An implementation of a unilateral constitutive law for a dynamic damage phase field modeling owing to a FEniCSx/MGIS association

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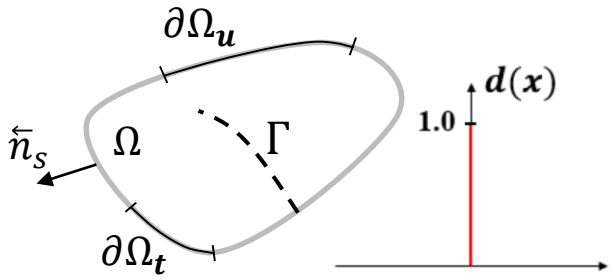
Outline

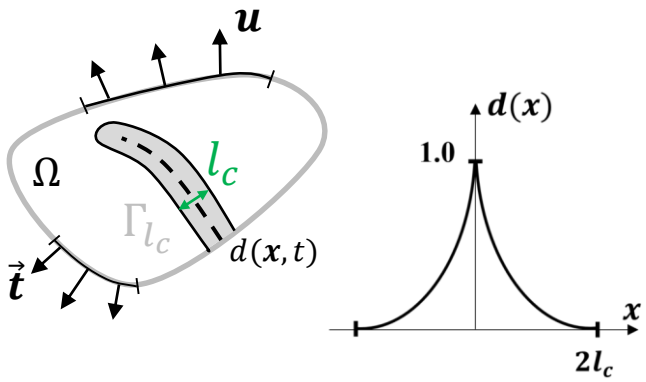
1. **Context and Motivations**
    - Phase field damage modelling for dynamic fracture
    - Unilateral contact condition at the crack lip
2. **Implementation framework**
    - Implementation of the constitutive law
    - FEniCSx – MGIS coupling
3. **Numerical applications**
4. **Discussions, conclusions and perspectives**

Outline

Phase-field mathematic background

- From a discontinous fracture problem to continous one through an additional field the damage variable « d » *[Francfort, Marigo and Bourdin]*





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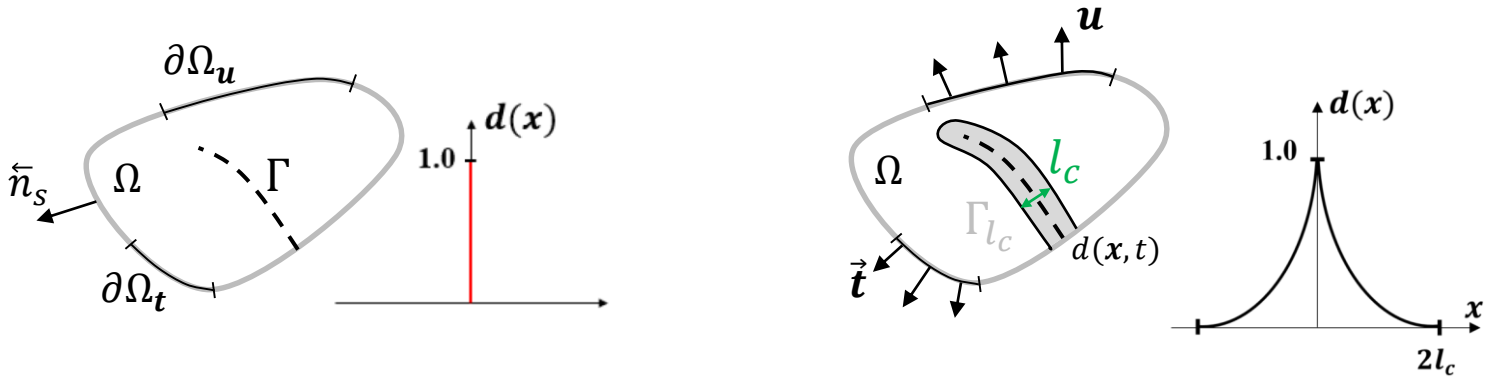

- Minimisation of the total energy of the system, the Lagrangian function, $\left(-L^{dyn}\right)$:

$$\mathrm{E}_{elastic}\,(\boldsymbol{u},d) \quad + \quad \mathrm{E}_{fracture}\,(d) \quad - \; P_{ext}(\boldsymbol{u}) \quad - \; \mathcal{K}_u\,(\dot{\boldsymbol{u}}) - \; D_d\,(\dot{d})$$

$$L^{dyn}\left(t,\boldsymbol{u},\dot{\boldsymbol{u}},d,\dot{d}\right) = \int \Psi([\boldsymbol{\varepsilon}(\boldsymbol{u})],d)\,d\tilde{x} \; + \int G_c\,\gamma(l_c,d,\vec{\boldsymbol{\nabla}}d)\,d\tilde{x} - P_{ext}(\boldsymbol{u}) - \int \frac{\rho_{\boldsymbol{u}}}{2}\dot{\boldsymbol{u}}^2\,d\tilde{x} - \int \frac{\rho_d}{2}\dot{d}^2\,d\tilde{x}$$

