

## Mfront user meeting 2022

# Damage mechanics implementations using MFront: orthotropic, anisotropic microplane and embedded strong-discontinuity models for quasi-brittle materials

Breno Ribeiro Nogueira <sup>a,c</sup>

Héloïse Rostagni <sup>a</sup>, Giuseppe Rastiello <sup>b</sup>, Cédric Giry <sup>a</sup>, Fabrice Gatuingt <sup>a</sup>, Carlo Callari <sup>c</sup>, Frédéric Ragueneau <sup>a</sup>, Ibrahim Bitar <sup>d</sup> and Benjamin Richard <sup>d</sup>

<sup>a</sup> Université Paris-Saclay, CentraleSupélec, ENS Paris-Saclay, CNRS, Laboratoire de Mécanique Paris-Saclay

<sup>b</sup> Université Paris-Saclay, CEA, Service d'études mécaniques et thermiques

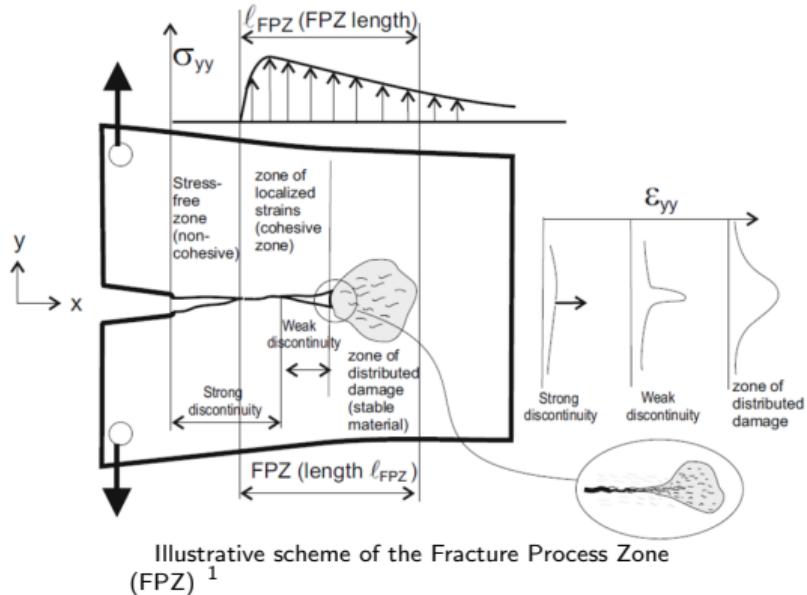
<sup>c</sup> Università degli Studi del Molise, Engineering Division, DiBT, Campobasso (Italie)

<sup>d</sup> Institut de Radioprotection et de Sûreté Nucléaire (IRSN), PSN-EXP/SES/LMAPS, France

October 2022

# Plan

- 1 Introduction
- 2 Microplane models
  - Basic principles
  - Disk simplification
  - Results and discussion
- 3 E-FEM
  - Problem definition
  - Numerical solution
  - Results
- 4 Masonry
  - Basic principles
  - Modelled mechanisms
  - Results
- 5 Conclusion
- 6 References



1. Huespe and Oliver, 2011.

# Introduction

# Introduction

## ■ Why study crack?

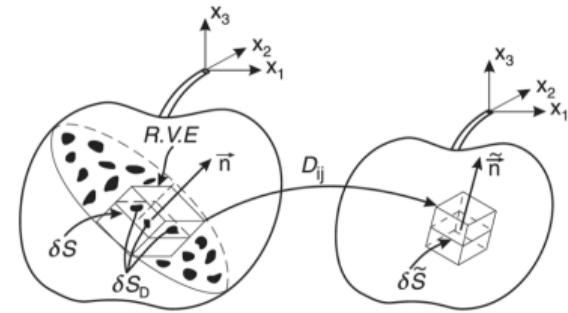
- Usual structural design codes  $\Rightarrow$  economic and primarily environmental costs
- Special constructions (e.g., nuclear reactor, dam, historical buildings, etc.)
- The need to provide a better comprehension of degradation mechanisms

## ■ Predict the cracking behavior

- Estimate crack opening
- Degradation level following a specific loading scenario

## ■ Study frameworks

- Continuum damage mechanics
- Explicit description of crack (strong discontinuities)

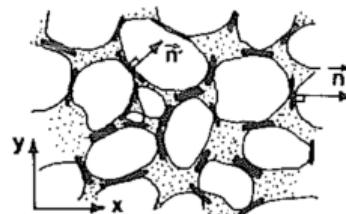
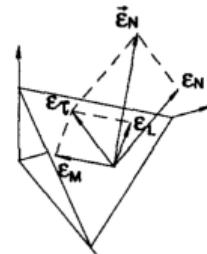
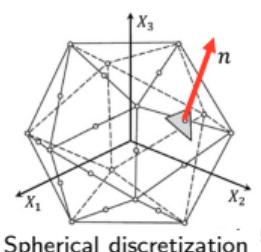


Illustrative scheme of damage volume element <sup>2</sup>

2. Lemaître and Desmorat, 2005.

# Microplane models

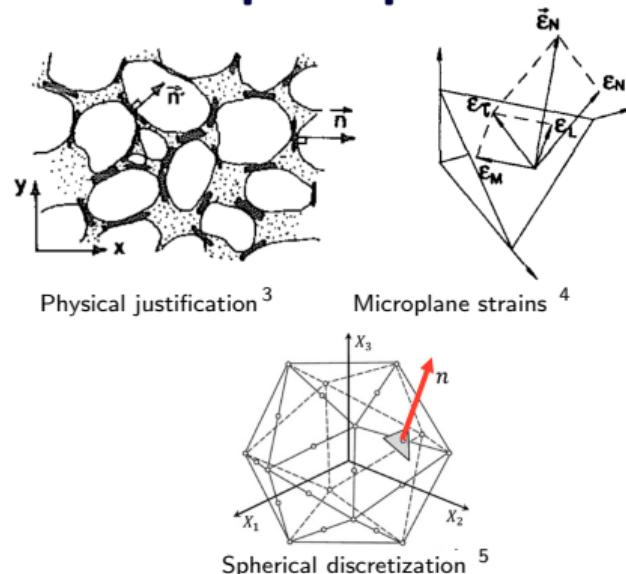
# Basic principles

Physical justification<sup>3</sup>Microplane strains<sup>4</sup>Spherical discretization<sup>5</sup>

- 3. Bažant and Gambarova, 1984.
- 4. Bažant et al., 1996.
- 5. Caner and Bažant, 2013.

- 1 Influenced by the theory introduced by Taylor in polycrystalline metals plasticity (Taylor, 1938)
- 2 Macroscopic strain tensor projected on each microplane (kinematic constraint)
- 3 Constitutive relations defined in a simple and practical way
- 4 Macroscopic stress tensor obtained by the means of the principle of virtual work
- 5 Microplane parameters obtained in function of macroscopic ones

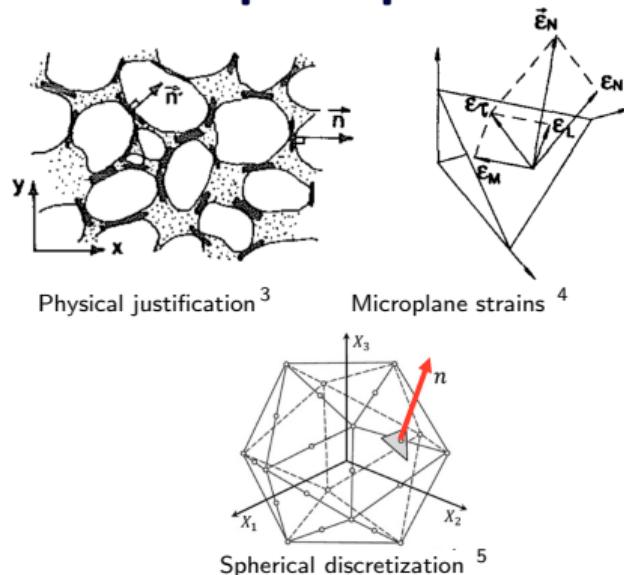
# Basic principles



- 
3. Bažant and Gambarova, 1984.  
 4. Bažant et al., 1996.  
 5. Caner and Bažant, 2013.

