Cosserat model in Small Deformation

Raffaele Russo & Tamara Dancheva

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Chapter 1

Cosserat Media in an Athermal and Small Deformation Framework

1.1 Notations

Throughout this report zero, first, second, third and fourth order tensors are used in an Euclicean space. An Einstein notation is employed in the whole report if not otherwise stated. Tensors are indicated either through an indicial notation or, in a more compact way, as reported in Table 1.1:

Tensor Order	Levi-Civita Notation	Compact Notation
0	A	A
1	A_i	$\underline{\boldsymbol{A}}$
2	A_{ij}	$\underset{\sim}{\boldsymbol{A}}$
3	A_{ijk}	$\underset{\sim}{\underline{\boldsymbol{A}}}$
4	A_{ijkl}	$\underset{\approx}{\boldsymbol{A}}$

Table 1.1: Tensorial Representation.

Besides the standard algebraic operations defined on the first order tensors, the double contraction is introduced, and it is defined as follows:

$$\underset{\approx}{A} : \underset{\approx}{B} = A_{ijkl}B_{mnkl};$$

which can be performed on tensors whose order is larger or equal to 2. The outer and double outer product can be defined as:

$$\underline{\boldsymbol{A}} \otimes \underline{\mathbf{B}} = \underset{\sim}{\mathbf{C}}; \Rightarrow A_i \otimes B_j = C_{ij} = A_i \cdot B_j; \tag{1.1}$$

$$\underset{\sim}{\boldsymbol{A}} \overset{\otimes}{\underset{\otimes}{}} \underset{\sim}{\mathbf{B}} = \underset{\approx}{\mathbf{C}}; \Rightarrow A_{ij} \overset{\otimes}{\underset{\otimes}{}} B_{kl} = C_{ijkl} = A_{ij} \cdot B_{kl}; \tag{1.2}$$

Divergence and gradient operators are performed by mean of the Nabla operator $\boldsymbol{\nabla}$. The gradient of scalars and first order tensors is indicated as:

$$\boldsymbol{\nabla} A = A_{,i}\underline{e}_i; \quad \underline{\boldsymbol{A}} \otimes \boldsymbol{\nabla} = A_{i,j}\underline{e}_i \otimes \underline{e}_j;$$

where $\underline{e}_i$ is the unit vector of the orthonormal basis $\{\underline{e}_1; \underline{e}_2; \underline{e}_3\}$ in $\mathbb{R}^3$. The divergence of first and second order tensors reads:

$$\mathrm{div}\underline{\boldsymbol{A}} = A_{i,i}; \quad \mathrm{div}\underset{\sim}{\boldsymbol{A}} = A_{ij,j}\underline{e}_i;$$

At some point we will need to use the following operator:

$$\underline{\boldsymbol{a}} = \mathrm{axl}\left(\underset{\sim}{\boldsymbol{A}}\right) \Rightarrow a_i = -\frac{1}{2}\epsilon_{ijk}A_{jk}; \tag{1.3}$$

which returns the vector with the three only components of a skew-symmetric tensor. The term $\underset{\sim}{\boldsymbol{\epsilon}}$ is the Levi-Civita permutation symbol:

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if (i,j,k) = (1,2,3),(2,3,1) or (3,1,2);} \\ -1, & \text{if (i,j,k) = (3,2,1),(2,1,3) or (1,3,2);} \\ 0, & \text{otherwise.} \end{cases}$$

1.2 Kinematic

The Cosserat media, introduced in 1909 by the Cosserat brothers [E. Cosserat and F. Cosserat 1909], is an enhanced continuum mechanical model which belongs to the family of the Generalized Continuum Mechanics models. The Cosserat brothers proposed the description of an enhanced continuum with three additional degrees of freedom in the three-dimensional space:

$$\{u_i,\, \theta_i\}, \quad i = 1, 2, 3 \tag{1.4}$$

where u_i is the translational displacement of every point in the domain, and θ_i is the additional degree of freedom, which represents the independent rotation of a triad of directors attached to the microstructure. In small deformation, the following quantities are used to quantify the deformation of the continua:

$$\underset{\sim}{\mathbf{e}} = \underline{\mathbf{u}} \otimes \boldsymbol{\nabla} + \underset{\sim}{\underline{\boldsymbol{\epsilon}}}.\underline{\boldsymbol{\theta}} \tag{1.5}$$

$$\underset{\sim}{\mathbf{k}} = \underline{\boldsymbol{\theta}} \otimes \boldsymbol{\nabla} \tag{1.6}$$