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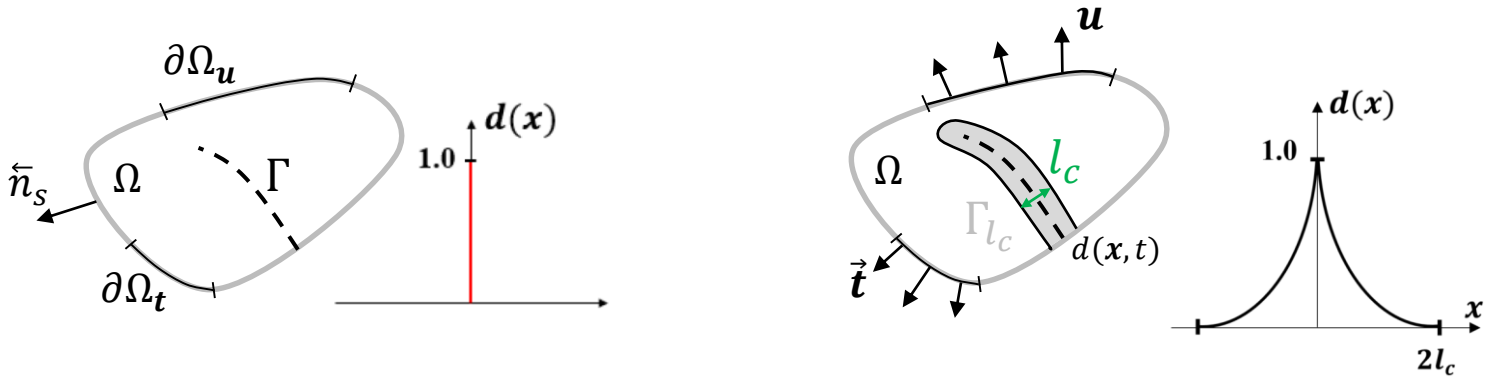

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Elastic energy density split $\Psi([\boldsymbol{\varepsilon}],d)$
(Traction-compression asymmetry)
$\Psi = \mathrm{g(d)}\Psi^+ + \Psi^-$
$\mathrm{g(d)}$: degradation function

Crack density function $\gamma(l_c,d,\vec{\boldsymbol{\nabla}}d)$
G_c : critical energy release rate
l_c : internal length

Inertia effects :
$\rho_{\boldsymbol{u}}$: mass density
ρ_d : micro-inertia related to microcrack evolution

Motivation

Admissible spaces :

$$\bar{\Omega}_{\boldsymbol{u}} = \left\{ \underline{\boldsymbol{v}} : \mathrm{T} \times \Omega \to H^1\left(\Omega, \boldsymbol{R}^{dim}\right),\ \underline{\boldsymbol{v}} = \underline{\boldsymbol{u}} \text{ on } \boldsymbol{\partial}_{\underline{\boldsymbol{u}}}\Omega \right\}, \qquad \bar{\Omega}_d = \{\beta : T \times \Omega \to [0,1], 0 < d < \beta < 1,\ \beta = d \text{ on } \partial_d \Omega\}$$

Weak formulation : $\left(\underline{\boldsymbol{u}}, \mathrm{d}\right) \in \bar{\Omega}_{\boldsymbol{u}} \times \bar{\Omega}_d,\ \forall\, (\delta\underline{\boldsymbol{u}}, \delta d) \in \bar{\Omega}_{\boldsymbol{u}} \times \bar{\Omega}_d$

$$\int_\Omega \int_t \left(-\rho_{\boldsymbol{u}} \underline{\ddot{\boldsymbol{u}}} + \boldsymbol{\nabla}.([\boldsymbol{\sigma}]) + \boldsymbol{b}\right) \delta\underline{\boldsymbol{u}}\, dt\, d\tilde{x} + \int_{\partial\Omega} -[\boldsymbol{\sigma}].\tilde{n}_s\, \delta\underline{\boldsymbol{u}} ds + \int_{\partial\Omega_n} \underline{\boldsymbol{t}}.\tilde{n}_s\, \delta\mathbf{u}\, ds = 0, \qquad \forall \delta\underline{\boldsymbol{u}} \in \partial\bar{\Omega}_{\boldsymbol{u}}$$

$$\int_\Omega \int_t \left(\rho_d \ddot{d} + \mu_d \dot{d} + \left(\frac{\partial \Psi}{\partial d} + G_c \frac{\partial \gamma}{\partial d} - G_c \boldsymbol{\nabla}.\left(\frac{\partial \gamma}{\partial \underline{\boldsymbol{\nabla d}}} \right) \right) \right) \delta d\, dt\, d\tilde{x} + \int_{\partial\Omega} \frac{\partial \gamma}{\underline{\boldsymbol{\nabla d}}} \delta d.\tilde{n}_s ds = 0, \qquad \forall\, \delta d \in \partial\bar{\Omega}_d$$

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- Energetic crack driving force : $Y = -\frac{\partial \Psi}{\partial d} = -g'(d)\frac{\partial \Psi}{\partial g} = -g'(d)H(\underline{\boldsymbol{u}})$, g a degradation function

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Outline

Energy density split method

Spherical/deviatoric decomposition *[Kotsovos, 1979; Ladeveze, 1983;…; Amor, 2008]*

$$\Psi^+ = \frac{1}{2}\lambda_0\langle tr(\boldsymbol{\varepsilon})\rangle_+^2 + \mu_0\, \boldsymbol{dev}(\boldsymbol{\varepsilon}) : \boldsymbol{dev}(\boldsymbol{\varepsilon}),$$

$$\Psi^- = \frac{1}{2}\lambda_0\langle tr(\boldsymbol{\varepsilon})\rangle_-^2$$

$$\boldsymbol{\sigma}(d,\boldsymbol{\varepsilon}) = g(d)\left(\lambda_0\langle tr(\boldsymbol{\varepsilon})\rangle_+ \boldsymbol{I} + \mu_0 \boldsymbol{dev}(\boldsymbol{\varepsilon})\right) + \left(\lambda_0\langle tr(\boldsymbol{\varepsilon})\rangle_-\right)$$

CONS :

- Developped from isotropic material
- Loss of the thermodynamical framework (loss of the symmetry of the stiffness, continuity of the stress)

Spectral decomposition on the strain (or stress)

[Mazars et al. 1989, Lemaitre et al. 2000,…Miehe, 2010 ; Cevera, 2021]

