Indicator frameworks

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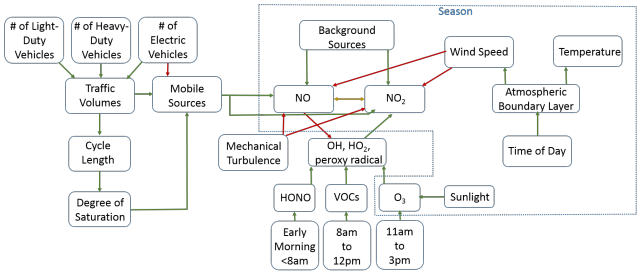
ABSTRACT

We develop a diagrammatic tool for constructing correlations between random variables, called an abstract indicator framework. Abstract indicator frameworks are modeled off operational (key performance) indicator frameworks as they are used in city planning and project governance, and give a rigorous, statistically-motivated process for constructing operational indicator frameworks.

1. INTRODUCTION

We take as our starting point a diagrams of simple correlations, as below:



This diagram correlates the measurement variables of an air pollution monitoring system with other, partially-observable variables like traffic composition, mechanical turbulence, and the presence of sunlight. It presents an intuitive and apparently useful description of a system at large. We would like to clarify the meaning of this diagram, and of others like it, by giving its interconnections a precise mathematical meaning. Clarifying the meaning of the diagram will not only make it more useful; it will allow us to connect this local, correlation-based picture of a system with other local pictures, as well as with more sophisticated scientific models of the world.

In this paper, we develop *abstract indicator frameworks*, a diagrammatic tool for constructing causally-linked sets of random variables and their correlations. Abstract indicator frameworks are modeled off operational (key performance) indicator frameworks, especially as they are used in city planning and project governance. Such operational indicator frameworks have three main uses: (1) to communicate quantitative information and strategic priorities to a wide audience, (2) to enable policy reactions to data, especially in the optimization of processes, and (3) to restrict attention to a set of 'relevant' indicators—thus discarding the information from many other, 'non-relevant' indicators.

In city planning, there are several strategy-setting frameworks for constructing operational indicator frameworks, from balanced scorecards [3] to SMART [2] to more specialized urban planning frameworks; in such frameworks, the indicators are often designed by mayors, chief strategy officers, and sizable expert committees in tandem with new projects, new policies, and new processes. Even assuming that the participants adhere to a framework, the process of choosing indicators is often ad hoc, the results do not account for statistical relationships between the indicators, and the generated data is hard to translate across localities.

We propose an alternative. Instead of constructing operational indicator frameworks expensively and internally, meaning indicator-by-indicator, we can specify them abstractly and externally, by means of their causal and statistical relationships to other, already extant sets of indicators. Such an approach is especially suited to the new 'smart' projects and smart cities where a primary challenge is to relate and integrate many large, heterogeneous data sets. This motivates the definition of abstract indicator frameworks, which we define as the objects of a certain "category of diagrams of random variables", Ind. We will apply *category theory*, originally developed to relate and analyze topological spaces, as an efficient language for relating and analyzing the causal and statistical aspects of indicator frameworks.

Let $\mathcal{X}$ stand for an indicator set. Abstract indicator frameworks have a notion of *process* that transforms one indicator set into another, a notion of *state* that represents the process of picking a specific indicator in $\mathcal{X}$, and a notion of *effect* that represents the process of "measuring" or computing the correlation with respect to a specific indicator in $\mathcal{X}$. Processes, states, and effects are represented, respectively:

These are needed to capture the operations of composing processes in sequence, called *composition*, and combining

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them in parallel, called *tensoring*. The composition $g \circ f$ (first f, then g) and tensor $f \otimes g$ are represented as:

We call any formalism with a notion of composition and tensoring a *process theory*. The semantics of process theories and their diagrams are governed by the theory of monoidal categories, which is surveyed in [11]. The goal of the paper is to specify an appropriate symmetric monoidal category, Rand, representing the appropriate operations on random variables, after which we can define a causal model as a strong monoidal functor from a causal theory into Rand. These causal models—essentially, diagrams in Rand—will be the promised abstract indicator frameworks. In Section 2, we will consider a preliminary version of Rand along with some of the possible alternatives. In Section 3, we will give the statistical justification for our choice of Rand, review the notion of a causal model from [4], and then give the full definition of abstract indicator frameworks.

