Value at Risk

For a given portfolio, *Value-at-Risk* (VAR) is defined as the number VAR such that:

$$\Pr(\text{Portfolio loses more than VAR within time period } t) < \alpha.$$

all this given:

- amount of time *t,* and
- probability level α (confidence level)
- all this under *normal* market conditions!

Example:

Probability ($1 million in S&P 500 Index will decline by more than 20% within a

year) < 10%

means that VAR = $200,000 (20% of $1,000,000) with $\alpha = 0.10$, $t = 1$ year (typically time period is much shorter, expressed in days). VAR is typically a dollar amount, not %.

Value at Risk is *only* about Market Risk under normal market conditions. VAR is important because it is used to allocate capital to market risk for banks, under their Risk Based Capital requirements. More precisely:

• The 1988 Bank for International Settlements (BIS) Accord defines how capital held for credit risk is calculated.

• The 1996 Amendment distinguishes the following:

- *Trading book:* loans not revalued on a regular basis.
- *Banking book:* different instruments (stocks, bonds, swaps, forwards, options, etc.) that *are* usually revalued daily.

• Capital for trading book calculated using VaR with $N = 10$ (trading days), and $\alpha = 0.01$, but usual notation used is $X = 1 - \alpha = 0.99$. Resulting VaR is multiplied by a coefficient *k,* where *k* varies by bank, but is at least 3.

Conditional VaR (C-VaR) is defined as the expected loss during an *N*-day period, conditional that we are in the (100 - X)% left tail of the distribution. This concept overcomes problems with distributions with two peaks, one of which is in the left tail. But it is much less popular than VaR (after all, regulators require VaR).

A short note from K.O.:

Basic VAR methodologies:

- Parametric;
- Historical;
- Simulation.

How is parametric done?

- Estimate historical parameters: asset returns, variances and covariances, for all asset classes, or assets comprising the portfolio;
- Calculate portfolio expected return and standard deviation;

- Estimate VAR assuming normal distribution of portfolio return. Typically assumes normality and serial independence. Wrong theoretically, but practitioners do not care. Problems with estimating parameters, especially volatility.

How is historical done?
- Assemble and maintain historical database;
- Use historical data as the future distribution.

Also wrong: What if the future isn't what it used to be? But … if generalized to the nonparametric method of bootstrap (resampling), may be the best there is. Of course, bootstrap is not used in practice, because practitioners generally do not know what it is.

How is simulation done?
- Specify distributions of model input factors,
- Use Monte Carlo simulation for factors,
- Combine them into global outcome, get a probability distribution.
- Assumptions on factors crucial. If one can get that distribution ideally, this may be an ideal method.

End of note.

Volatility per year versus volatility per day

$$\sigma_{yr} = \sigma_{day}\sqrt{252},$$

$$\sigma_{day} = \frac{\sigma_{yr}}{\sqrt{252}}.$$

There are 252 trading days, and studies indicate that volatility on non-trading days is minimal if nonexistent.

Consider \$10 million in IBM stock, N = 10 days (two trading weeks), and X = 99% confidence level. Assume daily volatility of 2%, i.e., daily standard deviation (SD) of \$200,000. Assume successive days' returns are independent, then over 10 days SD is

$$\sqrt{10}\cdot \$200{,}000 \approx \$632{,}456.$$

It is customary to assume in VAR calculations that the expected return over period considered is 0% (because in practice calculations are done over very short periods). It is also customary to assume normal distribution of returns. Because the 1st percentile of the standard normal distribution is -2.33 (and the 99$^{\text{th}}$ percentile is 2.33), VAR of this IBM stock portfolio is:

$$2.33\cdot \$632{,}456 = \$1{,}473{,}621.$$

Now consider a \$5 million in AT&T stock (symbol T). Assume its daily volatility is 1%. Then its 10 day SD is $\$50{,}000\sqrt{10}$. Its VAR is

$$2.33\cdot \$50{,}000\sqrt{10} \approx \$368{,}405.$$

Now combine the two assets in a portfolio and assume that the correlation of their returns is 0.7. We have

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y},$$

and we get $\sigma_{X+Y} = \$751,665.$

Thus 10-day 99% VAR of the combined IBM + T portfolio is

$$2.33 \cdot \$751,665 = \$1,751,379.$$

The amount

$$(\$1,473,621 + \$368,405) - \$1,751,379 = \$90,647.$$

is the VAR benefit of diversification.

