UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

Self Learning Material

M.Sc. Mathematics

Fourth Semester

(2019 Admission Onwards)

MTH4E05: ADVANCED COMPLEX ANALYSIS

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MTH4E05: ADVANCED COMPLEX ANALYSIS

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Preface

The Self Learning Material MTH4E05: ADVANCED COMPLEX ANALYSIS is prepared based on the syllabus for M.Sc. Mathematics (CBCSS) PG Programme of University of Calicut effective from 2019 admission onwards. The material is mainly intended for helping the students who are studying M.Sc. Mathematics course under the School of Distance of Education, University of Calicut.

The material is prepared based on the text book **JOHN B. CONWAY:** FUNCTIONS OF ONE COMPLEX VARIABLE (*2nd Edn.*) published by *Springer International Student Edition; 1992.*

In MTH4E05: ADVANCED COMPLEX ANALYSIS we continue the discussion of the theory of functions of one complex variables discussed in the study material MTH3C12: COMPLEX ANALYSIS. The prerequisites for reading this book are quite minimal; not much more than a stiff course in basic calculus and a few facts about partial derivatives.

Compactness and convergence in space of analytic functions is discussed in chapter 1. The Riemann mapping theorem is discussed in section 1.4 and the Weierstrass factorization theorem is discussed in section 1.5.

In section 2.1 in chapter 2, we discuss factorization of the sine function. The gamma function, the Riemann zeta function and Runge's theorem are also discussed in the remaining sections. Simple connectedness is the content of the section 2.5.

Mittag-Leffler's theorem is discussed in section 3.1 of chapter 3. Schwarz reflection principle, monodromy theorem, Jensen's formula and Hadamard factorization theorem are discussed in the remaining sections.

A quick review of stereographic projections and uniform convergence of improper integrals are done in the appendices.

Throughout this course we use the following notations:

$\mathbb{N} = \{1, 2, \ldots\}$, the set of natural numbers

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of integers

$\mathbb{R}$, the set of real numbers

$\mathbb{C}$, the set of complex numbers

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Chapter 1

Compactness and Convergence in Space of Analytic Functions

In this chapter a metric is put on the set of all analytic functions on a fixed region G, and compactness and convergence in this metric space is discussed. Among the applications obtained is a proof of the Riemann Mapping Theorem.

Actually some more general results are obtained which enable us to also study spaces of meromorphic functions.

1.1 The Space of Continuous Functions $C(G,\ \Omega)$

In this chapter $(\Omega,\, d)$ will always denote a complete metric space. Although much of what is said does not need the completeness of Ω, those results which hold the most interest are not true if $(\Omega,\, d)$ 1s not assumed to be complete.

Definition 1.1.1. If G is an open set in $\mathbb{C}$ and $(\Omega,\, d)$ is a complete metric space then designate by $C(G,\ \Omega)$ the set of all continuous functions from G to Ω. That is,

$$C(G,\ \Omega) = \{f : G \to \Omega,\ \ f \text{ is continuous.}\}$$

The following result tells that the set $C(G,\ \Omega)$ is never empty.

Proposition 1.1.2. $C(G,\ \Omega) \neq \Phi$.

Proof. Fix an element $\omega_0 \in \Omega$. Define $f : G \to \Omega$ by

$$f(g) = \omega_0$$

for $g \in G$. Then clearly (Why?) the constant function f is a continuous map from G to Ω and hence $f \in C(G,\ \Omega)$. This shows that $C(G,\ \Omega) \neq \Phi$. $\square$

Example 1.1.3. An example where $C(G,\ \Omega)$ contains *ONLY* constant functions. Suppose G is *connected* and $\Omega = \mathbb{N} = \{1,\, 2,\, \ldots\}$. If $f \in C(G,\ \Omega)$, then $f(G)$, being the continuous image of a connected set, is a connected subset of $\Omega = \mathbb{N}$, so [1] the only possibility is that $f(G) = \{n_0\}$ for some $n_0 \in \mathbb{N}$. This shows that

$$f(g) = n_0 \text{ for } g \in G.$$

Hence f is a constant map. Since the choice of $f \in C(G, \Omega)$ is arbitrary, this shows that every element in $C(G, \Omega)$ is a constant function.

Figure 1.1: Connected subsets of $\mathbb{N}$ are singleton sets consisting of natural numbers.

Hereinafter throughout our discussion we consider those Ω that is either $\mathbb{C}$ or $\mathbb{C}_\infty$. For these two choices of Ω, $C(G, \Omega)$ has many nonconstant elements. For example, each analytic function on G is in $C(G, \mathbb{C})$ and each meromorphic function on G is in $C(G, \mathbb{C}_\infty)$.

Proposition 1.1.4. *Each analytic function on G is in $C(G, \mathbb{C})$.*

Proof. This follows from the fact that each analytic function is a continuous function. $\square$

[1] Connected subsets of $\mathbb{N}$ are singleton sets consisting of natural numbers.

Proposition 1.1.5. *Each meromorphic function on G is in $C(G,\ \mathbb{C}_\infty)$.*

Proof. Let f be a meromorphic function on G. Then, by the definition of a meromorphic function, f is analytic everywhere except at the poles. Hence f is continuous every where except at the poles. We define

$$f : G \to \mathbb{C}_\infty$$

by setting

$$f(z) = \infty \ \text{ whenever } \ z \text{ is a pole of } f.$$

We claim that $f : G \to \mathbb{C}_\infty$ is continuous at poles also. At a point z, which is a pole of f,

$$\lim_{w \to z} f(w) = \infty$$

and already, by the definition of f,

$$f(z) = \infty$$

and hence, also at the point z,

$$\lim_{w \to z} f(w) = f(z)$$

showing that f is continuous at pole points also. Hence $f \in$

$C(G,\ \mathbb{C}_\infty)$. □

Proposition 1.1.6. *Proposition VII.1.2 (Page 142, Conway)* *If G is an open set in $\mathbb{C}$ then there is a sequence $\{K_n\}$ of compact subsets of G such that*

$$G = \bigcup_{n=1}^{\infty} K_n. \tag{1.1}$$

Moreover, the sets K_n can be chosen to satisfy the following conditions:

(a) $K_n \underset{\neq}{\subset} \mathrm{int}(K_{n+1})$

(b) $K \subset G$ *and* K *is compact implies* $K \subset K_n$ *for some* n;

(c) Every component of $\mathbb{C}_\infty -\ K_n$ *contains a component of* $\mathbb{C}_\infty -\ G$.

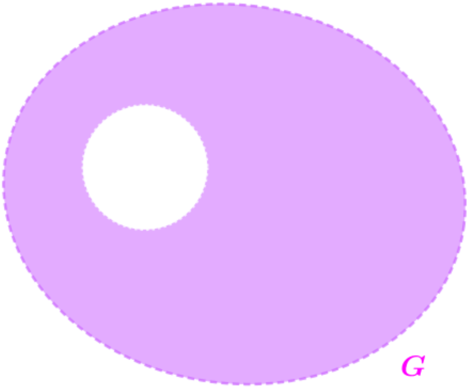

Figure 1.2: An example. G is an open connected set with a hole. $\mathbb{C} - G$ has two components, one is bounded and the other one unbounded. $\mathbb{C}_\infty - G$ has two components, one is bounded; and another one unbounded component containing ∞ in the extended complex plane.

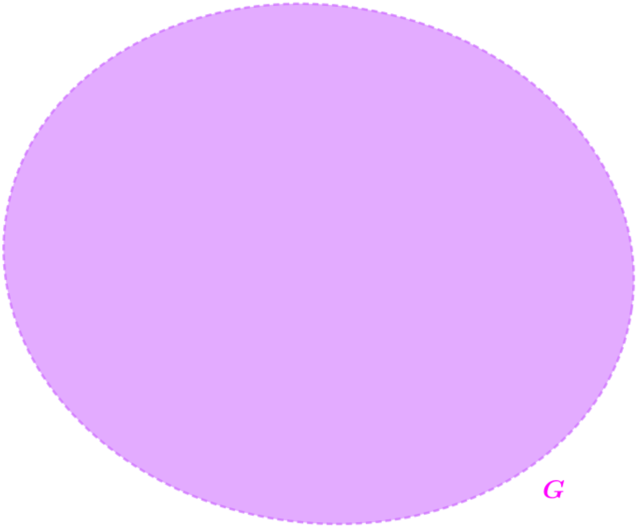

Figure 1.3: An example. G is an open connected set with no hole. $\mathbb{C} - G$ has only one component, which is unbounded. $\mathbb{C}_\infty - G$ has only one component, which is unbounded.

Proof. For each positive integer n, let

$$K_n = \{z : |z| \leqslant n\} \cap \{z : d(z,\ C - G) \geqslant 1/n\}; \qquad (1.2)$$

since K_n is clearly bounded and it is the intersection of two closed subsets of $\mathbb{C}$, K_n is compact.

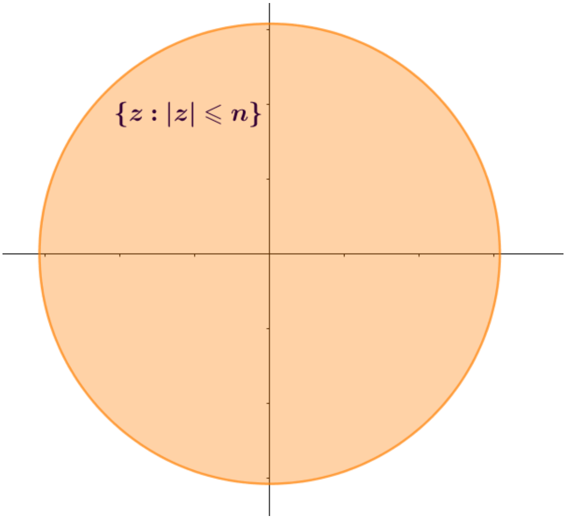

Figure 1.4: For each positive integer n, $\{z : |z| \leqslant n\}$, the closed disk centered at the origin and having radius n, is a closed and bounded subset of $\mathbb{C}$, and hence compact

`[Details:` For each positive integer n,

$$\{z : |z| \leqslant n\}$$

is bounded above by n and is a closed subset of $\mathbb{C}$. Hence

$$K_n = \{z : |z| \leqslant n\} \cap \{z : d(z,\ C - G) \geqslant 1/n\} \subset \{z : |z| \leqslant n\}$$

is also a bounded subset of $\mathbb{C}$.

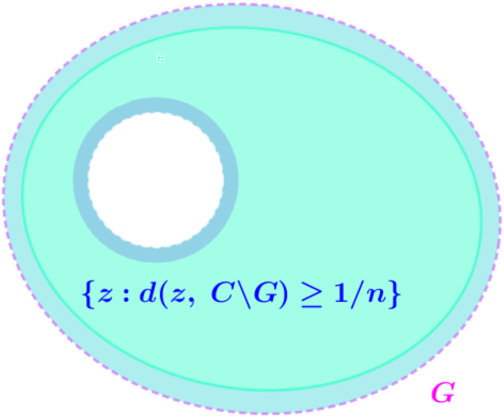

Figure 1.5: An example where G is an open set with a hole. Compare this figure with Figure 1.2. $\{z : d(z,\ \mathbb{C} - G) \geq 1/n\}$ is the set of all points in G that are at distance at most $1/n$ from the complement of G in $\mathbb{C}$.

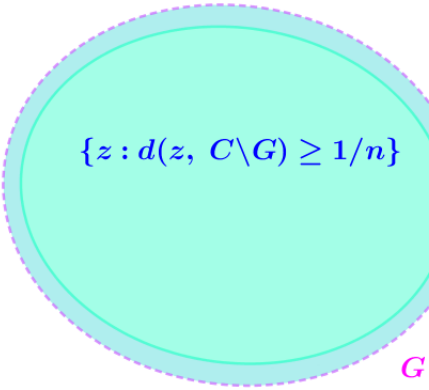

Figure 1.6: An example where G is an open set with no hole. Compare this figure with Figure 1.3. $\{z : d(z,\ \mathbb{C} - G) \geq 1/n\}$ is the set of all points in G that are at distance at most $1/n$ from the complement of G in $\mathbb{C}$.

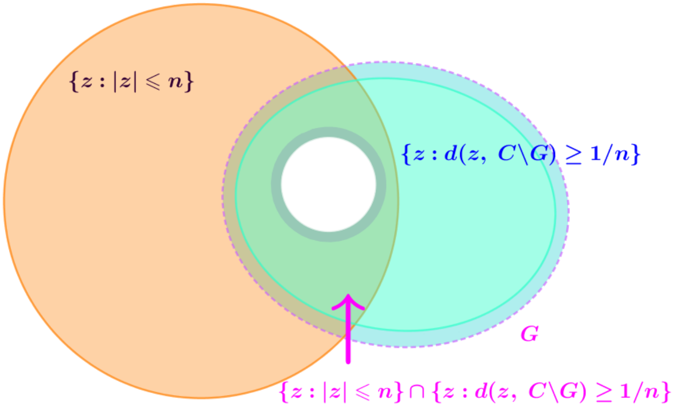

Figure 1.7: An example where G is an open set with a hole. $K_n = \{z : |z| \leq n\} \cap \{z : d(z,\ C - G) \geq 1/n\}$ is a closed and bounded subset of $\mathbb{C}$ and hence is compact. For each n, $\mathbb{C}_\infty - K_n$ has two components, one is bounded; and another one unbounded component containing ∞ in the extended complex plane. For each n, $\mathbb{C} - K_n$ also has two components; one bounded and other one unbounded.

Also

$$\underbrace{\{z : d(z,\ \mathbb{C} - G) \geqslant 1/n\}}_{=A_n,\ \text{a } \textit{closed} \text{ set } \textit{smaller} \text{ than } G}$$

is a closed subset of G because

$$\left\{z : d(z,\ \mathbb{C} - G) \geq \frac{1}{n}\right\} = \psi^{-1}\left([1/n,\ \infty)\right)$$

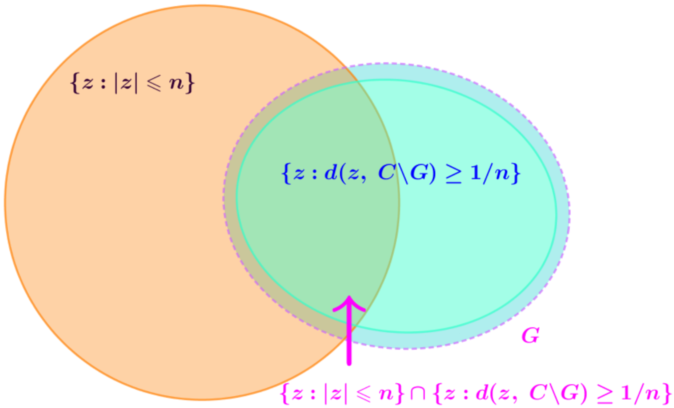

Figure 1.8: An example where G is an open set with no hole. $K_n = \{z : |z| \leq n\} \cap \{z : d(z,\ C - G) \geq 1/n\}$ is a closed and bounded subset of $\mathbb{C}$ and hence is compact. For each n, $\mathbb{C}_\infty - K_n$ has only one component, the unbounded component containing ∞ in the extended complex plane. For each n, $\mathbb{C} - K_n$ also has one component which is unbounded.

where $\psi : \mathbb{C} \to \mathbb{R}$ defined by

$$\psi(z) = d\ \underbrace{d(z,\ C - G)}_{\inf\{|z-w| : w \in C - G\}}$$

is a continuous function defined on $\mathbb{C}$ and hence

$$\underbrace{\psi^{-1}(\ \underbrace{[1/n,\ \infty)}_{\text{closed subset of } \mathbb{R}}\ )}_{\substack{\text{preimage of a closed set via inverse} \\ \text{of a continuous, function and hence closed in } \mathbb{C}}}$$

is a closed subset of $\mathbb{C}$.

Hence, being the intersection of these two closed sets, K_n is a closed subset of $\mathbb{C}$. Hence K_n is a closed and bounded subset of $\mathbb{C}$ and hence is compact. Define

$$V_{n+1} = \underbrace{\underbrace{\{z : |z| < n+1\}}_{open} \cap \underbrace{\{z : d(z,\ C - G) > 1/(n+1)\}}_{open}}_{open}$$

Then

$$K_n \underset{\neq}{\subset} V_{n+1} \underset{\neq}{\subset} K_{n+1}. \tag{1.3}$$

`[Details:`

$$\{z : |z| \leq n\} \subseteq \{z : |z| < n+1\}$$

and

$$\{z : d(z,\ C - G\} \geq n\} \subseteq \{z : d(z,\ C - G\} > n+1\}$$

so that

$$\underbrace{\{z : |z| \leq n\} \cap \{z : d(z,\ C - G\} \geq n\}}_{K_n}$$

is a subset of

$$\underbrace{\{z : |z| < n+1\} \cap \{z : d(z,\ C - G\} > n+1\}}_{V_{n+1}}.$$

Also

$$\{z : |z| < n+1\} \subseteq \{z : |z| \leq n+1\}$$

and

$$\{z : d(z,\ C\backslash G\} > n+1\} \subseteq \{z : d(z,\ C-G\} \geq n+1\}$$

gives

$$\underbrace{\{z : |z| < n+1\} \cap \{z : d(z,\ C-G\} > n+1\}}_{V_{n+1}}$$

is a subset of

$$\underbrace{\{z : |z| \leq n+1\} \cap \{z : d(z,\ C-G\} \geq n+1\}}_{K_{n+1}}.$$

]

Since $\mathrm{int}(K_{n+1})$ is the largest *open* subset of K_{n+1} and since

$$\underbrace{V_{n+1}}_{open} \underset{\neq}{\subset} K_{n+1},$$

it follows that

$$V_{n+1} \underset{\neq}{\subset} \mathrm{int} K_{n+1}$$

and hence (1.3) implies that $K_n \underset{\neq}{\subset} \mathrm{int}(K_{n+1})$. This proves Part (a).

It follows that (Why^1?)

$$G = \bigcup_{n=1}^{\infty} K_n$$

and from this we also get (Why^2?)

$$G = \bigcup_{n=1}^{\infty} \operatorname{int} K_n;$$

so if K is a compact subset of G the sets $\{\operatorname{int} K_n\}$ form an open cover of (G and in particular that of) K.

`Details:` Why^1:

$$K_n \subseteq G \ \forall n \Rightarrow \bigcup_{n=1}^{\infty} K_n \subseteq G.$$

Let $w \in G$. Then $\exists N_1 \in \mathbb{N}$ such that w is contained in the closed ball centered at 0 and having radius N_1 so that

$$|w| \leq N_1$$

(we can obtain this using the Archimedean property[2] of real numbers as follows: Suppose $|w| \geq n$ for all $n \in \mathbb{N}$, showing that $\mathbb{N}$ is bounded above by the real number $|w|$, which is a contradiction).

[2] **Archimedean Property (First Version):** *If $x \in \mathrm{R}$, then there exists $n_x \in \mathrm{N}$ such that $x < n_x$.*

Also, $w \in G$ implies $w \notin C - G = \overline{C - G}$ (as $C \backslash G$ is closed) which implies $d(w,\ C - G) > 0$. Hence, by the Archimedean property[3] of real numbers, $\exists N_2 \in \mathbb{N}$ such that

$$d(w,\ C - G) > \frac{1}{N_2}.$$

Take

$$N = \max\{N_1,\ N_2\}.$$

Then $N_1 < N$ and $1/N_2 \geq 1/N$ so that

$$|w| \leq N \text{ and } d(w,\ C - G) \geq \frac{1}{N}$$
$$\text{implies } w \in K_N.$$

As $K_N \subset \bigcup_{n=1}^{\infty} K_n$, we have

$$w \in \bigcup_{n=1}^{\infty} K_n.$$

Hence

$$G \subseteq \bigcup_{n=1}^{\infty} K_n.$$

Combining we obtain

$$G = \bigcup_{n=1}^{\infty} K_n.$$

---

[3] ***Archimedean Property (Second Version):*** If $y > 0$, there exists $n_y \in \mathrm{N}$ such that $0 < \frac{1}{n_y} < y$.

`Details:` As G is open, $G = \text{int}G$ and hence

$$G = \text{int}G = \text{int}\bigcup_{n=1}^{\infty} K_n. \tag{1.4}$$

The proof of Part (a) will be completed if we show that

$$\text{int}\bigcup_{n=1}^{\infty} K_n = \bigcup_{n=1}^{\infty} \text{int}K_n.$$

We know that[4]

$$\bigcup_{n=1}^{\infty} \text{int}K_n \subseteq \text{int}\bigcup_{n=1}^{\infty} K_n. \tag{1.5}$$

[This can be verified as follows: For each natural number n, $\text{int}K_n \subseteq K_n$ and hence

$$\bigcup_{n=1}^{\infty} \text{int}K_n \subseteq \bigcup_{n=1}^{\infty} K_n$$

so the open set $\bigcup_{n=1}^{\infty} \text{int}K_n$ is a subset of $\bigcup_{n=1}^{\infty} K_n$ and since $\text{int}\bigcup_{n=1}^{\infty} K_n$ is the largest open subset of $\bigcup_{n=1}^{\infty} K_n$, (1.5) follows.

[4] Theorem 1.24 (c) (in the Text *Foundations of Topology*, Second Edition, C. Wayne Patty): For any two subsets A and B of a topological space $(X, \mathcal{T})$,

$$\text{int}\,(A\,) \cup \text{int}\,(B) \subseteq \text{int}\,(A \cup B).$$

In general, if $\{A_n\}$ is a (finite or countable) collection of subsets of the topological space, then

$$\bigcup_{n} \text{int}A_n \subseteq \text{int}\bigcup_{n} A_n.$$

We now show that

$$\operatorname{int}\bigcup\nolimits_{n=1}^{\infty} K_n \subseteq \bigcup\nolimits_{n=1}^{\infty} \operatorname{int} K_n.$$

By Part (a), for each natural number n,

$$K_n \subseteq \operatorname{int} K_{n+1}$$

and hence

$$\bigcup_{n=1}^{\infty} K_n \subseteq \bigcup_{n=1}^{\infty} \operatorname{int} K_{n+1} = \bigcup_{n=2}^{\infty} \operatorname{int} K_n$$

and hence

$$\bigcup_{n=1}^{\infty} K_n \subseteq \operatorname{int} K_1 \cup \bigcup_{n=2}^{\infty} \operatorname{int} K_n = \underbrace{\bigcup_{n=1}^{\infty} \operatorname{int} K_n}_{\substack{\text{union of open sets}\\ \text{and hence is open}}}$$

This shows that

$$\operatorname{int}\bigcup_{n=1}^{\infty} K_n \subseteq \bigcup_{n=1}^{\infty} K_n \subseteq \bigcup_{n=1}^{\infty} \operatorname{int} K_n.$$

That is,

$$\operatorname{int}\bigcup_{n=1}^{\infty} K_n \subseteq \bigcup_{n=1}^{\infty} \operatorname{int} K_n. \tag{1.6}$$

Combining (1.5) and (1.6), we get

$$\bigcup_{n=1}^{\infty} \operatorname{int} K_n = \operatorname{int} \bigcup_{n=1}^{\infty} K_n.$$

Hence from (1.4), we obtain

$$G = \bigcup_{n=1}^{\infty} \operatorname{int} K_n. \tag{1.7}$$

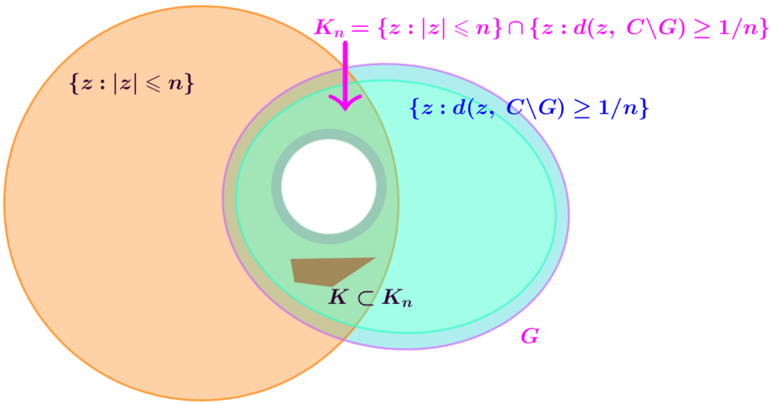

Figure 1.9: Given a compact set $K \subset G$, then $K \subset K_n$ for some n.

Hence the collection of open sets $\{\operatorname{int} K_n\}$ form an open cover for G. Hence if K is a **compact** subset of G, then

$$K \subseteq G = \bigcup_{n=1}^{\infty} \operatorname{int} K_n$$

also and hence $\{\operatorname{int} K_n\}$ form an open cover for the compact set K also, and hence K can be covered by finitely many

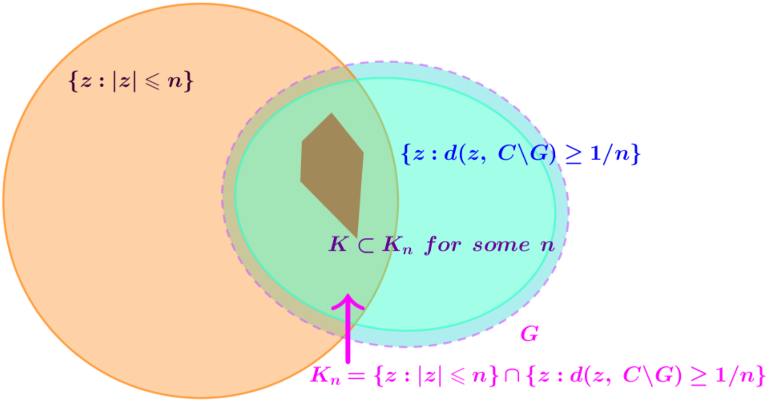

Figure 1.10: Given a compact set $K \subset G$, then $K \subset K_n$ for some n.

members of $\{\text{int}\, K_n\}$, say,

$$\{\text{int}\, K_{j_1},\ \text{int}\, K_{j_2},\ \ldots,\ \text{int}\, K_{j_m}\},\ \ \text{where}\ \ j_1 < j_2 < \cdots < j_m.$$

so that

$$K \subset \text{int} K_{j_1} \cup \cdots \cup \text{int} K_{j_m}.$$

Also, by Part (a), since $j_1 < j_2 < \cdots < j_m$, we have

$$K_{j_1} \subseteq \text{int} K_{j_2} \subseteq \cdots \subseteq \text{int} K_{j_m}$$

and hence

$$K \subset \text{int} K_{j_m} \subset K_{j_m}.$$

That is, we have shown that

$$K \subset K_{j_m}$$

and this proves Part (b).

To prove Part (c):

1. $$K_n \subseteq G \subseteq \mathbb{C}$$

   and hence $\infty \notin G$ and $\infty \notin K_n$; and so $C_\infty - G$ and $C_\infty - K_n$ contain ∞. Hence $C_\infty - G$ and $C_\infty \backslash K_n$ have components containing ∞. Moreover,

   $$K_n \subseteq G \Rightarrow C_\infty - K_n \supseteq \underbrace{C_\infty - G}_{\infty \in C_\infty - G}$$

   and therefore **the unbounded component of** $C_\infty \backslash K_n$ **that contains** ∞ also contain the component of $C_\infty \backslash G$ that contains ∞.

2. The unbounded component of $C_\infty - K_n$ contains $\{z : |z| > n\}$ `[Details:` From (1.2),

   $$C_\infty - K_n =$$

$$\underbrace{\{\,\infty\,\}\cup\{z:|z|>n\}}_{\substack{\text{connected set in } C_\infty - K_n \text{ and is} \\ \text{a subset of the component of} \\ C_\infty - K_n \text{ that contains } \infty}} \quad \cup\{z:d(z,\ C-\ G)<1/n\} \tag{1.8}$$

So **if D is a bounded component of** $C_\infty\backslash K_n$, then D is not a subset of $\{\,\infty\,\}\cup\{z:|z|>n\}$ and hence, by (1.8), D intersects with $\{u:d(u,\ C-\ G)<1/n\}$ so it is possible to choose a point z in D such that $z\in\{u:d(u,\ C-G)<1/n\}$ so that

$$d(z,\ C-G)<1/n.$$

As $d(z,\ C-G)=\inf\{|z-v|:v\in C-\ G\}$, $d(z,\ C-G)<1/n$ implies, if we let this infimum be r_0, then

$$\inf\{|z-v|:v\in C-G\}=r_0<1/n$$

and hence $1/n$ is not a lower bound of the set

$$\{|z-v|:v\in C-G\}$$

and hence there is a point w in $C-G$ with $|z-w|<1/n$; that is, that

$$z\in B\left(w;\ \frac{1}{n}\right).$$

We also note that

$$B\left(w;\ \frac{1}{n}\right) \subset \mathbb{C}_\infty - K_n$$

because of the following observation: $p \notin \mathbb{C}_\infty - K_n$ implies $p \in K_n$ implies in particular from (1.2) that $\underbrace{d(p,\ C\backslash\ G)}_{\inf\{\,|p-v|\ :v\ \in C-G\}} \geq 1/n$ and since $|\,p - w\,| \geq d(p,\ C - G)$ we have

$$|\,p - w\,| \geq \frac{1}{n}$$

implies $p \notin B\left(w;\ \frac{1}{n}\right)$ and hence

$$B\left(w;\ \frac{1}{n}\right) \subset \mathbb{C}_\infty - K_n.$$

Thus,

$$z \in B\left(w;\ \frac{1}{n}\right) \subset \mathbb{C}_\infty - K_n.$$

Since the disk $B\left(w;\ \frac{1}{n}\right)$ is connected and since z is in the (bounded) component D, $z \in B\left(w;\ \frac{1}{n}\right)$ implies

$$B\left(w;\ \frac{1}{n}\right) \subset D.$$

If D_1 is the component of $C_\infty - G$ that contains the above w it follows that $D_1 \subseteq D$. [Details: w is a point in $C - G$ and hence a point in $C_\infty - G$. Thus w must be an element in exactly one of the components of $C_\infty - G$ and we let that component be D_1. As D is a component of $C_\infty - K_n$ and since w is common element of D and D_1 (and also since $C_\infty - G \subseteq C_\infty - K_n$), it follows that $D_1 \subseteq D$.

This completes the proof of Part (c). □

1.1.1 Defining metric on $C(G,\ \Omega)$

By using compact subsets constructed as in Proposition 1.1.6 we are going to define a metric on $C(G,\ \Omega)$.

If

$$G = \bigcup_{n=1}^{\infty} K_n$$

where each K_n is compact and $K_n \subseteq \operatorname{int} K_{n+1}$, for $n = 1,\ 2,\ \ldots$ define $\rho_n : C(G,\ \Omega) \times C(G,\ \Omega) \to \mathbb{R}$ by

$$\rho_n(f,\ g) = \sup\{d(f(z),\ g(z)) : z \in K_n\} \tag{1.9}$$

for all functions f and g in $C(G,\ \Omega)$.

- Attention! ρ_n defined in (1.9) satisfies triangle inequality, because for any f, g and h in $C(G,\ \Omega)$, since d is a metric on Ω, for any $z \in K_n$,

$$d(f(z),\ g(z)) \leq d(f(z),\ h(z)) + d(h(z),\ g(z))$$

and hence

$$\sup\{d(f(z),\ g(z)) : z \in K_n\}$$

$$\begin{aligned} \leq\ & \sup\{d(f(z),\ h(z)) : z \in K_n\} \\ & + \sup\{d(h(z),\ g(z)) : z \in K_n\} \end{aligned}$$

so that

$$\rho_n(f,\ g) \leq \rho_n(f,\ h) + \rho_n(h,\ g).$$

But ρ_n is **not a metric** on $C(G,\ \Omega)$ because

$$\rho_n(f,\ g) = 0 \Rightarrow$$

$$f = g \text{ on } K_n \text{ but } f \text{ need not be equal to } g \text{ on } G.$$

Also define $\rho : C(G,\ \Omega) \times C(G,\ \Omega) \to \mathbb{R}$ by

$$\rho(f,\ g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,\ g)}{1 + \rho_n(f,\ g)}. \tag{1.10}$$

1. Whether ρ well defined? Yes. This is because of the following observation: Since

$$\frac{t}{1+t} \leq 1 \text{ for all } t \geq 0$$

and since

$$\rho_n(f,\ g) \geq 0 \ \text{ for all } \ f \text{ and } g \in C(G,\ \Omega)$$

we have

$$\frac{\rho_n(f,\ g)}{1+\rho_n(f,\ g)} \leq 1 \ \text{ for all } \ f \text{ and } g \in C(G,\ \Omega)$$

and hence

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \frac{\rho_n(f,\ g)}{1+\rho_n(f,\ g)} \leq \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \tag{1.11}$$

for all f and $g \in C(G,\ \Omega)$. Since the geometric series $\sum\limits_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$, with initial term 1/2 and common ratio 1/2, converges to $(1/2)/[1-(1/2)] = 1$, (1.11) shows (by Direct Comparison Test of Series of Non-Negative Real Numbers[5]) that the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \frac{\rho_n(f,\ g)}{1+\rho_n(f,\ g)}$$

converges for all f and g in $C(G,\ \Omega)$. Hence $\rho : G \times G \rightarrow$

[5] **Direct Comparison Test for Series of Non-Negative Terms** Let $\sum u_n$ and $\sum v_n$ be two series with non-negative terms such that $u_n \leq v_n$ for all $n > N$, for some integer N. Then

$$\text{if } \sum v_n \text{ is convergent, then } \sum u_n \text{ is also convergent.}$$

$\mathbb{R}$ is a well-defined function.

2. Whether ρ is a metric on $C(G, \Omega)$? Yes. This will be proved in Proposition 1.1.8.

Before proving that ρ is a metric on $C(G, \Omega)$ we recall:

Lemma 1.1.7. *If* (S, d) *is a metric space, then*

$$\mu(s, t) = \frac{d(s, t)}{1 + d(s, t)}$$

is also a metric on S*.*

- *A set is open in* (S, d) *if and only if it is open in* (S, μ)*.*
- *A sequence is a Cauchy sequence in* (S, d) *if and only if it is a Cauchy sequence in* (S, μ)*.*

Proof. μ is a metric on S.

(i)
- For any s and t in S, as d is a metric, $d(s, t) \geq 0$, and hence $\frac{d(s, t)}{1+d(s, t)} \geq 0$, and hence $\mu(s, t) \geq 0$.
- $$\begin{aligned} \mu(s, t) = 0 \quad &\Leftrightarrow \quad \frac{d(s, t)}{1 + d(s, t)} = 0 \\ &\Leftrightarrow \quad d(s, t) = 0 \underset{\text{as } d \text{ is a metric}}{\Leftrightarrow} s = t. \end{aligned}$$

(ii) *(Symmetry)*

$$\mu(s, t) = \frac{d(s, t)}{1 + d(s, t)} = \frac{d(t, s)}{1 + d(t, s)} = \mu(t, s).$$

(iii) *(Transitivity)* To prove the transitivity, we define

$$\varphi(t) = \frac{t}{1+t} \quad \text{for} \quad t \geq 0.$$

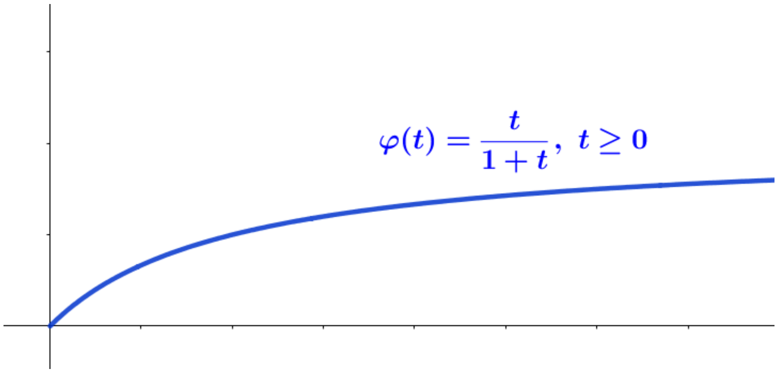

Figure 1.11: $\varphi(t) = \frac{t}{1+t}$ for $t \geq 0$ has derivative given by $\varphi'(t) = \frac{1}{(1+t)^2} > 0$ for $t \geq 0$.

Then, by the Quotient Rule of Differentiation,

$$\varphi'(t) = \frac{1}{(1+t)^2} > 0 \quad \text{for} \quad t \geq 0.$$

Thus φ is an increasing function for $t \geq 0$. That is,

$$0 \leq t_1 < t_2 \Rightarrow \varphi(t_1) \leq \varphi(t_2). \tag{1.12}$$

As d is a metric on S, it satisfies the triangle inequality and hence by

$$d(s,\ t) \le d(s,\ u) + d(u,\ t).$$

Taking $t_1 = d(s,\ t)$ and $t_2 = d(s,\ u) + d(u,\ t)$, then by (1.12), we have

$$\underbrace{d(s,\ t)}_{t_1} \le \underbrace{d(s,\ u) + d(u,\ t)}_{t_2}$$

implies

$$\underbrace{\frac{d(s,\ t)}{1 + d(s,\ t)}}_{\varphi(t_1)} \le \underbrace{\frac{d(s,\ u) + d(u,\ t)}{1 + d(s,\ u) + d(u,\ t)}}_{\varphi(t_2)}$$

implies

$$\frac{d(s,\ t)}{1 + d(s,\ t)} \le \frac{d(s,\ u)}{1 + d(s,\ u) + d(u,\ t)} + \frac{d(u,\ t)}{1 + d(s,\ u) + d(u,\ t)}$$

implies, since $\frac{1}{1+a+b} \le \frac{1}{1+a}$ if $a \ge 0$ and $b \ge 0$, that

$$\underbrace{\frac{d(s,\ t)}{1 + d(s,\ t)}}_{\mu(s,\ t)} \le \underbrace{\frac{d(s,\ u)}{1 + d(s,\ u)}}_{\mu(s,\ u)} + \underbrace{\frac{d(u,\ t)}{1 + d(u,\ t)}}_{\mu(u,\ t)}$$

That is for any s, u, t in S,

$$\mu(s,\ t) \le \mu(s,\ u) + \mu(u,\ t)$$

- Since $\mu(s,\ t) \leq d(s,\ t)$, it follows that for each $x \in S$ and $\varepsilon > 0$,

$$\underbrace{\{y \in S : d(y,\ x) < \varepsilon\}}_{=B_d(x,\varepsilon)} \subset \underbrace{\{y \in S : \mu(y,\ x) < \varepsilon\}}_{=B_\mu(x,\varepsilon)}.$$

As

$$\mu(s,\ t) = \frac{d(s,\ t)}{1 + d(s,\ t)} \Leftrightarrow d(s,\ t) = \frac{\mu(s,\ t)}{1 - \mu(s,\ t)}$$

corresponding to $\varepsilon > 0$, taking $\delta = \frac{\varepsilon}{1+\varepsilon}$ it follows that

$$\underbrace{\{y \in S : \mu(y,\ x) < \delta\}}_{=B_\mu(x,\ \delta)} \subset \underbrace{\{y \in S : d(y,\ x) < \varepsilon\}}_{=B_d(x,\varepsilon)},$$

since

$$\begin{aligned} & \mu(y,\ x) < \delta \Rightarrow 1 - \mu(y,\ x) > 1 - \delta \\ \Rightarrow\quad & \frac{1}{1 - \mu(y,\ x)} < \frac{1}{1 - \delta} \\ \Rightarrow\quad & (\text{as}\, \mu(y,\ x)\, \text{is a nonnegative number}) \\ & \underbrace{\frac{\mu(y,\ x)}{1 - \mu(y,\ x)}}_{=d(y,\ x)} < \frac{\mu(y,\ x)}{1 - \delta} < \frac{\delta}{1 - \delta} = \varepsilon \\ \Rightarrow\quad & d(y,\ x) < \varepsilon. \end{aligned}$$

Now if U is open in $(S,\ d)$ then $\forall x \in S\ \exists \varepsilon > 0$ such that $B_d(x,\ \varepsilon) \subseteq U \Rightarrow B_\mu(x,\ \varepsilon/(1+\varepsilon)) \subseteq U$ and hence U is open in $(S,\ \mu)$. Conversely, if U is open in $(S,\ \mu)$ then $\forall x \in S\ \exists \varepsilon > 0$ such that

$$B_\mu(x,\, \varepsilon) \subseteq U \Rightarrow B_d(x,\, \varepsilon) \subseteq B_\mu(x,\, \varepsilon) \subseteq U$$

and hence U is open in $(S,\ d)$.

- (s_n) is Cauchy in $(S,\, d)$ implies $\forall \varepsilon > 0 \exists N \in \mathbb{N}$ such that

$$d(s_n,\ s_m) < \varepsilon\ \ \forall n,\ m \geq N$$

implies

$$\mu(s_n,\ s_m) < \varepsilon\ \ \forall n,\ m \geq N$$

implies every Cauchy sequence in $(S,\ d)$ is a Cauchy sequence in $(S,\ \mu)$ also. *Conversely*, let (s_n) be a Cauchy sequence in $(S,\ \mu)$. We claim that (s_n) be a Cauchy sequence in $(S,\ d)$. Let $\varepsilon > 0$ be given. Since (s_n) is a Cauchy sequence in $(S,\ \mu)$ (by the definition of Cauchy sequence) corresponding to $\frac{\varepsilon}{1+\varepsilon}$ there is a positive integer N such that

$$\mu(s_n,\ s_m) < \frac{\varepsilon}{1+\varepsilon}\ \ \forall m,\, n \geq N.$$

Then

$$d(s_n,\ s_m) < \frac{\frac{\varepsilon}{1+\varepsilon}}{1 - \frac{\varepsilon}{1+\varepsilon}} = \varepsilon\ \ \forall m,\, n \geq N$$

and hence (s_n) is a Cauchy sequence in $(S,\ d)$, proving the claim.

□

Proposition 1.1.8. *$(C(G,\ \Omega),\ \rho)$ is a metric space.*

Proof. (i) •

$$\rho(f,\ g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,\ g)}{1+\rho_n(f,\ g)} \geq 0.$$

•

$$\rho(f,\ g) = 0$$

$$\begin{aligned}
&\Leftrightarrow \quad \frac{\rho_n(f,\ g)}{1+\rho_n(f,\ g)} = 0 \ \text{ for } \ n = 1,\ 2,\ \ldots \\
&\Leftrightarrow \quad \rho_n(f,\ g) = 0 \ \text{ for } \ n = 1,\ 2,\ \ldots \\
&\Leftrightarrow \quad \sup\{d(f(z),\ g(z)) : z \in K_n\} = 0 \ \text{ for } \ n = 1,\ 2,\ \ldots \\
&\Leftrightarrow \quad d(f(z),\ g(z)) = 0\,, \quad \forall z \in K_n, \quad \text{for } \ n = 1,\ 2,\ \ldots \\
&\Leftrightarrow \quad (\text{since } d \text{ is a metric}) \\
&\qquad\ f(z) = \ g(z)\,, \quad \forall z \in K_n, \quad \text{for } \ n = 1,\ 2,\ \ldots \\
&\Leftrightarrow \quad f(z) = \ g(z)\,, \quad \forall z \in G, \ \text{ because } \ G = \bigcup_{n=1}^{\infty} K_n \\
&\Leftrightarrow \quad f = \ g \ \text{ on } \ G.
\end{aligned}$$

(ii) Since d is a metric on Ω,

$$\begin{aligned}\rho_n(f,\ g) &= \sup\{d(f(z),\ g(z)) : z \in K_n\}\\ &= \sup\{d(g(z),\ f(z)) : z \in K_n\}\\ &= \rho_n(g,\ f),\end{aligned}$$

and hence

$$\rho(f,\ g) = \frac{\rho_n(f,\ g)}{1+\rho_n(f,\ g)} = \frac{\rho_n(g,\ f)}{1+\rho_n(g,\ f)} = \rho(g,\ f).$$

(iii) Since each ρ_n satisfies the triangle inequality on $C(G,\ \Omega)$ *(Attention! ρ_n is not a metric on $C(G,\ \Omega)$)*, referring the proof of triangle inequality in Lemma 1.1.7, we have for any n and for any f, g, and h in $C(G,\ \Omega)$,

$$\frac{\rho_n(f,\ g)}{1+\rho_n(f,\ g)} \le \frac{\rho_n(f,\ h)}{1+\rho_n(f,\ h)} + \frac{\rho_n(h,\ g)}{1+\rho_n(h,\ g)}$$

and hence

$$\left(\frac{1}{2}\right)^n \frac{\rho_n(f,\ g)}{1+\rho_n(f,\ g)}$$

$$\le \left(\frac{1}{2}\right)^n \frac{\rho_n(f,\ h)}{1+\rho_n(f,\ h)} + \left(\frac{1}{2}\right)^n \frac{\rho_n(h,\ g)}{1+\rho_n(h,\ g)}$$

which gives

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,\ g)}{1+\rho_n(f,\ g)}$$

$$\leq \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}\frac{\rho_n(f,\ h)}{1+\rho_n(f,\ h)}+\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}\frac{\rho_n(h,\ g)}{1+\rho_n(h,\ g)}$$

i.e., that

$$\rho(f,\ g)\leq \rho(f,\ h)+\rho(h,\ g).$$

Combining all the above ρ is a metric on $C(G,\ \Omega)$ and hence $(C(G,\ \Omega),\ \rho)$ is a metric space. □

The next lemma concerns about subsets of $C(G,\ \Omega)\times C(G,\ \Omega)$ and is very useful because it gives insight into the behavior of the metric ρ.

Lemma 1.1.9. *Let the metric ρ be defined as in (1.10):*

$$\rho(f,\ g)=\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{n}\frac{\rho_n(f,\ g)}{1+\rho_n(f,\ g)} \tag{1.10}$$

where ρ_n, $n=1,\ 2,\ \ldots$ are defined as in (1.9):

$$\rho_n(f,\ g)=\sup\{d(f(z),\ g(z)) : z\in K_n\}. \tag{1.9}$$

If $\varepsilon>0$ is given then there is a $\delta>0$ and a compact set $K\subset G$ such that for f and g in $C(G,\ \Omega)$,

$$\sup\{d(f(z),\ g(z)) : z\in K\}<\delta\quad\Rightarrow\quad\rho(f,\ g)<\varepsilon\,. \tag{1.13}$$

Conversely, if $\delta>0$ and a compact set $K\subset G$ are given, then *there is an* $\varepsilon > 0$ *such that for* f *and* g *in* $C(G,\ \Omega)$,

$$\rho(f,\ g) < \varepsilon \quad \Rightarrow \quad \sup\{d(f(z),\ g(z)) : z \in K\} < \delta\,. \qquad (1.14)$$

Proof. If $\varepsilon > 0$ is fixed let p be a positive integer such that

$$\sum_{n=p+1}^{\infty} \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{2}. \qquad (1.15)$$

`[Details:` $\sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a convergent series and hence its sequence of partial sums of the series converges to its sum that is also given by $\sum\limits_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. Hence corresponding to $\frac{\varepsilon}{2}$ there is a natural number p such that

$$\left| \underbrace{\sum_{n=1}^{m} \left(\frac{1}{2}\right)^n}_{m\text{th partial sum}} - \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \right| < \frac{\varepsilon}{2} \qquad \forall m \geq p.$$

In particular, putting $m = p$,

$$\left| \underbrace{\sum_{n=1}^{p} \left(\frac{1}{2}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n}_{-\sum\limits_{n=p+1}^{\infty} \left(\frac{1}{2}\right)^n} \right| < \frac{\varepsilon}{2}$$

and hence (1.15) is obtained. Also, corresponding to the above p there is a compact set K_p which is part of the sequence of compact subsets $\{K_n\}_{n=1}^{\infty}$ constructed in Proposition 1.1.6 whose union gives G. Let this compact set be K. That is, let

$$K = K_p.$$

Choose $\delta > 0$ such that

$$0 \leq t < \delta \text{ gives } \frac{t}{1+t} < \frac{\varepsilon}{2}. \tag{1.16}$$

(This is possible since $\varphi(t) = \frac{t}{1+t}$ is an increasing function for $t \geq 0$ (because $\varphi'(t) = \frac{1}{(1+t)^2}$, $t \geq 0$) and $\varphi(0) = 0$ (Fig. 1.12)).

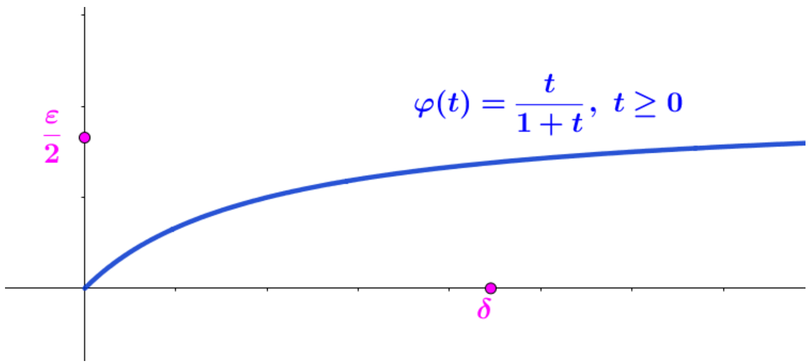

Figure 1.12: It is possible to find a $\delta > 0$ such that $0 \leq t < \delta$ gives $\frac{t}{1+t} < \frac{\varepsilon}{2}$.

Suppose f and g are functions in $C(G,\ \Omega)$ that satisfy

$$\sup\{d(f(z),\ g(z)) : z \in K\} < \delta. \tag{1.17}$$

By the construction in Proposition 1.1.6,

$$K_n \underset{\neq}{\subset} \operatorname{int}(K_{n+1}) \underset{\neq}{\subset} K_{n+1},$$

and hence in particular

$$K_n \subset K_p = K \ \text{ for } \ 1 \leq n \leq p,$$

and so it follows that for $1 \leq n \leq p$,

$$\begin{aligned} \rho_n(f,\ g) &= \sup\{d(f(z),\ g(z)) : z \in K_n\} \\ &\leq \sup\{d(f(z),\ g(z)) : z \in K\} \\ &< \delta, \ \text{ by the assumption (1.17)} \end{aligned}$$

This gives, using (1.16) with $t = \rho_n(f,\ g)$, that

$$\frac{\rho_n(f,\ g)}{1 + \rho_n(f,\ g)} < \frac{\varepsilon}{2} \ \text{ for } \ 1 \leq n \leq p. \tag{1.18}$$

Now

$$
\begin{aligned}
\rho(f,\ g) &= \sum_{n=1}^{p}\left(\frac{1}{2}\right)^{n}\frac{\rho_n(f,\ g)}{1+\rho_n(f,\ g)}+\sum_{n=p+1}^{\infty}\left(\frac{1}{2}\right)^{n}\underbrace{\frac{\rho_n(f,\ g)}{1+\rho_n(f,\ g)}}_{<1,\ \text{for all } n}\\
&< \underbrace{\sum_{n=1}^{p}\left(\frac{1}{2}\right)^{n}\left(\frac{\varepsilon}{2}\right)}_{<\varepsilon/2}+\underbrace{\sum_{n=p+1}^{\infty}\left(\frac{1}{2}\right)^{n}}_{<\varepsilon/2},\\
&\qquad \text{using (1.18) and (1.15)}\\
&< \varepsilon.
\end{aligned}
$$

That is, we have shown that if $\varepsilon > 0$ is given then there is a $\delta > 0$ and a compact set $K \subset G$ such that if f and g in $C(G,\ \Omega)$ are such that

$$\sup\{d(f(z),\ g(z)) : z \in K\} < \delta$$

then

$$\rho(f,\ g) < \varepsilon\,.$$

That is (1.13) is satisfied.

Conversely, suppose K and δ are given. Since, by the construction in Proposition 1.1.6,

$$G = \bigcup_{n=1}^{\infty} K_n = \bigcup_{n=1}^{\infty} \mathrm{int} K_n$$

it follows (since $K \subset G$) that

$$K \subset \bigcup_{n=1}^{\infty} \mathrm{int} K_n$$

showing that $\{\mathrm{int} K_n\}_{n=1}^{\infty}$ is an open cover for the compact set K; and hence K can be covered by finitely many members of $\{\mathrm{int} K_n\}_{n=1}^{\infty}$, say,

$$\{\mathrm{int}\, K_{j_1},\ \mathrm{int}\, K_{j_2},\ \ldots,\ \mathrm{int}\, K_{j_m}\},\ \text{ where } j_1 < j_2 < \cdots < j_m.$$

so that

$$K \subset \mathrm{int} K_{j_1} \cup \cdots \cup \mathrm{int} K_{j_m}. \tag{1.19}$$

Also, by Part (a) of Proposition 1.1.6, since $j_1 < j_2 < \cdots < j_m$, we have

$$K_{j_1} \subset \mathrm{int} K_{j_2} \subset \cdots \subset \mathrm{int} K_{j_m} \subset K_{j_m}$$

and hence, by 1.19,

$$K \subset K_{j_m}.$$

Let $p = j_m$. Then we have shown that $p \geq 1$ with

$$K \subset K_p;$$

this gives $K_p \supseteq K$ and

$$\rho_p(f,\ g) =$$

$$\sup\{d(f(z),\ g(z)) : z \in K_p\} \geq \sup\{d(f(z),\ g(z)) : z \in K\}. \tag{1.20}$$

Let $\varepsilon > 0$ be chosen so that

$$0 \leq s < 2^p\varepsilon \ \text{ implies } \ \frac{s}{1-s} < \delta.$$

[Such an ε exists, because $\psi(s) = \frac{s}{1-s}$, $s \neq 1$ is increasing on $[0,\ 1)$ because $\psi'(s) = \frac{1}{(1-s)^2} > 0$ when $0 < s < 1$] (Fig. 1.13).

Then, noting that

$$\frac{\frac{t}{1+t}}{1 - \frac{t}{1+t}} = t,$$

it follows that

$$\frac{t}{1+t} < 2^p\varepsilon \ \text{ implies } \ t < \delta. \tag{1.21}$$

So

$$\text{if } \ \rho(f,\ g) < \varepsilon$$

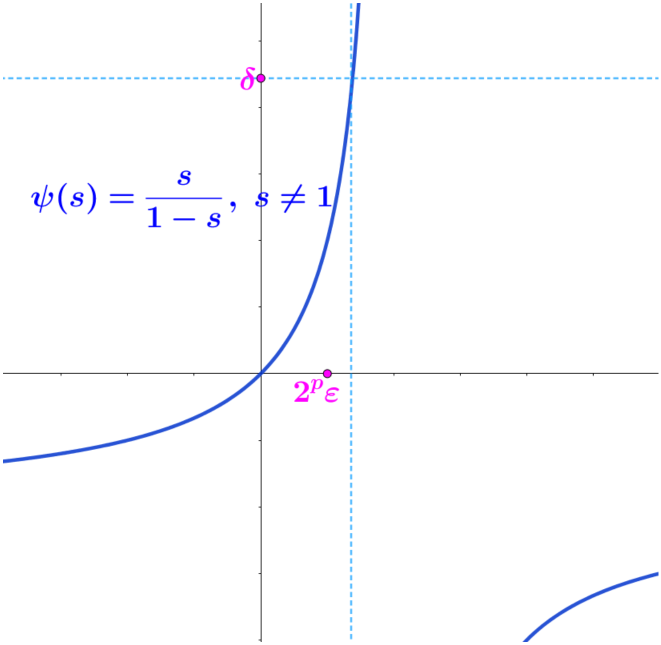

Figure 1.13: $\psi(s) = \frac{s}{1-s}$, $s \neq 1$ is increasing on $[0, 1)$ because $\psi'(s) = \frac{1}{(1-s)^2} > 0$ when $0 < s < 1$.

then

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,\ g)}{1 + \rho_n(f,\ g)} < \varepsilon$$

implies

$$\left(\frac{1}{2}\right)^n \frac{\rho_n(f,\ g)}{1 + \rho_n(f,\ g)} < \varepsilon \quad \text{for } n = 1,\ 2,\ \ldots$$

in particular (for $n = p$)

$$\left(\frac{1}{2}\right)^p \frac{\rho_p(f,\ g)}{1+\rho_p(f,\ g)} < \varepsilon$$

then

$$\frac{\rho_p(f,\ g)}{1+\rho_p(f,\ g)} < 2^p\varepsilon$$

and (using (1.21)) this gives $\rho_p(f,\ g) < \delta$. Now, using (1.20), it follows that

$$\sup\{d(f(z),\ g(z)) : z \in K\} < \delta.$$

But this is exactly the statement contained in (1.14). $\square$

Proposition 1.1.10.

(a) A set $\mathcal{O} \subset (C(G,\ \Omega),\ \rho)$ is open if and only if for each $f \in \mathcal{O}$ there is a compact set K and a $\delta > 0$ such that

$$\mathcal{O} \supset \{\, g : d(f(z),\ g(z)) < \delta,\ \ z \in K \,\}. \tag{1.22}$$

(b) A sequence $\{f_n\}$ in $(C(G,\ \Omega),\ \rho)$ converges to f if and only if $\{f_n\}$ converges to f ***uniformly on all compact subsets of*** *G.*

Proof. If $\mathcal{O}$ is open and $f \in \mathcal{O}$ then f is an interior point of

$\mathcal{O}$ and hence there is an $\varepsilon > 0$ such that $B_\rho(f, \varepsilon) \subseteq \mathcal{O}$. i.e.,

$$\{g : \rho(f,\ g) < \varepsilon\} \subseteq \mathcal{O}. \tag{1.23}$$

Now by First Part of Lemma 1.1.9 corresponding to this $\varepsilon > 0$ there exists $\delta > 0$ and a compact set K subset of G such that

$$\text{for } f,\ g \in C(G,\ \Omega), \text{ with } \sup\{d(f(z),\ g(z)) : z \in K\} < \delta$$

$$\Rightarrow \rho(f,\ g) < \varepsilon.$$

Hence

$$d(f(z),\ g(z)) < \delta \text{ for all } z \in K \Rightarrow \rho(f,\ g) < \varepsilon$$

implies

$$\{g : d(f(z),\ g(z)) < \delta\ ,\ z \in K\} \subseteq \{g : \rho(f,\ g) < \varepsilon\}.$$

Hence by Equation (1.23) it follows that

$$\{g : d(f(z),\ g(z)) < \delta\ ,\ z \in K\} \subseteq \mathcal{O}.$$

Conversely, assume that $\mathcal{O}$ satisfies the condition that for each $f \in \mathcal{O}$ there is a compact set K and a $\delta > 0$ (K and δ depend on the choice of f) such that

$$\mathcal{O} \supset \{\, g : d(f(z),\ g(z)) < \delta,\ \ z \in K \,\}.$$

Claim: $\mathcal{O}$ is an open subset of $C(G,\ \Omega)$ with respect to the metric ρ.

Let $f \in \mathcal{O}$ then by the assumption there is a compact set K (K depends on the choice of f) and a $\delta > 0$ such that

$$\mathcal{O} \supset \{\, g : d(f(z),\ g(z)) < \delta,\ \ z \in K \,\}.$$

Then by second part of Lemma 1.1.9, there is an $\varepsilon > 0$ such that for g in $C(G,\ \Omega)$,

$$\rho(f,\ g) < \varepsilon \quad \Rightarrow \quad \sup\{d(f(z),\ g(z)) : z \in K\} < \delta\,. \qquad (1.24)$$

Subclaim: $B_\rho(f,\ \varepsilon) \subset \mathcal{O}$ and hence f is an interior point of $\mathcal{O}$. As f is an arbitrary element of $\mathcal{O}$ this shows that each point of $\mathcal{O}$ is an interior point of $\mathcal{O}$ so that $\mathcal{O}$ is an open subset of $C(G,\ \Omega)$.

$$\begin{aligned} g \in B_\rho(f,\ \varepsilon) \quad &\Rightarrow \quad \rho(f,\ g) < \varepsilon \\ &\Rightarrow \quad (\text{by } (1.24))\ \sup\{d(f(z),\ g(z)) : z \in K\} < \delta \\ &\Rightarrow \quad g \in \{h : d(f(z),\ h(z)) < \delta,\ z \in K\} \\ &\Rightarrow \quad (\text{by } (2.55) \text{ in the assumption})\ g \in \mathcal{O} \end{aligned}$$

and this shows that $B_\rho(f,\ \varepsilon) \subset \mathcal{O}$, proving the subclaim. This completes the proof of the converse part.

Proof of Part (b) Assume that $f_n \to f$ in $(C(G,\ \Omega),\ \rho)$.

We prove that (f_n) *converges uniformly* to f *on all compact subsets of* G. For this, let K be a compact subset of G.

Claim: (f_n) *converges uniformly* to f *on* K. To prove the claim, let $\delta > 0$. Then by the converse part of Lemma 1.1.9, corresponding to K and δ there is an $\varepsilon > 0$ such that

$$f,\ g \in C(G,\ \Omega) \text{ and } \rho(f,\ g) < \varepsilon$$

$$\Rightarrow \sup\{d(f(z),\ g(z)) : z \in K\} < \delta. \tag{1.25}$$

As $f_n \to f$ in $(C(G,\ \Omega),\ \rho)$, by the definition of convergence of sequences, corresponding to $\varepsilon > 0$ there is a natural number $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\rho(f_n,\ f) < \varepsilon, \quad \forall n \geq N. \tag{1.26}$$

(We note that as ε depends on K and δ, the above N depends on K and δ, so that $N = N(\varepsilon) = N(K,\delta)$.)

Hence (1.25) shows that

$$\sup\{d(f_n(z),\ f(z)) : z \in K\} < \delta,\ \forall n \geq N. \tag{1.27}$$

i.e.,

$$d(f_n(z),\ f(z)) < \delta,\ \forall n \geq N,\ \forall z \in K. \tag{1.28}$$

As the choice of $\delta > 0$ is arbitrary, this shows that the sequence (f_n) converges to f *uniformly on the compact subset* K of G.

Since the choice of compact set $K \subset G$ is also arbitrary, this shows that (f_n) *converges uniformly to f on all compact subsets of G.*

Conversely, assume that (f_n) *converges uniformly to f* ***on all*** *compact subsets of G.*

Claim: $\{f_n\}$ converges to f in $(C(G,\ \Omega),\ \rho)$. To prove the claim, let $\varepsilon > 0$. Then, by the direct part of the Lemma 1.1.9 corresponding to $\varepsilon > 0$ there is a compact set K and $\delta > 0$ such that

$$f \text{ and } g \in C(G,\ \Omega) \text{ and } \sup\{d(f(z),\ g(z)) : z \in K\} < \delta$$

$$\Rightarrow \quad \rho(f,\ g) < \varepsilon\,. \tag{1.29}$$

By the assumption, the sequence of functions (f_n) *converges uniformly to f* on all compact subsets of G, and hence, in particular, on the compact set K so that there is natural number N such that

$$d(f_n(z),\ f(z)) < \delta,\ \ \forall n \geq N,\ \ \forall z \in K.$$

which implies

$$\sup\{d(f_n(z),\ f(z)) : z \in K\} < \delta,\ \ \forall n \geq N$$

implies, using (1.29), that

$$\rho(f_n,\ f) < \varepsilon\ \forall n \geq N.$$

Since the choice of $\varepsilon > 0$ is arbitrary, this shows that $\{f_n\}$ converges to f in $(C(G,\ \Omega),\ \rho)$. This completes the proof. $\square$

Corollary 1.1.11. *The collection of open sets on $C(G, \Omega)$ is independent of the choice of the sets $\{K_n\}$. That is, if*

$$G = \bigcup_{n=1}^{\infty} K'_n$$

where each K'_n is compact and

$$K'_n \subset \mathrm{int} K'_{n+1}$$

and if μ is the metric defined by the sets $\{K'_n\}$ then a set is open in $C((G, \Omega)\ \mu)$ if and only if it is open in $C((G, \Omega),\ \rho)$.