- 1 Influenced by the theory introduced by Taylor in polycrystalline metals plasticity (Taylor, 1938)
- 2 Macroscopic strain tensor projected on each microplane (kinematic constraint)
- 3 Constitutive relations defined in a simple and practical way
- 4 Macroscopic stress tensor obtained by the means of the principle of virtual work
- 5 Microplane parameters obtained in function of macroscopic ones

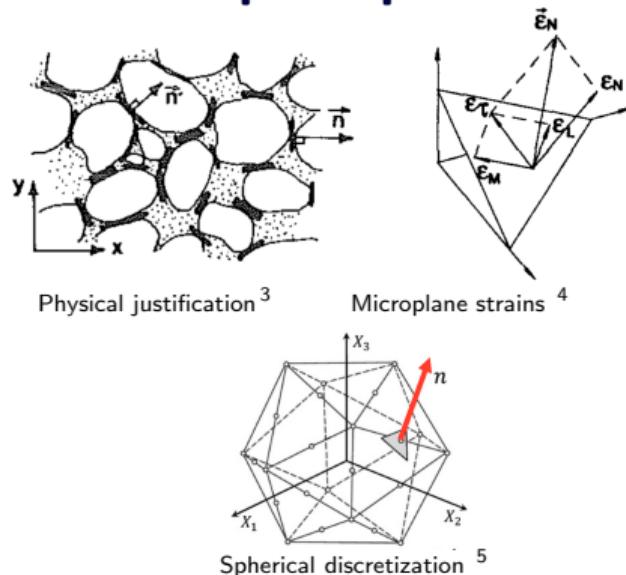
# Basic principles



- 
3. Bažant and Gambarova, 1984.  
 4. Bažant et al., 1996.  
 5. Caner and Bažant, 2013.

- 1 Influenced by the theory introduced by Taylor in polycrystalline metals plasticity (Taylor, 1938)
- 2 Macroscopic strain tensor projected on each microplane (kinematic constraint)
- 3 Constitutive relations defined in a simple and practical way
- 4 Macroscopic stress tensor obtained by the means of the principle of virtual work
- 5 Microplane parameters obtained in function of macroscopic ones

# Basic principles



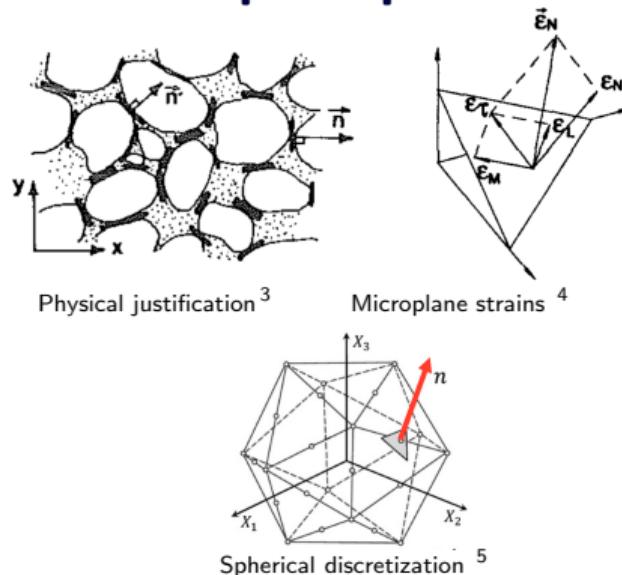
3. Bažant and Gambarova, 1984.

4. Bažant et al., 1996.

5. Caner and Bažant, 2013.

- 1 Influenced by the theory introduced by Taylor in polycrystalline metals plasticity (Taylor, 1938)
- 2 Macroscopic strain tensor projected on each microplane (kinematic constraint)
- 3 Constitutive relations defined in a simple and practical way
- 4 Macroscopic stress tensor obtained by the means of the principle of virtual work
- 5 Microplane parameters obtained in function of macroscopic ones

# Basic principles



3. Bažant and Gambarova, 1984.

4. Bažant et al., 1996.

5. Caner and Bažant, 2013.

- 1 Influenced by the theory introduced by Taylor in polycrystalline metals plasticity (Taylor, 1938)
- 2 Macroscopic strain tensor projected on each microplane (kinematic constraint)
- 3 Constitutive relations defined in a simple and practical way
- 4 Macroscopic stress tensor obtained by the means of the principle of virtual work
- 5 Microplane parameters obtained in function of macroscopic ones

# Disk simplification

(Kakarla et al., 2021; Kakarla, 2020; Park and Kim, 2003)

## Projection

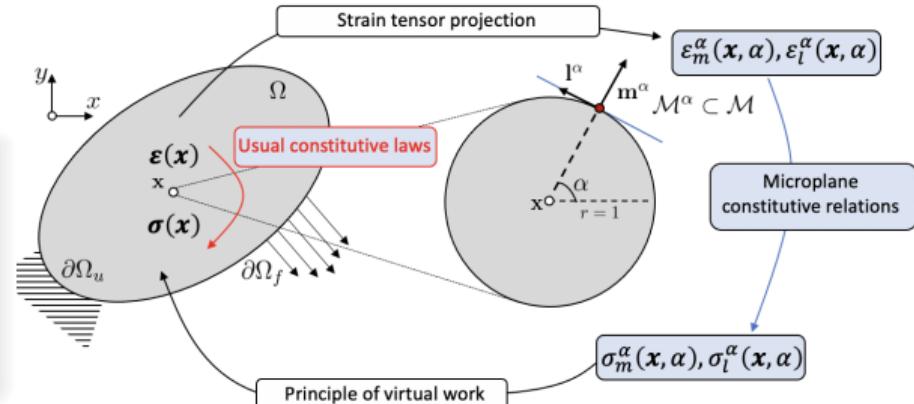
$$\epsilon_m^\alpha = m^\alpha \cdot \epsilon \cdot m^\alpha = M^\alpha : \epsilon = (m^\alpha \otimes m^\alpha) : \epsilon$$

$$\epsilon_l^\alpha = l^\alpha \cdot \epsilon \cdot m^\alpha = L^\alpha : \epsilon = (l^\alpha \otimes m^\alpha)^s : \epsilon$$

## Principle of virtual work

$$\pi \sigma : \delta \epsilon = \int_{\mathcal{M}} (\sigma_m^\alpha \delta \epsilon_m^\alpha + \sigma_l^\alpha \delta \epsilon_l^\alpha) dS$$

$$\sigma = \frac{2}{\pi} \sum_{N^\alpha} (\sigma_m^\alpha M^\alpha + \sigma_l^\alpha L^\alpha) W_\alpha$$



## Behaviour

$$\rho \psi^\alpha = (1 - \omega^\alpha)/2 [E_m(\epsilon_m^\alpha)^2 + E_l(\epsilon_l^\alpha)^2]$$

$$\sigma_{m,l}^\alpha = \frac{\partial \rho \psi^\alpha}{\partial \epsilon_{m,l}^\alpha} = (1 - \omega^\alpha) E_{m,l} \epsilon_{m,l}^\alpha$$

$$\mathbb{C}_s^\alpha = (1 - \omega^\alpha) [E_m(M^\alpha \otimes M^\alpha) + E_l(L^\alpha \otimes L^\alpha)]$$

# Disk simplification

(Kakarla et al., 2021; Kakarla, 2020; Park and Kim, 2003)