A linear model

Consider a portfolio of assets such that the changes in the values of those assets have a multivariate joint normal distribution. Let Δx_i be the change in value of asset *i* in one day, and α_i be the allocation to asset *i,* then for the change in value of the portfolio

$$\Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i$$

is normally distributed because of the multivariate joint normal distribution. Since $E(\Delta x_i) = 0$ is assumed for every *i*, $E(\Delta P) = 0.$ We also have:

$$\sigma_i = \sqrt{Var(\Delta x_i)}, \rho_{ij} = \mathrm{Corr}(\Delta x_i, \Delta x_j)$$

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \rho_{ij}\alpha_i\alpha_j\sigma_i\sigma_j$$

and the 99% VAR for *N* days is: 2.33 $\sigma_P\sqrt{N}$.

How bonds/interest rates are handled

Duration gives

$$\Delta P = -D \cdot P \cdot \Delta y.$$

Let σ_y be the yield volatility per day. What is it? One way to look at it: SD of $\Delta y.$ Then

$$\sigma_P = D \cdot P \cdot \sigma_y.$$

Another way of looking at it: SD of $\frac{\Delta y}{y},$ where *y* is the zero coupon bond yield for maturity *D*. Then

$$\Delta P = -D \cdot P \cdot y \cdot \frac{\Delta y}{y},$$

so that

$$\sigma_P = D \cdot P \cdot y \cdot \sigma_y.$$

Can include convexity in this approach (in addition to duration), but this still does not account for nonparallel yield curve shifts.

Cash flow mapping
Alternative approach in handling interest rates, dealing with the problem of not having data on volatility of most bonds, as most bonds are rarely traded. But data on certain Treasuries of standard maturities is available due to their frequent trading, especially on-the-run Treasuries. In this approach, we use prices of zeros with standard maturities (1/12 year, 0.25 year, 1, 2, 5, 7, 10, 30 year) as market variables. Note that any Treasury can be stripped into a packet of zeros. The mapping procedure is illustrated by this example:

Consider a \$1 million Treasury maturing in 0.8 years, with 10% semi-annual coupon. It can be viewed as a 0.3 year \$50K zero plus 0.8 year \$1050K zero. Suppose that the rates and zero prices are as follows:

	3 mos.	6 mos.	1 year
zero yield	5.50%	6.00%	7.00%
zero price volatillity (% per day)	0.06%	0.10%	0.20%

Assume the following daily returns correlations

	3 mos.	6 mos.	1 year
3 mos.	1.00	0.90	0.60
6 mos.	0.90	1.00	0.70
1 year	0.60	0.70	1.00

What is the rate for 0.80 years? We interpolate between 0.5 and 1.0 years and get yield of 6.60%. If you interpolate daily volatility, you get 0.16%. We now try to replicate volatility of the 0.8 year zero with 0.5 year zero and 1 year zero, by a position of α in the 6 mos. zero and 1 - α in the 1 year zero. Matching variances we get the following equation:

$$0.0016^2 = 0.001^2\alpha^2 + 0.002^2(1-\alpha)^2 + 2\cdot 0.7\cdot 0.001\cdot 0.002\cdot\alpha(1-\alpha)$$

This is a quadratic equation which gives α = 0.3203. The 0.8 year zero is worth

$$\frac{\$1{,}050\text{K}}{1.066^{0.8}} = \$997{,}662.$$

α is the portion of this amount, i.e., 0.3203($997,662) = $319,589 which is allocated to the six months zero, and the rest, i.e., 0.6797($997,662) = $678,073 is allocated to the one year zero.

Do the same calculations for the 0.3 year $50K zero. This is how that calculation goes.