Proof. This is a direct consequence of Part (a) of the preceding proposition since the characterization of open sets does not depend on the choice of the sets $\{K_n\}$. `[Details:` We choose a sequence $\{K_n\}$ of compact subsets of G such that

(Proposition 1.1.6)

(a) $G = \bigcup_{n=1}^{\infty} K_n$

(b) $K_n \underset{\neq}{\subset} \operatorname{int}(K_{n+1})$

and the metric ρ on $C(G,\ \Omega)$ is defined by

$$\rho(f,\ g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\rho_n(f,\ g)}{1 + \rho_n(f,\ g)}$$

where ρ_n is defined for all functions f and g in $C(G,\ \Omega)$ by

$$\rho_n(f,\ g) = \sup\{d(f(z),\ g(z)) : z \in K_n\}. \tag{1.9}$$

It is given that the metric μ is defined in terms of the sequence $\{K_n'\}$ of compact subsets of G such that

(a) $G = \bigcup_{n=1}^{\infty} K_n'$

(b) $K_n' \underset{\neq}{\subset} \operatorname{int}(K_{n+1}')$.

and the metric μ on $C(G,\ \Omega)$ is defined by

$$\mu(f,\ g) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{\mu_n(f,\ g)}{1 + \mu_n(f,\ g)}$$

where μ_n is defined for all functions f and g in $C(G,\ \Omega)$ by

$$\mu_n(f,\ g) = \sup\{d(f(z),\ g(z)) : z \in K_n'\}.$$

By Part (a) of Proposition 1.1.10, a set $\mathcal{O}$ is open in $(C(G,\ \Omega),\ \rho)$ if and only if for each $f \in \mathcal{O}$ there is a compact set K and a $\delta > 0$ such that

$$\mathcal{O} \supset \{\, g : d(f(z),\ g(z)) < \delta,\ \ z \in K \,\}$$

(*hence the characterisation of open sets does not depend on the choice of the sets* $\{K_n\}$, as the above K is need not be from $\{K_n\}$) and again using Proposition 1.1.10, this holds if and only if $\mathcal{O}$ is open in $(C(G,\ \Omega),\ \mu)$. *[Here the metric d on Ω is common to the definition of both ρ and μ; only $\{K_n\}$ together with d have role in determining ρ, and only $\{K'_n\}$ together with d have role in determining μ.]* □

Proposition 1.1.12. $(C(G,\ \Omega),\ \rho)$ *is a complete metric space.*

Proof. By utilizing Lemma 1.1.9, we show that every cauchy sequence in $(C(G,\ \Omega),\ \rho)$ converges in $(C(G,\ \Omega),\ \rho)$. Suppose $\{f_n\}$ is a Cauchy sequence in $(C(G,\ \Omega)$. Then *for each compact set* $K \subset G$ the restrictions of the functions f_n to K gives a Cauchy sequence in $(C(K,\ \Omega)$. With this, we note by Lemma 1.1.9 that, corresponding to a compact set $K \subset G$ and $\delta > 0$ there is an $\varepsilon > 0$ such that

$$f,\, g \in (C(G,\ \Omega),\, \rho) \,\text{and}\, \rho(f,\ g) < \varepsilon$$

$$\Rightarrow \sup\{d(f(z),\ g(z)) : z \in K\} < \delta. \tag{1.30}$$

As (f_n) is a Cauchy sequence in $(C(G,\ \Omega),\ \rho)$, corresponding to ε there is natural number N such that

$$\rho(f_n,\ f_m) < \varepsilon\ \forall\, m,\, n \geqslant N.$$

Hence (1.30) gives

$$\sup\{d(f_n(z),\ f_m(z)) : z \in K\} < \delta\ \forall\, n,\, m \geqslant N. \tag{1.31}$$

$$d(f_n(z),\ f_m(z)) < \delta\ \forall\, n,\, m \geqslant N,\ \forall z \in K. \tag{1.32}$$

This shows that for each $z \in K$, $(f_n(z))$ is a Cauchy sequence in the *complete* space Ω and hence converges to a point in Ω, we prefer to denote the limit point as $f(z)$, since $\lim\limits_{n\to\infty} f_n(z)$ depends on z. Thus for each $z \in K$, we can write

$$f(z) = \lim_{n\to\infty} f_n(z).$$

and f is a function from K to Ω (Clearly, (f_n) converges *point wise* to f on K. We will show shortly that f_n converges to f *uniformly* on K.) As G is the union of compact sets (Proposition 1.1.6), the above construction gives the function

$f : G \to \Omega$ defined by

$$f(z) = \lim_{n\to\infty} f_n(z), \;\; z \in G.$$

Claim: f is continuous on G and $\rho(f_n, \; f\,) \to 0\,.$

Let K be compact and fix $\delta > 0$, choose N so that (1.31) holds for n, $m \geq N$. If z is arbitrary in K but fixed then there is an integer $m \geq N$ so that

$$d(f(z), \;\; f_m(z)) < \delta. \tag{1.33}$$

(Such an m exists because for the fixed z, the sequence $(f_n(z))$ converges to $f(z)$ and hence corresponding to $\delta > 0$ there is an integer $m_1 \in \mathbb{N}$ so that

$$d(f(z), \;\; f_n(z)) < \delta \;\forall n \geq m_1.$$

We choose $m = \max\{m_1, \, N\}$. Then

$$d(f(z), \;\; f_n(z)) < \delta \;\forall n \geq m$$

and in particular, taking $n = m$, we obtain (1.33)).

Now (1.33) together with (1.31)gives that for $n \geq N$ (and also since $m \geq N$ already)

$$d(f(z),\ f_n(z)) < \underbrace{d(f(z),\ f_m(z))}_{<\delta} + \underbrace{d(f_m(z),\ f_n(z))}_{<\delta} < 2\delta. \tag{1.34}$$

Even though m depends on the choice of z, N is independent of the choice of z in K, and hence (1.34) gives

$$d(f(z),\ f_n(z)) < 2\delta,\ \forall z \in K\ \forall n \geq N$$

which gives

$$\underbrace{\sup\{d(f(z),\ f_n(z)) : z \in K\}}_{u_n \geq 0} < 2\delta \quad \forall n \geq N$$

i.e.,

$$|u_n - 0| < 2\delta \quad \forall n \geq N$$

and hence

$$\lim_{n\to\infty} u_n = 0$$

so that

$$\underbrace{\sup\{d(f(z),\ f_n(z)) : z \in K\}}_{u_n} \to 0 \ \text{ as } \ n \to \infty.$$

That is, $\{f_n\}$ converges to f **uniformly on the compact set** K. By considering all compact subsets of G, we can conclude that $\{f_n\}$ converges to f uniformly on *every* compact set in

G. In particular, noting that closed balls are compact sets, the convergence is uniform on all closed balls contained in G. Let z be a point on G. Then there exists $r > 0$ such that the closed ball $\overline{B(z,\ r)} \subset G$, and $\{f_n\}$ converges to f uniformly on $\overline{B(z;\ r)}$, and since each f_n is continuous, we have[6] f is continuous on $\overline{B(z;\ r)}$. In particular, f is continuous at z. Since z is an arbitrary point of G, this shows that f is continuous at each point on G. i.e.,f is continuous on G. i.e., $f \in C(G,\ \Omega)$.

We have now $\{f_n\}$ converges to $f \in C(G,\ \Omega)$ uniformly on all compact subsets of G. Hence, by Part (b) of Proposition 1.1.10, it follows that $\{f_n\}$ converges to f in $(C(G,\ \Omega),\ \rho)$; i.e., that $\rho(f_n,\ f\ ) \to 0$ as $n \to \infty$. We conclude that every Cauchy sequence in $(C(G,\ \Omega),\ \rho)$ converges to an element (a function) in $(C(G,\ \Omega),\ \rho)$; and hence $(C(G,\ \Omega),\ \rho)$ is complete.

□

Definition 1.1.13. A set $\mathcal{F} \subset C(G,\ \Omega)$ is **normal** if each sequence in $\mathcal{F}$ has a subsequence which converges to a function f in $C(G,\ \Omega)$.

Remark 1.1.14. ***Attention!*** If $\mathcal{F}$ is normal, a sequence in $\mathcal{F}$ has a subsequence which converges to a function f in

[6] **Theorem II.6.1**(Page 29, Conway) Suppose $f_n : (X,\ d) \to (\Omega,\ \rho)$ is continuous for each n and that f_n converges to f uniformly on $(X,\ d)$, then f is continuous.

$C(G,\ \Omega)$, this f **need not be in** $\mathcal{F}$. Hence **a normal set need not be sequentially compact.**[7]

Recall: [8]

Theorem 1.1.15. *Let $(X,\ d)$ be a metric space. The following are equivalent.*
(i) $(X,\ d)$ is compact
(ii) $(X,\ d)$ is sequentially compact (i.e., every sequence in X has a convergent subsequence in X).
(iii) $(X,\ d)$ is countably compact (i.e., every countable open cover of X has a finite subcover).
(iv) $(X,\ d)$ has Bolzano-Weierstrass Property (i.e., every infinite subset of X has a limit point in X).
(v) $(X,\ d)$ is complete and totally bounded.

Proposition 1.1.16. *A set $\mathcal{F} \subset C(G,\ \Omega)$ is normal if and only if its closure is compact.*

Proof. $\overline{\mathcal{F}}$ is compact if and only if (by Theorem 13)) $\bar{\mathcal{F}}$ is sequentially compact, if and only if every sequence in $\bar{\mathcal{F}}$ has a convergent subsequence. In particular, every sequence in $\mathcal{F}$ has a convergent subsequence. Thus, $\overline{\mathcal{F}}$ is compact *implies* $\mathcal{F}$ is normal.

[7] A metric space $(X,\ d)$ is sequentially compact if every sequence in X has a convergent subsequence in X.

[8] Ref. Text Book: *Foundations of Topology*, Wayne C. Patty (Theorem 4.10, Page 136)

Conversely, $\mathcal{F}$ is normal *implies* every sequence in $\mathcal{F}$ has a convergent subsequence (*limit of this convergent subsequence need not be an element in* $\mathcal{F}$.) We show that every sequence in $\overline{\mathcal{F}}$ also has a convergent subsequence (*here limit of the convergent subsequence must be an element in* $\overline{\mathcal{F}}$.)

Let (f_n) be a sequence in $\overline{\mathcal{F}}$. Then, by the property of closure of a subset of metric space[9], for each $n \in \mathbb{N}$, $B(f_n,\, 1/n) \cap \mathcal{F} \neq \emptyset$ and we choose an element $g_n \in B(f_n,\, 1/n) \cap \mathcal{F}$. Then (g_n) is a sequence in $\mathcal{F}$ such that

$$d(f_n,\ g_n) < \frac{1}{n},\ \ n = 1,\, 2,\, \ldots$$

As $\mathcal{F}$ is normal, the sequence (g_n) in $\mathcal{F}$ has a convergent subsequence, say, (g_{n_k}) that converges to some g (We note that $g \in \bar{\mathcal{F}}$, because $\bar{\mathcal{F}}$ is closed).

Claim: (f_{n_k}) converges to g in $\bar{\mathcal{F}}$ and hence $\bar{\mathcal{F}}$ is sequentially compact (and hence compact). By the construction,

$$\rho(f_{n_k},\ g_{n_k}) < \frac{1}{n_k}.$$

Let $\varepsilon > 0$ be given. By the Archimedean property of real

[9]Theorem 1.19, Page 28, C. Wayne Patty, *Foundations of Topology*: Let A be a subset of a topological space $(X,\ \tau)$, and let $x \in X$. Then $x \in \bar{A}$ if and only if every neighborhood of x has a nonempty intersection with A.

numbers[10], there exists $K \in \mathbb{N}$ such that

$$\frac{1}{n_k} < \frac{\varepsilon}{2}\ \forall k \geq K.$$

Also, since $(g_{n_k}) \to g$, there exists $M \in \mathbb{N}$ such that

$$\rho(g_{n_k},\ g) < \frac{\varepsilon}{2}\ \forall k \geq M$$

Choose $N = \max\{K,\ M\}$. Then

$$\rho(f_{n_k},\ g) \leq \underbrace{\rho(f_{n_k},\ g_{n_k})}_{<\frac{\varepsilon}{2}} + \underbrace{\rho(g_{n_k},\ g)}_{<\frac{\varepsilon}{2}} < \varepsilon \quad \forall k \geq N.$$

This shows that $(f_{n_k}) \to g$. Hence claim is proved. As we have shown that every sequence in $\bar{\mathcal{F}}$ has a convergent subsequence in $\bar{\mathcal{F}}$ that converges to an element in $\bar{\mathcal{F}}$, it follows that $\bar{\mathcal{F}}$ is sequentially compact. Hence (ii) implies (i) of Theorem 13, shows that $\bar{\mathcal{F}}$ is compact. □

NOTE: In the above, limit of the convergent subsequence need not be in $\mathcal{F}$, but in $\overline{\mathcal{F}}$, since $\overline{\mathcal{F}}$ is closed.

Definition 1.1.17. A metric space $(X,\ d)$ is **totally bounded,** if for any $\varepsilon > 0$, $(X,\ d)$ can be covered by finitely many balls of radius ε. That is, for any $\varepsilon > 0$, $(X,\ d)$ has an ε-net.

Remark 1.1.18. Some authors allow us to take balls of **ra-**

[10]See Page 17.

dius less than ε. But in our discussions, we consider balls having radius **equal to** ε.

Theorem 1.1.19. *[Ref.* Topology *Text Book]: A metric space* $(X,\ d)$ *is compact if and only if it is complete and totally bounded.*

Theorem 1.1.20. *[Ref.* Topology *Text Book]: If a metric space* $(X,\ d)$ *is totally bounded, then any subset of it also is totally bounded.*

Remark 1.1.21. If A is a subset of a totally bounded space $(X,\ d)$, then A is also totally bounded, and for any $\varepsilon > 0$, A can be covered by finitely many balls of radius ε **with centre of each ball is a point in** A.

Proposition 1.1.22. *A set* $\mathcal{F} \subset C(G,\ \Omega)$ *is normal if and only if for every compact set* $K \subset G$ *and* $\delta > 0$ *there are functions* $f_1, \ldots, f_n$ *in* $\mathcal{F}$ *such that for* f *in* $\mathcal{F}$ *there is at least one* k, $1 \leq k \leq n$, *with*

$$\sup\{\, d(f(z),\ f_k(z)) : z \in K\,\} < \delta. \tag{1.35}$$

Proof. Suppose $\mathcal{F}$ is normal and let compact set $K \subset G$ and $\delta > 0$ be given. By 1.1.9, there is an $\varepsilon > 0$ such that for f and g in $C(G,\ \Omega)$,

$$\rho(f,\ g) < \varepsilon \quad \Rightarrow \quad \sup\{d(f(z),\ g(z)) : z \in K\} < \delta \tag{1.36}$$

holds. By the previous proposition, $\bar{\mathcal{F}}$ is compact. Hence, by Theorem 1.1.19, $\bar{\mathcal{F}}$ is totally bounded. Hence, by Theorem 1.1.20, its subset $\mathcal{F}$ is totally bounded. So, by the definition of totally boundedness, corresponding to the above $\varepsilon > 0$, there are $f_1,\ \ldots,\ f_n$ in $\mathcal{F}$ such that $\mathcal{F}$ can be covered by balls centred at these points and having radius ε. That is,

$$\mathcal{F} \subset \bigcup_{k=1}^{n} \underbrace{\{f : \rho(f,\ f_k) < \varepsilon\}}_{B_\rho(f_k,\ \varepsilon)}. \tag{1.37}$$

So[11]

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[11] Details: Now, by the choice of ε, the relation (1.36) shows that

$$\rho(f,\ f_k) < \varepsilon \quad \Rightarrow \quad \sup\{d(f(z),\ f_k(z)) : z \in K\} < \delta$$

which implies

$$h \in \{f : \rho(f,\ f_k) < \varepsilon\} \quad \Rightarrow h \in \ \{\, f : \sup\{d(f(z),\ f_k(z)) : z \in K\} < \delta\}$$

which implies

$$\{f : \rho(f,\ f_k) < \varepsilon\} \subset \{\, f : \sup\{d(f(z),\ f_k(z)) : z \in K\} < \delta\}.$$

Hence, (1.37) can be written as

$$\mathcal{F} \subset \bigcup_{k=1}^{n} \{\, f : \sup\{d(f(z),\ f_k(z)) : z \in K\} < \delta\}.$$

Thus, $f \in \mathcal{F}$ implies $f \in \bigcup_{k=1}^{n} \{\, h : \sup\{d(h(z),\ f_k(z)) : z \in K\} < \delta\}$ implies

$$f \in \{\, h : \sup\{d(h(z),\ f_k(z)) : z \in K\} < \delta\} \ \text{ for some } k\ (k = 1,\ \ldots,\ n)$$

Hence

$$\sup\{d(f(z),\ f_k(z)) : z \in K\} < \delta \text{ for some } k\ (k = 1,\ \ldots,\ n).$$

$$h \in \mathcal{F} \quad \Rightarrow \quad h \in \{f : \rho(f,\ f_k) < \varepsilon\} \text{ for some } k \ (k = 1,\ \ldots,\ n).$$

$$\Rightarrow \quad \rho(h,\ f_k) < \varepsilon \text{ for some } k \ (k = 1,\ \ldots,\ n).$$

Now, using (1.36),

$$\rho(h,\ f_k) < \varepsilon \quad \Rightarrow \quad \sup\{d(h(z),\ f_k(z)) : z \in K\} < \delta$$

That is, we have shown that $h \in \mathcal{F}$ implies $\sup\{d(h(z),\ f_k(z)) : z \in K\} < \delta$ for some k $(k = 1,\ \ldots,\ n)$.

Conversely, suppose $\mathcal{F}$ has the stated property. We have to show that $\mathcal{F}$ is normal. We note that $\mathcal{F}$ has the stated property implies that $\bar{\mathcal{F}}$ has the stated property [12]

[12]Reason: Let K be a compact set $K \subset G$ and let $\delta > 0$. Then, by the second part of Lemma 1.1.9, there is an $\varepsilon > 0$ satisfying that for $f,\ g \in C(G,\ \Omega)$,

$$\rho(f,\ g) < \varepsilon \quad \Rightarrow \quad \sup\{d(f(z),\ g(z)) : z \in K\} < \delta. \tag{1.38}$$

Let $f \in \bar{\mathcal{F}}$. By the property of closure of a set, corresponding to the above ε, there exists $g \in \mathcal{F}$ such that $\rho(f,\ g) < \varepsilon$ which implies, by (1.38), that

$$\sup\{d(f(z),\ g(z)) : z \in K\} < \delta. \tag{1.39}$$

By the assumption of the present proposition, corresponding to the above K, and δ there functions $f_1,\ \ldots,\ f_n$ in $\mathcal{F}$ such that for the $g \in \mathcal{F}$ there is at least one k, $1 \le k \le n$, with

$$\sup\{d(g(z),\ f_k(z)) : z \in K\} < \delta \tag{1.40}$$

Now, applying triangle inequality and using the property of supremum, (1.39)

Hence, without loss of generality, **let us assume that $\mathcal{F}$ is closed.** Since $C(G,\ \Omega)$ is complete, its closed subset $\mathcal{F}$ is **complete.**

Claim:$\mathcal{F}$ has the stated property implies $\mathcal{F}$ is totally bounded *(We note that, here our assumption is that $\mathcal{F}$ is closed).*

$\mathcal{F}$ has the stated property implies by (1.35) and then using Lemma 1.1.9 that there is an $\varepsilon > 0$ and there are $f_1, \ldots, f_n$ in $\mathcal{F}$ such that for any f in $\mathcal{F}$, $\rho(f,\ f_k) < \varepsilon$ for some $k = 1, \ldots, 1$. Hence $f \in B(f_k,\ \varepsilon)$ for some $k = 1, \ldots, 1$, and so

$$\mathcal{F} \subset \bigcup_{k=1}^{n} B(f_k,\ \varepsilon)$$

(showing that $\mathcal{F}$ can be covered by balls centred at the points $f_1, \ldots, f_n$ and having radius ε, and hence) $\mathcal{F}$ is totally bounded.

Now $\mathcal{F}$ is complete and totally bounded. Then, by Theorem 13, $\mathcal{F}$ is compact. Hence, again by Theorem 13, $\mathcal{F}$ is sequentially compact. Thus every sequence in $\mathcal{F}$ has a convergent subsequence. So $\mathcal{F}$ is normal. $\square$

Proposition 1.1.23. *Let $(X_n,\ d_n)$ be a metric space for each*

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and (1.40) give

$$\begin{aligned} \sup\{d(f(z),\ f_k(z)) : z \in K\} &\leq \sup\{d(f(z),\ g(z)) : z \in K\} \\ &\quad + \sup\{d(g(z),\ f_k(z)) : z \in K\} \\ &< 2\delta \end{aligned}$$

showing that $\bar{\mathcal{F}}$ also has the stated property.

$n \geq 1$ and let $X = \prod_{n=1}^{\infty} X_n$ be their Cartesian product. That is[13],

$$X = \{\xi = \{x_n\} : x_n \in X_n, \ n = 1, \ 2, \ \ldots\}$$

For $\xi = \{x_n\}$ and $\eta = \{y_n\}$ in X, define

$$d(\xi, \ \eta) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \frac{d_n(x_n, \ y_n)}{1 + d_n(x_n, \ y_n)}. \tag{1.41}$$

Then d is a metric on X. Thus, $\left(\prod_{n=1}^{\infty} X_n, \ d\right)$ is a metric space.

Proof. Referring the argument in Page 28, the series on the right hand side of (1.41) converges to a real number. So d is a well-defined function.

(i) Being the sum of series of nonnegative terms, $d(\xi, \ \eta) \geq$

[13] $X = \left\{f : \mathbb{N} \to \bigcup_{n=1}^{\infty} X_n \text{ such that } f(n) \in X_n, \ n = 1, \ 2, \ \ldots\right\}$

0.

$$
\begin{aligned}
d(\xi,\ \eta) = 0 \quad &\Leftrightarrow \quad \sum_{n=1}^{\infty} \underbrace{\left(\frac{1}{2}\right)^{n} \frac{d_n(x_n,\ y_n)}{1 + d_n(x_n,\ y_n)}}_{\text{nonnegative terms}} \\
&\Leftrightarrow \quad \frac{d_n(x_n,\ y_n)}{1 + d_n(x_n,\ y_n)} = 0 \ \text{ for } n = 1,\ 2,\ \ldots \\
&\Leftrightarrow \quad d_n(x_n,\ y_n) = 0 \ \text{ for } n = 1,\ 2,\ \ldots \\
&\Leftrightarrow \quad \underbrace{x_n = y_n \ \text{ for } n = 1,\ 2,\ \ldots}_{\text{since each } d_n \text{ is a metric}} \\
&\Leftrightarrow \quad \xi = \eta
\end{aligned}
$$

(ii) Clearly, $d(\xi,\ \eta) = d(\eta,\ \xi)$.

(iii) Since each d_n $(n = 1,\ 2,\ \ldots)$ is a metric, it satisfies the triangle inequality. Hence referring the argument in Page 29 it can be seen that d satisfies the triangle inequality.

$\square$

Proposition 1.1.24. *With the assumptions of the previous proposition, we have the following.*

If $\xi^k = \{x_1^k,\ x_2^k,\ x_3^k,\ \ldots\} = \{x_n^k\}_{n=1}^{\infty}$ is a member in $X = \prod_{n=1}^{\infty} X_n$, then the sequence $\{\xi^k\}_{k=1}^{\infty} = \{\xi^1,\ \xi^2,\ \xi^3,\ \ldots\}$ in X converges to $\xi = \{x_n\}_{n=1}^{\infty} = \{x_1,\ x_2,\ x_3,\ \ldots\}$ in X as $k \to \infty$ if and only if $x_n^k \to x_n$ in X_k as $k \to \infty$ for each n.

Proof. Suppose the sequence $\{\xi^k\} = \{\xi^1,\ \xi^2,\ \xi^3,\ \ldots\}$ converges to $\xi = \{x_n\} = \{x_1,\ x_2,\ x_3,\ \ldots\}$ in $(X,\ d)$ as $k \to \infty$. i.e., $d(\xi^k,\ \xi) \to 0$ as $k \to \infty$. Since

$$d(\xi^k,\ \xi) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \underbrace{\frac{d_n(x_n^k,\ x_n)}{1 + d_n(x_n^k,\ x_n)}}_{\text{nonnegative terms}}$$

it follows that each nonnegative term

$$\left(\frac{1}{2}\right)^n \frac{d_n(x_n^k,\ x_n)}{1 + d_n(x_n^k,\ x_n)} \le d(\xi^k,\ \xi)$$

for $n = 1,\ 2,\ \ldots,$ and hence

$$\frac{d_n(x_n^k,\ x_n)}{1 + d_n(x_n^k,\ x_n)} \le 2^n d(\xi^k,\ \xi) \tag{1.42}$$

for $n = 1,\ 2,\ \ldots$ Since $d(\xi^k,\ \xi) \to 0$ as $k \to \infty$, so by (1.42), and using the Squeeze Theorem (the Sandwich Theorem)[14],

$$0 \le \lim_{k\to\infty} \frac{d_n(x_n^k,\ x_n)}{1 + d_n(x_n^k,\ x_n)} \le 2^n \lim_{k\to\infty} d(\xi^k,\ \xi) = 0$$

so that

$$\lim_{k\to\infty} \frac{d_n(x_n^k,\ x_n)}{1 + d_n(x_n^k,\ x_n)} = 0$$

[14]**The Squeeze Theorem/the Sandwich Theorem:** Suppose that $X = (x_n)$, $Y = (y_n)$, and $Z = (z_n)$ are sequences of real numbers such that $x_n \le y_n \le z_n$ for all $n \in \mathbb{N}$ and that $\lim(x_n) = \lim(z_n)$. Then $Y = (y_n)$ is convergent and $\lim(x_n) = \lim(y_n) = \lim(z_n)$.

for $n = 1,\ 2,\ \ldots$ Hence

$$\lim_{k\to\infty} d_n(x_n^k,\ x_n) = 0$$

for $n = 1,\ 2,\ \ldots$ That is, $x_n^k \to x_n$ as $k \to \infty$ in $(X_n,\ d_n)$ for $n = 1,\ 2,\ \ldots$

Conversely, suppose $x_n^k \to x_n$ as $k \to \infty$ in $(X_n,\ d_n)$ for $n = 1,\ 2,\ \ldots$

Claim: $\{\xi^k\} \to \xi = \{x_n\}_{n=1}^{\infty}$ as $k \to \infty$ in $(X,\ d)$.

Let $\varepsilon > 0$ be given. Then $\exists N_0 \in \mathbb{N}$ such that

$$\sum_{n=N_0}^{\infty} \left(\frac{1}{2}\right)^n < \frac{\varepsilon}{2}. \tag{1.43}$$

(This follows from the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$. Ref. Page 36).

Fix n where n is taken from $n = 1,\ \ldots,\ N_0 - 1$.

As $x_n^k \to x_n$ as $k \to \infty$ for each n, it follows that for each n, there exists $N_n \in \mathbb{N}$ such that

$$d_n(x_n^k,\ x_n) < \frac{2^n\varepsilon}{2(N_0 - 1)} \text{ for } k \geq N_n.$$

Let $N = \max\{N_1,\ \ldots\ ,\ N_{N_0-1}\}$. Then for $k \geq N$,

$$d(\xi^k,\ \xi)$$

$$= \underbrace{\sum_{n=1}^{N_0-1} \underbrace{\left(\frac{1}{2}\right)^n \underbrace{\frac{d_n(x_n^k,\ x_n)}{1+d_n(x_n^k,\ x_n)}}_{<\frac{2^n\varepsilon}{2(N_0-1)}\ \text{ for } k\geq N}}_{<\frac{\varepsilon}{2(N_0-1)}\ \text{ for } k\geq N}}_{<\frac{\varepsilon}{2}\ \text{ for } k\geq N} + \underbrace{\sum_{n=N_0}^{\infty} \underbrace{\left(\frac{1}{2}\right)^n \underbrace{\frac{d_n(x_n^k,\ x_n)}{1+d_n(x_n^k,\ x_n)}}_{\leq 1}}_{<\frac{1}{2^n}}}_{\sum\limits_{n=N_0}^{\infty}\left(\frac{1}{2}\right)^n<\frac{\varepsilon}{2}\quad \text{by } 1.43}$$

i.e.,

$$d(\xi^k,\ \xi) < \varepsilon \text{ for } k \geq N.$$

Hence $\{\xi^k\}$ converges to ξ as $k \to \infty$. $\square$

Proposition 1.1.25. *With the assumptions of Proposition 1.1.23, we have the following. If each $(X_n,\ d_n)$ is compact then X is compact.*[15]

Proof. Suppose each $(X_n,\ d_n)$ is compact. To show that $(X,\ d)$ is compact, by Theorem 13 it suffices to show that every sequence in X has a convergent subsequence; this is accomplished by the **Cantor diagonalization process.**
For each $k = 1,\ 2,\ \ldots,$ let

$$\xi^k = \{x_1^k,\ x_2^k,\ \ldots\} = \{x_n^k\}_{n=1}^{\infty}$$

[15] **Tychnoff Theorem:** The product of compact spaces is compact.

be an element in X. We list ξ^1, ξ^2, ... as below:

$$
\begin{aligned}
\xi^1 &= \{ \underset{\substack{\in X_1 \\ \text{(first} \\ \text{term)}}}{x_1^1}, \quad \underset{\substack{\in X_2 \\ \text{(second} \\ \text{term)}}}{x_2^1}, \ldots\} \\
\xi^2 &= \{ \underset{\substack{\in X_1 \\ \text{(first} \\ \text{term)}}}{x_1^2}, \quad \underset{\substack{\in X_2 \\ \text{(second} \\ \text{term)}}}{x_2^2}, \ldots\} \\
\xi^3 &= \{ \underset{\substack{\in X_1 \\ \text{(first} \\ \text{term)}}}{x_1^3}, \quad \underset{\substack{\in X_2 \\ \text{(second} \\ \text{term)}}}{x_2^3}, \ldots\} \\
&\vdots
\end{aligned}
$$

We then consider the sequence of first entries of each ξ^k, $k = 1, 2, \ldots$ given by

$$\{x_1^1,\ x_1^2,\ x_1^3,\ \ldots\} = \{x_1^k\}_{k=1}^{\infty}.$$

(Members of the new sequence are elements in X_1.) Since X_1 is compact, by Theorem 13, X_1 is sequentially compact, and hence the sequence $\{x_1^k\}_{k=1}^{\infty}$ of members of X_1 has a convergent subsequence, say,

$$\{x_1^k : k \in \mathbb{N}_1\}$$

where $\mathbb{N}_1$ is an **infinite** subset of $\mathbb{N}$ such that $\{x_1^k : k \in \mathbb{N}_1\}$ converges to some point $x_1 \in X_1$. That is,

$$\lim\{x_1^k : k \in \mathbb{N}_1\} = x_1.$$

Consider the sequence

$$\{x_2^k : k \in \mathbb{N}_1\}$$

of second entries of $\{\xi^k : k \in \mathbb{N}_1\}$ [Attention! Here we are not considering the sequence $\{x_2^k\}_{k=1}^{\infty}$ of second entries of each ξ^k, $k = 1, 2, \ldots$, but $\{x_2^k : k \in \mathbb{N}_1\}$ that is the sequence of second entries of ξ^k, $k \in \mathbb{N}_1$.] Now the sequence $\{x_2^k : k \in \mathbb{N}_1\}$ in the sequentially compact space X_2 has a convergent subsequence, say, $\{x_2^k : k \in \mathbb{N}_2\}$ where $\mathbb{N}_2$ is an **infinite** subset of $\mathbb{N}_1$ such that $\{x_2^k : k \in \mathbb{N}_2\}$ converges to some point $x_2 \in X_2$. That is,

$$\lim\{x_2^k : k \in \mathbb{N}_2\} = x_2.$$

Since $\lim\{x_1^k : k \in \mathbb{N}_1\} = x_1$ and since $\{x_1^k : k \in \mathbb{N}_2\}$ is a subsequence of $\{x_1^k : k \in \mathbb{N}_1\}$ it follows that

$$\lim\{x_1^k : k \in \mathbb{N}_2\} = x_1.$$

Continuing this process, we obtain a decreasing sequence of infinite subsets of $\mathbb{N}$ given by

$$\mathbb{N}_1 \supset \mathbb{N}_2 \supset \mathbb{N}_3 \supset \cdots$$

and points x_n in X_n such that

$$\lim\{x_n^k : k \in \mathbb{N}_n\} = x_n. \tag{1.44}$$

Let k_j be the jth integer in $\mathbb{N}_j$ and consider the sequence

$$\left\{\xi^{k_j}\right\}_{j=1}^{\infty} = \{\xi^{k_1},\ \xi^{k_2},\ \xi^{k_3},\ \ldots\}$$

Claim: $\left\{\xi^{k_j}\right\}_{j=1}^{\infty}$ converges to $\xi = \{x_1,\ x_2,\ \ldots\} = \{x_n\}_{n=1}^{\infty}$ as $j \to \infty$. To prove the claim, by Proposition 1.1.25, it suffices to show that for $n = 1,\ 2,\ \ldots,$

$$x_n = \lim_{j\to\infty} x_n^{k_j} \tag{1.45}$$

[Explanation:

$$\begin{aligned}
\xi^{k_1} &= \{\ \underset{\substack{\uparrow \\ \text{first} \\ \text{term}}}{x_1^{k_1}}\ ,\quad \underset{\substack{\uparrow \\ \text{second} \\ \text{term}}}{x_2^{k_1}}\ ,\ \ldots\} \\
\xi^{k_2} &= \{\ \underset{\substack{\uparrow \\ \text{first} \\ \text{term}}}{x_1^{k_2}}\ ,\quad \underset{\substack{\uparrow \\ \text{second} \\ \text{term}}}{x_2^{k_2}}\ ,\ \ldots\} \\
&\vdots \\
\xi &= \{x_1,\ x_2,\ \ldots\}
\end{aligned}$$

By looking the above display it is obvious that $\left\{\xi^{k_j}\right\}_{j=1}^{\infty}$ converges to $\xi = \{x_1,\ x_2,\ \ldots\} = \{x_n\}_{n=1}^{\infty}$ as $j \to \infty$ if and only if sequence of first terms $\{x_n^{k_1}\}_{n=1}^{\infty}$ converges to x_1, sequence of second terms $\{x_n^{k_2}\}_{n=1}^{\infty}$ converges to x_2, and so on. i.e., if and only if (1.45) holds.] But, by construction,

$$\mathbb{N}_j \subset \mathbb{N}_n$$

for $j \geq n$. Hence k_j, the jth integer in $\mathbb{N}_j$, is also in $\mathbb{N}_n$ for $j \geq n$. That is, when $j \geq n$, k_j, the jth integer in $\mathbb{N}_j$; k_{j+1}, the $(j+1)$th integer in $\mathbb{N}_{j+1}$; k_{j+2}, the $(j+2)$th integer in $\mathbb{N}_{j+2}$; etc. are in $\mathbb{N}_n$. So when $j \geq n$

$$\{k_j,\ k_{j+1},\ k_{j+2},\ \ldots\} \subseteq N_n$$

with

$$k_j < k_{j+1} < k_{j+2} < \cdots .$$

So

$$\{x_n^{k_j} : j \geq n\} = \{x_n^{k_j},\ x_n^{k_{j+1}},\ \ldots\}$$

is a subsequence of $\{x_n^k : k \in \mathbb{N}_n\}$. Thus,

$$\lim_{j\to\infty} x_n^{k_j} = \lim\{x_n^k : k \in \mathbb{N}_n\}$$

and so (1.45) follows from (1.44). We have shown that the sequence $\{\xi^k\}_{k=1}^{\infty}$ with members from X has a subsequence $\left\{\xi^{k_j}\right\}_{j=1}^{\infty}$ that converges to $\xi = \{x_1,\ x_2,\ \ldots\} = \{x_n\}_{n=1}^{\infty}$ as $j \to \infty$. Since $\{\xi^k\}_{k=1}^{\infty}$ is an arbitrary sequence in X, it follows that every sequence in X has a convergent subsequence in X, showing that X is sequentially compact. Since X is a metric space, this implies that X is compact. □

1.1.2 Equicontinuity of a Family of Continuous Functions

$C(G,\ \Omega)$ is the family of continuous functions from G to Ω. Hence, if $f \in C(G,\ \Omega)$ implies f is continuous at any point on G. Let $z_0 \in G$. Then for every $\varepsilon > 0$ there exists a $\delta = \delta(f,\ \varepsilon,\ z_0) > 0$ such that

$$z \in G \text{ and } |z - z_0| < \delta \quad \Rightarrow \quad d(f(z),\ f(z_0)) < \varepsilon.$$

Here $\delta = \delta(f,\ \varepsilon,\ z_0)$ depends on the function f in $C(G,\ \Omega)$, and may vary for different functions in $C(G,\ \Omega)$. We now search the possibility of having a common $\delta = \delta(\varepsilon,\ z_0)$ for all f in some subcollection of $C(G,\ \Omega)$.

Definition 1.1.26 (*Equicontinuity of a family of continuous functions at a point*). A set $\mathcal{F} \subset C(G,\ \Omega)$ is **equicontinuous at a point** $z_0 \in G$ if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon,\ z_0) > 0$ such that

$$z \in G \text{ and } \ |z - z_0| < \delta \quad \Rightarrow \quad d(f(z),\ f(z_0)) < \varepsilon \ \forall\ f\ \in \mathcal{F}.$$

Remark 1.1.27. In the above definition δ depends only on the choice of the point $z_0 \in G$ and $\varepsilon > 0$ and the same δ works for every f in the family $\mathcal{F}$.

Example 1.1.28. If $\mathcal{F} = \{f\}$ consists of the single function f alone, then the statement that $\mathcal{F}$ is equicontinuous at $z_0 \in G$ means for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon,\, z_0) > 0$ such that

$$z \in G \text{ and } |z - z_0| < \delta \quad \Rightarrow \quad d(f(z),\ f(z_0)) < \varepsilon$$

which is equivalent to the statement that f is continuous at z_0.

Definition 1.1.29 (*Equicontinuous over a set* $E \subset G$). A set $\mathcal{F} \subset C(G,\ \Omega)$ is **equicontinuous over a set** $E \subset G$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$z \,\text{and}\, z' \in E \text{ and } |z - z'| < \delta \;\Rightarrow\; d(f(z),\ f(z')) < \varepsilon \;\forall\; f\; \in \mathcal{F}.$$

Example 1.1.30. [$\{f\}$ **is equicontinuous over a set implies** f **is uniformly continuous on that set**] If $\mathcal{F} = \{f\}$ consists of the single function f, then the statement that $\mathcal{F}$ is equicontinuous over E means for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$z \,\text{and}\, z' \in E \text{ and } |z - z'| < \delta \;\Rightarrow\; d(f(z),\ f(z')) < \varepsilon$$

which is equivalent to the statement that f is uniformly continuous on E.

Proposition 1.1.31. *Suppose* $\mathcal{F} \subset C(G,\ \Omega)$ *is equicontin-*

uous at each point of G; then $\mathcal{F}$ is equicontinuous over each compact subset of G.

Proof. Let K be a compact subset of G.
Claim: $\mathcal{F}$ is equicontinuous on K.
Fix $\varepsilon > 0$. Then, by the pointwise equicontinuity of $\mathcal{F}$, for every $w \in K$ there exists $\delta_w = \delta_w(\varepsilon,\ w) > 0$ such that

$$w' \in G \text{ and } |w - w'| < \delta_w \quad \Rightarrow$$

$$d(f(w),\ f(w')) < \frac{\varepsilon}{2} \text{ for all } f \in \mathcal{F}. \tag{1.46}$$

Also we note that

$$K \subseteq \bigcup_{w \in K} B(w;\ \delta_w).$$

and hence

$$\{B(w;\ \delta_w) : w \in K\}$$

forms an open cover for the compact set K. Hence by the Lebesgue Covering Lemma[16] there is a $\delta > 0$ such that if $z \in K$,

$$B(z,\ \delta) \subset B(w,\ \delta_w) \text{ for some } w \in K.$$

[16] **[Lebesgue's Covering Lemma:]** If $(X,\ d)$ is a compact space and $\mathcal{U}$ is an open cover of X then there is an $\delta > 0$ such that if $x \in X$, there is a set U in $\mathcal{U}$ with $B(x,\ \delta) \subset U$. (In other words, every ball of radius δ is contained in some member of $\mathcal{U}$.

Thus, we have

$$z \text{ and } z' \text{ in } K \text{ and } \underbrace{|z - z'| < \delta}_{z' \in B(z,\ \delta)} \Rightarrow$$

$$\exists w \in K \text{ with } \underbrace{z' \in B(z,\ \delta)}_{\text{since} |z-z'|<\delta} \subset B(w;\ \delta_w). \tag{1.47}$$

From (1.47) we observe that $z \in B(w;\ \delta_w)$ (i.e., $|z - w| < \delta_w$) and $z' \in B(w;\ \delta_w)$ (i.e., $|z' - w| < \delta_w$) which then implies by (1.46) that

$$d(f(z),\ f(w)) < \frac{\varepsilon}{2} \text{ for all } f \in \mathcal{F}$$

and

$$d(f(z'),\ f(w)) < \frac{\varepsilon}{2} \text{ for all } f \in \mathcal{F}.$$

Hence by the triangle inequality,

$$d(f(z),\ f(z')) \leq d(f(z),\ f(w)) + d(f(w),\ f(z')) < \varepsilon$$

for all $f \in \mathcal{F}$. i.e., we have shown that

$$z \text{ and } z' \text{ in } K \text{ and } |z - z'| < \delta \quad \Rightarrow \quad d(f(z),\ f(z')) < \varepsilon\ \forall\ f \in \mathcal{F}.$$

Thus, $\mathcal{F}$ is equicontinuous over K. $\square$

Corollary 1.1.32. *Any $f \in C(G,\ \Omega)$ is uniformly continuous over each compact subset of G.*

Proof. Let $\mathcal{F} = \{f\}$, the singleton subfamily of $C(G,\ \Omega)$ consisting of f alone, then by Example 1.1.28, $\mathcal{F}$ is equicontinuous at each point of G. Then by Proposition 1.1.31, if K is a compact subset of G, then $\mathcal{F} = \{f\}$ is equicontinuous over K. Then, by Example 1.1.30, f is uniformly continuous over K. $\square$

Theorem 1.1.33. ***[Arzela-Ascoli Theorem]*** *A set $\mathcal{F} \subset C(G,\ \Omega)$ is normal if and only if the following two conditions are satisfied:*
(a). for each $z \in G$, $\{f(z) : f \in \mathcal{F}\}$ has compact closure in Ω;
(b). $\mathcal{F}$ is equicontinuous at each point of G.

Proof. First assume that $\mathcal{F}$ is normal. Notice that for each $z \in G$ the map $\varphi_z : C(G,\ \Omega) \to \Omega$ defined by

$$\varphi_z(f) = f(z), \quad f \in C(G,\ \Omega)$$

is continuous. [Details: Fix $z \in G$. Then the function φ_z is continuous at each $f \in C(G,\ \Omega)$ because of the following observation: We use The Sequential Criterion for the Continuity of a Function on Metric Spaces[17]: We show that φ_z

[17] **The Sequential Criterion for the Continuity of a Function on**

is continuous at f by verifying that for any sequence (f_n) in $C(G,\ \Omega)$

$$f_n \to f \text{ as } n \to \infty \text{ in } (C(G,\ \Omega),\ \rho)$$

$$\begin{aligned}
\Rightarrow\quad & \text{applying Part (b) of Proposition 1.1.10,} \\
& \text{for the compact singleton subset } \{z\}, \\
& f_n(z) \to f(z) \text{ as } n \to \infty \\
\Rightarrow\quad & \varphi_z(f_n) \to \varphi_z(f) \text{ as } n \to \infty \\
\Rightarrow\quad & \varphi_z \text{ is continuous at } f \quad]
\end{aligned}$$

Since $\mathcal{F}$ is normal, by Proposition **??**, $\bar{\mathcal{F}}$ is compact and hence its continuous image $\varphi_z(\bar{\mathcal{F}})$ is compact in Ω. i.e., for a fixed $z \in G$, $\{f(z) : f \in \bar{\mathcal{F}}\}$ is compact. Since

$$\{f(z) : f \in \mathcal{F}\} \subset \{f(z) : f \in \bar{\mathcal{F}}\},$$

the fact that $\{\ f(z) : f \in \bar{\mathcal{F}}\}$ is closed and bounded (since it is compact) implies that the closure of $\{f(z) : f \in \mathcal{F}\}$ is a subset of $\{\ f(z) : f \in \overline{\mathcal{F}}\}$.

Now being the closed subset of the compact set $\{\ f(z) : f \in$

Metric Spaces: Let $(X,\ d_X)$ and $(Y,\ d_Y)$ be metric spaces and let $f : X \to Y$. Then f is continuous at the point $x \in X$ if and only if for all sequences $\{x_n\}_{n=1}^{\infty}$ in X that converge to x we have that $\{f(x_n)\}_{n=1}^{\infty}$ in Y converges to $f(x)$. *This result is a special case of Theorem 1.55, Foundations of Topology, C. Wayne Patty.*

$\bar{\mathcal{F}}\}$, closure of $\{f(z) : f \in \mathcal{F}\}$ is also compact. Since $z \in G$ is arbitrary, it follows that for each $z \in G$, $\{f(z) : f \in \mathcal{F}\}$ has compact closure. This completes the proof of Part (a).

To show (b) fix a point $z_0 \in G$ and let $\varepsilon > 0$. Since G is open and $z_0 \in G$ it is possible to find $r > 0$ such that $B(z_0;\ \ r) \subset G$. Choose $0 < R < r$, then

$$K = \overline{B(z_0;\ \ R)} \subset B(z_0;\ \ r) \subset G.$$

Being closed and bounded K is compact. Now corresponding to the compact set K and the above $\varepsilon > 0$, by Proposition 1.1.22 (with $\frac{\varepsilon}{3}$ in place of δ,) there are functions $f_1, \ldots, f_n$ in $\mathcal{F}$ such that for each $f \in \mathcal{F}$ there is at least one f_k with

$$\sup\{d(f(z),\ f_k(z)) : z \in K\} < \frac{\varepsilon}{3}. \tag{1.48}$$

For each $j = 1, \ldots, n$, f_j is continuous at z_0, and hence corresponding to the above ε, there is a δ_j such that

$$|z - z_0| < \delta_j \ \ \Rightarrow \ \ d(f_j(z),\ f_j(z_0)) < \frac{\varepsilon}{3}$$

Then we let $\delta_0 = \min\{\delta_1,\ \ldots\ ,\ \delta_n\}$ and choose δ such that $0 < \delta < \min\{\delta_0,\ R\}$. Then

$$|z - z_0| < \delta \ \ \Rightarrow \ \ d(f_j(z),\ f_j(z_0)) < \frac{\varepsilon}{3} \ \ (j = 1,\ \ldots\ ,\ n).$$

Hence, if $|z - z_0| < \delta$, $f \in \mathcal{F}$ and k is chosen so that (1.48) holds, then

$$d(f(z),\ f(z_0))$$

$$\leq \underbrace{d(f(z),\ f_k(z))}_{<\frac{\varepsilon}{3}} + \underbrace{d(f_k(z),\ f_k(z_0))}_{<\frac{\varepsilon}{3}} + \underbrace{d(f_k(z_0),\ f(z_0))}_{<\frac{\varepsilon}{3}} < \varepsilon$$

That is, we have shown that corresponding to the point $z_0 \in G$ for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon,\ z_0) > 0$ such that

$$z \in G \text{ and } |z - z_0| < \delta \quad \Rightarrow \quad d(f(z),\ f(z_0)) < \varepsilon \,\forall f\ \in \mathcal{F}$$

and hence $\mathcal{F}$ is equicontinuous at z_0.

Now suppose $\mathcal{F}$ satisfies conditions (a) and (b). We **claim** that $\mathcal{F}$ is normal. Let $\{z_n\}$ be the sequence of **all points** in G with **rational** real and imaginary parts[18]. We note that the set of points in $\{z_n\}$ is dense in G.[19]

Then

$$\text{for } z \in G \text{ and } \delta > 0 \text{ there is a } z_n \text{ with } |z - z_n| < \delta \tag{1.49}$$

because $B(z,\ \delta)$ contains points in G with rational real and

[18] We consider *all points* in G *with rational real and imaginary parts* to form the sequence $\{z_n\}$; such a sequence can be constructed because we are considering points in G with **rational** real and imaginary parts; and the set of *such points* is countable, and hence we can enumerate them as $z_1, z_2, \ldots$ giving a sequence in G.

[19] $\mathbb{Q} \times \mathbb{Q}$ is dense in $\mathbb{R}$. $\{a + i\,b = (a,\ b) : a,\ b \in \mathbb{Q}\}$ is dense in $\mathbb{C}$.

imaginary parts.

For each $n \geq 1$, let

$$X_n = \overline{\{f(z_n) : f \in \mathcal{F}\}} \subset \Omega,$$

then from part (a), $(X_n,\ d)$ is a compact metric space. Thus, by Proposition 1.1.25, $X = \prod_{n=1}^{\infty} X_n$ is a compact metric space. For $f \in \mathcal{F}$ define $\tilde{f}$ in X by

$$\tilde{f} = (\underbrace{f(z_1)}_{\in X_1},\ \underbrace{f(z_2)}_{\in X_2},\ \ldots).$$

Let $\{f_k\}$ be a sequence in $\mathcal{F}$; so[20] $\{\tilde{f}_k\}$ is a sequence in the compact metric space X. Then, as X is sequentially compact, there is a $\xi \in X$ and a subsequence of $\{\tilde{f}_k\}$ which converges to ξ. For the sake of of convenient notation, we denote the convergent subsequence also by $\{\tilde{f}_k\}$ so that

$$\xi = \lim \tilde{f}_k.$$

Hence, using Proposition 1.1.24, if we let $\xi = \{\omega_n\}_{n=1}^{\infty} =$

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$$\tilde{f}_1 = (f_1(z_1),\ f_1(z_2),\ \ldots),$$
$$\tilde{f}_2 = (f_2(z_1),\ f_2(z_2),\ \ldots),$$
$$\vdots$$

$\{\omega_1, \omega_2, \ldots\}$, then for $n = 1, 2, \ldots,$

$$\lim_{k\to\infty} f_k(z_n) = \omega_n \tag{1.50}$$

Claim: $\{f_k\}$ (subsequence of the original sequence $\{f_k\}$) converges to a function $f \in C(G, \Omega)$.

By (1.50) the function f will have to satisfy

$$f(z_n) = \omega_n. \tag{1.51}$$

The importance of (1.50) is that it imposes control over the behavior of $\{f_k\}$ on a dense subset of G *(Note: Here the dense set $\{z_n : n \in \mathbb{N}\}$ of G, and f is defined on this dense subset of G by (1.51).* We will use the fact that $\{f_k\}$ is equicontinuous to spread this control to the rest of G.

To find the function f and show that $\{f_k\}$ converges to f it suffices (since $C(G, \Omega)$ is complete) to show that $\{f_k\}$ is a Cauchy sequence. i.e., we will show that for every $\varepsilon > 0$ there is a natural number J such that

$$\rho(f_k, f_j) < \varepsilon \quad \text{for all } k, j \geq J. \tag{1.52}$$

For this, by Lemma 1.1.9, it suffices to find an integer J such that for $k,\ j \geq J$,

$$\sup\{d(f_k(z),\ f_j(z)) : z \in K\} < \varepsilon + 1 \tag{1.53}$$

because then, with $\delta = \varepsilon + 1$, (1.52) follows.

So let $\varepsilon > 0$ and K be a compact set in G. Since K is a compact (and hence closed and bounded) subset of the open set G, boundary points of G are not members of K so that [21]

$$d(K,\ \partial G) = \inf\{|k-h| : k \in K,\ h \in \partial G\} > 0$$

and we let

$$R = d(K,\ \partial G).$$

Let

$$K_1 = \left\{ z : \underbrace{d(z,\ K)}_{\inf\{|z-k| : k \in K\}} \leq \frac{1}{2}R \right\}; \tag{1.54}$$

[21] $d(K,\ \partial G) = 0$ if and only if $\inf\{d(k,\ h) : k \in K,\ h \in \partial G\} =$ if and only there is an $h \in \partial G$ and a sequence $\{k_n\}$ in K such that $d(k_n,\ h) \to 0$ as $n \to \infty$. This shows that $h \in \bar{K} = K$ which is a contradiction to the fact that $K \cap \partial G = \emptyset$.

then [22] K_1 is compact[23] and

$$K \subset \operatorname{int} K_1 \subset K_1 \subset G.$$

Since $\mathcal{F}$ is equicontinuous at each point of G it is equicontinuous on the compact subset K_1 by Proposition 1.1.31. So choose δ, $0 < \delta < \frac{1}{2}R$, such that

$$z \text{ and } z' \text{ in } K_1 \text{ and } |z - z'| < \delta \Rightarrow d(f(z),\ f(z')) < \frac{\varepsilon}{3}\ \forall f \in \mathcal{F} \tag{1.55}$$

Now let D be the collection of points in $\{z_n\}$ which are also points in K_1; that is,

$$D = \{z_n : z_n \in K_1\}. \tag{1.56}$$

If $z \in K$ then (by (1.49)) there is a z_n with $|z - z_n| < \delta$; but $\delta < \frac{1}{2}R$ gives that[24]

$$d(z_n,\ K) < \frac{1}{2}R$$

or (using (1.54)) that $z_n \in K_1$ implies (using (1.56)) $z_n \in D$.

[22] $K \subset K_1$ because for any $z \in K$, $d(z, K) = 0$ and hence $z \in K_1$. Also, we note that $K \neq K_1$ as $K = K_1$ leads to the contradiction, because then it is impossible to find a z such that $d(z, K) > 0$ which shows that all points in G are in K. Hence $K \subset \operatorname{int} K_1 \subset K_1$.

[23] K_1 is closed and bounded and hence compact.

[24]
$$d(z_n,\ K) = \inf\{d(z_n, k) : k \in K\} \le |z_n - z| < \delta < \frac{1}{2}R.$$

That is we have shown that $z \in K$ implies $z \in B(z_n;\ \delta)$ for some $z_n \in D$. Hence

$$K \subseteq \bigcup_{w \in D} B(w,\ \delta)$$

so that $\{B(w;\ \delta) : w \in D\}$ is an open cover for the compact set K and hence this open cover has a finite sub collection that covers K. Let $w_1, \ldots, w_n \in D$ such that

$$K \subset \bigcup_{i=1}^{n} B(w_i;\ \delta). \tag{1.57}$$

$w_1, \ldots, w_n$ are in D, hence by (1.56) they are terms of the sequence $\{z_n\}$ and so by (1.50),

$$\lim_{k \to \infty} f_k(w_i) \text{ exists for } 1 \leq i \leq n.$$

So[25] there is an integer J such that for $j,\ k \geq J$,

$$d(f_k(w_i),\ f_j(w_i)) < \frac{\varepsilon}{3} \tag{1.58}$$

for $i = 1, \ldots, n$.

[25] $\lim_{k \to \infty} f_k(w_i)$ exists for $1 \leq i \leq n$ *implies* for $1 \leq i \leq n$, $\{f_k(w_i)\}_{k=1}^{\infty}$ is a Cauchy sequence; hence there are natural numbers $J_1, \ldots, J_n$ such that for $1 \leq i \leq n$,

$$d(f_k(w_i),\ f_j(w_i)) < \frac{\varepsilon}{3} \text{ for } j,\ k \geq J_i.$$

Take $J = \max\{J_1, \ldots, J_n\}$. Then (1.58) follows.

Let z be an arbitrary point in K and let w_i be such that $|w_i - z| < \delta$ (this is possible by (1.57)). If k and j are larger than J then (1.55) and (1.58) give

$$\begin{aligned} & d(f_k(z),\ f_j(z)) \\ \leq \ & \underbrace{d(f_k(z),\ f_k(w_i))}_{<\frac{\varepsilon}{3}} + \underbrace{d(f_k(w_i),\ f_j(w_i))}_{<\frac{\varepsilon}{3}} + \underbrace{d(f_j(w_i),\ f_j(z))}_{<\frac{\varepsilon}{3}} \\ < \ & \varepsilon. \end{aligned}$$

Since z was arbitrary this establishes that for $k,\ j \geq J$,

$$\sup\{d(f_k(z),\ f_j(z)) : z \in K\} \leq \varepsilon < \varepsilon + 1.$$

i.e., (1.53) is obtained and this completes the proof. □

1.2 Spaces of Analytic Functions

Let G be an open subset of the complex plane. $H(G)$ denotes the set of holomorphic (analytic) functions on G.

Theorem 1.2.1. [$H(G)$ is a closed subset of $C(G,\ \mathbb{C})$ and $\varphi : H(G) \to H(G)$ defined by $\varphi(f) = f'$ is a continuous function] *If $\{f_n\}$ is a sequence in $H(G)$ and f belongs to*

$C(G, \mathbb{C})$ such that f_n converges to f[26], then f is analytic on G (so $H(G)$ is a closed subset of $C(G, \mathbb{C})$) and $f_n^{(k)} \to f^{(k)}$ as $n \to \infty$ for each $k \geq 1$.

Proof. We will show that f is analytic by applying Morera's theorem[27]. So let T be a triangle contained inside a disk $D \subset G$. [*Note*: Here we take a disk D to ensure a region (because Morera's Theorem works on regions). Given G is just an open set, need not be a region. So G may have holes. If G contains holes then G is not not a region. So we take arbitrary disks in G and show that f is analytic on D; and since these disks cover G, we conclude that f is analytic on G.]

[26]By Proposition 1.1.10, a sequence $\{f_n\}$ in $(C(G, \Omega), \rho)$ converges to f if and only if $\{f_n\}$ converges to f ***uniformly on all compact subsets of*** G.. Hence, *in the present theorem,* f_n converges to f in $H(G)$ implies f_n converges **uniformly on every compact subset of** G

[27]**Morera's Theorem** THEOREM IV.5.10. . Let G be a region[28] and let $f : G \to \mathbb{C}$ be a continuous function such that

$$\int_T f = 0$$

for every closed triangular path T in G (i.e., T is a closed polygon with three sides); then f is analytic in G.

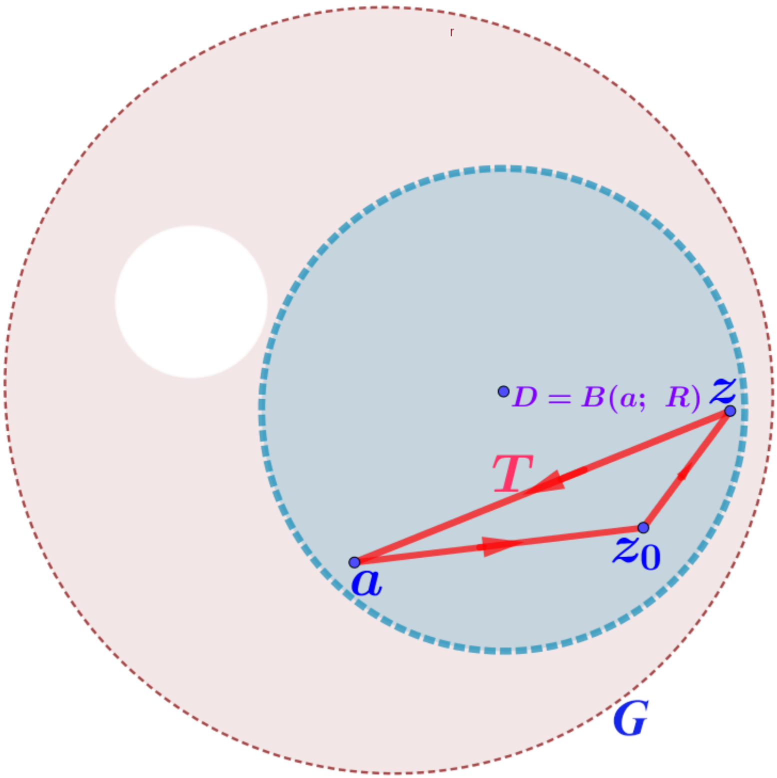

Figure 1.14: T is a triangle contained inside the disk $D \subset G$.

By assumption, f_n converges to f uniformly on every compact of subset of G, hence it follows, in particular that, $\{f_n\}$ converges to f uniformly over (*the closed and bounded, and hence compact triangle*) T. Hence (we can interchange limit and integration, and so)

$$\int_T f = \int_T \lim f_n = \lim \underbrace{\int_T f_n}_{\substack{=0,\text{ since each} \\ f_n \text{is, analytic, on } D}} = 0.$$

Thus, by Morera's theorem, f is analytic on D. Since $D \subset G$ is an arbitrary disk, this shows that f is analytic in every disk $D \subset G$; and this shows that f is analytic in G.

To show that for each $k \geq 1$, $f_n^{(k)} \to f^{(k)}$ as $n \to \infty$, let[29]

$$D = \overline{B(a;\ r)} \subset G$$

Then there is a number $R > r$ such that[30]

$$\overline{B(a;\ R)} \subset G.$$

[29]Existence of a point a as centre and r as radius possible, since G is open; reason: As G is open, if we let $a \in G$ then there is an $r_1 > 0$ such that $B(a;\ r_1) \subset G$. Now choose $0 < r < r_1$. Then $\overline{B(a;\ r)} \subset B(a;\ r_1) \subset G$.

[30]Existence of such an R, $(R > r)$ is guaranteed by the fact that $\overline{B(a;\ r)}$ is a closed subset of the open region G.

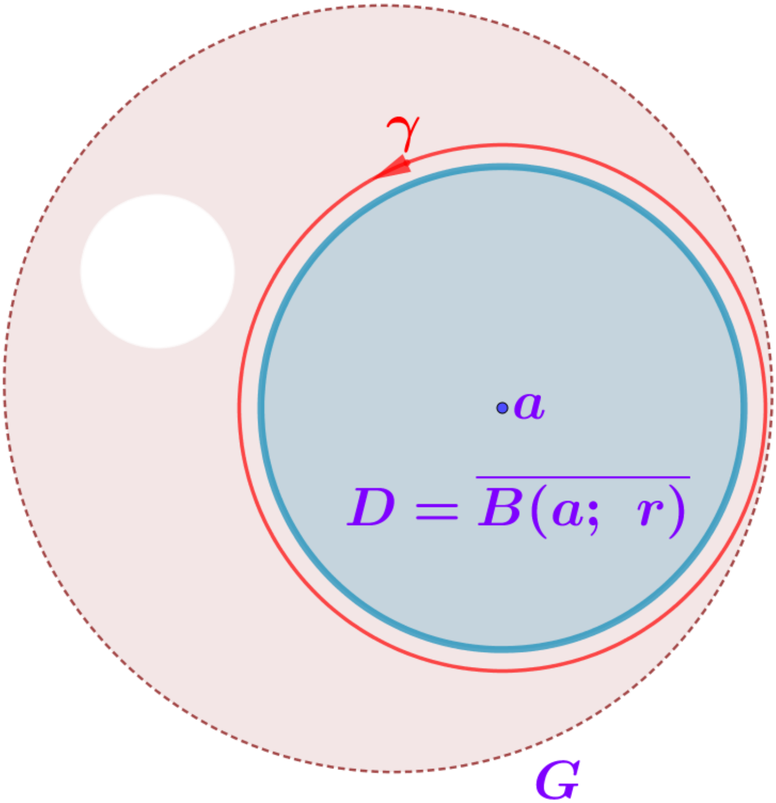

Figure 1.15: γ is the circle $|z-a| = R$ so that $z \in \{\gamma\}$ if and only if $|z-a| = R$.

If γ is the circle $|z-a| = R$ then for the function[31] $f_n - f$

[31] Being the difference of analytic functions, for each $n \geq 1$, $f_n - f$ is analytic on G.

Cauchy's Integral Formula[32] gives

$$f_n^{(k)}(z) - f^{(k)}(z) = (f_n - f)^{(k)}(z) = \frac{k!}{2\pi i} \int_\gamma \frac{f_n(w) - f(w)}{(w-z)^{k+1}} dw$$

so that[33]

$$\left| f_n^{(k)}(z) - f^{(k)}(z) \right| \leq \frac{k!}{2\pi} \int_\gamma \frac{|f_n(w) - f(w)|}{|w-z|^{k+1}} |dw| . \qquad (1.59)$$

Now we proceed as in the proof of Cauchy's Estimate (*Conway* Text Page 73). As w are points on the circle $\gamma : |z-a| = R$ they have the form

$$w = a + Re^{it}, \; 0 \leq t \leq 2\pi$$

[32]**Corollary to Cauchy's Integral Formula:** Let G be an open set and $f : G \to C$ an analytic function. If $\gamma_1, \; \gamma_2, \; \ldots, \; \gamma_m$ are closed rectifiable curves in G such that

$$n(\gamma_1;\, w) + n(\gamma_2;\, w) + \cdots + n(\gamma_n;\, w) = 0 \text{ for all } w \; \in \mathbb{C} \sim G,$$

then for $a \in G \sim \bigcup_{k=1}^{m} \{\, \gamma_k \,\}$ and $k \geq 1$

$$f^{(k)}(a) \sum_{j=1}^{m} n(\gamma_j;\; a) = k! \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{f(z)}{(z-a)^{k+1}} dz.$$

[33]Using Proposition 1.17 in Conway:

$$\left| \int_\gamma g \right| \leq \int_\gamma |g| \, |dz|.$$

and if z is a point in the closed disk $\overline{B(a;\, r)}$ it has the form

$$z = a + \eta\, e^{i\theta},$$

for some η and θ with $0 \leq \eta \leq r < R$ and $0 \leq \theta \leq 2\pi$.

Then

$$\begin{aligned} |w - z| &= \left| a + R\, e^{it} - (a + \eta\, e^{i\theta}) \right| \\ &= \left| R\, e^{it} - \eta\, e^{i\theta} \right| \\ &\geq \left| R\, e^{it} \right| - \left| \eta\, e^{i\theta} \right| \\ &= R - \eta \\ &\geq R - r \end{aligned}$$

so that

$$\frac{1}{|w - z|} \leq \frac{1}{R - r}$$

and hence (1.59) gives

$$\left| f_n^{(k)}(z) - f^{(k)}(z) \right| \leq \frac{k!}{(R - r)^{k+1} 2\pi} \int_{\gamma} |f_n(w) - f(w)| \underbrace{Rdt}_{|dw| = |Rie^{it}dt|}$$

and if for $n = 1,\, 2,\, \ldots$ we let

$$M_n = \sup\{|f_n(w) - f(w)| : |w - a| = R\}$$

it follows that

$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| \leq \frac{k!M_n R}{(R-r)^{k+1}2\pi} \underbrace{\int_\gamma dt}_{2\pi} \quad \text{for } |z-a| \leq r$$

That is,

$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| \leq \frac{k!M_n R}{(R-r)^{k+1}} \text{ for } |z-a| \leq r. \qquad (1.60)$$

But since f_n converges to f uniformly on every compact of subset of G, it follows, in particular that, f_n converges to f uniformly on $\{\gamma\}$ and hence for every $\varepsilon > 0$ there is a natural number N such that

$$|f_n(w) - f(w)| < \varepsilon \ \ \forall n \geq N, \ \forall w \in \{\gamma\}$$

so that

$$\sup\{|f_n(w) - f(w)| : w \in \{\gamma\}\} \leq \varepsilon \ \ \forall n \geq N.$$

Then

$$\lim M_n = 0.$$

Hence for every $\varepsilon > 0$ there is a natural number M such that

$$M_n < \frac{(R-r)^{k+1}}{k!R}\varepsilon \text{ for all } n \geq M$$

so it follows from (1.60) that

$$\left|f_n^{(k)}(z) - f^{(k)}(z)\right| < \varepsilon \text{ for all } n \geq M \text{ and } |z - a| \leq r.$$

As the choice of M depends only on ε it follows that $f_n^{(k)} \to f^{(k)}$ **uniformly on** $\overline{B(a;\ r)}$. Now if K is an arbitrary compact subset of G and $0 < r < d(K,\ \partial G)$ then K can be covered by finitely many balls of radius r having centres at points on K so that there are $a_1, \ldots, a_m$ in K such that[34]

$$K \subset \bigcup_{j=1}^{m} B(a_j;\ r). \tag{1.61}$$

Then by the discussion above, $f_n^{(k)} \to f^{(k)}$ uniformly on each of the closed balls $\overline{B(a_j;\ r)}$ $j = 1, \ldots, m$. In particular $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on each of the open balls $B(a_j;\ r),\ j = 1, \ldots, m$. Hence, by (1.61), it follows that[35] $f_n^{(k)}$ converges to

[34]Clearly, $K \subset \bigcup_{a \in K} B(a;\ r)$ and hence $\{B(a;\ r) : a \in K\}$ form an open cover for K and hence has a finite subcover.

