Riemann Sums: $\int_a^b f(x)dx=\sum_{k=1}^n\left(\frac{b-a}{n}\right)f\left(a+k\frac{b-a}{n}\right)$

FTC 1: $\int_a^b f(x)dx=F(b)-F(a)$

FTC 2: $\frac{d}{dx}\int_a^x f(t)dt=f(x)$

$\frac{d}{dx}\int_a^{u(x)}f(t)dt=f(u(x))u'(x)$, u is differentiable

Improper Integrals:

Type I: upper/lower bound tends to infinity

Type II:

1. *If $f(x)$ is continuous on $(a,b]$ and is discontinuous at a, then*

$$\int_a^b f(x)\,dx=\lim_{c\to a^+}\int_c^b f(x)\,dx.$$

2. *If $f(x)$ is continuous on $[a,b)$ and is discontinuous at b, then*

$$\int_a^b f(x)\,dx=\lim_{c\to b^-}\int_a^c f(x)\,dx.$$

3. *If $f(x)$ is discontinuous at c with $a<c<b$, then*

$$\int_a^b f(x)\,dx=\int_a^c f(x)\,dx+\int_c^b f(x)\,dx.$$

V between 2 curves then $\pi\int f(x)^2 - g(x)^2$

5 Applications of Integration

Area between 2 curves: $\int_a^b |f(x)-g(x)|dx$

Volume of Solid (Disk): $\pi\int_a^b f(x)^2dx$

In this, we are rotating about y-axis. opp for x-axis

Volume of Solid (Shell): $2\pi\int_a^b x|f(x)|\,dx$

Shell Method used when area is bounded in x/y direc^n BUT rotated in y/x direct.

Arc Length: $\int_a^b \sqrt{1+f'(x)^2}\,dx$

6 Sequences & Series

AP: $a+(n-1)d$

AP Sum: $\frac{n}{2}(2a+(n-1)d)$

GP: ar^{n-1}

GP Sum: $\frac{a(r^n-1)}{r-1}$

GP Sum to Infinity: $\frac{a}{1-r}$

only if $|r|<1$ (convergent)

Else,
1. $a>0, r\geq 1 \to \infty$
2. $a<0, r\geq 1 \to -\infty$
3. $r\leq 1 \to$ DNE

Limit laws and **sandwich theorem** apply for finding limits of sequences & series

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are **convergent**:

1. $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$ is convergent
2. $\sum_{n=1}^{\infty}(a_n+b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$ is convergent

→ convergent → limit is 0. Not converse

3. $\lim_{n\to\infty} a_n = 0$

contrapositive

If limit = 0 it could be convergent / divergent!

1. **n-th term Test**: if $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n \neq 0 \to \sum_{n=1}^{\infty} a_n$ is divergent

A series of non-negative terms converges iff there exists K s.t. $S_n < K, \forall n$ — Monotone Convergence Theorem

2. **Integral Test**: for $a_n = f(n)$, f is continuous, positive, decreasing f^n for $x \geq 1$

If $\int_1^{\infty} f(x)$ is convergent, $\sum_{n=1}^{\infty} a_n$ is convergent

If $\int_1^{\infty} f(x)$ is divergent, $\sum_{n=1}^{\infty} a_n$ is divergent

3. **p-series**: $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges iff $p>1$, $\sum_1^{\infty}\frac{1}{n}$ diverges

4. **Comparison Test**: $a_n \leq b_n, \forall n$

If $\sum_{n=1}^{\infty} b_n$ is convergent, $\sum_{n=1}^{\infty} a_n$ is convergent

If $\sum_{n=1}^{\infty} a_n$ is divergent, $\sum_{n=1}^{\infty} b_n$ is divergent

5. **Ratio/Root Test**: 6.

For $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ or $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L$:

$0\leq L<1$: $\sum_{n=1}^{\infty}|a_n|$ is cnvgt (absolute cnvgt)

$L>1$: $\sum_{n=1}^{\infty} a_n$ is divergent

NOTE: L can be finite or ∞.

$L=1$: test inconclusive

* Generally, root test is considered more powerful than ratio test.