2. BACKGROUND

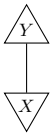
As mentioned above, there are a variety of approaches to choosing indicator frameworks as part of the process of strategic priorities. Of the many specialized approaches to choosing indicator frameworks in various fields, Niemeijer and de Groot [9] have suggested a similar methodology for choosing environmental indicator sets based on explicit causal networks of environmental forces and societal response; while their methodology is still largely qualitative rather than formal or statistical, their paper handily illustrates how (diagrams of) causal models can facilitate the selection of relevant indicator sets. In statistics, Horvath [6] also takes a compositional approach to correlation by focusing on weighted correlation networks, which represent random variables by nodes in a graph and edges between variables by a soft threshold on their correlation. These correlation networks have proved useful for analyzing high-dimensional data sets, especially gene expression data.

Even within the constraints of a process theory, there are still a number of diagrammatic approaches to probability. In this section, we will go over three examples: the traditional Hilbert space interpretation of random variables, the approach of Coecke and Spekkens [1] to Bayesian probability, and the original category of probabilistic mappings suggested by Lawvere [8]. We also briefly discuss graphical models such as those surveyed in [7].

The most straightforward approach, which we call Rand, uses the fact that real-valued random variables over some fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$ form a Hilbert space H where the inner product $\langle X, Y \rangle$ is just the covariance $\mathbb{E}(XY)$. Assuming that we restrict ourselves to standard variables with zero mean and unit variance, the covariance equals the correlation, and we can represent both by the inner product in Rand.

This inner product can be written as a process diagram, namely as the composition of a state and an effect in Rand into the following diagram:

$$\mathrm{Cor}(X,Y) = \mathrm{Cov}(X,Y) = \langle X, Y \rangle =$$



As we will see in the next section, Rand is actually already very close to what we want; the problem is that the obvious categorification of Rand does not give a natural way of analyzing the data of "intermediate" correlations.

Coecke and Spekkens [1] give a related graphical calculus for Bayesian inference. Restricting to standard probability, objects of the category Bayes are natural numbers, morphisms from m to n are $n \times m$ stochastic matrices, composition is matrix product, and the monoidal product is the matrix tensor product. Normalized states are probability distributions over the set $1, ..., n$:

$$: \mathrm{I} \to A = (p_1, p_2, \ldots, p_n) \text{ such that } \sum_{j=1}^{n} p_j = 1.$$

A joint state is a normalized state over the composite object $1, ..., mn$. A joint state is uncorrelated when it can be decomposed into a tensor product, and perfectly correlated when it can be represented as a delta function. In Bayes, uncorrelated and perfectly correlated joint states are depicted, respectively:

$$: \mathrm{I} \to A \otimes B = (p_i q_j | i \in \{1, \ldots, n\}, j \in \{1, \ldots, m\}),$$

$$: I \to A \otimes A = \mathbf{e}^T = (\delta_{i,i'} | i, i' \in \{1, \ldots, n\}).$$

A perfectly anti-correlated joint state in Bayes is just a cup with a NOT-gate attached to one end. More generally, any correlation can be obtained by attaching a suitable box to one of the ends of the cup.

The work of [1] is itself related to older work on the categorical foundations of probability initiated by Lawvere in [8] and developed in Giry [5], the category of probabilistic mappings, which we call Stoch, has objects Borel spaces (Ω, Σ_Ω) and morphisms stochastic kernels; a stochastic kernel $X : (\Omega, \Sigma_\Omega) \to (\Omega', \Sigma_{\Omega'})$ represents a function $X : \Omega \times \sigma_{\Omega'} \to [0, 1]$ that assigns to each $x \in \Omega$ and each measurable set $B \in \Sigma_{\Omega'}$ the probability of B given x, denoted $X(B|x)$; in this way, each random variable can be represented as a morphism between Borel spaces, and the composition of random variables corresponds to marginalization. The tensor product $X \otimes Y$ is simply the probability measure on the product of random variables, XY, i.e. it assigns the product measure to the pair $(x, y) \in (\Omega, \Omega')$.

Finally, in Bayesian statistics and machine learning, a variety of more generic graphical approaches, called graphical models, have been developed to model the conditional (in)dependence of multivariate random variables; the joint distribution over all the random variables in a graphical model is the product of their conditional distributions. Among the most familiar examples of graphical models are (directed)

Bayesian networks and (undirected) Markov networks. The upshot is that complex questions about joint distributions of many interrelated variables can be answered in terms of the topology of the graph. We mention these graphical models, and especially Bayesian networks, since they ground many current approaches to integrating causality with probability, including those of Bayes and Stoch.

3. INDICATOR FRAMEWORKS

In this section, we define the category Ind of abstract indicator frameworks and give an example, with diagrams, of a concrete indicator framework.