[35]As $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on each of the open balls $B(a_j;\ r),\ j = 1, \ldots, m$ for each $\varepsilon > 0$ there are natural numbers $N_1, \ldots, N_m$ such that

$$\left|f_n^{(k)}(w) - f^{(k)}(w)\right| < \varepsilon \quad \forall n \geq N_j,\ \forall w \in B(a_j;\ r).$$

Now let $N = \max\{N_1,\ \ldots\ ,\ N_m\}$. Then

$$\left|f_n^{(k)}(w) - f^{(k)}(w)\right| < \varepsilon \quad \forall n \geq N,\ \forall w \in \bigcup_{j=1}^{m} B(a_j;\ r).$$

Hence $f_n^{(k)}$ converges to $f^{(k)}$ uniformly on $\bigcup_{j=1}^{m} B(a_j;\ r)$.

$f^{(k)}$ uniformly on K. □

Remark 1.2.2. We assume that the metric on $H(G)$ is the metric which it inherits as a subset of $C(G,\ \Omega)$ [That is, the metrizable topology on $H(G)$ is the subspace topology on $H(G)$] inherits from the metrizable topology on $C(G,\ \Omega)$.

Corollary 1.2.3. *$H(G)$ is a complete metric space.*

Proof. The metric on $H(G)$ is the metric which it inherits as a subset of $C(G,\ \Omega)$. As $C(G,\ \Omega)$ is a complete metric space, and since, by Theorem 1.2.1, $H(G)$ is closed, it follows (noting that closed subset of a complete space is complete) that $H(G)$ is complete. □

Corollary 1.2.4. *If $f_n : G \to C$ is analytic and $\sum\limits_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to $f(z)$ then*

$$f^{(k)}(z) = \sum_{n=1}^{\infty} f_n^{(k)}(z)$$

Proof. For $m = 1,\ 2,\ \ldots,$ let

$$s_m(z) = \sum_{n=1}^{m} f_n(z) \ \text{ for } z \in G.$$

By assumption, the series of functions $\sum\limits_{n=1}^{\infty} f_n(z)$ converges uniformly on compact sets to $f(z)$ which is equivalent to saying that the sequence (s_m) of partial sums of the series of functions converges uniformly on compact sets to $f(z)$. Then, by Theorem 1.2.1, for each $k \geq 1$, the sequence $(s_m^{(k)}) \to f^{(k)}$ on G as $m \to \infty$. This is equivalent to saying that, $\sum_{n=1}^{m} f_n^{(k)}(z) \to f^{(k)}(z)$ or that

$$\sum_{n=1}^{\infty} f_n^{(k)}(z) = f^{(k)}(z)$$

□

Remark 1.2.5. [Attention!] There is no analogue for the above result in the theory of functions of real variable.

1. The Fourier series of the real valued function of real variable

$$f(x) = |x| = \begin{cases} -x, & -\pi \leq x < 0 \\ x, & 0 \leq x \leq \pi \end{cases}$$

and $f(x + 2\pi) = f(x)$ is

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + ...\right]$$

i.e., using the notations in the previous result

$$f(x) = \underbrace{\frac{\pi}{2}}_{f_1(x)} \underbrace{-\frac{4}{\pi} \cdot \frac{\cos x}{1^2}}_{f_2(x)} \underbrace{-\frac{4}{\pi} \cdot \frac{\cos 3x}{3^2}}_{f_3(x)} \underbrace{-\frac{4}{\pi} \cdot \frac{\cos 5x}{5^2}}_{f_4(x)} - \cdots$$

Though each term on the right hand side is differentiable, the function on the left side $f(x) = |x|$ is not differentiable.

2. As another case, it can be shown (using a Theorem of Weierstrass) that a continuous nowhere differentiable function on [0, 1] is the limit of a sequence of polynomials. Here each polynomial is infinitely differentiable, but its limit is not.

3. As another **example:** Let

$$f_n(x) = \frac{x^n}{n} \text{ for } 0 \leq x \leq 1.$$

Then (f_n) converges uniformly to 0 (the zero function). But, the sequence of derivatives $\{f_n'\}$ where

$$f_n'(x) = nx^{n-1} \text{ for } 0 \leq x \leq 1$$

does not converge uniformly on [0, 1] [Details: When $x = 1$, we have the sequence of *numbers* $(f_n'(1)) = (n)$ which diverges, when $x = 0$, we have the sequence of *numbers* $(f_n'(0)) = (0)$ which converges to 0 (different behaviour at various points of [0, 1], so convergence has no uniformity over [0, 1]).

Theorem 1.2.6. ***[Hurwitz's Theorem]***[36] *Let G be a region and suppose the sequence $\{f_n\}$ in $H(G)$ converges to f*[37]*. If $f \not\equiv 0$, $\overline{B(a;\ R)} \subset G$, and $f(z) \neq 0$ for $|z-a| = R$ then there is an integer N such that for $n \geq N$, f and f_n have the same number of zeros in $B(a;\ R)$.*

Proof. Since $f(z) \neq 0$ for $|z-a| = R$,[38]

$$\delta = \inf\{|f(z)| : |z-a| = R\} > 0$$

By assumption $f_n \to f$ uniformly on compact subsets of G, and hence in particular $f_n \to f$ uniformly on the compact set $|z-a| = R$, so corresponding to $\delta/2$ there is an integer N (independent of the choice of points on $|z-a| = R$) such that

$$|f(z) - f_n(z)| < \frac{\delta}{2} \text{ for all } n \geq N \text{ and } \quad |z-a| = R. \tag{1.62}$$

Since $\delta = \inf\{|f(z)| : |z-a| = R\}$ we have for any z such that $|z-a| = R$,

$$\frac{\delta}{2} < \delta \leq |f(z)|\,.$$

[36] Also see Ahlfors *Complex Analysis* Page 152

[37] By Proposition 1.1.10, the sequence $\{f_n\}$ in $(C(G,\ \Omega),\ \rho)$ converges to f implies $\{f_n\}$ converges to f ***uniformly on all compact subsets of*** G.

[38] Being the continuous image of the compact set $|z-a| = R$, $S = \{|f(z)| : |z-a| = R\}$ is compact and hence infimum of the set S is a member of S. By assumption, $f(z) \neq 0$ for $|z-a| = R$ so that $|f(z)| > 0$ for $|z-a| = R$; since $\inf S \in S$, this shows that $\inf S > 0$.

Also, for $n \geq 1$, adding the non-negative number $|f_n(z)|$ gives

$$\frac{\delta}{2} < |f(z)| \leq |f(z)| + |f_n(z)|$$

and so (1.62) gives

$$n \geq N \text{ and } |z-a| = R \Rightarrow |f(z) - f_n(z)| < \frac{\delta}{2} < |f(z)| + |f_n(z)|$$

i.e., for $n \geq N$ and $|z-a| = R$

$$|f(z) + (-f_n)(z)| < |f(z)| + |(-f_n)(z)|\,.$$

Hence Rouche's Theorem[39] implies that for $n \geq N$ if Z_f and Z_{-f_n} are the number of zeros f and $-f_n$ inside $|z-a| = R$ (Note that there are no poles, as f and f_n are analytic functions), then

$$Z_f = Z_{-f_n}$$

[39] **[Rouche's Theorem]** Suppose f and g are meromorphic in a neighborhood $\overline{B(a;\ R)}$ with no zeros or poles on the circle $\gamma = \{z : |z-a| = R\}$. If Z_f, Z_g are the number of zeros of f and g inside γ counted according to their multiplicities and P_f, P_g are the number of poles of f and g inside γ counted according to their multiplicities and if

$$|f(z) + g(z)| < |f(z)| + |g(z)|$$

on γ, then

$$Z_f - P_f = Z_g - P_g.$$

In the situation of proof of Hurwitz's Theorem, f and f_n are analytic functions and hence inside $|z-a| = R$, $P_f = P_{f_n} = 0$ so that

$$Z_f = Z_{f_n}.$$

Clearly, zeros of f_n are the zeros of $-f_n$, so the above gives that for $n \geq N$ if Z_f and Z_{f_n} are the number of zeros f and f_n inside $|z-a| = R$, then

$$Z_f = Z_{f_n}.$$

i.e., for $n \geq N$, f and f_n have the same number of zeros in $B(a;\; R)$. $\square$

Corollary 1.2.7. *If $\{f_n\} \subset H(G)$ converges to $f \in H(G)$and each f_n never vanishes on G then either $f \equiv 0$ or f never vanishes.*

Proof. **Case 1.** If $f \equiv 0$ there is nothing to prove.

Case 2. If $f \not\equiv 0$ we claim that f never vanishes. $f \not\equiv 0$ implies there is at least one $b \in G$ such that $f(b) \neq 0$.

We claim that $f(z) \neq 0$ for all $z \in G$. For this, we employ contradiction method.

We assume that there is an $a \in G$ such that $f(a) = 0$ (We will arrive at a contradiction). Since zeros of analytic functions are isolated, it is possible to find $R_1 > 0$ such that $B(a;\; R_1) \subset G$ and $B(a;\; R_1)$ doesn't contain zeros of f other than a. Choose R such that $0 < R < R_1$. Then $\overline{B(a;\; R)} \subset G$ and $f(z) \neq 0$ on $|z-a| = R$. Then by Hurwitz Theorem, there is a positive integer N such that for $n \geq N$, f and f_n have the same number of zeros in $B(a;\; R)$. By assumption, each f_n has no zeros in G and in particular no zeros in

$B(a;\ R)$. Hence it follows that f has no zeros in $B(a;\ R)$, a contradiction to the fact that $f(a) = 0$. This contradiction implies that $f(a) = 0$ is impossible. That is, there is no $z \in G$ such that $f(z) = 0$. Hence f has no zero, i.e., f never vanishes. □

Definition 1.2.8. A set $\mathcal{F} \subset H(G)$ is **locally bounded** if for each point $a \in G$ there are constants $M > 0$ and $r > 0$ *(these constants depend on the point* a) such that

$$|f(z)| \leq M, \text{ for } |z-a| < r \text{ and for all } f \in \mathcal{F}$$

Alternately,

Definition 1.2.9. A set $\mathcal{F} \subset H(G)$ is **locally bounded** if for each point $a \in G$ there is an $r > 0$ such that

$$\sup\left\{|f(z)| : |z-a| < r,\ f \in \mathcal{F}\right\} < \infty$$

Remark 1.2.10. $\mathcal{F}$ it locally bounded if about each point $a \in G$ there is a disk on which $\mathcal{F}$ is uniformly bounded (that is, same bound M works for all $f \in \mathcal{F}$.) The following Lemma extends this to the requirement that $\mathcal{F}$ be uniformly bounded on compact sets in G.

Lemma 1.2.11. *A set $\mathcal{F} \subset H(G)$ is locally bounded if and only if for each compact set $K \subset G$ there is a constant M*

such that

$$|f(z)| \leq M, \text{ for all } f \in \mathcal{F} \text{ and } z \in K.$$

Proof. Suppose $\mathcal{F} \subset H(G)$ is locally bounded. Then for each point $a \in G$ there are constants $M_a > 0$ and $r_a > 0$ such that

$$|f(z)| \leq M_a, \text{ for } |z - a| < r_a \text{ and for all } f \in \mathcal{F}.$$

Let K be a compact subset of G. Since

$$K \subseteq \bigcup_{a \in K} B(a;\ r_a),$$

the collection $\{B(a;\ r_a) :\ a \in K\}$ is an open cover for the compact set K; hence it follows that there are finite number of $a_1, \ldots, a_n$ in K such that the finite sub collection

$$\{B(a_1,\ r_{a_1}),\ \ \ldots\ ,\ B(a_n,\ r_{a_n})\}$$

is a cover for K. Take

$$M = \max\{M_{a_1},\ \ldots\ ,\ M_{a_n}\}.$$

Then, for all $j = 1, \ldots, n$,

$$|f(z)| \leq M \text{ for } |z - a_j| < r_j \text{ and for all } f \in \mathcal{F}. \tag{1.63}$$

If we take $z \in K$ then $z \in B(a_j,\ r_{a_j})$ for some $j = 1, \ldots, n$, and hence from (1.63) it follows that

$$|f(z)| \leq M \text{ for all } f \in \mathcal{F}.$$

As the choice of $z \in K$ is arbitrary, this shows that

$$|f(z)| \leq M \text{ for } z \in K \text{ and for all } f \in \mathcal{F}.$$

Conversely, assume that for each compact set $K \subset G$ there is a constant M such that

$$|f(z)| \leq M, \text{ for all } f \in \mathcal{F} \text{ and } z \in K.$$

To show that $\mathcal{F}$ is locally bounded at each point $a \in G$, fix $a \in G$. Since G is open there is a $R > 0$ such that $B(a,\ R) \subset G$. Let $0 < r < R$. Then

$$\overline{B(a,\ r)} \subset B(a,\ R) \subset G.$$

Being closed and bounded, $\overline{B(a,\ r)}$ is compact, and hence by assumption there is a constant M such that

$$|f(z)| \leq M \text{ for all } f \in \mathcal{F} \text{ and } z \in \overline{B(a,\ r)}.$$

In particular,

$$|f(z)| \leq M \text{ for all } f \in \mathcal{F} \text{ and } z \in B(a,\ r).$$

i.e., we have shown that corresponding to $a \in G$ there are constants $M > 0$ and $r > 0$ such that

$$|f(z)| \leq M_a, \text{ for } |z - a| < r_a \text{ and for all } f \in \mathcal{F}$$

showing that $\mathcal{F}$ is locally bounded at a. Since a is an arbitrary element in G it follows that $\mathcal{F}$ is locally bounded for each point $a \in G$. □

Theorem 1.2.12. ***[Montel's Theorem]*** *A family $\mathcal{F}$ in $H(G)$ is normal if and only if $\mathcal{F}$ is locally bounded.*

Suppose $\mathcal{F}$ in $H(G)$ is normal but fails to be locally bounded (we will arrive at a contradiction.) Then, by Lemma 1.2.11, there is a compact set $K \subset G$ such that it is **impossible** to find a constant M such that

$$|f(z)| \leq M$$

for all $f \in \mathcal{F}$ and $z \in K$. Hence for any constant $K > 0$ there is an $f \in \mathcal{F}$ and $z \in K$ such that

$$M < |f(z)|\,.$$

Thus, the set $\{|f(z)| : z \in K,\ f \in \mathcal{F}\}$ is not bounded from above and hence

$$\sup\{|f(z)| : z \in K,\ f \in \mathcal{F}\} = \infty.$$

That is,

$$\sup\{\sup\{|f(z)| : z \in K\},\ f \in F\} = \infty$$

Hence for each $n \in \mathbb{N}$, there is an $f_n \in \mathcal{F}$ such that

$$\sup\{|f_n(z)| : z \in K\} \geq n.$$

i.e., there is a sequence of functions $\{f_n\}$ in $\mathcal{F}$ such that

$$\sup\{|f_n(z)| : z \in K\} \geq n. \tag{1.64}$$

Since $\mathcal{F}$ is normal there is a function $f \in H(G)$ and a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ as $k \to \infty$. Now $f_{n_k} \to f$ as $k \to \infty$ in $H(G)$ with respect the metric $\rho_{|\,H(G)\times H(G)}$ inherited from the metric space $(C(G,\,\Omega),\,\rho)$, so it follows from Part (b) of Proposition 1.1.10 that f_{n_k} converges to f uniformly on all compact subsets of G. Hence, in particular, f_{n_k} converges to f uniformly on the compact subset K mentioned above. Hence for every $\varepsilon > 0$ there is a natural number N

such that

$$|f_{n_k}(z) - f(z)| < \varepsilon \quad \text{for all } z \in K \text{ and } n \geq N.$$

Hence

$$\sup\{|f_{n_k}(z) - f(z)| : z \in K\} \leq \varepsilon \quad \text{for all } n \geq N.$$

As the choice of $\varepsilon > 0$ is arbitrary, it follows that

$$\sup\{|f_{n_k}(z) - f(z)| : z \in K\} \to 0 \quad \text{as } k \to \infty$$

Using (1.64), we have

$$\begin{aligned} n_k &\leq \sup\{\underbrace{|f_{n_k}(z)|}_{\left|f_{n_k}(z)-f(z)+f(z)\right|} : z \in K\} \\ &\leq \sup\{|f_{n_k}(z) - f(z)| : z \in K\} \\ &\quad + \sup\{|f(z)| : z \in K\}. \end{aligned} \tag{1.65}$$

Now, if $|f(z)| \leq M$ for $z \in K$, then $\sup\{|f(z)| : z \in K\} \leq M$ so (1.65) gives

$$n_k \leq \sup\{|f_{n_k}(z) - f(z)| : z \in K\} + M.$$

Since the right hand side converges to M while the left hand converges to ∞ as $k \to \infty$ this gives a contradiction.

Conversely, suppose $\mathcal{F}$ is locally bounded; the Ascoli-Arzela

Theorem (Theorem 1.1.33) will be used to show that $\mathcal{F}$ is normal.

Fix a point $z \in G$. Since singleton subsets are compact, singleton subset $\{z\}$ of G is also compact and hence by Lemma 1.2.11, there is a constant M (depending on the point z) such that

$$|f(z)| \leq M, \text{ for all } f \in \mathcal{F}$$

Hence closure of $\{f(z) : f \in \mathcal{F}\}$ is both closed and bounded subset; and hence is compact. As z is an arbitrary element of G it follows that condition (a) of Ascoli-Arzela Theorem is satisfied[40].

To prove (b), we must show that $\mathcal{F}$ is equicontinuous at each point of G. For this, fix $a \in G$ and $\varepsilon > 0$. By assumption, $\mathcal{F}$ is locally bounded at a so there is an $M > 0$ and $R > 0$

---

[40] Alternative proof: Fix $a \in G$. Since $\mathcal{F}$ is locally bounded; there are constants $M > 0$ and $r > 0$ such that for all $f \in \mathcal{F}$,

$$|z - a| < r \Rightarrow |f(z)| \leq M$$

Hence

$$\{f(a) : f \in F\} \subseteq \underbrace{\overline{B(0,\ M)}}_{\text{compact (as closed and bounded)}}$$

So

$$\{f(a) : f \in F\} \subseteq \underbrace{\overline{\{f(a) : f \in F\}}}_{\substack{\text{compact, being closed subset} \\ \text{of the compact set on the right side}}} \subseteq \underbrace{\overline{B(0,\ M)}}_{\text{compact (as closed and bounded)}}$$

showing that $\{f(a) : f \in F\}$ has compact closure. Since a is an arbitrary element in G, condition (a) is satisfied.

such that

$$|f(z)| \leq M \text{ for } |z-a| < R \textit{ and } f \in \mathcal{F}.$$

Choose $0 < r < R$ such that $\overline{B(a;\ r)} \subset G$. Then

$$|f(z)| \leq M, \text{ for } |z-a| \leq r \text{ and } f \in \mathcal{F}.$$

i.e., that

$$|f(z)| \leq M, \text{ for } z \in \overline{B(a; r)} \text{ and } f \in \mathcal{F}.$$

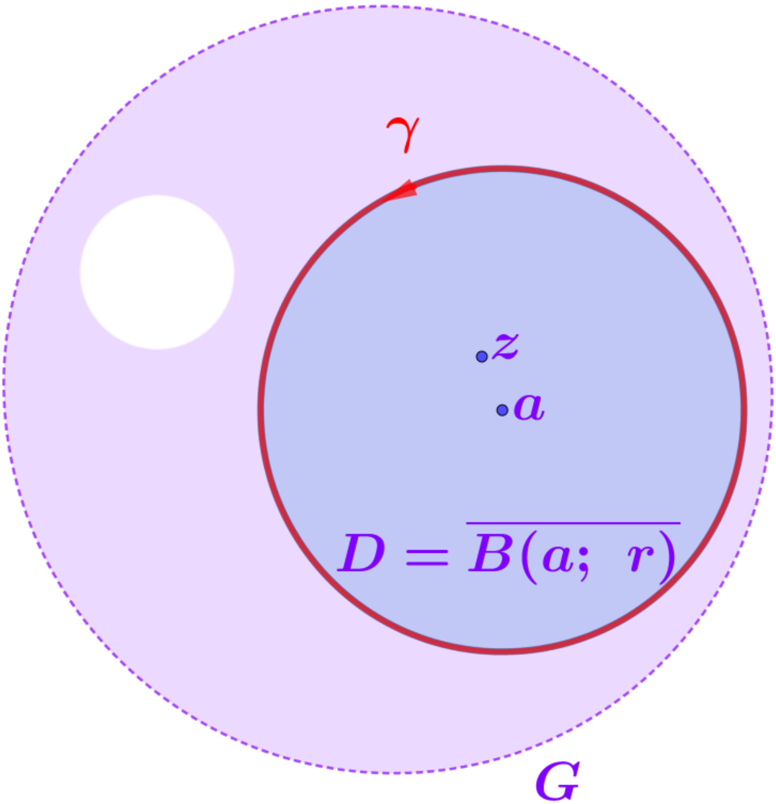

Figure 1.16: z is chosen such that $|z-a| < \frac{1}{2}r$. Also, $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$ is the boundary of the closed disk $\overline{B(a;\ r)}$.

Let $|z-a| < \frac{1}{2}r$ and $f \in \mathcal{F}$; then using Cauchy's Integral formula[41] with $\gamma(t) = a + re^{it},\ 0 \leq t \leq 2\pi$,

$$|f(a) - f(z)| = \left| \frac{1}{2\pi i} \left(\int_\gamma \frac{f(w)}{w-a} dw - \int_\gamma \frac{f(w)}{w-z} dw \right) \right|$$

$$\leq \frac{1}{2\pi} \left| \int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)} dw \right| \tag{1.66}$$

Now $w \in \{\gamma\}$ implies $|w-a| = |re^{it}| = r$ so that $\frac{1}{|w-a|} = \frac{1}{r}$ and

$$\underbrace{|w-a|}_{r} \leq |w-z| + \underbrace{|z-a|}_{< r/2}$$

so that

$$r < |w-z| + \frac{r}{2}$$

and hence

$$|w-z| > \frac{r}{2}$$

[41] **Cauchy's Integral Formula (First Version)** Let G be an open subset of the plane and $f : G \to C$ an analytic function. If γ is a closed rectifiable curve in G such that $n(\gamma;\, w) = 0$ for all $w \in C \sim G$, then for $a \in G \sim \{\gamma\}$

$$n\,(\gamma;\ a\,)f(a) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz.$$

Using these, (1.66) gives

$$|f(a) - f(z)| \leq |a - z| \frac{M}{\pi r^2} \underbrace{\int_\gamma |dw|}_{2\pi r};$$

That is,

$$|f(a) - f(z)| \leq \frac{4M}{r} |a - z| \tag{1.67}$$

Letting $\delta < \min\left\{\frac{r}{2},\ \frac{r}{4M}\varepsilon\right\}$ it follows from (1.67) that

$$|a - z| < \delta \ \Rightarrow \ |f(a) - f(z)| < \varepsilon \text{ for all } f \in \mathcal{F}.$$

This shows that $\mathcal{F}$ is equicontinuous at $a \in G$. Since a is an arbitrary element in G it follows that $\mathcal{F}$ equicontinuous at each point of G. Thus conditions (a) and (b) of Ascoli-Arzela Theorem hold, and hence $\mathcal{F}$ is normal.

Corollary 1.2.13. *A set $\mathcal{F} \subset H(G)$ is compact if and only if it is closed and locally bounded.*

Proof. $\mathcal{F} \subset H(G)$ is compact implies it is closed and bounded. Since $\mathcal{F}$ is closed its closure is $\mathcal{F}$ itself. Hence we have closure of $\mathcal{F}$ is compact. Hence by Proposition[42] $\mathcal{F}$ is normal. Then, by Montel's Theorem (Theorem 1.2.12), $\mathcal{F}$ is locally bounded. Thus, $\mathcal{F}$ is closed and locally bounded.

[42] A set $\mathcal{F} \subset C(G,\ \Omega)$ is normal if and only if its closure is compact.

Conversely, assume that $\mathcal{F} \subset H(G)$ is closed and locally bounded. Then, by Montel's Theorem, $\mathcal{F}$ is normal. Then by Proposition closure of $\mathcal{F}$ is compact. Since $\mathcal{F}$ is closed, its closure is $\mathcal{F}$ itself, so it follows that $\mathcal{F}$ is compact. $\square$

1.3 Spaces of Meromorphic Functions

If G is a region and f is a meromorphic function on G, and if

$$f(z) = \infty \text{ whenever } z \,\text{is a pole of}\, G$$

then $f : G \to C_\infty$ is a continuous function If $M(G)$ is the set of all meromorphic functions on G then consider $M(G)$ as a subset of $C(G, C_\infty)$ and endow it with the metric of $C(G, C_\infty)$. In this section this metric space will be discussed as $H(G)$ was discussed in the-previous section.

Recall that[43] the metric d is defined on C_∞ as follows: for z_1 and z_2 in C

$$d(z_1,\ z_2) = \frac{2\,|z_1 - z_2|}{[(1 + |z_1|^2)(1 + |z_2|^2)]^{1/2}};$$

and for z in C

$$d(z,\ \infty) = \frac{2}{(1 + |z_1|^2)^{1/2}}.$$

[43]Ref. Appendix A: *The Extended Plane and its Spherical Representation*

Notice that for non zero complex numbers z_1 and z_2,

$$\begin{aligned} d\left(\frac{1}{z_1},\ \frac{1}{z_2}\right) &= \frac{2\left|\frac{1}{z_1}-\frac{1}{z_2}\right|}{[(1+\left|\frac{1}{z_1}\right|^2)(1+\left|\frac{1}{z_2}\right|^2)]^{1/2}} \\ &= \frac{2\,|z_1-z_2|}{[(1+|z_1|^2)(1+|z_2|^2)]^{1/2}} = d(z_1,\ z_2) \end{aligned}$$

that is,

$$d(z_1,\ z_2) = d\left(\frac{1}{z_1},\ \frac{1}{z_2}\right); \tag{1.68}$$

and for $z \neq 0$

$$d\left(\frac{1}{z},\ \infty\right) = \frac{2}{\left[1+\left|\frac{1}{z_1}\right|^2\right]^{1/2}}$$

and so

$$d(z,\ 0) = d\left(\frac{1}{z},\ \infty\right). \tag{1.69}$$

Also recall that if $\{z_n\}$ 1s a sequence in $\mathbb{C}$ and $z \in \mathbb{C}$ that satisfies $d(z,\, z_n) \to 0$ then $|z - z_n| \to 0$.

Some facts about the relationship between the metric spaces $\mathbb{C}$ and $\mathbb{C}_\infty$ are summarized in the next proposition. In order to avoid confusion $B(a;\, r)$ will be used to designate a ball in $\mathbb{C}$ and $B_\infty(a;\, r)$ to designate a ball in $\mathbb{C}_\infty$.

Proposition 1.3.1. *(a) If a is in $\mathbb{C}$ and $r > 0$ then there is a number $\rho > 0$ such that $B_\infty(a;\, \rho) \subset B(a;\, r)$.*

(b) *Conversely, if* $\rho > 0$ *is given and* a *is in* $\mathbb{C}$ *then there is a number* $r > 0$ *such that* $B(a;\, r) \subset B_\infty(a;\, \rho)$.

(c) *If* $\rho > 0$ *is given then there is a compact set* $K \subset \mathbb{C}$ *such that* $\mathbb{C}_\infty - K \subset B_\infty(\infty;\, \rho)$.

(d) *Conversely, if a compact set* $K \subset \mathbb{C}$ *is given, there is a number* $\rho > 0$ *such that* $B_\infty(\infty;\, \rho) \subset \mathbb{C}_\infty - K$.

The first observation is that $M(G)$ is not complete. In fact if $f_n(z) \equiv n$ then $\{f_n\}$ is a Cauchy sequence in $M(G)$. But $\{f_n\}$ converges to the function which is identically ∞ in $C(G, \mathbb{C}_\infty)$ and this is not meromorphic. However this is the worst that can happen.

Theorem 1.3.2. *Let* $\{f_n\}$ *be a sequence in* $M(G)$ *and suppose* $f_n \to f$ *in* $C(G, \mathbb{C}_\infty)$. *Then either* f *is meromorphic or* $f \equiv \infty$. *If each* f_n *is analytic then either* f *is analytic or* $f \equiv \infty$.

Proof. Suppose there is a point a in G with $f(a) \neq \infty$ and put $M = |f(a)|$. Using Part (a) of the previous Proposition we can find a number $\rho > 0$ such that

$$B_\infty(f(a);\, \rho) \subset B(f(a);\, M).$$

But since $f_n \to f$ there is an integer n_0 such that

$$d(f_n(a),\ f(a)) < \frac{1}{2}\rho \quad \text{for all } n \geq n_0.$$

Also $\{f,\ f_1,\ f,\ \ldots\}$ is compact in $C(G, \mathbb{C}_\infty)$ so that it is equicontinuous. That is, there is an $r > 0$ such that

$$|z-a| < r \ \text{ implies } \ d(f_n(a),\ f(a)) < \frac{1}{2}\rho.$$

That gives that

$$d(f_n(z),\ f(a)) \le \rho \ \text{ for } \ |z-a| \le r \ \text{ andfor } n \ge n_0.$$

But by the choice of ρ,

$$|f_n(z)| \le |f_n(z) - f(a)| + |f(a)| \le 2M$$

for all $z \in \overline{B(a;\ r)}$ and $n \ge n_0$. But then (from the formula for the metric d)

$$\frac{2}{1+4M^2}\,|f_n(z) - f(z)| \le d(f_n(z),\ f(z))$$

for z in $\overline{B(a;\ r)}$ and $n \ge n_0$. Since $d(f_n(z),\ f(z)) \to 0$ uniformly for z in $\overline{B(a;\ r)}$. Since the tail end of hte sequece $\{f_n\}$ is bounded on $B(a;\ r)$, f_n has no poles and must be analytic near $z = a$ for $n \ge n_0$. It follows that f is anlaytic in a disk about a.

Now suppose there isa point a in G with $f(a) = \infty$. For a function g in $C(G, \mathbb{C}_\infty)$ define $1/g$ by

$$\left(\frac{1}{g}\right)(z) = \begin{cases} \frac{1}{g(z)} \text{ if } & g(z) \neq 0 \text{ or } \infty, \\ 0 \text{ if } & g(z) = \infty, \\ \infty \text{ if } & g(z) = 0\,. \end{cases}$$

It follows that $\frac{1}{g} \in C(G, \mathbb{C}_\infty)$. Also, since $f_n \to f$ in $C(G, \mathbb{C}_\infty)$ it follows from (1.68) and (1.69) that

$$\frac{1}{f_n} \to \frac{1}{f} \text{ in } C(G, \mathbb{C}_\infty).$$

Now each function $\frac{1}{f_n}$ is meromorphic on G; so the preceding paragraph gives a number $r > 0$ and an integer n_0 such that $\frac{1}{f}$ and $\frac{1}{f_n}$ are analytic on $B(a; r)$ for $n \geq n_0$ and $\frac{1}{f_n} \to \frac{1}{f}$ uniformly on $B(a; r)$. From Hurwitz's Theorem either $\frac{1}{f} \equiv 0$ or $\frac{1}{f}$ has isolated zeros in $B(a; r)$. So if $f \not\equiv$ then $\frac{1}{f} \not\equiv 0$ and f must be meromorphic in $B(a; r)$. Combining this with the first part of the proof we have that f is meromorphic in G if f is not identically infinite.

If each f_n is analytic then $\frac{1}{f_n}$ has no zeros in $B(a; r)$. It follows from Corollary to Hurwitz's Theorem that either $\frac{1}{f} \equiv 0$ or $\frac{1}{f}$ never vanishes. But since $f(a) = \infty$ we have that $\frac{1}{f}$ has at least one zero; thus $f \equiv \infty$ in $B(a; r)$. Combining this with the first part of the proof we see that $f \equiv \infty$ or f is analytic. $\square$

Corollary 1.3.3. *$M(G) \cup \{\infty\}$ is a complete metric space.*

Corollary 1.3.4. *$H(G) \cup \{\infty\}$ is closed in $C(G, \mathbb{C}_\infty)$*

To discuss normality in $M(G)$ one must introduce the quantity

$$\frac{2\,|f'(z)|}{1 + |f(z)|^2},$$

for each meromorphic function f. However if z is a pole of f then the above expression is meaningless since $f'(z)$ has no meaning. To rectify this take the limit of the above expression as z approaches the pole. To show that the limit exists let a be a pole of f of order $m \geq 1$; then

$$f(z) = g(z) + \frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{z-a}$$

for z in some disk about a and g analytic in that disk. For $z \neq a$

$$f'(z) = g'(z) - \left[\frac{mA_m}{(z-a)^{m+1}} + \cdots + \frac{A_1}{(z-a)^2}\right].$$

Thus

$$\frac{2\,|f'(z)|}{1 + |f(z)|^2}$$

$$= \frac{2\left|\frac{mA_m}{(z-a)^{m+1}} + \cdots + \frac{A_1}{(z-a)^2} - g'(z)\right|}{1 + \left|\frac{A_m}{(z-a)^m} + \cdots + \frac{A_1}{z-a} + g(z)\right|^2}$$
$$= \frac{2\,|z-a|^{m+1}\,|mA_m + \cdots + A_1(z-a)^{m-1} - g'(z)(z-a)^{m+1}|}{|z-a|^{2m} + |A_m + \cdots + A_1(z-a)^{m-1} + g(z)(z-a)^m|^2}$$

So if $m \geq 2$
$$\lim_{z\to a} \frac{2\,|f'(z)|}{1+|f(z)|^2} = 0$$

If $m = 1$ then
$$\lim_{z\to a} \frac{2\,|f'(z)|}{1+|f(z)|^2} = \frac{2}{|A_1|}.$$

Definition 1.3.5. If f is a meromorphic function on the region G then define $\mu(f)) : G \to \mathbb{R}$ by

$$\mu(f)(z) = \frac{2\,|f'(z)|}{1+|f(z)|^2}$$

whenever z is not a pole of f, and

$$\mu(f)(a) = \lim_{z\to a} \frac{2\,|f'(z)|}{1+|f(z)|^2}$$

if a is a pole of f.

It follows that $\mu(f) \in C(G, \mathbb{C})$.

The reason for introducing $\mu(f)$ is as follows: If $f : G \to C_\infty$ is meromorphic then for z close to z' we have that $d(f(z),\ f(z'))$ is approximated by $\mu(f)(z)\,|z-z'|$. So if a bound can be obtained for $\mu(f)$ then f is a Lipschitz function. If f belongs to a family of functions and $\mu(f)$ is uniformly bounded for f in this family, then the family is a uniformly Lipschitz set of functions. This is made precise in the following proof.

Theorem 1.3.6. *A family $\mathcal{F} \subset M(G)$ is normal in $C(G, C_\infty)$ if and only if $\mu(\mathcal{F}) \equiv \{ \mu(f) : f \in \mathcal{F}\}$ is locally bounded.*

Note. If $f_n(z) = nz$ for $n \geq 1$ then

$$\mu(f_n)(z) = \frac{2n}{1 + n^2 |z|^2}.$$

Thus $\mathcal{F} = \{f_n\}$ is normal in $C(G, C_\infty)$ and $\mu(\mathcal{F})$ is locally bounded. However, $\mathcal{F}$ is not normal in $M(G)$ since the sequence $\{f_n\}$ converges to the constantly infinite function which does not belong to $M(G)$.

Proof. We will assume that $\mu(\mathcal{F})$ is locally bounded and prove that $\mathcal{F}$ is normal by applying the Arzela-Ascoli Theorem. Since C_∞ is compact it suffices to show that $\mathcal{F}$ is equicontinuous at each point of G. So let K be an arbitrary closed disk contained in G and let M be a constant with $\mu(f)(z) \leq M$ for all z in K and all f in $\mathcal{F}$. Let z and z' be arbitrary points in K.

Suppose neither z nor z' are poles of a fixed function f in

$\mathcal{F}$ and let $\alpha > 0$ be an arbitrary number. Choose points

$$w_0 = z,\, w_1,\, \ldots,\, w_n = z'$$

in K which satisfy the following conditions:

$\diamondsuit$ w is in the line segment $[w_{k-1},\, w_k]$ implies (1.70)
w is not a pole of f;

(1.71)

$\diamondsuit$ $$\sum_{k=1}^{n} |w_k - w_{k-1}| \leq 2\,|z - z'| \tag{1.72}$$

$\diamondsuit$ $$\frac{1 + |f(w_k)|^2}{[(1 + |f(w_k)|^2)(1 + |f(w_{k-1})|^2)^{1/2}} - 1 < \alpha, \tag{1.73}$$
$1 \leq k \leq n;$

$\diamondsuit$ $$\left| \frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) \right| < \alpha, \tag{1.74}$$
$1 \leq k \leq n;$

To see that such points can be found select a polygonal path P in K satisfying (1.71) and (1.72). Cover P by small disks in which conditions similar to (1.73) and (1.74) hold, choose a finite subcover, and then pick points $w_0,\, \ldots,\, w_n$ on P such that each segment $[w_{k-1},\, w_k]$ lies in one of these disks. Then $\{w_0,\, \ldots,\, w_n\}$ will satisfy all of these conditions. If

$$\beta_k = [(1 + |f(w_{k-1})|^2\,(1 + |f(w_k)|^2)]^{1/2}$$

then

$$
\begin{aligned}
& d(f(z),\ f(z')) \\
\leq & \sum_{k=1}^{n} d(f(w_{k-1}),\ f(w_k)) \\
= & \sum_{k=1}^{n} \frac{2}{\beta_k} \left| f(w_k) - f(w_{k-1}) \right| \\
\leq & \sum_{k=1}^{n} \frac{2}{\beta_k} \left| \frac{f(w_k) - f(w_{k-1})}{w_k - w_{k-1}} - f'(w_{k-1}) \right| |w_k - w_{k-1}| \\
& + \sum_{k=1}^{n} \frac{2}{\beta_k} |f'(w_{k-1})| \, |w_k - w_{k-1}|.
\end{aligned}
$$

Using the fact that $2\,|f'(w_{k-1})| \leq M(1 + |f(w_k)|^2)$ and the conditions on $w_0, \ldots, w_n$ this becomes

$$
\begin{aligned}
d(f(z),\ f(z')) & \leq 2\alpha \sum_{k=1}^{n} \frac{1}{\beta_k} |w_k - w_{k-1}| \\
& \quad + M \sum_{k=1}^{n} \frac{2}{\beta_k} \left(\frac{1 + |f(w_{k-1})|^2}{\beta_k} \right) |w_k - w_{k-1}| \\
& \leq (4\alpha + 2\alpha M) |z - z'| + \sum_{k=1}^{n} M \, |w_k - w_{k-1}| \\
& \leq 4\alpha + 2\alpha M + 2M) |z - z'| .
\end{aligned}
$$

Since $\alpha > 0$ was arbitrary this gives that if z and z' are not poles of f then

$$d(f(z),\ f(z')) \leq 2M\,|z - z'|\,. \tag{1.75}$$

Now suppose z' is a pole of f but z is not. If w is in K and is not a pole then it follows from (1.75) that

$$\begin{aligned} d(f(z),\ \infty) &\leq d(f(z),\ f(w)) + d(f(w),\ \infty\,) \\ &\leq 2M\,|z - w| + d(f(w),\ \infty). \end{aligned}$$

Since it is possible to let w approach z' without w ever being a pole of f (poles are isolated!), this gives that $f(w) \to f(z') = \infty$ and $|z - w| \to |z - z'|$. Thus (1.75) holds if at most one of z and z' is a pole. But a similar procedure gives that (1.75) holds for all z and z' in K. So if $K = B(a;\,r)$ and $\varepsilon > 0$ are given that for $\delta < \min\{r,\ \varepsilon/2M\}$ we have that $|z - a| < \delta$ implies $d(f(z),\ f(a)) < \varepsilon$, and δ is independent of f in $\mathcal{F}$. This gives that $\mathcal{F}$ is equicontinuous at each point a in G. The proof of the converse is left as an exercise. $\square$

1.4 The Riemann Mapping Theorem

We wish to define an equivalence relation between regions in $\mathbb{C}$. After doing this it will be shown that all proper simply connected regions in $\mathbb{C}$ are equivalent to the open disk $D = \{z : |z| < 1\}$, and hence are equivalent to one another.

Definition 1.4.1. A region G_1 is **conformally equivalent** to G_2 if there is an analytic function $f : G_1 \to \mathbb{C}$ such that f is one-one and $f(G_1) = G_2$.

Proposition 1.4.2. *On the collection $\mathcal{R}$ of all regions in $\mathbb{C}$ the relation $\sim$ defined by $G_1 \sim G_2$ if G_1 is conformally equivalent to G_2 is an equivalence relation.*

Proof. **Reflexivity** For each G in $\mathcal{R}$, $G \sim G$ because of the following observations. The function $i : G \to G$ defined by $i(g) = g$ for $g \in G_1$ is the identity map and is both one-to-one and onto. Also i is analytic.

Symmetry For G_1 and G_2 in $\mathcal{R}$, $G_1 \sim G_2$ implies there is an analytic function $f : G_1 \to G_2$ that is both one-to-one and onto. Then $f^{-1} : G_2 \to G_1$ is also[44] an analytic function that is both one-to-one and onto. Hence $G_2 \sim G_1$.

Transitivity For G_1, G_2 and G_3 in $\mathcal{R}$, $G_1 \sim G_2$ and $G_2 \sim G_3$ implies there are analytic functions $f : G_1 \to G_2$ and $g : G_2 \to G_3$ that are both one-to-one and onto. Then the composition $g \circ f : G_1 \to G_3$ is also an analytic function that is both one-to-one and onto. Hence $G_1 \sim G_3$.

Hence $\sim$ is an equivalence relation on $\mathcal{R}$. $\square$

[44]**Theorem***(Page 99, Conway)* Suppose $f : G \to \Omega$ is both one-to-one, onto and analytic. Then $f^{-1} : \Omega \to G$ is also analytic.

- $\mathbb{C}$ is not equivalent to any bounded region. Suppose B is a bounded region and $\mathbb{C} \sim B$. Then there is an analytic function $f : \mathbb{C} \to B$ that is both one-to-one and onto. Then f is a bounded entire function so, by Liouville's Theorem, f is a constant and hence f is not one-to-one, which is a contradiction. Hence $\mathbb{C}$ is not equivalent to B.

- It is easy to show from the definitions that if G_1 is simply connected and G_1 is equivalent to G_2 then G_2 must be simply connected.

- If f is the principal branch of the square root[45] then

$$f(z) = r^{(1/2)} e^{i(\theta/2)}$$

[45]When $z \neq 0$ and the exponent c is a complex number; z^c is defined by means of the equation

$$z^c = e^{c \log z},$$

where $\log z$ denotes the multiple valued logarithmic function. Hence powers of z are in general multiple-valued. If $z = re^{i\theta}$ and α is any real number, the **branch**

$$\log z = \ln r + i\,\theta \qquad (r > 0,\ \ \alpha < \theta < \alpha + 2\pi)$$

of the logarithmic function is single valued and analytic in the indicated domain. When that branch of $\log z$ is used, it follows that the function

$$z^c = \exp(c \log z)$$

is *single-valued and analytic* in the same domain. If we put $\alpha = -\pi$, then we get the **principal branch** of $\log z = \ln r + i\,\theta \qquad (r > 0,\ \ -\pi < \theta < \pi)$ and is usually denoted by

$$\operatorname{Log} z = \ln r + i\,\theta \qquad (r > 0,\ \ -\pi < \theta < \pi).$$

where $r > 0$, $-\pi < \theta < \pi$. f is a one-to-one analytic function from $\mathbb{C} - \{z : z \le 0\}$ onto the right half plane (Fig. 1.17). Hence $\mathbb{C} - \{z : z \le 0\}$ is equivalent to the right half plane.

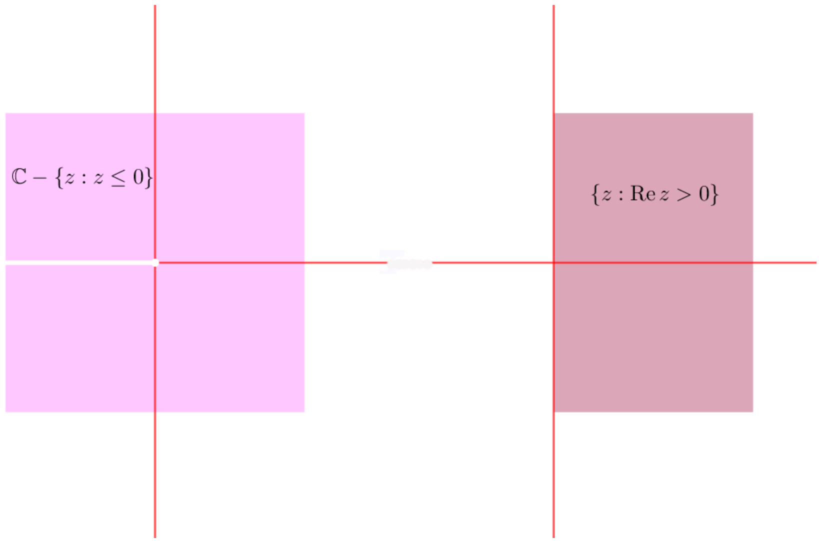

Figure 1.17: $f(z) = r^{(1/2)} e^{i(\theta/2)}$ where $r > 0$, $-\pi < \theta < \pi$. f is a one-to-one analytic function from $\mathbb{C} - \{z : z \le 0\}$ onto the right half plane. The left side is the domain of f and the right side is the range of f.

Hence the principal branch of $z^{1/2}$ is given by

$$z^{1/2} = e^{c\,\mathrm{Log}z} = e^{(1/2)(\ln r + i\,\theta)} = \underbrace{e^{(1/2)(\ln r)}}_{\exp(\ln r^{(1/2)})} e^{(1/2)(i\,\theta)} = r^{(1/2)} e^{i(\theta/2)}$$

where $r > 0$, $-\pi < \theta < \pi$.

Theorem 1.4.3. Riemann Mapping Theorem *Let G be a simply connected region which is not the whole plane and let $a \in G$. Then there is a unique analytic function $f : G \to \mathbb{C}$ having the properties:*

(a) $f(a) = 0$ and $f'(a) > 0$;

(b) f is one-one;

(c) $f(G) = \{z : |z| < 1\}$.

Proof. The proof that the function f is unique is rather easy. In fact, if g has the properties (a) to (c) above (with g in place of f) and if

$$D = \{z : |z| < 1\}$$

then (g^{-1} is analytic, one-one, and onto and hence the composition)

$$f \circ g^{-1} : D \to D$$

is analytic, one-one, and onto. Also $f \circ g^{-1}(0) = f(a) = 0$ so Theorem VI.2.5 (Conway, Text Page 132)[46] implies (noting that $(f \circ g^{-1})(\underbrace{0}_{a}) = \underbrace{0}_{\alpha})$ there is a constant c with $|c| = 1$

[46] Theorem VI.2.5: Let $f : D \to D$ be a one to one analytic function of D onto D and suppose $f(a) = 0$. Then there exists a complex number c with $|c| = 1$ such that $f = c\varphi_a$ where

$$\varphi_a(z) = \frac{z - a}{1 - \bar{a}z}$$

and

$$f \circ g^{-1}(z) = c\varphi_0(z) = c\left(\frac{z-0}{1-0}\right) = cz$$

for all z. But then

$$f(z) = cg(z) \tag{1.76}$$

gives $f'(z) = cg'(z)$ so that $0 < f'(a) = cg'(a)$; since $g'(a) > 0$ it follows that both $f'(a)$ and $g'(a)$ are positive real numbers with $f'(a) = cg'(a)$ which implies $c = 1$. Hence from (1.76) we obtain

$$f(z) = g(z)$$

i.e., $f = g$. The existence of f is contained in the following Lemma 1.4.4. $\square$

To motivate the proof of the existence of f, consider the family $\mathcal{F}$ of all analytic functions f having properties (a) and (b) and satisfying $|f(z)| < 1$ for z in G. The idea is to choose a member of $\mathcal{F}$ having property (c). Suppose $\{K_n\}$ is a sequence of compact subsets of G such that $\bigcup_{n=1}^{\infty} K_n = G$, $K_n \subset \text{int } K_{n+1}$ and $a \in K_n$ for each n. Then $\{f(K_n)\}$ is a sequence of compact subsets of $D = \{z : |z| < 1\}$. Also, as n becomes larger K_n becomes larger and larger and tries to fill out G so their images $f(K_n)$ becomes larger and larger and tries to fill out the disk D. By choosing a function f in $\mathcal{F}$ with the largest possible derivative at a (so that the rate of change is largest), we choose the function which starts out the fastest at $z = a$. It thus has the best possible chance of finishing first; that is, of having

$$\bigcup_{n=1}^{\infty} f(K_n) = D.$$

Before carrying out this proof, it is necessary for future developments to point out that[47] **the only property of a simply connected region which will be used is the fact that every non-vanishing analytic function has an analytic square root.** So *the Riemann Mapping Theorem will be completely proved by proving the following.*

Lemma 1.4.4. *Let G be a region which is not the whole plane and such that every non-vanishing analytic function on G has an analytic square root. If $a \in G$ then there is an analytic function $f : G \to \mathbb{C}$ having the properties:*

(a) $f(a) = 0$ *and* $f'(a) > 0$;

(b) f *is one-one;*

(c) $f(G) = D = \{z : |z| < 1\}$.

Proof. Define $\mathcal{F}$ by letting

$$\mathcal{F} =$$

$$\{ f \in H(G) : f \text{ is one} - \text{one},\ f(a) = 0,\ f'(a) > 0,\ f(G) \subset D\}.$$

[47] Theorem 1.5.27 in the footnote in Page 178.

Since $f(G) \subset D$, $\sup\{|f(z)| : z \in G\} \leq 1$ for f in $\mathcal{F}$; so in particular, for any compact subset K of G

$$|f(z)| \leq 1, \text{ for all } f \in \mathcal{F} \text{ and } z \in K.$$

Hence, by Lemma 1.2.11, $\mathcal{F}$ is locally bounded. It follows by Montel's Theorem $\mathcal{F}$ is normal if it is non-empty. So the first fact to be proved is

$$\mathcal{F} \neq \emptyset. \tag{1.77}$$

It will be shown that

$$\overline{\mathcal{F}} = \mathcal{F} \cup \{0\}. \tag{1.78}$$

Once these facts are known the proof can be completed. Indeed, suppose (1.77) and (1.78) hold and consider the function $\varphi : H(G) \to C$ defined by

$$\varphi(f) = f'(a).$$

φ is a continuous function follows from sequential criterion for continutiy of a function by the following observation: $f_n \to f$ in $H(G)$ implies (by Theorem 1.2.1 with $k = 1$) that $f_n' \to f'$, that is that $\rho(f_n',\ f') \to 0$ and hence by Part (b) of Proposition 1.1.10 it follows that $f_n' \to f'$ on all compact subsets of G and hence in particular (on the compact singleton set $\{a\}$)

$$\underbrace{f_n'(a)}_{\varphi(f_n)} \to \underbrace{f'(a)}_{\varphi(f)}.$$

As $\mathcal{F}$ is normal, $\overline{\mathcal{F}}$ is compact. Hence the continuous image $\varphi(\overline{\mathcal{F}})$ is compact. That is,

$$\{\varphi(f) : f \in \mathcal{F} \ or \ f = 0\}$$

is compact. That is,

$$\{f'(a) : f \in \mathcal{F} \ or \ f = 0\}$$

is compact. By the definition of $\mathcal{F}$, $f \in \mathcal{F}$ implies $f'(a) > 0$ showing that $f'(a)$ is a positive real number. Also, $0'(a) = 0$. Thus,

$$\{f'(a) : f \in \mathcal{F} \ or \ f = 0\} \subset \mathbb{R}$$

and, being compact, $\{f'(a) : f \in \mathcal{F} \ or \ f = 0\}$ is closed and bounded also. Hence the supremum of the set belongs to the set, and hence there is an f in $\overline{\mathcal{F}}$ with

$$f'(a) \geq g'(a) \text{ for all } g \text{ in } \overline{\mathcal{F}}.$$

In particular,

$$f'(a) \geq g'(a) \text{ for all } g \text{ in } \mathcal{F}.$$

Now $f \in \overline{\mathcal{F}}$ implies (from (1.78) that) either $f \in \mathcal{F}$ or $f = 0$.

Because $\mathcal{F} \neq \emptyset$, $f = 0$ is **not** possible. So $f \in \mathcal{F}$.

It remains to show that $f(G) = D$. Suppose $\omega \in D$ such that $\omega \notin f(G)$. Then the function

$$\frac{f(z) - \omega}{1 - \bar{\omega} f(z)}$$

is analytic in G and never vanishes[48]. As G is simply connected, using Part (a) implies (h) of Theorem 1.5.27 in the footnote in Page 178, corresponding to the non-vanishing analytic function f there is an analytic function $h : G \to \mathbb{C}$ such that

$$[h(z)]^2 = \frac{f(z) - \omega}{1 - \bar{\omega} f(z)}. \tag{1.79}$$

Since the Mobius transformation

$$T\zeta = \frac{\zeta - \omega}{1 - \bar{\omega}\zeta}$$

maps D onto D [49], $h(G) \subset D$[50]. Define $g : G \to \mathbb{C}$ by

$$g(z) = \frac{|h'(a)|}{h'(a)} \frac{h(z) - h(a)}{1 - \overline{h(a)} h(z)}.$$

[48]Details: The function is well-defined because the denominator never becomes 0 (the denominator becomes 0 means $\bar{\omega} f(z) = 1$ which never happens). Also, being the quotient of two analaytic functions, and since the denominator never becomes 0, $\frac{f(z)-\omega}{1-\bar{\omega} f(z)}$ is analytic. Also, the function never vanishes, for numerator is equal to 0 means $\omega = f(z)$ for some $z \in G$ which is a contradiction to the fact that $\omega \notin f(G)$.

[49]Details are given in first half of Page 131 of Conway.

[50]Why $h(G) \subset D$?

Then[51] $g(G) \subset D$, $g(a) = 0$, and g is one-one[52]. Also, by Quotient Rule of Differentiation,

$$g'(z) = \frac{|h'(a)|}{h'(a)} \cdot \frac{[1 - \overline{h(a)}h(z)]h'(z) + [h(z) - h(a)]\overline{h(a)}h'(z)}{[1 - \overline{h(a)}h(z)]^2}$$

and putting $z = a$ we obtain

$$\begin{aligned} g'(a) &= \frac{|h'(a)|}{h'(a)} \cdot \frac{h'(a)[1 - |h(a)|^2]}{[1 - |h(a)|^2]^2} \\ &= \frac{|h'(a)|}{1 - |h(a)|^2}. \end{aligned}$$

But $|h(a)|^2 = |-\omega| = |\omega|$ and differentiating (1.79) gives (since $f(a) = 0$) that

$$2h(a)h'(a) = f'(a)(1 - |\omega|^2).$$

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[51] Why $g(G) \subset D$?

[52] Why g is one-one? **We show that** $z_1 \neq z_2$ **implies** $g(z_1) \neq g(z_2)$. $z_1 \neq z_2$ implies (since $f \in \mathcal{F}$, f is one-to-one so $f(z_1) \neq f(z_2)$ implies

$$\frac{f(z_1) - \omega}{1 - \bar{\omega} f(z_1)} \neq \frac{f(z_2) - \omega}{1 - \bar{\omega} f(z_2)}$$

(for otherwise it can be seen that $\underbrace{[f(z_1) - f(z_2)]}\ \underbrace{(1 - |h(a)|^2)}_{\neq 0,\text{ as } h(a) \in h(G) \subset D} = 0$ which implies $f(z_1) - f(z_2) = 0$ which is a contradiction to the fact that $f(z_1) \neq f(z_2)$) implies $[h(z_1)]^2 \neq [h(z_1)]^2$ implies

$$\frac{h(z_1) - h(a)}{1 - \overline{h(a)}h(z_1)} \neq \frac{h(z_2) - h(a)}{1 - \overline{h(a)}h(z_2)}$$

(for other wise we will get $h(z_1) - h(z_2) = 0$ which is a contradiction to the fact that $h(z_1) \neq h(z_2)$) implies $g(z_1) \neq g(z_2)$.

Therefore

$$\begin{aligned} g'(a) &= \frac{f'(a)(1-|\omega|^2)}{2\sqrt{|\omega|}} \cdot \frac{1}{1-|\omega|} \\ &= f'(a)\left(\frac{1+|\omega|}{2\sqrt{|\omega|}}\right) \\ &> f'(a). \end{aligned}$$

This gives that g is in $\mathcal{F}$ and contradicts the choice of f. **Thus it must be that $f(G) = D$.**

Now to establish (1.77) and (1.78). Since $G \neq \mathbb{C}$, let $b \in \mathbb{C} - G$ and let g be a function analytic on G such that $[g(z)]^2 = z - b$.

If z_1 and z_2 are points in G and $g(z_1) = \pm g(z_2)$ (i.e., $g(z_1) = g(z_2)$ or $g(z_1) = -g(z_2)$) then $[g(z_1)]^2 = [g(z_2)]^2$ implies $z_1 - b = z_2 - b$ implies $z_1 = z_2$. In particular, g is one-one.

By the Open Mapping Theorem[53] $g(G)$ is open and hence the element $g(a)$ in G is an interior point of G so that there is a number $r > 0$ such that

$$g(G) \supset B(g(a); r). \tag{1.80}$$

So if there is a point z in G such that $g(z) \in B(-g(a); r)$

[53]**Open Mapping Theorem** Let G be a region and suppose that f is a non-constant analytic function in G. Then for any open set U in G, $f(U)$ is open.

then (the distance between $g(z)$ and $-g(a)$ is less than r so that)

$$r > |g(z) + (-g(a))| = |g(z) + g(a)| = | - g(z) - g(a).|$$

This shows that the distance between $-g(z)$ and $g(a)$ is less than r so that $-g(z) \in B(g(a); r)$ and hence by (1.80) $-g(z) \in g(G)$ so there is a w in G with $g(w) = -g(z)$; then (as in remarks preceding (1.80)) $[g(w)]^2 = [g(z)]^2$ implies $w - b = z - b$ implies $w = z$.

Now $g(w) = -g(z)$ and $w = z$ implies $g(z) = -g(z)$ implies $2g(z) = 0$ implies $g(z) = 0$. But then

$$z - b = [g(z)]^2 = 0$$

implies $b = z$ and since $z \in G$ this shows that b is also in G, a contradiction. Hence there is **no** $z \in G$ such that $g(z) \in B(-g(a); r)$ so that $g(G) \cap B(-g(a); r) = \emptyset$ which implies

$$g(G) \cap \underbrace{\{\zeta : |\zeta + g(a)| < r\}}_{B(-g(a);r)} = \emptyset. \tag{1.81}$$

Let U be the disk $\{\zeta : |\zeta + g(a)| < r\} = B(-g(a); r)$. There is a Mobius transformation[54] T such that $T(\mathbb{C}_\infty - \bar{U}\,) = D$.

[54] A mapping of the form

$$w = S(z) = \frac{az + b}{cz + d}$$

Let $g_1 = T \circ g$; then g_1 is analytic and $g_1(G) \subset D$. If $\alpha = g_1(a)$ then let $g_2(z) = \varphi_\alpha \circ g_1(z)$; so we still have that $g_2(G) \subset D$ and g_2 is analytic, but we also have that[55] $g_2(a) = 0$. Now it is a simple matter[56] to find a complex number c, $|c| = 1$, such

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is called a **linear fractional transformation** and it is a **Mobius transformation** when $ad - bc \neq 0$. If $ad - bc = 0$ then $ad = bc$, then the rational function is a constant since

$$S(z) = \frac{az+b}{cz+d} = \frac{a(cz+d)}{c(cz+d)} - \underbrace{\frac{ad-bc}{c(cz+d)}}_{0} = \frac{a}{c}$$

and is thus not considered a Mobius transformation. Also, $ad - bc \neq 0$ ensures that $\frac{dw}{dz} \neq 0$ and hence the trnasformation is a conformal mapping.

The following are simiple Mobius transformations:

- $S(z) = z + b \quad (a = 1,\ c = 0,\ d = 1)$ is a **translation**;
- $S(z) = az \quad (b = 0,\ c = 0,\ d = 1)$ is a combination of homothety and rotation and if $|a| = 1$ this is a rotation and if $a \in R$ it is a homothety;
- $S(z) = \frac{1}{z} \quad (a = 0,\ b = 1,\ c = 1,\ d = 0)$ is an **inversion in the unit circle**)

[55]

$$g_2(a) = (\varphi_\alpha \circ g_1)(a) = \varphi_\alpha(\alpha) = \frac{\alpha - \alpha}{1 - \bar{\alpha}\alpha} = 0.$$

[56]Now it is a simple matter to find a complex number c, $|c| = 1$, such that $g_3(z) = cg_2(z)$ has positive derivative at $z = a$. Explanation: $g_2'(z)$ is a complex valued function so $g_2'(a)$ is a complex number. We choose complex number c on the unit circle (so that $|c| = 1$) and that the product $cg_2'(a)$ is a positive real number. We define $g_3(z) = cg_2(z)$; then $g_3'(a) = cg_2'(a)$ is a positive real number. i.e., $g_3(z)$ has positive derivative at $z = a$.

that $g_3(z) = cg_2(z)$ has positive derivative at $z = a$. Hence

$$g_3 \in H(G),\ \ g_3 \text{ is one}-\text{one},\ \ g_3(a) = 0,\ \ g_3'(a) > 0,\ \ g_3(G) \subset D$$

so that $g_3 \in \mathcal{F}$ so that $\mathcal{F}$ is nonempty. **This establishes (1.77).**

Suppose $\{f_n\}$ is a sequence in $\mathcal{F}$ and $f_n \to f$ in $H(G)$. Then by Part (b) of Proposition 1.1.10 it follows that $f_n \to f$ on the singleton set $\{a\}$ so that $f_n(a) \to f(a)$. As f_n are members of $\mathcal{F}$, $f_n(a) = 0$ and hence the limit value is $f(a) = 0$.

$f_n \to f$ in $H(G)$ implies (by Theorem 1.2.1 with $k = 1$) that $f_n' \to f'$, that is that $\rho(f_n',\ f') \to 0$ and hence by Part (b) of Proposition 1.1.10 it follows that $f_n' \to f'$ on the compact singleton set $\{a\}$) so that $f_n'(a) \to f'(a)$. This shows that $f'(a)$ is a limit of a sequence $\{f_n'(a)\}$ of positive real numbers[57] and hence

$$f'(a) \geq 0. \tag{1.82}$$

Let z_1 be an arbitrary element of G and put

$$\zeta = f(z_1);$$

let

$$\zeta_n = f_n(z_1).$$

[57] As f_n are members of $\mathcal{F}$, $f_n'(a) > 0$. i.e., for $n = 1,\ 2,\ \ldots\ f_n'(a)$ are positive real numbers. Moreover, from *Real Analysis*, we know limit of a sequence of positive real numbers is greater than or equal to 0.

Let $z_2 \in G$, $z_2 \neq z_1$ and let K be a closed disk centered at z_2 such that $z_1 \notin K$ (Whether such a K exists?[58] Then $f_n(z) - \zeta_n$ never vanishes on K since[59] f_n is one- one. But $f_n(z) - \zeta_n \to f(z) - \zeta$ uniformly on K, so Hurwitz's Theorem gives that $f(z) - \zeta$ never vanishes on K or $f(z) \equiv \zeta$[60]. **Case 1.** If $f(z) \equiv \zeta$ on K then f is the constant function ζ throughout K; As K is an arbitrary closed subset of G this shows that

$$f(z) \equiv \zeta \text{ on } G.$$

Since $f(a) = 0$ we have that $\zeta = 0$ which implies

$$f(z) \equiv 0 \text{ on } G$$

or that $f = 0$ or that $f \in \{0\}$ in (1.78).

Case 2. If $f(z) - \zeta$ never vanishes on K, as K is an arbitrary closed subset of G, this shows that $f(z) - \zeta$ never vanishes on G. As $\zeta = f(z_1)$ this shows that for any $z_2 \neq z_1$ in G, $f(z_2) - f(z_1) \neq 0$; i.e., that $f(z_2) \neq f(z_1)$. As z_1 is also an arbitrary element of G, this shows that

$$z_2 \neq z_1 \text{ implies } f(z_2) \neq f(z_1).$$

[58]There are lots of such closed disks. For example, take $r = \frac{1}{2}|z_1 - z_2|$. Then the closed disk $K = \overline{B(z_2;\ r)}$ doesn't contain z_1.

[59]for otherwise, $f_n(z) = \zeta_n$ implies $f_n(z) = f_n(z_1)$, now since $z \in K$ and $z_1 \notin K$, $z \neq z_1$. This contradicts the fact that f_n is one-to-one.

[60]Details: **Case 1.** If $f(z) - \zeta \equiv 0$ then $f(z) \equiv \zeta$ and we are done. **Case 2.** If $f(z) - \zeta \not\equiv 0$

that is, f is one-one. But if f is one-one then f' can never vanish; so in particular $f'(a) \neq 0$ so (1.82)implies that $f'(a) > 0$. All these observations show that f is in $\mathcal{F}$.

Case 1 and Case 2 shows that $f \in \overline{\mathcal{F}}$ implies either $f = 0$ or $f \in \mathcal{F}$. This proves (1.78) and the proof of the lemma is complete. □

Corollary 1.4.5. *Among the simply connected regions there are only two equivalence classes; one consisting of $\mathbb{C}$ alone and the other containing all the proper simply connected regions.*

Proof. If G_1 and G_2 are proper simply connected regions, then by the Riemann mapping theorem, and using the definition of conformally equivalent regions, it follows (with $D = \{z : |z| < 1\}$) that

$$G_1 \cong D$$

and

$$G_2 \cong D.$$

By symmetry, $G_2 \cong D$ gives $D \cong G_2$. Hence, by transitivity, $G_1 \cong D$ and $D \cong G_2$ gives $G_1 \cong G_2$. This shows that proper simply connected regions are conformally equivalent to each other. On $\mathbb{C}$ a simply connected region is (i) either $\mathbb{C}$ itself (ii) or a proper simply connected region. Hence the equivalence relation "conformally equivalent to " partitions the set of all simply connected regions into two equivalence classes; one containing $\mathbb{C}$ alone and the other containing all the proper simply connected regions.