The 0.25-year and 0.50 year rates are 5.50% and 6.00%, respectively. Linear interpolation gives the 0.30-year rate as 5.60%. The present value of $50,000 received at time 0.30 years is

$$\frac{\$50{,}000}{1.056^{0.3}} = \$49{,}189.32.$$

The volatilities of 0.25-year and 0.50-year zero-coupon bonds are 0.06% and 0.10%per day, respectively. Using linear interpolation we get the volatility of a 0.30-year zero-coupon bond as 0.068% per day. Assume that α is the value of the 0.30-year cash flow allocated to a 3-month zero-coupon bond, and 1 - α is allocated to a six-month zero-coupon bond. We match variances obtaining the equation:

$$0.00068^2 = 0.0006^2\alpha^2 + 0.001^2(1-\alpha)^2 + 2\cdot 0.9\cdot 0.0006\cdot 0.001\cdot\alpha(1-\alpha)$$

which simplifies to

$$0.28\alpha^2 - 0.92\alpha + 0.5376 = 0.$$

This is a quadratic equation, and its solution is:

$$\alpha = \frac{-0.92 + \sqrt{0.92^2 - 4\cdot 0.28\cdot 0.5376}}{2\cdot 0.28} = 0.760259.$$

This means that a value of

$$0.760259\cdot\$49{,}189.32 = \$37{,}397$$

is allocated to the three-month bond and a value of

$$0.239741\cdot\$49{,}189.32 = \$11{,}793$$

is allocated to the six-month bond.

This way the entire bond is “mapped” into positions in standard maturity zero Treasury bonds. Since we are given the volatilities and correlations of those bonds, and since we are assuming zero return in a short period of time, we just take the portfolio as mapped, and calculate its standard deviation.

The portfolio consists of $678,073 in 1-year bond, $11,793 + $319,589 = $331,382 in 0.5-year bond, $37,397 in 0.25-year bond.

The 10 day 99% VAR of the bond is then 2.33 times $\sqrt{10}$ times the SD calculated.

When the linear model can be used
The linear model starts with the equation

$$\Delta P = \sum_{i=1}^{n}\alpha_i\Delta x_i$$

which means that the change in the value of the portfolio is a linear function of the changes in the values of the underlying. This is not the case for many derivatives, especially options. Is there a case when derivatives can be handled with the linear model? Here are some examples.

Assets denominated in foreign currency can be accommodated, by measuring them in U.S dollars. Forward contract on a foreign currency can be regarded as an exchange of a foreign zero coupon bond maturing at contract maturity for a domestic zero maturing at the same time.

Interest rate swap: it can be viewed as the exchange of a floating rate bond for a fixed rate bond. The floater can be regarded as a zero with maturity equal the next reset date. Thus this is a bond portfolio and can be handled by the linear model.

When the portfolio contains options, the linear model can be used as an approximation. Consider a portfolio of options on a single stock with price *S*. Suppose the delta of the portfolio is δ, so that: $\delta \approx \frac{\Delta P}{\Delta S}$. Define $\Delta x = \frac{\Delta S}{S}$. Then $\Delta P \approx S\delta\Delta x$. If the portfolio consists of many such instruments, we get $\Delta P \approx \sum_{i=1}^{n} S_i \delta_i \Delta x_i$, which is essentially the linear model.

Example. A portfolio consists of options on IBM, with delta of 1,000 and options on T, with delta of 20,000. You are given IBM share price of \$120 and T share price of \$30. Then

$$\Delta P = 120 \cdot 1000 \cdot \Delta x_1 + 30 \cdot 20{,}000 \cdot \Delta x_2 = 120{,}000 \Delta x_1 + 600{,}000 \Delta x_2.$$

If daily volatility of IBM is 2% and daily volatility of T is 1%, with correlation 0.70, the standard deviation of ΔP is

$$\sqrt{(120 \cdot 0.02)^2 + (600 \cdot 0.01)^2 + 2 \cdot 120 \cdot 0.02 \cdot 600 \cdot 0.01 \cdot 0.7} = 7{,}869.$$