## Projection

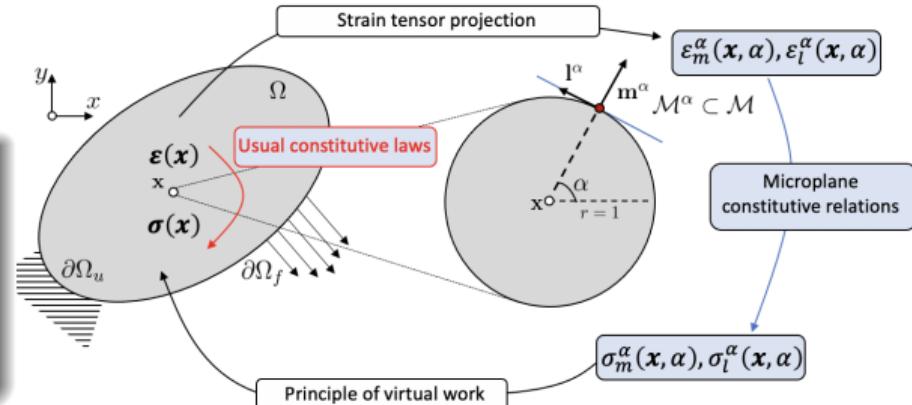
$$\epsilon_m^\alpha = \mathbf{m}^\alpha \cdot \boldsymbol{\epsilon} \cdot \mathbf{m}^\alpha = \mathbf{M}^\alpha : \boldsymbol{\epsilon} = (\mathbf{m}^\alpha \otimes \mathbf{m}^\alpha) : \boldsymbol{\epsilon}$$

$$\epsilon_l^\alpha = \mathbf{l}^\alpha \cdot \boldsymbol{\epsilon} \cdot \mathbf{m}^\alpha = \mathbf{L}^\alpha : \boldsymbol{\epsilon} = (\mathbf{l}^\alpha \otimes \mathbf{m}^\alpha)^s : \boldsymbol{\epsilon}$$

## Principle of virtual work

$$\pi \boldsymbol{\sigma} : \delta \boldsymbol{\epsilon} = \int_{\mathcal{M}} (\sigma_m^\alpha \delta \epsilon_m^\alpha + \sigma_l^\alpha \delta \epsilon_l^\alpha) dS$$

$$\boldsymbol{\sigma} = \frac{2}{\pi} \sum_{N^\alpha} (\sigma_m^\alpha \mathbf{M}^\alpha + \sigma_l^\alpha \mathbf{L}^\alpha) W_\alpha$$



## Behaviour

$$\rho \psi^\alpha = (1 - \omega^\alpha)/2 [E_m(\epsilon_m^\alpha)^2 + E_l(\epsilon_l^\alpha)^2]$$

$$\sigma_{m,l}^\alpha = \frac{\partial \rho \psi^\alpha}{\partial \epsilon_{m,l}^\alpha} = (1 - \omega^\alpha) E_{m,l} \epsilon_{m,l}^\alpha$$

$$\mathbb{C}_s^\alpha = (1 - \omega^\alpha) [E_m(\mathbf{M}^\alpha \otimes \mathbf{M}^\alpha) + E_l(\mathbf{L}^\alpha \otimes \mathbf{L}^\alpha)]$$

# Disk simplification

(Kakarla et al., 2021; Kakarla, 2020; Park and Kim, 2003)

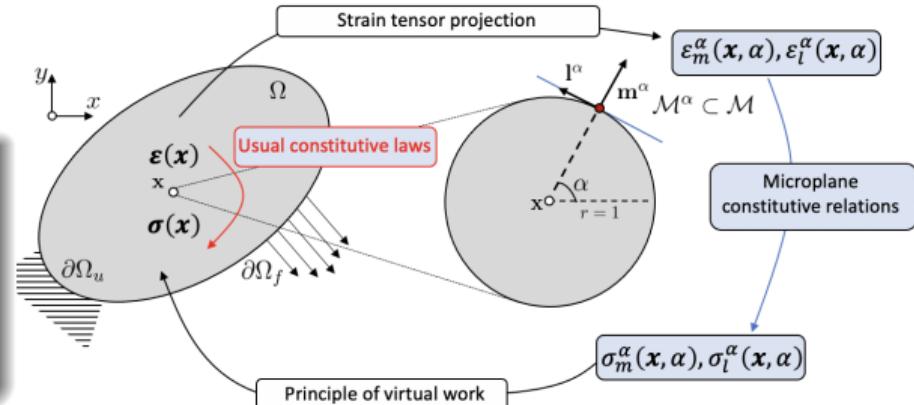
## Projection

$$\epsilon_m^\alpha = \mathbf{m}^\alpha \cdot \boldsymbol{\epsilon} \cdot \mathbf{m}^\alpha = \mathbf{M}^\alpha : \boldsymbol{\epsilon} = (\mathbf{m}^\alpha \otimes \mathbf{m}^\alpha) : \boldsymbol{\epsilon}$$

$$\epsilon_l^\alpha = \mathbf{l}^\alpha \cdot \boldsymbol{\epsilon} \cdot \mathbf{m}^\alpha = \mathbf{L}^\alpha : \boldsymbol{\epsilon} = (\mathbf{l}^\alpha \otimes \mathbf{m}^\alpha)^s : \boldsymbol{\epsilon}$$

## Principle of virtual work

$$\begin{aligned}\pi \boldsymbol{\sigma} : \delta \boldsymbol{\epsilon} &= \int_{\mathcal{M}} (\sigma_m^\alpha \delta \epsilon_m^\alpha + \sigma_l^\alpha \delta \epsilon_l^\alpha) dS \\ \boldsymbol{\sigma} &= \frac{2}{\pi} \sum_{N^\alpha} (\sigma_m^\alpha \mathbf{M}^\alpha + \sigma_l^\alpha \mathbf{L}^\alpha) W_\alpha\end{aligned}$$



## Behaviour

$$\rho\psi^\alpha = (1 - \omega^\alpha)/2 [E_m(\epsilon_m^\alpha)^2 + E_l(\epsilon_l^\alpha)^2]$$

$$\sigma_{m,l}^\alpha = \frac{\partial \rho\psi^\alpha}{\partial \epsilon_{m,l}^\alpha} = (1 - \omega^\alpha) E_{m,l} \epsilon_{m,l}^\alpha$$

$$\mathbb{C}_s^\alpha = (1 - \omega^\alpha) [E_m(\mathbf{M}^\alpha \otimes \mathbf{M}^\alpha) + E_l(\mathbf{L}^\alpha \otimes \mathbf{L}^\alpha)]$$

# Disk simplification

(Kakarla et al., 2021; Kakarla, 2020; Park and Kim, 2003)

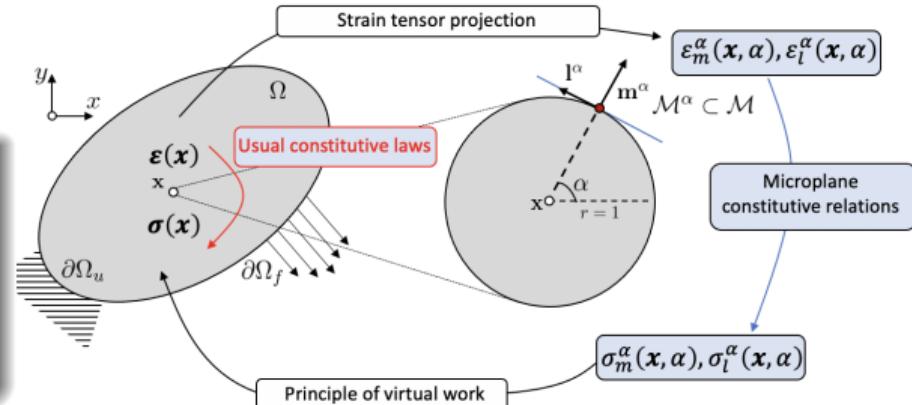
## Projection

$$\epsilon_m^\alpha = \mathbf{m}^\alpha \cdot \boldsymbol{\epsilon} \cdot \mathbf{m}^\alpha = \mathbf{M}^\alpha : \boldsymbol{\epsilon} = (\mathbf{m}^\alpha \otimes \mathbf{m}^\alpha) : \boldsymbol{\epsilon}$$

$$\epsilon_l^\alpha = \mathbf{l}^\alpha \cdot \boldsymbol{\epsilon} \cdot \mathbf{m}^\alpha = \mathbf{L}^\alpha : \boldsymbol{\epsilon} = (\mathbf{l}^\alpha \otimes \mathbf{m}^\alpha)^s : \boldsymbol{\epsilon}$$

## Principle of virtual work

$$\begin{aligned}\pi \boldsymbol{\sigma} : \delta \boldsymbol{\epsilon} &= \int_{\mathcal{M}} (\sigma_m^\alpha \delta \epsilon_m^\alpha + \sigma_l^\alpha \delta \epsilon_l^\alpha) dS \\ \boldsymbol{\sigma} &= \frac{2}{\pi} \sum_{N^\alpha} (\sigma_m^\alpha \mathbf{M}^\alpha + \sigma_l^\alpha \mathbf{L}^\alpha) W_\alpha\end{aligned}$$