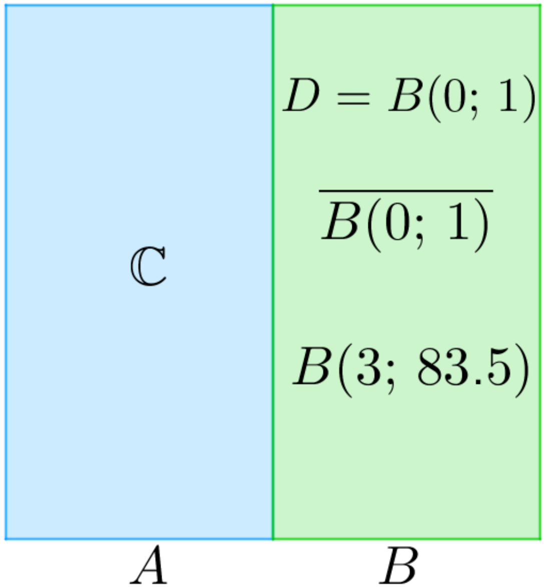

Figure 1.18: The equivalence relation "conformally equivalent to " partitions the set of all simply connected regions into two equivalence classes A and B. A consists of only one element, the set $\mathbb{C}$ of complex numbers. B consists of all proper simply connected regions. Only some proper simply connected regions are marked in B; they are $D = \{z : |z| < 1\}$ and some open and closed balls.

$\square$

1.5 The Weierstrass Factorization Theorem

Problem: Given a sequence $\{a_k\}$ in G **which has no limit point i**n G and a sequence of integers $\{m_k\}$, is there a function f which is analytic on G and such that the **only zeros of f are at the points a_k, with the multiplicity of the zero at a_k equal to m_k?** The answer to this question is **yes** and the result is due to Weierstrass [The German mathematician Karl Theodor Wilhelm Weierstrass (31 October 1815 – 19 February 1897) is known as the "father of modern analysis". Despite leaving university without a degree, he studied mathematics and trained as a teacher, eventually teaching mathematics, physics, botany and gymnastics.]

Karl Weierstrass (1814-1897)

Proposition 1.5.1. *In continuation to the discussion above, if* ***there are only a finite number of points*** a_1, ... a_n, *and finite number of integers m_1, ... m_n, then*

$$f(z) = (z - a_1)^{m_1} \cdots (z - a_n)^{m_n}$$

is the desired function.

What happens if there are *infinitely many points* in this sequence? To answer this we must discuss the convergence of infinite products of numbers and functions.

Definition 1.5.2. If $\{z_n\}$ is a sequence of complex numbers and if

$$z = \lim_{n\to\infty} \prod_{k=1}^{n} z_k$$

exists, then z is the **infinite product** of the numbers z_n and is denoted by

$$z = \prod_{n=1}^{\infty} z_n.$$

Example 1.5.3. If any one the z_n is zero, then the limit is zero, regardless of the behaviour of the remaining terms of the sequence.

Proof: Suppose $z_K = 0$ for some natural number K. Then for $n \geq K$, $\prod\limits_{k=1}^{n} z_k = 0$ and hence $z = \lim\limits_{n\to\infty} \prod\limits_{k=1}^{n} z_k = 0$. [More details: Let $\varepsilon > 0$ be given. Then choose $K \in \mathbb{N}$ for which

$z_K = 0$ (NOTE: The same K works for any $\varepsilon > 0$.) Then

$$\left| \underbrace{\underbrace{\prod_{k=1}^{n} z_k}_{0 \text{ for } n \geq K} - 0}_{0} \right| < \varepsilon \quad \text{for } n \geq K$$

showing that

$$\lim_{n \to \infty} \prod_{k=1}^{n} z_k = 0$$

Proposition 1.5.4. ***[A necessary condition for the convergence of an infinite product]***[61] *Except for the case where zero appears, a necessary condition for the convergence of an infinite product with limit of the product is nonzero is that the n-th term must go to 1.*

Proof. Suppose that no one of the numbers z_n is zero, and

[61] Recall the situation in the case of infinite series: If the series

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + \cdots + u_n + \cdots$$

converges, then

$$\lim_{n \to +\infty} u_n = 0.$$

The converse is not true. For example, for the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} + \ldots$, nth term is $u_n = \frac{1}{n}$ and $\lim_{n \to +\infty} u_n = 0$; but the series doesn't converge.

that $z = \prod_{n=1}^{\infty} z_n$ exists and is **also not zero.** Let

$$p_n = \prod_{k=1}^{n} z_k \text{ for } n \geq 1;$$

then no p_n is zero (for if any one of p_n is zero then (noting that being a field, $\mathbb{C}$ is an integral domain so) one of $z_1, \ \ldots \ z_n$, is zero, which is a contradiction to our assumption) and

$$\frac{p_n}{p_{n-1}} = \frac{\prod_{k=1}^{n} z_k}{\prod_{k=1}^{n-1} z_k} = z_n. \tag{1.83}$$

Since $z \neq 0$ and $p_n = \prod_{k=1}^{n} z_k \to z$, using (1.83) it follows that

$$\lim_{n\to\infty} z_n = \frac{\lim_{n\to\infty} p_n}{\lim_{n\to\infty} p_{n-1}} = \frac{z}{z} = 1.$$

$\square$

Remark 1.5.5. Attention! The condition is not necessary if limit of the product is 0. For example, if $z_n = a$ for all n and $|a| < 1$, then the infinite product converges with

$$\prod_{k=1}^{\infty} z_k = \lim_{n\to\infty} \prod_{k=1}^{n} z_k = \lim_{n\to\infty} \underbrace{a \cdots a}_{n \text{ times}} = \lim_{n\to\infty} a^n = 0,$$

but $\lim_{n\to\infty} z_n = a \neq 1$.

Definition 1.5.6. Principal Branch of Logarithm Function The **principal branch** of the *multiple valued* logarithmic function is defined as: For $z = re^{i\,\Theta}$ $(r > 0,\ -\pi < \Theta < \pi)$, an element in the open connected set $G = \mathbb{C} - \{z : z \leq 0\}$ (that is, G is the complex plane with points on the ray from origin along the negative real axis are removed) (Here Θ is the **principal amplitude** of z),

$$\log z = \underbrace{\ln r}_{\substack{\text{natural logarithm of the}\\ \text{nonnegative real number } r}} + i\,\Theta \quad (r > 0,\ -\pi < \Theta \leq \pi).$$

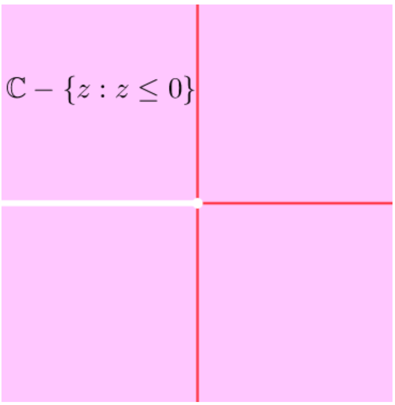

Figure 1.19: The domain of principal branch of the logarithmic function $\log z = \ln r + i\,\Theta \quad (r > 0,\ -\pi < \Theta \leq \pi.)$.

By Proposition 1.5.4, if the product $\prod_{n=1}^{\infty} z_n$ is nonzero, then

$z_n \to 1$. So it is no restriction to suppose that $\mathrm{Re}\, z_n > 0$ for all n.

Proposition 1.5.7. *Let* $\mathrm{Re}\, z_n > 0$ *for all* $n \geq 1$. *Then* $\prod_{n=1}^{\infty} z_n$ *converges to a nonzero number if and only if the series* $\sum_{n=1}^{\infty} \log z_n$ *converges.*

Proof. Assume that $\prod_{n=1}^{\infty} z_n$ converges to a **nonzero** number $z = re^{i\theta}, \ -\pi < \theta \leq \pi$

Claim: $\sum_{n=1}^{\infty} \log z_n$ converges. Let $p_n = \prod_{k=1}^{n} z_k$ and

$$l(p_n) =$$

$$\underbrace{\log|p_n|}_{\substack{\text{natural logarithm of the} \\ \text{nonnegative real number } |p_n|}} + i \underbrace{\theta_n}_{\substack{\theta_n \text{ is chosen such that} \\ \theta-\pi<\theta_n\leq\theta+\pi}}, \quad \theta - \pi < \theta_n \leq \theta + \pi \tag{1.84}$$

(NOTE: Since $\mathrm{Re} z_n > 0$ forall $n \geq 1$ none of z_n is zero and hence $p_n = \prod_{k=1}^{n} z_k \neq 0$ for any any $n \geq 1$).

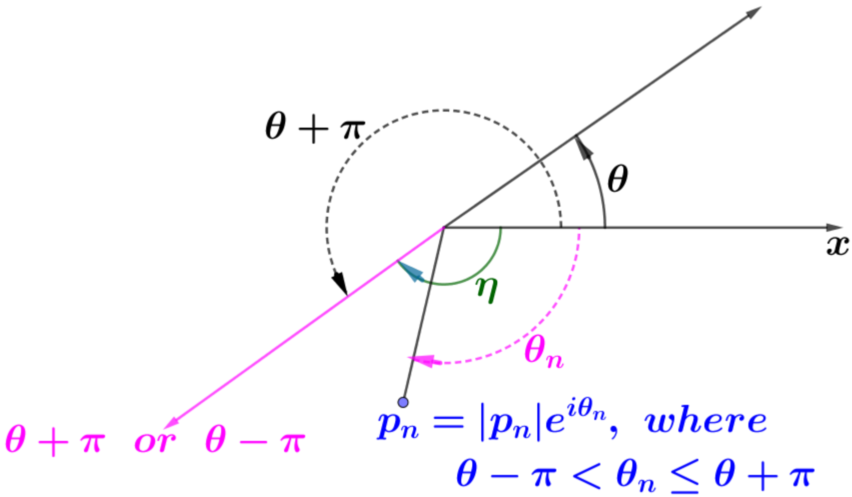

Figure 1.20: We consider the branch of logarithm function for which the branch cut is the ray $\eta = \theta - \pi$ where θ is the (fixed one and is the) **principal amplitude** of z above (the nonzero z is the limit of the series, by assumption.)

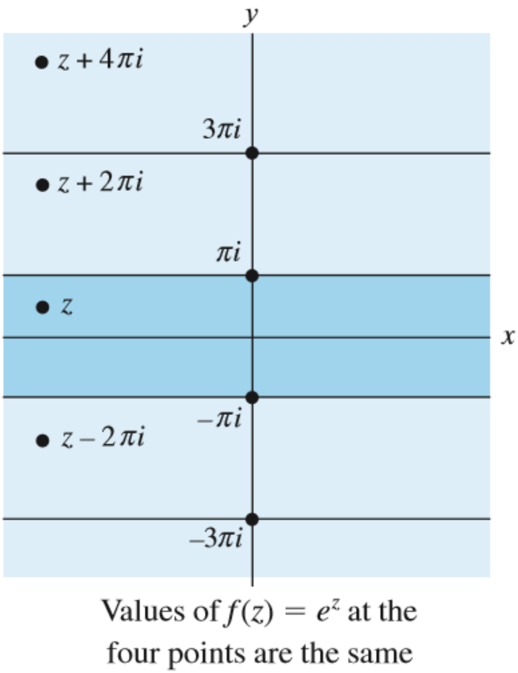

Figure 1.21: Figure shows the domain of the exponential function (the domain is the entire complex plane) and four points are marked whose imaginary parts differ by integer multiples of $2\pi i$ that gives same exp value $\exp(z)$. Similarly, we can see that $\exp s_n = p_n = \exp l(p_n)$ so that s_n and $l(p_n)$ differ by $2\pi i k_n$ for some integer k_n.

In the above for any $n \geq 1$ we have considered ***the* branch** of logarithm of p_n where the branch cut is the ray $\eta = \theta + \pi$ where θ is the (fixed one and is the) **principal amplitude** of z

above (the nonzero z is the limit of the series, by assumption) (Figure 1.20). [62] If

$$s_n = \log z_1 + \cdots + \log z_n = \log z_1 \cdots z_n = \log p_n \qquad (1.85)$$

then

$$\exp s_n = \exp \log p_n = p_n$$

so that

$$s_n = l(p_n) + 2\pi i k_n \qquad (1.86)$$

for some integer k_n [63]

Moreover, $p_n \to z$ implies $p_{n-1} \to z$ and (since l is analytic, and hence in particular continuous function)

$$l(p_n) \to l(z) \qquad (1.87)$$

---

[62] Attention/Notation: l in (1.84) is **NOT** the principal branch of the logarithm function, **but** the branch of the logarithm function with branch cut $\eta = \theta + \pi$ and branch point 0.

[63] Reason: Since exp function is $2\pi i$ periodic (Figure 1.21), the equations $\exp s_n = p_n$ and $\exp l(p_n) = p_n$ (the last equation follows from (1.84)) implies s_n and $l(p_n)$ differ by $2\pi i k_n$ for some integer k_n. ***Another Argument***:

$$s_n = l(p_n) + 2\pi i \underbrace{k_n}_{\substack{\text{This } k_n \in \mathbb{Z} \text{ is due to the fact that} \\ \theta_n \equiv t_n \bmod (2\pi) \text{ where } t_n = \arg z_1 + \cdots + \arg z_n, \\ \theta - \pi < \arg z_1 \le \theta + \pi,\ \ldots,\ \theta - \pi < \arg z_n \le \theta + \pi}}$$

By assumption $\prod_{k=1}^{n} z_k$ converges to the nonzero number z as $n \to \infty$. i.e., $p_n \to z$, so, by Proposition 1.5.4, $z_n \to 1$ as $n \to \infty$, so $\log z_n \to 0$ as $n \to \infty$, so by (1.85),

$$s_n - s_{n-1} = \log z_n \to 0.$$

and

$$l(p_{n-1}) \to l(z)$$

so that

$$l(p_n) - l(p_{n-1}) \to \underbrace{l(z) - l(z)}_{=0} \text{ as } n \to \infty$$

[i.e., $\lim_{n\to\infty} l(p_n) - \lim_{n\to\infty} l(p_{n-1}) = l(z) - l(z) = 0.$] From (1.86), we have

$$s_{n-1} = l(p_{n-1}) + 2\pi i k_{n-1}. \tag{1.88}$$

Subtracting (1.88) from (1.86), we obtain

$$s_n - s_{n-1} = l(p_n) - l(p_{n-1}) + 2\pi i(k_n - k_{n-1})$$

and when $n \to \infty$ we obtain left hand side converges to 0 and $l(p_n) - l(p_{n-1})$ converges to 0, so that $2\pi i(k_n - k_{n-1})$ converges to 0. So, $k_n - k_{n-1} \to 0$ as $n \to \infty$. Since each k_n is an integer, $k_n - k_{n-1}$ is also an integer, and we have this integer value can be made very small as we please by taking n very large. This shows, in particular, that corresponding to $\varepsilon = \frac{1}{2}$ there is an integer n_0 such that

$$\left| \underbrace{k_n - k_{n-1}} - 0 \right| < \frac{1}{2} \quad \forall n \geq n_0.$$

As $k_n - k_{n-1}$ is an integer, the only possibility is that

$$k_n - k_{n-1} = 0 \quad \forall n \geq n_0$$

which implies

$$k_n = k_{n-1} \quad \forall n \geq n_0$$

We let

$$k_n = k \text{ for } n \geq n_0.$$

Then from (1.86) and (1.87), it follows that

$$\lim_{n\to\infty} s_n = \underbrace{\lim_{n\to\infty} l(p_n)}_{l(z)} + \underbrace{\lim_{n\to\infty} 2\pi i k_n}_{2\pi i \lim\limits_{n\to\infty} k_n = 2\pi i k}$$

i.e., that

$$\lim_{n\to\infty} s_n = l(z) + 2\pi i k$$

or that

$$s_n \to l(z) + 2\pi i k$$

that is, the n-th partial sum of the series $\sum \log z_n$ converges, and hence the series itself is convergent.

Conversely, suppose that the series $\sum \log z_n$ converges. If

$$s_n = \sum_{k=1}^{n} \log z_k$$

(i.e., s_n is the n-th partial sum of the series $\sum \log z_n$) and $s_n \to s$ then

$$\exp s_n = \exp \sum_{k=1}^{n} \log z_k = \exp \log \prod_{k=1}^{n} z_k = \prod_{k=1}^{n} z_k$$

so that $\prod_{k=1}^{n} z_k$ converges to

$$\lim_{n\to\infty} \exp s_n \underset{\substack{\uparrow \\ \text{as exp is a} \\ \text{continuous function}}}{=} \exp \lim_{n\to\infty} s_n = \exp s \neq 0.$$

($\exp s \neq 0$ since there is no z such that $\exp z = 0$). That is,

$$\prod_{n=1}^{\infty} z_n = e^s \neq 0.$$

This completes the proof. $\square$

1.5.1 Power Series Expansion of $\log(1+z)$ about $z = 0$.

We first consider the power series expansion of $\log(1+z)$ about $z = 0$.

Proposition 1.5.8. *The function* $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ *given by*

$$f(z) = \log z = \ln|z| + i\,\mathrm{Arg}(z)$$

is continuous at all z *except those along the negative real axis* $\{z : z \leq 0\}$ *(Fig. 1.19).*

Remark 1.5.9. The function f defined above is continuous at all z except those along the branch cut $\{z : z \leq 0\}$ (Fig. 1.19).

Proposition 1.5.10. *The function* $f : \mathbb{C}\backslash\{z : z \leq 0\} \to \mathbb{C}$ *given by*

$$f(z) = \log z = \ln|z| + i\,\mathrm{Arg}(z)$$

is analytic on $\mathbb{C}\backslash\{z : z \leq 0\}$.

The above result also gives the following:

Proposition 1.5.11. *The function* $f : \mathbb{C}\backslash\{z : z \leq -1\} \to \mathbb{C}$ *given by*

$$f(z) = \log(1 + z)$$

is analytic on $\mathbb{C}\backslash\{z : z \leq -1\}$.

In view of Proposition 1.5.11, using Taylor's Theorem[64]

[64]**Taylor's Theorem:** Suppose that a function f is analytic throughout an open disk $|z - z_0| < R_0$, centered at z_0 and radius R_0. Then, at each point z in that disk, $f(z)$ has the series representation (called **Taylor series** about

$\log(1+z)$ has the power series representation[65] at $z = 0$ (which has a radius of convergence[66] 1 (Ref. Figure 1.22)) given by

$$\log(1+z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n} = z - \frac{z^2}{2} + \cdots.$$

---

the point z_0)

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad (|z - z_0| < R_0),$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!} \qquad (n = 0, \quad 1, \quad 2, \quad \ldots).$$

That is, the power series here converges to $f(z)$ when $|z - z_0| < R_0$.

[65]Using Taylor series we expand textbflog $(1+z)$.

Take $f(z) = \log(1+z)$ so that $f(0) = \log(1+0) = \text{Ln}1 = 0$;

$$f'(z) = [\log(1+z)]' = \frac{1}{1+z}; f'(0) = \frac{1}{1+0} = 1;$$

$$f''(z) = -\frac{1}{(1+z)^2}; f''(0) = -1; f'''(z) = \frac{2}{(1+z)^3}; f'''(0) = 2;$$

$$f^{(4)}(z) = -\frac{6}{(1+z)^4}; f^{(4)}(0) = -6; f^{(5)}(z) = \frac{24}{(1+z)^5}; f^{(5)}(0) = 24.$$

Substituting these values in Taylor series (with $z_0 = 1$) we get

$$\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \frac{z^5}{5} - \ \ldots$$

[66]The radius of convergence

$$R = \frac{1}{L} = \frac{1}{\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{\lim\limits_{n\to\infty} \left|\frac{(-1)^n/(n+1)}{(-1)^{n-1}/n}\right|} = 1.$$

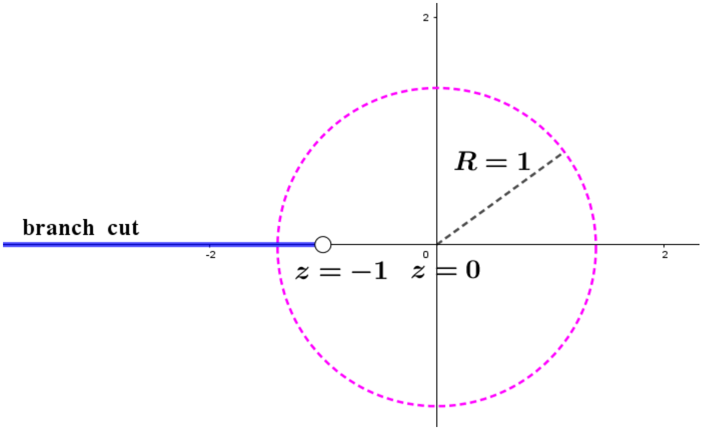

Figure 1.22: The branch cut of of $\log(1+z)$

i.e., if $|z|<1$ then the power series expansion of $\log(1+z)$ given above is valid.
If $|z|<1$, then

$$\begin{aligned}\left|1-\frac{\log(1+z)}{z}\right| &= \left|1-\frac{z-\frac{z^2}{2}+\cdots}{z}\right| = \left|\frac{1}{2}z-\frac{1}{3}z^2+\cdots\right| \\ &\leq \frac{1}{2}|z|+\frac{1}{3}|z|^2+\cdots \qquad (1.89) \\ &\leq \frac{1}{2}\left(\underbrace{|z|+|z|^2+\cdots}_{\substack{\text{geometric series of real numbers with} \\ \text{common ratio } |z| \text{ and initial term } |z|}}\right) \\ &= \frac{1}{2}\cdot\frac{|z|}{1-|z|} \qquad (1.90)\end{aligned}$$

If we further require $|z| < \frac{1}{2}$ then $-|z| > -\frac{1}{2} \Rightarrow 1 - |z| > 1 - \frac{1}{2} = \frac{1}{2}$ so that $\frac{|z|}{1-|z|} < \frac{(1/2)}{(1/2)} = 1$ and hence (1.90) gives

$$\left|1 - \frac{\log(1+z)}{z}\right| < \frac{1}{2}$$

i.e., for $|z| < \frac{1}{2}$,

$$|z - \log(1+z)| < \frac{1}{2}|z|$$

Hence, for $|z| < \frac{1}{2}$,

$$|z| = |z - \log(1+z) + \log(1+z)|$$

$$\leq \underbrace{|z - \log(1+z)|}_{<(1/2)|z| \;\; for \;\; |z|<(1/2)} + |\log(1+z)|$$

so that

$$\frac{1}{2}|z| \leq |\log(1+z)|\,.$$

Also, for $|z| < \frac{1}{2}$,

$$|\log(1+z)| =$$

$$|z - (z - \log(1+z))| \leq |z| + \underbrace{|z - \log(1+z)|}_{<(1/2)|z| \;\; for \;\; |z|<(1/2)} \leq \frac{3}{2}|z|$$

Combining[67], we obtain, for $|z| < \frac{1}{2}$,

$$\frac{1}{2}|z| \leq |\log(1+z)| \leq \frac{3}{2}|z|. \tag{1.91}$$

Proposition 1.5.12. *Let* $\mathrm{Re} z_n > -1$; *then the series*

$$\sum_{n=1}^{\infty} \log(1+z_n)$$

converges absolutely if and only if the series $\sum_{n=1}^{\infty} z_n$ *converges absolutely.*

Proof. $\sum_{n=1}^{\infty} z_n$ converges absolutely (i.e, $\sum_{n=1}^{\infty} |z_n|$ converges) implies[68]. $\sum_{n=1}^{\infty} z_n$ converges. Then the n-th term converges to 0; i.e., $z_n \to 0$ as $n \to \infty$. Hence, in particular, taking $\varepsilon = \frac{1}{2}$ there is $N \in \mathbb{N}$ such that

$$|z_n - 0| < \frac{1}{2} \quad \forall n \geq N$$

i.e.,

$$|z_n| < \frac{1}{2} \quad \forall n \geq N. \tag{1.92}$$

---

[67] $\log(1+z)$ is a complex number, while $|\log(1+z)|$ is a nonnegative real number.

[68] Every absolutely convergent series of complex numbers is convergent. `General Result:` (Page126, *Functional Analysis*, B. V. Limaye) In a Banach space X every absolutely convergent series of elements of X is convergent.

i.e., eventually $|z_n| < \frac{1}{2}$. Then, using (1.91), for $n \geq N$,

$$|\log(1+z_n)| \leq \frac{3}{2}|z_n|.$$

Hence

$$\sum_{n=N}^{\infty} |\log(1+z_n)| \leq \frac{3}{2} \sum_{n=N}^{\infty} |z_n|$$

i.e., the series $\sum\limits_{n=N}^{\infty} |\log(1+z_n)|$ is dominated by the convergent series $\frac{3}{2} \sum\limits_{n=N}^{\infty} |z_n|$ (the series $\sum\limits_{n=N}^{\infty} |z_n|$ is the N-tail of the convergent series $\sum\limits_{n=1}^{\infty} |z_n|$) Hence the series

$$\sum_{n=1}^{\infty} |\log(1+z_n)| = \underbrace{\sum_{n=1}^{N-1} |\log(1+z_n)|}_{\text{finite real number}} + \underbrace{\sum_{n=N}^{\infty} |\log(1+z_n)|}_{<\infty}$$

converges.

Conversely, assume that $\sum\limits_{n=1}^{\infty} |\log(1+z_n)|$ converges. Then, nth term of the series converges to 0 as $n \to \infty$. i.e., $|\log(1+z_n)| \to 0$ as $n \to \infty$. So $1+z_n \to 1$ as $n \to \infty$[69].

[69] $\log(1+z_n) = \underbrace{\ln|1+z_n|}_{\text{positve real number}} + i \operatorname{Arg}(1+z_n)$ so $|\log(1+z_n)| \to 0$ as $n \to \infty$ implies $\ln|1+z_n| \to 0$ as $n \to \infty$ implies $|1+z_n| \to 1$ as $n \to \infty$ implies $1+z_n \to 1$ as $n \to \infty$ (Fig. 1.23).

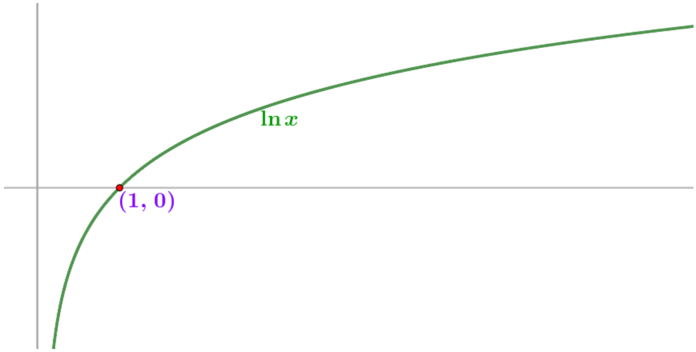

Figure 1.23: The real valued function $\ln x$ of real variable converges to 0 as $x \to 1$.

Then $z_n \to 0$ as $n \to \infty$. Hence, in particular, taking $\varepsilon = \frac{1}{2}$ there is $N \in \mathbb{N}$ such that

$$|z_n - 0| < \frac{1}{2} \quad \forall n \geq N$$

Then, using (1.91), for $n \geq N$,

$$\frac{1}{2} |z_n| \leq |\log (1 + z_n)| \,.$$

Hence

$$\frac{1}{2} \sum_{n=N}^{\infty} |z_n| \leq \sum_{n=N}^{\infty} |\log(1 + z_n)|.$$

Being tail part of a convergent series, the series $\sum\limits_{n=N}^{\infty} |\log(1 + z_n)|$ converges and hence by Comparison Test, $\frac{1}{2} \sum\limits_{n=N}^{\infty} |z_n|$ converges.

Thus, the series

$$\frac{1}{2}\sum_{n=1}^{\infty}|z_n| = \underbrace{\frac{1}{2}\sum_{n=1}^{N-1}|z_n|}_{\text{finite real number}} + \underbrace{\frac{1}{2}\sum_{n=N}^{\infty}|z_n|}_{<\infty}$$

converges. □

Definition 1.5.13. If $\mathrm{Re} z_n > 0$ for all n, then the infinite product $\prod_{n=1}^{\infty} z_n$ is said to **converge absolutely** if the series $\sum_{n=1}^{\infty} \log z_n$ converges absolutely.

Remark 1.5.14. [**Attention!**] Naturally one may expect that the infinite product $\prod_{n=1}^{\infty} z_n$ converges absolutely if $\prod_{n=1}^{\infty} |z_n|$ converges. BUT THIS IS FALSE. For example, let $z_n = -1$ for all n. Then $|z_n| = 1$ for all n so that

$$\prod_{n=1}^{\infty}|z_n| = \lim_{m\to\infty}\underbrace{\prod_{n=1}^{m}|z_n|}_{1} = 1$$

and hence the series $\prod_{n=1}^{\infty}|z_n|$ converges to 1. But,

$$\prod_{n=1}^{\infty} z_n = \lim_{m\to\infty}\underbrace{\prod_{n=1}^{m} z_n}_{1 \text{ or } -1}$$

does not converge. So in order to get that absolute convergence imply convergence, we must seek Definition 1.5.13.

Proposition 1.5.15. ***[Absolute convergence imply convergence]*** *Absolute convergence of a product imply convergence.*

Proof. $\mathrm{Re} z_n > 0$ for all n and $\prod_{n=1}^{\infty} z_n$ converges absolutely implies (by Definition 1.5.13) $\sum_{n=1}^{\infty} |\log z_n|$ converges. Since every absolutely convergent series of complex numbers is convergent, it follows that $\sum_{n=1}^{\infty} \log z_n$ converges. Then, by Proposition 1.5.7, $\prod_{n=1}^{\infty} z_n$ converges to a nonzero number. Hence absolute convergence of the product imply convergence of the product. □

Proposition 1.5.16. *If a product converges absolutely then any rearrangement of the terms of the product results in a product which is still absolutely convergent.*

Proof. $\mathrm{Re} z_n > 0$ for all n and $\prod_{n=1}^{\infty} z_n$ converges absolutely implies (by Definition 1.5.13) $\sum_{n=1}^{\infty} |\log z_n|$ converges. Being an absolutely convergent series, any rearrangement of $\sum_{n=1}^{\infty} |\log z_n|$ also converges. This implies (again by Definition 1.5.13) that any rearrangement of $\prod_{n=1}^{\infty} z_n$ is also absolutely convergent. □

Corollary 1.5.17. ***[Fundamental Criterion for Convergence of an Infinite Product in terms of Infinite Series]*** *If* $\mathrm{Re} z_n > 0$ *for all* n*, then the infinite product* $\prod_{n=1}^{\infty} z_n$ *converges absolutely if and only if the series* $\sum_{n=1}^{\infty} (z_n - 1)$ *converges absolutely.*

Using Propositions 1.5.7 and 1.5.12. $\mathrm{Re} z_n > 0$ for all n and $\prod_{n=1}^{\infty} z_n$ converges absolutely implies (by Definition 1.5.13)

$$\sum_{n=1}^{\infty} \underbrace{|\log z_n|}_{|\log(1+z_n-1)|}$$

converges. Then, by Proposition 1.5.12, $\sum_{n=1}^{\infty} (z_n - 1)$ converges absolutely.

Conversely, assume that $\sum_{n=1}^{\infty} (z_n - 1)$ converges absolutely. Then, by Proposition 5.4, $\sum_{n=1}^{\infty} \underbrace{|\log(1 + z_n - 1)|}_{|\log z_n|}$ converges absolutely and by Proposition 1.5.7 $\prod_{n=1}^{\infty} z_n$ converges absolutely. $\square$

1.5.2 Convergence of Products of Functions

Question: Suppose $\{f_n\}$ is a sequence of functions on a set X and $f_n(x) \to f(x)$ uniformly for $x \in X$, when will $\exp f_n(x) \to \exp f(x)$ uniformly for $x \in X$? Below is a partial answer.

Lemma 1.5.18. *Let X be a set and let $f,\ f_1,\ f_2,\ \ldots$ be functions from X into $\mathbb{C}$ such that $f_n(x) \to f(x)$ uniformly for $x \in X$. If there is a constant a such that $\mathrm{Re} f(x) \le a$ for all $x \in X$ then $\exp f_n(x) \to \exp f(x)$ uniformly for $x \in X$.*

Proof. We know that $\exp z$ is continuous at $z = 0$, so if $\varepsilon > 0$ is given then it is possible to choose $\delta > 0$ such that

$$|z - 0| < \delta \quad \Rightarrow \quad \left|e^z - e^0\right| < \varepsilon e^{-a}$$

i.e., that

$$|z| < \delta \quad \Rightarrow \quad |e^z - 1| < \varepsilon e^{-a}. \tag{1.93}$$

Since $f_n(x) \to f(x)$ uniformly for $x \in X$, corresponding to the above $\delta > 0$ there is a natural number n_0 (depends only on δ and works for all $x \in X$) such that

$$|f_n(x) - f(x)| < \delta \text{ for all } x \in X \text{ and } n \ge n_0.$$

Thus, for $n \ge n_0$ for any $x \in X$, $|f_n(x) - f(x)| < \delta$ so from

(1.93), we have

$$\varepsilon e^{-a} > \left|\exp(\underbrace{f_n(x) - f(x)}_{|f_n(x)-f(x)|<\delta \ \forall n \geq n_0}) - 1\right| = \left|\frac{\exp f_n(x)}{\exp f(x)} - 1\right|.$$

It follows that for any $x \in X$ and for $n \geq n_0$,

$$|\exp f_n(x) - \exp f(x)| < \varepsilon e^{-a} \left|\exp f(x)\right|. \tag{1.94}$$

We note that[70]

$$\left|\underbrace{\exp}_{\substack{\text{complex}\\ \text{exp function}}} \underbrace{f(x)}_{\in \mathbb{C}}\right| = \left|\underbrace{\exp}_{\substack{\text{real exp function}\\ \text{in calculus}}} \underbrace{\mathrm{Re} f(x)}_{\in \mathbb{R}}\right| = \underbrace{\exp \mathrm{Re} f(x)}_{\text{as real } \exp(y)>0 \,\forall y} \tag{1.95}$$

By assumption,

$$\mathrm{Re} f(x) \leq a \text{ for all } x \in X.$$

[70] If $z = \underbrace{x}_{\mathrm{Re} z} + i \underbrace{y}_{\mathrm{Im} z}$, then $e^z = e^{x+i\,y} = \underbrace{e^x}_{>0} \underbrace{e^{i\,y}}_{\substack{\text{complex}\\ \text{number}}}$ and $|e^z| = e^x \underbrace{\left|e^{i\,y}\right|}_{1} = e^x$

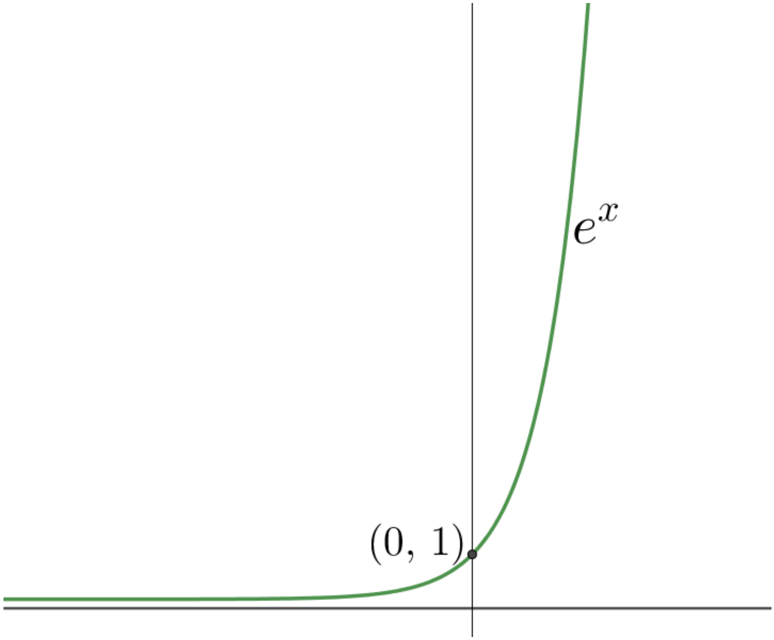

Figure 1.24: Real valued exp function of real variable is a monotonic increasing one-to-one function that maps $\mathbb{R}$ onto $(0, \infty)$.

This implies, noting that real valued exp function is a monotonic increasing function (Fig. 1.19),

$$\exp \mathrm{Re} f(x) \leq \exp a \;\; \forall x \in X$$

Thus, from (1.95), we have

$$|\exp f(x)| \leq e^a \;\; \forall x \in X$$

and hence from (1.94), we obtain for any $x \in X$ and for $n \geq n_0$,

$$|\exp f_n(x) - \exp f(x)| < \varepsilon e^{-a} e^a = \varepsilon.$$

Thus, $\exp f_n(x) \to \exp f(x)$ uniformly for $x \in X$. $\square$

Lemma 1.5.19. *Let (X, d) be a compact metric space and let $\{g_n\}$ be a sequence of continuous functions from X into $\mathbb{C}$ such that $\sum_{n=1}^{\infty} g_n(x)$ converges absolutely and uniformly for $x \in X$. Then the product*

$$f(x) = \prod_{n=1}^{\infty} (1 + g_n(x)) \tag{1.96}$$

converges absolutely and uniformly for $x \in X$. Also there is an integer n_0 such that $f(x) = 0$ if and only if $g_n(x) = -1$ for some n, $1 \leq n \leq n_0$.

Proof. Since the series of functions $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly for $x \in X$, the n-th term function $g_n(x)$ converges uniformly to 0 for $x \in X$ as $n \to \infty$. Hence, in particular, corresponding to $\varepsilon = \frac{1}{2}$ there is a natural number n_0 such that

$$|g_n(x) - 0| < \frac{1}{2} \quad \forall x \in X, \quad \forall n > n_0;$$

$$|g_n(x)| < \frac{1}{2} \quad \forall x \in X, \quad \forall n > n_0. \tag{1.97}$$

This implies, noting that

$$|g_n(x)| = |\mathrm{Re}g_n(x) + i\mathrm{Im}g_n(x)| = \sqrt{[\mathrm{Re}g_n(x)]^2 + [\mathrm{Im}g_n(x)]^2},$$

$$[\mathrm{Re}g_n(x)]^2 + [\mathrm{Im}g_n(x)]^2 < \frac{1}{4} \ \forall x \in X, \quad \forall n > n_0$$

implies

$$[\mathrm{Re}g_n(x)]^2 < \frac{1}{4} \ \forall x \in X, \quad \forall n > n_0$$

implies

$$-\frac{1}{2} < \mathrm{Re}g_n(x) < \frac{1}{2} \ \forall x \in X, \quad \forall n > n_0$$

Hence $\forall x \in X, \quad \forall n > n_0$

$$\mathrm{Re}[1 + g_n(x)] = 1 + \mathrm{Re}g_n(x) > 0.$$

As (1.97) holds for all $x \in X$ and for all $n > n_0$, second inequality of (1.91) gives that

$$|\log(1 + g_n(x))| \leq \frac{3}{2}\,|g_n(x)| \ \ \forall x \in X, \quad \forall n > n_0.$$

Since $\sum\limits_{n=n_0+1}^{\infty} g_n(x)$ converges absolutely $\forall x \in X$, we have

$\sum_{n=n_0+1}^{\infty} |g_n(x)|$ converges $\forall x \in X$. Now, since the series of nonnegative terms $\sum_{n=n_0+1}^{\infty} |\log(1+g_n(x))|$ is dominated by the convergent series

$$\sum_{n=n_0+1}^{\infty} |g_n(x)|$$

it follows that $\sum_{n=n_0+1}^{\infty} |\log(1+g_n(x))|$ is also convergent. Since every absolutely convergent series is convergent, it follows that (the series of complex numbers) $\sum_{n=n_0+1}^{\infty} \log(1+g_n(x))$ converges $\forall x \in X$. Thus

$$h(x) = \sum_{n=n_0+1}^{\infty} \log(1+g_n(x) \text{ for } x \in X$$

converges. Also, since n_0 is independent of points on X it follows that the convergence is uniform on X.

Being the composition of continuous functions log and $1+g_n(x)$, the function $\log(1+g_n(x)$ continuous, and also since the series converges *uniformly* for $x \in X$, it follows that limit function h of the series is a continuous function on X. Since X is compact, its continuous image $h(X)$ is also compact, and hence is closed and bounded. So h is a bounded function on X. In particular, $\operatorname{Re} h$ is bounded on X, so there is a constant a such that

$$\operatorname{Re} h(x) < a \quad \forall x \in X.$$

Thus, Lemma 1.5.18 applies and gives that

$$\exp \sum_{n=n_0+1}^{\infty} \log(1 + g_n(x) \to \exp h(x)$$

uniformly for $x \in X$. That is[71],

$$\exp \log \prod_{n=n_0+1}^{\infty} (1 + g_n(x) \to \exp h(x)$$

uniformly for $x \in X$. That is,

$$\prod_{n=n_0+1}^{\infty} (1 + g_n(x) \to \exp h(x)$$

uniformly for $x \in X$. That is,

$$\exp h(x) = \prod_{n=n_0+1}^{\infty} (1 + g_n(x)$$

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$$\begin{aligned}
\sum_{n=n_0+1}^{\infty} \log(1 + g_n(x) &= \lim_{m\to\infty} \sum_{n=n_0+1}^{m} \log(1 + g_n(x) = \lim_{m\to\infty} \log \prod_{n=n_0+1}^{m} (1 + g_n(x) \\
&= \log \lim_{m\to\infty} \prod_{n=n_0+1}^{m} (1 + g_n(x) = \log \prod_{n=n_0+1}^{\infty} (1 + g_n(x).
\end{aligned}$$

converges uniformly for $x \in X$. Thus, from (1.96), we have

$$\begin{aligned} f(x) &= [1+g_1(x)]\cdots[1+g_{n_0}(x)] \prod_{n=n_0+1}^{\infty} (1+g_n(x) \\ &= [1+g_1(x)]\cdots[1+g_{n_0}(x)] \exp h(x) \end{aligned}$$

(converges uniformly) for $x \in X$ and $\exp h(x) \neq 0$ $\forall x \in X$. So if $f(x) = 0$ it must be that $1 + g_n(x) = 0$ for some n, $1 \leq n \leq n_0$. i.e., that $g_n(x) = -1$ for some n, $1 \leq n \leq n_0$. □

Theorem 1.5.20. *Let G be a region in $\mathbb{C}$ and let $\{f_n\}$ be a sequence in $H(G)$ such that no f_n is identically zero. If $\sum_{n=1}^{\infty} [f_n(z) - 1]$ converges absolutely and uniformly on compact subsets of G then $\prod_{n=1}^{\infty} f_n(z)$ converges in $H(G)$ to an analytic function $f(z)$. If a is a zero of f then a is a zero of only a finite number of the functions f_n, and the multiplicity of the zero of f at a is the sum of the multiplicities of the zeros of the functions f_n at a.*

Proof. Since $\sum_{n=1}^{\infty} [f_n(z) - 1]$ converges absolutely and uniformly on compact subsets of G, it follows from the preceding theorem that

$$f(z) = \underbrace{\prod_{n=1}^{\infty} f_n(z)}_{\prod_{n=1}^{\infty} [1+f_n(z)-1]}$$

converges absolutely and uniformly on compact subsets of G. As the convergence is uniform and since each $\{f_n\}$ is analytic, the limit function is also analytic. That is, the infinite product converges in $H(G)$.

Suppose a is a zero of f; i.e., $f(a) = 0$. Since $a \in G$ and G is open, we have there is an open ball $B(a,\ R) \subset G$. By choosing $0 < r < R$, we obtain $\underbrace{\overline{B(a,\ r)}}_{\substack{\text{closed and bounded,} \\ \text{so compact}}} \subset G$. By hypothesis, $\sum\limits_{n=1}^{\infty} [f_n(z) - 1]$ converges absolutely and uniformly on compact subsets of G, and hence, in particular, $\sum\limits_{n=1}^{\infty} [f_n(z) - 1]$ converges absolutely and uniformly on $\overline{B(a,\ r)}$. Then, according to the previous Lemma, there is an integer n_0 with

$$f(z) = [\underbrace{f_1(z)}_{1+g_1(x)}] \cdots [\underbrace{f_{n_0}(z)}_{1+g_{n_0}(x)}] \prod_{n=n_0+1}^{\infty} f_n(z)$$

such that $f(z) = 0$ if and only if $\underbrace{f_n(z) = 0}_{1+g_n(x)=0}$ for some n, $1 \leq$

$n \leq n_0$. By taking[72], $g(z) = \underbrace{\prod_{n=n_0+1}^{\infty} f_n(z)}_{\neq 0 \ \ \forall z \in \overline{B(a,\ r)}}$ we have

$$f(z) = f_1(z) \cdots f_{n_0}(z) g(z)$$

where g does not vanish in $\overline{B(a,\ r)}$ and

$$f(z) = 0 \text{ if and only if } \underbrace{f_n(z) = 0}_{1+g_n(x)=0} \text{ for some } n,\ 1 \leq n \leq n_0. \tag{1.98}$$

Hence the multiplicity of the zero of f at a is the sum of the multiplicities of the zeros of the functions f_n at a. [73]

□

Definition 1.5.21. An elementary factor is one of the following functions $E_p(z)$ for $p = 0,\ 1,\ \ldots$:

$$E_0(z) = 1 - z$$

---

[72]Why $g(z) \neq 0$ on $\overline{B(a,\ r)}$? Because $f(z) = 0$ if and only if $\underbrace{f_n(z) = 0}_{1+g_n(x)=0}$ for some n, $1 \leq n \leq n_0$.

[73]Details: a is a zero of multiplicity $m \geq 1$ of f iff (Ref. Page 76, Conway) there is an analytic function $h : \overline{B(a,\ r)} \to C$ such that $f(z) = (z-a)^m h(z)$ where $h(a) \neq 0$ iff (using (1.98)) $(z-a)^m$ divides $f_1(z) \cdots f_{n_0}(z)$ but $(z-a)^{m+1}$ does not divide $f_1(z) \cdots f_{n_0}(z)$. Again, using (1.98), this is iff $f_1(z) = (z-a)^{m\,1} g_1(z),\ \ldots\ ,\ f_{n_0}(z) = (z-a)^{m_{n_0}} g_{n_0}(z)$ with $m_1 + \cdots + m_{n_0} = m$ and $g_1(a) \neq 0,\ \ldots\ ,\ g_{n_0}(a) \neq 0$.

$$E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right), \quad p \geq 1 \quad (1.99)$$

Remark 1.5.22. For $p \geq 0$, the function[74] $E_p(z/a)$ has $1 - \frac{z}{a}$ as a factor and this factor becomes 0 when $z = a$ and the remaining part never becomes zero (due to the presence of exp function in the case $p > 0$.) Hence $E_p(z/a),\ p \geq 0$ has a simple zero at $z = a$ and no other zero.

Remark 1.5.23. If $b \in \mathbb{C}\backslash G$, then

$$E_p\left(\frac{a-b}{z-b}\right) = \begin{cases} 1 - \left(\frac{a-b}{z-b}\right), & p = 0, \\ \left\{1 - \left(\frac{a-b}{z-b}\right)\right\} \underbrace{\exp(\cdots)}_{\neq 0\ \forall z}, & p \geq 1 \end{cases}$$

is analytic on G (as the point b lie out side G) and has a simple zero at $z = a$.

The above functions will be used to manufacture analytic functions with prescribed zeros of prescribed multiplicity (as we are asked at the beginning of this section), but first an inequality must be proved which will enable us to apply Theorem 1.5.20 and obtain a convergent infinite product.

Lemma 1.5.24. *If* $|z| \leq 1$ *and* $p \geq 0$ *then* $|1 - E_p(z)| \leq |z|^{p+1}$.

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$$E_0\left(\frac{z}{a}\right) = 1 - \frac{z}{a}$$
$$E_p\left(\frac{z}{a}\right) = \left(1 - \frac{z}{a}\right)\exp\left(\frac{z}{a} + \frac{z^2}{2a^2} + \cdots + \frac{z^p}{pa^p}\right), \ p \geq 1$$

Proof. **Case 1):** When $p = 0$, the result is true because

$$|1 - E_0(z)| = |1 - (1 - z)| = |z| \leq |z|$$

Case 2): $p \geq 1$: For a fixed p let

$$E_p(z) = 1 + \sum_{k=1}^{\infty} a_k z^k \tag{1.100}$$

be its power series expansion[75] about $z = 0$. By differentiating the power series (1.100) we obtain

$$E_p'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1}$$

and differentiating the original expression (1.99) for $E_p(Z)$ we obtain

$$\begin{aligned} E_p'(z) &= -\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) \\ &+ (1-z)\exp\left(\underbrace{z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}}_{u}\right)\left(\underbrace{1 + \frac{2z}{2} + \cdots + \frac{pz^{p-1}}{p}}_{du/dz}\right) \end{aligned}$$

[75]Since $E_p(z)$ is analytic on $\mathbb{C}$ power series representation exists at each point on $\mathbb{C}$ and in particular there is a power series representation at the point $z = 0$.

$$
\begin{aligned}
&= -\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right) \\
&\quad +(1-z)\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)\left(1+z+\cdots+z^{p-1}\right) \\
&= -\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right) \\
&\quad +\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right) \\
&\qquad \times\left(1+z+\cdots+z^{p-1}-z\left\{1+z+\cdots+z^{p-1}\right\}\right) \\
&= -\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right) \\
&\quad +\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right) \\
&\qquad \times\left(1+z+\cdots+z^{p-1}-\left\{z+\cdots+z^{p-1}+z^p\right\}\right) \\
&= -\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right) \\
&\quad +\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right)(1-z^p) \\
&= -z^p\exp\left(z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right).
\end{aligned}
$$

Comparing the two expressions gives two pieces of information about the coefficients a_k. First,

$$a_1 = a_2 = \cdots = a_p = 0;$$

second, since the coefficients of the expansion of

$$\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right)$$

are all positive, $a_k \le 0$ for $k \ge p+1$. Thus,

$$|a_k| = -a_k \ \text{ for } k \ge p+1. \tag{1.101}$$

The above discussion gives, using (1.100), that

$$E_p(z) = 1 + \sum_{k=p+1}^{\infty} a_k z^k. \tag{1.102}$$

Now using (1.99) and (1.102) it follows that

$$0 = E_p(1) = 1 + \sum_{k=p+1}^{\infty} a_k.$$

From this, and using (1.101), we have

$$\sum_{k=p+1}^{\infty} |a_k| = -\sum_{k=p+1}^{\infty} a_k = 1.$$

Hence, using (1.102), for $|z| \leq 1$,

$$
\begin{aligned}
|E_p(z) - 1| &= \left|\sum_{k=p+1}^{\infty} a_k z^k\right| = |z|^{p+1}\left|\sum_{k=p+1}^{\infty} a_k z^{k-p-1}\right| \\
&\leq |z|^{p+1} \sum_{k=p+1}^{\infty} |a_k| \underbrace{|z|^{k-p-1}}_{\leq 1} = |z|^{p+1} \underbrace{\sum_{k=p+1}^{\infty} |a_k|}_{1} \\
&= |z|^{p+1}
\end{aligned}
$$

and this completes the proof. □

Theorem 1.5.25. *Let $\{a_n\}$ be a sequence in $\mathbb{C}$ such that $\lim_{n\to\infty} |a_n| = \infty$ and $a_n \neq 0$ for all $n \geq 1$.* (This is not a sequence of distinct points; but, by hypothesis, no point is repeated an infinite number of times.) *If $\{p_n\}$ is any sequence of integers such that*

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \tag{1.103}$$

for all $r > 0$ then

$$f(z) = \prod_{n=1}^{\infty} E_{p_n}(z/a_n)$$

*converges in $H(\mathbb{C})$. The function f is an **entire function** with zeros only at the points a_n. If z_0 occurs in the sequence*

$\{a_n\}$ exactly m times then f has a zero at $z = z_0$ of multiplicity m. Furthermore, if $p_n = n - 1$ then (1.103) will be satisfied.

Proof. Suppose there are integers p_n such that (1.103) is satisfied. Then, according to previous Lemma, whenever $|z| \le r$ and $r \le |a_n|$ (so that $|z| \le |a_n|$ so that $\left|\frac{z}{a_n}\right| \le 1$)

$$|1 - E_{p_n}(z/a_n)| \le \left|\frac{z}{a_n}\right|^{p_n+1} \le \left(\frac{r}{|a_n|}\right)^{p_n+1}. \tag{1.104}$$

By assumption $\lim\limits_{n\to\infty} |a_n| = \infty$ and so, by the definition of limit of sequence having limit value ∞, for a fixed $r > 0$ there is an integer N such that

$$|a_n| \ge r$$

for all $n \ge N$. i.e., for $n \ge N$

$$r \le |a_n|$$

Now if $z \in \overline{B(0;\ r)}$ then

$$|z| \le r$$

and (since $r \le |a_n|$ for $n \ge N$) it follows that

$$|z| \le r \le |a_n| \text{ for } n \ge N,$$

and hence we have

$$\left|\frac{z}{a_n}\right| \leq 1 \text{ for } n \geq N,$$

and hence by (1.104) and by assumption, we obtain

$$\sum_{n=1}^{\infty} |1 - E_{p_n}(z/a_n)|$$

is dominated by the convergent series

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_k|}\right)^{p_n+1}$$

and hence $\sum\limits_{n=1}^{\infty} |1 - E_{p_n}(z/a_n)|$ converges for $z \in \overline{B(0;\ r)}$. So

$$\sum_{n=1}^{\infty} [1 - E_{p_n}(z/a_n)]$$

converges absolutely and uniformly[76] on the compact set $\overline{B(0;\ r)}$. Since $\overline{B(0;\ r)}$ is an arbitrary compact subset of C it follows that $\sum\limits_{n=1}^{\infty} [1 - E_{p_n}(z/a_n)]$ converges absolutely and uniformly on compact subsets of $\mathbb{C}$ i.e., that $\sum\limits_{n=1}^{\infty} [E_{p_n}(z/a_n) - 1]$ con-

[76] Converges uniformly on the compact set $\overline{B(0;\ r)}$ because the convergence of dominated series $\sum\limits_{n=1}^{\infty} \left(\frac{r}{|a_k|}\right)^{p_n+1}$ is independent of the choice of points in $\overline{B(0;\ r)}$.

verges absolutely and uniformly on compact subsets of $\mathbb{C}$ and so converges absolutely in $H(\mathbb{C})$. Hence, by Theorem 1.5.20, $\prod_{n=1}^{\infty} E_{p_n}(z/a_n)$ converges in $H(C)$ to analytic function $f(z)$. Since the convergence is in $H(\mathbb{C})$ it follows that f is an entire function. If z_0 occurs in the sequence $\{a_n\}$ exactly m times then there will be exactly m elementary functions of the form $E_{p_n}(z/z_0)$ (having z_0 a simple zero) (only p_n changes) showing that f has a zero at $z = z_0$ of multiplicity m.

It remains to show that $\{p_n\}$ can be found so that (1.103) holds for all r *(We will see shortly that this happens if we let $p_n = n-1$).* Since $\lim_{n\to\infty} |a_n| = \infty$, by the definition of limit of sequence with limit value ∞, corresponding to $r > 0$ there is an integer N such that

$$|a_n| > 2r \quad \forall n \geq N.$$

This gives that

$$\frac{r}{|a_n|} < \frac{1}{2} \quad \forall n \geq N.$$

So, if $p_n = n-1$ for all n, then

$$\left(\frac{r}{|a_n|}\right)^{p_n+1} = \left(\frac{r}{|a_n|}\right)^{n} < \left(\frac{1}{2}\right)^{n} \quad \forall n \geq N$$

Hence the tail end $\sum_{n=N}^{\infty}\left(\frac{r}{|a_n|}\right)^{p_n+1}$ of the series in (1.103) is dominated by the convergent geometric series $\sum\limits_{n=N}^{\infty}\left(\frac{1}{2}\right)^n$. Thus, the series $\sum\limits_{n=N}^{\infty}\left(\frac{r}{|a_n|}\right)^{p_n+1}$ converges and hence

$$\sum_{n=1}^{\infty}\left(\frac{r}{|a_n|}\right)^{p_n+1} = \underbrace{\sum_{n=1}^{N-1}\left(\frac{r}{|a_n|}\right)^{p_n+1}}_{\text{finite real number}} + \underbrace{\sum_{n=N}^{\infty}\left(\frac{r}{|a_n|}\right)^{p_n+1}}_{<\infty},$$

the series in (1.103), converges. This completes the proof. $\square$

Theorem 1.5.26. ***[Characterization of Simple Connectedness]***[77] *Let G be an open connected subset of $\mathbb{C}$. Then*

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Theorem 1.5.27. *Let G be an open connected subset of $\mathbb{C}$. Then the following are equivalent:*

(a) G is simply connected;

(b) $n(\gamma; a) = 0$ for every closed rectifiable curve γ in G and every point a in $\mathbb{C} - G$;

(c) $\mathbb{C}_\infty - G$ is connected;

(d) For any f in $H(G)$ there is a sequence of polynomials that converges to f in $H(G)$;

(e) For any f in $H(G)$ and any closed rectifiable curve γ in G, $\int\limits_\gamma f = 0$;

(f) Every function f in $H(G)$ has a primitive;

(g) For any f in $H(G)$ such that $f(z) \neq 0$ for all z in G there is a function g in $H(G)$ such that $f(z) = \exp g(z)$;

(h) For any f in $H(G)$ such that $f(z) \neq 0$ for all z in G there is a function g in $H(G)$ such that

$$f(z) = [g(z)]^2;$$

That is, every non-vanishing analytic function has an analytic square root.

(i) G is homeomorphic to the unit disk;

the following are equivalent:
(i). G is simply connected;
(ii). For any $f \in H(G)$ such that $f(z) \neq 0$ for all $z \in G$ there is a function $g \in H(G)$ such that

$$f(z) = \exp g(z)$$

Since $\mathbb{C}$ is simply connected, and since $f \in H(\mathbb{C})$ means f is an entire function, we have following corollary.

Corollary 1.5.28. *For an entire function f that has no zeros, there is an entire function g such that*

$$f(z) = \exp g(z)$$

Proof. With $G = \mathbb{C}$, a simply connected region, f is an entire function that has no zeros means $f \in H(\mathbb{C})$ and $f(z) \neq 0$ implies (by the above theorem) there is a function $g \in H(\mathbb{C})$ (i.e., g is an entire function) such that $f(z) = \exp g(z)$. □

Theorem 1.5.29. ***[The Weierstrass Factorization Theorem]*** *Let f be an entire function and let $\{a_n\}$ be the non-zero zeros of f repeated according to multiplicity; suppose f has a zero at $z = 0$ of order $m \geq 0$* (a zero of order $m = 0$ at $z = 0$

(j) If $u : G \to \mathbb{R}$ is harmonic then there is a harmonic function $v : G \to \mathbb{R}$ such that $f = u + i\,v$ is analytic in G.

This result is detailed in Theorem 2.5.4and proved in Page 320.

means $f(0) \neq 0$.) *Then there is an entire function g and a sequence of integers $\{p_n\}$ such that*

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

Proof. f has a zero at $z = 0$ of order $m \geq 0$ if and only if z^m is a factor (and z^{m+1} is NOT a factor f). $\{a_n\}$ be the *non-zero* zeros of f repeated according to multiplicity implies (by Theorem 1.5.25) that it is possible to find a sequence $\{p_n\}$ of integers such that

$$g(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

is an entire function with zeros only at the points a_n (if z_0 occurs in the sequence $\{a_n\}$ exactly m times then g has a zero at $z = z_0$ of multiplicity m. [78] Then

$$h(z) = z^m \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

has the same zeros as f with the same multiplicities (this is

[78]Note: In Theorem 1.5.25 it is required that $\lim_{n\to\infty} |a_n| = \infty$, but here it is NOT mentioned?

because of the presence of z^m and $E_{p_n}\left(\frac{z}{a_n}\right)$.) Thus,

$$\frac{f(z)}{h(z)}$$

is defined and analytic everywhere except the points $z = 0,\ a_1,\ a_2,\ \ldots$ that are zeros of the analytic function $h(z)$ in the denominator. As the zeros of an analytic function are isolated, the points $z = 0,\ a_1,\ a_2,\ \ldots$ are isolated singular points points of f/h. Also,

$$\lim_{z\to 0}(z-0)\underbrace{\frac{f(z)}{g(z)}}_{\substack{\text{defined and nonzero in a}\\ \text{deleted neighborhood of } 0}} = \underbrace{\lim_{z\to 0}(z-0)}_{0}\ \underbrace{\lim_{z\to 0}\frac{f(z)}{g(z)}}_{\text{finite number}} = 0$$

and for each n

$$\lim_{z\to a_n}(z-a_n)\underbrace{\frac{f(z)}{g(z)}}_{\substack{\text{defined and nonzero in a}\\ \text{deleted neighborhood of } a_n}}$$

$$= \underbrace{\lim_{z\to a_n}(z-a_n)}_{0}\ \underbrace{\lim_{z\to a_n}\frac{f(z)}{g(z)}}_{\text{finite number}} = 0$$

Hence, by Theorem V.1.2 (Page 102, Conway)[79], $z = 0,\ a_1,\ a_2,\ \ldots$

[79] THEOREM V.1.2. **If f has an isolated singularity at a then**

$z = a$ is a removable singularity if and only if $\lim_{z\to a}(z-a)f(z) = 0$.

are removable singular points. By cancelling out the common factors in the numerator and denominator (This cancellation is possible initially when z is different from $0,\ a_1,\ a_2,\ \ldots$ then we extend it to all of $\mathbb{C}$) f/h is an *entire function and*, furthermore, *has no zeros.* Since $\mathbb{C}$ is simply connected, by Corollary 1.5.28, it follows that there is an entire function g such that

$$\frac{f(z)}{h(z)} = \exp g(z).$$

Thus,

$$f(z) = h(z) \exp g(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right).$$

□

Theorem 1.5.30. *Let G be a region and let $\{a_j\}$ be a sequence of distinct points in G with no limit point in G; and let $\{m_j\}$ be a sequence of integers. Then there is an analytic function f defined on G whose only zeros are at the points a_j; furthermore, a_j is a zero of f of multiplicity m_j.*

Proof. **STAGE 1:** We begin by showing (in STAGE 1b) that it suffices to prove this theorem for the special case where there is a number $R > 0$ such that

$$\{z : |z| > R\} \subset G \quad \text{and} \quad |a_j| \leq R \quad \text{for all } j \geq 1. \tag{1.105}$$

STAGE 1a: It must be shown (in **STAGE 2**) that with this hypothesis (in STAGE 1) there is a function f in $H(G)$ with the a_j's as its only zeros and m_j = the multiplicity of the zero at $z = a_j$; and with the further property that

$$\lim_{z \to \infty} f(z) = 1. \tag{1.106}$$

STAGE 1b: In fact, if such an f can always be found for a set satisfying (1.105), let G_1 be an arbitrary open set in C with $\{\alpha_j\}$ be a sequence of distinct points in G_1 with no limit point, and let $\{m_j\}$ be a sequence of integers. Now if $\overline{B(a;\ r)}$ is a disk in G_1 such that $\alpha_j \notin B(a;\ r)$ forall $j \geq 1$, (it is possible to find a closed ball like this: $\{\alpha_j\}$ is a sequence of distinct points in G_1 with no limit point implies no point of G is a limit point of $\{\alpha_j\}$ so it is possible to find $a \in G$ and $L > 0$ such that $B(a;\ L) \cap \{\alpha_j : j \geq 1\} = \emptyset$ and by taking $0 < r < L$ we obtain $\overline{B(a;\ r)} \cap \{\alpha_j : j \geq 1\} = \emptyset$).

With the above a, consider the Mobius transformation

$$Tz = \frac{1}{z - a}.$$

Put

$$G = T(G_1).$$

It is easy to see that G satisfies condition (1.105) where

$$a_j = T\alpha_j = \frac{1}{\alpha_j - a}.$$

[DETAILS: The choice of r such that

$$\overline{B(a;\ r)} \cap \{\alpha_j : j \geq 1\} = \emptyset$$

helps us to say that

$$|\alpha_j - a| \geq r,\ \forall\ j \geq 1$$

which imply

$$|a_j| = |T\alpha_j| = \left|\frac{1}{\alpha_j - a}\right| \leq \frac{1}{r},\ \forall\ j \geq 1.$$

Also,

$$\lim_{z \to a} Tz = \lim_{z \to a} \frac{1}{z - a} = \infty$$

so by the definition of infinite limit at a finite point, we have for any $R > 0$ there is a $\delta > 0$ such that

$$\left|\frac{1}{z - a}\right| > R \quad \text{whenever} \quad 0 < |z - a| < \delta.$$

We observe that (noting $z = a$ gives $\frac{1}{z-a} = \infty$)

$$\left|\frac{1}{z-a}\right| > R \Leftrightarrow |z-a| < \frac{1}{R} \quad .$$

Hence with $R = \frac{1}{r}$ and by taking $w = T(z)$, we have

$$|w| > R(= 1/r) \Leftrightarrow |z-a| < \frac{1}{R}(= r)$$

$$\Leftrightarrow \quad z \in B\left(a,\ \frac{1}{R}\right) = B\left(a,\ r\right) \Leftrightarrow T(z) \in T(B(a;\ r))$$

Since $B(a;\ r) \subset G_1$ from the above we have

$$\{w : |w| > R = (1/r)\} = T(B(a;\ r)) \subset T(G_1) = G$$

i.e., we have shown that

$$\{z : |z| > R\} \subset G \quad \text{and} \quad |a_j| \leq R \quad \text{forall } j \geq 1.] \qquad (1.107)$$

If there is a function f in $H(G)$ with a zero at each a_j of multiplicity m_j, with no other zeros, and such that f satisfies (1.106); then

$$g(z) = f(Tz)$$

is analytic in $G_1 - \{a\}$ with a removable singularity at $z = a$.