Before giving the definition of the category Ind of abstract indicator frameworks, we will go through some of the statistical justification. Suppose that we have a correlation between random variables X and Y and another one between Y and Z. What can we say about the correlation between X and Z? One obvious guess would be

$$\mathrm{Cor}(X,Z) = \mathrm{Cor}(X,Y)\mathrm{Cor}(Y,Z). \tag{1}$$

Of course we know that Equation 1 is, in general, false.[1] But it is, under certain conditions, still the best guess. The following result is a standard exercise in statistics.

Lemma 3.1 *If $a = \mathrm{Cor}(X,Y)$ and $b = \mathrm{Cor}(Y,Z)$, then*

$$\mathrm{Cor}(X,Z) \geq ab - \sqrt{1-a^2}\sqrt{1-b^2} \tag{2}$$

$$\mathrm{Cor}(X,Z)\rangle \leq ab + \sqrt{1-a^2}\sqrt{1-b^2} \tag{3}$$

Proof. WLOG, assume that A, B, C are standard variables with zero mean and unit variance, since the correlation is invariant under changes to mean and variance. We can write $X = aY + E_{Y,X}$ and $Z = bY + E_{Y,Z}$ where, by assumption, $E_{Y,X}, E_{Y,Z}$ are random variables uncorrelated with Y.

Then $\langle X,Z\rangle = \mathrm{Cor}(X,Z) = \langle aY + E_{Y,X}, bY + E_{Y,Z}\rangle = ab + \langle E_{Y,X}, E_{Y,Z}\rangle$, and we can use the Cauchy-Schwarz inequality to bound $\langle E_{Y,X}, E_{Y,Z}\rangle$ from above and from below, giving the lemma. □

The lemma tells us that there is a range of possible values, centered around $\mathrm{Cor}(X,Y)\mathrm{Cor}(Y,Z)$, for the composite correlation; unfortunately, in practice that range can so large as to be useless. In such a situation, we may ask what is the obstruction, given $\mathrm{Cor}(X,Y)$ and $\mathrm{Cor}(Y,Z)$, to knowing the canonical or 'true' correlation of their composite, and whether we can reduce or get around that obstruction. Reading the proof of the lemma, we know the obstruction is just the correlation $\langle E_{Y,X}, E_{Y,Z}\rangle$; that is, if $\langle E_{Y,X}, E_{Y,Z}\rangle$ were 0, our guess would be valid.

One may also derive the above result from the definition of the partial correlation of X and Z, fixing Y. Recall that the partial correlation $\rho_{XZ\cdot Y}$ is defined as the correlation between the residuals of X and of Z, fixing Y. In terms of their component correlations,

$$\rho_{XZ\cdot Y} = \frac{\mathrm{Cor}(X,Z) - \mathrm{Cor}(X,Y)\mathrm{Cor}(Y,Z)}{\sqrt{1-\mathrm{Cor}(X,Y)^2}\sqrt{1-\mathrm{Cor}(Y,Z)^2}}.$$

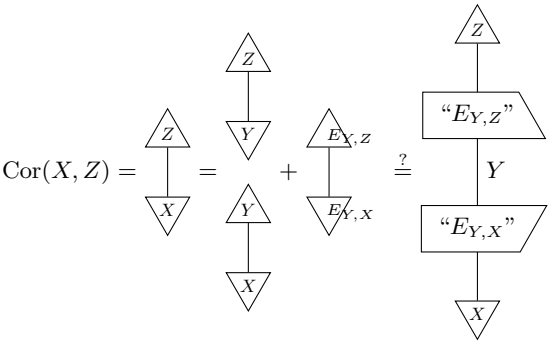
Thus

$$\langle E_{Y,X}, E_{Y,Z}\rangle = \rho_{XZ\cdot Y}\sqrt{1-\mathrm{Cor}(X,Y)^2}\sqrt{1-\mathrm{Cor}(Y,Z)^2}.$$

In other words, Equation 1 is correct just when the partial correlation $\rho_{XZ\cdot Y} = 0$, when $\mathrm{Cor}(X,Y) = 1$ or -1 (i.e. X and Y are linear functions of each other), or when $\mathrm{Cor}(Y,Z) = 1$ or -1. This allows us to produce another guess:

$$\text{“}\mathrm{Cor}(X,Z)\text{”} = Y \text{ s.t. } \rho_{XZ\cdot Y} = 0 \tag{4}$$

Explicitly, $E_{Y,X}$ measures the nonlinear component of the relation between X and Y. But one may also think of it as a measure of the 'noise' or 'error' between X and Y, at least as it concerns the correlation. The idea of Equation 4 is that, if we are lucky in choosing Y, then the noise factors $E_{Y,X}$ and $E_{Y,Z}$ will "cancel out" to produce the true correlation $\mathrm{Cor}(X,Z) = \mathrm{Cor}(X,Y)\mathrm{Cor}(Y,Z)$. Heuristically, we can represent this process as below:



That is, the correlation between X and Z can be computed by applying a transformation "$E_{Y,X}$", representing some sort of noise factor, then applying a transformation "$E_{Y,Z}$" that reverses the noise introduced by "$E_{Y,X}$".