$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^+ + \boldsymbol{\varepsilon}^-$, with $\boldsymbol{\varepsilon}^\pm = \sum_{n=1}^{3}\langle \varepsilon_n\rangle_\pm \underline{p_n}\otimes\underline{p_n}$,

$(\varepsilon_n, \underline{p_n})$: nth eigenvalue, eigenvector

$\mathbb{D}^\pm(\boldsymbol{\varepsilon}) = \partial_{\boldsymbol{\varepsilon}}[\boldsymbol{\varepsilon}^\pm]$ (projection tensor)

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, d) = \mathbb{C}(d) : \boldsymbol{\varepsilon},$$

$$\text{with } \mathbb{C}(d) = g(d)\,\mathbb{D}^+ : (\mathbb{C} : \mathbb{D}^+) + \mathbb{D}^- : (\mathbb{C} : \mathbb{D}^-)$$

Isotropic elasticity assumption : $\mathbb{C} = 2\boldsymbol{\mu}\mathbb{I} + \boldsymbol{\lambda}\mathbf{I}\otimes\mathbf{I}$,

$$\Psi^\pm = \frac{1}{2}\lambda_0\langle tr(\boldsymbol{\varepsilon})\rangle_\pm^2 + \mu_0\boldsymbol{\varepsilon}^\pm : \boldsymbol{\varepsilon}^\pm$$

$$\boldsymbol{\sigma}(d,\boldsymbol{\varepsilon}) = g(d)\,(\lambda_0\langle tr(\boldsymbol{\varepsilon})\rangle_+ \boldsymbol{I} + \mu_0\boldsymbol{\varepsilon}^+) + (\lambda_0\langle tr(\boldsymbol{\varepsilon})\rangle_- + \mu_0\boldsymbol{\varepsilon}^-)$$

Energy density split method

Orthogonal decomposition on the elastic tensor $\mathbb{C}$

[Marc François, 1998, Desmorat, 2000; He and Shao, 2019; …]

$$\Psi^{\pm} = \frac{1}{2}\boldsymbol{\varepsilon}^{\pm}\colon\mathbb{C}\colon\boldsymbol{\varepsilon}^{\pm} \text{ with } \boldsymbol{\varepsilon}^{+}\colon(\mathbb{C}\colon\boldsymbol{\varepsilon}^{-}) = \mathbf{0}$$

Kelvin decomposition of the stiffness tensor :

$\mathbb{C} = \sum_{i=1}^{3}\Lambda_i\boldsymbol{\omega_i}\otimes\boldsymbol{\omega_i}$, 4th order elastic tensor $\mathbb{C}$

$\Lambda_i, \boldsymbol{\omega_i}$: eigenvalue, 2nd order eigentensor of $\mathbb{C}$

- $\mathbb{C}^{\pm 1/2} = \sum_{i=1}^{3}\Lambda_i{}^{\pm 1/2}\boldsymbol{\omega_i}\otimes\boldsymbol{\omega_i}$
- $\tilde{\boldsymbol{\varepsilon}} = \mathbb{C}^{1/2}\boldsymbol{\varepsilon}$ (transformed strain),
- $\tilde{\boldsymbol{\varepsilon}}^{\pm}$: positive and negative part from the spectral decomposition of $\tilde{\boldsymbol{\varepsilon}}$
- $\widetilde{\mathbb{P}}^{\pm}(\tilde{\boldsymbol{\varepsilon}}) = \partial_{\tilde{\boldsymbol{\varepsilon}}}[\tilde{\varepsilon}^{\pm}]$ (projection tensor) for implicit solver
- $\boldsymbol{\varepsilon}^{\pm} = \mathbb{C}^{-1/2}\tilde{\boldsymbol{\varepsilon}}^{\pm}$

$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, d) = \mathbb{C}(d)\colon\boldsymbol{\varepsilon},$$

with $\mathbb{C}(d) = g(d)\mathbb{P}^{+}\colon(\mathbb{C}:\mathbb{P}^{+}) + \mathbb{P}^{-}\colon(\mathbb{C}:\mathbb{P}^{-})$ and

$\mathbb{P}^{\pm} = \mathbb{C}^{-1/2}\colon(\widetilde{\mathbb{P}}^{\pm}\colon\mathbb{C}^{+1/2})$

PROS : Anisotropic material, explicit resolution no need of the projectors $\mathbb{P}^{\pm}$

Outline

MFront Implementation : Orthogonal split

➢ Variable definition

```
Yg.setGlossaryName("YoungModulus");
@MaterialProperty real nu;
nu.setGlossaryName("PoissonRatio");
@ComputeStiffnessTensor<UnAltered> { "Yg" , "nu" };

@Parameter real kres = 1e-6;  // residual stiffness

@StateVariable real H;
H.setEntryName("HistoryFunction");
@StateVariable real psi_p;
psi_p.setEntryName("PositiveEnergyDensity");

@ExternalStateVariable real psi_0;
psi_0.setEntryName("ThresholdEnergyDensity");
@ExternalStateVariable real d;
d.setGlossaryName("Damage");
```