Equation 1.5 indicates the Cosserat deformation tensor, whereas Equation 1.6 refers to the Cosserat wryness tensor. It might be here useful to expand the Cosserat strain and wryness tensor formulation to fully comprehend their structure:

$$\underset{\sim}{\mathbf{e}} = \frac{1}{2}\begin{pmatrix} 2\,u_{1,1} & u_{1,2}+u_{2,1} & u_{1,3}+u_{3,1} \\ u_{2,1}+u_{1,2} & 2\,u_{2,2} & u_{2,3}+u_{3,2} \\ u_{3,1}+u_{1,3} & u_{3,2}+u_{2,3} & 2\,u_{3,3} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & u_{1,2}-u_{2,1} & u_{1,3}-u_{3,1} \\ u_{2,1}-u_{1,2} & 0 & u_{2,3}-u_{3,2} \\ u_{3,1}-u_{1,3} & u_{3,2}-u_{2,3} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{pmatrix} \tag{1.7}$$

$$\underset{\sim}{\mathbf{k}} = \begin{pmatrix} \theta_{1,1} & \theta_{1,2} & \theta_{1,3} \\ \theta_{2,1} & \theta_{2,2} & \theta_{2,3} \\ \theta_{3,1} & \theta_{3,2} & \theta_{3,3} \end{pmatrix} \tag{1.8}$$

It can be observed that in case the Cosserat rotation, *e.g.* θ_3, is equal and opposite to the material rotation, *e.g.* $0.5\,(u_{1,2} - u_{2,1})$, the skew symmetric part of the stress tensors is not governed by any degree of freedom of the system and the theory degenerates in the *Indeterminate Couple Stress Theory* [Eringen 1967; Toupin 1964]. A complete thermodynamically-compatible viscoplastic version of the Cosserat media under a Small Deformation assumption can be found in a recently published manuscript [Russo et al. 2020].

1.3 Balance Equations

By imposing that the external power is equal to the internal power, the equilibrium equations may be derived:

$$P^{(i)} = P^{(e)} \tag{1.9}$$

$$\int_{\Omega} p^{(i)} dV = \int_{\Omega} \left(\underline{\mathbf{f}}.\dot{\underline{\mathbf{u}}} + \underline{\mathbf{c}}.\dot{\underline{\boldsymbol{\theta}}}\right) dV + \int_{\partial\Omega} \left(\underline{\mathbf{t}}.\dot{\underline{\mathbf{u}}} + \underline{\mathbf{m}}.\dot{\underline{\boldsymbol{\theta}}}\right) dS \tag{1.10}$$

where $\underline{\mathbf{f}}$, $\underline{\mathbf{c}}$, $\underline{\mathbf{t}}$, $\underline{\mathbf{m}}$ are first order tensors indicating respectively the external body forces, the external body couples, the external surface traction and the external surface couple traction. The internal power density is defined as:

$$p^{(i)} = \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\mathbf{e}}} + \underset{\sim}{\boldsymbol{\mu}} : \dot{\underset{\sim}{\mathbf{k}}} \tag{1.11}$$

and by plugging Equation 1.11 into Equation 1.10 and reordering the result:

$$\int_{\Omega} \left(\underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\mathbf{e}}} + \underset{\sim}{\boldsymbol{\mu}} : \dot{\underset{\sim}{\mathbf{k}}}\right) dV = \int_{\Omega} \left(\underline{\mathbf{f}}.\dot{\underline{\mathbf{u}}} + \underline{\mathbf{c}}.\dot{\underline{\boldsymbol{\theta}}}\right) dV + \int_{\partial\Omega} \left(\underline{\mathbf{t}}.\dot{\underline{\mathbf{u}}} + \underline{\mathbf{m}}.\dot{\underline{\boldsymbol{\theta}}}\right) dS; \tag{1.12}$$

