① $y'' - 3y' + 2y = 0$ $\quad y(0) = 1 \quad y'(0) = 0$

$r^2 - 3r + 2 = 0$ $\qquad r = \frac{3 \pm \sqrt{9 - 4 \cdot 1 \cdot 2}}{2 \cdot 1}$

$y(x) = Ce^{2x} + De^{x}$ $\qquad = \frac{3 \pm 1}{2}$

$y'(x) = 2Ce^{2x} + De^{x}$

$r_1 = 2 \qquad r_2 = 1$

$\cancel{y(0) = 1 = Ce^{2} + De}$

$y(0) = 1 = C + D$

$y'(0) = 0 = 2C + D$

$C = 1 - D$ $\qquad C = 1 - 2 = \underline{-1}$

$2(1-D) + D = 0$

$2 - 2D + D = 0$

$\underline{D = 2}$

$\cancel{y(x) = 2(-1)e^{2x} + 2e^{x} = 2e^{x} - 2e^{2x}}$

$y(x) = \underline{2e^{x} - e^{2x}}$

Svar: E

2 $f(x) = x^2 - 1$ $\quad x_0 = 3$ $\quad f'(x) = 2x$

Newtons metode:

$$x_n = x_{n-1} + \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$x_1 = x_0 + \frac{f(x_0)}{f'(x_0)}$$

$$= 3 + \frac{f(3)}{f'(3)}$$

$$= 3 + \frac{8}{6}$$

$$= \frac{9}{3} - \frac{4}{3}$$

$$= \frac{5}{3}$$

Svar: C

③ formell:

$$\frac{f(a+h)-2f(a)+f(a-h)}{h^2}$$

$f(x)=x^3$, $a=1$

$$\frac{(a+h)^3-2\cdot a^3+(a-h)^3}{h^2}$$

$$=\frac{(1+h)^3-2\cdot 1^3+(1-h)^2}{h^2}$$

$$=\frac{(1+3h+3h^2+h^3)-2+(1-3h+3h^2-h^3)}{h^2}$$

$$=\frac{(1+3h^2)+(1-3h^2)-2}{h^2}$$

$$=\frac{6h^2}{h^2}$$

$$=\underline{6}$$

<u>Svar: C</u>

(4) $P_2(x) = c_0 + c_1(x-x_0) + c_2(x-x_0)(x-x_1)$

$f(x) = x^2 \quad x_0 = 0, \; x_1 = 1, \; x_2 = 2$

$c_0 = f(x_0) = f(0) = \underline{0}$

$$c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(1) - f(0)}{1-0} = \frac{1-0}{1} = \underline{1}$$

$$c_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$$

$$= \underline{\frac{\frac{3-1}{2}}{}} = 1$$

$P_2(x) = (x-0) + (x-0)(x-1)$

$= \underline{x + x(x-1)}$

$= x(1 + (x-1))$

$= x^2$

Svar: B

Del 2

(1) $\sum_{k=1}^{n} (3k^2+k) = n(n+1)^2$

Sjekker om likningen gir samme svar for $n=1$

$\sum_{k=1}^{1} (3k^2+k) = 3\cdot 1^2+1 = \underline{4}$

$1(1+1)^2 = 2^2 = \underline{4}$

antar at likningen stemmer for alle $n \geq 1$, da skal $n+1$ være

$(n+1)((n+1)+1)^2 = (n+1)(n+2)^2$

$$\sum_{k=1}^{n} (3k^2+k) + (3(n+1)^2+(n+1)) = n(n+1)^2 + (3(n+1)^2+(n+1))$$

$$\cancel{= (n+1)(n+3(n+1)+}$$

$$= (n+1)(n(n+1)+3(n+1)+1)$$

$$= (n+1)(n^2+n+3n+3+1)$$

$$= (n+1)(n^2+4n+4)$$

$$\underline{\underline{= (n+1)(n+2)^2}}$$

Del 2

(2) $3x_{n+2} - 7x_{n+1} + 2x_n = -6 \qquad x_0 = 2,\ x_1 = \frac{8}{3}$

a) Karakteristisk homogen likning:

$3r^2 - 7r + 2 = 0 \qquad r = \frac{7 \pm \sqrt{49 - 4 \cdot 3 \cdot 2}}{6}$

$x_n^h = C2^n + D3^{-n} \qquad = \frac{7 \pm \sqrt{49-24}}{6}$

$= \frac{7 \pm 5}{6}$

$r_1 = 2 \quad r_2 = \frac{1}{3}$

$x_n^p = 3A - 7A + 2A = -6$

$2A = 6$

$A = 3$

$\underline{x_n = 3 + C2^n + D3^{-n}}$

$x_0 = 2 = 3 + C + D$

$x_1 = \frac{8}{3} = 3 + C2 + \frac{D}{3}$

$D = -1 - C$

$3 + C2 + \frac{(-1-C)}{3} = \frac{8}{3}$

$9 + C6 - 1 - C = 8$

$C5 = 0$

$\underline{C = 0}$

$D = -1 - 0 = -1$

$x_n = \underline{\underline{3 - 3^{-n}}}$

b) Initialbetingelsen $\frac{8}{3}$ kan ikke representeres eksakt på en maskin; så det blir avrundingsfeil. Den beregnede løsningen blir da:

$3 - (1 + e_1)3^{-n} + e_2 \cdot 2^n$

$e_2 \cdot 2^n$ vil gjøre at løsningen divergerer, noe den eksakte løsningen ikke gjør.

Del 2

(3) a) $\int_0^{\pi} x^{\frac{1}{2}} \sin x \, dx \qquad f(x) = \sqrt{x} \sin x$

$= \int_0^{\pi} f(x)\, dx \qquad h = \frac{\pi}{3}$

$\approx \frac{\pi}{3}\left(\frac{f(0) + f(\pi)}{2} + f\left(\frac{\pi}{3}\right) + f\left(\frac{2\pi}{3}\right)\right)$

$\approx 2{,}2405$

b) ~~T_5 av $\sin x = \sin a + (x-a)\cos a + \frac{(x-a)^2}{2!} +$~~

T_5 a $\sin x = \sin a + (x-a)\cos a - \frac{(x-a)^2}{2!}\sin a - \frac{(x-a)^3}{3!}\cos a$
$+ \frac{(x-a)^4}{4!}\sin a + \frac{(x-a)^5}{5!}\cos a$

$a = 0$

$T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} = x - \frac{x^3}{6} + \frac{x^5}{120}$

$g(x) = \sqrt{x}\; T_5(x)$

$\int_0^{\pi} \sqrt{x} \cdot T_5(x) \approx \frac{\pi}{3}\left(\frac{g(0) + g(\pi)}{2} + g\left(\frac{\pi}{3}\right) + g\left(\frac{2\pi}{3}\right)\right)$

$\underline{\underline{\approx 2{,}7772}}$

Del 2

(4) $x' = e^{-x} \cos t \qquad x(0) = 0$

$\frac{dx}{dt} = e^{-x} \cos t$

$e^{x}\, dx = \cos t\, dt$

$\int e^{x}\, dx = \int \cos t\, dt$

$e^{x} = \sin t + C$

$\underline{x = \ln|\sin t + C|}$

$0 = \ln|\sin(0) + C|$

$0 = \ln|C|$

$e^{0} = e^{\ln|C|}$

$1 = C$

Eulers metode: $\qquad h = 1$

$x_1 = x_0 + h f(t_0, x_0)$

$x_1 = 0 + 1 \cdot (e^{-0} \cdot \cos(0))$

$= 1 \cdot 1 = \underline{1}$

Eulers midtpunktsmetode:

$x_{\frac{1}{2}} = x_0 + \frac{h f(t_0, x_0)}{2}$

$= 0 + 0{,}5 (e^{-0} \cdot \cos(0))$

$= 0{,}5 \cdot 1 = \frac{1}{2}$

$x_1 = x_0 + h f(t_{\frac{1}{2}}, x_{\frac{1}{2}})$

$= 0 + 1 (e^{-\frac{1}{2}} \cos(\frac{1}{2}))$

$= \frac{\cos(\frac{1}{2})}{\sqrt{e}}$

$\approx \frac{0{,}8776}{1{,}6487} \approx \underline{0{,}5323}$

$\underline{\underline{x = \ln|\sin t + 1| \qquad t \neq \pi}}$

Eulers midtpunktsmytode gir minst avvik fordi den har en feilmargin på $O(h^2)$, imens ~~ete~~ eulers metode har en feilmargin på $O(h)$.

Eksakt løsning:

~~$x(\frac{\pi}{4}) = \ln|\sin(\frac{\pi}{4})$~~

$x(\frac{\pi}{4}) = \ln|\sin(\frac{\pi}{4}) + 1|$

$= \ln\left|\frac{\sqrt{2} + 2}{2}\right|$

$\approx 0{,}5348$

Programmet som regnet ut svarene på oppgave 3 a og b del 2:

```
import math

f = lambda x: math.sqrt(x) * math.sin(x)

print((math.pi/3)*((f(0) + f(math.pi))/2 + f(math.pi/3) + f(2*math.pi/3)))


g = lambda x: math.sqrt(x) * (x - (x**3/math.factorial(3)) + ((x**5)/math.factorial(5)))

print((math.pi/3)*((g(0) + g(math.pi))/2 + g(math.pi/3) + g(2*math.pi/3)))
```