Isotrop linear elastic

$$\mathbb{C} = \frac{\mathrm{E}}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & (1-2\nu)/2 & 0 & 0 \\ & & & & (1-2\nu)/2 & 0 \\ & & & & & (1-2\nu)/2 \end{bmatrix}$$

$$H = \max_{0\le\tau\le T}(0, \Psi^{+}(\varepsilon,\tau) - \Psi_0)$$

https://thelfer.github.io/mgis/web/mgis_fenics_phase_field.html

MFront Implementation : Integrator block


```
@Integrator {
  constexpr const strain emin = 1.e-12;
  constexpr const auto id4 = Stensor4::Id();
  constexpr const auto esolver = StrainStensor::
   GTESYMMETRICQREIGENSOLVER;

  // call diagonalize() function
  auto [vp, ns, Dp1_2, Dn1_2] = diagonalize(D);

  // positive part
  const auto f = [](const real x) { return x > 0 ? x : 0; };
  // derivative of the positive part
  const auto df = [&emin](const real x) {
    return std::abs(x) < emin ? 0.5 : ((x < 0) ? 0 : 1); };

  // total strain
  const auto e = eto + deto;
  // transformed strain tensor
  const auto et = eval(Dp1_2 * e);
  // Positive part of 'et' tensor and its projector
  const auto [etp,Ptp] = et.template
   computeIsotropicFunctionAndDerivative<esolver>(
      f, df, emin * 0.1);
  // Negative part of 'et' tensor
  const auto etn = et - etp;
  // Positive and Negative of the strain
  const auto ep = Dn1_2 * etp;
  const auto en = Dn1_2 * etn;

  // Degradation function
  const auto g = ((1-d)*(1-d)) +kres ;
  // Compute stress
  sig = g * D * ep + D * en;

  // positive elastic energy density and history H
  psi_p = 0.5*(ep|(D*ep));
  const auto h0 = max(psi_p-psi_0, real(0));
  H = max(H, h0);
  const strain tr  = trace(e);

  // Tangent operator
  static_cast<void>(smt);
  //static_cast<void>(computeTangentOperator_);
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$$\tilde{\boldsymbol{\varepsilon}} = \mathbb{D}^{1/2}\boldsymbol{\varepsilon}$$

$$\tilde{\boldsymbol{\varepsilon}}^{+}, \tilde{\mathbb{P}}^{+}(\tilde{\boldsymbol{\varepsilon}})$$

$$\tilde{\boldsymbol{\varepsilon}}^{-} = \tilde{\boldsymbol{\varepsilon}} - \tilde{\boldsymbol{\varepsilon}}^{+}$$

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$$\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, d) = g(d)\mathbb{C}\boldsymbol{\varepsilon}^{+} + \mathbb{C}\boldsymbol{\varepsilon}^{-}$$

$$H = \max_{0 \le \tau \le T}(0, \Psi^{+}(\varepsilon, \tau) - \Psi_0)$$

MFront Implementation : Kelvin decomposition (1/2)


```
@OrthotropicBehaviour<Pipe>;

@Includes {

    #ifndef TFEL_MATH_ST2TOST2_DIAGONALIZE
    #define TFEL_MATH_ST2TOST2_DIAGONALIZE

    namespace tfel::math {
      /*!
      * return the eigenvalues and eigen tensors of the of a
   given orthotropic
      * stiffness tensor, as well as its positive square root
   and inverse of the positive square roots
      */
      template <unsigned short N, typename ValueType>
      std::tuple<
          tvector<StensorDimeToSize<N>::value, ValueType>,
          tvector<StensorDimeToSize<N>::value,
                stensor<N, base_type<ValueType>>>,
          const st2tost2<N, ValueType>,
          const st2tost2<N, ValueType>>
```

```
    diagonalize(const st2tost2<N, ValueType> &C) {
      using real = base_type<ValueType>;
      auto e = [](const unsigned short i) constexpr {
              auto s = stensor<N, real>(real(0));
              s[i] = 1;
              return s;
              };
      constexpr const auto cste = Cste<ValueType>::sqrt2;
      constexpr auto e1 = e(0);
      constexpr auto e2 = e(1);
      constexpr auto e3 = e(2);
      auto tmp =
      stensor<3u, ValueType>{C(0, 0), C(1, 1),
                             C(2, 2), cste * C(0, 1),
                              cste * C(0, 2), cste * C(1, 2)
};
      const auto [vp, m] = tmp.template computeEigenVectors<
          stensor_common::FSESJACOBIEIGENSOLVER>();
      auto ns = tvector<StensorDimeToSize<N>::value, stensor<
N, real>>{};
      ns[0] = m(0, 0) * e1 + m(1, 0) * e2 + m(2, 0) * e3;
      ns[1] = m(0, 1) * e1 + m(1, 1) * e2 + m(2, 1) * e3;
      ns[2] = m(0, 2) * e1 + m(1, 2) * e2 + m(2, 2) * e3;
```

Eigenvectors, ns[0], ns[1], ns[2]

MFront Implementation : Kelvin decomposition (2/2)


```
      if constexpr (N == 1) {
        // positive and negative square root
        // of stiffness tensor C
        const auto Cp1_2 =
        power<1,2>(vp(0)) * (ns[0] ^ ns[0]) +
        power<1,2>(vp(1)) * (ns[1] ^ ns[1]) +
        power<1,2>(vp(2)) * (ns[2] ^ ns[2]);
        const auto Cn1_2 =
        power<-1, 2>(vp(0)) * (ns[0] ^ ns[0]) +
        power<-1, 2>(vp(1)) * (ns[1] ^ ns[1]) +
        power<-1, 2>(vp(2)) * (ns[2] ^ ns[2]);
        return {vp, ns, Cp1_2, Cn1_2};
      } else if constexpr (N == 2) {
        ns[3] = e(3);
        const auto Cp1_2 =
        power<1, 2>(vp(0)) * (ns[0] ^ ns[0]) +
        power<1, 2>(vp(1)) * (ns[1] ^ ns[1]) +
        power<1, 2>(vp(2)) * (ns[2] ^ ns[2]) +
        power<1, 2>(C(3, 3)) * (ns[3] ^ ns[3]);
        const auto Cn1_2 =
        power<-1, 2>(vp(0)) * (ns[0] ^ ns[0]) +
        power<-1, 2>(vp(1)) * (ns[1] ^ ns[1]) +
        power<-1, 2>(vp(2)) * (ns[2] ^ ns[2]) +
        power<-1, 2>(C(3, 3)) * (ns[3] ^ ns[3]);
        return {tvector<4u, ValueType>{
          vp(0), vp(1), vp(2), C(3, 3)}, ns, Cp1_2, Cn1_2};
```

$$\mathbb{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 \\ & C_{22} & C_{23} & 0 \\ & & C_{33} & 0 \\ & & & C_{44} \end{bmatrix}$$