The 5th percentile of standard normal distribution is -1.65, and so the 5 day 95% VAR is

$$1.65 \cdot \sqrt{5} \cdot 7{,}869 = \$29{,}033.$$

A quadratic model

When a portfolio includes options, its gamma (γ) should be included in the analysis. We have

$$\Delta P = \delta \Delta S + \frac{1}{2}\gamma (\Delta S)^2,$$

and with $\Delta x = \frac{\Delta S}{S}$, we can write this as

$$\Delta P = S\delta \Delta x + \frac{1}{2} S^2 \gamma (\Delta x)^2.$$

The problem is that ΔP is not normally distributed, although we assume Δx is, $\Delta x \sim N\left(0,\sigma^2\right)$. The moments of ΔP are

$$E(\Delta P)=\frac{1}{2}S^2\gamma\sigma^2,$$

$$E\left((\Delta P)^2\right)=S^2\delta^2\sigma^2+\frac{3}{4}S^4\gamma^2\sigma^4,$$

and

$$E\left((\Delta P)^3\right)=\frac{9}{2}S^4\delta^2\gamma\sigma^4+\frac{15}{8}S^6\gamma^3\sigma^6.$$

We can pretend that ΔP is normal and fit a normal distribution to the first two moments. The alternative is to use Cornish-Fischer expansion .

For a portfolio in which each instrument depends on one market variable, we get

$$\Delta P=\sum_{i=1}^{n}S_i\delta_i\Delta x_i+\sum_{i=1}^{n}\frac{1}{2}S_i^2\gamma_i\left(\Delta x_i\right)^2.$$

If pieces of the portfolio can depend on more than one variable then we get a more complicated picture:

$$\Delta P=\sum_{i=1}^{n}S_i\delta_i\Delta x_i+\sum_{i=1}^{n}\sum_{i=1}^{n}\frac{1}{2}S_iS_j\gamma_{ij}\Delta x_i\Delta x_j,$$

where $\gamma_{ij}=\dfrac{\partial^2 P}{\partial S_i S_j}$. This can be used to estimate moments of ΔP.

Monte Carlo Simulation

One day VAR calculation

1. Value the portfolio today in the usual way using the current values of market variables.
2. Sample once from the multivariate normal probability distribution of the Δx_i's.
3. Use the values of the Δx_i's that are sampled to determine the value of each market variable at the end of one day.
4. Revalue the portfolio at the end of the day in the usual way.
5. Subtract the value calculated in step one from the value in step four to determine a sample ΔP.
6. Repeat steps two to five many times to build up a probability distribution for ΔP.

VAR is then calculated as the appropriate percentile of the probability distribution so obtained.

An alternative approach, lowering the number of calculations is to assume that

$$\Delta P=\sum_{i=1}^{n}S_i\delta_i\Delta x_i+\sum_{i=1}^{n}\sum_{i=1}^{n}\frac{1}{2}S_iS_j\gamma_{ij}\Delta x_i\Delta x_j,$$

and skip steps 3, 4 above. This is called a partial simulation.

Historical Simulation
Create a database of daily movements of all market variables for several years. Use the database as the probability distribution (the book says first day in database is first day in your simulation, and so on, that's not really true, you simulate from the empirical distribution given by the sample). ΔP is then calculated for each simulation trial, and the empirical distribution of ΔP has the percentiles determining VAR.

Strength: no artificial assumption of normal distribution
Weaknesses:

- Limited by data set available (that's not really true, you can do bootstrap/resampling, but the book says so, thus you must remember), both in length of time and data availability.
- Sensitivity analysis difficult.
- Can't use volatility updating schemes (volatility changes over time, but schemes have been designed to account for that).

Stress testing and back testing

- Stress testing: Estimating how the portfolio would have performed under the most extreme market conditions. For example: five standard deviation move in a market variable in a day. This is next to impossible under normal distribution assumption (happens once in 7000 years) but in reality it happens about once every 10 years (and some researchers say that one such move in some market happens every year).
- Back testing: checking how well our model did in predicting things in the past. For example: how often did a one day loss exceed 1-day 99% VAR. If this happens roughly one percent of the time, well then we are in business.