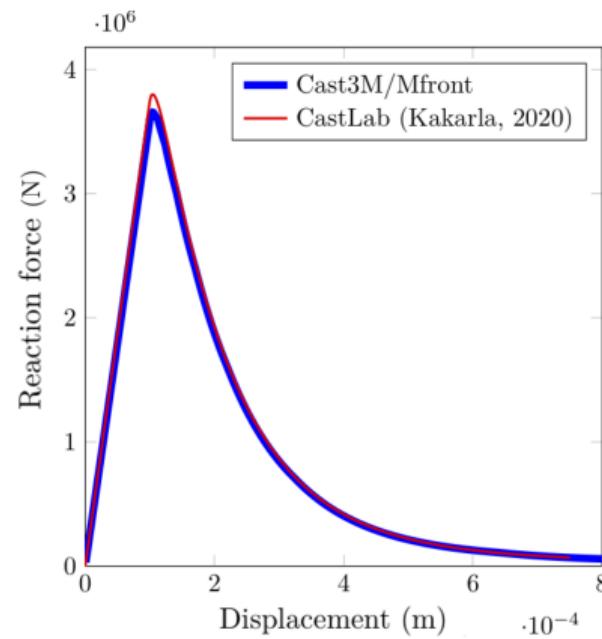
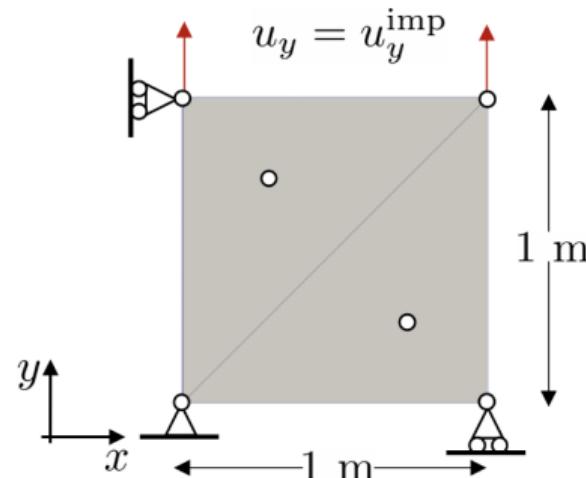
## Behaviour

$$\rho \psi^\alpha = (1 - \omega^\alpha)/2 [E_m(\epsilon_m^\alpha)^2 + E_l(\epsilon_l^\alpha)^2]$$

$$\sigma_{m,l}^\alpha = \frac{\partial \rho \psi^\alpha}{\partial \epsilon_{m,l}^\alpha} = (1 - \omega^\alpha) E_{m,l} \epsilon_{m,l}^\alpha$$

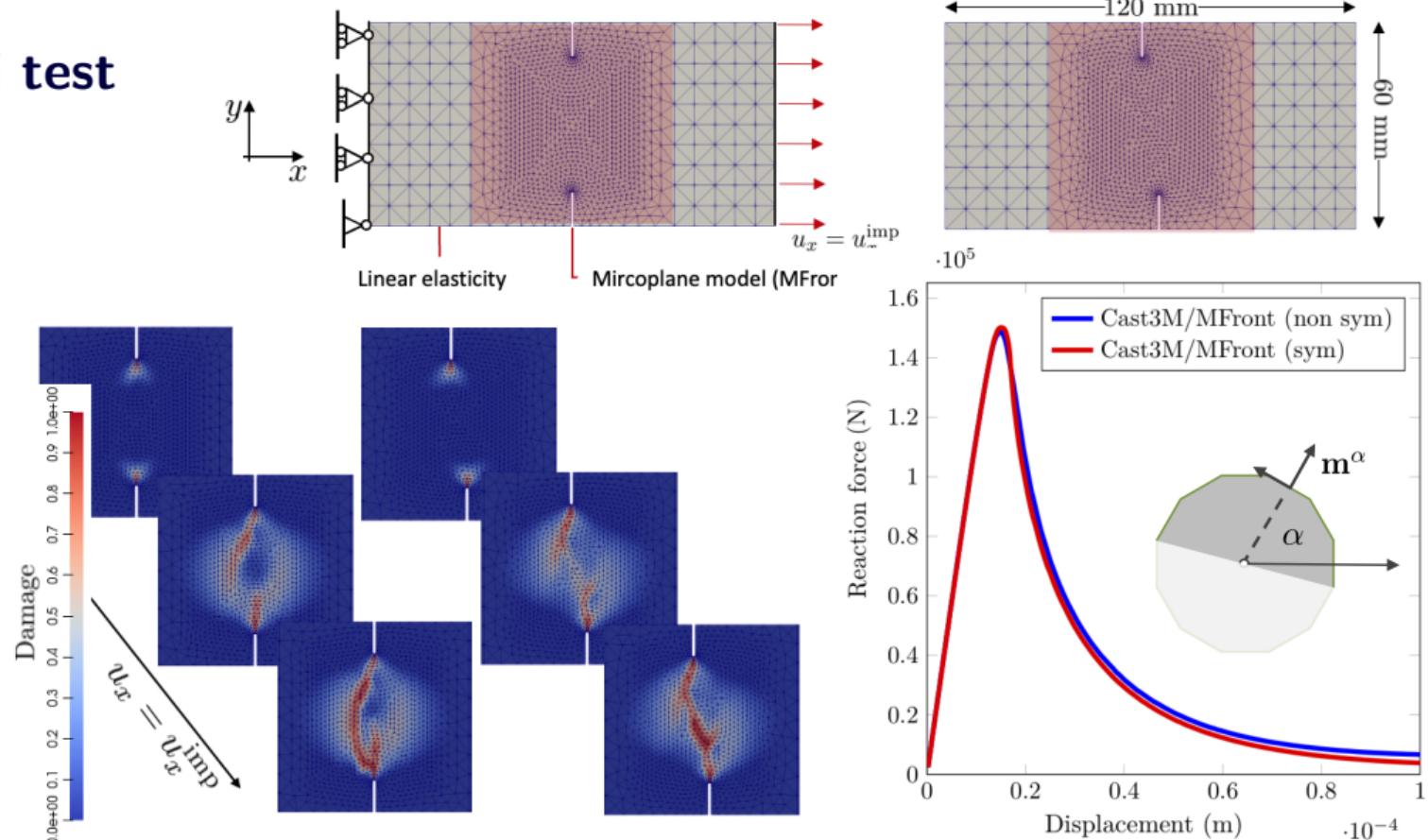
$$\mathbb{C}_s^\alpha = (1 - \omega^\alpha) [E_m(\mathbf{M}^\alpha \otimes \mathbf{M}^\alpha) + E_l(\mathbf{L}^\alpha \otimes \mathbf{L}^\alpha)]$$

## 2 CST example

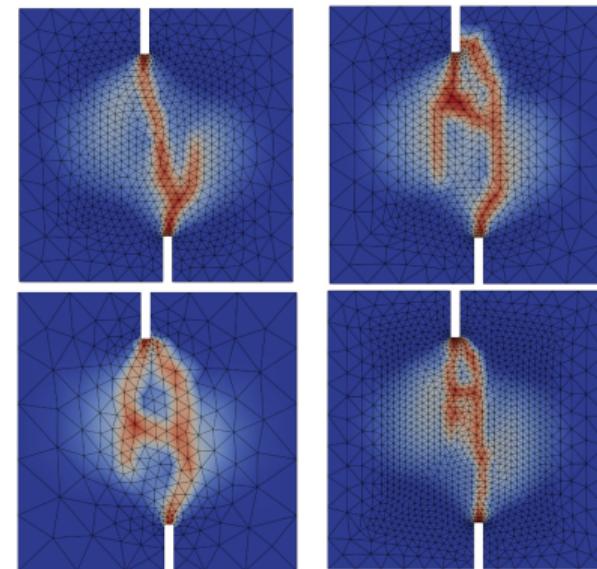
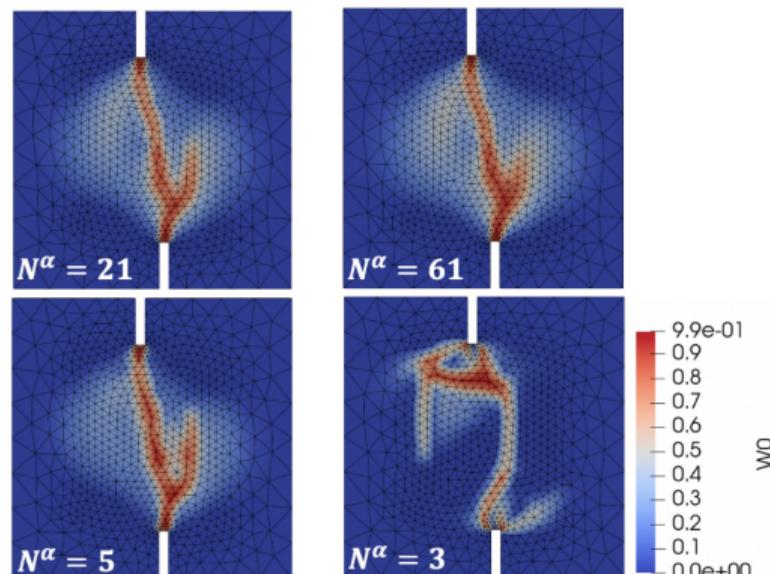