Recall that real-valued, square-integrable random variables over a given probability space form a Hilbert space $L^2(\Omega, \mathcal{F}, \mathbb{P})$ whose inner product is just the covariance.

Definition 3.2 *The symmetric monoidal category of random variables,* Rand*, is defined by the following data:*

1. *objects are finite-dimensional Hilbert spaces*

   $$\mathcal{X} = L^2(\Omega_{\mathcal{X}}, \mathcal{F}_{\mathcal{X}}, \mathbb{P}_{\mathcal{X}})$$

   of square-integrable random variables (under the equivalence relation $X_1 \sim X_2$ if $\mathbb{P}_{\mathcal{X}}(X_1 = X_2) = 1$) over probability spaces $(\Omega_{\mathcal{X}}, \mathcal{F}_{\mathcal{X}}, \mathbb{P}_{\mathcal{X}})$, with an associated basis $B_{\mathcal{X}} = \{X_1, X_2, ..., X_n\}$

2. *morphisms $F : \mathcal{X} \to \mathcal{Y}$ are bounded linear operators*

3. *the composition is the usual composition of bounded linear operators*

4. *the tensor product of $\mathcal{X}$ and $\mathcal{Y}$ is the pushout over their joint support in $\Omega_{\mathcal{X}} \times \Omega_{\mathcal{Y}}$*

[1] Correlations are rarely composed in practice because (1) the computation is usually false and (2) because we can usually compute the composite correlation directly from the data. It is only when we lack the data (which is quite often in studies of complex systems) that we use a causal model to infer the correlation. Unfortunately, causal models are often invoked in the process of imposing a learned model such as a Kalman filter or a dynamical Bayesian network, which will often conflate the statistical and causal contributions.

The main feature of this definition, over the traditional

We will sometimes refer to the space of random variables $\mathcal{X}$ as a set of variables or indicators; in such cases, we always mean the basis set of random variables in $\mathcal{X}$.

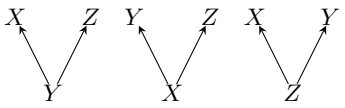
It will help to think of random variables as representing column vectors or "dimensions" of data in a table of such data, where row vectors in that table represent particular data points. The correlation between two column vectors is just their sample correlation. This has several benefits: it makes the inner product (correlation) and tensor product (entity resolution) very concrete, it is what a data analyst actually looks at, and it highlights the restrictions and challenges imposed by the presence and absence of data. In fact, we can define a category Rand^* explicitly in such terms:

Definition 3.3 *The symmetric monoidal category of random variables,* Rand^**, is defined by the following data:*

1. *objects* $\mathcal{X} = L^2(\Omega_\mathcal{X}, \mathcal{F}_\mathcal{X}, \mathbb{P}_\mathcal{X})$ *of* Rand^* *are* $m \times n$ *tables of* $\mathbb{R}$*-valued data vectors whose columns,* X*, represent indicators*
2. *morphisms* $F : \mathcal{X} \to \mathcal{Y}$ *are* $n \times k$ $\mathbb{R}$*-valued matrices mapping indicator sets to indicator sets*
3. *the composition is just the matrix product*
4. *the tensor product of* $\mathcal{X} \otimes \mathcal{Y}$ *is the entity-resolved table of their data values over a table of linkages,* $S \subset \Omega_\mathcal{X} \times \Omega_\mathcal{Y}$

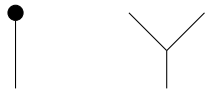
For example, if $\mathcal{X}$ represents time-series data, then S represents linkages of data that occur at the same time.