From which, the equilibrium equations can be written:

$$\mathrm{div}\underset{\sim}{\boldsymbol{\sigma}} + \underline{\mathbf{f}} = \underline{\mathbf{0}} \tag{1.13}$$

$$\mathrm{div}\underset{\sim}{\boldsymbol{\mu}} + \mathrm{axl}\left(\underset{\sim}{\boldsymbol{\sigma}}^{skew}\right) + \underline{\mathbf{c}} = \underline{\mathbf{0}} \tag{1.14}$$

and it can be appreciated that the symmetry of the stress tensor (a result that usually comes from the solution of the balance of angular momentum) is in general not ensured.

The discretization of Equation 1.12 will be used to retrieve the typical $\underset{\sim}{\mathbf{K}} \cdot \underline{\mathbf{u}} = \underline{\mathbf{f}}$ Equation for a finite element procedure in a quasi-static case and the solution will be obtain through the Newton-Raphson algorithm.

1.4 Elastic Behavior

In order to define the elastic behavior of the media using a Hyper-Elastic material model, a Helmholtz free energy potential must be assigned, and it can be written in the following quadratic form:

$$\Psi\left(\underset{\sim}{\mathbf{e}}^{e}, \underset{\sim}{\mathbf{k}}^{e}\right) = \frac{1}{2}\underset{\sim}{\mathbf{e}}^{e} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{e}}^{e} + \frac{1}{2}\underset{\sim}{\mathbf{k}}^{e} : \underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{k}}^{e} + \frac{1}{2}\underset{\sim}{\mathbf{e}}^{e} : \underset{\approx}{\mathbf{D}} : \underset{\sim}{\mathbf{k}}^{e} \tag{1.15}$$

where $\underset{\approx}{\boldsymbol{\Lambda}}$, $\underset{\approx}{\mathbf{D}}$ and $\underset{\approx}{\mathbf{C}}$ are their fourth orders elasticity tensors. The last term in the RHS of Equation 1.15 is a coupling term which vanishes under the assumption of point symmetry. From the potential defined in Equation 1.15, hypothesizing point-symmetry, the stress and couple stress tensors can be respectively written as:

$$\underset{\sim}{\boldsymbol{\sigma}} = \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{e}}^{e}} = \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{e}}^{e} \tag{1.16}$$

$$\underset{\sim}{\boldsymbol{\mu}} = \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{k}}^{e}} = \underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{k}}^{e} \tag{1.17}$$

in case of isotropic material, it could be demonstrated that the Cosserat stress and couple stress tensors assume the following forms:

$$\underset{\sim}{\boldsymbol{\sigma}} = \lambda\, \mathrm{trace}\left(\underset{\sim}{\mathbf{e}}^{e}\right) \underset{\sim}{\mathbf{I}} + 2\mu \left(\underset{\sim}{\mathbf{e}}^{e}\right)^{sym} + 2\mu_c \left(\underset{\sim}{\mathbf{e}}^{e}\right)^{skew} \tag{1.18}$$

$$\underset{\sim}{\boldsymbol{\mu}} = \alpha\, \mathrm{trace}\left(\underset{\sim}{\mathbf{k}}\right) \underset{\sim}{\mathbf{I}} + 2\beta \left(\underset{\sim}{\mathbf{k}}^{e}\right)^{sym} + 2\gamma \left(\underset{\sim}{\mathbf{k}}^{e}\right)^{skew} \tag{1.19}$$

The stability condition expressed through the elastic coefficients reads:

$$\begin{cases} 3\lambda + 2\mu \geq 0 \\ \mu \geq 0 \\ \mu_c \geq 0 \end{cases} \tag{1.20}$$

$$\begin{cases} 3\alpha + 2\beta \geq 0 \\ \beta \geq 0 \\ \gamma \geq 0 \end{cases} \tag{1.21}$$