- 10 times faster than the results obtained by Kakarla et al., 2021, Kakarla, 2020

# Shi test



## Shi test: crack paths



■ Cracking convergence upon mesh refinement

■ Energy regularization  $\Rightarrow$  different crack paths for different meshes

# Strong discontinuities

## E-FEM 2D

(e.g. Ortiz et al., 1987; Simo et al., 1993; Oliver, 1996; Oliver et al., 2004; Jirasek and Zimmermann, 2001; Simone et al., 2003)

# Kinematics

**Displacement field :**

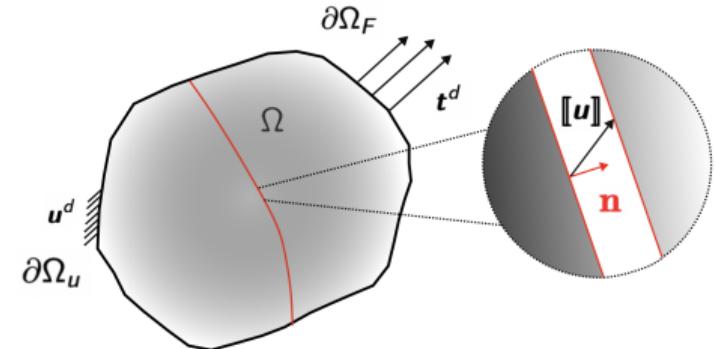
$$\mathbf{u}(\mathbf{x},t) = \bar{\mathbf{u}}(\mathbf{x},t) + \mathcal{H}_\Gamma(\mathbf{x})[\![\mathbf{u}]\!](\mathbf{x},t)$$

**Strain field :**

$$\boldsymbol{\varepsilon} = \nabla^s \bar{\mathbf{u}}(\mathbf{x},t) + \mathcal{H}_\Gamma(\mathbf{x}) \nabla^s [\![\mathbf{u}]\!](\mathbf{x},t) + \delta_\Gamma ([\![\mathbf{u}]\!] \otimes \mathbf{n})^s$$

**Hu-Washizu variational principle (weak formulation with three fields) :**

$$\int_{\Omega} \boldsymbol{\sigma} : \nabla^s \mathbf{u}^\star d\Omega + \int_{\Omega} \boldsymbol{\sigma}^\star : (\nabla^s \mathbf{u} - \boldsymbol{\varepsilon}) d\Omega + \int_{\Omega} \boldsymbol{\varepsilon}^\star : (\tilde{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}) - \boldsymbol{\sigma}) = \int_{\partial\Omega_F} \mathbf{t}^d \cdot \mathbf{u}^\star dS$$



# Enrichment and numerical solutions

Different approaches available (Jirásek, 2000) : SOS, KOS and SKON

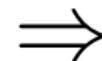
$$\mathbf{u} = \mathbf{N}\mathbf{d} + \mathbf{N}_c\mathbf{d}_c$$

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{d} + \mathbf{G}\mathbf{e}$$

$$\boldsymbol{\varepsilon}^* = \mathbf{B}\mathbf{d}^* + \mathbf{G}^*\mathbf{e}^*$$

$$\boldsymbol{\sigma} = \mathbf{S}\mathbf{s}$$

into the variational  
formulation



$$\int_{\Omega} \mathbf{B}^T \tilde{\boldsymbol{\sigma}} (\mathbf{B}\mathbf{d} + \mathbf{G}\mathbf{e}) d\Omega = \mathbf{f}_{\text{ext}}$$

$$\int_{\Omega} \mathbf{G}^* \tilde{\boldsymbol{\sigma}} (\mathbf{B}\mathbf{d} + \mathbf{G}\mathbf{e}) d\Omega = \mathbf{0}$$

$$\mathbf{G}^* = \left( \delta_{\Gamma} - \frac{\text{meas}(\Gamma_e)}{\text{meas}(\Omega_e)} \right) \mathcal{N}^T$$

# Numerical solution (Alfaiate et al., 2002 for traction-separation law)

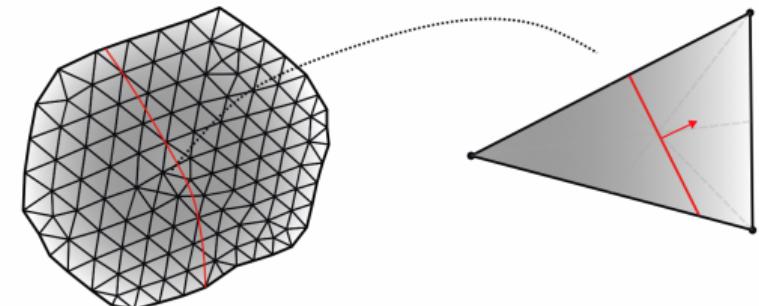
**Continuity of the traction vector for each element:**

$$\int_{\Omega_e} \left( \delta_\Gamma - \frac{\text{meas}(\Gamma_e)}{\text{meas}(\Omega_e)} \right) \mathcal{N}^T \tilde{\sigma} d\Omega \Rightarrow t_\Gamma = \mathcal{N}^T \tilde{\sigma} \quad \text{CST elements}$$

**Problem to solve:**

$$\mathbf{r}(\mathbf{d}, \mathbf{e}) = \mathbf{f}_{\text{int}}(\mathbf{d}, \mathbf{e}) - \mathbf{f}_{\text{ext}} = \mathbf{0}$$

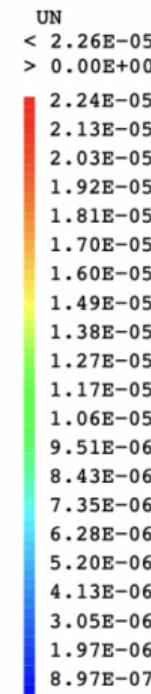
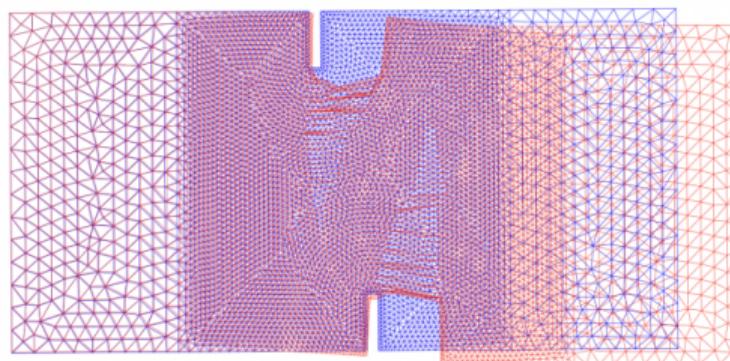
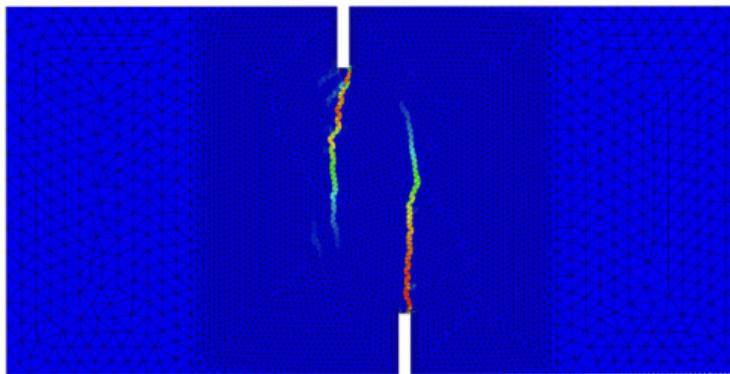
$$\mathbf{h}^e(\mathbf{d}, \mathbf{e}) = \mathcal{N}^T \tilde{\sigma}(\mathbf{d}, \mathbf{e}) - \mathbf{t}_\Gamma(\mathbf{e}) = \mathbf{0} \quad \forall \Omega_e \in E_{\text{fiss}}$$