[80] Hence there is an analytic function h on G_1 such that $h(z) = g(z)$ for $z \neq a$ and $h(a) = \lim_{z\to a} g(z) = \underbrace{\lim_{z\to a} f(Tz)}_{\lim_{w\to\infty} f(w)=1} = 1.$ Then h has the prescribed zeros at each α_j of multiplicity m_j.) We treat g itself as h that is analytic on G_1. Then, g has the prescribed zeros at each α_j of multiplicity m_j.

STAGE 2: So assume that G satisfies (1.105). Define a second sequence $\{z_n\}$ **consisting of the points in** $\{a_j\}$, **but such that each** a_j **is repeated according to its multiplicity** m_j. Now, for each $n \geq 1$ there is a point $w_n \in C - G$ such that

$$|w_n - z_n| = \underbrace{d(z_n,\ C - G)}_{\substack{>0,\text{ since } z_n \notin C-G \\ \text{and } C-G \text{ is closed}}}$$

[81]

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[80] Details: Clearly g is not defined at $z = a$ and hence g is analytic on G_1 except at the point $z = a$, showing that $z = a$ is an isolated singularity of g. Moreover,

$$\lim_{z\to a}(z-a)g(z) = \underbrace{\lim_{z\to a}(z-a)}_{0}\underbrace{\lim_{z\to a} g(z)}_{} = \underbrace{\lim_{z\to a}(z-a)}_{0}\ \underbrace{\lim_{z\to a} f(Tz)}_{\lim_{w\to\infty} f(w)=1} = 0$$

Hence, by Theorem Theorem V.1.2 (Page 102, Conway), $z = a$ is an isolated singularity of g.

[81] Details: $\underbrace{C - G}_{\substack{\text{closed subset of} \\ \text{compact set, hence} \\ \text{compact}}} \subset \underbrace{\{z : |z| \leq R\}}_{\text{compact set}}$ so, by Theorem **??**, for each

Notice that the hypothesis (1.105) **excludes the possibility that** $G = C$ unless the sequence $\{a_j\}$ were finite [82]. In fact, if $\{a_j\}$ were finite the theorem could be easily proved (see Proposition 1.5.1). (The next lines say that if $\{a_j\}$ are not finite, then $C - G$ is nonempty so that $G = C$ is impossible.)

So it suffices to assume that $\{a_j\}$ **is infinite.** Since

$$|a_j| \leq R \ \text{ for all } j \geq 1$$

we have $\{a_j\}$ is a bounded infinite subset of C and hence[83] has a limit point. By assumption this limit point is not in G so the only possibility is that the limit point belongs to $C - G$ showing that $C - G$ is nonempty. We have observed above that $C - G$ is compact also. That is, $C - G$ is nonempty as well as compact.

Also,

$$\lim_{n \to \infty} |w_n - z_n| = 0.$$

$z_n \in G$ there is a $w_n \in C - G$ such that $|w_n - z_n| = d(z_n,\ C - G)$.

[82] If $\{a_j\}$ is a finite set, there is no limit point for $\{a_j\}$ and we cannot argue as above (as in the case of infinite $\{a_j\}$) to say that $C - G$ is nonempty; so in this case there is a possibility that $C - G$ is empty and $G = C$ when $\{a_j\}$ is finite. i.e., the case of $G = C$ (if there is any) happens only when $\{a_j\}$ is finite.

[83] **Bolzano-Weierstrass:** Every bounded infinite set of real or complex numbers has at least one limit point

Consider the functions

$$E_n\left(\frac{z_n - w_n}{z - w_n}\right)$$

(that are obtained from elementary functions); each has a simple zero at $z = z_n$. It must be shown that the infinite product of these functions converges in $H(G)$.

To do this, let K be a compact subset of G so that $d(C - G,\ K) > 0$. For any point $z \in K$,

$$|z - w_n| \geq \underbrace{d(w_n,\ K\,)}_{\inf\{\,|w_n - w\,|\,:w\in K\}} \geq \underbrace{d(C - G,\ K\,)}_{\inf\{\,|p - w\,|\,:p\in C-G\ ,\ w\in K\}}$$

and this implies

$$\frac{1}{|z - w_n|} \leq \frac{1}{d(w_n,\ K\,)} \leq \frac{1}{d(C - G,\ K\,)}$$

so that

$$\left|\frac{z_n - w_n}{z - w_n}\right| \leq \frac{|z_n - w_n|}{d(w_n,\ K)} \leq \frac{|z_n - w_n|}{d(C - G,\ K)}. \tag{1.108}$$

It follows that for any δ, $0 < \delta < 1$, there is an integer N such that

$$\left|\frac{z_n - w_n}{z - w_n}\right| < \delta$$

for all $z \in K$ and $n \geq N$. [84] But then Lemma 1.5.24 gives that

$$\left| E_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1 \right| \leq \left| \frac{z_n - w_n}{z - w_n} \right|^{n+1} \leq \delta^{n+1} \tag{1.109}$$

for all $z \in K$ and $n \geq N$. This shows that for all $z \in K$, the series

$$\sum_{n=1}^{\infty} \left| E_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1 \right|$$

has a majorant

$$\sum_{n=1}^{\infty} \delta^{n+1}$$

(a convergent geometric series with common ratio δ) and hence

[84] Details: since $\lim\limits_{n\to\infty} |w_n - z_n| = 0$ there is an integer N such that $||w_n - z_n| - 0| < \delta \;\; \forall n \geq N$ and hence reading from right of (1.108) gives

$$\left| \frac{z_n - w_n}{z - w_n} \right| < \delta d(C - G,\ K)$$

for all $z \in K$ and $n \geq N$.

by Weierstrass M-Test[85], it follows that the series

$$\sum_{n=1}^{\infty}\left[E_n\left(\frac{z_n-w_n}{z-w_n}\right)-1\right]$$

converges uniformly and absolutely on K. Then according to Theorem 1.5.20,

$$f(z)=\prod_{n=1}^{\infty}E_n\left(\frac{z_n-w_n}{z-w_n}\right)$$

converges in $H(G)$, so that f is an analytic function on G. Also, Theorem 1.5.20 implies that the points $\{a_j\}$ are the only zeros of f and m_j is the order of the zero at $z=a_j$ (because a_j occurs m_j times in the sequence $\{z_n\}$). To show that $\lim\limits_{z\to\infty} f(z)=1$, let $\varepsilon>0$ be an arbitrary number and let $R_1>R$ (R_1 will be further specified shortly). If $|z|\geq R_1$

[85] Weierstrass M-Test If for all $z\in K$

$$\sum_{n=1}^{\infty} f_n(z)$$

has the series with positive terms

$$\sum_{n=1}^{\infty} a_n$$

as a majorant and if the majorant converges, then the minorant $\sum\limits_{n=1}^{\infty} f_n(z)$ converges uniformly on K.

then, because $|z_n| \leq R$ and $w_n \in C - G \subset B(0;\, R)$,

$$\left| \frac{z_n - w_n}{z - w_n} \right| \leq \frac{2R}{R_1 - R}.$$

[86]

So, if we choose $R_1 > R$ so that

$$2R < \delta(R_1 - R)$$

for some δ, $0 < \delta < 1$, (1.109) holds for $|z| \geq R_1$ and for **all** $n \geq 1$. i.e.,

$$\left| E_n \left(\frac{z_n - w_n}{z - w_n} \right) - 1 \right| \leq \left| \frac{z_n - w_n}{z - w_n} \right|^{n+1} \leq \delta^{n+1} \qquad (1.110)$$

holds for $|z| \geq R_1$ and for **all** $n \geq 1$. In particular,

$$\mathrm{Re} E_n \left(\frac{z_n - w_n}{z - w_n} \right) > 0$$

---

[86] Details:

$$|z_n| \leq R \ \text{ and } \ |w_n| \leq R \Rightarrow \underbrace{|z_n - w_n| \leq |z_n| + |w_n|}_{\text{by triangle inequality}} \leq 2R$$

and

$$|z| \geq R_1 > R \ \text{ and } \ |w_n| \leq R \Rightarrow |z - w_n| \geq R_1 - R$$

(To get a geometric idea of the last concept, you may draw concentric circles of radii R_1 and R centered at the origin.)

for all n and $|z| \geq R_1$; (for otherwise, there is an n such that

$$\mathrm{Re} E_n\left(\frac{z_n - w_n}{z - w_n}\right) = 0$$

which implies that for this n

$$1 \leq \delta^{n+1} < 1$$

a contradiction.) Then

$$|f(z) - 1| = \left|\exp\left(\sum_{n=1}^{\infty} \log E_n\left(\frac{z_n - w_n}{z - w_n}\right)\right) - 1\right| \tag{1.111}$$

is a meaningful equation (because for all n, $\mathrm{Re} E_n\left(\frac{z_n - w_n}{z - w_n}\right) > 0$ hence $\left|E_n\left(\frac{z_n - w_n}{z - w_n}\right)\right| > 0$ and so $\underbrace{\ln\left|E_n\left(\frac{z_n - w_n}{z - w_n}\right)\right|}_{\text{natural logarithm of a positve real number}}$ is meaningful, which is used to define complex logarithm

$$\log E_n\left(\frac{z_n - w_n}{z - w_n}\right).$$

)

On other hand 1.91 (with $z = E_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1$) [87] and

[87] Whether this $|z| < \frac{1}{2}$?

(1.109) give that, for $|z| \geq R_1$,

$$\begin{aligned}
\left|\sum_{n=1}^{\infty} \log E_n\left(\frac{z_n - w_n}{z - w_n}\right)\right| &\leq \sum_{n=1}^{\infty}\left|\log E_n\left(\frac{z_n - w_n}{z - w_n}\right)\right| \\
&\leq \sum_{n=1}^{\infty} \frac{3}{2}\left|E_n\left(\frac{z_n - w_n}{z - w_n}\right) - 1\right| \\
&\leq \sum_{n=1}^{\infty} \frac{3}{2}\delta^{n+1} \\
&= \frac{3}{2} \cdot \frac{\delta^2}{1 - \delta}
\end{aligned}$$

If we further restrict δ so that

$$\left| e^w - \underbrace{1}_{e^0} \right| < \varepsilon$$

whenever

$$|w| < \frac{3}{2} \cdot \frac{\delta^2}{1 - \delta}$$

(this is possible, because e^z is continuous at $z = 0$), then equation (1.111) gives that

$$|f(z) - 1| < \varepsilon \quad \text{whenever} \quad |z| \geq R_1.$$

That is,

$$\lim_{z \to \infty} f(z) = 1.$$

$\square$

Corollary 1.5.31. *If f is a meromorphic function on an open set G then there are analytic functions g and h on G such that*

$$f = \frac{g}{h}.$$

Proof. f is a meromorphic function on G implies f is analytic on G except at the poles. Let $\{a_j\}$ be the poles of f and let m_j be the order of the pole at a_j. According to the preceding theorem there is an analytic function h with a zero of multiplicity m_j at each $z = a_j$ and with no other zeros. Then hf has removable singularities at each point a_j. [Details: a_j is a pole of order m_j means in a **very small neighborhood of** a_j there is an analytic function $\varphi(z)$ such that $\varphi(a) \neq 0$ and

$$f(z) = \frac{\varphi(z)}{(z-a_j)^{m_j}}$$

Then

$$\lim_{z \to a_j} (z - a_j) \underbrace{h(z)}_{(z-a_j)^{m_j}\psi(z)} f(z)$$

$$\begin{aligned}
&= \lim_{z \to a_j} (z - a_j) \lim_{z \to a_j} (z - a_j)^{m_j} \psi(z) \frac{\varphi(z)}{(z-a_j)^{m_j}} \\
&= \underbrace{\lim_{z \to a_j} (z - a_j)}_{0} \underbrace{\lim_{z \to a_j} \psi(z)\varphi(z)}_{\text{finite real number}} \\
&= 0.
\end{aligned}$$

Hence a_j is a removable singular point.] (By cancelling out the common factors, and considering the continuous extension) it follows that $g = hf$ is analytic in G. Hence $f = \frac{g}{h}$ with g and h analytic functions on G. $\square$

Chapter 2

Factorization of the Sine Function, Runge's Theorem and Simple Connectedness

2.1 Factorization of the Sine Function

In this section an application of the Weierstrass Factorization Theorem to $\sin \pi z$ is given. If an infinite sum or product is followed by a prime (apostrophe) (i.e., $\sum'$ or $\prod'$), then the sum or product is to be taken over a11 the indicated indices n except $n = 0$. For example,

$$\sum_{n=-\infty}^{\infty}{}' a_n = \sum_{n=1}^{\infty} a_{-n} + \sum_{n=1}^{\infty} a_n.$$

The zeros of

$$\sin \pi z = \frac{1}{2\,i} \left(e^{i\pi z} - e^{-i\pi z} \right)$$

are precisely the integers; moreover, each zero is simple. Since

$$\sum_{n=-\infty}^{\infty}{}' \left(\frac{r}{n} \right)^2 < \infty$$

for all $r > 0$, one can ((1.103) in page 174)

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|} \right)^{p_n+1} < \infty \tag{1.103}$$

choose $p_n = 1$ for all n in the Weierstrass Factorization Theorem. Thus

$$\sin \pi z = [\exp g(z)]\, z \prod_{n=-\infty}^{\infty}{}' \left(1 - \frac{z}{n} \right) e^{z/n}$$

or, because the terms of the infinite product can be rearranged,

$$\sin \pi z = [\exp g(z)]\, z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right) \tag{2.1}$$

for some entire function $g(z)$. If $f(z) = \sin \pi z$ then, according to Theorem 1.2.1 in page 84,

$$\begin{aligned}\pi \cot \pi z &= \frac{f'(z)}{f(z)} \\ &= g'(z) + \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}\end{aligned}$$

and the convergence is uniform over compact subsets of the plane that contain no integers (actually, a small additional argument is necessary to justify this-details are left as an exercise). But,

$$\pi \cot \pi z = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

for z not an integer (details are left as an exercise). So it must be that g is a constant, say $g(z) = a$ for all z. It follows from (2.1) that for $0 < |z| < 1$

$$\frac{\sin \pi z}{\pi z} = \frac{e^a}{\pi} \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Letting z approach zero gives that $e^a = \pi$. This gives the following:

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) \tag{2.2}$$

and the convergence is uniform over compact subsets of $\mathbb{C}$.

2.2 The Gamma Function

Let G be an open set in the plane and let $\{f_n\}$ be a sequence of analytic functions on G. If $\{f_n\}$ converges in $H(G)$ to f and f is not identically zero, then it easily follows that $\{f_n\}$ converges to f in $M(G)$. Since

$$d(z_1,\ z_2) = d\left(\frac{1}{z_1},\ \frac{1}{z_2}\right),$$

where d is the spherical metric on $\mathbb{C}_\infty$ (Ref. Appendix A), it fo11ows that $\left\{\frac{1}{f_n}\right\}$ converges to $\frac{1}{f}$ in $M(G)$. It is an easy exercise to show that $\left\{\frac{1}{f_n}\right\}$ converges uniformly to $\frac{1}{f}$ on any compact set K on which no f_n vanishes. (What does Hurwitz's Theorem have to say about this situation). Since, according to Theorem 1.5.25, the infinite product

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n}$$

converges in $H(\mathbb{C})$ to an entire function which only has simple zeros at $z = -1,\ -2,\ \ldots$ the above discussion yields that

$$\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{z/n} \tag{2.3}$$

converges on compact subsets of $\mathbb{C}-\{-1,\ -2,\ \ldots\}$ to a function with simple poles at $z = -1,\ -2,\ \ldots$.

Definition 2.2.1. The **gamma function,** $\Gamma(z)$, is the meromorphic function on $\mathbb{C}$ with simple poles at $z = 0, -1, \ldots$ defined by

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n}, \tag{2.4}$$

where γ is a constant chosen so that $\Gamma(1) = 1$.

The first thing that must be done is to show that the constant γ exists; this is an easy matter. Substituting $z = 1$ In (2.3) yields a finite number

$$c = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-1} e^{1/n}$$

which is clearly positive. Let $\gamma = \log c$; it follows that with this choice of γ, equation (2.4) for $z = 1$ gives $\Gamma(1) = 1$. This constant γ is called **Euler's constant** and it satisfies

$$e^{\gamma} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{-1} e^{1/n}. \tag{2.5}$$

Since both sides of (2.5) involve only real positive numhers and the real logarithm is continuous, we may apply the logarithm function to both sides of (2.5) and obtain

$$\begin{aligned}
\gamma &= \sum_{k=1}^{\infty} \log\left[\left(1+\frac{1}{k}\right)^{-1} e^{1/k}\right] \\
&= \sum_{k=1}^{\infty}\left[\frac{1}{k} - \log(k+1) + \log k\right] \\
&= \lim_{n\to\infty} \sum_{k=1}^{n}\left[\frac{1}{k} - \log(k+1) + \log k\right] \\
&= \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - \log(n+1)\right].
\end{aligned}$$

Adding and subtracting $\log n$ to each term of this sequence we obtain

$$\begin{aligned}
\gamma &= \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - \log(n+1) + \log n - \log n\right] \\
&= \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - \{\log(n+1) - \log n\} - \log n\right] \\
&= \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - \log\left(\frac{n+1}{n}\right) - \log n\right]
\end{aligned}$$

and using the fact that $\lim \log\left(\frac{n+1}{n}\right) = 0$ we obtain

$$\gamma = \lim_{n\to\infty}\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right) - \log n\right]. \tag{2.6}$$

This last formula can be used to approximate γ. Equation (2.6) is also used to derive another expression for $\Gamma(z)$. From the definition of $\Gamma(z)$ it follows that

$$\begin{aligned}
\Gamma(z) &= \frac{e^{-\gamma z}}{z} \lim_{n\to\infty} \prod_{k=1}^{n} \left(1+\frac{z}{n}\right)^{-1} e^{z/k} \\
&= \frac{e^{-\gamma z}}{z} \lim_{n\to\infty} \prod_{k=1}^{n} \frac{ke^{z/k}}{z+k} \\
&= \lim_{n\to\infty} \frac{e^{-\gamma z} n!}{z(z+1)\cdots(z+n)} \exp\left(z\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right).
\end{aligned}$$

However

$$e^{-\gamma z}\exp\left[\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)z\right]$$

$$= n^z \exp\left[z\left(-\gamma+1+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right)\right].$$

So that the following is obtained.

Theorem 2.2.2. Gauss's Formula *For* $z \neq 0, -1, \ldots$

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}. \tag{2.7}$$

The formula of Gauss yields a simple derivation of the functional equation saifsfled by the gamma function.

Theorem 2.2.3. Functional Equation *For* $z \neq 0, -1, \ldots$

$$\Gamma(z+1) = z\Gamma(z). \tag{2.8}$$

To obtain this important equation substitute $z+1$ for z

in (2.7); this gives

$$\begin{aligned}\Gamma(z+1) &= \lim_{n\to\infty} \frac{n!n^{z+1}}{(z+1)\cdots(z+n)(z+n+1)} \\ &= z \lim_{n\to\infty} \left[\frac{n!n^z}{z(z+1)\cdots(z+n)}\right]\left[\frac{n}{z+n+1}\right] \\ &= z\Gamma(z)\end{aligned}$$

since $\lim\left(\frac{n}{z+n+1}\right) = 1$.

Now consider $\Gamma(z+2)$; we have $\Gamma(z+2) = \Gamma((z+1)+1) = (z+1)\Gamma(z+1)$ by the functional equation. A second application of (2.8) gives

$$\Gamma(z+2) = z(z+1)\Gamma(z).$$

In fact, by reiterating this procedure

$$\Gamma(z+n) = z(z+1)\cdots(z+n-1)\Gamma(z) \tag{2.9}$$

for n a non negative integer and $z \neq 0, -1, \ldots$. In particular setting $z = 1$ gives that

$$\Gamma(n+1) = n! \tag{2.10}$$

That is, the Γ function is analytic in the right half plane and agrees with 1he factorial function at the integers. We may therefore consider the gamma function as an extension of the factorial to the complex plane; alternately, if $z \neq -1, -2, \ldots$ then letting $z! = \Gamma(z+1)$ is a justifiable definition of $z!$.

As has been pointed out, Γ has simple poles at $z = 0, -1, \ldots;$ we wish to find the residue of Γ at each of its poles. To do this recall from the result mentioned in the footnote[1] that

$$\operatorname{Res}(\Gamma;\ -n) = \lim_{z \to -n} (z+n)\Gamma(z)$$

for each non negative integer n. But from (2.9)

$$(z+n)\Gamma(z) = \frac{\Gamma(z+n+1)}{z(z+1)\cdots(z+n-1)}.$$

So letting z approach $-n$ gives that

$$\operatorname{Res}(\Gamma;\ -n) = \frac{(-1)^n}{n!}, \quad n \geq 0. \tag{2.11}$$

We can calculate Γ'/Γ by

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)} \tag{2.12}$$

---

[1] Proposition V.2.4 in Conway: Suppose f has a pole of order m at $z = a$ and put

$$g(z) = (z-a)^m f(z).$$

Then

$$\operatorname{Res}(f;\ a) = \frac{1}{(m-1)!} g^{(m-1)}(a)$$

$$= \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left\{(z-a)^m f(z)\right\}\Big|_{z=a}$$

for $z \neq 0, -1, \ldots$ and convergence is uniform on every compact subset of $\mathbb{C} - \{0, -1, \ldots\}$. It follows from Theorem 1.2.1 (page 84) that to calculate the derivative of Γ'/Γ we may differentiate the series (2.12) term by term. Thus when z is not a negative integer

$$\left(\frac{\Gamma'(z)}{\Gamma(z)}\right)' = \frac{1}{z^2} + \sum_{n=1}^{\infty} \frac{1}{(n+z)^2} \tag{2.13}$$

At this time the reader may well be asking when this process will stop. Will we calculate the second derivative of Γ'/Γ? The answer to this question is no. The answer to the implied question of why anyone would want to derive formulas (2.12) and (2.13) is that they allow us to characterize the gamma function in a particularly beautiful way.

Notice that the definition of $\Gamma(z)$ gives that $\Gamma(x) > 0$ if $x > 0$. Thus, $\log \Gamma(x)$ is well defined for $x > 0$ and, according to formula (2.13), the second derivative of $\log \Gamma(x)$ is always positive. This implies that the gamma function is logarithmically convex on $(0, \infty)$; that is, $\log \Gamma(x)$ is convex there. It turns out that this property together with the functional equation and the fact that $\Gamma(1) = 1$ completely characterize the gamma function.

Theorem 2.2.4. Bohr-Mollerup Theorem *Let f be a function defined on $(0, \infty)$ such that $f(x) > 0$ for all $x > 0$. Suppose that f has the following properties:*

(a). $\log f(x)$ *is a convex function;*

(b). $f(x+1) = xf(x)$ *for all* x*;*

(c). f(1)=1.

Then $f(x) = \Gamma(x)$ *for all* x*.*

Proof. Begin by noting that since f has properties (b) and (c), the function also satisfies

$$f(x+n) = x(x+1)\cdots(x+n-1)f(x) \tag{2.14}$$

for every non-negative integer n. So if $f(x) = \Gamma(x)$ for $0 < x \leq 1$, this equation will give that f and Γ are everywhere identical. Let $0 < x \leq 1$ and let n be an integer larger than 2. It can be seen that

$$\frac{\log f(n-1) - \log f(n)}{(n-1)-n}$$

$$\leq \frac{\log f(x+n) - \log f(n)}{(x+n)-n} \leq \frac{\log f(n+1) - \log f(n)}{(n+1)-n}$$

Since (2.14) holds we have that $f(m) = (m-1)!$ for every integer $m \geq 1$. Thus the above inequalities become

$$-\log(n-2)! + \log(n-1)!$$

$$\leq \frac{\log f(x+n) - \log(n-1)!}{x} \leq \log n! \ - \log(n-1)! \ ;$$

or

$$x \log(n-1) \le \log f(x+n) - \log(n-1)! \le x \log n \,.$$

Adding $\log(n-1)!$ to each side of this inequality and applying the exponential (exp is a monotone increasing function and therefore preserves inequalities) gives

$$(n-1)^x (n-1)! \le f(x+n) \le n^x (n-1)!$$

Applying (2.14) to calculate $f(x+n)$ yields

$$\begin{aligned} \frac{(n-1)^x (n-1)!}{x(x+1)\cdots(x+n-1)} &\le f(x) \le \frac{n^x (n-1)!}{x(x+1)\cdots(x+n-1)} \\ &= \frac{n^x n!}{x(x+1)\cdots(x+n)} \left[\frac{x+n}{n}\right] \end{aligned}$$

Since the term in the middle of this sandwich, $f(x)$, does not involve the integer n and since the inequality holds for all integers $n \ge 2$, we may vary the integers on the left and right hand sided independently of one another and preserve the inequality. In particular, $n+1$ may be substituted for n on the left while allowing the right hand side to remain unchanged. This gives

$$\frac{n^x n!}{x(x+1)\cdots(x+n)} \le f(x) \le \frac{n^x n!}{x(x+1)\cdots(x+n)} \left[\frac{x+n}{n}\right]$$

for all $n \geq 2$ and x in $[0, 1)$. Now take the limit as $n \to \infty$. Since $\lim \left(\frac{x+n}{n}\right) = 1$, Gauss' formula implies that $\Gamma(x) = f(x)$ for $0 < x \leq 1$. The result now follows by applying (2.14) and the Functional Equation. $\square$

Theorem 2.2.5. *If* $\mathrm{Re} z > 0$ *then*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt. \tag{2.15}$$

Proof is given in Page 215.

The integrand in (2.15) behaves badly at $t = 0$ and $t = \infty$, so that the meaning of the above equation becomes explicitly stated. Rather than give a formal definition of the convergence of an improper integral, the properties of this particular integral are derived in Lemma 2.2.6 below.

Lemma 2.2.6. *Let*

$$S = \{z : a \leq \mathrm{Re} z \leq A\} \text{ where } 0 < a < A < \infty.$$

(a) For every $\varepsilon > 0$ *there is a* $\delta > 0$ *such that for all* z *in* S

$$\left| \int_\alpha^\beta e^{-t} t^{z-1} dt \right| < \varepsilon$$

whenever $0 < \alpha < \beta < \delta$.

(b) *For every* $\varepsilon > 0$ *there is a number* κ *such that for all* z *in* S

$$\left| \int_{\alpha}^{\beta} e^{-t} t^{z-1} dt \right| < \varepsilon$$

whenever $\beta > \alpha > \kappa$.

Proof. To prove (a) note that if $0 < t \leq 1$ and z is in S then

$$(\mathrm{Re} z - 1) \log t \leq (a - 1) \log t \ ;$$

since $e^{-t} \leq 1$,

$$\left| e^{-t} t^{z-1} \right| \leq t^{\mathrm{Re} z - 1} \leq t^{a-1}.$$

So if $0 < \alpha < \beta < 1$ then

$$\begin{aligned} \left| \int_{\alpha}^{\beta} e^{-t} t^{z-1} dt \right| &\leq \int_{\alpha}^{\beta} t^{a-1} dt \\ &= \frac{1}{a} (\beta^a - \alpha^a) \end{aligned}$$

for all z in S. If $\varepsilon > 0$ then we can choose δ, $0 < \delta < 1$, such that

$$\frac{1}{a} (\beta^a - \alpha^a) < \varepsilon \text{ for } |\alpha - \beta| < \delta.$$

This proves part (a).

To prove part (b) note that for z in S and $t \geq 1$, $|t^{z-1}| \leq$

t^{A-1}. Since $t^{A-1}\exp(-\frac{1}{2}t)$ is continuous on $[1, \infty)$ and converges to zero as $t \to \infty$, there is a constant c such that $t^{A-1}\exp(-\frac{1}{2}t) \leq c$ for all $t \geq 1$. This gives that

$$\left|e^{-t}t^{z-1}\right| \leq ce^{-\frac{1}{2}t}$$

for all z in S and $t \geq 1$. 1. If $\beta > \alpha > 1$ then

$$\begin{aligned}\left|\int_{\alpha}^{\beta} e^{-t}t^{z-1}dt\right| &\leq c\int_{\alpha}^{\beta} e^{-\frac{1}{2}t}dt \\ &= 2c(e^{-\frac{1}{2}\alpha} - e^{-\frac{1}{2}\beta})\,.\end{aligned}$$

Again, for any $\varepsilon > 0$ there is a number $\kappa > 1$ such that

$$\left|2c(e^{-\frac{1}{2}\alpha} - e^{-\frac{1}{2}\beta})\right| < \varepsilon$$

whenever α, $\beta > \kappa$, giving part (b). $\square$

The results of the preceding lemma embody exactly the concept of a uniformly convergent integral. In fact, if we consider the integrals

$$\int_{\alpha}^{1} e^{-t}t^{z-1}dt$$

for $0 < \alpha < 1$, then part (a) of Lemma 2.2.6 says that these integrals satisfy a Cauchy criterion as $\alpha \to 0$. That is, the difference between any two will be arbitrarily small if α and β are taken sufficiently close to zero. A similar interpretation is available for the integrals

$$\int_1^\alpha e^{-t}t^{z-1}dt$$

for $\alpha > 1$. The next proposition formalizes this discussion.

Proposition 2.2.7. *If* $G = \{z : \text{Re}z > 0\}$ *and*

$$f_n(z) = \int_{1/n}^n e^{-t}t^{z-1}dt$$

for $n \geq 1$ *and* z *in* G, *then each* f_n *is analytic on* G *and the sequence is convergent in* $H(G)$.

Proof. Think of $f_n(z)$ as the integral of $\varphi(t,\ z) = e^{-t}t^{z-1}$ along the straight line segment $\left[\frac{1}{n},\ n\right]$ and we can conclude that $f_n$is analytic. Now if K is a compact subset of G there are positive real numbers a and A such that $K \subset \{z : a \leq \text{Re}z \leq A\}$. Since for $m > n$,

$$f_m(z) - f_n(z) \quad = \quad \int_{1/m}^m e^{-t}t^{z-1}dt - \int_{1/n}^n e^{-t}t^{z-1}dt$$

$$= \int_{1/m}^{1/n} e^{-t}t^{z-1}dt + \int_{n}^{m} e^{-t}t^{z-1}dt$$

Lemma 2.2.6 and Lemma 1.1.9 imply that $\{f_n\}$ is a Cauchy sequence in $H(G)$. But $H(G)$ is complete (Corollary 1.2.3) so that $\{f_n\}$ must converge. $\square$

If f is the limit of the functions $\{f_n\}$ from the above proposition then define the integral to be this function. That is,

$$f(z) = \int_{0}^{\infty} e^{-t}t^{z-1}dt, \quad \mathrm{Re}z > 0. \tag{2.16}$$

To show that this function $f(z)$ is indeed the gamma function for Re$z > 0$ we only have to show that $f(x) = \Gamma(x)$ for $x \geq 1$. Since $[1, \infty)$ has limit points in the right half plane and both f and Γ are analytic then it follows that f must be Γ. [2] Now observe that successive performing of integration by parts on $(1 - t/n)^n t^{x-1}$ yields

$$\int_{0}^{n} \left(1 - \frac{t}{n}\right)^n t^{x-1}dt = \frac{n!n^x}{x(x+1)\cdots(x+n)}$$

which converges to$\Gamma(x)$ as $n \to \infty$ by Gauss's formula, If

[2] Here we have used the **Identity Theorem:** If f and g are analytic on a region G (i.e., G is an open connected set), then $f \equiv g$ if and only if $\{z \in G : f(z) = g(z)\}$ has a limit point in G.

we can show that the integral in this equation converges to $\int\limits_0^\infty e^{-t}t^{x-1}dt = f(x)$ as $n \to \infty$ then Theorem 2.2.5 is proved. This is indeed the case and it follows from the following lemma.

Lemma 2.2.8. *(a)* $\left\{\left(1+\frac{z}{n}\right)^n\right\}$ *converges to* e^z *in* $H(\mathbb{C})$.

(b) If $t \geq 0$ *then* $\left(1-\frac{t}{n}\right)^n \leq e^{-t}$ *for all* $n \geq t$.

Proof. (a) Let K be a compact subset of the plane. Then $|z| < n$ for all z in K and n sufficiently large. It suffices to show that

$$\lim_{n\to\infty} n \log\left(1+\frac{z}{n}\right) = z$$

uniformly for z in K by Lemma 5.7. Recall that

$$\log(1+w) = \sum_{k=1}^{\infty} (-1)^{k-1}\frac{w^k}{k}$$

for $|w| < 1$. Let $n > |z|$ for all z in K; if z is any point in K then

$$n \log\left(1+\frac{z}{n}\right) = z - \frac{1}{2}\cdot\frac{z^2}{n} + \frac{1}{3}\cdot\frac{z^3}{n^2} - \cdots .$$

So

$$n \log\left(1+\frac{z}{n}\right) = z\left[1 - \frac{1}{2}\left(\frac{z}{n}\right) + \frac{1}{3}\left(\frac{z}{n}\right)^2 - \cdots\right] \quad (2.17)$$

taking absolute values gives that

$$\begin{aligned}
\left|n\log\left(1+\frac{z}{n}\right)-z\right| &\le |z|\sum_{k=2}^{\infty}\frac{1}{k}\left|\frac{z}{n}\right|^{k-1} \\
&\le |z|\sum_{k=2}^{\infty}\left|\frac{z}{n}\right|^{k-1}, \\
&\qquad \text{since } \frac{1}{k}<1 \quad \text{for} \quad k=2,\ 3,\ \ldots \\
&\le |z|\sum_{k=1}^{\infty}\left|\frac{z}{n}\right|^{k} \\
&= \frac{|z|^2}{n}\frac{1}{1-|z/n|} \\
&\le \frac{R^2}{n-R}
\end{aligned}$$

where $R \ge |z|$ for all z in K. If $n \to \infty$ then this difference goes to zero uniformly for z in K.

(b) Now let $t \ge 0$ and substitute $-t$ for z in (2.17) where $t \le n$. This gives

$$n\log\left(1-\frac{t}{n}\right)+t=-t\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{t}{n}\right)^{k-1}\le 0.$$

Thus

$$n\log\left(1-\frac{t}{n}\right)\le -t;$$

and since exp is a monotone function part (b) is proved.

$\square$

Proof of Theorem 2.2.5 Fix $x > 1$ and let $\varepsilon > 0$. According to Part (b) of Lemma 2.2.6 we can choose $\kappa > 0$ such that

$$\int_{\kappa}^{r} e^{-t} t^{x-1} dt < \frac{\varepsilon}{4} \tag{2.18}$$

whenever $r > \kappa$. Let n be any integer larger than κ and let f_n be the function defined in Proposition 2.2.7. Then

$$f_n(x) - \int_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1} dt$$

$$\begin{aligned} = \quad & - \int_0^{1/n} \left(1 - \frac{t}{n}\right)^n t^{x-1} dt \\ & + \int_{1/n}^{n} \left[e^{-t} - \left(1 - \frac{t}{n}\right)^n\right] t^{x-1} dt \end{aligned}$$

Now by Part (b) of Lemma 2.2.8 and by Part (a) of Lemma 2.2.6

$$\int_0^{1/n} \left(1 - \frac{t}{n}\right)^n t^{x-1} dt \leq \int_0^{1/n} e^{-t} t^{x-1} dt < \frac{\varepsilon}{4} \tag{2.19}$$

for sufficiently large n. Also, if n is sufficiently large, part (a) of the
preceding lemma gives

$$\left|\left(1-\frac{t}{n}\right)^n - e^{-t}\right| \leq \frac{\varepsilon}{4M\kappa}$$

for t in $[0,\, \kappa]$ where $M = \int\limits_0^{\kappa} t^{x-1}dt$. Thus

$$\left|\int\limits_{1/n}^{\kappa}\left[e^{-t}-\left(1-\frac{t}{n}\right)^n\right]t^{x-1}dt\right| \leq \frac{\varepsilon}{4} \tag{2.20}$$

Using Lemma 2.2.8 (b) and (2.18)

$$\left|\int\limits_{\kappa}^{n}\left[e^{-t}-\left(1-\frac{t}{n}\right)^n\right]t^{x-1}dt\right| \leq 2\int\limits_0^n e^{-t}t^{x-1}dt \leq \frac{\varepsilon}{2}$$

for $n > \kappa$. If we combine this inequality with (2.19) and (2.20), we get

$$\left|f_n(x) - \int\limits_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}dt\right| < \varepsilon$$

for n sufficiently large. That is

$$\begin{aligned} 0 &= \lim \left[f_n(x) - \int\limits_0^n \left(1 - \frac{t}{n}\right)^n t^{x-1} dt \right] \\ &= \lim \left[f_n(x) - \frac{n! n^x}{x(x+1)\cdots(x+n)} \right] \\ &= f(x) - \Gamma(x). \end{aligned}$$

This completes the proof of Theorem 2.2.5. $\square$

As an application of Theorem 2.2.5 and the fact that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

notice that

$$\sqrt{\pi} = \int\limits_0^\infty e^{-t} t^{(1/2)-1} dt = \int\limits_0^\infty e^{-t} t^{-(1/2)} dt.$$

Performing a change of variables by putting $t = s^2$ gives

$$\sqrt{\pi} = \int\limits_0^\infty e^{-s^2} s^{-1}(2s) ds = 2 \int\limits_0^\infty e^{-s^2} ds.$$

That is,

$$\int\limits_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2}.$$

This integral is often used in probability theory.

2.3 The Riemann Zeta Function

Let z be a complex number and n a positive integer. Then

$$n^z = \underbrace{\exp}_{\substack{\textit{complex}\text{ valued}\\ \text{function of}\\ \text{complex variable}}} (z \log n)$$

is a single valued function, because $\log n \in \mathbb{R}$ [3]

$$|n^z| = \left| \underbrace{\exp}_{\substack{\textit{complex}\text{ valued}\\ \text{function of}\\ \text{complex variable}}} (z \log n) \right|$$

[3] $\log n \in \mathbb{R}$ as, for the natural number n, $\log n = \underbrace{\ln n}_{\text{the natural logarithm of the positive integer } n}$.

$$
\begin{aligned}
&= \left|\exp((\mathrm{Re}z + i\mathrm{Im}z)\underbrace{\log n}_{\in\mathbb{R}})\right|, \\
&= |\exp((\mathrm{Re}z \log n + i\mathrm{Im}z \log n)| \\
&= \underbrace{|\exp(\mathrm{Re}z \log n)|}_{\exp(\mathrm{Re}z \log n)} \underbrace{|\exp(i\mathrm{Im}z \log n)|}_{1} \text{ (See Footnote 3)} \\
&= \underbrace{\underbrace{\exp}_{\substack{\text{real valued}\\ \text{function of}\\ \text{real variable}}} (\underbrace{\mathrm{Re}z}_{\in R} \underbrace{\log n}_{\in R})}_{>0}.
\end{aligned}
$$

[4] so that

$$
\begin{aligned}
\left|n^{-z}\right| &= \exp(\mathrm{Re}(-z)\log n) \\
&= \underbrace{\exp}_{\substack{\text{real valued}\\ \text{function of}\\ \text{real variable}}} (\underbrace{-\mathrm{Re}z}_{\in R} \underbrace{\log n}_{\in R})
\end{aligned}
$$

---

[4] We note that

$$
\underbrace{\exp}_{\substack{\textit{complex}\text{ valued}\\ \text{functionof}\\ \text{complexvariable}}} \underbrace{(\underbrace{\mathrm{Re}z}_{\in R} \underbrace{\log n}_{\in R})}_{\in R} = \underbrace{\underbrace{\exp}_{\substack{\textit{real}\text{ valued}\\ \text{functionof}\\ \text{complexvariable}}} (\mathrm{Re}z \log n)}_{>0}
$$

while (since $|e^{i\,\theta}| = 1$ for any real θ,)

$$
\left|\exp(i\,\underbrace{\mathrm{Im}z \log n}_{\in R}\right| = 1.
$$

$$
\begin{aligned}
&= \exp(\log n^{-\mathrm{Re}z}) \\
&= n^{-\mathrm{Re}z}.
\end{aligned}
$$

Thus,

$$\sum_{k=1}^{n} \left|k^{-z}\right| = \sum_{k=1}^{n} k^{-\mathrm{Re}z}.$$

So if $\mathrm{Re}z \geq 1+\varepsilon$ then

$$-\mathrm{Re}z \leq -(1+\varepsilon)$$

so that

$$\sum_{k=1}^{n} \left|k^{-z}\right| = \sum_{k=1}^{n} k^{-\mathrm{Re}z} \leq \sum_{k=1}^{n} k^{-(1+\varepsilon)}.$$

Then, with $M_k = k^{-(1+\varepsilon)}$ and noting that

$$\left|k^{-z}\right| \leq M_k = k^{-(1+\varepsilon)} \text{ for } k = 1,\ 2,\ \ldots$$

and since [5] $\sum\limits_{k=1}^{\infty} k^{-(1+\varepsilon)}$ converges, it follows, using Weierstrass M-Test,[6] that the series

$$\sum_{k=1}^{\infty} \left|k^{-z}\right|$$

converges uniformly on $\{z : \mathrm{Re}z \geq 1+\varepsilon\}$. Hence the series

$$\sum_{k=1}^{\infty} k^{-z} \tag{2.21}$$

converges **uniformly** and **absolutely** on $\{z : \mathrm{Re}z \geq 1+\varepsilon\}$. In particular, the series (2.21) **converges** in $H\left(\{z : \mathrm{Re}z > 1\}\right)$

---

[5] We know that the $p-$ series

$$\sum_{k=1}^{\infty} k^{-p} = \sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges for $p > 1$. Here $1+\varepsilon > 1$, so $\sum\limits_{k=1}^{\infty} k^{-(1+\varepsilon)}$ converges.

[6] **II.6.2, Weierstrass M-Test:** Let X be a set and $u_n : X \to \mathbb{C}$ be a function such that

$$|u_n(x)| \leq M_n \text{ for every } x \in X$$

and suppose the constants M_n (that are nonnegative real numbers) satisfy

$$\sum_{n=1}^{\infty} M_n < \infty.$$

Then the series of functions

$$\sum_{n=1}^{\infty} u_n$$

is **uniformly convergent.**

to an analytic function $\zeta(z)$. [7]

Definition 2.3.1. The Riemann zeta function is defined for $\text{Re}z > 1$ by the equation

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}.$$

The zeta function, as well as the gamma function, has been the subject of an enormous amount of mathematical research since their introduction. The analysis of the zeta function has had a profound effect on number theory and this has, in turn, inspired more work on ζ. In fact, one of the most famous unsolved problems in Mathematics is the location of the zeroes of the zeta function.

We wish to demonstrate a relationship between the zeta function and the gamma function.

[7] *Explanation*: We note that $\bigcup_{\varepsilon>0} (1+\varepsilon,\ \infty) = (1,\ \infty)$. On $\{z : \text{Re}z > 1\}$, for each $k = 1,\ 2,\ \ldots$ the functions $k^{-z} = \frac{1}{k^z}$ are analytic and hence they are members of $H\left(\{z : \text{Re}z > 1\}\right)$. Hence, by the uniform convergence, the series $\sum_{k=1}^{\infty} k^{-z}$ converges to an analytic function. i.e., the the series $\sum_{k=1}^{\infty} k^{-z}$ converges in $H\left(\{z : \text{Re}z > 1\}\right)$ to an analytic function, say, to the function $\zeta(z)$.

We let[8]

$$\Gamma(z) = \int\limits_0^\infty e^{-t} t^{z-1} dt \text{ for } \mathrm{Re} z > 0.$$

Performing a change of variable in this integral by letting $t = nu$ gives

$$\begin{aligned}
\Gamma(z) &= \int\limits_0^\infty e^{-(nu)} (nu)^{z-1} d(nu) \\
&= n^{z-1} \int\limits_0^\infty e^{-nu} u^{z-1} n du \\
&= n^{z} \int\limits_0^\infty e^{-nu} u^{z-1} du \\
&= n^{z} \int\limits_0^\infty e^{-nt} t^{z-1} dt,
\end{aligned}$$

that is,

$$n^{-z}\ \Gamma(z) = \int\limits_0^\infty e^{-nt} t^{z-1} dt.$$

---

[8]This is possible by: **Theorem VII.7.15** If $\mathrm{Re} z > 0$, then

$$\Gamma(z) = \int\limits_0^\infty e^{-t} t^{z-1} dt.$$

If Re$z > 1$ and we sum this equation over all positive n, then

$$\zeta(z)\Gamma(z) = \underbrace{\sum_{n=1}^{\infty} n^{-z}}_{\zeta(z)} \Gamma(z) = \sum_{n=1}^{\infty} n^{-z}\Gamma(z) = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-nt} t^{z-1} dt \ . \tag{2.22}$$

We wish to show that this infinite sum can be taken inside the integral sign (i.e., to show that

$$\sum_{n=1}^{\infty} \int_0^{\infty} e^{-nt} t^{z-1} dt \ = \int_0^{\infty} \sum_{n=1}^{\infty} e^{-nt} t^{z-1} dt \, .$$

But first, an analogue of Lemma 2.2.6.[9] is considered.

[9]Before that we recall Cauchy criterion for convergence of series of real numbers and Cauchy criterion for **uniform** convergence that of **series of functions:Lemma 2.2.6** Let

$$S = \{z : a \leq \mathrm{Re} z \leq A\} \text{ where } 0 < a < A < \infty.$$

(a) For every $\varepsilon > 0$ there is a $\delta > 0$ such that for all z in S

$$\left| \int_{\alpha}^{\beta} e^{-t} t^{z-1} dt \right| < \varepsilon$$

whenever $0 < \alpha < \beta < \delta$.

(b) For every $\varepsilon > 0$ there is a number κ such that for all z in S

$$\left| \int_{\alpha}^{\beta} e^{-t} t^{z-1} dt \right| < \varepsilon$$

whenever $\beta > \alpha > \kappa$.

(Also see Appendix in Page 422.)

Before discussing the next lemma, we note that the integral of function of two variables (in which integration is with respect to one variable t alone) given by

$$\int_0^1 (e^t - 1)^{-1} t^{z-1} dt$$

is an improper integral, because the integrand behaves badly at $t = 0$, but for a given z the integral converges. We are going to show that this integral (which gives the value, a function of z, is uniformly convergent on some given set of z values.) Also,

$$\int_1^\infty (e^t - 1)^{-1} t^{z-1} dt$$

is an improper integral, because the integrand behaves badly at $t = \infty$.

Uniform Convergence of the Improper Integrals is discussed in Page 428.

Lemma 2.3.2. ***[(a) Cauchy Criterion for Uniform Convergence of the improper integral*** $\int_0^1 (e^t - 1)^{-1} t^{z-1} dt$ ***on some given set*** S.***]***

Let $S = \{z : \mathrm{Re} z \geq a\}$ *where* $a > 1$ *and*

$$S_\eta(z) = \int_\eta^1 (e^t - 1)^{-1} t^{z-1} dt.$$

If $\varepsilon > 0$ *then there is a number* $\delta,\ 0 < \delta < 1,$ *such that for all* $z \in S$

$$\left| \underbrace{\int_\alpha^\beta (e^t - 1)^{-1} t^{z-1} dt}_{S_\beta(z) - S_\alpha(z)} \right| < \varepsilon \qquad (2.23)$$

whenever $\alpha < \beta < \delta$.

(b) [Cauchy Criterion for Uniform Convergence of the improper integral $\int_1^\infty (e^t - 1)^{-1} t^{z-1} dt$ ***on some given set*** *S.]*

Let

$$S = \{z : \mathrm{Re} z \leq A\}$$

where $-\infty < A < \infty$ *and*

$$S_\eta(z) = \int_1^\eta (e^t - 1)^{-1} t^{z-1} dt.$$

If $\varepsilon > 0$ then there is a number $\kappa > 1$ such that for all $z \in S$

$$\left| \underbrace{\int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt}_{S_\beta(z) - S_\alpha(z)} \right| < \varepsilon \tag{2.24}$$

whenever $\beta > \alpha > \kappa$.

Proof. (a) Since (the real valued exponential function of real variable)

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

we have

$$e^t - 1 = t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$$

so that

$$e^t - 1 = t + \underbrace{\frac{t^2}{2!} + \frac{t^3}{3!} + \cdots}_{\geq 0, \text{ for all } t \geq 0}$$

and hence

$$e^t - 1 \geq t$$

for all $t \geq 0$. Thus, we have that for $0 < t \leq 1$ and $z \in S$

$$\begin{aligned}
\left|(e^t-1)^{-1}t^{z-1}\right| &= \left|(e^t-1)^{-1}\right|\left|t^{z-1}\right| \\
&\leq \left|t^{-1}\right|\left|t^{z-1}\right| = \left|t^{z-2}\right| = \left|e^{(z-2)\log t}\right| \\
&= \left|e^{(z-2)(\ln t + i\underbrace{2n\pi}_{0})}\right| \\
&= \left|e^{(\mathrm{Re}z-2+i\mathrm{Im}z)\ln t}\right| = \left|e^{(\mathrm{Re}z-2)\ln t}\right|\underbrace{\left|e^{i\mathrm{Im}z\ln t}\right|}_{1} \\
&= e^{(\mathrm{Re}z-2)\ln t}.
\end{aligned}$$

$$\mathrm{Re}z \geq a$$

$$\begin{aligned}
&\Rightarrow\ \mathrm{Re}z-2 \geq a-2 \\
&\Rightarrow\ (\text{for } 0<t<1,\ \ \ln t<0,\ \text{so})\ (\mathrm{Re}z-2)\ln t \leq (a-2)\ln t \\
&\Rightarrow\ (\text{since real valued exponential function of real variable is} \\
&\qquad \text{an increasing function}) \\
&\quad\ e^{(\mathrm{Re}z-2)\ln t} \leq e^{(a-2)\ln t} = t^{a-2}
\end{aligned}$$

Also, when $t=1$, $\ln t = 0$, so for that value of t,

$$(\mathrm{Re}z-2)\underbrace{\ln t}_{0} = (a-2)\underbrace{\ln t}_{0}.$$

Hence, for $0 < t \leq 1$ and $z \in S$

$$\left|(e^t - 1)^{-1} t^{z-1}\right| \leq t^{a-2}.$$

Since $a > 1$, by Example B.1.1 in page 425, the integral

$$\int_0^1 t^{a-2} dt = \int_0^1 \frac{dt}{t^{2-a}} = \int_0^1 \frac{dt}{(t-0)^{2-a}}$$

is finite.[10]

Now, $\int_0^1 t^{a-2} dt$ converges for $a > 1$ implies (by Cauchy Criterion for Convergence of improper integral) that for a given (arbitrary choice) $\varepsilon > 0$ there is a δ, $0 < \delta < 1$, such that

$$\int_\alpha^\beta t^{a-2} dt < \varepsilon$$

whenever $\alpha < \beta\delta$. Hence, using,

$$\left|(e^t - 1)^{-1} t^{z-1}\right| \leq t^{a-2},$$

[10] Explanation: When $a \geq 2$, then $b = a - 2 \geq 0$, so that the integral

$$\int_0^1 t^b dt = \left[\frac{t^{b+1}}{b+1}\right]_0^1 = \frac{1}{b+1}$$

is finite. When $1 < a < 2$, then $-1 > -a > -2$, so that $2 - 1 > 2 - a > 2 - 2$ i.e., $1 > 2 - a > 0$ so that, by Example B.1.1 in page 425, the integral

$$\int_0^1 t^{a-2} dt = \int_0^1 \frac{dt}{t^{2-a}}$$

is finite.

the Direct Comparison Test implies that

$$\int_{\alpha}^{\beta} \left|(e^t-1)^{-1}t^{z-1}\right| dt \leq \int_{\alpha}^{\beta} t^{a-2}dt < \varepsilon$$

whenever $\alpha < \beta < \delta$. Hence

$$\left|\int_{\alpha}^{\beta} (e^t-1)^{-1}t^{z-1}dt\right| \leq \int_{\alpha}^{\beta} \left|(e^t-1)^{-1}t^{z-1}\right| dt < \varepsilon$$

whenever $\alpha < \beta < \delta$ and this is (2.23). This proves part (a).

(b) If $t \geq 1$ and z is any point in S then,

$$\begin{aligned} \left|t^{z-1}\right| &= \left|e^{(z-1)\log t}\right| = \left|e^{(z-1)\ln t}\right| \\ &= \left|e^{(\mathrm{Re}z-1)\ln t + i\,\mathrm{Im}z\ln t}\right| = e^{(\mathrm{Re}z-1)\ln t} \end{aligned}$$

and since for $t \geq 1$, $\ln t > 0$ so that

$$(\mathrm{Re}z-1)\ln t \leq (A-1)\ln t$$

and by the increasing behaviour of the exponential function,

$$e^{(\mathrm{Re}z-1)\ln t} \leq e^{(A-1)\ln t}$$

so that for $t \geq 1$ and z is any point in S,

$$\left|t^{z-1}\right| \leq e^{(A-1)\ln t}.$$

i.e., for $t \geq 1$ and z is any point in S,

$$\left|t^{z-1}\right| \leq t^{A-1}.$$

Since $t^{A-1}\exp(-\frac{1}{2}t)$ is a continuous real valued function on $[1, \infty)$ that converges to 0 as $t \to \infty$, there is a constant c such that

$$t^{A-1}\exp(-\frac{1}{2}t) \leq c \;\; \forall t \geq 1.$$

[11]

Hence

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[11]Explanation: Since

$$\lim_{t\to\infty} t^{A-1}\exp(-\frac{1}{2}t) = 0$$

for a given $\varepsilon > 0$ there is a $K > 0$ such that

$$\left|t^{A-1}\exp(-\frac{1}{2}t) - 0\right| \leq \varepsilon \;\; \forall t > K.$$

In particular, for $\varepsilon = 1$ (or any fixed positive real number) there is a $K_1 > 0$ such that

$$\left|t^{A-1}\exp(-\frac{1}{2}t) - 0\right| \leq \varepsilon \;\; \forall t > K_1.$$

Now $t^{A-1}\exp(-\frac{1}{2}t)$ is continuous on the closed and bounded (hence compact) set $[1, \; K_1]$ implies it is bounded on that set. That is, there is $C > 0$ such that

$$\left|t^{A-1}\exp(-\frac{1}{2}t)\right| \leq C$$

on $[1, \; K_1]$. Take

$$c = \max\{C, \; 1\}.$$

Then,

$$\left|t^{A-1}\exp(-\frac{1}{2}t)\right| \leq c$$

for all $t \geq 1$.

$$\begin{aligned}\left|(e^t-1)^{-1}t^{z-1}\right| &= \underbrace{\left|(e^t-1)^{-1}\right|}_{(e^t-1)^{-1}}\left|t^{z-1}\right| \\ &\leq (e^t-1)^{-1}\left|t^{z-1}\right| \\ &\leq c\exp(\frac{1}{2}t)(e^t-1)^{-1}.\end{aligned}$$

We note that

$$\exp(\frac{1}{2}t)(e^t-1)^{-1}$$

is integrable on $[1, \infty)$. Now

$$\int_1^\infty e^{(1/2)t}(e^t-1)^{-1}dt < \infty$$

[12]implies (by the Cauchy Convergence Criterion) for a given $\varepsilon > 0$, there exists a a number $\kappa > 1$ such that

$$\underbrace{\int_{\alpha}^{\beta} e^{(1/2)t}(e^t-1)^{-1}dt}_{\int_1^{\beta} e^{(1/2)t}(e^t-1)^{-1}dt - \int_1^{\alpha} e^{(1/2)t}(e^t-1)^{-1}dt} \qquad < \varepsilon$$

---

[12]Explanation: Putting $u = e^{(1/2)t}$,

$$\begin{aligned}
\int_1^{\infty} e^{(1/2)t}(e^t-1)^{-1}dt &= \lim_{b\to\infty}\int_1^{b} e^{(1/2)t}(e^t-1)^{-1}dt \\
&= \lim_{b\to\infty}\frac{1}{2}\int_{e^{(1/2)}}^{e^{(1/2)b}}\frac{du}{u^2-1} \\
&= \lim_{b\to\infty}\frac{1}{2}\int_{e^{(1/2)}}^{e^{(1/2)b}}\left\{\frac{1/2}{u-1}-\frac{1/2}{u+1}\right\}du \\
&= \frac{1}{4}\lim_{b\to\infty}\left[\ln(u-1)-\ln(u+1)\right]_{e^{(1/2)}}^{e^{(1/2)b}} \\
&= \frac{1}{4}\lim_{b\to\infty}\left[\ln\frac{u-1}{u+1}\right]_{e^{(1/2)}}^{e^{(1/2)b}} \\
&= \frac{1}{4}\lim_{b\to\infty}\left[\ln\frac{1-(1/u)}{1+(1/u)}\right]_{e^{(1/2)}}^{e^{(1/2)b}} \\
&= \frac{1}{4}\left\{\underbrace{\lim_{b\to\infty}\ln\frac{1-\left(1/e^{(1/2)b}\right)}{1+(1/e^{(1/2)b})}}_{0}-\underbrace{\ln\frac{1-\left(1/e^{(1/2)}\right)}{1+(1/e^{(1/2)})}}_{\text{finite real number}}\right\}
\end{aligned}$$

whenever $\beta > \alpha > \kappa$. Hence by comparison test, for all $z \in S$,

$$\int_{\alpha}^{\beta} \left|(e^t - 1)^{-1} t^{z-1}\right| dt < \varepsilon$$

whenever $\beta > \alpha > \kappa$. Thus,

$$\left|\int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{z-1} dt\right| \leq \int_{\alpha}^{\beta} \left|(e^t - 1)^{-1} t^{z-1}\right| dt < \varepsilon$$

whenever $\beta > \alpha > \kappa$ so that (2.24) is obtained, proving Part (b). $\square$

Corollary 2.3.3. *(a) If* $S = \{z : a \leq \mathrm{Re} z \leq A\}$ *where* $1 < a < A < \infty$ *then the integral*

$$\int_{0}^{\infty} (e^t - 1)^{-1} t^{z-1} dt$$

converges uniformly on S.

(b) If $S = \{z : \mathrm{Re} z \leq A\}$ *where* $-\infty < A < \infty$ *then the integral*

$$\int_{1}^{\infty} (e^t - 1)^{-1} t^{z-1} dt$$

converges uniformly on S.

Proof. Let $S_1 = \{z : a \leq \mathrm{Re} z\}$ and $S_2 = \{z : \mathrm{Re} z \leq A\}$. Then $S = S_1 \cap S_2$. By Part (a) of the previous Lemma,

$\int_0^1 (e^t - 1)^{-1} t^{z-1} dt$ satisfies the Cauchy Criterion for Uniform Convergence on the set S_1 and hence converges uniformly on S_1. By Part (b) of the previous Lemma, $\int_1^\infty (e^t - 1)^{-1} t^{z-1} dt$ satisfies the Cauchy Criterion for Uniform Convergence on the set S_2 and hence converges uniformly on S_2. Hence, both $\int_0^1 (e^t - 1)^{-1} t^{z-1} dt$ and $\int_1^\infty (e^t - 1)^{-1} t^{z-1} dt$ converge uniformly on $S = S_1 \cap S_2$. Hence

$$\int_0^1 (e^t - 1)^{-1} t^{z-1} dt + \int_1^\infty (e^t - 1)^{-1} t^{z-1} dt$$

converges uniformly on $S = S_1 \cap S_2$. This proves Part (a).

(b). By Part (b) of the previous result, $\int_1^\infty (e^t - 1)^{-1} t^{z-1} dt$ satisfies the Cauchy Criterion for Uniform Convergence on the set $S = \{z : \text{Re} z \leq A\}$ and hence $\int_1^\infty (e^t - 1)^{-1} t^{z-1} dt$ converges uniformly on S. $\square$

Example 2.3.4. [NBHM Scholarship Test, October 22, 2005] Let D_n be the open disc of radius n with centre at the point $(n, 0) \in \mathbb{R}^2$. Does there exist a function $f : \mathbb{R}^2 \to \mathbb{R}$ of the form

$$f(x,\ y) = ax + by$$

such that

$$\bigcup_{n=1}^{\infty} D_n = \{(x,\ y)|f(x,\ y) > 0\}?$$

If the answer is *Yes*, give the values of a and b.

Solution Yes. For $n \in \mathbb{N}$, we have

$$D_n = \{(x,\ y) \in \mathbb{R}^2|(x-n)^2 +\ y^2 < n^2\}.$$

In particular,

$$D_1 = \{(x,\ y) \in \mathbb{R}^2|(x-1)^2 +\ y^2 < 1\}.$$

$$D_2 = \{(x,\ y) \in \mathbb{R}^2|(x-2)^2 +\ y^2 < 4\}.$$

and so on (Fig. (2.1)). Then

$$\bigcup_{n=1}^{\infty} D_n = \{(x,\ y) \in R^2|x > 0\} = \{z \in \mathbb{C}|\mathrm{Re}z > 0\}. \tag{2.25}$$

From this, we define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,\ y) = x$$

such that

$$\bigcup_{n=1}^{\infty} D_n = \{(x,\ y) \in \mathbb{R}^2|\underbrace{x}_{f(x,\ y)} > 0\} = \{(x,\ y)|f(x,\ y) > 0\}.$$

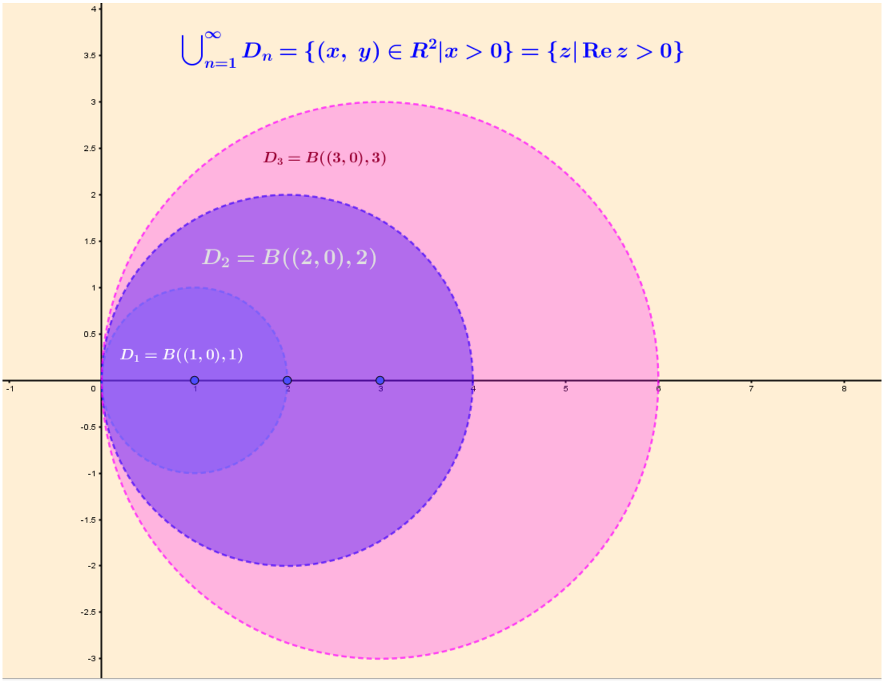

Figure 2.1:

Hence with $a = 1$ and $b = 0$ we have a function of the desired form.

