# Advanced Autonomous Systems

## Improving the Localization Process: Estimating the bias of the angular rate measurements

In our project, we implemented an estimator for estimating the states $\mathbf{X} = [x \quad y \quad \phi]^T$, for which we considered the following process model for the platform (a discrete time version of the continuous kinematic model of a tricycle or a car),

$$\mathbf{X}(k+1) = f\left(\mathbf{X}(k), \mathbf{u}(k)\right)$$

$$\Downarrow$$

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \phi(k+1) \end{bmatrix} = \begin{bmatrix} x(k) \\ y(k) \\ \phi(k) \end{bmatrix} + T \cdot \begin{bmatrix} v(k) \cdot \cos(\phi(k)) \\ v(k) \cdot \sin(\phi(k)) \\ \omega(k) \end{bmatrix}$$

$$\mathbf{X}(k) = \begin{bmatrix} x(k) \\ y(k) \\ \phi(k) \end{bmatrix}, \quad \mathbf{u}(k) = \begin{bmatrix} v(k) \\ \omega(k) \end{bmatrix}$$

One of the inputs of this model is the angular rate, $\omega(k)$. We know its values because we measure them (using a gyroscope), although not perfectly, because these measurements are polluted by noise. We initially assumed that the noise was WGN (white Gaussian noise); however, the uncertainty about $\omega(k)$ is far for being white, particularly because there is usually an unknown offset (the "bias") in the measurements. The offset is a constant or slowly time varying error, which, from our perspective, is random noise, but it is definitely non-white.

In previous project tasks we were able to estimate the offset through a basic approach; however, we can estimate it simultaneously with the estimation of the platform's states. For that we consider the following "augmented" state vector:

$$\mathbf{X} = [x \quad y \quad \varphi \quad b]^T$$

In which the new component $b$ is the unknown bias, which pollutes the angular rate measurements.

The new process model is then defined as follows,

$$\mathbf{X}(k+1) = f\left(\mathbf{X}(k), \mathbf{u}(k)\right)$$

$$\Downarrow$$

$$\begin{bmatrix} x(k+1) \\ y(k+1) \\ \phi(k+1) \\ b(k+1) \end{bmatrix} = \begin{bmatrix} x(k) \\ y(k) \\ \phi(k) \\ b(k) \end{bmatrix} + T \cdot \begin{bmatrix} v(k) \cdot \cos(\phi(k)) \\ v(k) \cdot \sin(\phi(k)) \\ \omega_b(k) - b(k) \\ 0 \end{bmatrix}$$

In this model we assume that the bias ***b*** is constant, i.e. $b(k+1) = b(k)$

The rate of the heading is now defined by the biased measurement from the gyro, $\omega_b$, minus the bias (***b***, which would also be estimated by the filter), $\omega_b(k)-b(k)$.
In our experiment, provided that we have available observations of range or/and bearing, the system would be observable, i.e. the full state vector, including the bias, would be estimated.
The observation model does not change, respect to the previous cases in which we did not estimate the bias. The observations related to each observed landmark are range and bearing:

$$obs = \begin{bmatrix} r & \alpha \end{bmatrix}^T \qquad \text{E2}$$

Their observation models are:

$$h(\mathbf{X}) = \begin{bmatrix} h_1(x,y,\phi,b) \\ h_2(x,y,\phi,b) \end{bmatrix} = \begin{bmatrix} \sqrt{(x_a-x)^2+(y_a-y)^2} \\ \tan^{-1}(y_a-y, x_a-x)-\phi+\pi/2 \end{bmatrix}$$
$$\Downarrow \qquad \text{E3}$$
$$= \begin{bmatrix} \sqrt{(x_a-x)^2+(y_a-y)^2} \\ \tan^{-1}(y_a-y, x_a-x)-\phi+\pi/2 \end{bmatrix} = \begin{bmatrix} r \\ \alpha \end{bmatrix}$$

Note: The function $\tan^{-1}(y,x)$ is actually implemented by the function $\text{atan2}(y,x)$ in Matlab and other programming languages. It is actually the argument (angle) of the 2D vector $(x,y)$.