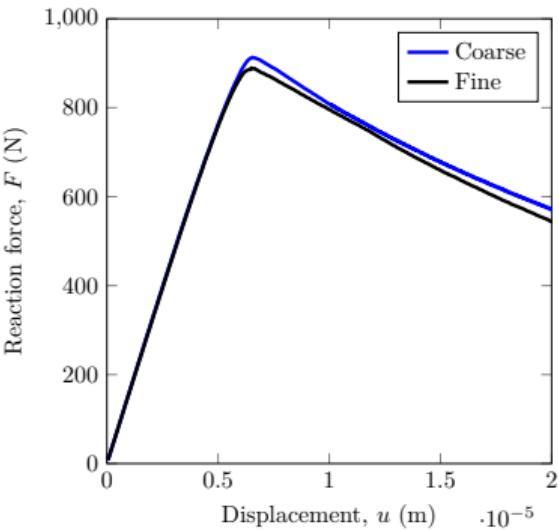
→ Equivalent to a non-linear behaviour law  $\implies$  Mfront  
*(need to add matrix operations extensions...)*

# 1D bar response

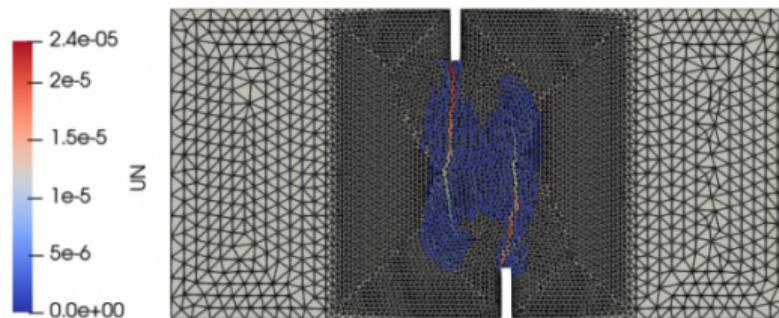
# Shi-test



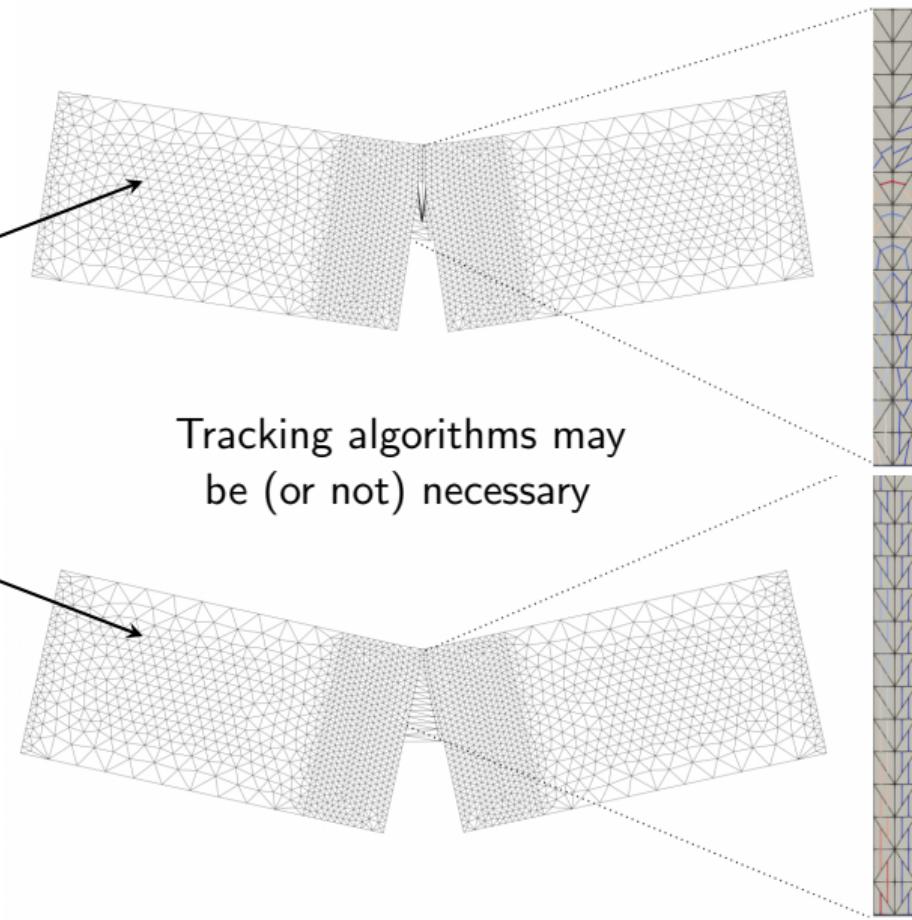
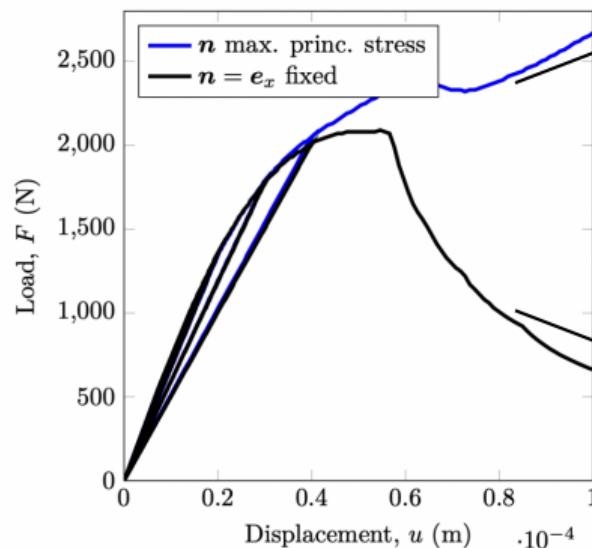
Structural response of the Shi-test with E-FEM method



# Shi-test



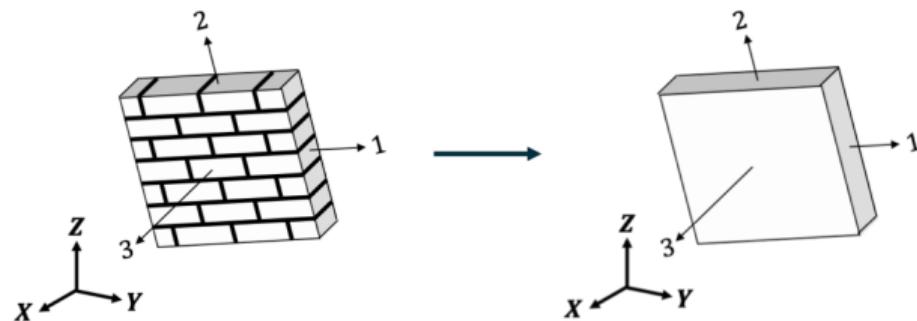
## 3-point bending



# Modelling of dissipative mechanisms for masonry-like materials

# Basic principles

- Homogenised material



- Framework of the thermodynamics of irreversible processes

## *Decomposition of the Gibbs free enthalpy density*

$$\rho\Psi^* = \rho\Psi^{*n}(\sigma_i, d_i) + \rho\Psi^{*s}(\sigma_k, d_i, \sigma_k^\pi, \alpha_k), \quad (i, k) \in \llbracket 1; 3 \rrbracket \times \llbracket 4; 6 \rrbracket$$