Important Result

Alternating series:

$\sum_{n=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \cdots$

is convergent if b_n is decreasing and $\lim_{n\to\infty} b_n = 0$

Absolute Convergence:

$\sum_{n=1}^{\infty}|a_n|$ convergent $\to \sum_{n=1}^{\infty} a_n$ convergent

∞ series ∞ variable x centered at a

Power Series: $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$,

Radius of Convergence $R = \frac{1}{L}$ (L from ratio/root test) ↳ $L = \left|\frac{c_{n+1}}{c_n}\right|$

$R=0$: Series converges at $x=a$ ①

$R=\infty$: Series converges for all x ②

Else: converges absolutely if $|x-a|<R$, ③ Concept

diverges if $|x-a|>R$

we get $\frac{1}{R}|x-c|$ from ratio test, so introduce <1 for converging

If power series $R>0$, then:

→ $f(x)$ differentiable on $|x-a|<R$

→ $f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$ for $|x-a|<R$

→ $\int f(x)dx = \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1} + C$ for $|x-a|<R$

Taylor Series: $\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty}\frac{f^n(a)}{n!}(x-a)^n$

Maclaurin Series: $a=0$, $\sum_{n=0}^{\infty}\frac{f^n(0)}{n!}x^n$

7 Vectors

Unit vector of a, $u = \frac{a}{||a||}$

$a\cdot b = \|\|a\|\|\,\|\|b\|\| \cos\theta$	$a\times b = \|\|a\|\|\,\|\|b\|\| \sin\theta$
a, b are orthogonal if $a\cdot b = 0$	$a\times b$ is orthogonal to both a and b
$a\cdot b = b\cdot a$	$a\times b = -b\times a$
$a\cdot a = \|\|a\|\|^2$	$\|\|a\times a\|\| = 0$

Projection of b onto a: $\frac{a\cdot b}{||a||}$

Distance between skew lines:
→ Find vector from a point in l_1 to l_2, AP
→ unit vec $= \frac{d_1\times d_2}{|d_1\times d_2|}$
→ $AP\cdot(d_1\times d_2)$

Magnitude of Projection: $\left(\frac{a\cdot b}{||a||}\right)\frac{a}{||a||}$

Distance from point to line: $\frac{||a\times b||}{||a||}$ — a – direc^n vector of line, b – vector from point to some point on line

Distance from point to plane: $\frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$

(For point $P(x_0,y_0,z_0)$, to plane $ax+by+cz=d$)

$\langle a,b,c\rangle$ is normal vector

8 Functions of Several Variables

For $r(t) = \langle f(t), g(t), h(t)\rangle$, its derivative at $t=a$ is $r'(a) = \langle f'(a), g'(a), h'(a)\rangle$

Arc Length: $\int_a^b\sqrt{f'(t)^2+g'(t)^2+h'(t)^2}dt = \int_a^b||r'(t)||dt$

Partial Derivative: differentiate with respect to smth and treat the rest of the independent vars as constants

Can look like f_x or $\frac{\partial y}{\partial x}$

$a\times b = (a_2b_3 - a_3b_2)\underset{\sim}{i} - (a_1b_3 - a_3b_1)\underset{\sim}{j} + (a_1b_2 - a_2b_1)\underset{\sim}{k}$

Clairaut's Theorem: $f_{xy}(a,b) = f_{yx}(a,b)$,

$f_{xyy} = f_{yxy} = f_{yyx}$ ↳ any permutation is same

Normal vector to tangent plane: Special case

$\langle f_x(a,b), f_y(a,b), -1\rangle$ Generalis^n of $\nabla F(x,y,z)$ for $z=f(x,y)$

Equation of tangent plane for $f(x,y)$:

$f_x(a,b)(x-a) + f_y(a,b)(y-b) - (z - f(a,b)) = 0$

Chain rule: If $x=g(t)$, $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$ or — tree method

If $x = g(s,t)$,

$\frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$, $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$

If $F(x,y,z)$: $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}$, $-\frac{F_y}{F_z}$ for $\frac{\partial z}{\partial y}$

note $F(x,y,z) = a$

Increments: $\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y)$

$dz = f_x(x,y)dx + f_y(x,y)dy$

If $\Delta x, y$ small, $\Delta z \approx dz = f_x(a,b)\Delta x + f_y(a,b)\Delta y$