```
      } else {
        ns[3] = e(3);
        ns[4] = e(4);
        ns[5] = e(5);
        const auto Cp1_2 =
        power<1, 2>(vp(0)) * (ns[0] ^ ns[0]) +
        power<1, 2>(vp(1)) * (ns[1] ^ ns[1]) +
        power<1, 2>(vp(2)) * (ns[2] ^ ns[2]) +
        power<1, 2>(C(3, 3)) * (ns[3] ^ ns[3]) +
        power<1, 2>(C(4, 4)) * (ns[4] ^ ns[4]) +
        power<1, 2>(C(5, 5)) * (ns[5] ^ ns[5]);
        const auto Cn1_2 =
        power<-1, 2>(vp(0)) * (ns[0] ^ ns[0]) +
        power<-1, 2>(vp(1)) * (ns[1] ^ ns[1]) +
        power<-1, 2>(vp(2)) * (ns[2] ^ ns[2]) +
        power<-1, 2>(C(3, 3)) * (ns[3] ^ ns[3]) +
        power<-1, 2>(C(4, 4)) * (ns[4] ^ ns[4]) +
        power<-1, 2>(C(5, 5)) * (ns[5] ^ ns[5]);
        return {tvector<6u, ValueType>{
          vp(0), vp(1), vp(2), C(3, 3), C(4, 4), C(5, 5)
          }, ns, Cp1_2, Cn1_2};
      }
    }  // end of diagonalize

  }  // end of namespace tfel::math

  #endif
}
```

$$\mathbb{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}$$

Outline

Implementation of a flexible environment within a toolbox

- **Toolbox** implemented to test multiple resolution strategy through a channel

(**) https://github.com/FEniCS

Displacement
Damage

Field
- Space function
- Dirichlet, Load
- Mesh features
- parameters

Elliptic
Parabolic
Hyperbolic

Formulation

Implicit method
(Newton Raphson)
Explicit integrator

Numerical resolution strategy

Metric error estimation
Balance energy
Method of crack tracking a(t)

Post-processing

Implementation of a flexible environment within a toolbox

➢ **Toolbox** implemented to test multiple resolution strategy through a channel

(**) https://github.com/FEniCS

Field
- Space function
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- Mesh features
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Displacement
Damage

Elliptic
Parabolic
Hyperbolic

Formulation

Implicit method
(Newton Raphson)
Explicit integrator

Numerical resolution strategy

Metric error estimation
Balance energy
Method of crack tracking a(t)

Post-processing

Constitutive law (*) :
- **traction/compression splitting law**

(Deviatoric/volumetric, spectral and orthogonal decomposition)

(*) https://github.com/thelfer/

$[\boldsymbol{\nabla u}]^{k-1}, [\boldsymbol{\varepsilon}]^{k-1}$

Solver

MFront/MGIS

$\Psi([\varepsilon], d)^k, \left(\frac{\partial \Psi}{\partial [\boldsymbol{\varepsilon}]}\right)^k, \left(\frac{\partial^2 \Psi}{\partial [\boldsymbol{\varepsilon}]^2}\right)^k$

(∗) MFront/MGIS : A code generator for material constitutive models knowledge

Benchmark with reference solution and **sensitivity analysis**
- Mean square error estimation on physical quantities

Implementation Coupling with FEniCSx


```python
def set_mfront_environnement(self):
    hypothesis = mgis_bv.Hypothesis.PlaneStrain
    split_model_path = self.problem.parameters.mfront.ngy
    self.mfront_behaviour = mgis_bv.load(
                        split_model_path.lib,
                        split_model_path.behavior_label,
                        hypothesis)
    local_ncells = self.problem.msh.topology.index_map(
                self.problem.msh.topology.dim).size_local
    num_ghost_cell = self.problem.msh.topology.index_map(
                self.problem.msh.topology.dim).num_ghosts
    ngauss  = 4*(local_ncells + num_ghost_cell)
    self.mfront_mdm = mgis_bv.MaterialDataManager(
                self.mfront_behaviour, ngauss)


    for s in [self.mfront_mdm.s0, self.mfront_mdm.s1]:
        mgis_bv.setMaterialProperty(s, "YoungModulus",
                                    self.problem.mp.E)
        mgis_bv.setMaterialProperty(s, "PoissonRatio",
                                    self.problem.mp.nu)
        mgis_bv.setExternalStateVariable(s,"Temperature",
                                         293.15e0)
        mgis_bv.setExternalStateVariable(s, "Damage",0e0)
        psi_0 = self.thresold_psi0()
        mgis_bv.setExternalStateVariable(s,
                        "ThresholdEnergyDensity", psi_0)
```

https://github.com/thelfer/MFrontGenericInterfaceSupport/tree/master/bindings/python/mgis