Principal component analysis
Dealing with correlated market variables – the most important example being interest rates at various maturities being correlated, but not perfectly correlated. The example in text deals with the yield curve and factors (variables) developed to model its changes. Variables are called PC1 (Principal Component 1) through PC10. Interest rate changes observed on a given day are expressed as a linear combination of the factors by solving a set of ten equations. Interest rate move for a particular factor is known as factor loading. Factor scores are the amounts of the factors in the rate movement. Importance of the factor is measured by SD of the factor score. Sum of squares of SD of factor scores is the total variance.

How does one use PC analysis to calculate VAR?
In this illustration, only two factors are assumed (PC1 and PC2). Suppose that we have a portfolio with the following exposures to the interest rates movements (change in portfolio value for a 1 bp rate move in $millions):

1 year rate: +10,
2 year rate: +4,

3 year rate: -8,
4 year rate: -7,
5 year rate: +2

The first factor PC1 has loadings for these Treasury rates:
1 year: 0.32,
2 year: 0.35,
3 year: 0.36,
4 year: 0.35,
5 year: 0.36

The second factor PC2 has loadings for these Treasury rates:
1 year: -0.32,
2 year: -0.10,
3 year: 0.02,
4 year: 0.14,
5 year: 0.17

PC1 measures parallel shifts in the curve, PC2 measures twist of the yield curve (steepening or becoming flatter).

Exposure to PC1 is:

$$10 \cdot 0.32 + 4 \cdot 0.35 - 8 \cdot 0.36 - 7 \cdot 0.36 + 2 \cdot 0.36 = -0.08,$$

Exposure to PC2 is:

$$10 \cdot (-0.32) + 4 \cdot (-0.10) - 8 \cdot 0.02 - 7 \cdot 0.14 + 2 \cdot 0.17 = -4.40.$$

If f_1, f_2 are factor scores (in bps) then the change in the portfolio value is approximated by

$$\Delta P \approx -0.08 f_1 - 4.40 f_2.$$

The factor scores are assumed uncorrelated and SD's of factors are given as 17.49 for PC1 and 6.05 for PC2. We get the SD of ΔP as

$$\sqrt{0.08^2 \cdot 17.49^2 + 4.40^2 \cdot 6.05^2} = 26.66.$$

The 1 day 99% VAR is calculated as

$$26.66 \cdot 2.33 = 62.12.$$

This example was chosen intentionally to have relatively low exposure to parallel shifts (PC1) versus twists (PC2) so that using standard duration analysis would result in significant error.

Key ideas in PC analysis is to replace dependent variables driving returns (such as Treasury rates) by uncorrelated principal component factors.

Exercise
Consider a position consisting of \$1,000,000 investment in asset *X* and \$1,000,000 investment in asset *Y*. Assume that the daily volatilities of both assets are 0.1% and that the correlation coefficient between their returns is 0.30. What is the 5-day 95% Value at

Risk for this portfolio, assuming a parametric model with zero expected return? The 95th percentile of the standard normal distribution is 1.645.

Solution.
The standard deviation of the daily dollar change in the value of each asset is \$1,000. The variance of the portfolio's daily change is:

$$1000^2 + 1000^2 + 2 \cdot 0.3 \cdot 1000 \cdot 1000 = 2,600,000.$$

The standard deviation of the portfolio's daily change in value is the square root of 2,600,000, i.e., \$1,612.45. The standard deviation of the five-day change in the portfolio value is:

$$\$1,612.45 \cdot \sqrt{5} = \$3,605.55.$$

The 95th percentile of the standard normal distribution is 1.645. Therefore (assuming zero mean), the five-day 95% Value at Risk is:

$$1.645 \cdot \$3,605.55 = \$5,931.$$

Exercise
A pension plan has a position in bonds worth \$4 million. The effective duration of the portfolio is 3.70 years. Assume that the yield curve changes only in parallel shifts and that the volatility of the yield (standard deviation of the daily shift size) is 0.09%. Use the duration model for estimating volatility of the portfolio and estimate the 20-day Value at Risk for the portfolio. The 90th percentile of the standard normal distribution is 1.282, assume the parametric VAR model.