Directional Derivatives:

In direction of unit vector $u = \langle a,b\rangle$ *

$D_uf(x,y) = f_x(x,y)a + f_y(x,y)b = \langle f_x, f_y\rangle\cdot u$

Gradient: $\nabla f(x,y) = \langle f_x, f_y\rangle$

$\nabla f(x,y)$ is the normal to the level curve

① Max gradient $= ||\nabla f(P)||$

$\nabla F(x,y,z)$ is the normal to the level surface

② Direc^n of max gradient $= \nabla f(P)$

Equation of tangent plane for $F(x,y,z)$:

$\nabla F(x_0,y_0,z_0)\cdot\langle x-x_0, y-y_0, z-z_0\rangle = 0$

Maximizing $D_uf(P) = ||\nabla f(P)||\cos\theta$:

when $\nabla f(P)$ points in the direction of u

Max $D_uf(P) = ||\nabla f(P)||$, Min $D_uf(P) = -||\nabla f(P)||$

Local extrema → $f_x(a,b) = f_y(a,b) = 0$ (1 way)

$f_x(a,b) = f_y(a,b) = 0$ or one of the partial derivatives does not exist → Critical Point

Local Extrema → Critical Point (1 way)

Second Derivative Test: to find max/min/saddle/none

$D = f_{xx}(a,b)\times f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$D>0, f_{xx}(a,b)>0$, then $f(a,b)$ is local minimum

$D>0, f_{xx}(a,b)<0$, then $f(a,b)$ is local maximum

$D<0$, then $f(a,b)$ is a saddle point

$D=0$, no conclusion

9 Double Integrals

Volume of solid above rectangle $R_{[a,b]\times[c,d]}$:

(Fubini's Theorem)

$\int\int_R f(x,y)dA = \int_a^b\int_c^d f(x,y)dydx = \int_c^d\int_a^b f(x,y)dx\,dy$

$\int\int_R g(x)h(y)dA = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$

Volume of solid above area bound by 2 curves:

Type I: $\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$ (bounded in x-dir) or Type II: $\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy$ (bounded in y-dir)

$x = r\cos\theta$, $y = r\sin\theta$

Volume of solid above polar function:

$\int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$

Surface Area over area:

$\int\int_D\sqrt{f_x^2+f_y^2+1}dA$

$0\leq a\leq r\leq b$, $\alpha\leq\theta\leq\beta$

10 ODE

Separable Form:

$\frac{dy}{dx} = f(x)g(y)$

$\int\frac{1}{g(y)}dy = \int f(x)dx + c$

Reduction to Separable:

Form 1: $\frac{dy}{dx} = g\left(\frac{y}{x}\right)$, let $v = \frac{y}{x}$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then $g\left(\frac{y}{x}\right) = v + x\frac{dv}{dx}$

$\frac{1}{g(v)-v}dv = \frac{1}{x}dx$

Form 2: $\frac{dy}{dx} = f(ax+by+c)$,

let $v = ax+by+c$

Integrating Factor: * coefficient of $\frac{dy}{dx}$ must be 1

$\frac{dy}{dx} + P(x)y = Q(x)$, $I(x) = e^{\int P(x)dx}$

$\frac{d}{dx}(y\cdot I(x)) = Q(x)\cdot I(x)$ (Product Rule, FTC)

$y\cdot I(x) = \int Q(x)\cdot I(x)dx$

Bernoulli Equation:

$y' + p(x)y = q(x)y^n$, let $u = y^{1-n}$

$u' + (1-n)p(x)u = (1-n)q(x)$

(now use IF)

ODE techniques

	method
① Linear $y' + P(x)y = Q(x)$	Integrating factor
② Separable $y' = f(x)g(y)$	separate & integrate
③ Make separable using sub^n	$v = \frac{y}{x}$ or $v = ax+by+c$
④ Bernoulli $y' + P(x)y = Q(x)y^n$	Transform to linear & integrate

Small Angle

$\sin x \approx x - \frac{x^3}{6} + \ldots$

$\cos x \approx 1 - \frac{x^2}{2} + \ldots$

$\tan x \approx x + \frac{x^3}{3} + \ldots$