By linearization of the observation function, $h(x,y,\phi,b)$, at the PRIOR expected value of the estimated $(x,y,\phi,b)$, we obtain the associated **H** matrix.
Note that, now, we have 4 states, consequently the **H** matrix would have a size 2x4 (in place of 2x3, as we had considered in the previous projects, where the state vector had dimensionality =3).

$$\mathbf{H} = \left.\frac{\partial h(\mathbf{X})}{\partial \mathbf{X}}\right|_{\mathbf{X}=\hat{\mathbf{X}}^-} = \left.\begin{bmatrix} \frac{\partial h_1}{\partial x} & \frac{\partial h_1}{\partial y} & \frac{\partial h_1}{\partial \phi} & \frac{\partial h_1}{\partial b} \\ \frac{\partial h_2}{\partial x} & \frac{\partial h_2}{\partial y} & \frac{\partial h_2}{\partial \phi} & \frac{\partial h_1}{\partial b} \end{bmatrix}\right|_{\mathbf{X}=\hat{\mathbf{X}}^-}$$

$$\mathbf{H} = \left.\begin{bmatrix} -\frac{(x_a-x)}{\sqrt{(x_a-x)^2+(y_a-y)^2}} & -\frac{(y_a-y)}{\sqrt{(x_a-x)^2+(y_a-y)^2}} & 0 & 0 \\ \frac{(y_a-y)}{(x_a-x)^2+(y_a-y)^2} & \frac{-(x_a-x)}{(x_a-x)^2+(y_a-y)^2} & -1 & 0 \end{bmatrix}\right|_{\mathbf{X}=\hat{\mathbf{X}}^-} \qquad \text{E4}$$

Note_1 for students: You should verify that the partial derivatives in Equation E4 are correct.
Question for students: The 4$^{th}$ column is populated by zeroes, why? Does it mean that we could not observe its associated state, ***b*** ?
Note_2: Bearing equation may differ, depending on the axes conventions you are using.

### *Jacobian of Process model*

As usually, for performing the prediction step, in the EKF, we need to evaluate the Jacobian matrix of the process model. The Jacobian matrix, of the process model as function of X, is similar to the standard case, but now it is 4x4, because the state vector is 4x1. We obtain the following matrix.

$$\mathbf{J} = \frac{\partial f}{\partial[\mathrm{x},\mathrm{y},\phi,\mathrm{b}]} = \left[ \begin{array}{cc|c|c} 1 & 0 & -T \cdot v(k) \cdot \sin(\phi(k)) & 0 \\ 0 & 1 & +T \cdot v(k) \cdot \cos(\phi(k)) & 0 \\ 0 & 0 & 1 & -T \\ 0 & 0 & 0 & 1 \end{array} \right]$$

which, as before, it is evaluated at the current expected value.
We notice that, due to the particular function, the 3x3 submatrix J(1:3,1:3) is identical to J in the standard case; and that the new elements are constants.

### *Initializing the covariance matrix P*

As our initial knowledge about the bias *b(0)* is not perfect we need to express that degree of confidence, in the initial covariance matrix. Element $p_{4,4}(0)$ has to be properly set for that purpose. (Why $p_{4,4}(0)$? Because it is the covariance of the marginal PDF about the state number four, which is the bias, at the initial time.)
Suppose we believe that our initial guess about the bias (expressed by the component $x_4(0)$) may be wrong in about 2 degrees/sec, then we can define a variance for that estimate, to be $p_{4,4}(0) = \left(2 \cdot \pi \cdot 180^{-1}\right)^2$

$$\mathbf{P}(0) = \begin{bmatrix} .. & .. & .. & 0 \\ .. & .. & .. & 0 \\ .. & .. & .. & 0 \\ 0 & 0 & 0 & p_{4,4}(0) \end{bmatrix}$$

The rest of the matrix is assumed $p_{4,i}(0) = p_{i,4}(0) = 0 \quad (i \in \{1,2,3\})$ (we assume there is no statistical dependency between our uncertainty about *b(0)* and the rest of the assumed initial states)
And for $\mathbf{P}_{1:3,1:3}(0)$, the assumptions are the ones you considered before, in the previous work.
Note: Usually we try to be a bit more pessimistic (better being conservative than being overconfident), we should set $p_{4,4}(0) = \left(4 \cdot \pi \cdot 180^{-1}\right)^2$ (which means that we assume twice the standard deviation we had proposed)

### *The* **Q** *matrix*

As we assume that the bias is constant, and we are sure that that assumption is correct, we consider that its part in the process model is perfect, so $q_{4,4} = 0$. This is because we are 100% confident about the model $b(k+1) = b(k)$.

$$\mathbf{Q}=\begin{bmatrix} .. & .. & .. & 0 \\ .. & .. & .. & 0 \\ .. & .. & .. & 0 \\ 0 & 0 & 0 & q_{4,4} \end{bmatrix}$$

### Case in which the bias can change (slowly time varying)

Suppose the case in which we suspect that the bias may change, we know that $b(k+1)=b(k)$ is not perfectly correct. However, we do not know the dynamics of the variation of $b$, so we can not include any better process model for describing the evolution of the state ***b***. Consequently, we keep our nominal model $b(k+1)=b(k)$; however, in this case, we do not assume that it is "perfect". In order to include certain distrust about our model about ***b*** we set a proper value for $q_{4,4}>0$. We are saying now that the model is $b(k+1)=b(k)+\xi_b(k)$, where the variable $\xi_b(k)$ is assumed to be a random variable, whose time sequence is WHITE (WGN).