Solution.
The duration model says $\Delta B = -D \cdot B \cdot \Delta y,$ where B is the bond portfolio value, D is the effective duration, and y is the yield. We know that D = 3.70, and that the standard deviation of Δy is 0.09%. Thus the standard deviation of the return of the portfolio $\frac{\Delta B}{B} = -D \cdot \Delta y$ is (0.09%)(3.70) = 0.3332%. The portfolio value is \$4 million. The standard deviation of its daily change in value is \$4,000,000(0.3332%) = \$13,320. The 90th percentile of the standard normal distribution is 1.282, and thus our estimate of the 20-day 90% Value at Risk is: $\$13,320 \cdot \sqrt{20} \cdot 1.282 = \$76,367.$

Exercise
A bank owns a portfolio of options on the U.S. dollar – Pound Sterling exchange rate. The delta of the portfolio is given as 56.00. Current exchange rate is \$1.50 per Pound Sterling. You are given that the daily volatility of the exchange rate is 0.70%. What is the approximate linear relationship between the change in the portfolio value and the proportional change in the exchange rate? Estimate the 10-day 99% Value at Risk.

Solution.
Given the value of delta, the approximate relationship between the daily change in the portfolio value, ΔP, and the daily change in the exchange rate, $\Delta S,$ is $\Delta P = 56 \Delta S.$ Let

Δx be the proportional daily change in the exchange rate. The $\Delta x = \frac{\Delta S}{1.5}$. Therefore $\Delta P = 56 \cdot 1.5 \Delta x = 84 \Delta x$. The standard deviation of Δx equals the daily volatility of the exchange rate, i.e., 0.70%. The standard deviation of ΔP therefore is 84(0.70%)=0.588. The 10-day 99% Value at Risk is thus estimated as: $0.588 \cdot 2.33 \cdot \sqrt{10} = 4.33$.

Exercise
We know that option portfolios are not easily represented by a linear model. In fact, the portfolio gamma for the previous problem is 16.2. How does this change the estimate of the relationship between the change in the portfolio value and the proportional change in the exchange rate? Calculate an update of the 10-day 99% Value at Risk based on estimate of the first two moments of the change in the portfolio value.

Solution.
Based on the Taylor series expansion

$$\Delta P = 56 \cdot 1.5 \cdot \Delta x + \frac{1}{2} \cdot 1.5^2 \cdot 16.2 \cdot (\Delta x)^2 .$$

This simplifies to

$$\Delta P = 84 \Delta x + 18.225 (\Delta x)^2 .$$

The first two moments of ΔP are

$$E(\Delta P) = E\left(\frac{1}{2} \cdot 1.5^2 \cdot 16.2 \cdot (\Delta x)^2 \right) = \frac{1}{2} \cdot 1.5^2 \cdot 16.2 \cdot 0.007^2 = 0.000893$$

and

$$E\left((\Delta P)^2 \right) = 1.5^2 \cdot 56^2 \cdot 0.007^2 + \frac{3}{4} \cdot 1.5^4 \cdot 16.2^2 \cdot 0.007^4 = 0.346.$$

The standard deviation of ΔP is

$$\sqrt{0.346 - 0.000893^2} = 0.588.$$

We use the mean and standard deviation so calculated and pretend that ΔP has normal distribution, fitting a normal distribution with the same mean and variance. The ten-day 99% Value at Risk is calculated as:

$$\sqrt{10} \cdot 2.33 \cdot 0.588 - 10 \cdot 0.000893 = 4.3235.$$

Exercise
Assume that the daily change in the value of a portfolio is well approximated by a linear combination of two factors calculated from a principal components analysis. The delta of the portfolio with respect to the first factor is 6 and the delta of the portfolio with respect to the second factor is - 4. The standard deviations of the two factors are 20 and 8, respectively. What is the 5-day 90% Value at Risk?