In a code, second order tensors are often distributed in vectors, and a specific order is followed in filling this vector with the entries of the second order tensors in order to retain compatibility. In case of TFEL, the following order is adopted:

$$\underset{\sim}{\mathbf{A}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \Longrightarrow \begin{bmatrix} A_{11} \\ A_{22} \\ A_{33} \\ A_{12} \\ A_{21} \\ A_{13} \\ A_{31} \\ A_{23} \\ A_{32} \end{bmatrix} \tag{1.22}$$

and by following the same reasoning, also the fourth-orders elastic operators of the stress and couple stress for an isotropic material can be rearranged in a bi-dimensional array as:

$$\underset{\approx}{\mathbf{\Lambda}} = \begin{bmatrix} (\lambda+2\mu) & \lambda & \lambda & & & & & & \\ \lambda & (\lambda+2\mu) & \lambda & & & & & & \\ \lambda & \lambda & (\lambda+2\mu) & & & & & & \\ & & & (\mu+\mu_c) & (\mu-\mu_c) & & & & \\ & & & (\mu+\mu_c) & (\mu-\mu_c) & & & & \\ & & & & & (\mu+\mu_c) & (\mu-\mu_c) & & \\ & & & & & (\mu+\mu_c) & (\mu-\mu_c) & & \\ & & & & & & & (\mu+\mu_c) & (\mu-\mu_c) \\ & & & & & & & (\mu+\mu_c) & (\mu-\mu_c) \end{bmatrix}; \tag{1.23}$$

$$\underset{\approx}{\mathbf{C}} = \begin{bmatrix} (\alpha+2\beta) & \alpha & \alpha & & & & & & \\ \alpha & (\alpha+2\beta) & \alpha & & & & & & \\ \alpha & \alpha & (\alpha+2\beta) & & & & & & \\ & & & (\beta+\gamma) & (\beta-\gamma) & & & & \\ & & & (\beta+\gamma) & (\beta-\gamma) & & & & \\ & & & & & (\beta+\gamma) & (\beta-\gamma) & & \\ & & & & & (\beta+\gamma) & (\beta-\gamma) & & \\ & & & & & & & (\beta+\gamma) & (\beta-\gamma) \\ & & & & & & & (\beta+\gamma) & (\beta-\gamma) \end{bmatrix}; \tag{1.24}$$

where only the non-null entries of the matrix have been written. Unlike in the isotropic classical continuum model where two material parameters define the elastic response of the material, in the isotropic Cosserat media six parameters are required to describe its elastic behavior. From a simple dimensional analysis, one of the possible elastic characteristic length emerges from Equations 1.18 and 1.19. In fact, assuming that $\gamma = \beta$ the following can be found:

$$l_{el} = \sqrt{\frac{\beta}{\mu}} \tag{1.25}$$

It is important to remember that this characteristic length is not derived from any physical consideration. Following Equation 1.18 it can be observed that the stress tensor is not symmetric as in the classical continuum mechanics.

1.5 Elasto-Plastic Behavior

In a small deformation framework, the following additive decompositions hold:

$$\dot{\underset{\sim}{\mathbf{e}}} = \dot{\underset{\sim}{\mathbf{e}}}^e + \dot{\underset{\sim}{\mathbf{e}}}^p \tag{1.26}$$

$$\dot{\underset{\sim}{\mathbf{k}}} = \dot{\underset{\sim}{\mathbf{k}}}^e + \dot{\underset{\sim}{\mathbf{k}}}^p \tag{1.27}$$

The plastic evolution of the Cosserat media may be described by a single or a double plastic multiplier. In the former case, one only plastic multiplier exists both for plastic strain and plastic wryness, whereas in the latter case there exist two distinct plastic multiplier for strain and curvature. In this report we consider a single multiplier:

$$\dot{\underset{\sim}{\mathbf{e}}}^p = \dot{p}\frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}}, \quad \dot{\underset{\sim}{\mathbf{k}}}^p = \dot{p}\frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\mu}}}, \tag{1.28}$$