Rand and Rand^* supply the basic statistical foundation for a theory of indicator frameworks. We would now like to incorporate a causal foundation. There are several reasons for doing so. First: many statistical arguments, e.g. partial correlation, actually rely on an implicit choice of causal model—see discussions related to confounding and mediating variables by Pearl [10], among others. For example, given three random variables X, Y, Z; the partial correlations $\rho_{XZ \cdot Y}$, $\rho_{XY \cdot Z}$, or $\rho_{YZ \cdot X}$ are all equally valid; which one we take as 'true' depends on which of the following causal structures we believe is true:

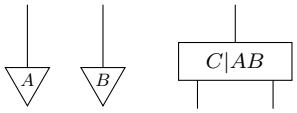


Second, many operational indicator frameworks are constructed based experts' causal models of the indicators, e.g. as in [9] or as in the air pollution diagram shown at the very beginning of this paper. Any story of indicator frameworks would be incomplete without mentioning causation. And third, the pattern developed here for causation will be useful later, when we want to incorporate not only causal models but arbitrary scientific models (such as those in Bayesian networks) into our indicator frameworks.

We recall the definition of causal theory from Fong [4], as a certain symmetric monoidal category induced from a directed acyclic graph (i.e. the causal structure), such as any of the three graphs above. Without going into the details, the idea is that given such a causal structure, we can specify a symmetric monoidal category whose objects are collections of the letters $\{X, Y, Z\}$, and whose morphisms are generated by the counit (representing 'deletion') and comultiplication (representing 'copying'), depicted respectively by



and by a set of causal mechanisms generated from the causal structure, $[A] : \varnothing \to A$, $[B] : \varnothing \to B$, and $[C|AB] : AB \to C$, depicted as



Given a causal theory, i.e. a symmetric monoidal category, we can define a model of that causal theory $\mathcal{C}$ in Rand as a strong monoidal functor $F : \mathcal{C} \to \mathsf{Rand}$. To specify such a functor, it suffices to define its behavior on every atomic variable and every generating map in $\mathcal{C}$, i.e. the counit, comultiplication, and causal mechanisms of $\mathcal{C}$, since the values of the functor on the rest of $\mathcal{C}$ is specified up to isomorphism by the definition of a strong monoidal functor.

What is the functor doing? If A is an atomic causal variable of the causal theory $\mathcal{C}$, then F sends A to a one-dimensional Hilbert space, e.g. one with basis set $\{X\}$. Then on tensor products of atomic causal variables, $F(A \otimes B)$ gives the tensor product $F(A) \otimes F(B)$, i.e. the integrated random variable with basis set $\{F(A), F(B)\}$ and probabilities inherited from the product measure. On morphisms, $F([A]) : \mathbb{R} \to F(A)$ is just a random variable, and a causal mechanism $[C|AB]$ becomes a transformation $F([C|AB]) : F(A \otimes B) \to F(C)$.

We can now state the definition of Ind.

Definition 3.4 *The category* Ind *of abstract indicator frameworks is defined by the following data:*

1. *an object* I *of* Ind *is a strong symmetric monoidal functor* $\mathcal{C} \to \mathsf{Rand}$ *from a causal theory* $\mathcal{C}$ *to the category of random variables.*
2. *a morphism* η *between abstract indicator frameworks is a natural transformation of strong symmetric monoidal functors*

One may compare Ind with the category of stochastic causal models in [4], which are generalizations of Bayesian networks.

4. CONCLUSION

In this paper, we sought to give a rigorous mathematical alternative to the traditional, indicator-by-indicator process of constructing indicator frameworks, especially in city planning and project governance. We proposed that indicator frameworks could be defined (and optimized) by means of their relationships to other indicator frameworks. These relationships were both correlational as well as causal. Therefore, we sought to develop a semantics for the problem of constructing indicator frameworks that clearly differentiated between the correlational and causal modes of reasoning.

We examined several options for the semantics of probability, including Bayes [1] and Stoch [8]. After reflecting on the practical necessities of data analysis, we decided to base our

construction on something less related to probability spaces and more directly in terms of random variables and correlations, and defined the symmetric monoidal category Rand random variables and correlations. We then introduced the idea of a causal model from [4], and used this to motivate the definition of the category Ind of abstract indicator frameworks as models of a causal theory in Rand.

We then used Ind as the setting for an optimization problem: how to construct a mediating indicator framework that best explains the relationship between a given set of indicators (e.g. those of a local project) and another set of indicators (e.g. those of broad interest to the public). The mediating framework can be used to answer the question, "what are the secondary impacts of my project"?

This research is very much a work in progress. This preliminary paper recommends a particular mathematical definition; in future versions, we will demonstrate an example of an abstract indicator framework, along with a corresponding optimization, on real-world data. It will also be interesting to replace the simple causal models examined here with other models of the world, ranging from Bayesian models to dynamical system models.

5. ACKNOWLEDGEMENTS

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