Solution.
The factors used in a principal components analysis are assumed to be uncorrelated. Therefore, the daily variance of the portfolio is:

$$6^2 \cdot 20^2 + (-4)^2 \cdot 8^2 = 15,424.$$

The daily standard deviation is the square root of that, i.e., \$124.19. Since the 90th percentile of the standard normal distribution is 1.282, the 5-day 90% value at risk is estimated as:

$$124.19 \cdot \sqrt{5} \cdot 1.282 = \$356.01.$$

Exercise

Suppose a company has a portfolio consisting of positions in stocks, bonds, foreign exchange and commodities. Assume that there are no derivatives in the portfolio. Explain the assumptions underlying

(a) the linear model, and;
(b) the historical simulation model,

for calculating Value at Risk.

Solution.

The linear model:

- It assumes that the percentage daily change in each market variable has a normal probability distribution.

The historical simulation model:

- It assumes that the probability distribution observed for the percentage daily changes in the market variables in the past is the probability distribution that will rule over the next day (or whatever period is under consideration).

Exercise

Explain how an interest rate swap is mapped into a portfolio of zero-coupon bonds with standard maturities for the purposes of Value at Risk calculations.

Solution.

When a final exchange of principal is added in, the floating side of a swap is equivalent to a zero coupon bond with a maturity date equal to the date of the next payment. The fixed side is a regular coupon-bearing bond, which is equivalent to a portfolio of zero-coupon bonds. The swap can therefore be mapped into a portfolio of zero-coupon bonds with maturity dates corresponding to the payment dates. Each of the zero-coupon bonds can then be mapped into positions in the adjacent standard-maturity zero-coupon bonds.

Exercise

Explain why the linear model can provide only approximate estimates of Value at Risk for a portfolio containing options.

Solution.

The change in the value of an option is not linearly related to the change in the value of the underlying variables. When the change in the values of underlying variables is normal, the change in the value of the option is non-normal. The linear model assumes that the distribution considered is normal. Using it for options produces only an approximation, and can produce potentially significant error.

Exercise
Suppose that the 5-year rate is 6%, the seven year rate is 7%, both expressed with annual compounding. Also assume that the daily volatility of a 5-year zero-coupon bond is 0.5%, and the daily volatility of a 7-year zero-coupon bond is 0.58%. The correlation coefficient between daily returns of the two bonds is 0.60. Map a cash flow of $1,000 received at time 6.5 years into a position in a five-year bond and a position in a seven-year bond.

Solution.
The 6.5-year cash flow is mapped into a 5-year zero-coupon bond and a 7-year zero-coupon bond. The 5-year and 7-year rates are 6% and 7%, respectively. Using linear interpolation we get the 6.5-year rate as 6.75%. The present value of $1,000 received in 6.5 years is

$$\frac{\$1{,}000}{1.0675^{6.5}} = 654.05.$$

The volatility of 5-year and 7-year zero-coupon bonds are 0.50% and 0.58% per day, respectively. We interpolate the volatility of a 6.5-year zero-coupon bond as 0.56% per day. Assume that the fraction α is allocated to a 5-year zero-coupon bond and 1 - α is allocated to a 7-year zero-coupon bond. To match variances we solve the equation:

$$0.56^2 = 0.50^2\alpha^2 + 0.58^2(1-\alpha)^2 + 2\cdot 0.6\cdot 0.5\cdot 0.58\alpha(1-\alpha)$$

which simplifies to:

$$0.2384\alpha^2 - 0.3248\alpha + 0.0228 = 0.$$

This is a quadratic equation with solution:

$$\alpha = \frac{0.3248 - \sqrt{0.3248^2 - 4\cdot 0.2384\cdot 0.0228}}{2\cdot 0.2384} = 0.0742443.$$