```python
def sigma_from_mfront(self):
    mgis_bv.update(self.mfront_mdm)
    self.quadrature_local_projection(self.problem.msh,
            epsilon_mfront(self.fp.u_cg),
            self.fp.strain_quad.function_space,
            self.fp.strain_quad)

    self.mfront_mdm.s1.gradients[:,:] =
        self.fp.strain_quad.x.array.reshape(
        (self.mfront_mdm.n,
        self.fp.stress_quad.ufl_shape[0]))

    res = mgis_bv.integrate(self.mfront_mdm,
                    optTgOp,
                    self.problem.time_stepper.dt,
                    0,
                    self.mfront_mdm.n)

    self.set_stress(
        self.mfront_mdm.s1.thermodynamic_forces.flatten()
)
```

```python
def set_external_data_into_mfront(self):
    self.quadrature_local_projection(self.problem.msh,
                    self.fp.alpha_cg,
                    self.fp.alpha_quad.function_space,
                    self.fp.alpha_quad)
    mgis_bv.setExternalStateVariable(self.mfront_mdm.s1,
                'Damage', self.fp.alpha_quad.x.array,
    mgis_bv.MaterialStateManagerStorageMode.LocalStorage)
```

Implementation Coupling with FEniCSx


```
    def set_mfront_environnement(self):
        hypothesis = mgis_bv.Hypothesis.PlaneStrain
        split_model_path = self.problem.parameters.mfront.ngy
        self.mfront_behaviour = mgis_bv.load(
                            split_model_path.lib,
                            split_model_path.behavior_label,
                            hypothesis)
        local_ncells = self.problem.msh.topology.index_map(
                    self.problem.msh.topology.dim).size_local
        num_ghost_cell = self.problem.msh.topology.index_map(
                    self.problem.msh.topology.dim).num_ghosts
        ngauss  = 4*(local_ncells + num_ghost_cell)
        self.mfront_mdm = mgis_bv.MaterialDataManager(
                    self.mfront_behaviour, ngauss)


        for s in [self.mfront_mdm.s0, self.mfront_mdm.s1]:
            mgis_bv.setMaterialProperty(s, "YoungModulus",
                                        self.problem.mp.E)
            mgis_bv.setMaterialProperty(s, "PoissonRatio",
                                        self.problem.mp.nu)
            mgis_bv.setExternalStateVariable(s,"Temperature",
                                             293.15e0)
            mgis_bv.setExternalStateVariable(s, "Damage",0e0)
            psi_0 = self.thresold_psi0()
            mgis_bv.setExternalStateVariable(s,
                            "ThresholdEnergyDensity", psi_0)
```

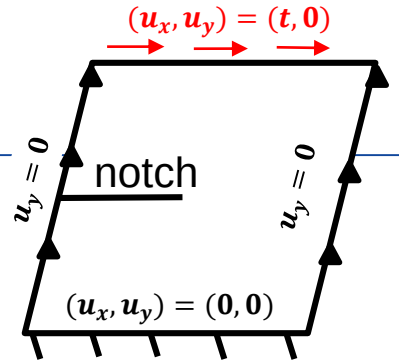
https://github.com/thelfer/MFrontGenericInterfaceSupport/tree/master/bindings/python/mgis

```
    def sigma_from_mfront(self):
        mgis_bv.update(self.mfront_mdm)
        self.quadrature_local_projection(self.problem.msh,
                epsilon_mfront(self.fp.u_cg),
                self.fp.strain_quad.function_space,
                self.fp.strain_quad)

        self.mfront_mdm.s1.gradients[:,:] =
            self.fp.strain_quad.x.array.reshape(
            (self.mfront_mdm.n,
            self.fp.stress_quad.ufl_shape[0]))

        res = mgis_bv.integrate(self.mfront_mdm,
                        optTgOp,
                        self.problem.time_stepper.dt,
                        0,
                        self.mfront_mdm.n)

        self.set_stress(
            self.mfront_mdm.s1.thermodynamic_forces.flatten()
    )
```

```
    def tangent_matrix(self):
        self.fp.K_tangent_quad.x.array[:]= self.mfront_mdm.K.
    flatten()