In case of plastic behavior we could re-write the Helmholtz free energy adding the influence of the plastic deformation in the formulation:

$$\Psi\left(\underset{\sim}{\mathbf{e}}^e, \underset{\sim}{\mathbf{k}}^e, p\right) = \frac{1}{2}\underset{\sim}{\mathbf{e}}^e : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{e}}^e + \frac{1}{2}\underset{\sim}{\mathbf{k}}^e : \underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{k}}^e + \Psi_p(p) \tag{1.29}$$

where the last term is the Helmholtz free energy that is stored due to hardening phenomena. With not-elastic responses being modeled, it might result useful to check the thermodynamic of the model to verify its compatibility, and this can be done by writing the second thermodynamical principle, or Clausius-Duhem inequality for an isothermal case:

$$\dot{\Psi} + \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\mathbf{e}}} + \underset{\sim}{\boldsymbol{\mu}} : \dot{\underset{\sim}{\mathbf{k}}} \geq 0; \tag{1.30}$$

and given the dependencies of the Helmholtz free energy on the elastic and plastic deformations:

$$\dot{\Psi} = \left[\frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{e}}^e}\dot{\underset{\sim}{\mathbf{e}}}^e + \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{k}}^e}\dot{\underset{\sim}{\mathbf{k}}}^e + \frac{\partial \Psi}{\partial p}\dot{p}\right]; \tag{1.31}$$

the Clausius-Duhem inequality assumes the following form:

$$\left(\underset{\sim}{\boldsymbol{\sigma}} - \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{e}}^e}\right) : \dot{\underset{\sim}{\mathbf{e}}}^e + \left(\underset{\sim}{\boldsymbol{\mu}} - \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{k}}^e}\right) : \dot{\underset{\sim}{\mathbf{k}}}^e + \underset{\sim}{\boldsymbol{\sigma}} : \dot{\underset{\sim}{\mathbf{e}}}^p + \underset{\sim}{\boldsymbol{\mu}} : \dot{\underset{\sim}{\mathbf{k}}}^p - \frac{\partial \Psi}{\partial p}\dot{p} \geq 0; \tag{1.32}$$

from which, assuming that the elastic deformations are fully recoverable, the constitutive equations can be derived as:

$$\underset{\sim}{\boldsymbol{\sigma}} = \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{e}}^e}; \tag{1.33}$$

$$\underset{\sim}{\boldsymbol{\mu}} = \frac{\partial \Psi}{\partial \underset{\sim}{\mathbf{k}}^e}; \tag{1.34}$$

and the remaining terms of the Clausius-Duhem inequality define the plastic dissipation. It might useful to identify:

$$\frac{\partial \Psi}{\partial p} = A; \tag{1.35}$$

as a generalized stress thermodynamically associated to the plastic multiplier. The yield function governing the plastic evolution of the Cosserat media can incorporate several behaviors such as isotropic hardening, kinematic hardening and so on. Assuming, for the sake of simplicity, only linear isotropic hardening to be present, the plastic part of the Helmholtz free energy and the yield function can be expressed as:

$$\Phi(p) = H\frac{p^2}{2}; \tag{1.36}$$

$$f = f(\underset{\sim}{\boldsymbol{\sigma}}, \underset{\sim}{\boldsymbol{\mu}}, A) = \sigma_{eq} - R(A); \tag{1.37}$$

$$R = Hp - \sigma_y = A - \sigma_y; \tag{1.38}$$

where R is the radius of the yield surface in the stress space, H is the isotropic hardening modulus, σ_y is the initial yield stress and σ_{eq} is an equivalent measure of the local state of stress of the material. One method to express the equivalent stress in the Cosserat media is to extends a J-2 plasticity theory to a Cosserat continuum [Borst 1991; Lippmann 1969; Mühlhaus and Vardoulakis 1987], and it reads:

$$\sigma_{\mathrm{eq}} = \sqrt{\frac{3}{2}\left(\mathrm{a}_1\, \underset{\sim}{\boldsymbol{\sigma}}' : \underset{\sim}{\boldsymbol{\sigma}}' + \mathrm{a}_2\, \underset{\sim}{\boldsymbol{\sigma}}' : \underset{\sim}{\boldsymbol{\sigma}}'^{\mathrm{T}} + \mathrm{b}\, \underset{\sim}{\boldsymbol{\mu}} : \underset{\sim}{\boldsymbol{\mu}}\right)} \tag{1.39}$$