This means that amount of

$$0.0742443\cdot \$654.05 = \$48.56$$

is allocated to the 5-year bond and amount of

$$0.925757\cdot \$654.05 = \$605.49$$

is allocated to the 7-year bond. Note that the equivalent 5-year and 7-year cash flows are

$$\$48.56\cdot 1.06^5 = \$64.98$$

and

$$\$605.49\cdot 1.07^7 = \$972.28.$$

Exercise
A company has entered into a six-month forward contract to buy 1 million Pound Sterling for $1.5 million. The daily volatility of a six-month zero-coupon Pound Sterling bond (when its price is translated to U.S. dollars) is 0.06% and the daily volatility of the six-month zero-coupon dollar bond is 0.05%. The correlation between returns from the two bonds is 0.80. Current exchange rate is 1.53. Calculate the standard deviation of the

change in the dollar value of forward contract in one day. What is the 10-day 99% Value at Risk? Assume that the six-month interest rate in both Pound Sterling and dollars is 5% per annum with continuous compounding.

Solution.

The contract if a long position in a Pound Sterling bond combined with a short position in a dollar bond. The value of the Pound Sterling bond is

$$\$1.53 \cdot e^{-0.05 \cdot 0.5} \text{ million} = \$1.492 \text{ million.}$$

The value of the dollar bond is

$$\$1.5 \cdot e^{-0.05 \cdot 0.5} \text{ million} = \$1.463 \text{ million.}$$

The variance of the change in the value of the contract in one day is, based on the formula for $\operatorname{Var}(X - Y)$,

$$1.492^2 \cdot 0.0006^2 + 1.463^2 \cdot 0.0005^2 -$$
$$- 2 \cdot 0.8 \cdot 1.492 \cdot 0.0006 \cdot 1.463 \cdot 0.0005 = 0.000000288.$$

The square root of this quantity, 0.000537, is the standard deviation, in millions of dollars. Therefore, the 10-day Value at Risk is

$$0.000537 \cdot \sqrt{10} \cdot 2.33 = \$0.00396 \text{ million.}$$

Casualty Actuarial Society May 2005 Course 8 Examination, Problem No. 37

Consider an investment portfolio consisting of the following

Asset	Market Value	Daily Volatilities
Aluminum	\$100,000	0.70%
Zinc	\$400,000	0.20%

i) The coefficient of correlation is 0.80.

ii) The 99% one-tailed *Z*-value is 2.33.

a) Calculate the 15-day, 99% value at risk (VaR) of the portfolio. Assume the change in portfolio value is normally distributed.

b) Calculate the impact of diversification on the portfolio VaR.

c) Suppose this portfolio also includes options and the gamma of the portfolio is 10. Without doing any calculations, state an alternative method one could use to estimate VaR.

Solution.

Let us write *A* and *Z* for the two assets, and use these subscripts to indicate the parameters of the distributions of returns of these two assets. Let us also write *P* for items referring to the portfolio. We have

$$\sigma_P = \sqrt{\sigma_A^2 + \sigma_Z^2 + 2 \cdot 0.80 \cdot \sigma_A \cdot \sigma_B} = \sqrt{700^2 + 800^2 + 2 \cdot 0.80 \cdot 700 \cdot 800} \approx 1423.38.$$

Therefore, the portfolio 15-day, 99%, VaR is

$$2.33 \cdot 1423.38 \cdot \sqrt{15} \approx 12,844.62.$$

On the other hand, the individual VaR's are:

$$2.33 \cdot 700 \cdot \sqrt{15} \approx 6316.84 \text{ for } A, \text{ and}$$

$$2.33 \cdot 800 \cdot \sqrt{15} \approx 7129.24 \text{ for } Z.$$

Therefore, the diversification gain for VaR is

$$6316.84 + 7129.24 - 12{,}844.62 = 691.46.$$

This is the interpretation of diversification gain in Hull's book. The CAS model solution used a different interpretation. Instead, it asked the question: what would VaR be if the two assets were perfectly correlated? It would be

$$2.33 \cdot (700 + 800) \cdot \sqrt{15} \approx 13{,}536.08.$$

The gain is therefore,

$$13{,}536.08 - 12{,}844.62 = 691.46,$$

same as before. If we were to incorporate the gamma of the option in the portfolio, we would fit a quadratic model with

$$\Delta P = \delta \cdot \Delta S + \frac{1}{2} \cdot \gamma \cdot (\Delta S)^2 .$$