    def history_miehe(self):
        self.fp.H_quad.x.array[:] = self.mfront_mdm.s1.
    internal_state_variables[:,0]
```

Outline

First numerical results (Quasi-static regime)



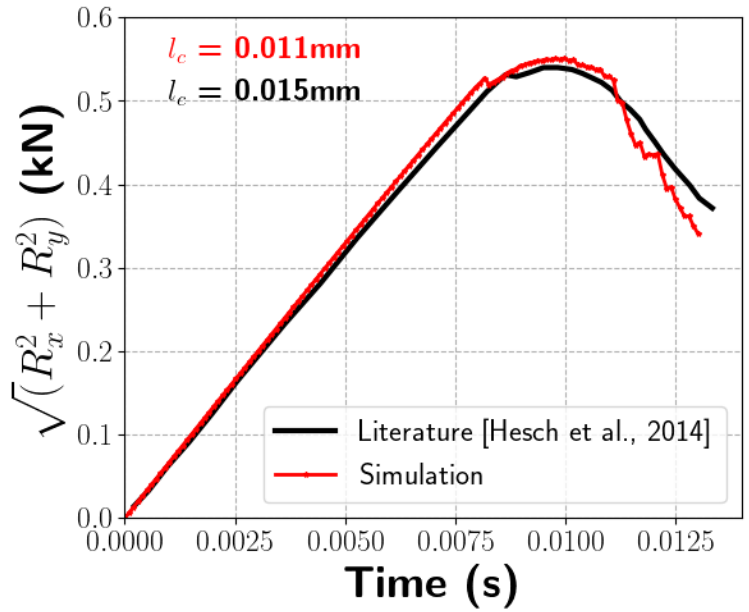
➢ Single edge notch shear test (SENS)

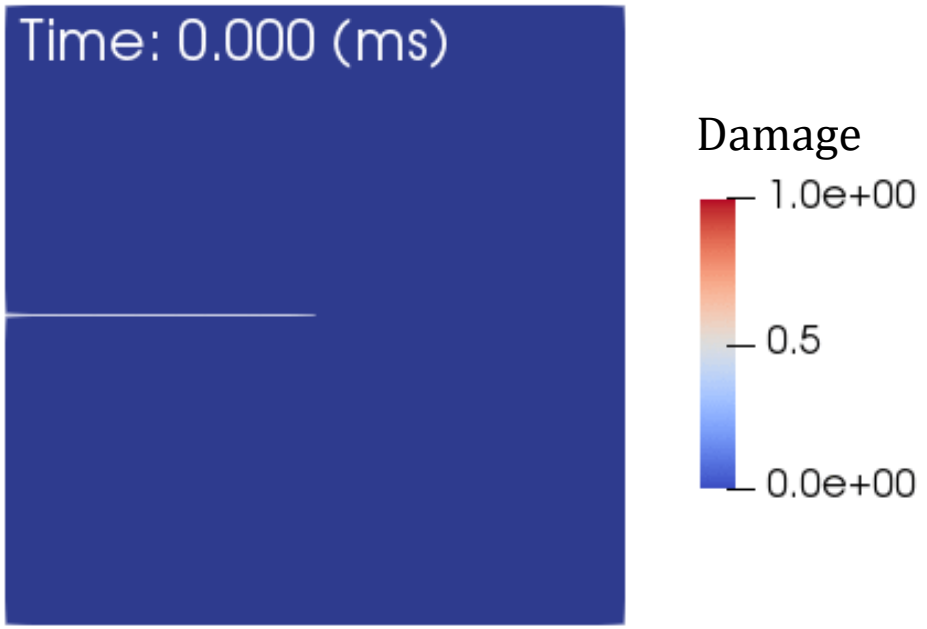
Strategy : Elliptic PDE (u-problem) and Elliptic PDE (d-problem) ➔ Implicit / Implicit

Mesh features : 256*256 elements,

Loading step : $\Delta u_x = 10^{-4} mm$, $\Delta u_x = 10^{-6} mm$

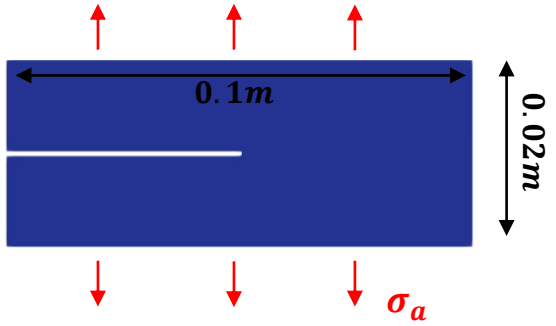
Spectral decomposition split *[Miehe et al., 2010]*





Dynamic simulation cases

Crack branching



$Mesh$: ~ 1.02 million of elements, $h_{mesh} \approx 10^{-5}m$

Constant given strength : $\sigma_a = 300$ MPa

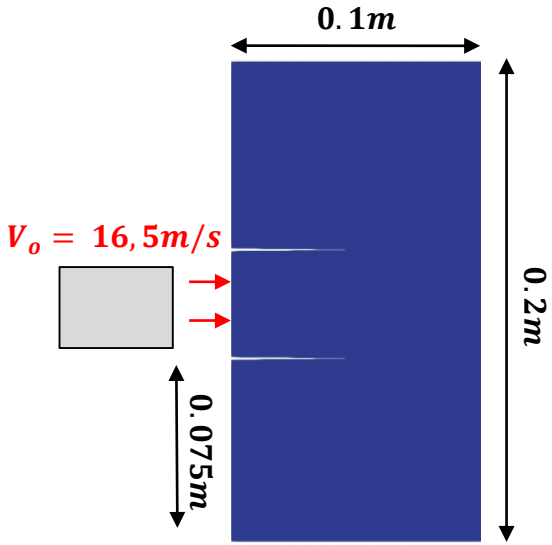
Steel material parameters :

$E = 190GPa$, $\rho = 8000kg/m^3$,
$\nu = 0.3$, $G_c = 22130J/m$

Traction/compression asymmetry : Orthogonal decomposition

Internal length $l_c = 3\, h_{mesh}$

Kalthoff and Winkler test *[Kalthoff et al., 2000]*

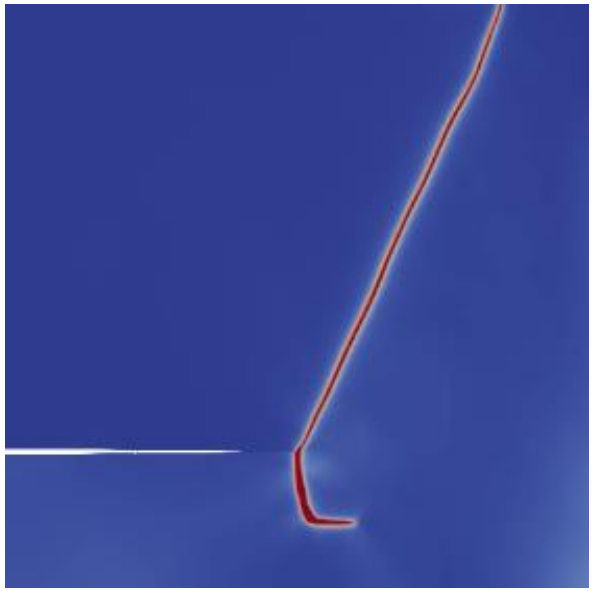


$Mesh$: ~ 1 million of elements, $h_{mesh} \sim 10^{-4}m$,