Example 2.3.5. For $m = 2, 3, \ldots,$ let E_m be the open disc of radius $m - 1$ with centre at the point $(m, 0) \in \mathbb{R}^2$. Then

$$\bigcup_{m=2}^{\infty} E_m = \{(x,\ y) \in R^2 | x > 1\} = \{z \in C | \mathrm{Re} z > 1\}.$$

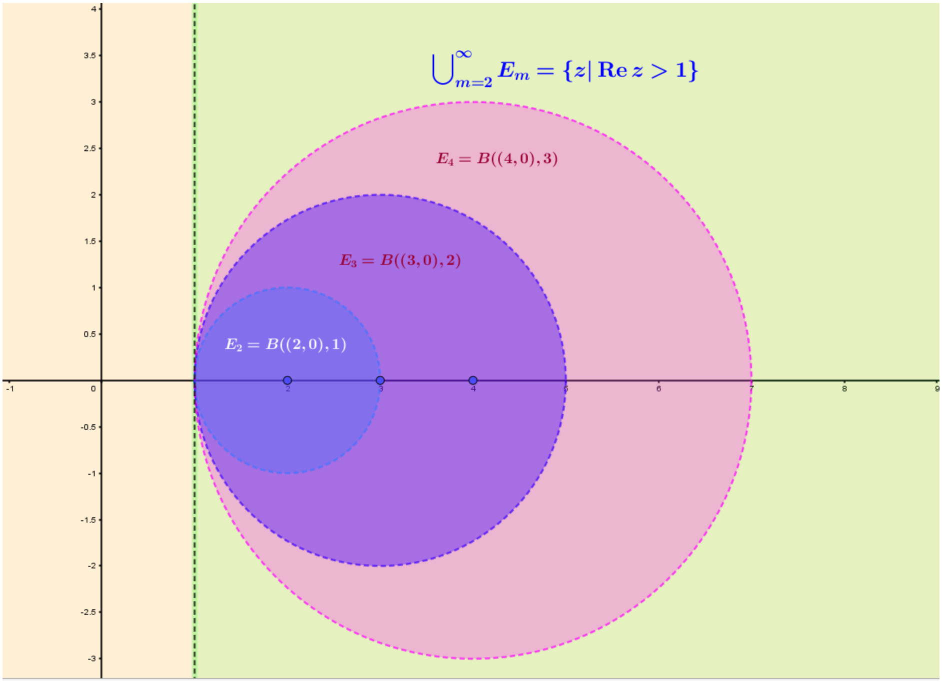

Figure 2.2:

Proposition 2.3.6. *For* $\mathrm{Re} z > 1$

$$\zeta(z)\Gamma(z) = \int_0^\infty (e^t - 1)^{-1} t^{z-1} dt. \qquad (2.26)$$

Proof. According to the above Corollary, the integral $\int_0^\infty (e^t - 1)^{-1} t^{z-1} dt$ is an analytic function on the region $\{z \in \mathbb{C} | \mathrm{Re} z > 1\}$ [**Explanation**: By the above Corollary, for any a and A such that $1 <$

$a < A < \infty$, $\int\limits_0^\infty (e^t - 1)^{-1} t^{z-1} dt$ is uniformly convergent on $S_{a,\,A} = \{z \in \mathbb{C} | a \leq \text{Re}z \leq A\}$. In order to show that $\int\limits_0^\infty (e^t - 1)^{-1} t^{z-1} dt$ is analytic on $\{z \in \mathbb{C} | \text{Re}z > 1\}$ we show that it is analytic at each point of $\{z \in \mathbb{C} | \text{Re}z > 1\}$. For this, take $w \in \{z \in \mathbb{C} | \text{Re}z > 1\}$. We choose $R > 0$, a and A such that $1 < a < A < \infty$ and $B(w, R) \subseteq \{z \in C | 1 < a \leq \text{Re}z \leq A < \infty\}$. $\int\limits_0^\infty (e^t - 1)^{-1} t^{z-1} dt$ is uniformly convergent

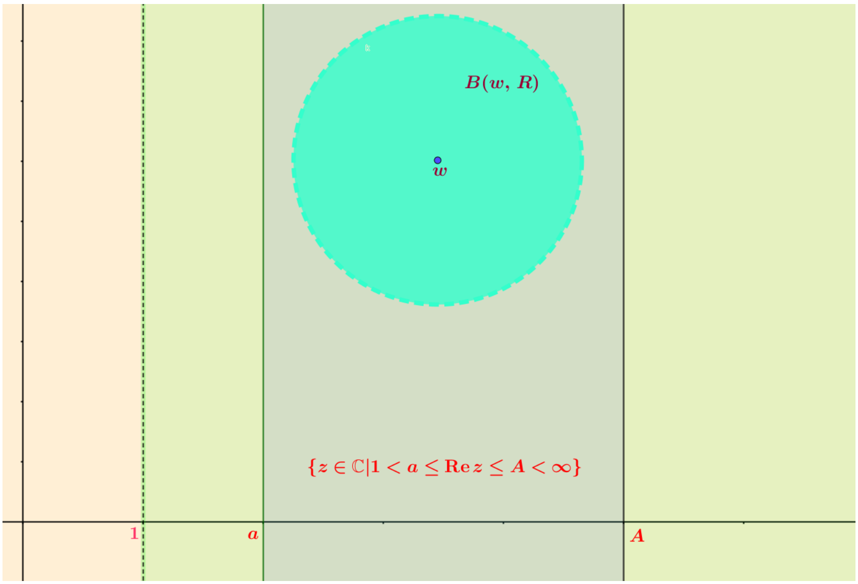

Figure 2.3:

on $\{z \in C | 1 < a \leq \text{Re}z \leq A < \infty\}$ and hence on its subset

$B(w, R)$. Thus, on $B(w, R)$

$$\int_0^\infty (e^t-1)^{-1}t^{z-1}dt = \underbrace{\lim_{\substack{\alpha\to 0\\ \beta\to\infty}} \underbrace{\int_\alpha^\beta (e^t-1)^{-1}t^{z-1}dt}_{\text{analytic on } B(w,R)}}_{\substack{\text{analytic on } B(w,R) \text{ as } \int_0^\infty (e^t-1)^{-1}t^{z-1}dt \\ \text{is uniformly convergent on } B(w,R)}}$$

is analytic. By varying w over $\{z \in \mathbb{C}|\mathrm{Re}z > 1\}$ it follows that $\int_0^\infty (e^t-1)^{-1}t^{z-1}dt$ is analytic on $\{z \in \mathbb{C}|\mathrm{Re}z > 1\}$.

Thus, to obtain (2.26) it suffices to show that $\zeta(z)\Gamma(z)$ equals the integral $\int_0^\infty (e^t-1)^{-1}t^{z-1}dt$ for $z = x > 1$ (i.e., to show the equality for all real numbers greater than 1.)[13]

The integral on the right hand side of (2.26) (which is a function z) is analytic on $\{z : \mathrm{Re}z > 1\}$ and hence for any $c \in \{z : \mathrm{Re}z > 1\}$ there is a $R_c > 0$ such that the integral (which is a function z) can be expressed by the Taylor series

$$\sum_{n=0}^\infty a_n(z-c)^n \quad , \quad |z-c| < R_c.$$

centred at c and having radius R_c such that for each z in

[13] Why is this enough?

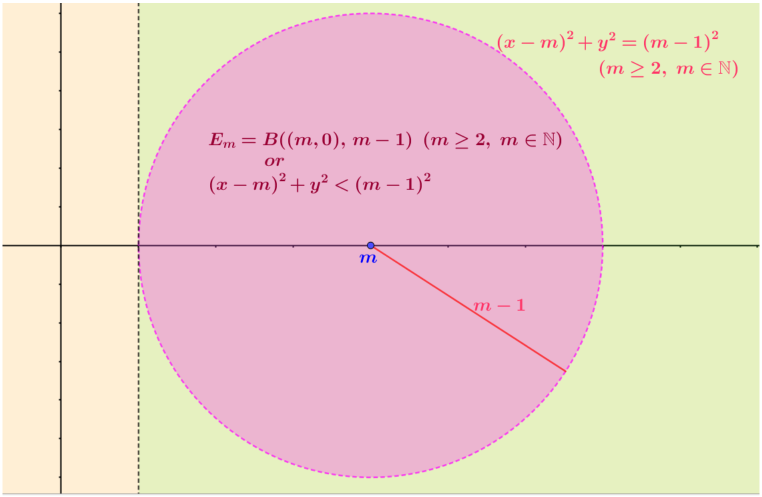

Figure 2.4:

$B(c,\ R_c)$,

$$\int_0^\infty (e^t-1)^{-1}t^{z-1}dt = \sum_{n=0}^{\infty} a_n(z-c)^n \quad , \quad |z-c| < R_c. \tag{2.27}$$

For $m = 2, 3, \ldots,$ let E_m be the open disc of radius $m-1$ with centre at the point $(m,\ 0) \in \mathbb{R}^2$. Then

$$\bigcup\nolimits_{m=2}^{\infty} E_m = \{(x,\ y) \in R^2 | x > 1\} = \{z \in C | \mathrm{Re} z > 1\}. \tag{2.28}$$

In particular, in (2.27), if we choose $z = x = m \geq 2$ where

$m \in \mathbb{N}$ then for the real number $R_m = m - 1 > 0$ then

$$\int_0^\infty (e^t - 1)^{-1} t^{z-1} dt = \sum_{n=0}^\infty a_n (z-m)^n \quad , \quad |z - m| < R_m. \tag{2.29}$$

We note that (Fig. 2.4)

$$B(m,\ R_m) \subset \{z : \mathrm{Re} z > 1\}\,.$$

Also, the function $\zeta(z)\Gamma(z)$ is analytic on $\{z : \mathrm{Re} z > 1\}$ and hence for any $a \in \{z : \mathrm{Re} z > 1\}$ there is an $R_a > 0$ such that $\zeta(z)\Gamma(z)$ can be represented by the Taylor series

$$\zeta(z)\Gamma(z) = \sum_{n=0}^\infty b_n (z-a)^n, \ \ |z - a| < R_a. \tag{2.30}$$

In particular, for $z = x = m \geq 2$ $(m \in \mathbb{N})$ we have $R_m = m - 1 > 0$ such that

$$\zeta(z)\Gamma(z) = \sum_{n=0}^\infty b_n (z-m)^n, \ \ |z - m| < R_m. \tag{2.31}$$

If

$$\zeta(x)\Gamma(x) = \int_0^\infty (e^t - 1)^{-1} t^{x-1} dt$$

for all $x > 1$ (i.e., for all real numbers greater than 1), then for those values of x, the left hand sides of (2.29) and (2.31) are the same and hence

$$\sum_{n=0}^{\infty} a_n(x-m)^n = \sum_{n=0}^{\infty} b_n(x-m)^n, \qquad x > 1,\ |x-m| < R_m. \tag{2.32}$$

Equating constant terms (or putting $x = m$ in the series on both sides), we obtain

$$a_0 = b_0.$$

Differentiating both sides of (2.32) with respect to x, the above gives (term wise differentiation is possible because the functions $\zeta(z)\Gamma(z)$ and $\int\limits_0^\infty (e^t-1)^{-1}t^{z-1}dt$ are analytic on the region $\{z : \mathrm{Re}z > 1\}$) and hence, in particular, analytic at each point on $\{x : x > 1\}$)

$$\sum_{n=1}^{\infty} na_n(x-m)^{n-1} = \sum_{n=1}^{\infty} nb_n(x-m)^{n-1} \tag{2.33}$$

and equating constant terms (or putting $x = m$ in the series on both sides of the above equation) we get

$$a_1 = b_1.$$

Differentiating both sides of (2.33) with respect to x, we obtain

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-m)^{n-2} = \sum_{n=2}^{\infty} n(n-1)b_n(x-m)^{n-2} \tag{2.34}$$

and equating constant terms (or putting $x = m$ in the series on both sides of the above equation) we get

$$a_2 = b_2.$$

Continuing like this, we obtain

$$a_n = b_n$$

for all $n = 0, 1, \ldots$. Hence

$$\begin{aligned} \zeta(z)\Gamma(z) &= \sum_{n=0}^{\infty} b_n(z-a)^n = \sum_{n=0}^{\infty} a_n(z-a)^n \\ &= \int\limits_0^{\infty} (e^t-1)^{-1}t^{z-1}dt, \ \ |z-m| < R_m. \end{aligned}$$

i.e., we have shown that

$$\zeta(z)\Gamma(z) = \int\limits_0^{\infty} (e^t-1)^{-1}t^{z-1}dt$$

on $B(m, \ R_m)$ (Fig. (2.4)). Take the union of all $B(m, \ R_m)$

where m is a natural number such that $m \geq 2$. **The union so obtained is the whole of the region** $\{z : \mathrm{Re}z > 1\}$ (Ref. Example 2.3.5). Hence if we take any $w \in \{z \in C|\mathrm{Re}z > 1\}$ then $w \in B(m,\ R_m)$ for some integer $m \geq 2$. As

$$\zeta(z)\Gamma(z) = \int_0^\infty (e^t - 1)^{-1} t^{z-1} dt$$

on $B(m,\ R_m)$ it follows that

$$\zeta(w)\Gamma(w) = \int_0^\infty (e^t - 1)^{-1} t^{w-1} dt.$$

As the choice of $w \in \{z \in C|\mathrm{Re}z > 1\}$ is arbitrary, it follows that

$$\zeta(w)\Gamma(w) = \int_0^\infty (e^t - 1)^{-1} t^{w-1} dt$$

for any $w \in \{z \in C|\mathrm{Re}z > 1\}$. i.e.,

$$\zeta(z)\Gamma(z) = \int_0^\infty (e^t - 1)^{-1} t^{z-1} dt$$

on $\{z \in C|\mathrm{Re}z > 1\}$.

Hence it remains to show that

$$\zeta(z)\Gamma(z) = \int_0^\infty (e^t - 1)^{-1} t^{z-1} dt$$

for $z = x > 1$ (i.e., we show the equality for all real numbers greater than 1.) For this, we fix $z = x > 1$. Then, clearly $1 < \underbrace{x}_{=a, \text{ say}} < \infty$ and hence by Lemma 2.3.2 it follows that

$$\int_0^1 (e^t - 1)^{-1} t^{z-1} dt$$

is uniformly convergent on $S = \{z : \mathrm{Re} z \geq x\}$ and hence corresponding to $\varepsilon > 0$ there is an $\alpha > 0$ such that (for $z \in S$)

$$\left| \underbrace{\int_0^1 (e^t - 1)^{-1} t^{z-1} dt}_{\substack{\text{Limit of the sequence} \\ \text{of definite integrals}}} - \underbrace{\int_\alpha^1 (e^t - 1)^{-1} t^{z-1} dt}_{\substack{\text{Definite integral that is} \\ \text{an approximation to the limit}}} \right| < \frac{\varepsilon}{4}$$

and in particular,

$$\left| \underbrace{\int_0^1 (e^t-1)^{-1} t^{x-1} dt}_{\geq 0} - \underbrace{\int_\alpha^1 (e^t-1)^{-1} t^{x-1} dt}_{\geq 0} \right| < \frac{\varepsilon}{4}$$

i.e.,

$$\int_0^\alpha (e^t-1)^{-1} t^{x-1} dt < \frac{\varepsilon}{4}. \tag{2.35}$$

Also,

$$\int_1^\infty (e^t-1)^{-1} t^{z-1} dt$$

converges uniformly on $S_1 = \{z : \text{Re}z \leq A\}$ where $-\infty < 1 < A < \infty$ and hence there is a $\beta > 0$ such that $0 < \alpha < \beta < \infty$ and

$$\left| \underbrace{\int_1^\infty (e^t-1)^{-1} t^{z-1} dt}_{\substack{\text{Limit of the sequence} \\ \text{of definite integrals}}} - \underbrace{\int_1^\beta (e^t-1)^{-1} t^{z-1} dt}_{\substack{\text{Definite integral that is} \\ \text{an approximation to the limit}}} \right| < \frac{\varepsilon}{4}$$

for all $z \in S_1 \cap \{z : \mathrm{Re} z > 1\}$ and in particular

$$\left| \underbrace{\int_1^\infty (e^t - 1)^{-1} t^{x-1} dt}_{\geq 0} - \underbrace{\int_1^\beta (e^t - 1)^{-1} t^{x-1} dt}_{\geq 0} \right| < \frac{\varepsilon}{4}$$

so that

$$\int_\beta^\infty (e^t - 1)^{-1} t^{x-1} dt < \frac{\varepsilon}{4}. \tag{2.36}$$

Since $|e^{-t}| < 1$ when $t > 0$ (Fig. 2.5),

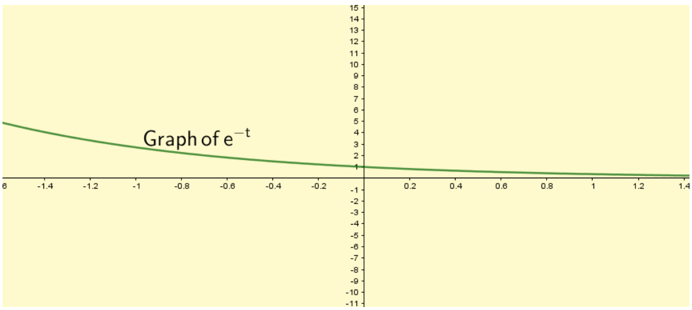

Figure 2.5:

$$\sum_{k=1}^\infty e^{-kt}$$

is a geometric series with first term e^{-t} and common ratio e^{-t} with its modulus < 1, so that the geometric series converges to

$$\frac{e^{-t}}{1-e^{-t}} = \frac{1}{e^t - 1} = (e^t - 1)^{-1}$$

so the nth partial sum

$$\sum_{k=1}^{n} e^{-kt} \leq \sum_{k=1}^{\infty} e^{-kt} = (e^t - 1)^{-1}$$

for all $n \geq 1$. From

$$\sum_{k=1}^{n} e^{-kt} \leq (e^t - 1)^{-1}$$

(multiplying with $t^{x-1} > 0$) we have

$$\sum_{k=1}^{n} e^{-kt} t^{x-1} \leq (e^t - 1)^{-1} t^{x-1}$$

which gives

$$\int_0^\alpha \left(\sum_{k=1}^{n} e^{-kt} t^{x-1} \right) dt \leq \int_0^\alpha (e^t - 1)^{-1} t^{x-1} dt.$$

By the Sum Rule of Integration of Definite Integrals,

$$\int_0^\alpha \left(\sum_{k=1}^{n} e^{-kt} t^{x-1} \right) dt = \sum_{k=1}^{n} \int_0^\alpha e^{-kt} t^{x-1} dt$$

and hence

$$\sum_{k=1}^{n} \int_{0}^{\alpha} e^{-kt} t^{x-1} dt \leq \int_{0}^{\alpha} (e^t - 1)^{-1} t^{x-1} dt$$

and letting $n \to \infty$, the above gives

$$\sum_{k=1}^{\infty} \int_{0}^{\alpha} e^{-kt} t^{x-1} dt \leq \int_{0}^{\alpha} (e^t - 1)^{-1} t^{x-1} dt.$$

Hence, using (2.35),

$$\sum_{k=1}^{\infty} \int_{0}^{\alpha} e^{-kt} t^{x-1} dt < \frac{\varepsilon}{4}.$$

Similarly, using (2.36),

$$\sum_{k=1}^{\infty} \int_{\beta}^{\infty} e^{-kt} t^{x-1} dt < \frac{\varepsilon}{4}.$$

Using Equation (2.22) yields (in the below we have used the fact that for a convergent series of *positive terms*, interchange of terms will not affect the sum)

$$\left| \zeta(x)\Gamma(x) - \int_{0}^{\infty} (e^t - 1)^{-1} t^{x-1} dt \right|$$

$$
\begin{aligned}
&= \left| \sum_{k=1}^{\infty} \int_0^{\infty} e^{-kt} t^{x-1} dt - \int_0^{\infty} (e^t - 1)^{-1} t^{x-1} dt \right| \\
&= \left| \sum_{k=1}^{\infty} \int_0^{\alpha} e^{-kt} t^{x-1} dt + \sum_{k=1}^{\infty} \int_{\alpha}^{\beta} e^{-kt} t^{x-1} dt \right. \\
&\quad + \sum_{k=1}^{\infty} \int_{\beta}^{\infty} e^{-kt} t^{x-1} dt - \int_0^{\alpha} (e^t - 1)^{-1} t^{x-1} dt \\
&\quad \left. - \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{x-1} dt - \int_{\beta}^{\infty} (e^t - 1)^{-1} t^{x-1} dt \right| \\
&\leq \underbrace{\left| \sum_{k=1}^{\infty} \int_0^{\alpha} e^{-kt} t^{x-1} dt \right|}_{<\varepsilon/4} + \underbrace{\left| \sum_{k=1}^{\infty} \int_{\beta}^{\infty} e^{-kt} t^{x-1} dt \right|}_{<\varepsilon/4} \\
&\quad + \underbrace{\left| \int_0^{\alpha} (e^t - 1)^{-1} t^{x-1} dt \right|}_{<\varepsilon/4} + \underbrace{\left| \int_{\beta}^{\infty} (e^t - 1)^{-1} t^{x-1} dt \right|}_{<\varepsilon/4} \\
&\quad + \left| \sum_{k=1}^{\infty} \int_{\alpha}^{\beta} e^{-kt} t^{x-1} dt - \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{x-1} dt \right| \\
&\leq \varepsilon + \left| \sum_{k=1}^{\infty} \int_{\alpha}^{\beta} e^{-kt} t^{x-1} dt - \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{x-1} dt \right|
\end{aligned}
$$

But $\sum\limits_{k=1}^{\infty} e^{-kt}$ converges to $(e^t - 1)^{-1}$ uniformly on the compact set $[\alpha,\ \beta]$, so that (interchange of summation and integration are justifiable)

$$\sum_{k=1}^{\infty} \int_{\alpha}^{\beta} e^{-kt} t^{x-1} dt = \int_{\alpha}^{\beta} \sum_{k=1}^{\infty} e^{-kt} t^{x-1} dt = \int_{\alpha}^{\beta} (e^t - 1)^{-1} t^{x-1} dt$$

and hence

$$\left| \zeta(x)\Gamma(x) - \int_{0}^{\infty} (e^t - 1)^{-1} t^{x-1} dt \right| \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary this shows that

$$\zeta(x)\Gamma(x) = \int_{0}^{\infty} (e^t - 1)^{-1} t^{x-1} dt.$$

This completes the proof. $\square$

2.3.1 Extending the Domain of the Riemann Zeta Function ζ to $\{z : \mathrm{Re} z > 0\}$

We wish to use Proposition 2.3.6 to extend the domain of definition of ζ to $\{z : \mathrm{Re} z > -1\}$ (and eventually to all of the complex plane $\mathbb{C}$.) . To do this, consider the Laurent expansion of $(e^z - 1)^{-1}$ given by

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} a_n z^n \tag{2.37}$$

for some constants $a_1, a_2, \ldots$. [14] The right hand side is a Taylor series and hence continuous. Hence if take a closed neighborhood K of 0, the right hand side is bounded by some $M > 0$ on the neighborhood K. In particular, the right hand side is bounded on the neighborhood K^0 (interior of K). Hence, in particular, $\frac{1}{e^t - 1} - \frac{1}{t}$ remains bounded in a neighborhood of $t = 0$ (an interval containing 0). But this implies that

[14] Explanation: The Maclaurin Series of e^z (that is valid for all complex numbers z) is

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$$

which implies

$$e^z - 1 = \frac{z}{1!} + \frac{z^2}{2!} + \cdots.$$

Hence (for $z \neq 0$)

$$\begin{aligned}
\frac{1}{e^z - 1} &= \frac{1}{z\left(\frac{1}{1!} + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)} \\
&= \frac{1}{z}\left[1 + \left(\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)\right]^{-1} \\
&= \frac{1}{z}\left\{1 - \left(\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right) + \left(\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)^2 - \left(\frac{z}{2!} + \frac{z^2}{3!} + \cdots\right)^3 + \cdots\right\} \\
&= \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} a_n z^n
\end{aligned}$$

the integral

$$\int_0^1 \left(\frac{1}{e^t-1}-\frac{1}{t}\right)t^{z-1}dt$$

[15]

---

[15] (In the above we need to be careful about the convergence of the series expansion of $\left[1+\left(\frac{z}{2!}+\frac{z^2}{3!}+\cdots\right)\right]^{-1}$. **Aliter:** Let $f(z)=(e^z-1)^{-1}$ which has a simple pole at $z=0$ (easy enough to see). Consider

$$h(z)=\frac{e^z-1}{z}=\frac{1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots}{z}=1+\frac{z}{2!}+\frac{z^2}{3!}+\cdots=\sum_{n=0}^{\infty}\frac{z^n}{(n+1)!}.$$

h is an entire function (prove it to yourself!). Now let

$$g(z)=\frac{1}{h(z)}$$

which is analytic over some area. Now

$$f(z)=\frac{1}{zh(z)}=\frac{g(z)}{z}=\underbrace{\sum_{n=-\infty}^{\infty}a_nz^n}_{\substack{\text{Laurent series}\\ \text{expansion}}}\,. \tag{2.38}$$

Furthermore, a_n can be found from

$$a_n=\frac{1}{2\pi i}\int_{|z|=R}\frac{f(z)}{z^{n+1}}dz=\frac{1}{2\pi i}\int_{|z|=R}\frac{g(z)}{z^{n+2}}dz\,.$$

g is analytic on the inside of $|z|=R$. Hence by Cauchy's Theorem, $a_n=0$ for $n\le -2$ (because when $n\le -2$, $\frac{g(z)}{z^{n+2}}$ is analytic). Hence (2.38) gives

$$f(z)=\sum_{n=-1}^{\infty}a_nz^n$$

where for $n\ge -1$,

$$a_n=\frac{1}{2\pi i}\int_{|z|=R}\frac{f(z)}{z^{n+1}}dz=\frac{1}{2\pi i}\int_{|z|=R}\frac{g(z)}{z^{n+2}}dz=\frac{g^{(n+1)}(0)}{(n+1)!}.$$

converges uniformly on compact subsets of the right half

Now compute the first few a_n. To find the derivative of g, we should first find the derivative of h. For $k \geq 0$,

$$h^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n! z^{n-k}}{(n-k)!(n+1)!}$$

and hence for all $k \geq 0$

$$h^{(k)}(0) = \frac{1}{k+1}. \tag{2.39}$$

It can be seen that

$$1 = g(z)h(z)$$

so, differentiating

$$0 = g'h + gh'$$

and on successive differentiation

$$0 = \sum_{i=0}^{k} \binom{k}{i} h^{(k-i)} g^{(i)}.$$

Using (2.39), $h^{(k-i)}(0) = \frac{1}{k-i+1}$ and hence the above equation becomes **(Verify)**

$$0 = \sum_{i=0}^{k} \binom{k}{i} \frac{1}{k-i+1} g^{(i)}(0) = \sum_{i=0}^{k} \binom{k+1}{i} g^{(i)}(0) \tag{2.40}$$

Going back to the coefficients a_n, we have for $n \geq -1$,

$$a_n = \frac{1}{2\pi i} \int_{|z|=R} \frac{g(z)}{z^{n+2}} dz = \frac{g^{(n+1)}(0)}{(n+1)!}. \tag{2.41}$$

Hence (2.40) becomes

$$0 = \sum_{k=0}^{n} \frac{a_{k-1}}{(n-(k-1))!}. \tag{2.42}$$

From (2.41),

$$a_{-1} = \frac{g^{(0)}(0)}{(-1+1)!} = \frac{g(0)}{0!} = \frac{1}{h(0)} = 1$$

and using this, (2.42) gives

$$\frac{a_{-1}}{2!} + \frac{a_0}{1} = 0 \Rightarrow a_0 = -\frac{1}{2}.$$

plane $\{z : \mathrm{Re} z > 0\}$ [17] and therefore represents an analytic function there. Hence, noting that for $z \neq 1$,

$$\int_0^1 t^{z-2} dt = \left[\frac{t^{z-1}}{z-1}\right]_0^1 = \frac{1}{z-1} \quad (Justify\ this!)$$

All positive odd terms are zero[16] Let

$$B_k = (-1)^{k-1}(2k!)a_{2k-1}$$

be Bernoulli numbers. Note that

$$F(z) = \frac{1}{e^z - 1} - \frac{1}{z} + \frac{1}{2}$$

is an odd function. Therefore,

$$f(z) = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} a_{2k-1} z^{2k-1} = \frac{1}{z} - \frac{1}{2} + \sum_{k=1}^{\infty} (-1)^k \frac{B_k}{(2k!)} z^{2k-1}.$$

As

$$\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} a_n z^n$$

and hence

$$\frac{1}{e^z - 1} - \frac{1}{z} = -\frac{1}{2} + \sum_{n=1}^{\infty} a_n z^n.$$

[17] Why?

we obtain

$$\begin{aligned}\zeta(z)\Gamma(z) &= \int_0^1 \frac{t^{z-1}}{e^t-1}dt + \int_1^\infty \frac{t^{z-1}}{e^t-1}dt \\ &= \underbrace{\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}dt}_{\text{analytic}} + \underbrace{(z-1)^{-1}}_{\text{NOT analytic at } z=1} \\ &\quad + \underbrace{\int_1^\infty \frac{t^{z-1}}{e^t-1}dt}_{\text{analytic}} \end{aligned} \tag{2.43}$$

In the above (using Corollary 2.3.3 (b)) each of these summands, except $(z-1)^{-1}$, is analytic in the right half plane. Thus one may define $\zeta(z)$ for $\{z : \mathrm{Re} z > 0\}$ by setting it equal to $[\Gamma(z)]^{-1}$ times the right hand side of (2.43). That is,

$$\zeta(z) =$$

$$\frac{1}{\Gamma(z)}\left\{\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}dt + (z-1)^{-1} + \int_1^\infty \frac{t^{z-1}}{e^t-1}dt\right\}, \tag{2.44}$$

$z \neq 1,\ \ \mathrm{Re} z > 0.$

In this manner ζ is meromorphic in the right half plane with a simple pole at $z = 1$ ($\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} n^{-1}$ diverges) whose residue is 1 (being the coefficient of $(z-1)^{-1}$) in the expansion of $\zeta(z)$ in 2.44 (Fig. 2.6).

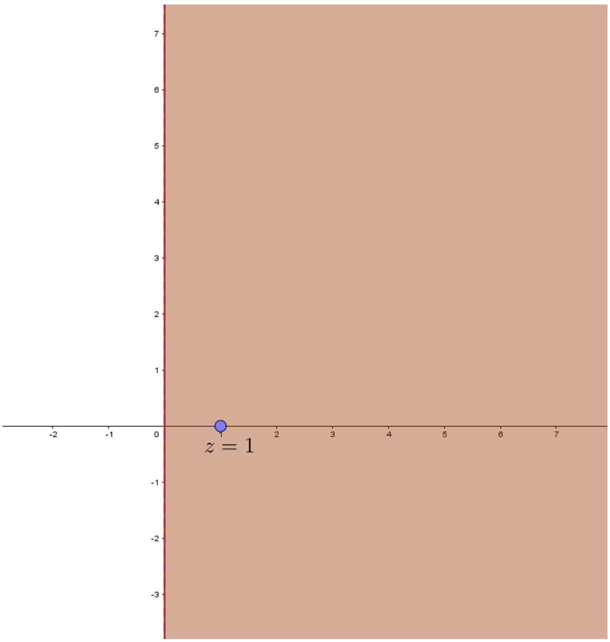

Figure 2.6: ζ is mereomorphic function on the right half plane with a simple pole at $z = 1$ ($\zeta(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} n^{-1}$ diverges) whose residue is 1

2.3.2 Extending the Domain of the Riemann Zeta Function ζ to $\{z : \mathrm{Re}z > -1\}$

Now suppose $0 < \mathrm{Re}z < 1$; then

$$\begin{aligned}\int_1^\infty t^{z-2}dt &= \lim_{b\to\infty}\left[\frac{t^{z-1}}{z-1}\right]_1^b = \frac{\lim\limits_{b\to\infty} e^{(z-1)\ln b} - 1}{z-1} \\ &= \frac{\lim\limits_{b\to\infty} e^{(\mathrm{Re}z-1+i\ \mathrm{Im}z)\ln b} - 1}{z-1}\end{aligned}$$

Now

$$e^{(\mathrm{Re}z-1+i\ \mathrm{Im}z)\ln b} = e^{(\mathrm{Re}z-1)\ln b}e^{i\ \mathrm{Im}z\ln b}$$

and (from $0 < \mathrm{Re}z < 1$) $-1 < (\mathrm{Re}z - 1) < 0$ so that

$$\lim_{b\to\infty} e^{(\mathrm{Re}z-1)\ln b} = 0$$

Informally, the above is $\lim\limits_{d\to\infty} e^{-d} = 0$. Also,

$$\left|e^{i\ \mathrm{Im}z\ln b}\right| = 1.$$

Thus,

$$\lim_{b\to\infty} e^{(\mathrm{Re}z-1+i\ \mathrm{Im}z)\ln b} = 0$$

and hence

$$\int_1^\infty t^{z-2}dt = -\frac{1}{z-1}.$$

Applying this to Equation (2.43) gives

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}dt - \int_1^\infty t^{z-2}dt + \int_1^\infty \frac{t^{z-1}}{e^t-1}dt$$

$$= \int_0^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1}dt \quad (0 < \mathrm{Re} z < 1). \qquad (2.45)$$

Again considering the Laurent expansion of $(e^z - 1)^{-1}$ in (2.37) we see that

$$\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} \leq ct$$

for some constant c and all t in the unit interval $[0, 1]$ [18] Thus the integral

$$\int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t} - \frac{1}{2}\right) t^{z-1}dt$$

is uniformly convergent on compact subsets of $\{z : Rez > -1\}$ [19]

[18] Why? Explain!

[19] *Explanation:* On $\{z : \mathrm{Re} z > -1\}$,

$$\left|\left(\frac{1}{e^t-1} - \frac{1}{t} - \frac{1}{2}\right) t^{z-1}\right| \leq c\,|t|\,\left|t^{z-1}\right| = c\,|t^z| \leq ct^{\mathrm{Re} z}$$

Also, since[20]

$$\lim_{t\to\infty} t\left(\frac{1}{e^t-1}-\frac{1}{t}\right)=1$$

there is a constant c' such that

$$\left(\frac{1}{e^t-1}-\frac{1}{t}\right)\le\frac{c'}{t},\quad t\ge 1.$$

as

$$\begin{aligned}
|t^z| &= \left|t^{\mathrm{Re}z+i\ \mathrm{Im}z}\right| = \left|t^{\mathrm{Re}z}t^{i\ \mathrm{Im}z}\right| = \left|t^{\mathrm{Re}z}\right|\left|t^{i\ \mathrm{Im}z}\right| \\
&= \left|\underbrace{t^{\mathrm{Re}z}}_{\ge 0}\right|\underbrace{\left|e^{i\ \mathrm{Im}z\ln t}\right|}_{1} = t^{\mathrm{Re}z}.
\end{aligned}$$

We know that $\int\limits_a^b \frac{dx}{(x-a)^p}$ converges if $p<1$ and diverges if $p\ge 1$. In particular, $\int\limits_0^1 \frac{dt}{t^p}$ converges if $p<1$. Hence for $a>-1$, $(-a<1)$

$$\int\limits_0^1 t^a dt=\int\limits_0^1 \frac{dt}{t^{-a}}$$

converges. Hence

$$\int\limits_0^1 t^{\mathrm{Re}z}$$

converges when $\mathrm{Re}z > -1$. Hence by Weierstrass M-Test, $\int\limits_0^1\left(\frac{1}{e^t-1}-\frac{1}{t}-\frac{1}{2}\right)t^{z-1}dt$ is uniformly convergent on compact subsets of $\{z:\mathrm{Re}z>-1\}$ (Explain!)

[20]Why? Explain.

[21]

The above is the definition of finite limit at infinity of complex valued functions. Using the analogue for the real valued function of real variable,

$$\lim_{t\to\infty} t\left(\frac{1}{e^t-1}-\frac{1}{t}\right)=1$$

if and only if for a given $C>0$ there is a $\delta>0$ such that

$$\left|\underbrace{t\left(\frac{1}{e^t-1}-\frac{1}{t}\right)}_{f(t)}-\underbrace{1}_{L}\right|<C \quad \forall t\geq\delta.$$

$$-C<\underbrace{t\left(\frac{1}{e^t-1}-\frac{1}{t}\right)}_{f(t)}-\underbrace{1}_{L}<C \quad \forall t\geq\delta$$

$$-C+1<t\left(\frac{1}{e^t-1}-\frac{1}{t}\right)<C+1 \quad \forall t\geq\delta$$

---

[21]**Limit at Infinity:** The statement

$$\lim_{z\to\infty} f(z)=w_0$$

means that, for each $\varepsilon>0$ there is a $\delta>0$ such that

$$|f(z)-w_0|<\varepsilon \quad \text{whenever} \quad |z|>\frac{1}{\delta}.$$

That is, the point $w=f(z)$ lies in the ε neighborhood of w_0 whenever z lies in the deleted neighborhood $|z|>\frac{1}{\delta}$ of ∞.

$$-C' < t\left(\frac{1}{e^t-1}-\frac{1}{t}\right) < C' \quad \forall t \geq \delta$$

where

$$C' = \max\{|-C+1|\,,\ C+1\}$$

so that

$$\left|t\left(\frac{1}{e^t-1}-\frac{1}{t}\right)\right| < C' \quad \forall t \geq \delta.$$

Choose

$$C'' = \sup\left\{\left|t\left(\frac{1}{e^t-1}-\frac{1}{t}\right)\right| : t \in [1,\ \delta]\right\}.$$

Take

$$c' = \max\{C',\ C''\}$$

then

$$\left|t\left(\frac{1}{e^t-1}-\frac{1}{t}\right)\right| \leq c', \quad t \geq 1.$$

Hence

$$\left|\frac{1}{e^t-1}-\frac{1}{t}\right| \leq \frac{c'}{t}, \quad t \geq 1.$$

This gives that the integral

$$\int_1^\infty \left(\frac{1}{e^t-1}-\frac{1}{t}\right) t^{z-1} dt$$

converges uniformly on compact subsets of $\{z : \mathrm{Re} z < 1\}$.

Using these last two integrals with equation (2.45) gives

$$\zeta(z)\Gamma(z) = \int_0^1 \left(\frac{1}{e^t-1} - \frac{1}{t} - \frac{1}{2}\right) t^{z-1} dt - \frac{1}{2z} + \int_1^\infty \left(\frac{1}{e^t-1} - \frac{1}{t}\right) t^{z-1} dt \quad (2.46)$$

for $0 < \text{Re} z < 1$. But since both integrals converge in the strip $-1 < \text{Re} z < 1$ (2.46) can be used to define $\zeta(z)$ in $\{z : -1 < \text{Re} z < 1\}$. What happens at $z = 0$? Since the term $(2z)^{-1}$ appears on the right hand side of (2.46) **will ζ have a pole at $z = 0$?. The answer is no.** To define $\zeta(z)$ we must divide (2.46) by $\Gamma(z)$. When this happens the term $\frac{1}{2z}$ in question becomes

$$\frac{1}{2z\Gamma(z)} = \frac{1}{2\Gamma(z+1)}$$

(since $\Gamma(z+1) = z\Gamma(z)$) which is analytic at $z = 0$ (See the discussion of the gamma function). Thus,if ζ is so defined in the strip $\{z : -1 < \text{Re} z < 1\}$ it is analytic there (Fig. 2.7). If this is combined with (2.43), $\zeta(z)$ is defined for $\text{Re} z > -1$ with a simple pole at $z = 1$ (Fig. 2.8).

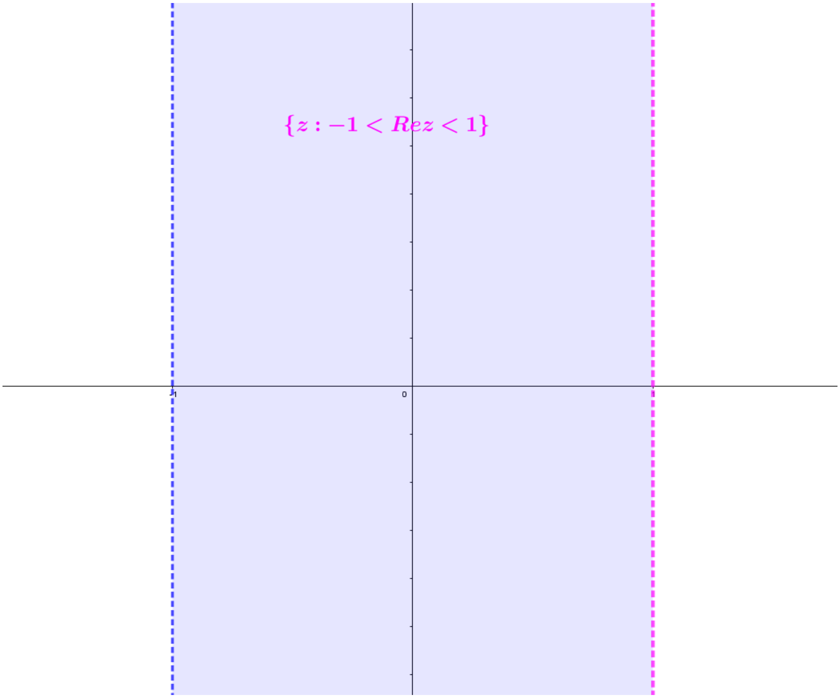

Figure 2.7:

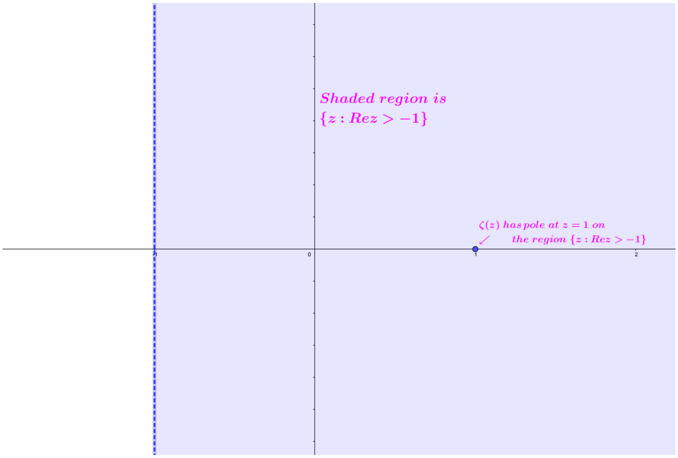

Figure 2.8:

Theorem 2.3.7. ***Riemann's Functional Equation.***

$$\zeta(z)\Gamma(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin(\tfrac{1}{2}\pi z) \tag{2.47}$$

for $-1 < \mathrm{Re} z < 0$.

Proof. We prove (2.47) for x real and in $(-1, 0)$; but since both sides of (2.47) are analytic in the strip $-1 < \mathrm{Re} z < 0$, (2.47) follows [22]
Now if $-1 < \mathrm{Re} z < 1$ then

$$\int_1^\infty t^{z-1} dt = -\frac{1}{z}. \tag{2.48}$$

[22] Exercise: Give details.

Inserting (2.48) in (2.46) gives

$$\zeta(z)\Gamma(z) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} - \frac{1}{2}\right) t^{z-1} dt, \quad -1 < \mathrm{Re} z < 0. \tag{2.49}$$

But

$$\frac{1}{e^t - 1} + \frac{1}{2} = \frac{2 + e^t - 1}{(e^t - 1)(2)} = \frac{1}{2}\left(\frac{e^t + 1}{e^t - 1}\right) = \frac{i}{2} \cot\left(\frac{1}{2} it\right).$$

A straightforward computation gives

$$\cot\left(\frac{1}{2} it\right) = \frac{2}{it} - 4it \sum_{n=1}^\infty \frac{1}{t^2 + 4n^2\pi^2}$$

for $t \neq 0$. Thus

$$\left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) \frac{1}{t} = 2 \sum_{n=1}^\infty \frac{1}{t^2 + 4n^2\pi^2}.$$

Applying this to (2.49) gives

$$\begin{aligned}\zeta(z)\Gamma(z) &= 2\int_0^\infty \left(\sum_{n=1}^\infty \frac{1}{t^2+4n^2\pi^2}\right) t^z dt \\ &= 2\sum_{n=1}^\infty \int_0^\infty \frac{t^z}{t^2+4n^2\pi^2} dt = 2\sum_{n=1}^\infty (2\pi n)^{z-1} \int_0^\infty \frac{t^z}{t^2+1} dt \\ &= 2(2\pi)^{z-1}\zeta(1-z)\int_0^\infty \frac{t^z}{t^2+1} dt \qquad (2.50)\end{aligned}$$

for $-1 < \text{Re}z < 0$ [23] Now for x a real number with $-1 < x < 0$, the change of variable gives [24]

$$\begin{aligned}\int_0^\infty \frac{t^x}{t^2+1} dt &= \frac{1}{2}\int_0^\infty \frac{s^{\frac{1}{2}(x-1)}}{s+1} dt \\ &= \frac{1}{2}\pi \text{cosec}[\tfrac{1}{2}\pi(1-x)] \\ &= \frac{1}{2}\pi \sec(\tfrac{1}{2}\pi x) \qquad (2.51)\end{aligned}$$

Using

$$\sqrt{\pi}\,\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma(z+\tfrac{1}{2})$$

[23] Exercise: In the above, justify the interchanging of the sum and the integral.

[24] Exercise: Give details.

we obtain

$$\frac{1}{\Gamma(x)} = \frac{\Gamma(1-x)}{\pi}\sin \pi x = \frac{\Gamma(1-x)}{\pi}\left[2\sin(\tfrac{1}{2}\pi x)\cos(\tfrac{1}{2}\pi x)\right].$$

Combining this with (2.50) and (2.51) yields the result. □

Theorem 2.3.8. *The zeta function can be defined to be meromorphic in the plane with only a simple pole at $z = 1$ and* $\mathrm{Res}\,(\zeta,\ 1) = 1$. *For $z \neq 1$, ζ satisfies Riemann's functional equation.*

Proof. The same type of reasoning as in the proof of the above result gives that (2.47) holds for $-1 < \mathrm{Re} z < 1$ (what happens at $z = 0$?). But we wish to do more than this. We notice that the right hand side of (**??**)is analytic in the left hand plane $\mathrm{Re} z < 0$. Thus, use (2.47) to extend definition of $\zeta(z)$ to $\mathrm{Re} z < 0$. This completes the proof. □

Since the gamma function $\Gamma(1-z)$ has a pole at $z = 1, 2, \ldots$ and since ζ is analytic at $z = 2, 3, \ldots$ we know, from Riemann's functional equation that

$$\zeta(1-z)\sin(\tfrac{1}{2}\pi z) = 0 \tag{2.52}$$

for $z = 2, 3, \ldots$ Furthermore, since the pole of $\Gamma(1-z)$ at $z = 2, 3, \ldots$ is simple, each of the zeroes of (2.52) must be simple. Since

$$\sin(\tfrac{1}{2}\pi z) = 0$$

whenever z is an even integer,

$$\zeta(1-z) = 0$$

for $z = 3, 5, \ldots$. That is,

$$\zeta(z) = 0$$

for $z = -2, -4, -6, \ldots$. for . Similar reasoning gives that ζ has no other zeroes outside the closed strip $\{z : 0 \leq \mathrm{Re}z \leq 1\}$.

Definition 2.3.9. The points $z = -2, -4, -6, \ldots$ are called the **trivial zeroes** of ζ and the strip $\{z : 0 \leq \mathrm{Re}z \leq 1\}$ is called the **critical strip.**

We now are in a position to state one of the most celebrated open questions in all of Mathematics. Is the following true?

Hypothesis 2.3.10. ***The Riemann Hypothesis.*** *If z is a zero of the zeta function ζ in the critical strip $\{z : 0 \leq \mathrm{Re}z \leq 1,\}$ then* $\mathrm{Re}z = \frac{1}{2}$.

It is known that there are no zeroes of ζ on the line $\mathrm{Re}z = 1$ (and hence none on $\mathrm{Re}z = 0$ by the functional equation) and there are an infinite number of zeroes on the line $\mathrm{Re}z = \frac{1}{2}$. But no one has been able to show that ζ has any zeroes off the line $\mathrm{Re}z = \frac{1}{2}$ and ***no one has been able to show that***

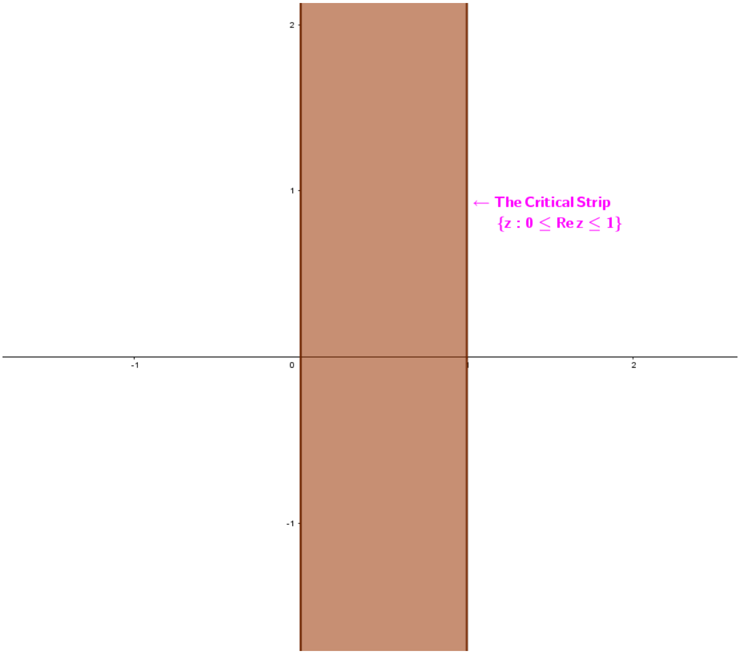

Figure 2.9:

all zeroes must lie on the line.

A positive resolution of the Riemann Hypothesis will have numerous beneficial effects on number theory. Perhaps the best way to realize the connection between the zeta function and number theory is to prove the following theorem.

Theorem 2.3.11. ***Euler's Theorem.*** *If* $\mathrm{Re}z > 1$ *then*

$$\zeta(z) = \prod_{n=1}^{\infty} \left(\frac{1}{1 - p_n^{-z}} \right) \tag{2.53}$$

where $\{p_n\}$ *is the sequence of prime numbers.*

Proof. For $\mathrm{Re}z > 1$, and for $n = 1, 2, \ldots,$

$$\left| p_n^{-z} \right| = \left| e^{-(\mathrm{Re}z + i\,\mathrm{Im}z)\ln p_n} \right| = \underbrace{\left| e^{-\mathrm{Re}z \ln p_n} \right|}_{0 < p_n^{-\mathrm{Re}z} < 1} \underbrace{\left| e^{i\,(-\mathrm{Im}z \ln p_n\,)} \right|}_{1} < 1$$

so that the geometric series

$$\sum_{m=0}^{\infty} p_n^{-mz} = \sum_{m=0}^{\infty} (p_n^{-z})^m$$

with common ratio p_n^{-z} (having modulus less than) converges to $\frac{1}{1-p_n^{-z}}$. That is,

$$\frac{1}{1 - p_n^{-z}} = \sum_{m=0}^{\infty} p_n^{-mz} \tag{2.54}$$

for all $n \geq 1$. Now if $n \geq 1$ and we take the product of the terms $(1 - p_k^{-z})^{-1}$ for $1 \leq k \leq n$, then by the *distributive law of multiplication* and by (2.54),

$$\prod_{k=1}^{n} \left(\frac{1}{1 - p_k^{-z}} \right) = \prod_{k=1}^{n} \sum_{j=0}^{\infty} p_k^{-jz}$$

$$= \sum_{j=0}^{\infty} \prod_{k=1}^{n} p_k^{-jz} = \sum_{j=0}^{\infty} \underbrace{\left(\prod_{k=1}^{n} p_k^{j} \right)}_{n_j}^{-z} = \sum_{j=1}^{\infty} n_j^{-z}$$

when the integers $n_1, n_2, \ldots$ **are all the integers** (noting that $n_0 = \left(\prod_{k=1}^{n} p_k^0 \right) = 1$) **which can be factored as a product of powers of the prime numbers** $p_1, \ldots, p_n$ **alone.** (The reason that no numbers has a coefficient in this expansion other than 1 is that the factorisation of n_j into the product of primes is unique[25].) By letting $n \to \infty$ the result is achieved. $\square$

2.4 Runge's Theorem

We know that an analytic function in an open disk is given by a power series. Furthermore, on proper subdisks the power series converges uniformly to the function. As a corollary to this result, we have:

[25] Explain!

[26] [27]

Corollary 2.4.1. *An analytic function on a disk D is the limit in $H(D)$ of a sequence of polynomials.*

Proof. Let D be an open disk. Then there is an $a \in D$ and $R > 0$ such that $D = B(a; R)$. Then, by THEOREM IV.2.8 (in the footnote 1 above),

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{for} \quad |z-a| < R$$

[26]**Theorem IV.2.8: [Power series Representation of an Analytic Function over an *Open Ball* in $\mathbb{C}$]** Let f be analytic in $B(a; R)$; then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{for} \quad |z-a| < R$$

where the coefficients of the power series on the right hand side are given by

$$a_n = \frac{1}{n!} \underbrace{f^{(n)}(a)}_{\substack{n\text{thderivativeof} f \\ \text{evaluatedat } a}} \qquad n = 0, 1, \ldots$$

and **the power series has radius of convergence** $\geq R$.

[27]This follows from Part (c) of the following Theorem: **Theorem III.1.3:** For the power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ with radius of convergence R given by

$$\frac{1}{R} = \limsup |a_n|^{1/n},$$

(a) if $|z-a| < R$, the series converges absolutely;

(b) if $|z-a| > R$, the terms of the series becomes unbounded and so the series diverges;

(c) if $0 < r < R$ then the series converges uniformly on the closed disk $\{z : |z-a| \leq r\}$.

and, by Part (c) of THEOREM III.1.3 (in the footnote, and also refer Fig. 2.10 to see an illustration), the series on the right hand side converges uniformly on all compact subsets of D. That is, the sequence of polynomials $\{\sum_{n=0}^{k} a_n(z-a)^n\}$ converges to f uniformly on all compact subsets of D. Hence, by Part (b) of Proposition VII.1.10,[28] the sequence of polynomials $\{\sum_{n=0}^{k} a_n(z-a)^n\}$ converges to f in $(C(D, \mathbb{C}), \rho)$. As $H(D)$ is a closed subspace[29] of $(C(D, \mathbb{C}), \rho)$ it follows that the sequence of polynomials converges to f in $H(D)$. In other words, f is the limit in $H(D)$ of a sequence of polynomials[30]. $\square$

We ask the question : Can the result in the above Corollary be generalized to arbitrary regions G? The answer is no. As one might expect the counter-example is furnished by (Fig. 2.11)

$$G = \{z : 0 < |z| < 2\}.$$

If there exists a sequence of polynomials $\{p_n(z)\}$ that con-

[28] PROPOSITION VII.1.10:

(a) A set $\mathcal{O} \subset (C(G, \Omega), \rho)$ is open if and only if for each $f \in \mathcal{O}$ there is a compact set K and a $\delta > 0$ such that

$$\mathcal{O} \supset \{ g : d(f(z), g(z)) < \delta, \ z \in K \}. \tag{2.55}$$

(b) A sequence $\{f_n\}$ in $(C(G, \Omega), \rho)$ converges to f if and only if $\{f_n\}$ converges to f ***uniformly on all compact subsets of*** G.

[29] This follows by COROLLARY VII.2.3: ($H(G)$ is a complete metric space.

[30] Note that each polynomial is analytic and hence belongs to $H(D)$.

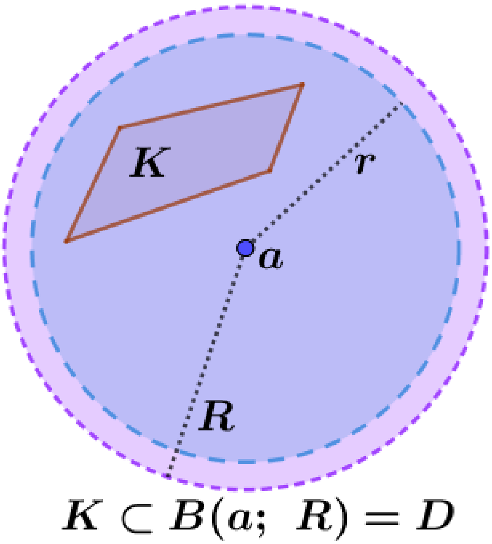

Figure 2.10: K is a compact subset of D and hence there is r, $0 < r < R$ such that $K \subset \overline{B(a;\, r)} \subset D$.

verges to an analytic function f on $G = \{z : 0 < |z| < 2\}$ (Fig. 2.11)(so that f is the limit in $H(G)$ of the sequence $\{p_n(z)\}$ of polynomials; so the convergence is uniform on compact subsets of G) and γ is the circle $|z| = 1$ (Fig. 2.11) then

$$\begin{aligned}
\int\limits_\gamma f &= \int\limits_\gamma \lim p_n \\
&= \lim \int\limits_\gamma p_n, \quad \text{interchange of limit and integration} \\
&\quad \text{is justified by the uniform convergence on} \\
&\quad \text{compact subsets of } G \text{ and for each } n, \text{ applying} \\
&\quad \text{Cauchy's theorem, } \int\limits_\gamma p_n = 0, \\
&\quad \text{since } p_n \text{ is analytic on } \mathbb{C} \\
&= 0.
\end{aligned}$$

The above **cannot be true** for every analytic function defined on G. For example, z^{-1} is in[31] $H(G)$ and $\int\limits_\gamma z^{-1} = 2\pi \neq 0$,[32]

[31] z^{-1} is in $H(G)$ because it is the ratio of the polynomials 1 and z and it is well-defined on $G = \{z : 0 < |z| < 2\}$ because it never becomes 0.

[32]

$$\int_\gamma (z)^{-1} dz = 2\pi i,$$

where γ is the curve given by the parametrization (being points on the unit circle $r = 1$)

$$\gamma(t) = e^{2\pi i\, t} \quad (0 \le t \le 1)$$

$dz = 2\pi i\; e^{2\pi i\, t} dt$ and therefore,

$$\int_\gamma \frac{dz}{z} = \int_0^1 \frac{2\pi i\; e^{2\pi i\, t}}{e^{2\pi i\, t}} dt \;=\; 2\pi i \int_0^1 dt \;=\; 2\pi i.$$

$\square$

so that there is no sequence of polynomials that converges to z^{-1} in $H(G)$.

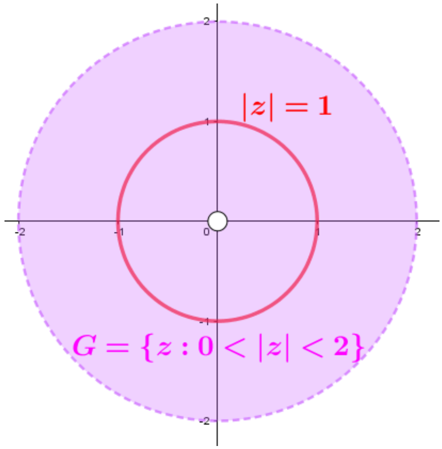

Figure 2.11: $G = \{z : 0 < |z| < 2\}$ is the annular disk centered at 0 and having radius 2. $\gamma(t) = r\, e^{2\pi i\, t}$ $(0 \leq t \leq 1)$ is the parametrization of the unit circle $|z| = 1$.

The fact that functions analytic on a disk are limits of polynomials is due to the fact that disks are ***simply connected****.*[33] If G is a punctured disk then the Laurent series development[34] shows that each analytic function on G is the

[33] An open set G is simply connected if G is connected and every closed curve in G is homotopic to zero. A simply connected set does not have a hole. An open set with hole is not simply connected.

[34] Laurent Series Development: Let f be analytic in the annulus

uniform limit of rational functions[35] whose poles lie outside G (in fact at the center of G). That is, each f in $H(G)$ is the limit of a sequence of rational functions which also belongs to $H(G)$. This is what can be generalized to arbitrary regions, and it is part of the content of Runge's theorem.

We begin by proving a version of the Cauchy Integral Formula. Unlike the former version, however, the next proposition says that there exists curves such that the formula holds; not that the formula holds for every curve.

Proposition 2.4.2. *Let K be a compact subset of the region G; then there are straight line segments $\gamma_1, \ \ldots \ , \gamma_n$ in $G-K$ such that for every function f in $H(G)$,*

$$f(z) = \sum_{k=1}^{n} \frac{1}{2\pi i} \int\limits_{\gamma_k} \frac{f(w)}{w-z} dw \text{ for all } z \text{ in } K. \tag{2.57}$$

---

ann$(a;\ R_1,\ R_2)$. Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$$

where the convergence in absolute and uniform over the closed annulus ann$(a;\ r_1,\ r_2)$ if $R_1 < r_1 < r_2 < R_2$. Also the coefficients a_n are given by the formula

$$a_n = \frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(z)}{(z-a)^{n+1}} dz \tag{2.56}$$

where γ is the circle $|z-a| = r$ for any r, $R_1 < r < R_2$. Moreover, this series is unique.

[35] of the form $\sum\limits_{n=-k}^{m} a_n (z-a)^n$

The line segments form a finite number of closed polygons.

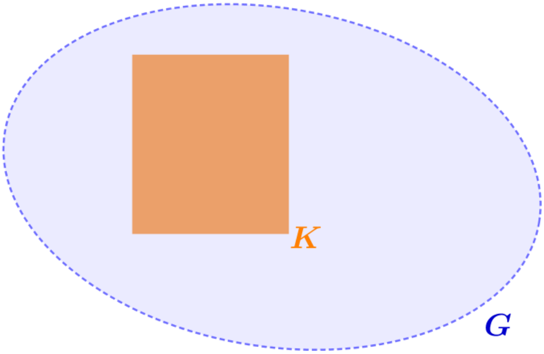

Figure 2.12: A case where $K = \overline{\text{int}K}$.

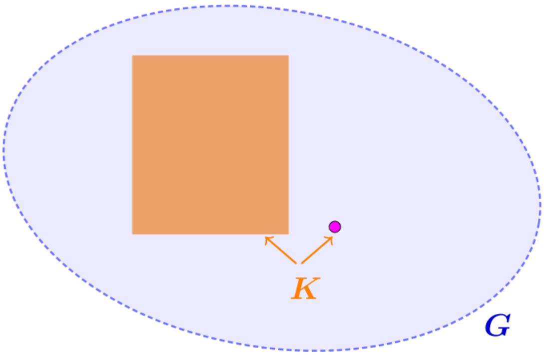

Figure 2.13: A case where $K \neq \overline{\text{int}K}$. This happens because interior of the point in magenta colour is empty. If we begin with a K as in this figure, then we may enlarge the same as in Fig. 2.14 or as in Fig. 2.15 to ensure that $K = \overline{\text{int}K}$.

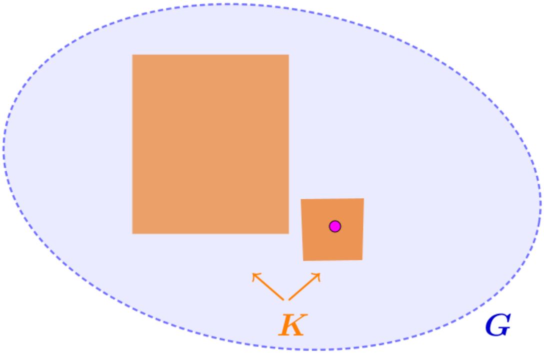

Figure 2.14: Ref. caption in Fig. 2.13

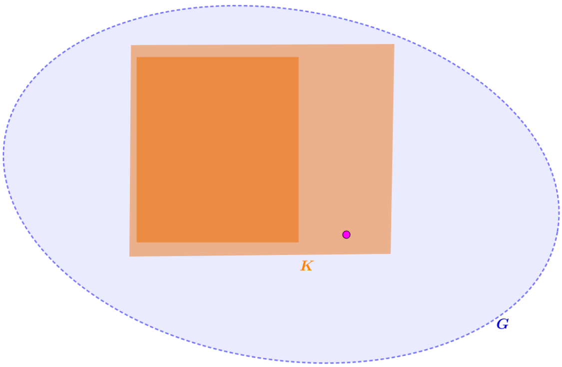

Figure 2.15: Ref. caption in Fig. 2.13

Proof. Observe that by enlarging[36] K a little we may assume

[36] A case where $K = \overline{\mathrm{int} K}$ is given in Fig. 2.12. A case where $K \neq \overline{\mathrm{int} K}$ is given in Fig. 2.13. The shortcoming in Fig. 2.13 can be solved as in Fig. 2.14 or as in Fig. 2.15.

that $K = \overline{\mathrm{int}K}$. Let[37] $0 < \delta < \frac{1}{2}d(K,\ C - G)$ [38] and place a *grid* of horizontal and vertical lines in the plane such that consecutive lines are less than a distance δ apart (A case is shown in Fig. 2.16). Let $R_1, \ldots, R_m$ be the resulting rectangles that intersect K (there are only a finite number of them because K is compact[39]). Also let ∂R_j be the boundary of R_j, $1 \leq j \leq m$, considered as polygon with the counter-clockwise direction (Fig. 2.17).

[37] G is open so $\mathbb{C} - G$ is closed; K is compact so it is closed and bounded and hence K is a proper closed subset of G and hence $(\mathbb{C} - G) \cap K = \emptyset$ and $d(K, \mathbb{C} - G) > 0$. For $d(K, \mathbb{C} - G) = 0$ will lead to the contradiction to the fact that K is a proper closed subset of the open set G.

[38] δ can be chosen such that $0 < \delta < \frac{1}{2}d(K,\ C - G)$ because $d(K, \mathbb{C} - G) > 0$ and using the density theorem of real numbers.

[39] K can be covered by rectangular laminas and collection of interior of such rectangular laminas form an open cover for the compact set K, and hence has a finite subcover, say, $\{\mathrm{int}R_1, \ldots, \mathrm{int}R_m\}$ so that $K \subset \bigcup_{i=1}^{m} \mathrm{int}R_i \subset \bigcup_{i=1}^{m} R_i$. Thus, $\{R_1, \ldots, R_m\}$ covers K.

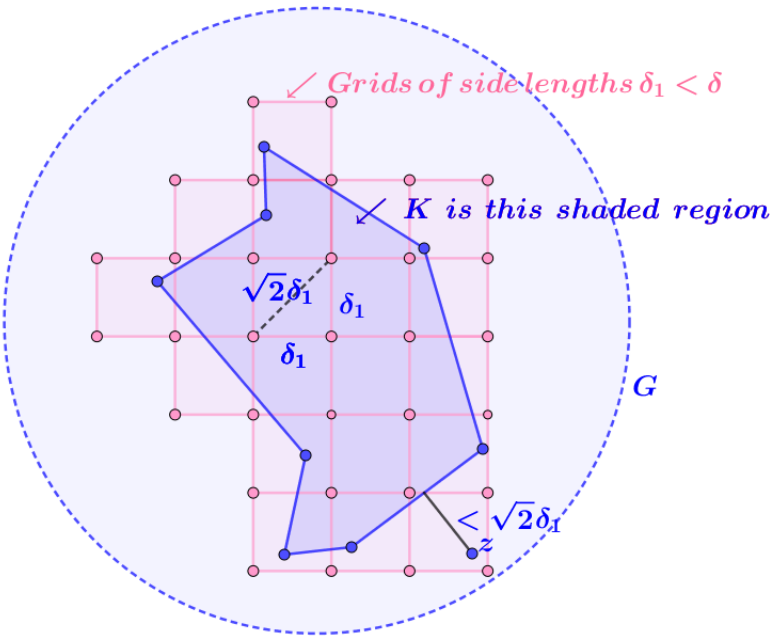

Figure 2.16: Compact set K is covered by grids $R_1, \ldots, R_n$ of side lengths $\delta_1 < \delta$ (there is no need to use the same δ_1 for all grids, any (positive) length less than δ is okay). Choose $z \in R_j$ (for some $j = 1, \ldots, n$). In the figure $z \in R_j$ but not in K (We have the freedom to choose z such that $z \in K \cap R_j$. In that case clearly $z \in G$.) If we consider the z as in the figure, then $d(z, K) < \sqrt{2}\delta_1$ (where $\sqrt{2}\delta_1$ is the length of the diagonal of the grid), and hence z not belongs to $\mathbb{C} - G$. Why? Because $\delta_1 < \delta < \frac{1}{2}d(K, C\backslash G)$ and since $d(z, K) < \sqrt{2}\delta_1 < \sqrt{2}\delta < \frac{\sqrt{2}}{2}d(K, C\backslash G) = \frac{1}{\sqrt{2}}d(K, C\backslash G) < d(K, C\backslash G)$ and $d(K, C\backslash G) = \inf\{|u - v| : u \in K, v \in C\backslash G\}$ and so z not belongs to $C\backslash G$. Thus, $z \in G$. That is every element of R_j is in G. This shows that each grid $R_j \subset G$

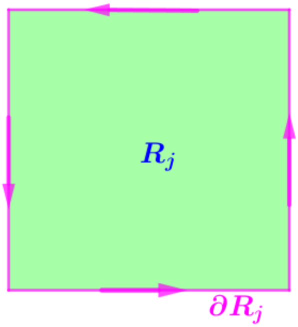

Figure 2.17:

If $z \in R_j$, $1 \leq j \leq m$, then[40] $d(z,\ K) < \sqrt{2}\delta$ so that $R_j \subset G$ by the choice of δ [41]. Also, many of the sides of the rectangles $R_1, \ldots, R_m$ will intersect. Suppose R_j and R_i have common side and let σ_j and σ_i be the line segments in ∂R_j, and ∂R_i respectively, such that $R_i \cap R_j = \{\sigma_j\} = \{\sigma_i\}$ (Fig. 2.18).[42]

[40] $d(z,\ K) < \sqrt{2}\delta < \frac{1}{\sqrt{2}} d(K,\ C - G) < d(K,\ C - G)$

[41] Details: **Case 1:** $z \in R_j$ and $z \in K$ then $d(z,\ K) = 0$ and then the inequality is obvious. **Case 2:** $z \in R_j$ and $z \notin K$, then situation as in Figure 2.16 occurs. It can be seen that $d(z,\ K)$ is less than the diagonal of the square lamina, which is less than $\sqrt{2}\delta_1$ (where δ_1 is the length of a side of the square lamina), which is less than $\sqrt{2}\delta$.