From this equation, assuming that $a_1 = a_2 = a$, another characteristic length can be identified, namely, the plastic Cosserat characteristic length:

$$l_p = \sqrt{\frac{a}{b}}; \tag{1.40}$$

A more complete formulation of the equivalent stress is given by the following [Forest and Sievert 2003]:

$$\sigma_{\text{eq}} = \sqrt{\frac{3}{2}\left(\text{a}_1\, \underset{\sim}{\boldsymbol{\sigma}}' : \underset{\sim}{\boldsymbol{\sigma}}' + \text{a}_2\, \underset{\sim}{\boldsymbol{\sigma}}' : \underset{\sim}{\boldsymbol{\sigma}}'^{\text{T}} + \text{b}_1\, \underset{\sim}{\boldsymbol{\mu}} : \underset{\sim}{\boldsymbol{\mu}} + \text{b}_2\, \underset{\sim}{\boldsymbol{\mu}} : \underset{\sim}{\boldsymbol{\mu}}^{\text{T}}\right)}; \tag{1.41}$$

This is a valid choice as well, as long as the quantities used to evaluate the equivalent stress state of the media do not vary with the frame of reference, *i.e.* they are invariant. However, it has been proven that due to the terms associated with the coefficients a_2 and b_2 in Equation 1.41, a mis-coupling between in-plane and out-of-plane couple stresses in a 2D framework rises, therefore, in order to avoid this not-physical response, these coefficients are set as zero. Assuming:

- Single plastic multiplier;
- Normality rule;
- Associated flow rule;
- Equivalent Stress as in Equation 1.41;
- Yield function as in Equation 1.37.

the evolution of the plastic deformation (plastic multiplier) in the Cosserat media are determined by enforcing the condition on the yield radius (isotropic plasticity) to follow the equivalent stress measure. This is ensured by solving the consistency condition:

$$\begin{cases} f = 0 \\ \dot{f} = 0 \end{cases} \tag{1.42}$$

From the second condition and considering Equation 1.37:

$$\dot{f} = d\,f = \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}} : d\underset{\sim}{\boldsymbol{\sigma}} + \frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\mu}}} : d\underset{\sim}{\boldsymbol{\mu}} + \frac{\partial f}{\partial A} d\,A = 0; \tag{1.43}$$

considering that thanks to the normality rule we can write:

$$\frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\sigma}}} = \underset{\sim}{\mathbf{n}} = \frac{3}{2}\,\frac{a_1 \underset{\sim}{\boldsymbol{\sigma}}' + a_2 \underset{\sim}{\boldsymbol{\sigma}}'^T}{\sigma_{eq}}; \tag{1.44}$$

$$\frac{\partial f}{\partial \underset{\sim}{\boldsymbol{\mu}}} = \underset{\sim}{\mathbf{n}}_c = \frac{3}{2}\,\frac{b_1 \underset{\sim}{\boldsymbol{\mu}} + b_2 \underset{\sim}{\boldsymbol{\mu}}^T}{\sigma_{eq}}; \tag{1.45}$$

the consistency condition assume the following form:

$$\underset{\sim}{\mathbf{n}} : \underset{\approx}{\boldsymbol{\Lambda}} : (\dot{\underset{\sim}{\mathbf{e}}} - \dot{p}\underset{\sim}{\mathbf{n}}) + \underset{\sim}{\mathbf{n}}_c : \underset{\approx}{\mathbf{C}} : \left(\dot{\underset{\sim}{\mathbf{k}}} - \dot{p}\underset{\sim}{\mathbf{n}}_c\right) - \frac{\partial A}{\partial p}\,\dot{p} = 0; \tag{1.46}$$

and by solving for $\dot{p}$:

$$\dot{p} = \frac{\underset{\sim}{\mathbf{n}} : \underset{\approx}{\boldsymbol{\Lambda}} : \dot{\underset{\sim}{\mathbf{e}}} + \underset{\sim}{\mathbf{n}}_c : \underset{\approx}{\mathbf{C}} : \dot{\underset{\sim}{\mathbf{k}}}}{\dfrac{\partial A}{\partial p} + \underset{\sim}{\mathbf{n}}_c : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{n}} + \underset{\sim}{\mathbf{n}}_c : \underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{n}}}; \tag{1.47}$$