Normal contribution

Shear contribution

→ coupling between orthotropic elasticity, damage and plasticity

# Modelled mechanisms: cracking to damage

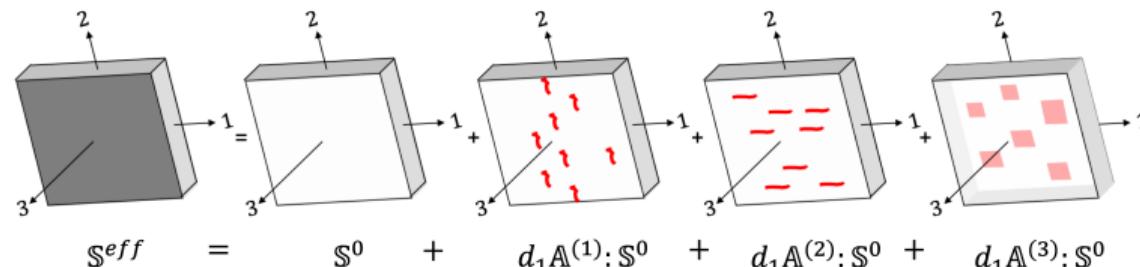
- Damage evolution → mainly driven by cracks in mortar joints



Dais et al., 2021

## Decomposition of the compliance tensor decomposition

$$\mathbb{S}^{eff} = \mathbb{S}^0 + \Delta\mathbb{S} = \mathbb{S}^0 + \sum_{i=1,3} \left( d_i \mathbb{A}^{(i)} : \mathbb{S}^0 \right)$$

 $d_i$  scalar damage variable $\mathbb{A}^{(i)}$  damage effect tensor $\mathbb{S}^0$  initial compliance tensor

Crack families and effect  
on the compliance tensor  
Tisserand et al., 2022

# Modelled mechanisms: unilateral effect

- Seismic analyses involve alternating loading → crack closure

*Normal stress partition into a positive and negative part*

based on Ladevèze, 1983

$$\sigma_i = \langle \sigma_i \rangle_+ + \langle \sigma_i \rangle_-, \quad i \in \{1,2,3\}$$

*Coupling between orthotropic elasticity and damage for normal direction*

$$\rho \Psi^{\star n} = \frac{1}{2} \left[ (1 + d_i) \frac{\langle \sigma_i \rangle_+^2}{E_i} + \frac{\langle \sigma_i \rangle_-^2}{E_i} - 2 \frac{\nu_{il}}{E_i} \sigma_i \sigma_l - 2 \frac{\nu_{im}}{E_i} \sigma_i \sigma_m \right], \quad (i,l,m) = (1,2,3)$$

*Normal strain*

$$\varepsilon_i = \rho \frac{\partial \Psi^{\star n}}{\partial \sigma_i} = (1 + d_i) \frac{\langle \sigma_i \rangle_+}{E_i} + \frac{\langle \sigma_i \rangle_-}{E_i} - \frac{\nu_{il}}{E_i} \sigma_l - \frac{\nu_{im}}{E_i} \sigma_m, \quad (i,l,m) = (1,2,3)$$

# Modelled mechanisms: internal sliding and friction

- Frictional sliding at crack surfaces → hysteretic dissipation and permanent strains

## Coupling between friction and damage for shear direction

based on Desmorat et al., 2007

$$\rho \Psi^{*s} = \frac{1}{4G_{pq}} \left[ \frac{(\sigma_k - \sigma_k^\pi)^2}{1 - g_k(d_p, d_q)} + \frac{(\sigma_k^\pi)^2}{g_k(d_p, d_q)} - w_k^s \right], \quad k(\leftrightarrow pq) \in [4; 6]$$

## Shear stress

$$\sigma_k = 2G_{pq} [1 - g_k(d_p, d_q)] \varepsilon_k + 2G_{pq} g_k(d_p, d_q) [\varepsilon_k - \varepsilon_k^\pi], \quad k(\leftrightarrow pq) \in [4; 6]$$

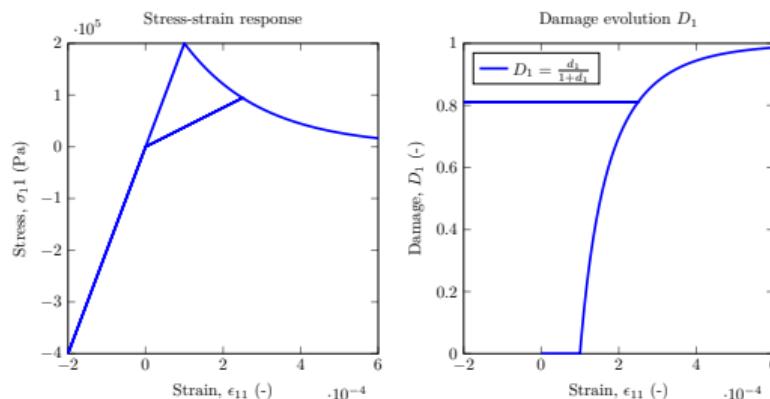
## Threshold functions (frictional resistance and confinement effects)

$$f_k = |\sigma_k^\pi - X_k| + \mu_k [\langle \sigma_p \rangle_- + \langle \sigma_q \rangle_-] \leq 0, \quad k(\leftrightarrow pq) \in [4; 6]$$

- Plastic flow develops independently along the orthotropic directions of masonry

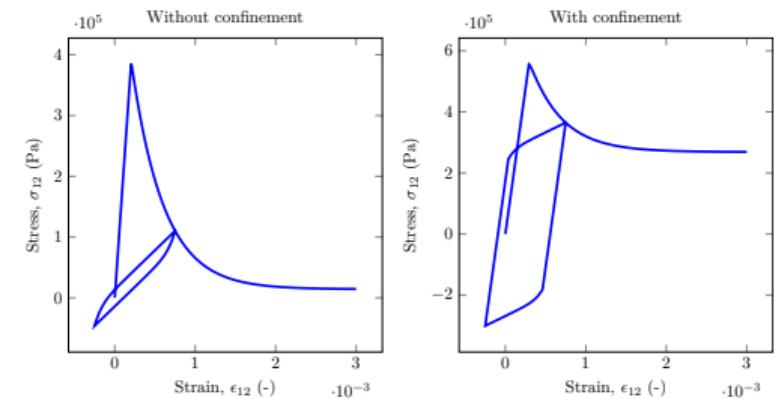
# Validation: MTest

## Unidirectional traction/compression test (damage + unilateral effect)



Unidirectional tensile/compressive test along direction 1  
(left) Stress-strain response (right) Evolution of damage variable D1

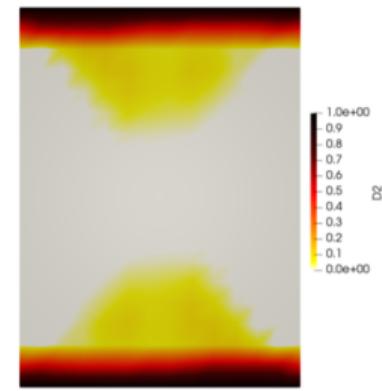
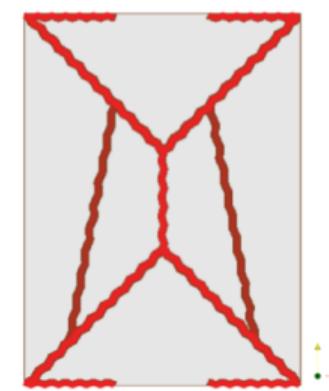
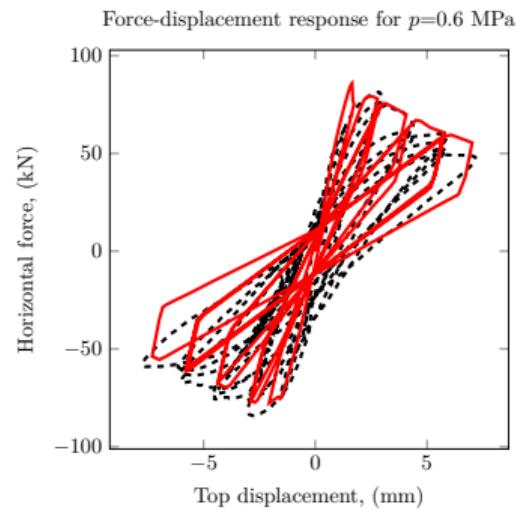
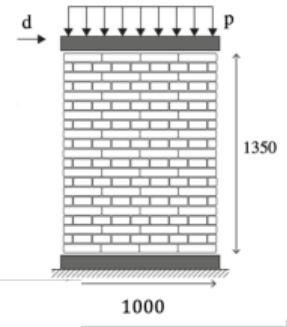
## Cyclic shear test (damage + friction)