[42] $R_i \cap R_j = \{\sigma_j\} = \{\sigma_i\}$ means the trace of σ_j and the trace of σ_i are the same and is equal to $R_i \cap R_j$. *Do not read* $\{\sigma_j\}$ be the singleton set containing σ_j, but as trace of σ_j.

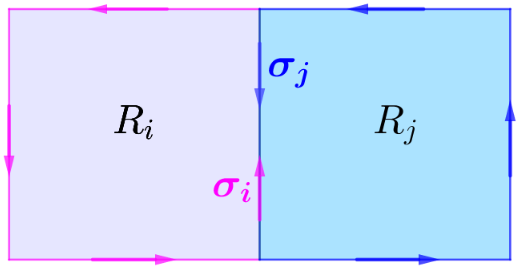

Figure 2.18: σ_j and σ_i are directed in the opposite sense

From the direction given to ∂R_j and ∂R_i, σ_j and σ_i are directed in the opposite sense. So if φ is any continuous function on $\{\sigma_j\}$, [43]

$$\int_{\sigma_j} \varphi + \int_{\sigma_i} \varphi = 0. \tag{2.58}$$

Let $\gamma_1, \ldots, \gamma_n$ be *those directed line segments that constitute a side of exactly one of* the R_j, $1 \leq j \leq m$ (Fig. 2.19). Thus

$$\sum_{k=1}^{n} \int_{\gamma_k} \varphi = \sum_{j=1}^{n} \int_{\partial R_j} \varphi \tag{2.59}$$

for every continuous function φ on $\bigcup_{j=1}^{m} \partial R_j$.[44]

[43] $\int_{\sigma_i} \varphi = -\int_{\sigma_j} \varphi$ since σ_i and σ_j give the same line segment but with opposite orientations.

[44] Reason: Sum of line integrals on other line segments will be zero due to (2.58).

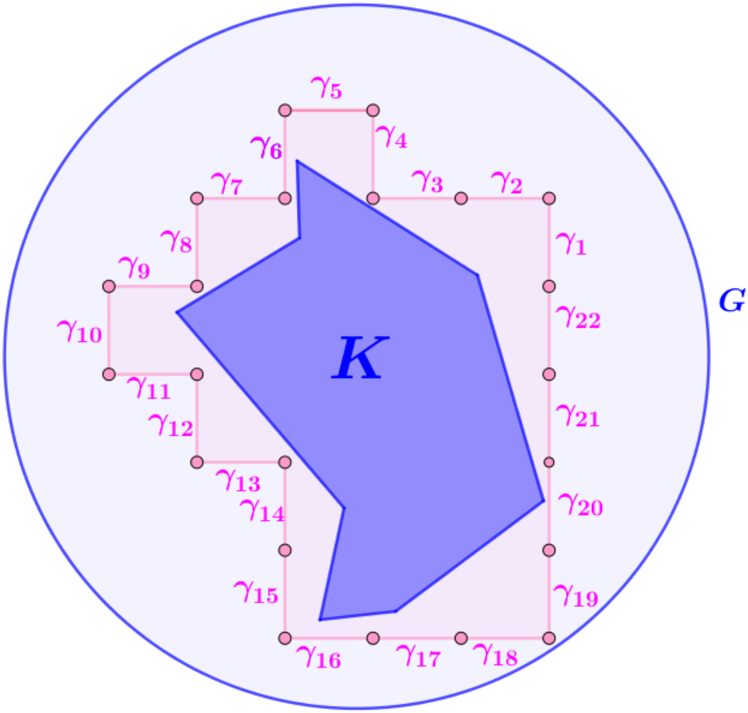

Figure 2.19: In continuation of Fig. 2.16. In the present figure, $\gamma_1, \ldots, \gamma_{22}$ are the *directed line segments that constitute a side of exactly one of* the R_j, $1 \leq j \leq m$.

Claim: Each γ_k, $k = 1, \ldots, n$ is in $G - K$ (Fig. 2.19).

Suppose not. If one of the γ_k, $k = 1, \ldots, n$ intersects K, by drawing a figure it can be seen that *there are two rectangles in the grid* with γ_k as a side and so both grids meet K. That is, γ_k is the common side of two of the rectangles

$R_1, \ldots, R_m$ and *this contradicts the choice of* γ_k. Hence claim is proved.

If z belongs to K and is not on the boundary of any R_j, then[45]

$$\varphi(w) = \frac{1}{2\pi i} \cdot \frac{f(w)}{w - z}$$

is continuous on $\bigcup_{j=1}^{m} \partial R_j$ for f in $H(G)$. Hence it follows from (2.59) that

$$\sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\partial R_j} \frac{f(w)}{w - z} dw = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} dw. \tag{2.60}$$

But z belongs to the interior of exactly one R_j, $j = 1, \ldots, m$, let it be R_p (for p is exactly one of the numbers $1, \ldots, m$). i.e., $z \in R_p$. Then, by Cauchy's Integral Formula,

$$\frac{1}{2\pi i} \int_{\partial R_p} \frac{f(w)}{w - z} dw = f(z).$$

[45] f is continuous (because $f \in H(G)$), $w - z$ is continuous and $w - z \neq 0$ on $\bigcup_{j=1}^{m} \partial R_j$ because z is not on the boundary of any R_j. So the quotient $\frac{f(w)}{w-z}$ is a continuous function.

Also, then for $j \neq p$ and $j = 1, \ldots, m$, $z \notin R_j$, so[46] [47]

$$\frac{1}{2\pi i} \int\limits_{\partial R_j} \frac{f(w)}{w-z} dw = 0.$$

Then sum of the m summands on the left hand side of (2.60) becomes $f(z)$ so (2.60) becomes

$$f(z) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int\limits_{\gamma_k} \frac{f(w)}{w-z} dw \tag{2.61}$$

whenever $z \in K - \bigcup\limits_{j=1}^{m} \partial R_j$. But both sides of (2.61) are con-

---

[46] Cauchy-Goursat Theorem states that "if a function f is analytic at all points interior to and on a simple closed contour C, then $\int\limits_C f(z)dz = 0$." Here $\frac{f(w)}{w-z}$ is analytic at all points interior to and on ∂R_j when $j \neq p$, so for $j \neq p$, $\int\limits_{\partial R_j} \frac{f(w)}{w-z} dw = 0$.

Alternative proof using Cauchy′s Theorem (First Version): $G - \{z\}$ is an open set and $\frac{f(w)}{w-z}$ is analytic on that set. Also, for $j \neq p$, $n(\partial R_j;\ w) = 0$ for all $w \in \mathbb{C} - G$, so by Cauchy's Theorem (First Version), $\int\limits_{\partial R_j} \frac{f(w)}{w-z} dw = 0$.

[47] **Cauchy′s Theorem (First Version)** Let G be an open subset of the plane and $f : G \to \mathbb{C}$ is analytic. If $\gamma_1,\ \gamma_2,\ \ldots,\ \gamma_m$ are closed rectifiable curves in G such that

$$n(\gamma_1;\ w) + n(\gamma_2;\ w) + \cdots + n(\gamma_n;\ w) = 0 \text{ for all } w\ \in \mathbb{C} \sim G,$$

then

$$\sum_{k=1}^{m} \int\limits_{\gamma_k} f(z)\, dz = 0.$$

tinuous functions on K (because each γ_k misses K)[48] and they agree on the dense subset $K - \bigcup_{j=1}^{m} \partial R_j$ of K. i.e., the functions $f(z)$ and $\sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} dw$ that are continuous on K are equal (by (2.61)) on the dense subset[49] $K - \bigcup_{j=1}^{m} \partial R_j$ of K; hence these two functions are equal on the set K also.[50] That is, (2.61) holds for all z in K. That is,

$$f(z) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} dw \tag{2.62}$$

---

[48] f is continuous on K because $f \in H(G)$. To see that the function on the R.H.S. of (2.61) is also a continuous function of K, we use the Lemma: Let γ be a rectifiable curve and suppose φ is a function defined and continuous on $\{\gamma\}$. For each $m \geq 1$, let

$$F_m(z) = \int_{\gamma} \frac{\varphi(w)}{(w-z)^m} dw \quad \text{for} \quad z \notin \{\ \gamma\ \}.$$

Then each F_m is analytic on $\mathbb{C} - \{\gamma\}$ and

$$F'_m(z) = mF_{m+1}(z).$$

In the present situation, f is continuous on each set $\{\gamma_k\}$ and hence $\int_{\gamma_k} \frac{f(w)}{w-z} dw$ is continuous on $\mathbb{C} - \{\gamma_k\}$ and since γ_k misses K it follows that the function of (the variable z) $\int_{\gamma_k} \frac{f(w)}{w-z} dw$ is continuous on K. Hence (the sum of continuous functions) $\sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w-z} dw$ is also continuous on K.

[49] $K - \bigcup_{j=1}^{m} \partial R_j$ is a dense subset of K, since for any point z in K there is a neighborhood of z that has nonempty intersection with $K - \bigcup_{j=1}^{m} \partial R_j$.

[50] here the idea of continuous extension is used.

for all $z \in K$. That is (2.57) is proved.

Also it is obvious that the line segments form a finite number of closed polygons. So the remainder of the proof follows. □

The next lemma provides the first step in obtaining approximation by rational functions.

Lemma 2.4.3. *Let γ be a rectifiable curve*[51] *and let K be a compact set such that $K \cap \{\gamma\} = \emptyset$. If f is a continuous function on $\{\gamma\}$ and $\varepsilon > 0$ then there is a rational function $R(z)$ having all its poles on $\{\gamma\}$ and such that*

$$\left| \int_{\gamma} \frac{f(w)}{w-z} dw - R(z) \right| < \varepsilon$$

for all z in K.

Proof. Since K and $\{\gamma\}$ are disjoint there is a number r with $0 < r < d(K, \{\gamma\})$.[52] If γ is defined on $[0, 1]$ then for $0 \leq$

[51] γ need not be a *closed* curve

[52]
$$d(K, \{\gamma\}) = \inf\{|z-w| : z \in K, \ w \in \{\gamma\}\} \geq 0$$
and since K and $\{\gamma\}$ closed sets and are disjoint strict inequality holds; so that
$$d(K, \{\gamma\}) > 0.$$
(Why "$= 0$" is not possible? If "$= 0$" then there are two sequences $\{k_n\}$ in K and $\{d_n\}$ in $\underbrace{\{\gamma\}}_{\text{trace of } \gamma}$ such that $k_n - d_n \to 0$ as $n \to \infty$. This implies

$s,\ t \leq 1$ and z in K [53]

$$\left| \frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(s))}{\gamma(s) - z} \right|$$

$$\begin{aligned} &\leq \frac{1}{r^2} \left| f(\gamma(t))\gamma(s) - f(\gamma(s))\gamma(t) - z[f(\gamma(t)) - f(\gamma(s))] \right| \\ &\leq \frac{1}{r^2} \left| f(\gamma(t)) \right| \left| \gamma(s) - \gamma(t) \right| \\ &\quad + \frac{1}{r^2} \left| \gamma(t) \right| \left| f(\gamma(s)) - f(\gamma(t)) \right| \\ &\quad + \frac{|z|}{r^2} \left| f(\gamma(s)) - f(\gamma(t)) \right| \end{aligned}$$

There is a constant $c > 0$ such that[54] $|z| \leq c$ for all z in K,

---

$\lim_{n\to\infty} k_n = \lim_{n\to\infty} d_n$. Since K and $\{\gamma\}$ are closed sets, it follows that $\lim_{n\to\infty} k_n \in K$ and $\lim_{n\to\infty} d_n \in \{\gamma\}$ so that $\lim_{n\to\infty} k_n = \lim_{n\to\infty} d_n \in K \cap \{\gamma\}$, showing that $K \cap \{\gamma\} \neq \emptyset$, a contradiction.)

Now $d(K,\ \{\gamma\}) > 0$ hence by the density theorem on $\mathbb{R}$, there is a real number r such that $0 < r < d(K,\ \{\gamma\})$.

[53] If $\gamma : [0,\ 1] \to \mathbb{C}$ then for $z \in K$ and $0 \leq t \leq 1$, $|\gamma(t) - z| > r$ (also for $0 \leq s \leq 1$), $|\gamma(s) - z| > r$) because

$$0 < r < d(K,\ \{\gamma\}) = \inf\{|p - q| : p \in K,\ q \in \{\gamma\}\} \leq |\gamma(t) - z|$$

since $|\gamma(t) - z|$ is an element of the set $\{|p - q| : p \in K,\ q \in \{\gamma\}\}$. Hence for $0 \leq s,\ t \leq 1$ and z in K

$$\frac{1}{|\gamma(t) - z|} < \frac{1}{r} \text{ and } \frac{1}{|\gamma(s) - z|} < \frac{1}{r}.$$

[54] Reason: K, being compact, is (closed and) bounded hence there is a constant $M_1 > 0$ such that $|z| \leq M_1$ for all $z \in K$. γ is continuous on $[0,\ 1]$ implies γ is bounded on $[0,\ 1]$ (i.e., the set $\gamma([0,\ 1])$ is bounded), so there is a constant $M_2 > 0$ such that $|\gamma(t)| \leq M_2$ for all $t \in [0,\ 1]$. Also, f and γ

$|\gamma(t)| \leq c$ and $|f(\gamma(t))| \leq c$ for all t in $[0, 1]$. This gives that for all s and t in and z in K,

$$\left|\frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(s))}{\gamma(s)-z}\right| \leq \frac{c}{r^2}\,|\gamma(s)-\gamma(t)| + \frac{2c}{r^2}\,|f(\gamma(s)) - f(\gamma(t))|\,.$$

Since both γ and $f \circ \gamma$ are uniformly continuous[55] on $[0, 1]$, there is partition $\{0 = t_0 < t_1 < \cdots < t_n < 1\}$ such that[56]

---

are continuous implies the composition $f \circ \gamma$ is continuous on $[0, 1]$. Hence $f \circ \gamma$ is continuous on $[0, 1]$, and so there is a constant $M_3 > 0$ such that $|f \circ \gamma(t)| \leq M_3$ for all $t \in [0, 1]$. Take $c = \max\{M_1,\ M_2,\ M_3\}$.

[55] $f \circ \gamma(t)$ and $\gamma(t) - z$ are continuous functions on $[0, 1]$, and since $K \cap \underbrace{\{\gamma\}}_{\text{trace of } \gamma} = \emptyset$, $\gamma(t) - z \neq 0$ for any $t \in [0, 1]$; it follows that the quotient $\frac{f\circ\gamma(t)}{\gamma(t)-z}$ is a continuous function on $[0, 1]$. Thus, $\frac{f\circ\gamma(t)}{\gamma(t)-z}$ is uniformly continuous on $[0, 1]$.

Since γ is a rectifiable curve, $V(\gamma) < +\infty$ (Recall: γ is a rectifiable curve, if γ is a function of bounded variation, and hence $V(\gamma) < +\infty$.)

[56] Since $\frac{f\circ\gamma(t)}{\gamma(t)-z}$ is uniformly continuous on $[0, 1]$, for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{(f\circ\gamma)(t)}{\gamma(t)-z} - \frac{(f\circ\gamma)(t')}{\gamma(t')-z}\right| < \frac{\varepsilon}{V(\gamma)} \quad \forall\ t,\ t' \in [0,\ 1] \ \text{ with } \ |t - t'| < \delta.$$

The compact interval $[0, 1]$ can be covered by the collection of open intervals $\{(t - \delta/4,\ t + \delta/4) : t \in [0,\ 1]\}$ and hence has a finite subcover

$$\{(t^0 - \delta/4,\ t^0 + \delta/4),\ (t^1 - \delta/4,\ t^1 + \delta/4),\ \ldots,\ (t^n - \delta/4,\ t^n + \delta/4)\}$$

where $t^0 < t^1 < \cdots < t^n$ (renumber the superscript if needed). Choose $t_0 = 0$, t_1 is an element in $(t^1,\ t^2)$, t_2 is an element in $(t^2,\ t^3)$, $\ldots$, t_{n-1} is an element in $(t^{n-1},\ t^n)$, and $t_n = 1$. Then $|t_{j-1} - t_j| < \delta$. Note: We ensure that $t_0 = 0 < t_1 < \cdots < t_n = 1$ so that $\{t_0,\ t_1,\ \ldots,\ t_n\}$ is a partition of $[0, 1]$ (Fig. 2.20). Thus, if we take t such that $t_{j-1} \leq t \leq t_j$ then $|t - t_j| < \delta$, and

Figure 2.20: $\{t_0,\ t_1,\ \ldots,\ t_n\}$ is a partition of $[0,\ 1]$.

$$\left| \frac{f(\gamma(t))}{\gamma(t) - z} - \frac{f(\gamma(t_j))}{\gamma(t_j) - z} \right| < \frac{\varepsilon}{V(\gamma)} \tag{2.63}$$

for $t_{j-1} \leq t \leq t_j$, $1 \leq j \leq n$, and z in K. Define $R(z)$ to be the rational function

$$R(z) = \sum_{j=1}^{m} f(\gamma(t_{j-1})) \cdot \frac{\gamma(t_j) - \gamma(t_{j-1})}{\gamma(t_{j-1}) - z}.$$

The poles[57] of $R(z)$ are[58] $\gamma(0),\ \gamma(t_1),\ \ldots,\ \gamma(t_{n-1})$. Using (2.63)

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hence by the above

$$\left| \frac{(f \circ \gamma)(t)}{\gamma(t) - z} - \frac{(f \circ \gamma)(t_j)}{\gamma(t_j) - z} \right| < \frac{\varepsilon}{V(\gamma)}$$

for $t_{j-1} \leq t \leq t_j$, $1 \leq j \leq n$, and z in K.

[57] $f(\gamma(t_{j-1}))\gamma(t_j) - \gamma(t_{j-1})$ and $\gamma(t_{j-1})$ are complex numbers and so the poles of $R(z)$ are those values α for which $\gamma(t_{j-1}) - \alpha = 0$.

[58] We note that $\gamma(t_n) = \gamma(1) = \gamma(0) = \gamma(t_0)$. So the pole $\gamma(0)$ is the same point $\gamma(n)$, so there is no need to consider the same again.

yields that[59], [60]

[59]

$$
\begin{aligned}
R(z) &= \sum_{j=1}^{n} \frac{f(\gamma(t_{j-1}))\left[\gamma(t_j)-\gamma(t_{j-1})\right]}{\gamma(t_{j-1})-z} \\
&= \sum_{j=1}^{n} \frac{f(\gamma(t_{j-1}))}{\gamma(t_{j-1})-z} \underbrace{\int_{t_{j-1}}^{t_j} d\gamma(t)}_{\gamma(t)]_{t_{j-1}}^{t_j}=\gamma(t_j)-\gamma(t_{j-1})} \\
&= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{f(\gamma(t_{j-1}))}{\gamma(t_{j-1})-z} d\gamma(t).
\end{aligned}
$$

As $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = 1$,

$$
\underbrace{\{\gamma\}}_{\text{trace of } \gamma} = \gamma\left([t_0,\ t_1]\right) \cup \gamma\left([t_1,\ t_2]\right) \cup \cdots \cup \gamma\left([t_{n-1},\ t_n]\right).
$$

[60] Thus,

$$
\begin{aligned}
\int_{\gamma} \frac{f(w)}{w-z} dw &= \int_{t=t_0=0}^{t=t_n=1} \frac{f(\gamma(t))}{\gamma(t)-z} d\gamma(t) \\
&= \int_{t_0}^{t_1} \frac{f(\gamma(t))}{\gamma(t)-z} d\gamma(t) + \int_{t_1}^{t_2} \frac{f(\gamma(t))}{\gamma(t)-z} d\gamma(t) + \cdots + \int_{t_{n-1}}^{t_n} \frac{f(\gamma(t))}{\gamma(t)-z} d\gamma(t) \\
&= \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \frac{f(\gamma(t))}{\gamma(t)-z} d\gamma(t)
\end{aligned}
$$

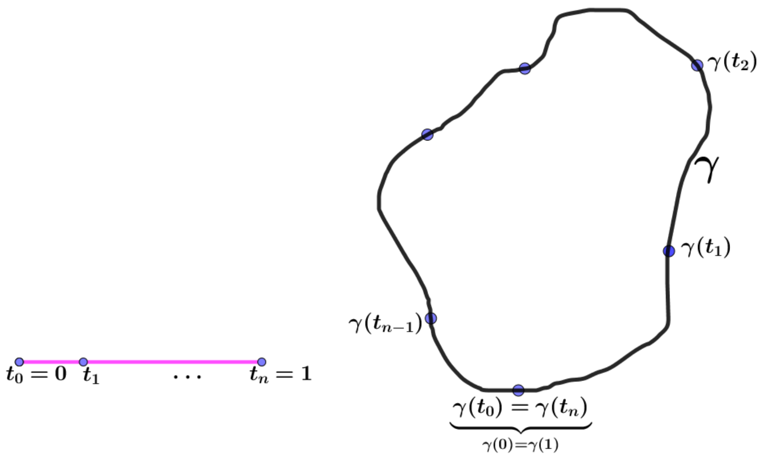

Figure 2.21:

$$
\begin{aligned}
& \left| \int_\gamma \frac{f(w)}{w-z} dw - R(z) \right| \\
= & \left| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \left[\frac{f(\gamma(t))}{\gamma(t)-z} - \frac{f(\gamma(t_{j-1}))}{\gamma(t_{j-1})-z} \right] d\gamma(t) \right| \\
\leq & \frac{\varepsilon}{V(\gamma)} \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} d\left|\gamma\right|(t) \\
= & \varepsilon
\end{aligned}
$$

for all z in K. [61] □

Before stating Runge's Theorem let us agree to say that a polynomial is a rational function with a pole at[62] ∞. It is easy to see that a rational function whose only pole is at ∞ is a polynomial.

Theorem 2.4.4. Runge′s Theorem *Let K be a compact subset of $\mathbb{C}$ and let E be a subset of $\mathbb{C}_\infty - K$ that meets each component of $\mathbb{C}_\infty - K$. If f is analytic in an open set containing K and $\varepsilon > 0$ then there is a rational function $R(z)$ whose only poles lie in E and such that*

$$|f(z) - R(z)| < \varepsilon$$

for all z in K.

The proof that will be given here was obtained by S. Grabiner. For this proof we place the result in a different setting. On the space $C(K, \mathbb{C})$ we define a distance function ρ by[63]

$$\rho(f,\ g) = \sup\{|f(z) - g(z)| : z \in K\}$$

for f and g in $C(K, \mathbb{C})$. It is easy to see that $\rho(f_n,\ f) \to 0$ if

[61] In the above we have used the fact that $\sum\limits_{j=1}^{n} d\,|\gamma|\,(t) = V(\gamma)$.

[62] as $\lim\limits_{z\to\infty} |p(z)| = \infty$ where p is a nonconstant polynomial.

[63] $\sup\{|f(z) - g(z)| : z \in K\}$ exists because the continuous function $|f - g|$ on the compact set K is bounded.

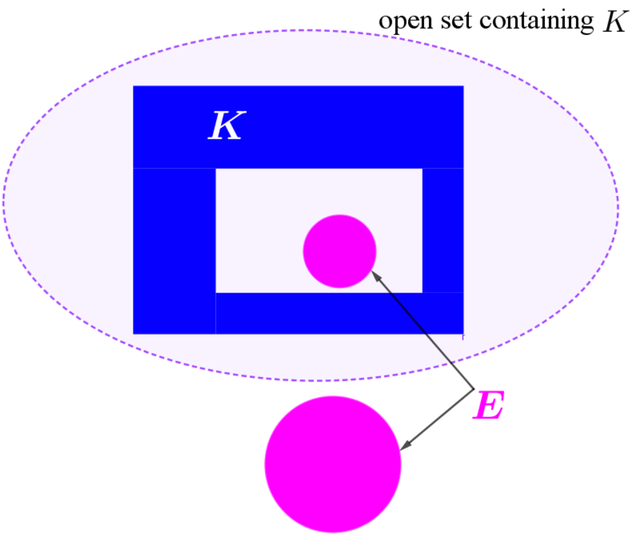

Figure 2.22: An illustration of Runge's Theorem. In the figure, E (the union of two circular disks) is a subset of $\mathbb{C}_\infty - K$ that meets each component (both bounded and unbounded) of $\mathbb{C}_\infty - K$. Note: For the K in the figure, $\mathbb{C}_\infty - K$ has exactly two components; one is a bounded set and another one is an unbounded set containing ∞.

and only if f_n converges uniformly to f on K. Hence $C(K, \mathbb{C})$ is a complete metric space.

So Runge's Theorem says that if f is analytic on a neighborhood of K and $\varepsilon > 0$ then there is a rational function $R(z)$ with poles in E such that

$$|f(z) - R(z)| < \varepsilon$$

for all z in K. Hence

$$\rho(f,\ R) = \sup\{|f(z) - R(z)| : z \in K\} \leq \varepsilon$$

By taking $\varepsilon = \frac{1}{n}$, corresponding to each $n = 1,\ 2,\ \ldots,$ there are rational functions R_n such that

$$\rho(f,\ R_n) \leq \frac{1}{n}$$

That is, that the sequence $\{R_n(z)\}$ of rational functions with poles in E is such that $\rho(f,\ R_n) \to 0$; that is, such that $\{R_n\}$ converges to f uniformly[64] on K.

Remark 2.4.5. Let

$$B(E)$$

be the set of all functions f in $C(K,\ \mathbb{C})$ such that there is a sequence $\{R_n\}$ of rational functions with poles in E such that $\{R_n\}$ converges to f uniformly on K. Runge's Theorem states that if f is analytic in a neighborhood of K then $f|_K$, the restriction of f to K, is in $B(E)$.

[64] $\rho(f_n,\ f) \to 0$ if and only if f_n converges to f uniformly on K because $\rho(f_n,\ f) \to 0$

$$\begin{aligned}
&\Leftrightarrow \quad \forall\varepsilon > 0\ \exists\ N \in N \quad \sup\{|f_n(z) - f(z)| : z \in K\} < \varepsilon \forall n \geq N \\
&\Leftrightarrow \quad \forall\varepsilon > 0\ \exists\ N \in N \quad \sup\{: z \in K\}\,|f_n(z) - f(z)| < \varepsilon\ \ \forall z \in K,\ n \geq N \\
&\Leftrightarrow \quad f_n \to f \text{ uniformlyon } \quad K
\end{aligned}$$

Lemma 2.4.6. *$B(E)$ is a closed subalgebra of $C(K, \mathbb{C})$ that contains every rational function with a pole in E.*

Proof. To say that $B(E)$ is an algebra is to say that if f and g are in $B(E)$ and $\alpha \in \mathbb{C}$ then αf, $f+g$, and fg are in $B(E)$.

$f,\ g \in B(E)$ implies (by the definition of $B(E)$) that there are sequences (f_n) and (g_n) of rational functions with poles in E such that $f_n \to f$ and $g_n \to g$ uniformly on K. This implies $f_n + g_n$ converges uniformly to $f+g$ on K which implies (again by the definition of $B(E)$) that $f+g \in B(E)$. Similarly, it can be proved that αf and fg are in $B(E)$ and hence $B(E)$ is an algebra.

Next we show that $B(E)$ is closed. Let (f_n) be a sequence in $B(E)$ that converges to f. We claim that $f \in B(E)$.

$f_n \to f$ implies for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $\rho(f_n,\ f) < \varepsilon \ \forall n \geq N$. Fix $\varepsilon > 0$. Then there exists $N = N(\varepsilon) \in \mathbb{N}$ such that

$$\rho(f_n,\ f) < \frac{\varepsilon}{2} \ \forall n \geq N.$$

In particular,

$$\rho(f_N,\ f) < \frac{\varepsilon}{2}\,.$$

As $f_N \in B(E)$ there is a sequence (g_m) of rational functions with poles in E such that

$$\rho(g_m,\ f_N) < \frac{\varepsilon}{2} \ \forall m \geq K = K(\varepsilon).$$

Take

$$H = \max\{N,\ K\}.$$

Then

$$\rho(g_m,\ f) \leq \rho(g_m,\ f_N) + \rho(f_N,\ f) < \varepsilon\,.\ \forall m \geq H.$$

This shows that the sequence (g_m) of rational functions with poles in E is such that $g_m \to f$. Hence, by the definition of $B(E)$, $f \in B(E)$. Thus $B(E$ is closed. This completes the proof. □

Lemma 2.4.7. *Let V and U be open subsets of $\mathbb{C}$ with $V \subset U$ and $\partial V \cap U = \emptyset$. If H is a component of U and $H \cap V \neq \emptyset$, then $H \subset V$.*

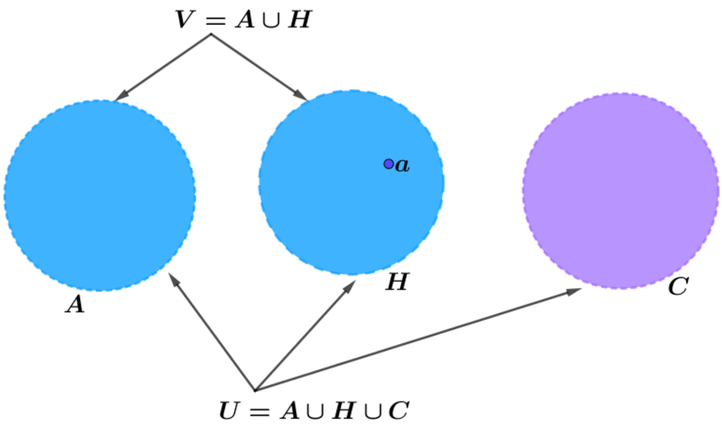

Figure 2.23: An illustration of Lemma 2.4.7. In the figure, V is the union of two open balls A and H. U is the union of three open balls A, H and C. Boundary of V is the boundaries (that are circles) of A and H and they do not intersect with the open set U. Hence $\partial V \cap U = \emptyset$. H is a component of U (for the U in the figure there are three components A, H and C) and also $H \cap V \neq \emptyset$. Also, it can be seen that $H \subset V$.

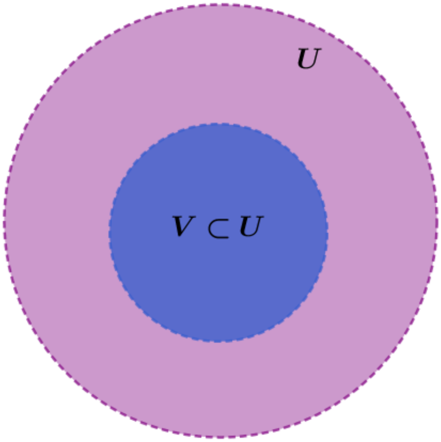

Figure 2.24: Lemma 2.4.7 cannot be applied when $V \subset U$ is as in the figure. Here the boundary of V is a circle, and is contained in U so that $\partial V \cap U \neq \emptyset$.

Proof. Let[65] $a \in H \cap V$ and let G be the component of V such that $a \in G$ (In Fig. 2.23, $G = H$. There are other situations also). Now a is a common element of the connected sets G and H (G and H are connected as they are components). Then[66] $H \cup G$ is connected and contained[67] in U. Since H is a component of U and $H \cup G$ is connected, it follows from $H \subset H \cup G \subset U$ that $H \cup G = H$ which implies that $G \subset H$. But[68] $\partial G \subset \partial V$ and so the hypothesis of the lemma that

[65] $H \cap V \neq \emptyset$ so it contains elements.

[66] The connected sets H and G have a common element a so that $H \cap G \neq \emptyset$ and hence $H \cup G$ is connected.

[67] $H \cup G \subset U$ since $H \subset U$ and $G \subset V \subset U$.

[68] $\partial G \subset \partial V$ because G is a component (maximal connected subset) of V.

$\partial V \cap U = \emptyset$, says that $\partial G \cap H = \emptyset$. [69] This implies (noting that $\mathbb{C} - G = (\mathbb{C} - \bar{G}) \cup \partial G$, we take like this to get an open set $\mathbb{C} - \bar{G}$ in the fourth step below) that

$$\begin{aligned}
H - G &= H \cap (\mathbb{C} - G) = H \cap [(\mathbb{C} - \bar{G}) \cup \partial G] \\
&= \left\{H \cap (\mathbb{C} - \bar{G})\right\} \cup \underbrace{\left\{H \cap \partial G\right\}}_{\partial G \cap H = \Phi} \\
&= \left\{H \cap (\mathbb{C} - \bar{G})\right\} \cup \Phi \\
&= \underbrace{\underbrace{H}_{open\ in\ H} \cap \underbrace{(C - \bar{G})}_{open\ in\ C}}_{open\ in\ H}
\end{aligned}$$

so that $H - G$ is open in H. But G is open implies that $\mathbb{C} - G$ is closed in $\mathbb{C}$. Hence

$$H - G = \underbrace{H \cap \underbrace{(C - G)}_{closed\ in\ C}}_{closed\ in\ H}$$

i.e., $H - G = H \cap (\mathbb{C} - G)$ is closed in H. Since H is connected,

---

[69] By assumption, $\partial V \cap U = \emptyset$, and hence

$$\partial V \cap U = \Phi \text{ and } \partial G \subset \partial V \Rightarrow \partial G \cap U = \Phi$$

which implies (in particular, since H is a component of U and hence $H \subset U$) that

$$\partial G \cap H = \Phi. \tag{2.64}$$

it cannot have a non-empty proper subset of H that is both open and So either $H - G = \emptyset$ or $H - G = G$. As $G \neq \emptyset$, $H - G = \emptyset$ is not possible. So $H - G = \emptyset$. As $G \subset H$, this implies $H = G$. So $H = G \subset V$. $\square$

Lemma 2.4.8. *If $a \in \mathbb{C} - K$ then $\frac{1}{z-a} \in B(E)$.*

Proof. [70] **Case 1:** $\infty \notin E$. Let $U = \mathbb{C} - K$ and let

$$V = \{a \in C : \frac{1}{z-a} \in B(E)\}$$

so[71]

$$E \subset V \subset U.$$

[72]**Claim:**

$$\text{If } a \in V \text{ and } |b-a| < d(a, K) \text{ then } b \in V. \tag{2.65}$$

[70] E is a subset of $\mathbb{C}_\infty - K$ that meets each component of $\mathbb{C}_\infty - K$. Hence E may or may not contain ∞.

[71] $E \subset V$: $a \in E$ and a is a pole of $\frac{1}{z-a}$, so by the definition of $B(E)$, $\frac{1}{z-a} \in B(E)$, because the sequence $\left\{\frac{1}{z-a}\right\}$ of rational functions with poles in E converges uniformly to $\frac{1}{z-a}$. Hence $a \in V$, by the definition of V. Also $V \subset U$ because: $a \in V \Rightarrow \frac{1}{z-a} \in B(E)$ and since $B(E)$ is a subalgebra (and hence a subset) of $C(K, \mathbb{C})$ it follows that $\frac{1}{z-a} \in C(K, \mathbb{C})$ and hence $z \neq a$ for all $z \in K$ so that $a \notin K$ implies $a \in \mathbb{C} - K = U$.

[72] We will show that $V = U$. Then, $\mathbb{C} - K = \{a \in C : \frac{1}{z-a} \in B(E)\}$ so that $a \in \mathbb{C} - K$ implies $\frac{1}{z-a} \in B(E)$.

i.e.,

$$a \in V, \ \ b \in \underbrace{B(a,\ d(a,\ K))}_{\substack{\text{ball centered at } a \text{ and} \\ \text{having radius } d(a,\ K)}} \Rightarrow b \in V$$

i.e., if $a \in V$, then any point in $B(a,\ d(a,\ K))$ is in V. i.e.,

$$a \in V \Rightarrow B(a,\ d(a,\ K)) \subset V$$

that is, V is an open subset of U.

Proof of the claim:

$|b-a| < d(a,\ K)$ implies (by the density theorem of $\mathbb{R}$) that there exists $r,\ 0 < r < 1$ such that[73]

$$|b-a| < r\, d(a,\ K) < d(a,\ K)$$

i.e.,

$$|b-a| < r\ \underbrace{\inf\{|z-a| : z \in K\}}_{d(a,\ K)}$$

i.e.,

$$|b-a| < r|z-a| \text{ for all } z \in K. \tag{2.66}$$

Hence

$$|b-a|\,|z-a|^{-1} < r < 1 \text{ for all } z \in K. \tag{2.67}$$

[73]We choose r such that $0 < r < 1$ and the positive real number $r\, d(a,\ K)$ lies between $|b-a|$ and $d(a,\ K)$.

Let

$$u_n(z) = \left(\frac{b-a}{z-a}\right)^n \quad \text{for } z \in K.$$

Then

$$|u_n(z)| \le r^n \quad \text{for } n = 0,\ 1, \quad \text{for } z \in K.$$

Hence, by Weierstrass M-Test[74], $\sum\limits_{n=0}^{\infty} u_n(z)$ converges uniformly[75] on K. i.e.,

$$\sum_{n=0}^{\infty} \left(\frac{b-a}{z-a}\right)^n$$

converges uniformly on K.

Hence

$$\frac{|b-a|}{|z-a|} < r < 1 \text{ for all } z \in K$$

gives that[76]

$$\left[1 - \frac{b-a}{z-a}\right]^{-1} = \sum_{n=0}^{\infty} \left(\frac{b-a}{z-a}\right)^n \tag{2.68}$$

---

[74] Weierstrass M-Test: Let $u_n : X \to \mathbb{C}$ be a function such that $|u_n(x)| \le M_n$ for every x in X and suppose that the constants satisfy $\sum\limits_{n=0}^{\infty} M_n < \infty$. Then $\sum\limits_{n=0}^{\infty} u_n$ is uniformly convergent.

[75] Here we take $M_n = r^n$ for $n = 0, 1, 2, \ldots$ and since $0 < r < 1$ the geometric series $\sum\limits_{n=0}^{\infty} M_n = \sum\limits_{n=0}^{\infty} r^n$ converges. Hence by Weierstrass M-Test $\sum\limits_{n=0}^{\infty} \left(\frac{b-a}{z-a}\right)^n$ converges uniformly on K.

[76] $\frac{1}{1-x} = 1 + x + x^2 + \cdots, \quad \text{if} \quad |x| < 1.$

converges uniformly on K by Weierstrass M-test. If

$$Q_n(z) = \sum_{k=0}^{n} \left(\frac{b-a}{z-a}\right)^k$$

($Q_n(z)$ is the nth partial sum of the (uniformly convergent) series $\sum\limits_{n=0}^{\infty} \left(\frac{b-a}{z-a}\right)^n$) then $\frac{Q_n(z)}{z-a} \in B(E)$ since $a \in V$ and $B(E)$ is an algebra.[77]

Since

$$(z-b)^{-1} = (z-a)^{-1}\left[1 - \frac{b-a}{z-a}\right]^{-1} \tag{2.69}$$

and since $B(E)$ is closed, the uniform convergence of (2.68)

---

[77]Details: Already, $a \in V$ (see the assumption of the claim) and hence by the definition of V, $\frac{1}{z-a} \in B(E)$. $B(E)$ is a subalgebra and hence (any scalar multiple of $\frac{1}{z-a}$, product of $\frac{1}{z-a}$, sums of members of $B(E)$ are all in $B(E)$. So, scalar multiple,

$$(b-a) \cdot \frac{1}{z-a} = \frac{b-a}{z-a} \in B(E)$$

product of $\frac{b-a}{z-a}$'s,

$$\left(\frac{b-a}{z-a}\right) \cdot \left(\frac{b-a}{z-a}\right) = \left(\frac{b-a}{z-a}\right)^2 \in B(E)$$

and so on, and the finite sum of above elements in $B(E)$ given by

$$1 + \left(\frac{b-a}{z-a}\right) + \cdots + \left(\frac{b-a}{z-a}\right)^n \in B(E)$$

Hence $Q_n(z) \in B(E)$. Also the product of $\frac{1}{z-a}$ and $Q_n(z)$ given by $\frac{1}{z-a} \cdot Q_n(z) \in B(E)$.

imply that[78] $\frac{1}{z-b} \in B(E)$. That is, by the definition of V, $b \in V$. **Hence the claim is proved.**

Note that (2.65) implies that[79] V is open. If $b \in \partial V$ (i.e., if b is an interior point of V) it is possible to consider a sequence $\{a_n\}$ in V with $b = \lim a_n$. Since $b \notin V$ [80] it follows from (2.65) that[81]

$$|b - a_n| \geq d(a_n,\ K). \tag{2.70}$$

[78] $\lim_{n\to\infty} Q_n(z) \in B(E)$ and

$$\lim_{n\to\infty} \underbrace{\frac{1}{z-a}}_{\in B(E)} \cdot Q_n(z) = \frac{1}{z-a} \lim_{n\to\infty} Q_n(z) = \frac{1}{z-a}\left[1 - \frac{b-a}{z-a}\right]^{-1}$$

is an element of $B(E)$, that is, $(z-b)^{-1} \in B(E)$, by (2.69). i.e., $b \in V$.

[79] By (2.65), $a \in V$ implies

$$\underbrace{B(a,\ d(a,\ K))}_{\substack{ball\ centered\ at\ a\ and \\ having\ radius\ d(a,\ K)}} \quad \subset V$$

implies each a is an interior point of V. Hence each point of V is an interior point of V. Hence V is open.

[80] $b \notin V$ as $b \in \partial V$ and V is open, boundary points of V are not members of V.

[81] For otherwise, $\left|b - \underbrace{a_n}_{\in V}\right| < d(a_n,\ K)$ implies (by (2.65)) $b \in V$, a contradiction to the fact that $b \notin V$,

$$\begin{aligned}
\lim_{n\to\infty} |b - a_n| &\geq \lim_{n\to\infty} d(a_n,\ K) \\
\underbrace{\left| b - \lim_{n\to\infty} a_n \right|}_{|b-b|=0} &\geq d(\lim_{n\to\infty} a_n,\ K)
\end{aligned}$$

so that

$$0 \geq d(b,\ K).$$

We know that $d(b,\ K) \geq 0$ and hence

$$d(b,\ K) = 0.$$

Hence[82] $b \in \bar{K}$ which implies $b \in K$ (as K is compact, $\bar{K} = K$).

i.e., we have shown that if $b \in \partial V$ then $b \in K$ (this together with the fact that $U = \mathbb{C} - K$) implies that $b \notin U$.

Thus $\partial V \cap U = \emptyset$.

If H is component of $U = \mathbb{C} - K$ then $H \cap E \neq \emptyset$, (because E is a subset of $\mathbb{C}_\infty - K$ that meets each component of $\mathbb{C}_\infty - K$). As $E \subset V \subset U,\ \ H \cap E \neq \Phi,\ H \cap E \neq \emptyset$ implies that $H \cap V \neq \emptyset$. That is,

$$\text{every component of } U \text{ intersects with } V. \tag{2.71}$$

i.e., if H is component of U, then $H \cap V \neq \emptyset$ (where $\partial V \cap U = \emptyset$

[82] Let A be a subset of the metric space X. Then, $d(x,\ A) = 0$ if and only if $x \in \bar{A}$.

also), hence, by Lemma 2.4.7,

$$H \subset V. \tag{2.72}$$

Since H is an arbitrary component of U (which always intersects with V, by (2.71)), it follows by (2.72) that every component of U is contained in V. Hence[83]

$$U \subset V.$$

Already $V \subset U$. Hence

$$V = U.$$

By the definition of V this shows that if $a \in \mathbb{C} - K$ then $\frac{1}{z-a} \in B(E)$.

Case 2: $\infty \in E$. Let $d=$ the metric on $\mathbb{C}_\infty$ (Details are given in Appendix *The Extended Plane and its Spherical Representation* in page 412). Choose a_0 in the unbounded component of[84] $\mathbb{C} - K$ such that

$$d(a_0,\ \infty) \leq \frac{1}{2} \underbrace{d(\infty,\ K)}_{\inf\{d(\infty,\ z):z\in K\}}$$

[83] noting that U is the union of its components.

[84] a_0 in the unbounded component of $\mathbb{C} - K$ so $a_0 \neq \infty$ (Note that $\infty \notin \mathbb{C} - K$).

and

$$|a_0| > 2\max\{\,|z| : z \in K\}. \tag{2.73}$$

Let[85]

$$E_0 = (E - \{\infty\}) \cup \{a_0\}; \tag{2.74}$$

[86] so E_0 meets each component of[87] $\mathbb{C}_\infty - K$. As $\infty \notin E_0$, we can apply Case 1 for the set E_0.

If $a \in \mathbb{C} - K$ then Case 1 gives that[88]

$$\frac{1}{z-a} \in B(E_0). \tag{2.75}$$

If we can show that $\frac{1}{z-a_0} \in B(E)$ then it follows that[89] $B(E_0) \subset B(E)$ and so $\frac{1}{z-a} \in B(E)$ for each a in $\mathbb{C} - K$. [90]

---

[85]By taking $E_0 = (E - \{\infty\}) \cup \{a_0\}$, we remove the element ∞ in E and instead include the element a_0 that is not in E. Note that both ∞ and a_0 are in the unbounded component of E.

[86]so a rational function with poles in E_0 implies such poles either belongs to $E - \{\infty\}$ or pole $= a_0$.

[87]Reason: Though ∞ is missing in E_0, the point a_0 in the unbounded component of $C - K$ is in E_0.

[88] We can apply Case 1 because $\infty \notin E_0$. Also note that by saying $\frac{1}{z-a} \in B(E_0)$ we are not saying that $\frac{1}{z-a} \in B(E)$.

[89]$B(E_0) =$ the set of all functions f in $C(K, \mathbb{C})$ such that there is a sequence $\{R_n\}$ of rational functions with poles in E_0 such that $\{R_n\}$ converges to f uniformly on K. (By this, the case of the point a_0 that is not in E is also considered).

[90]$(z-a_0)^{-1}$ is a rational function with pole at a_0. If $(z-a_0)^{-1} \in B(E)$, since $B(E)$ is an algebra, all possible linear combinations and products involving $(z-a_0)^{-1}$ are in $B(E)$. So all rational functions with poles at a_0 are in $B(E)$. Already $(z-a_0)^{-1} \in B(E_0)$. (Note: The only element in E_0 that is not in E is a_0, that is the reason why we consider the point a_0 separately). Hence along with (2.74), $B(E_0) \subset B(E)$. By (2.75), $(z-a)^{-1} \in B(E_0)$ which together with $B(E_0) \subset B(E)$ implies $(z-a)^{-1} \in B(E)$. So in any case, $(z-a)^{-1} \in B(E)$

Now $\left|\frac{z}{a_0}\right| \leq \frac{1}{2}$ for all z in K (since, using (2.73), $|a_0| > 2\max\{|z| : z \in K\}$) and so

$$\frac{1}{z-a_0} = -\frac{1}{a_0(1-z/a_0)} = -\frac{1}{a_0}\sum_{n=0}^{\infty}\left(\frac{z}{a_0}\right)^n$$

converges uniformly[91] on K. So (its nth partial sum $-\frac{1}{a_0}\sum_{k=0}^{n}\left(\frac{z}{a_0}\right)^k$ (which is a polynomial on K) converges uniformly to $\frac{1}{z-a_0}$. For $n = 1, 2, \ldots,$ we let

$$Q_n(z) = -\frac{1}{a_0}\sum_{k=0}^{n}\left(\frac{z}{a_0}\right)^k.$$

Then the sequence of functions $\{Q_n\}$ converges uniformly to $\frac{1}{z-a_0}$ on K. Since Q_n (being a polynomial) has its only pole at ∞, (and since $\infty \in E$ by assumption), $Q_n \in B(E)$. Thus,[92] $\frac{1}{z-a_0} \in B(E)$. □

The proof of Runge's Theorem: If f is analytic on an open set G and $K \subset G$ then for each $\varepsilon > 0$ Proposition 2.4.2 and Lemma 2.4.3 imply the existence of an rational function $R(z)$ with poles in $\mathbb{C} - K$ such that $|f(z) - R(z)| < \varepsilon$ for all z

for any $a \in \mathbb{C} - K$. Hence it remains to show that $(z - a_0)^{-1} \in B(E)$.

[91] This follows by applying Weierstrass M- Test, with $\left|\frac{z}{a_0}\right| \leq \frac{1}{2}$ for all $z \in K$ and noting that $\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} \frac{1}{2^n}$ converges.

[92] $\{Q_n\}$ is a sequence in $B(E)$ that converges uniformly to $\frac{1}{z-a_0}$. Since $B(E)$ is closed, this shows that $\frac{1}{z-a_0} \in B(E)$.

in K.[93] But Lemma 2.4.8 and the fact that $B(E)$ is an algebra gives that $R \in B(E)$.[94]

Corollary 2.4.9. *Let G be an open subset of the plane and let E be a subset of $\mathbb{C}_\infty - K$ such that E meets every component*

[93]If f is analytic on an open set G and $K \subset G$ then for each $\varepsilon > 0$ (using Proposition 2.4.2 and Lemma 2.4.3) (note that f is continuous on $\{\gamma\} = \{\gamma_1\} \cup \cdots \cup \{\gamma_n\}$ because $f \in H(G)$)

$$\left| \underbrace{\underbrace{\frac{1}{2\pi i} \int\limits_{\gamma} \frac{f(w)}{w-z} dw}_{\sum_{k=1}^{n} \frac{1}{2\pi i} \int\limits_{\gamma_k} \frac{f(w)}{w-z} dw}}_{f(z)} - R(z) \right| < \varepsilon \ \ \forall z \in K$$

for some rational function $R(z)$ with poles in $\mathbb{C} - K$ such that

$$|f(z) - R(z)| < \varepsilon \ \ \forall z \in K.$$

$R(z)$ has poles (if any) in $C - K$ and $R(z)$ has the form

$$R(z) = \frac{p(z)}{(z-a_1)\cdots(z-a_k)}$$

where $a_1, \ldots, a_k \in C - K$.

[94]We have noted above that $R(z)$ has all its poles in $\mathbb{C} - K$. Let the poles be $a_1, \ldots, a_k$. Then, $a_1, \ldots, a_k \in C - K$. Now, by Lemma 2.4.8, $a_1, \ldots, a_k \in C - K$ implies $(z-a_1)^{-1}, \ \ldots \ , \ (z-a_k)^{-1}$ are in $B(E)$. $B(E)$ is a subalgebra, thus this implies $(z-a_1)^{-1} \cdots (z-a_k)^{-1}$ are in $B(E)$. Also, by the definition of $B(E)$, the polynomial $p(z)$ also belongs to $B(E)$. Hence

$$p(z)(z-a_1)^{-1} \cdots (z-a_k)^{-1} \in B(E).$$

This implies $R(z) \in B(E)$ (i.e., R is a rational function whose only poles lies in E).

of $\mathbb{C}_\infty - G$. *Let* $R(G,\, E)$ *be the set of rational functions with poles in* E *and consider* $R(G,\, E)$ *as a subspace of* $H(G)$. *If* $f \in H(G)$ *then there is a sequence* $\{R_n\}$ *in* $R(G,\, E)$ *such that* $f = \lim R_n$. *That is,* $R(G,\, E)$ *is dense in* $H(G)$.

Proof. Let K be a compact subset of G and $\varepsilon > 0$; it must be shown that there is an R in $R(G,\, E)$ such that

$$|f(z) - R(z)| < \varepsilon \text{ for all } z \text{ in } K. \tag{2.76}$$

According to Proposition VII. 1.2 (in Conway) in Page 6 [95] there is a compact set K_1 such that $K \subset K_1 \subset G$ and each component of $\mathbb{C}_\infty - K_1$ contains a component of $\mathbb{C}_\infty - G$. Hence, E meets each component of $\mathbb{C}_\infty - K_1$. The Corollary now follows from Runge's Theorem as follows:

Now K_1 is a compact subset of $\mathbb{C}$ and E be a subset of $\mathbb{C}_\infty - K_1$ that meets each component of $\mathbb{C}_\infty - K_1$. $K_1 \subset G$. Hence $f \in H(G)$ and $\varepsilon > 0$ implies (by Runge's theorem)

[95]**Proposition VII. 1.2:** If G is an open set in $\mathbb{C}$ then there is a sequence $\{K_n\}$ of compact subsets of G such that

$$G = \bigcup_{n=1}^{\infty} K_n. \tag{2.77}$$

Moreover, the sets K_n can be chosen to satisfy the following conditions:

(a) $K_n \underset{\neq}{\subset} \operatorname{int}(K_{n+1})$

(b) $K \subset G$ and K is compact implies $K \subset K_n$ for some n;

(c) Every component of $\mathbb{C}_\infty -\ K_n$ contains a component of $\mathbb{C}_\infty -\ G$.

there is a rational function $R(z) \in R(G, E)$ such that

$$|f(z) - R(z)| < \varepsilon \;\; \forall z \in K_1.$$

Since $K \subset K_1$, it follows that (2.76) holds. □

The next corollary follows by letting $E = \{\infty\}$ and using the fact that a rational function whose only pole is at ∞ is a polynomial .

Corollary 2.4.10. *If G is an open subset of $\mathbb{C}$ such that $\mathbb{C}_\infty - G$ is connected then for each analytic function f on G there is a sequence of polynomials $\{p_n\}$ such that $f = \lim p_n$ in $H(G)$.*

Proof. Take $E = \{\infty\}$. Then E meets the unbounded component of $\mathbb{C}_\infty - G$ (Note: As G is an open subset of $\mathbb{C}$ and $\mathbb{C}_\infty - G$ is connected, it has only one component and it is the unbounded component that contains ∞). Let $f \in H(G)$. Then by Corollary 2.4.9, there is a sequence $\{R_n\}$ in $R(G, E)$ such that

$$f = \lim R_n. \tag{2.78}$$

$R(G, E)$ is the set of rational functions with poles in $E = \{\infty\}$. i.e., $R(G, E)$ is the set of rational functions that have pole at ∞. Hence, by the fact that a rational function whose only pole is at ∞ is a polynomial, $R(G, E)$ is the set of polynomials. Hence (2.78) gives that

$$f = \lim p_n$$

where (p_n) is a sequence of polynomials. $\square$

Corollary 2.4.9 can be strengthened a little by requiring only that $\overline{E}$ meets each component of $\mathbb{C}_\infty - G$.

The condition that $\overline{E}$ meet every component of $\mathbb{C}_\infty - G$ cannot be relaxed. This can be seen by considering the punctured plane $\mathbb{C} - \{0\} = G$. So $\mathbb{C}_\infty - G = \{0, \infty\}$. Suppose that for this case we could weaken Runge's Theorem/Corollary 2.4.9 by assuming that E consisted of ∞ alone. Then for each integer $n \geq 1$ we could find a polynomial $p_n(z)$ [96]such that

$$\left| \frac{1}{z} - p_n(z) \right| < \frac{1}{n} \tag{2.79}$$

for $\frac{1}{n} \leq |z| \leq n$. Then

$$|1 - zp_n(z)| < \frac{|z|}{n} \leq 1 \text{ for } \frac{1}{n} \leq |z| \leq n\,.$$

But if $|z| = n$ then

$$|p_n(z)| = \frac{1}{n}\,|zp_n(z)| \leq \frac{1}{n}\,|zp_n(z) - 1| + \frac{1}{n} \leq \frac{2}{n}.$$

[96]because by Corollary 2.4.9, $R_n \in R(G, E)$ so the only poles are in E, i.e., at ∞ alone. So R_n's are polynomials.

By the Maximum Modulus Theorem,

$$|p_n(z)| \leq \frac{2}{n} \quad \text{for} \quad |z| \leq n\,.$$

In particular,

$$p_n(z) \to 0 \text{ uniformly on } |z| \leq n\,.$$

This clearly contradicts (2.79) and shows that E must be the set $\{0, \infty\}$.

Of course, the point in the above paragraph could have been made by appealing to what was said about this Same example at the beginning of this section. However, this further exposition gives an introduction to a concept whose connection with Runge's Theorem is quite intimate.

Definition 2.4.11. Let K be a compact subset of the plane; the **polynomially convex hull** of K, denoted by $\widehat{K}$, is defined to be the set of all points w such that for every polynomial p

$$|p(w)| \leq \max\{\, |p(z)| : z \in K\,\}\,.$$

That is, if the right hand side of this inequality is denoted by $||p||_K$, then

$$\widehat{K} = \{w : |p(w)| \leq \|p\|_K \ \text{ for all polynomials } p\}$$

If K is an annulus then $\widehat{K}$ is the disk obtained by filling in the interior hole. In fact, if K is any compact set the Maximum Modulus Theorem gives that $\widehat{K}$ is obtained by filling in any "holes" that may exist in K.[97]

2.5 Simple Connectedness

Recall that an open connected set G is simply connected if and only if every closed rectifiable curve in G is homotopic to zero. The purpose of this section is to prove some equivalent formulations of simple connectedness.

Definition 2.5.1. Let X and Ω be metric spaces; a **homeomorphism** between X and Ω is a continuous map $f : X \to \Omega$ which is one-one, onto, and such that $f^{-1} : \Omega \to X$ is also continuous.

Proposition 2.5.2. *Let X and Ω be metric spaces. If $f : X \to \Omega$ is one-one, onto, and continuous then f is a homeomorphism if and only if f is open (or, equivalently, f is closed)*[98].

[97] Polynomials are analytic on $\mathbb{C}$(with poles at ∞.) In particular, polynomials are analytic on any subset of K. Taking polynomials $p(z)$, $\max\{p(z) : z \in K\}$ attains on boundary. So if there is any hole inside K, and w belong that hole, considering the compact set filling all holes it can be seen that

$$|p(w)| \leq \max\{p(z) : z \in K\}.$$

[98] Proof of this result can be seen in Topology text books.

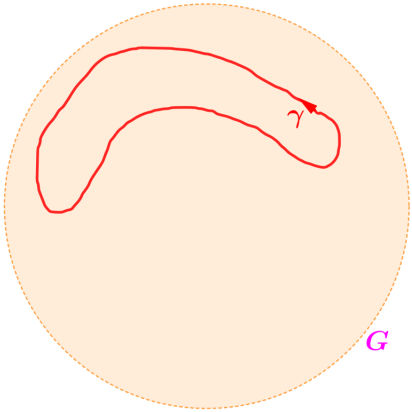

Figure 2.25: The open connected set G in the figure is simply connected, because any rectifiable curve in G is not homotopic to 0. For example, the closed rectifiable curve γ in G is homotopic to 0.

Definition 2.5.3. Let X and Ω be metric spaces. If there is a homeomorphism between X and Ω then the metric spaces X and Ω are **homeomorphic.**

We claim that $\mathbb{C}$ and $D = \{z : |z| < 1\}$ are homeomorphic. In fact

$$f(z) = \frac{z}{1 + |z|}$$

maps $\mathbb{C}$ onto D in a one-one fashion and its inverse,

$$f^{-1}(\omega) = \frac{\omega}{1 - |\omega|},$$

is clearly continuous. Also, if f is a one-one analytic function on an open set G (Proof is given in the proof of the following

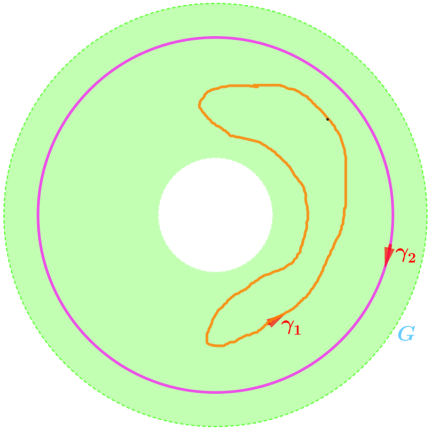

Figure 2.26: The open connected set G in the figure is not simply connected, because the closed rectifiable curve γ_2 in G is not homotopic to 0. (Note: But, γ_1 in G is homotopic to 0. $n(\gamma_2, a) \neq a$, if a is a point in hole. G is not simply connected (This also follows from Part (a) if and only if Part (b) of Theorem 2.5.4.

theorem) and $\Omega = f(G)$ then G and Ω are homeomorphic. Finally, all annuli are homeomorphic to the punctured plane.

Theorem 2.5.4. *Let G be an open connected subset of $\mathbb{C}$. Then the following are equivalent:*

(a) G is simply connected;

(b) $n(\gamma; a) = 0$ for every closed rectifiable curve γ in G and every point a in $\mathbb{C} - G$;

(c) $\mathbb{C}_\infty - G$ is connected;

(d) *For any f in $H(G)$ there is a sequence of polynomials that converges to f in $H(G)$;*

(e) *For any f in $H(G)$ and any closed rectifiable curve γ in G, $\int_\gamma f = 0$;*

(f) *Every function f in $H(G)$ has a primitive;*

(g) *For any f in $H(G)$ such that $f(z) \neq 0$ for all z in G there is a function g in $H(G)$ such that $f(z) = \exp g(z)$;*

(h) *For any f in $H(G)$ such that $f(z) \neq 0$ for all z in G there is a function g in $H(G)$ such that*

$$f(z) = [g(z)]^2;$$

That is, every non-vanishing analytic function has an analytic square root.

(i) *G is homeomorphic to the unit disk;*

(j) *If $u : G \to \mathbb{R}$ is harmonic then there is a harmonic function $v : G \to \mathbb{R}$ such that $f = u + i\,v$ is analytic in G.*

Proof. The plan is to show that $(a) \Rightarrow (b) \Rightarrow \ldots \Rightarrow (i) \Rightarrow (a)$ and $(h) \Rightarrow (j) \Rightarrow (g)$. Many of these implications have already been done.

$(a) \Rightarrow (b)$ If a is a point in the complement of G then (for any $z \in G$, $z - a \neq 0$ so that)

$$f(z) = \frac{1}{z-a}, \quad z \in G$$

is analytic on G, and if γ is a closed rectifiable curve in G, then by Cauchy's theorem,

$$\int_{\gamma} \frac{1}{z-a} dz = 0$$

so that

$$n(\gamma;\ a) = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z-a} dz = 0.$$

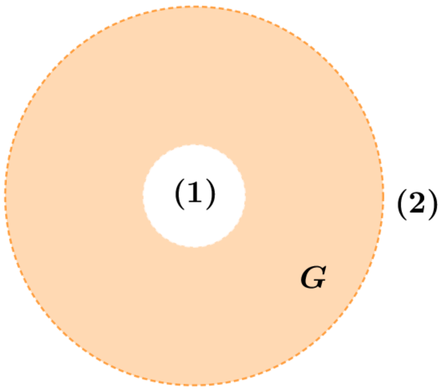

Figure 2.27: G is not simply connected. $\mathbb{C}_\infty - G$ is not connected, since it is the union of the region (1) and the unbounded region (2) containing ∞ in the extended complex plane.

$(b) \Rightarrow (c)$ Suppose $\mathbb{C}_\infty - G$ is not connected; then $\mathbb{C}_\infty - G = A \cup B$ where A and B are disjoint, non-empty, *closed* subsets of $\mathbb{C}_\infty$. Since $\infty \in \mathbb{C}_\infty - G$, [99]it must be either in A or in B. Suppose that ∞ is in B; thus, A must be a compact subset of[100] $\mathbb{C}$ (A is compact in $\mathbb{C}_\infty$ [101] and does not contain ∞). But then

$$G_1 = G \cup A = \mathbb{C}_\infty - B$$

is an open set in $\mathbb{C}$ and contains[102] A. According to last part of Proposition 2.4.2 there are a finite number of *polygons*[103] $\gamma_1, \ldots, \gamma_m$ in $G_1 - A = G$ such that for every analytic function f on G_1

$$f(z) = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} dw$$

for all z in A. In particular, if $f(z) \equiv 1$ (a constant analytic

[99]Note: If G is not simply connected, then $\mathbb{C}_\infty - G$ is not connected (Fig. 2.27).

[100] $\infty \notin A$, so A is a bounded region in $\mathbb{C}_\infty$ (Ref. the region (1) in Fig. 2.27), so A is a bounded region in $\mathbb{C}$ also. Already A is closed, this together with the fact that A is bounded implies A is a compact subset of $\mathbb{C}$.

[101]A is compact in $\mathbb{C}_\infty$ because it a closed subset of the compact space $\mathbb{C}_\infty$.

[102]$\mathbb{C}_\infty - B$ is open in $\mathbb{C}_\infty$. But $\infty \in B$, so $\mathbb{C}_\infty - B \subset \mathbb{C}$. Hence $\mathbb{C}_\infty - B$ is an open set in $\mathbb{C}$, because $\mathbb{C}_\infty - B = \underbrace{\mathbb{C}_\infty - B}_{\text{open in } \mathbb{C}} \cap \mathbb{C}$, being the element in the subspace topology of $\mathbb{C}$ induced by the topology on $\mathbb{C}_\infty$.