The evaluation of the plastic multiplier obviously depends on the form of the yield function, that is, on the form of the plastic part of the Helmholtz free energy, that is, on A. A different plastic behavior would induce a different solution for the plastic multiplier.

The evaluation of the consistent material tangent matrices in case of plasticity can be performed by starting from the following:

$$d\underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{\boldsymbol{\Lambda}} : d\underset{\sim}{\mathbf{e}}^{el} = \underset{\approx}{\boldsymbol{\Lambda}} : (d\underset{\sim}{\mathbf{e}} - d\underset{\sim}{\mathbf{e}}^{p}) = \underset{\approx}{\boldsymbol{\Lambda}} : d\underset{\sim}{\mathbf{e}}^{e}l - dp\,\underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{n}}; \tag{1.48}$$

$$d\underset{\sim}{\boldsymbol{\mu}} = \underset{\approx}{\mathbf{C}} : d\underset{\sim}{\mathbf{k}}^{el} = \underset{\approx}{\mathbf{C}} : (d\underset{\sim}{\mathbf{k}} - d\underset{\sim}{\mathbf{k}}^{p}) = \underset{\approx}{\mathbf{C}} : d\underset{\sim}{\mathbf{k}}^{e}l - dp\,\underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{n}}_c; \tag{1.49}$$

from which we can evaluate the consistent tangent matrices as:

$$\frac{\partial \underset{\sim}{\boldsymbol{\sigma}}}{\partial \underset{\sim}{\mathbf{e}}} = \underset{\approx}{\boldsymbol{\Lambda}} - \frac{\partial \left(d\lambda \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{n}} \right)}{\partial \mathbf{e}}; \tag{1.50}$$

$$\frac{\partial \underset{\sim}{\boldsymbol{\mu}}}{\partial \underset{\sim}{\mathbf{k}}} = \underset{\approx}{\mathbf{C}} - \frac{\partial \left(d\lambda \underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{n}}_c \right)}{\partial \mathbf{k}}; \tag{1.51}$$

and the form of these derivatives can be easily obtained by resorting to the index notation. The final form of the elasto-plastic consistent material tangent matrices is:

$$\Lambda^{ep}_{ijkl} = \Lambda_{ijkl} - \frac{\Lambda_{ijrs} n_{rs} n_{pq} \Lambda_{pqkl}}{\dfrac{\partial A}{\partial p} + n_{tu} \Lambda_{tuvz} n_{vz} + n_{ctu} \Lambda_{tuvz} n_{c_{vz}}}; \tag{1.52}$$

$$C^{ep}_{ijkl} = C_{ijkl} - \frac{C_{ijrs} n_{c_{rs}} n_{c_{pq}} C_{pqkl}}{\dfrac{\partial A}{\partial p} + n_{tu} \Lambda_{tuvz} n_{vz} + n_{ctu} \Lambda_{tuvz} n_{c_{vz}}}; \tag{1.53}$$

$$\tag{1.54}$$

or in a compact form:

$$\underset{\approx}{\boldsymbol{\Lambda}}^{ep} = \underset{\approx}{\boldsymbol{\Lambda}} - \frac{\left(\underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{n}} \right) \overset{\otimes}{\otimes} \left(\underset{\sim}{\mathbf{n}} : \underset{\approx}{\boldsymbol{\Lambda}} \right)}{\underset{\sim}{\mathbf{n}} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{n}} + \underset{\sim}{\mathbf{n}}_c : \underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{n}}_c + \dfrac{\partial A}{\partial p}}; \tag{1.55}$$