Stress-strain response for (left) Cyclic shear test without stress confinement (right) Cyclic shear test with stress confinement

## Example: Cyclic shear test

- Masonry panel under cyclic loading (Anthoine et al., 1995)



(left) Comparisons between experimental (black dashed line) and numerical (red solid line) force-displacement response curve under cyclic loading  
 (middle) Experimental crack pattern (Anthoine et al., 1995) (right) Numerical damage map (Tisserand et al., 2022)

# Conclusion

# Conclusion

## ■ Work done

- Three different models implemented with Mfront
- Additional matrix operations required in `TFELMathExtension.hxx` for E-FEM

## ■ Perspectives

- Couple formulations with **nonlocal** regularization
- Extension to bars, plates and shells
- Extension to **3D**: microplane and E-FEM

Merci de votre attention

# References I

- Alfaiate, J., G.N. Wells, and L.J. Sluys (Apr. 2002). "On the use of embedded discontinuity elements with crack path continuity for mode-I and mixed-mode fracture". en. In: *Engineering Fracture Mechanics* 69.6, pp. 661–686. ISSN: 00137944.
- Anthoine, A., Georges Magonette, and Guido Magenes (1995). "Shear-compression testing and analysis of brick masonry walls". In: *Tenth European Conference on Earthquake Engineering* 3, pp. 1657–1662.
- Bažant, Zdeněk P. and Pietro G. Gambarova (Sept. 1984). "Crack Shear in Concrete: Crack Band Microplane Model". In: *Journal of Structural Engineering* 110.9, pp. 2015–2035. ISSN: 0733-9445, 1943-541X. (Visited on 09/26/2020).
- Bažant, Zdeněk P., Yuyin Xiang, and Pere C. Prat (Mar. 1996). "Microplane Model for Concrete. I: Stress-Strain Boundaries and Finite Strain". en. In: *Journal of Engineering Mechanics* 122.3, pp. 245–254. ISSN: 0733-9399, 1943-7889. (Visited on 09/26/2020).
- Caner, F.C. and Z.P. Bažant (2013). "Microplane Model M7 for Plain Concrete. I: Formulation". In: *Journal of Engineering Mechanics* 139(12), pp. 1724–1735.
- Dais, Dimitris, İhsan Engin Bal, Eleni Smyrou, and Vasilis Sarhosis (2021). "Automatic crack classification and segmentation on masonry surfaces using convolutional neural networks and transfer learning". In: *Automation in Construction* 125, p. 103606. ISSN: 0926-5805.
- Desmorat, Rodrigue, F. Ragueneau, and H. Pham (2007). "Continuum damage mechanics for hysteresis and fatigue of quasi-brittle materials and structures". In: *International Journal for Numerical and Analytical Methods in Geomechanics* 31, pp. 307–329. DOI: 10.1002/nag.532.
- Huespe, Alfredo E. and Javier Oliver (2011). "Crack Models with Embedded Discontinuities". In: *Numerical Modeling of Concrete Cracking*. Ed. by Günter Hofstetter and Günther Meschke. Vienna: Springer Vienna, pp. 99–159. ISBN: 978-3-7091-0897-0. DOI: 10.1007/978-3-7091-0897-0\_3. URL: [https://doi.org/10.1007/978-3-7091-0897-0\\_3](https://doi.org/10.1007/978-3-7091-0897-0_3).
- Jirasek, Milan and Thomas Zimmermann (2001). "Embedded crack model. Part II: combination with smeared cracks". In: *International Journal for Numerical Methods in Engineering* 50.6, pp. 1291–1305. ISSN: 0029-5981, 1097-0207.

## References II

- Jirásek, Milan (2000). "Comparative study on F finite elements with embedded discontinuities". In: *Comput. Methods Appl. Mech. Engrg.* 188, pp. 307–330.
- Kakarla, S., G. Rastiello, B. Richard, and C. Giry (2021). "Coupled continuous–discrete formulation based on microplane and strong discontinuity models for representing non-orthogonal intersecting cracks". In: *Engineering Fracture Mechanics* 245.
- Kakarla, Santosh (2020). "From anisotropic damage to multiple cracks by coupling a microplane model and a strong discontinuity formulation in the Embedded Finite Element Method". PhD thesis. Université Paris-Saclay - CEA - Service d'études mécaniques et thermiques - France. ENS Paris-Saclay.
- Ladevèze, Pierre (1983). *Sur une théorie de l'endommagement anisotrope*. Laboratoire de Mécanique et Technologie.
- Lemaitre, J. and R. Desmorat (2005). *Engineering Damage Mechanics: ductile, creep, fatigue and brittle failures*. Springer Nature Switzerland AG.
- Oliver, J. (Nov. 1996). "Modelling strong discontinuities in solid mechanics via strain softening constitutive equations. Part 1: fundamentals". en. In: *International Journal for Numerical Methods in Engineering* 39.21, pp. 3575–3600. ISSN: 0029-5981, 1097-0207. DOI: 10.1002/(SICI)1097-0207(19961115)39:21<3575::AID-NME65>3.0.CO;2-E. URL: [https://onlinelibrary.wiley.com/doi/10.1002/\(SICI\)1097-0207\(19961115\)39:21<3575::AID-NME65>3.0.CO;2-E](https://onlinelibrary.wiley.com/doi/10.1002/(SICI)1097-0207(19961115)39:21<3575::AID-NME65>3.0.CO;2-E) (visited on 07/08/2022).
- Oliver, J., A. Huespe, M.D.G. Pulido, and S. Blanco (Feb. 2004). "Computational modeling of cracking of concrete in strong discontinuity settings". en. In: *Computers and Concrete* 1.1, pp. 61–76. DOI: 10.12989/CAC.2004.1.1.061. URL: <https://doi.org/10.12989/CAC.2004.1.1.061> (visited on 02/18/2022).

# References III

- Ortiz, Michael, Yves Leroy, and Alan Needleman (Mar. 1987). "A finite element method for localized failure analysis". en. In: *Computer Methods in Applied Mechanics and Engineering* 61.2, pp. 189–214. ISSN: 00457825. DOI: 10.1016/0045-7825(87)90004-1. URL: <https://linkinghub.elsevier.com/retrieve/pii/0045782587900041> (visited on 07/08/2022).
- Park, Honggun and Hakjun Kim (Mar. 2003). "Microplane Model for Reinforced-Concrete Planar Members in Tension-Compression". en. In: *Journal of Structural Engineering* 129.3, pp. 337–345. ISSN: 0733-9445, 1943-541X. (Visited on 09/28/2020).
- Simo, J. C., J. Oliver, and F. Armero (1993). "An analysis of strong discontinuities induced by strain-softening in rate-independent inelastic solids". en. In: *Computational Mechanics* 12.5, pp. 277–296. ISSN: 0178-7675, 1432-0924. DOI: 10.1007/BF00372173. URL: <http://link.springer.com/10.1007/BF00372173> (visited on 07/08/2022).
- Simone, Angelo, Garth N. Wells, and Lambertus J. Sluys (2003). "From continuous to discontinuous failure in a gradient-enhanced continuum damage model". In: *Computer Methods in Applied Mechanics and Engineering* 192.41-42, pp. 4581–4607. ISSN: 00457825.
- Taylor, G. I. (1938). "Plastic strain in metals". In: *J. Inst. Metals* 62, pp. 307–324.
- Tisserand, Pierre-Jean, Héloïse Rostagni, Cédric Giry, Thi Thanh Huyen Nguyen, Rodrigue Desmorat, and Frédéric Ragueneau (2022). "An orthotropic damage model with internal sliding and friction for masonry-like material". In: *Engineering Fracture Mechanics* 267, p. 108397. ISSN: 0013-7944.