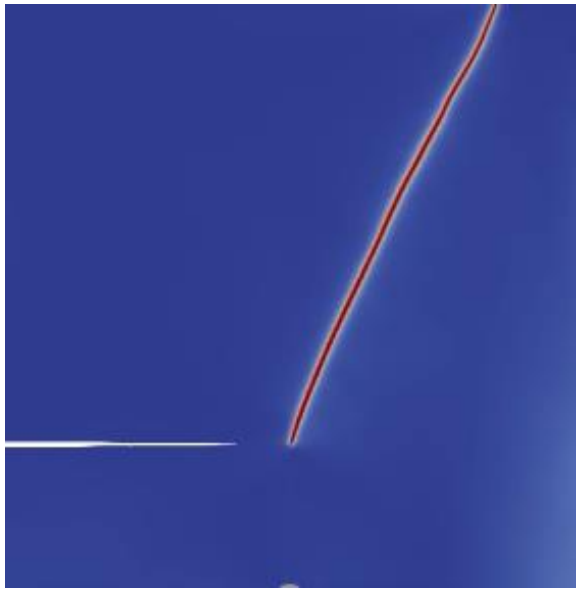
Given displacement :

$$\overrightarrow{u_a}.\vec{x} = \begin{cases} \frac{1}{2}\frac{t^2}{T_o}V_o\,, t < T_o \\ V_o(t - T_o) + \frac{1}{2}T_oV_o, \quad t \geq T_o \end{cases}$$

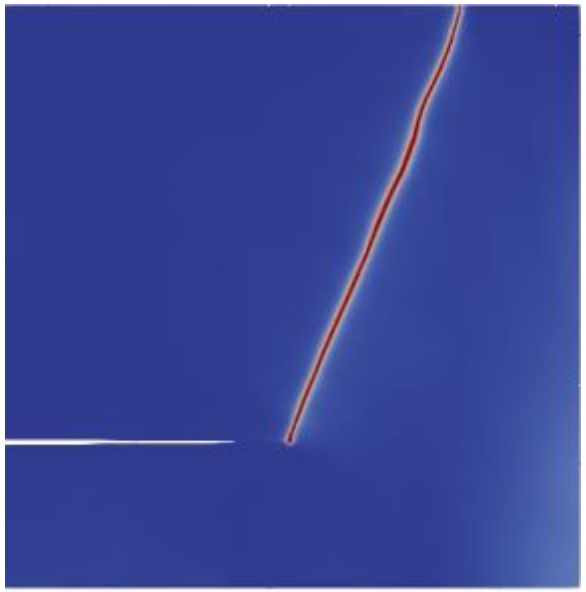
Comparison of the energy density splits



Spherical/Deviatoric split

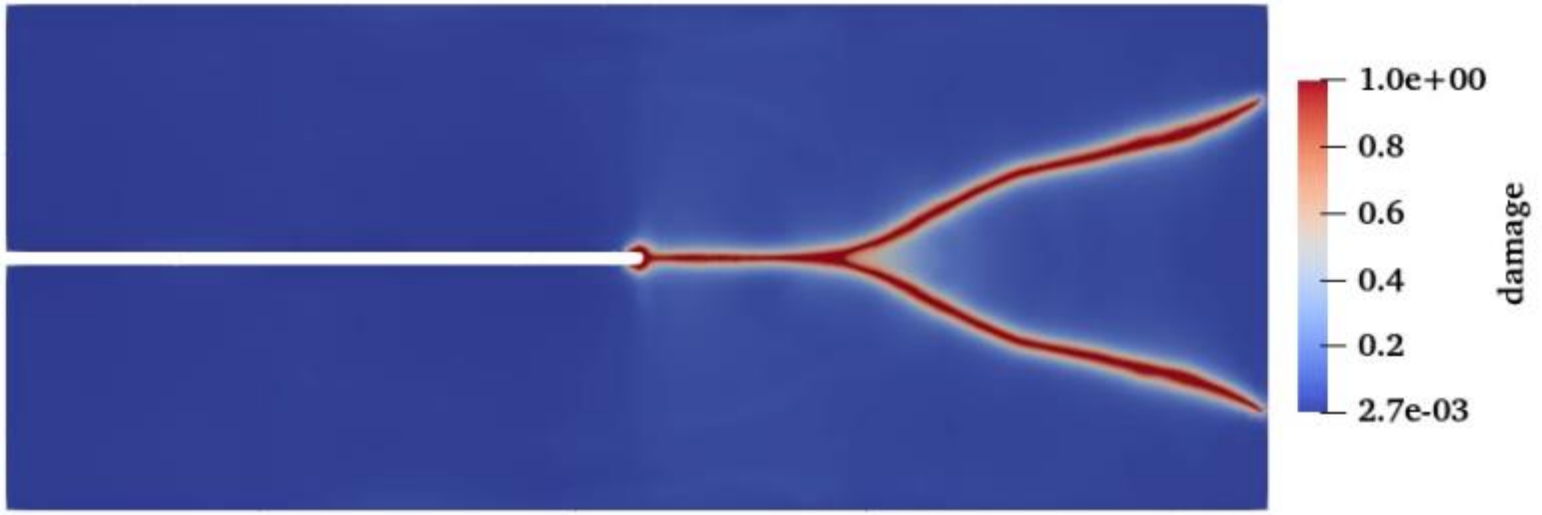


Spectral decomposition on the strain



Orthogonal decomposition (Spectral decomposition on the elastic tensor)

Crack branching – orthogonal split



Discussion and perspectives

- Implementation on FEniCSx (0.6 version) coupling with Mfront/MGIS codes (4.2.0 version)

Extension or improvement :

- Extension of the orthogonal split to higher order of material anisotropy (triclinic, monoclinic …)
- Only a unique damage scalar used ➔ Damage anisotropy (tensorial form) not involve
- Implicit resolution not perform (Tangent operator not computed in my case)

Challenge

- Big challenge mixing transient regime and anisotropic material

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Thank you for your attention

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