$$\underset{\approx}{\mathbf{C}}^{ep} = \underset{\approx}{\mathbf{C}} - \frac{\left(\underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{n}}_c \right) \overset{\otimes}{\otimes} \left(\underset{\sim}{\mathbf{n}}_c : \underset{\approx}{\mathbf{C}} \right)}{\underset{\sim}{\mathbf{n}} : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{n}} + \underset{\sim}{\mathbf{n}}_c : \underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{n}}_c + \dfrac{\partial A}{\partial p}}; \tag{1.56}$$

where the operator $\overset{\otimes}{\otimes}$ can be found in the notation section (Equation 1.2). In case this must implemented in a software, following the demotion of the second/fourth order tensors to first/second order tensors, these matrices can be written as second order tensors with a more comprehensible format:

$$\underset{\sim}{\boldsymbol{\Lambda}}^{ep} = \underset{\sim}{\boldsymbol{\Lambda}} - \frac{\left(\underset{\sim}{\boldsymbol{\Lambda}} \cdot \underset{\sim}{\mathbf{n}} \right) \otimes \left(\underset{\sim}{\mathbf{n}} : \underset{\sim}{\boldsymbol{\Lambda}} \right)}{\underset{\sim}{\mathbf{n}} : \underset{\sim}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{n}} + \underset{\sim}{\mathbf{n}}_c : \underset{\sim}{\mathbf{C}} : \underset{\sim}{\mathbf{n}}_c + \dfrac{\partial A}{\partial p}}; \tag{1.57}$$

$$\underset{\sim}{\mathbf{C}}^{ep} = \underset{\sim}{\mathbf{C}} - \frac{\left(\underset{\sim}{\mathbf{C}} \cdot \underset{\sim}{\mathbf{n}}_c \right) \otimes \left(\underset{\sim}{\mathbf{n}}_c \cdot \underset{\sim}{\mathbf{C}} \right)}{\underset{\sim}{\mathbf{n}} \cdot \underset{\sim}{\boldsymbol{\Lambda}} \cdot \underset{\sim}{\mathbf{n}} + \underset{\sim}{\mathbf{n}}_c \cdot \underset{\sim}{\mathbf{C}} \cdot \underset{\sim}{\mathbf{n}}_c + \dfrac{\partial A}{\partial p}}; \tag{1.58}$$

where the operator $\otimes$ is the outer product acting on two vectors.

Material Integration

In a numerical procedure, the integration of the material behavior could be done either through an explicit of implicit solution. In this case we are going to adopt an implicit, Euler-backward integration scheme. The internal variables which will be integrate are the elastic parts of the deformation tensors ($\underset{\sim}{\mathbf{e}}^{el}$ and $\underset{\sim}{\mathbf{k}}^{el}$), and the plastic multiplier p. As usual, the current increment is assumed to be fully elastic, and then a plastic adjustment is performed if the yield condition is met. Therefore, the rates to be provided would be:

$$\dot{p} = \frac{\underset{\sim}{\mathbf{n}} : \underset{\approx}{\boldsymbol{\Lambda}} : \dot{\underset{\sim}{\mathbf{e}}} + \underset{\sim}{\mathbf{n}}_c : \underset{\approx}{\mathbf{C}} : \dot{\underset{\sim}{\mathbf{k}}}}{\dfrac{\partial A}{\partial p} + \underset{\sim}{\mathbf{n}}_c : \underset{\approx}{\boldsymbol{\Lambda}} : \underset{\sim}{\mathbf{n}} + \underset{\sim}{\mathbf{n}}_c : \underset{\approx}{\mathbf{C}} : \underset{\sim}{\mathbf{n}}}; \tag{1.59}$$

$$\dot{\underset{\sim}{\mathbf{e}}}^{el} = \dot{\underset{\sim}{\mathbf{e}}} - \dot{p} \underset{\sim}{\mathbf{n}}; \tag{1.60}$$

$$\dot{\underset{\sim}{\mathbf{k}}}^{el} = \dot{\underset{\sim}{\mathbf{k}}} - \dot{p} \underset{\sim}{\mathbf{n}}_c; \tag{1.61}$$

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