Example, if we know that the bias can change up to 1 degree/second in 10 minutes, then we can define its variance $q_{4,4}=\left(T\cdot 1\cdot(\pi/180)\cdot 1/(10\cdot 60)\right)^2$ ( a covariance associated to a standard deviation twice this value would be also adequate); where T is the sample time of the discrete process model. You can note that we set the covariance entries $q_{4,i}=q_{i,4}=0 \quad (\forall \quad i\in\{1,2,3\})$, because we are sure that the uncertainty in the bias process model does not have dependency with other sources of uncertainty in the rest of the process model.

### What performance should we expect?

In the next figure we can see a usual case about how the estimation process is able to estimate the bias which was present in the angular rate measurements. The off-line estimate (because we had the unusual opportunity of allowing the platform to stay stationary for a number of seconds) provided a value close to 1 degree/second. The on-line estimation, whose initial guess was b=0, was able to converge to a value, very close to the off line one.

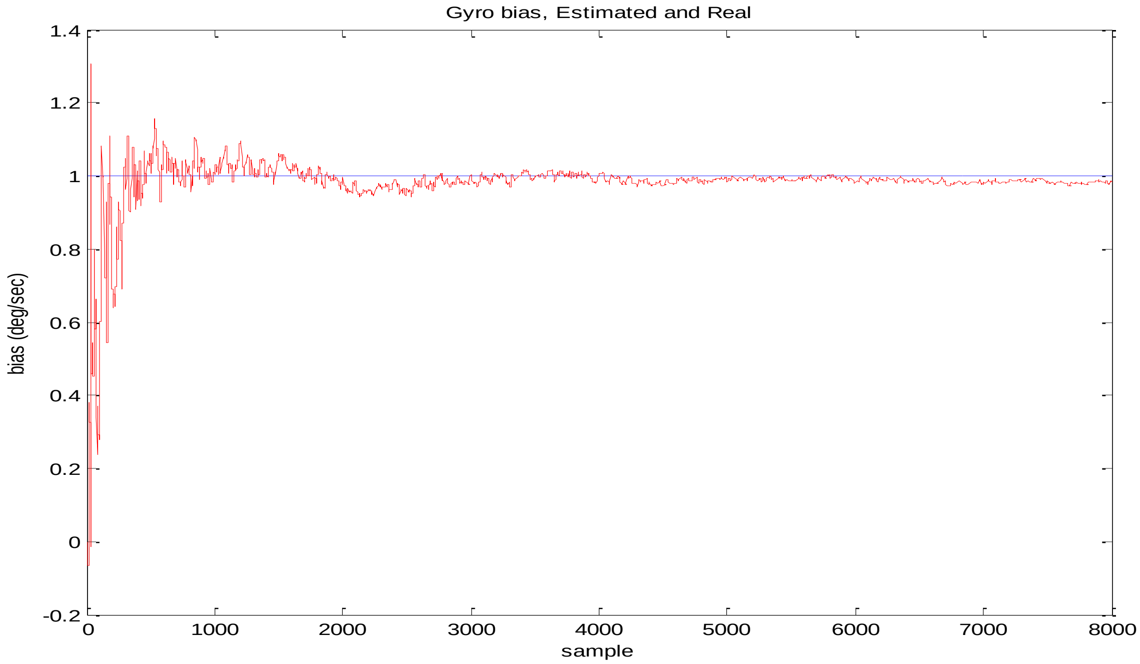


Figure showing the evolution of the estimated bias, for the case of gyro measurements polluted by a constant bias of magnitude 1degree/second. The initial expected bias was set =0 and its initial variance for a standard deviation of 2 deg/sec. The estimation process achieves a good estimate of the bias, as it can be seen in the figure, where the estimated bias (in red) converges to the real offset (in blue).

## Generalizing the concept

The problem of estimating the gyroscope's bias is a particular case of a more general problem, which is about estimating parameters of the process and/or observation models. This is usual, in many applications, in which we know the structure of the model, but we do not exactly know the value of certain parameters that are part of the model. In our current problem the parameter is a scalar, it is a simple bias. I will mention some examples, in class. There exists relevant theory about estimating parameters, it is usually called "Parameter identification".

Questions about this lecture: Ask the lecturer, Jose Guivant, via Moodle or by email (j.guivant@unsw.edu.au)