[103]not line segments, but closed polygons.

function on G_1) then

$$1 = \sum_{k=1}^{m} \frac{1}{2\pi i} \int_{\gamma_k} \frac{f(w)}{w - z} dw = \sum_{k=1}^{m} n(\gamma_k;\ z)$$

for all z in A. L.H.S. is 1 and R.H.S is sum of integers implies not all integers on the R.H.S. is zero, implies $n(\gamma_k;\ z) \neq 0$ for some closed polygon γ_k in G. That is, for any z in A there is at least one polygon γ_k in G such that $n(\gamma_k;\ z) \neq 0$. This contradicts (b). Hence (a) implies (b) is proved.

$(c) \Rightarrow (d)$ See Corollary 2.4.10.

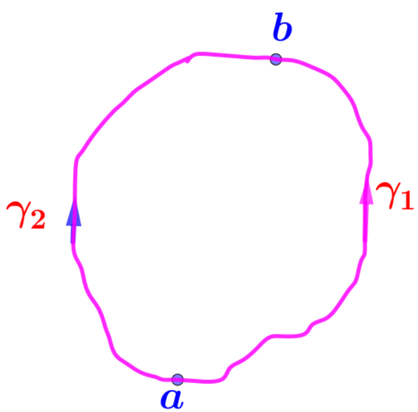

Figure 2.28:

$(d) \Rightarrow (e)$ Let γ be a closed rectifiable curve in G, and let f be an analytic function on G. Then by assumption, it is possible to find a sequence $\{p_n\}$ of polynomials such that

$f = \lim p_n$ in $H(G)$. Since each polynomial is analytic in $\mathbb{C}$ and[104] $\gamma \sim 0$ in $\mathbb{C}$, so by Cauchy's theorem, $\int\limits_\gamma p_n = 0$ for every n. But $\{p_n\}$ converges to f uniformly on the set[105] $\{\gamma\}$ so that

$$\int\limits_\gamma f = \int\limits_\gamma \lim p_n = \lim \int\limits_\gamma p_n = \lim 0 = 0.$$

$(e) \Rightarrow (f)$ Fix a in G. From condition (e) it follows that there is a function $F : G \to \mathbb{C}$ defined by letting

$$F(z) = \int\limits_\gamma f$$

[106] where γ is *any* rectifiable curve in G from a to z. It follows that (by referring the proof[107] of corollary IV.6.16 in Conway)

[104] $\gamma \sim 0$ in $\mathbb{C}$, since every closed rectifiable curve in $\mathbb{C}$ is homotopic to zero. i.e.,

[105] $\{\gamma\}$ is the trace of the curve γ.

[106] Even though the function F is defined with the help of a rectifiable curve from a to z, the function is well-defined, because of the following observations. There are infinitely many rectifiable curves from a to z. We have to show that the value of $F(z)$ is independent of the choice of the rectifiable curve from a to z. For this purpose, we consider two rectifiable curves γ_1 and γ_2 from a to z as in Fig. 2.28. Then $\gamma = \gamma_1 - \gamma_2$ is a closed rectifiable curve and hence by the assumption, $\int\limits_\gamma f = 0$. That is, $\int\limits_{\gamma_1 - \gamma_2} f = 0$ which implies $\int\limits_{\gamma_1} f - \int\limits_{\gamma_2} f = 0$ so that $\int\limits_{\gamma_1} f = \int\limits_{\gamma_2} f$, showing that the value of $F(z)$ is independent of the choice of the rectifiable curve from a to z. Hence F is a well-defined function.

[107] **Corollary IV.6.16 (Conway, Page 94):** If G is simply connected $f : G \to \mathbb{C}$ is analytic in G then f has a primitive in G.

Proof. Fix a point a in G and let γ_1 and γ_2 be any two rectifiable curves in

G from a to a point z in G. (Since G is open and connected there is always a path from a to any other point of G.) Then,

$$0 = \int_{\gamma_1 - \gamma_2} f = \int_{\gamma_1} f - \int_{\gamma_2} f.$$

where $\gamma_1 - \gamma_2$ is the curve which goes from a to z along γ_1 and then back to a along $-\gamma_2$. That is,

$$\int_{\gamma_1} f = \int_{\gamma_2} f$$

and this shows that line integral of f from a to z is independent of path from a to z. That is, $\int_\gamma f$ is a constant for any rectifiable curve γ from a to z. Hence we can get a *well defined* function $F : G \to \mathbb{C}$ by setting

$$F(z) = \int_\gamma f$$

where γ is any rectifiable curve from a to z. We **claim** that F is a primitive of f.

If $z_0 \in G$ and $r > 0$ is such that $B(z_0;\, r) \subset G$, then let γ be a path from a to z_0. For z in $B(z_0;\, r)$ let

$$\gamma_z = \gamma + [z_0,\ z];$$

that is, γ_z is the path γ followed by the straight line segment from z_0 to z. Hence

$$\begin{aligned}
\frac{F(z) - F(z_0)}{z - z_0} &= \frac{1}{(z - z_0)} \left(\int_{\gamma_z} f - \int_\gamma f \right) \\
&= \frac{1}{(z - z_0)} \int_{\gamma_z - \gamma} f \\
&= \frac{1}{(z - z_0)} \int_{[z_0,\ z]} f.
\end{aligned}$$

Now proceed as in the proof of Morera's theorem to show that

$$F'(z_0) = f(z_0).$$

that F is a primitive of f and hence $F' = f$.

$(f) \Rightarrow (g)$ If $f(z) \neq 0$ for all z in G then f'/f (being the quotient of two analytic functions with denominator is different from zero) is analytic on G. Part (f) implies that the analytic function f'/f has a primitive. That is, there is a function F such that $F' = f'/f$. It follows (see the proof[108] of Corollary IV.6.17) that there is an appropriate constant c

$\square$

[108]**Corollary IV.6.17 (Conway, Page 94):**Let G be simply connected and let $f : G \to \mathbb{C}$ be an analytic function such that $f(z) \neq 0$ for any z in G. Then there is an analytic function $g : G \to \mathbb{C}$ such that $f(z) = \exp g(z)$. If $z_0 \in G$ and $e^{w_0} = f(z_0)$, we may choose g such that $g(z_0) = w_0$.

Proof. Since f never vanishes, $\frac{f'}{f}$ is analytic on G; so, by the preceding corollary, it must have a primitive g_1. If $h(z) = \exp g_1(z)$ then h is analytic and never vanishes. So, $\frac{f}{h}$ is analytic and its derivative is

$$\frac{h(z)f'(z) - h'(z)f(z)}{h(z)^2}.$$

But

$$h' = \underbrace{\exp g_1(z)}_{h(z)} \frac{d}{dz} g_1(z) = g_1' h$$

so that

$$hf' - fh' = 0.$$

Hence $\frac{f}{h}$ is a constant c for all z in G. That is

$$f(z) = c \exp g_1(z) = \exp[g_1(z) + c']$$

for some c' (with $c = \exp c'$). By letting

$$g(z) = g_1(z) + c' + 2\pi i k$$

for an appropriate k, $g(z_0) = w_0$ and the theorem is proved. $\square$

such that

$$g = F + c$$

satisfies $f(z) = \exp g(z)$ for all z in G.

$(g) \Rightarrow (h)$ This is trivial[109].

$(h) \Rightarrow (i)$ **Case 1.** If $G = \mathbb{C}$ then the function

$$\frac{z}{1+|z|}$$

is a homeomorphism from $\mathbb{C}$ to the open unit disk $\{z : |z| < 1\}$. **Case 2.** If $G \neq \mathbb{C}$ then [110] Lemma VII.4.3 (Lemma 1.4.4 in Page 126) implies that there is an analytic mapping f of G onto D which is one-one. Such a map is a homeomorphism.

$(i) \Rightarrow (a)$ Assume G is homeomorphic to the unit disk. We have to show that G is simply connected.

G is homeomorphic to the unit disk allows us to consider a homeomorphism $h : G \to D = \{z : |z| < 1\}$. Also, let γ be a

---

[109]

$$f(z) = \exp g(z) = \exp[\tfrac{1}{2}g(z) + \tfrac{1}{2}g(z)] = \left\{ \underbrace{\exp[\tfrac{1}{2}g(z)]}_{=h(z),\ \text{say}} \right\}^2 = \{h(z)\}^2 .$$

[110]**Lemma VII.4.3:** Let G be a region which is not the whole plane and such that every non-vanishing analytic function on G has an analytic square root. If $a \in G$ then there is an analytic function $f : G \to \mathbb{C}$ having the properties:

(a) $f(a) = 0$ and $f'(a) > 0$;

(b) f is one-one;

(c) $f(G) = D = \{z : |z| < 1\}$.

closed curve in G (note that γ is not assumed to be rectifiable. We **claim** that $\gamma \sim 0$.). Then

$$\sigma(s) = (h \circ \gamma)(s) = h(\gamma(s))$$

is a closed curve[111] in the simply connected domain D. [112]Thus, there is a continuous function $\Lambda : I^2 \to D$ such that

$$\Lambda(s,\, 0) = \sigma(s) \text{ for } 0 \le s \le 1,$$

$\Lambda(s,\, 1) = 0$ for $0 \le s \le 1$ and $\Lambda(0,\, t) = \Lambda(1,\, t)$ for $0 \le t \le 1$.

It follows that $\Gamma = h^{-1} \circ \Lambda$ is a continuous map of I^2 into G and demonstrates that γ is homotopic to the curve which is constantly equal to[113] $h^{-1}(0)$.

$(h) \Rightarrow (j)$ Suppose that $G \neq \mathbb{C}$; then the Riemann Mapping Theorem[114] implies there is an analytic function h on

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[111] σ is a closed curve because

$$\sigma(0) = (h \circ \gamma)(0) = h(\underbrace{\gamma(0)}_{\gamma(1)}) = h(\gamma(1)) = \sigma(1).$$

[112] D has no hole. Hence any closed curve in it is homotopic to 0.

[113] Here $h^{-1}(0)$ means $h^{-1}(0)$.

[114] **Riemann Mapping Theorem** Let G be a simply connected region *which is not the whole plane* and let $a \in G$. Then there is a unique analytic function $f : G \to \mathbb{C}$ having the properties:

(a) $f(a) = 0$ and $f'(a) > 0$;

(b) f is one-one;

(c) $f(G) = \{z : |z| < 1\}$.

G such that h is one-one and $h(G) = D$. If $u : G \to \mathbb{R}$ is harmonic then $u_1 : D \to \mathbb{R}$ defined by

$$u_1 = u \circ h^{-1}$$

is a harmonic function on D.[115] By Theorem III.2.30[116] there is a harmonic function $v_1 : D \to \mathbb{R}$ such that $f_1 = u_1 + i\, v_1$ is analytic on D. Let $f = f_1 \circ h$. Then f is analytic on G and u is the real part of f. Thus $v = \mathrm{Im} f = v_1 \circ h$ is the sought after harmonic conjugate. Since Theorem III.2.30 also applies to $\mathbb{C}$, (j) follows from (h).

$(j) \Rightarrow (g)$: Suppose $f : G \to \mathbb{C}$ is analytic and never vanishes, and let $u = \mathrm{Re} f$, $v = \mathrm{Im} f$. If the real valued function $U : G \to \mathbb{R}$ is defined by[117] [118]

$$U(x,\, y) = \log|f(x + i\, y)| = \log[u(x,\, y)^2 + v(x,\, y)^2]^{1/2}$$

then a computation shows that U is harmonic[119]. Let V be a

[115] Verify that $\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0$.

[116] **Theorem III.2.30:** Let G be either the whole plane $\mathbb{C}$ or some open disk. If $u : G \to \mathbb{R}$ is a harmonic function then u has a harmonic conjugate.

[117] $\underbrace{\log \underbrace{|f(x + i\, y)|}_{\text{realnumber}}}_{>0}$

[118] $|f(x + i\, y)| = |u(x,\ y) + i\, v(x,\ y)| = \sqrt{u(x,\ y)^2 + v(x,\ y)^2}$

[119] Verify that the real valued function U of two variables x and y is harmonic by showing that $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$.

harmonic function on G such that

$$g = U + iV$$

is analytic on G and let $h(z) = \exp g(z)$. Then h is analytic , never vanishes[120], and $\left|\frac{f(z)}{h(z)}\right| = 1$ for all z in G.[121]

That is, f/h is an analytic function whose range is not open.[122]

[120]since $h(z) = \exp g(z) \neq 0$ for all z.

[121] $\left|\frac{f}{g}(z)\right| = \left|\frac{f(z)}{h(z)}\right| = 1$ for all $z \in G$. Hence $\frac{f}{g}(z)$ are points on the unit circle when $z \in G$. So range of $\frac{f}{g} \subset$ the unit circle.

[122]f/h is an analytic function whose range is not open: Because $\left|\frac{f(z)}{h(z)}\right| = 1$ for all z in G, f/h is a function such that Range of f/h is a subset of the unit circle.

Case 1. Range of f/h is the unit circle implies Range of f/h is not open.

Case 2. Range of f/h is a proper subset of the unit circle implies no point of Range of f/h is an interior point of Range of f/h (Ref. Fig. 2.29

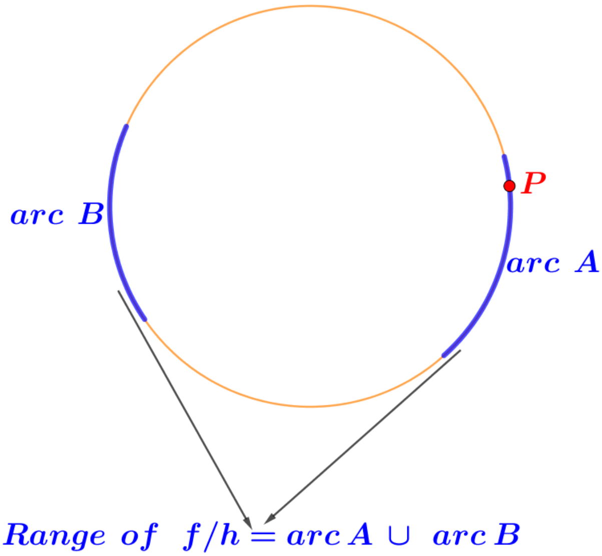

Figure 2.29: Case 2: $Range\ of\ \ f/h$ is the union of two arcs: $arc\,A$ and $arcB$ and hence Range of f/h is a proper subset of the unit circle. In the figure P is not an interior point of Range of f/h because there is no ball centered at P and wholly contained in Range of f/h.

It follows that there is a constant c such that

$$f(z) = ch(z) = c\exp g(z) = \exp[g(z) + c_1].$$

Thus, $g(z) + c_1$ is a branch of $\log f(z)$. This completes the proof of the theorem. $\square$

This theorem constitutes an aesthetic peak in mathematics. Notice that it says that a topological condition (simple connectedness) is equivalent to analytical conditions (e.g., the existence of harmonic conjugates and Cauchy's Theorem) as well as an algebraic condition (the existence of a square root) and other topological conditions. This certainly was not expected when simple connectedness was first defined. Nevertheless, the value of the theorem is somewhat limited to the fact that simple connectedness implies these nine properties. Although it is satisfying to have the converse of these implications, it is only the fact that the connectedness of $\mathbb{C}_\infty - G$ implies that G is simply connected which finds wide application. No one ever verifies one of the other properties in order to prove that G is simply connected.

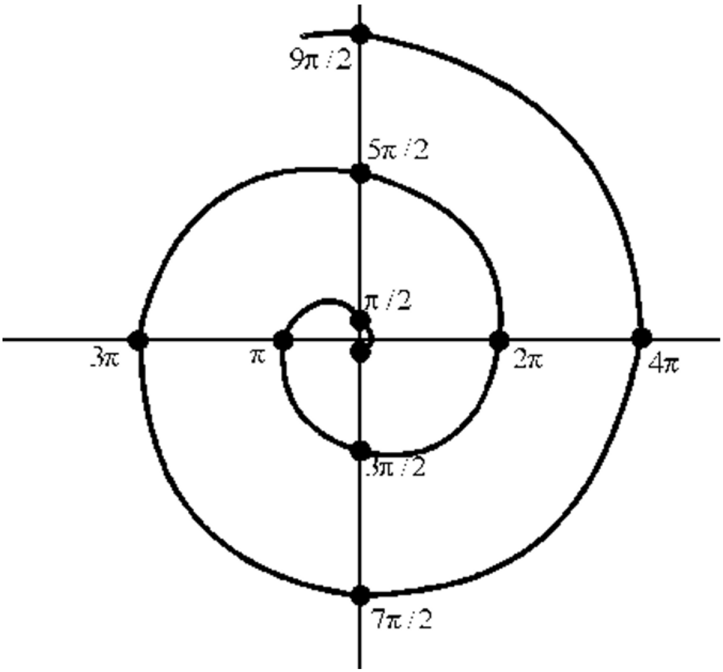

Figure 2.30:

For an example consider the set $G = \mathbb{C} - \{z = re^{ir} : 0 \leq r < \infty\}$; that is, G is the complement of the infinite spiral (See Example 2.5.5) $r = \theta$, $0 \leq \theta < \infty$. Then $\mathbb{C}_\infty - G$ is the spiral together with the point at infinity. Since this is connected, G is simply connected.

Example 2.5.5. We sketch the graph of the polar equation $r = \theta \quad (\theta \geq 0)$ by plotting points.
We first note some points on the curve:

- When $\theta = 0$, $r = \theta = 0$and this corresponds the point $(0,\ 0)$in polar coordinates and this point is the pole.

- When $\theta = \pi/4$, $r = \theta = \pi/4$and this corresponds the point $(\pi/4,\ \pi/4)$in polar coordinates, and this point is marked in polar graph (Fig. 2.31)

- When $\theta = \pi/2$, $r = \theta = \pi/2$and this corresponds the point $(\pi/2,\ \pi/2)$in polar coordinates, and this point is marked in polar graph (Fig. 2.31)

- When $\theta = \pi$, $r = \theta = \pi$and this corresponds the point $(\pi,\ \pi)$in polar coordinates, and this point is marked in polar graph (Fig. 2.31)

- When $\theta = 2\pi$, $r = \theta = 2\pi$ and this corresponds the point $P(2\pi,\ 2\pi)$in polar coordinates, and this point is at 2πdistance from the pole and OP makes angle 2π with the polar axis (i.e., OP lies on the polar axis). The point is marked in polar graph (Fig. 2.31)

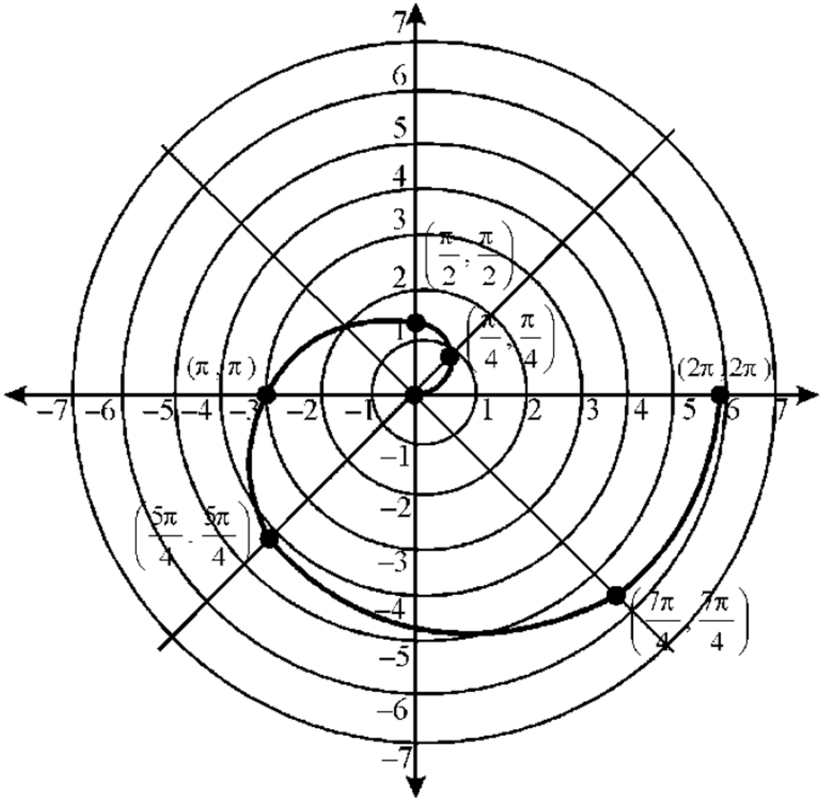
7
6
5
4
3
2
1
−1
−2
−3
−4
−5
−6
−7
−7 −6 −5 −4 −3 −2 −1 1 2 3 4 5 6 7
$\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$
$\left(\frac{\pi}{4}, \frac{\pi}{4}\right)$
(π, π)
$(2\pi, 2\pi)$
$\left(\frac{5\pi}{4}, \frac{5\pi}{4}\right)$
$\left(\frac{7\pi}{4}, \frac{7\pi}{4}\right)$

Figure 2.31:

Chapter 3

Mittag - Leffler's Theorem and Hadamard Factorization Theorem

3.1 Mittag - Leffler's Theorem

Consider the following problem: Let G be an open subset of $\mathbb{C}$ and let $\{a_k\}$ be a sequence of distinct points in G such that $\{a_k\}$ has no limit point in G. For each integer $k \geq 1$ consider the rational function

$$S_k(z) = \sum_{k=1}^{m_k} \frac{A_{jk}}{(z - a_k)^j}, \tag{3.1}$$

where m_k is some positive integer and $A_{1k}, \ldots, A_{mk}$ are arbitrary complex coefficients. Is there a meromorphic function f on G whose poles are exactly the points $\{a_k\}$ and such that the singular part of f at $z = a_k$ is $S_k(z)$? The answer is yes and this is the content of Mittag-Leffler's Theorem.

Theorem 3.1.1. Mittag - Leffler's Theorem *Let G be an open set, $\{a_k\}$ a sequence of distinct points in G without a limit point in G, and let $\{S_k(z)\}$ be the sequence of rational functions given by equation (3.1). Then there is a meromorphic function f on G whose poles are exactly the points $\{a_k\}$ and such that the singular part of f at a_k is $S_k(z)$.*

Proof. Although the details of this proof are somewhat cumbersome, the idea is simple. We use Runge's Theorem to find rational functions $\{R_k(z)\}$ with poles in $\mathbb{C}_\infty - G$ such that the sequence $\left\{\sum_{k=1}^{n} S_k(z) - R_k(z)\right\}$ is a Cauchy sequence in[1] $M(G)$. The resulting limit is the sought after meromorphic function. (Actually we must do a little more than this.)[2]

[1] $M(G)$ is the set of meromorphic functions on G.

[2] Details: $R_k(z)$ has no pole in G. Also, $S_k(z)$ given by (3.1) has only pole at a_k. Hence $S_k(z) - R_k(z)$ has the only pole a_k. Hence $\sum_{k=1}^{n} [S_k(z) - R_k(z)]$ has poles only at the points $a_1, \ldots, a_n$. Thus,

$$\sum_{k=1}^{\infty} [S_k(z) - R_k(z)]$$

has poles only at the points $a_1, a_2, \ldots, \ldots$ i.e., $\sum_{k=1}^{\infty} [S_k(z) - R_k(z)]$ is a mero-

We use Proposition[3] 1.1.6 in Page 6 to find compact subsets of G such that

$$G = \bigcup_{n=1}^{\infty} K_n, \quad K_n \subset \mathrm{int} K_{n+1},$$

and each component of $\mathbb{C}_\infty - K_n$ contains a component of $\mathbb{C}_\infty - G$. Since each K_n is compact and $\{a_k\}$ has no limit point in G, there are only a finite number of points a_k in each K_n.[4]

Define the sets of integers I_n as follows:

morphic function with poles only at the points $a_1, a_2, \ldots, \ldots$. This is the meromorphic function that we need.

[3] **Proposition VII.1.2:** If G is an open set in $\mathbb{C}$ then there is a sequence $\{K_n\}$ of compact subsets of G such that

$$G = \bigcup_{n=1}^{\infty} K_n. \tag{3.2}$$

Moreover, the sets K_n can be chosen to satisfy the following conditions:

(a) $K_n \underset{\neq}{\subset} \mathrm{int}(K_{n+1})$

(b) $K \subset G$ and K is compact implies $K \subset K_n$ for some n;

(c) Every component of $\mathbb{C}_\infty - K_n$ contains a component of $\mathbb{C}_\infty - G$.

[4] If K_n contains infinitely many points of $\{a_k\}$, then that collection is an infinite bounded subset of $\mathbb{C}$ (the collection is bounded as it is a subset of the compact set K_n which is (closed and) bounded) and hence, by Bolzano Weierstrass Theorem, it has a limit points in K_n and hence in G. Thus the sequence $\{a_k\}$ has a limit point in G which is a contradiction.

$$\begin{aligned} I_1 &= \{k : a_k \in K_1\}, \\ I_n &= \{k : a_k \in K_n - k_{n-1} \quad \text{for } n \geq 2.\}, \end{aligned}$$

Define functions f_n by

$$f_n(z) = \sum_{k \in I_n} S_k(z)$$

for $n \geq 1$. Then f_n (being finite sum of rational functions) is a rational function and its poles are the points[5]

$$\{a_k : k \in I_n\} \subset K_n - K_{n+1}.$$

(If I_n is empty let $f_n = 0$). Since f_n has no poles in K_{n-1} (for $n \geq 2$) (see (3.3) in the footnote) it is analytic in a neighborhood of K_{n-1}.[6] According to Runge's Theorem there is a rational function $R_n(z)$ with its poles in $\mathbb{C}_\infty - G$ and that satisfies

$$|f_n(z) - R_n(z)| < \left(\frac{1}{2}\right)^n$$

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[5]

$$f_n(z) = \underbrace{\sum_{k \in I_n} \underbrace{\sum_{j=1}^{m_k} \frac{A_{j\,k}}{(z-a_k)^j}}_{\text{pole at} a_k}}_{\text{poles at} a_k,\ k \in I_n,\ \text{where } I_n = \{k:\ a_k \in K_n - K_{n-1}\}} \tag{3.3}$$

[6]This means f_n is analytic in an open set containing K_{n-1}.

for all z in K_{n-1}. We claim that

$$f(z) = f_1(z) + \sum_{n=2}^{\infty} [f_n(z) - R_n(z)] \tag{3.4}$$

is the desired meromorphic function. It must be shown that f is a meromorphic function and that it has the desired properties. Start by showing the series $\sum_{n=2}^{\infty} [f_n(z) - R_n(z)]$ in (3.4) converges uniformly on every compact subset of $G - \{a_k : k \geq 1\}$. This will give that f is analytic on $G - \{a_k : k \geq 1\}$ and it will only remain to show that each a_k is a pole with singular part $S_k(z)$.

So let K be a compact subset of $G - \{a_k : k \geq 1\}$; then K is a compact subset of G and, therefore, there is an integer N such that

$$K \subset K_N.$$

If $n \geq N$ then

$$|f_n(z) - R_n(z)| < \left(\frac{1}{2}\right)^n \text{ for all } z \text{ in } K.$$

That is, the series (3.4) is dominated on K by a convergent series of numbers; by the Weierstrass M-test the series (3.4) converges uniformly on K. Thus[7] f is analytic on $G - \{a_k : k \geq 1\}$.

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Now consider a fixed integer $k \geq 1$; there is a number $R > 0$ such that[8]

$$|a_j - a_k| > R \text{ for } j \neq k\,.$$

That is, the members of $\{a_j\}$ that is not a_k is at distance greater than R from a_k.

Thus,

$$f(z) = S_k(z) + g(z) \text{ for } 0 < |z - a_k| < R, \tag{3.5}$$

where g is analytic in $B(a_k;\, R)$. Hence, $z = a_k$ is a pole of f and $S_k(z)$ is its singular part.

Take

$$g_n(z) = f_n(z) - R_n(z) \text{ and } M_n = \left(\frac{1}{2}\right)^n$$

then since for each $n = 1,\, 2,\, \ldots,$

$$|g_n(z)| = |f_n(z) - R_n(z)| \leq M_n \;\forall z \in K$$

and since $\sum M_n = \sum \left(\frac{1}{2}\right)^n$ converges, it follows from Weierstrass M-test that the series (3.4) converges uniformly on K. Now, being the uniform limit of analytic functions, f is analytic on K. Hence f is analytic on $G - \{a_k : k \geq 1\}$ (Reason: K is an arbitrary compact subset of $G - \{a_k : k \geq 1\}$ and f is analytic on K implies (by varying K) that f is analytic at each point of $G - \{a_k : k \geq 1\}$. Thus, f is analytic on $G - \{a_k : k \geq 1\}$.)

[8]Why this is possible? Reason: As $\{a_j\}$ has no limit point, a_k is not a limit point of $\{a_j\}$ and hence a_k has a neighbourhood (of radius $R+1$) such that $B(a_k;\, R_{k+1})$ doesn't contain a point of $\{a_j\}$ other than a_k. Thus,

$$|a_j - a_k| \geq R + 1 > R \quad \text{for} \quad j \neq k$$

(3.5) shows that f has no pole other than $z = a_k$ in the neighbourhood $B(a_k;\, R)$.[9] i.e., f is analytic on $G - \{a_k : k \geq 1\}$ (this is observed earlier) and by the discussion just above (by varying the integers k ($k \geq 1$)) f has poles at the points $a_1,\, a_2,\, \ldots$[10] There exists $R_1 > 0$ such that f has only one pole at $z = a_1$ on $B(a_1;\, R_1)$, there exists $R_2 > 0$ such that f has only one pole at $z = a_2$ on $B(a_2;\, R_2)$, and so on. Thus, on G, the points $a_1,\, a_2,\, \ldots$ are the only pole points.[11] Hence f is a meromorphic function f on G whose poles are exactly the points $\{a_k : k \geq 1\}$ and such that the singular part of f at a_k is $S_k(z)$. □

Example 3.1.2. (The simplest meromorphic function in the plane with pole at every integer n)

$$\frac{1}{z-n}$$

has singularity at $z = n$. That is, the simplest singular part (of a meromorphic function with pole at $z = n$ is) $\frac{1}{z-n}$. But

$$\sum_{n=-\infty}^{\infty} \frac{1}{z-n}$$

[9]On $B(a_k;\, R)$, f has two parts: $g(z)$ is analytic on $B(a_k;\, R)$ and $S_k(z)$, the singular part, has only one pole at $z = a_k$ (See 3.1) on $B(a_k;\, R)$, .

[10] See footnote

[11]Reason: f is analytic on $G - \{a_1,\, a_2,\, \ldots\}$. So the possible singular points of f on G are $a_1,\, a_2,\, \ldots$. By the observation, $a_1,\, a_2,\, \ldots$ are poles of f on G. Thus, the only poles of f on G are $a_1,\, a_2,\, \ldots$.)

doesn't converge in the (set of meromorphic functions) $M(\mathbb{C})$. However,

$$\frac{1}{z-n}+\frac{1}{z+n}=\frac{2z}{z^2-n^2}$$

and

$$\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

does converge in $M(\mathbb{C})$. The singular part of this function at $z=m$ is $\frac{1}{z-m}$.[12]

3.2 Analytic Continuation and Riemann Surfaces: Introduction

Consider the following problem. Let f be an analytic function on a region G; when can f be extended to an analytic function f_1 on an open set G_1 which properly contains G?

Case 1. If G_1 is obtained by adjoining to G a ***disjoint*** open set so that G becomes a component of G_1, f can be ex-

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$$\frac{1}{z}+\sum_{\substack{n=1\\ n\neq m}}^{\infty}\frac{2z}{z^2-n^2}+\left(\frac{2z}{z+m}\right)\left(\frac{1}{z-m}\right) \tag{3.6}$$

and

$$\pi\cot\pi z=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}$$

for z not an integer. That is, the meromorphic function in (3.6) converges to $\pi\cot\pi z$.

tended to G_1 **by defining it in any way we wish** on $G_1 - G$ so long as the result is analytic.

Case 2. So **to eliminate such trivial cases** (as in Case

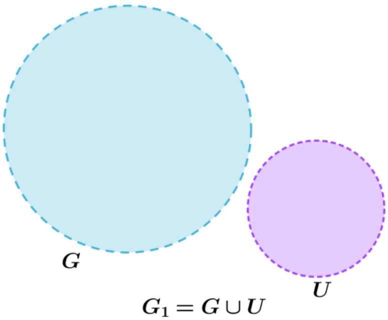
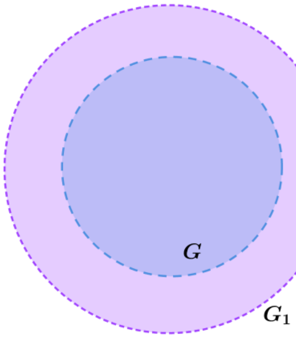


Figure 3.1: $G_1 = G \cup U$ and G is a component of G_1. Hence the analytic function f on G can be extended to G_1 by defining it in any way we wish on $G_1 - G$ (here U) so long as the result is analytic.

1) it is required that G_1 also be a region (i.e., G_1 also an open connected region so that G cannot be a component of G_1 and so f defined on G cannot be "easily" extended to G_1 to get an analytic function.)

Actually, this process has already been encountered. Recall that in the discussion of the Riemann Zeta function (Page 218) $\zeta(z)$ was initially defined for $\mathrm{Re}z > 1$. Using various identities, principal among which was Riemann's functional equation, ζ was extended so that it was defined and analytic in $\mathbb{C} - \{1\}$ with a simple pole at $z = 1$. That is, ζ was analytically continued from a smaller region to a larger one.



$G \subset G_1$

Figure 3.2: In the figure, $G \subset G_1$, but G is not a component of G_1. Hence the analytic function f on G cannot be easily extended to G_1 by defining it in any way we wish on $G_1 - G$ to get an analytic function on G_1.

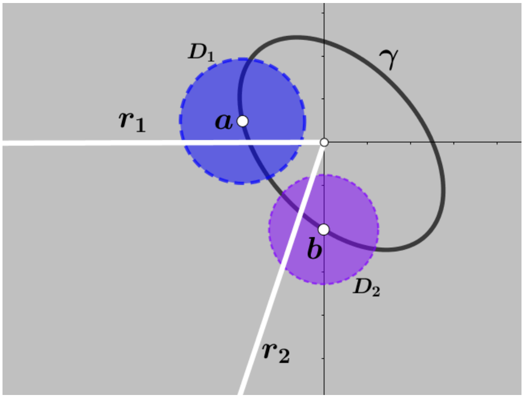


Figure 3.3: γ is a closed curve. If we consider a as the initial point (also terminal point) then for the disk D_1 about a we choose a branch l_1 of the complex logarithmic function $\log f$ with branch cut as the ray r_2. For the disk D_2 about b we choose a branch l_2 of the complex logarithmic function $\log f$ with branch cut as the ray r_1. This is to ensure that concerned branches of logarithms are analytic on the disks D_1 and D_2. We proceed as this at every point.

Another example was in the discussion that followed the proof of the Argument Principle.[13] There a meromorphic function f and a closed

[13][**Argument Principle**] Let f be meromorphic in G with poles $p_1,\ p_2,\ \ldots\ ,\ p_m$ and zeros $z_1,\ z_2,\ \ldots\ ,\ z_n$ counted according to multiplicity. If γ is a closed rectifiable curve in G with $\gamma \approx 0$ and not passing through $p_1,\ p_2,\ \ldots\ ,\ p_m;\ z_1,\ z_2,\ \ldots\ ,\ z_n$; then

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} n(\gamma;\ z_k) - \sum_{j=1}^{m} n(\gamma;\ p_j). \qquad (3.7)$$

Note: The right hand side of (3.7) is an integer.

Proof.

$$\frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \frac{1}{z-a_k} - \sum_{j=1}^{m} \frac{1}{z-p_j} + \frac{g'(z)}{g(z)}$$

where g is analytic and never vanishes in G. [NOTE: g'/g is analytic on G as g has no poles and no zeros.] Thus,

$$\int_{\gamma} \frac{f'(z)}{f(z)} = \sum_{k=1}^{n} \underbrace{\int_{\gamma} \frac{1}{z-a_k}}_{2\pi i\, n(\gamma;\ z_k)} - \sum_{j=1}^{m} \underbrace{\int_{\gamma} \frac{1}{z-p_j}}_{2\pi i\, n(\gamma;\ p_j)} + \underbrace{\int_{\gamma} \frac{g'(z)}{g(z)}}_{\substack{=0,\text{ by Cauchy's Theorem,}\\ \text{since } g'/g \text{ is analytic on } G}}$$

which gives

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} = \sum_{k=1}^{n} n(\gamma;\ z_k) - \sum_{j=1}^{m} n(\gamma;\ p_j).$$

□

Since no zero or pole of f lies on γ there is a disk $B(a;\ r)$, for each a in $\{\gamma\}$, the trace of γ, such that a branch of $\log f(z)$ can be defined on $B(a;\ r)$ [this is possible by taking r sufficiently small that (We choose r to ensure that like this $\log f(z)$ can be defined on $B(a;\ r)$.) (Fig. 3.3) $f(z) \neq 0$ or ∞ in $B(a;\ r)$.] This r depends on the choice of a i.e., $r = r_a$. Then the collection of these type of balls, i.e.,

$$\{B(a;\ r_a) : a \in \{\gamma\}\}$$

form an open cover for the compact set $\{\gamma\}$ [$\{\gamma\}$ is closed and bounded, hence compact] and hence by Lebesgue's Covering Lemma, [[**Lebesgue's Covering Lemma:**] If $(X,\ d)$ is a compact space and $\mathcal{U}$ is an open cover of X then there is an $\delta > 0$ such that if $x \in X$, there is a set U in $\mathcal{U}$ with $B(x,\ \delta) \subset U$. (In other words, every ball of radius δ is contained in some member of $\mathcal{U}$).] there is a positive number $\varepsilon > 0$ such that

$$\text{for each } a \in \{\gamma\},\ \ B(a;\ \varepsilon) \subseteq B(a;\ r_a),$$

and hence by our construction this shows that, for each $a \in \{\gamma\}$ we can define a branch of $\log f(z)$ on $B(a;\ \varepsilon)$. Suppose the rectifiable curve (continuous function) γ is defined over the compact set $[0,\ 1]$. Then, being a continuous function over a compact set, γ is uniformly continuous on $[0,\ 1]$ so that corresponding rectifiable curve γ not passing through any zero or pole of f was given. If $z = a$ is the initial point of γ (and also the final point, because γ is a closed curve), we put a disk D_1 about a on which it was possible to define a branch ℓ_1 of the

---

to the so obtained ε, there is a $\delta > 0$ such that

$$|\gamma(t) - \gamma(s)| < \varepsilon \quad \text{whenever} \quad |t - s| < \delta. \tag{3.8}$$

Starting with $t_0 = 0$ and choosing points $t_0 < t_1 < t_2 < \cdots < t_k = 1$ such that two consecutive points are at distance less than $\delta/2$ it follows that $0 = t_0 < t_1 < \cdots < t_k = 1$ is partition of $[0,\, 1]$ such that [When $1 \le j \le k$, $t_{j-1} \le t \le t_j$, i.e., $t \in [t_{j-1},\, t_j]$ implies $|t - t_{j-1}| < \delta$ implies (using (3.8)) $|\gamma(t) - \gamma(t_{j-1})| < \varepsilon$ implies $\gamma(t) \in B(\gamma(t_{j-1});\ \varepsilon)$.]

$$\gamma(t) \in B(\gamma(t_{j-1});\ \varepsilon) \tag{3.9}$$

for $t_{j-1} \le t \le t_j$ and $1 \le j \le k$. Let l_j be a branch of $\log f$ defined on $B(\gamma(t_{j-1});\ \varepsilon)$ for $1 \le j \le k$. Also, since the j-th sphere $B(\gamma(t_j);\ \varepsilon)$ and $(j+1)$th sphere $B(\gamma(t_{j+1});\ \varepsilon)$ both contain $\gamma(t_j)$ [this is possible by (3.9)] we can choose the branches $l_1,\, \ldots, l_k$ so that [$l_1(\gamma(t_1)) = l_2(\gamma(t_1))$ means that l_1 and l_2 are chosen in such a way that their arguments are the same at $\gamma(t_1)$.]

$$l_1(\gamma(t_1)) = l_2(\gamma(t_1));\ l_2(\gamma(t_2)) = l_3(\gamma(t_2));\ \ \ldots\ ;$$

$$l_{k-1}(\gamma(t_{k-1})) = l_k(\gamma(t_{k-1})).$$

If γ_j is the path restricted to $[t_{j-1},\, t_j]$ then, since $l_j' = f'/f$, we have (by Theorem: Let G be open in $\mathbb{C}$ and let γ be a rectifiable path in G with initial and end points α and β respectively. If $f : G \to \mathbb{C}$ is a continuous function with a primitive (F is a primitive of f when $F' = f$.) $F : G \to \mathbb{C}$, then $\int\limits_\gamma f = F(\beta) - F(\alpha)$.) that

$$\int\limits_{\gamma_j} \frac{f'}{f} = l_j(\gamma(t_j)) - l_j(\gamma(t_{j-1}))$$

for $1 \le j \le k$. Summing the both sides of k number of the above equations,

$$\int\limits_{\gamma_1} \frac{f'}{f} + \int\limits_{\gamma_2} \frac{f'}{f} + \cdots + \int\limits_{\gamma_k} \frac{f'}{f}$$

complex logarithmic function $\log f$ (Fig. 3.3). Continuing, we covered γ by a finite number of disks D_1, D_2, $\ldots$, D_n, where consecutive disks intersect and such that there is a branch ℓ_j of $\log f$ on D_j. Furthermore, the functions ℓ_j were chosen so that

$$\ell_j(z) = \ell_{j-1}(z) \text{ for } z \in D_{j-1} \cap D_j, \ 2 \leq j \leq n.$$

$$\begin{aligned} &= \underbrace{\underbrace{l_1(\gamma(t_1))}_{l_2(\gamma(t_1))} - l_1(\gamma(t_0))} + \underbrace{\underbrace{l_2(\gamma(t_2))}_{l_3(\gamma(t_2))} - l_2(\gamma(t_1))} \\ &\quad + \cdots + \underbrace{\underbrace{l_k(\gamma(t_k))}_{l_k(\gamma(t_0))} - l_k(\gamma(t_{k-1}))} \end{aligned}$$

Telescoping the right hand sides gives us

$$\int\limits_\gamma \frac{f'}{f} = l_k(\gamma(t_0)) - l_1(\gamma(t_0)) = l_k(a) - l_1(a)$$

where we take $a = \gamma(t_0)) = \gamma(t_1))$. That is,

$$l_k(a) - l_1(a) = 2\pi i K$$

for some integer K, because using the Argument Principle,

$$\frac{1}{2\pi i} \int\limits_\gamma \frac{f'(z)}{f(z)} = K \Rightarrow \int\limits_\gamma \frac{f'(z)}{f(z)} = 2\pi i K.$$

Because $2\pi i K$ is purely imaginary we get

$$\mathrm{Im} l_k(a) - \mathrm{Im} l_1(a) = 2\pi K.$$

That is,

$$\arg l_k(a) - \arg l_1(a) = 2\pi K.$$

The process analytically continues ℓ_1 to $D_1 \cup D_2$, then $D_1 \cup D_2 \cup D_3$, and so on. However, *an unfortunate thing (for this continuation) happened when the last disk D_n was reached.* According to the Argument Principle it is distinctly possible that

$$\ell_n(z) \neq \ell_1(z) \text{ for } z \in D_n \cap D_1.$$

In fact,

$$\ell_n(z) - \ell_1(z) = 2\pi i K$$

for some (possibly zero) integer K.

This last example is a particularly fruitful one. *This process of continuing a function along a path will be examined and a criterion will be derived which ensures that continuation around a closed curve results in the same function that begins the continuation.* Also, the fact that continuation around a closed path can lead to a different function than the one started with, will introduce us to the concept of a Riemann Surface.

This chapter begins with the Schwarz Reflection Principle which is more like the process used to continue the zeta function than the process of continuing along an arc.

3.3 Schwarz Reflection Principle

If G is a region and (some examples are given in Fig. 3.4)

$$G^* = \{z : \bar{z} \in G\}$$

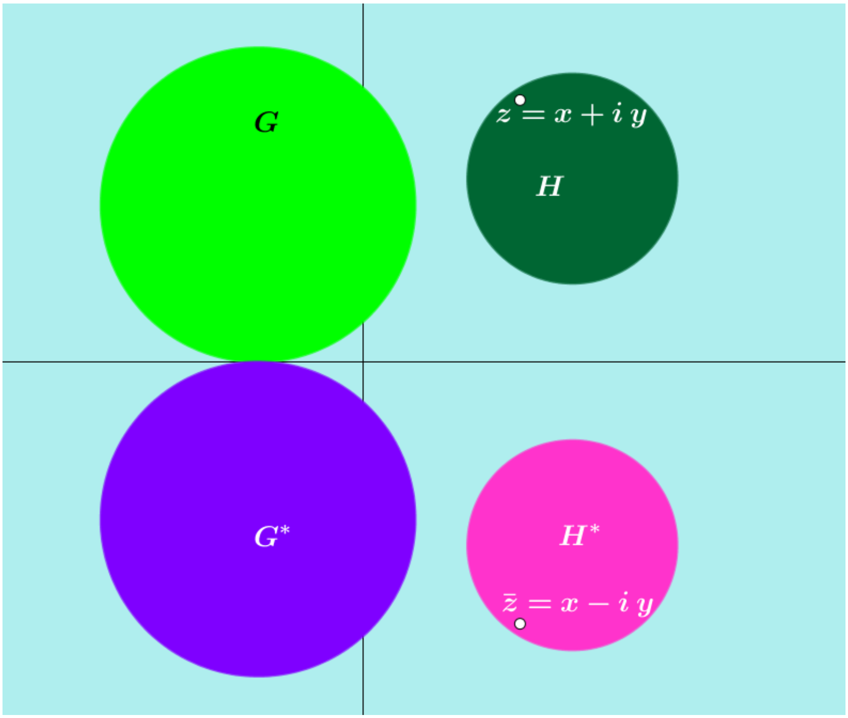


Figure 3.4: Figures showing G, G^*, H and H^*. In general, a set G need not be an open disk as shown in the figure. *Exercise:* Draw four different regions K and its K^*.

and if f is an analytic function on G, then

$$f^* : G^* \to C$$

defined by

$$f^*(z) = \overline{f(\bar{z})}$$

is also analytic[14]. Now suppose that (an example is given in Fig. 3.5)

$$G = G^*;$$

that is, G is symmetric with respect to the real axis.

[14]Note: f^* is a well defined function, because for each $z \in G^*$, $\bar{z} \in G$, and hence $f(\bar{z})$ is defined and so its conjugate is defined.

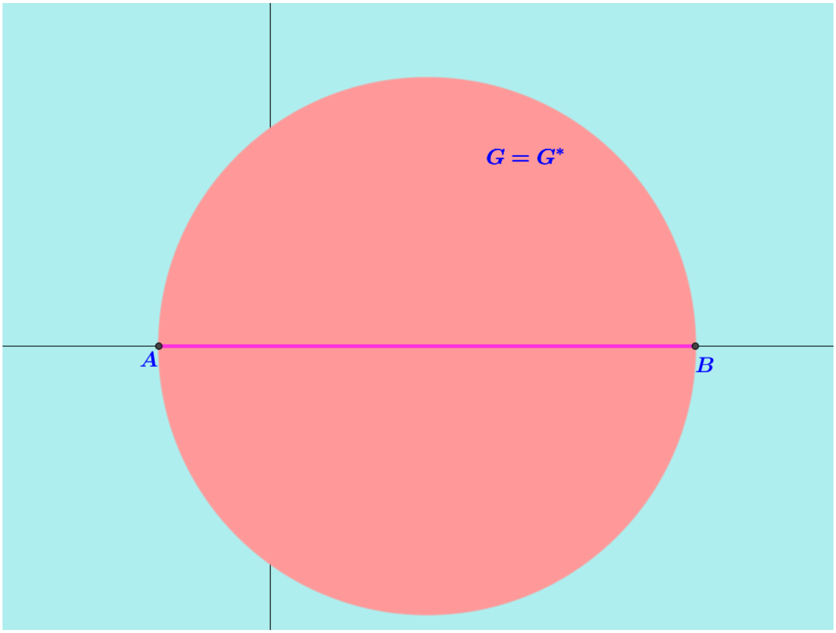


Figure 3.5: An example where $G = G^*$. It contains an open interval of the real line with end points A and B. Any subinterval of the same is also a choice. Note: In general G may not be a disk like the above. There are cases where intersection of certain set $H = H^*$ with the real line will give union of disjoint intervals. *Exercise:* Draw four different $H = H^*$ and observe this.

Then

$$g(z) = f(z) - \underbrace{\overline{f(\bar{z})}}_{f^*(z)}$$

is analytic on G. Since G is connected it must be that G contains an open interval of the real line (An example is given in Fig.3.5). Suppose $f(x)$ is real for all x in $G \cap \mathbb{R}$; then

$$g(x) \equiv f(x) - \overline{f(\bar{x})} \equiv f(x) - f(x) \equiv 0 \text{ for } x \in G \cap \mathbb{R}.$$

But $G \cap \mathbb{R}$ has a limit point in G so that[15]

$$f(z) = \overline{f(\bar{z})} \text{ for all } z \in G.$$

The fact that f must satisfy this equation is used to extend a function defined on $G \cap \{z : \text{Im} z \geq 0\}$ to all of G.

If G is a symmetric region (i.e., $G = G^*$) then let

$$G_+ = \{z \in G : \text{Im} z > 0\}$$

$$G_- = \{z \in G : \text{Im} z < 0\}$$

and

$$G_0 = \{z \in G : \text{Im} z = 0\}.$$

Theorem 3.3.1. ***[Schwarz Reflection Principle]*** *Let G be a region such that $G = G^*$. If*

$$f : G_+ \cup G_0 \to \mathbb{C}$$

[15] Why? This follows using the Identity Theorem: "If g and h are analytic on a region G, then $g \equiv h$ if and only if $\{z \in G : g(z) = h(z)\}$ has a limit point in G." In the present situation, $\{z \in G : g(z) = 0\}$ has a limit point in G. So, $g \equiv 0$ on G, so that $f(z) = \overline{f(\bar{z})}$ for all $z \in G$.

is a continuous function which is analytic on G_+ *and if* $f(x)$ *is real for* $x \in G_0$*, then there is an analytic function*

$$g : G \to \mathbb{C}$$

such that

$$g(z) = f(z)$$

for $z \in G_+ \cup G_0$.

Proof. Define[16]

$$g(z) = \begin{cases} \overline{f(\bar{z})} \ , & z \in G_- \\ f(z) \ , & z \in G_+ \cup G_0 \end{cases}$$

Clearly g is continuous on $G_+ \cup G_0$. It is continuous on G_- also: for $z \in G_-$ and any sequence (z_n) in G_- that converges to z, we have

$$(\overline{z_n}) \to \bar{z}$$

in G_+, and by the continuity of f on $G_+ \cup G_0$ it follows (by the Sequential Criterion for Continuity) that

$$(f(\overline{z_n})) \to f(\bar{z})$$

[16] Clearly g is a well-defined function on $G_+ \cup G_0$ because f is a function there. It is also well-defined on G_- also: for $z \in G_-$, $\bar{z} \in G_+$ and hence we get the value $f(\bar{z})$ and we take its conjugate and get $\overline{f(\bar{z})}$.

and so

$$(\overline{f(\overline{z_n})}) \to \underbrace{\overline{f(\bar{z})}}_{g(z)}$$

and by the definition of g, it follows that

$$(g(z_n)) \to g(z)$$

and hence g is continuous at z. Since z is an arbitrary point on G_- it follows that g is continuous on G_- and hence g is continuous on $G = G_+ \cup G_0 \cup G_-$.

lt is trivial that g is analytic on $G_+ \cup G_-$. [17] It remains to show that f is analytic on G_0. For this, fix a point[18] $x_0 \in G_0$ and let $R > 0$ with

$$B(x_0;\ R) \subset G.$$

It suffices to show that g is analytic on $B(x_0;\ R)$; to do this

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[17] [g is analytic on G_+ follows from the fact that f is analytic on G_+. Also, for any $z \in G_-$, $\bar{z} \in G_+$, and hence f is differentiable at $\bar{z}$. i.e.,

$$\lim_{h \to 0} \frac{f(\bar{z}+h) - f(\bar{z})}{h}$$

exists, so that

$$\lim_{h \to 0} \frac{\overline{f(\bar{z}+h)} - \overline{f(\bar{z})}}{\bar{h}} = \lim_{\bar{h} \to 0} \frac{g(z+\bar{h}) - g(z)}{\bar{h}}$$

exists, showing that g is differentiable at z. Since z is an arbitrary point on G_- it follows that g is differentiable on the open connected region G_- and hence g is analytic on G_-.

[18] $x_0 \in G_0$ is a real number.

we apply Morera's Theorem.

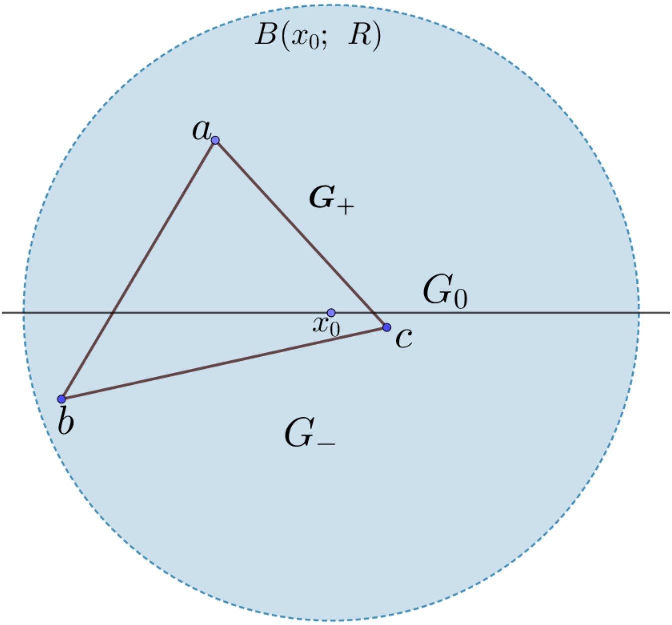

Figure 3.6: $B(x_0;\, R) \subset G$ and the triangle $T = [a,\, b,\, c,\, a] \subset B(x_0;\, R)$.

Let $T = [a,\, b,\, c,\, a]$ be a triangle in $B(x_0;\ R)$. To show that[19]

$$\int_T f = 0$$

[19]Instead of proving that $\int_T g = 0$ why we show that $\int_T f = 0$? This will be obvious from equation (3.10).

it is sufficient to show that

$$\int_P f = 0$$

whenever P is a triangle or a quadrilateral lying entirely in $G_+ \cup G_0$ or $G_- \cup G_0$.

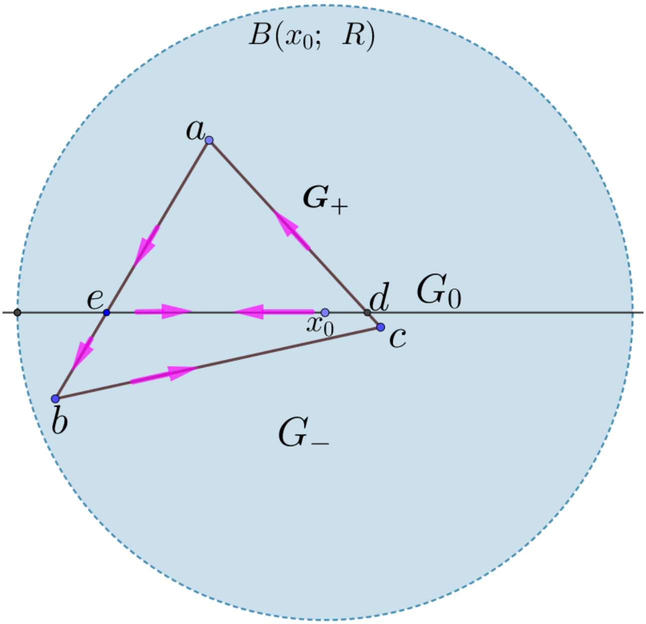

Figure 3.7: $B(x_0;\, R) \subset G$ and the triangle $T = [a,\, b,\, c,\, a] \subset B(x_0;\, R)$. Also if we let $T_1 = [e,\, d,\, a,\, e]$ and P be the quadrilateral $P = [e,\, b,\, c,\, d,\, e]$ then $\int\limits_T f = \int\limits_{T_1} f + \int\limits_P f$ and hence to show that $\int\limits_T f = 0$ it is enough to show that each integral on the right hand side give the value 0. Note T_1 is a subset of $G_+ \cup G_0$ while P is a subset of $G_- \cup G_0$. Similarly, various cases can be considered.

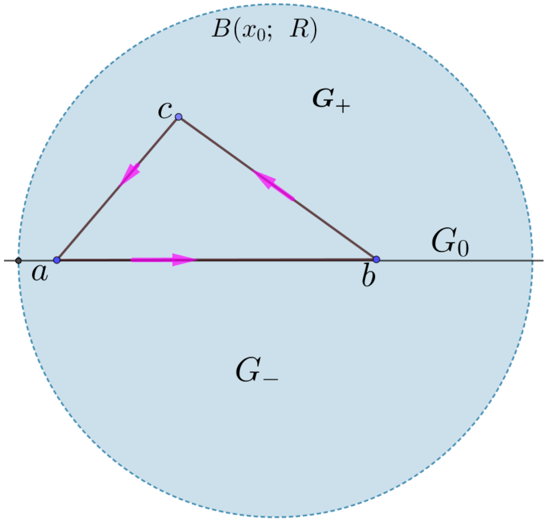

Figure 3.8:

In fact, this is easily seen by considering various pictures given as in Fig. 3.7. Therefore assume that $T \subset G_+ \cup G_0$ and $[a,\ b] \subset G_0$ (An example is given in Fig. 3.8). The proof of the other cases is similar.

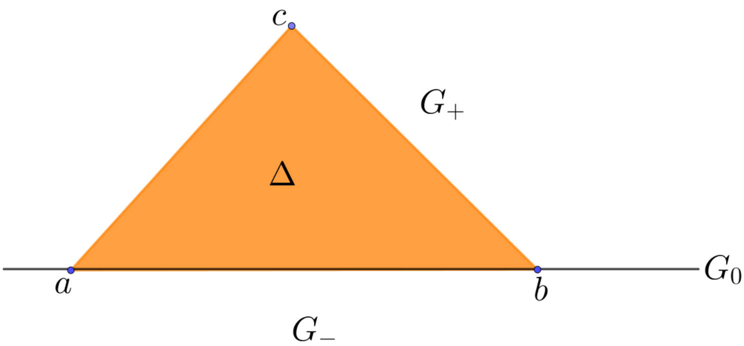

Figure 3.9:

Let Δ designate T together with its inside (Fig. 3.9); then

$$g(z) = f(z) \text{ for all } z \in \Delta. \tag{3.10}$$

By hypothesis f is continuous on $G_+ \cup G_0$ and so f is **uniformly continuous** on (the closed and bounded, hence compact set) Δ. So if $\varepsilon > 0$ (then by the uniform continuity of f there is a $\delta > 0$ (depends only on ε and independent of the choice of points on Δ) such that when z and z' are elements in Δ and $|z - z'| < \delta$ then

$$|f(z) - f(z')| < \varepsilon. \tag{3.11}$$

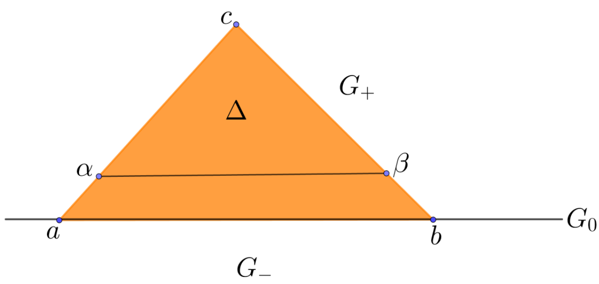

Figure 3.10:

Now choose α and β on the line segments $[c,\ a]$ and $[b,\ c]$ respectively (Fig. 3.10), so that $|\alpha - a| < \delta$ and $|\beta - b| < \delta$. Let

$$T_1 = [\alpha,\ \beta,\ c,\ \alpha]$$

and

$$Q = [a,\ b,\ \beta,\ \alpha,\ a].$$

Then

$$\int_T f = \int_{T_1} f + \int_Q f$$

but T_1 and its inside are contained in G_+ and f is analytic there; hence $\int_{T_1} f = 0$ by Cauchy's theorem, and hence

$$\int_T f = \int_Q f. \tag{3.12}$$

We note that:

- a parametric form of the line segment $[a, b]$ is

$$z = tb + (1-t)a, \ \ 0 \leq t \leq 1 \tag{3.13}$$

  and hence

$$dz = (b-a)dt \tag{3.14}$$

- a parametric form of the line segment $[\alpha, \beta]$ is

$$z = t\beta + (1-t)\alpha, \ \ 0 \leq t \leq 1 \tag{3.15}$$

  and hence

$$dz = (\beta - \alpha)dt \tag{3.16}$$

  so parametric form of the line segment $[\beta, \alpha]$ is

$$z = -t\beta - (1-t)\alpha, \ \ 0 \leq t \leq 1 \tag{3.17}$$

  and hence

$$dz = -(\beta - \alpha)dt \tag{3.18}$$

But if $0 \leq t \leq 1$ then

$$\left| \underbrace{[(t\beta + (1-t)\alpha] - [(tb + (1-t)a]}_{t(\beta - b) + (1-t)(\alpha - a)} \right|$$

$$\leq t \underbrace{|\beta - b|}_{<\delta} + (1-t)\underbrace{|\alpha - a|}_{<\delta} < \delta \tag{3.19}$$

so that (using (3.11) with $z = t\beta + (1-t)\alpha$ and $z' = tb + (1-t)a$)

$$|f(t\beta + (1-t)\alpha) - f(tb + (1-t)a)| < \varepsilon. \tag{3.20}$$

If

$$M = \max\{|f(z)| : z \in \Delta\} \tag{3.21}$$

(such a maximum exists, since $|f|$ is continuous and hence bounded on the compact set Δ) and

$$l = \text{the perimeter of } T.$$

We obtain (using (3.13) to (3.18))

$$\left| \int_{[a,\, b]} f + \int_{[\beta,\, \alpha]} f \right|$$

$$= \left| (b-a) \int_0^1 f(tb + (1-t)a)dt - (\beta - \alpha) \int_0^1 f(t\beta + (1-t)\alpha)dt \right|$$

$$\begin{aligned}
&\leq |b-a|\left|\int_0^1 [f(tb+(1-t)a)-f(t\beta+(1-t)\alpha)]\,dt\right| \\
&\quad +|(b-a)-(\beta-\alpha)|\left|\int_0^1 f(t\beta+(1-t)\alpha)dt\right|,
\end{aligned}$$

by adding and subtracting $(b-a)f(t\beta+(1-t)\alpha)$ and then applying triangle inequality

$$\begin{aligned}
&\leq \varepsilon\,|b-a|+M\,|(b-\beta)+(\alpha-a)| \text{ details in footnote} \\
&\leq \varepsilon l+2M\delta \qquad (3.22)
\end{aligned}$$

[20] Also, when $t=0$ (3.19) gives that $|a-\alpha|<\delta$ so that

$$\left|\int_{[\alpha,\,a]} f\right| \leq M\left|\int_\alpha^a dz\right| = M\,|a-\alpha| \leq M\delta \qquad (3.23)$$

[20] Using (3.20)

$$\left|\int_0^1 [f(tb+(1-t)a)-f(t\beta+(1-t)\alpha)]\,dt\right| \leq \varepsilon \underbrace{\int_0^1 dt}_{1} = \varepsilon$$

and using (3.21)

$$\left|\int_0^1 f(t\beta+(1-t)\alpha)dt\right| \leq M \underbrace{\int_0^1 dt}_{1} = M$$

and

$$\left| \int\limits_{[b,\ \beta]} f \right| \leq M\delta. \tag{3.24}$$

Combining these last two inequalities (3.23) and (3.24) with (3.12) and (3.22) give that[21]

$$\left| \int\limits_{T} f \right| \leq \varepsilon l + 4M\delta.$$

Since it is possible to choose $\delta \leq \varepsilon$ and since ε is arbitrary, it follows that

$$\int\limits_{T} f = 0.$$

Hence f is analytic. $\square$

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$$\begin{aligned} \left|\int\limits_{T} f\right| = \left|\int\limits_{Q} f\right| &= \left| \int\limits_{[a,\ b]} f + \int\limits_{[b,\ \beta]} f + \int\limits_{[\beta,\ \alpha]} f + \int\limits_{[\alpha,\ a]} f \right| \\ &\leq \left| \int\limits_{[a,\ b]} f + \int\limits_{[\beta,\ \alpha]} f \right| + \left| \int\limits_{[\alpha,\ a]} f \right| + \left| \int\limits_{[b,\ \beta]} f \right| \end{aligned}$$

3.4 Analytic Continuation Along a Path

Let us begin the section by *recalling the definition of a function.* We use the somewhat imprecise statement that a function is a triple

$$(f,\, G,\, \Omega)$$

where G and Ω are sets and f is a "rule" which assigns to each element of G a unique element of Ω. Thus, for two functions to be the same not only must the rule to be the same but that domains and the ranges must coincide.[22] If we enlarge the range Ω to a set Ω_1 then $(f,\, G,\, \Omega_1)$ is a *different function.* However, this point should not be emphasized here; we do wish to emphasize that a change in the domain results in a new function. Indeed the purpose of analytic continuation is to enlarge the domain. Thus, let (Fig. 3.11)

$$G = \{z : \mathrm{Re} z > -1\}$$

and

$$f(z) = \log(1+z) \text{ for } z \text{ in } G,$$

where log is the principal branch of the logarithm. Let $D = B(0;\, 1)$ (the open ball centered at the origin 0 and radius 1)

[22]In calculus also this situation is there. For example, $f(x) = x^2$, $x \geq 0$ and $g(x) = x^2$, $x \in \mathbb{R}$ are two different functions.

(Fig. 3.11) and let

$$g(z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

for z in D (g is the Taylor series of $\log z$ on D with center at 0. g is defined only on D.) Then $(f, G, \mathbb{C}) \neq (g, D, \mathbb{C})$ even though $f(z) = g(z)$ for all z in D.

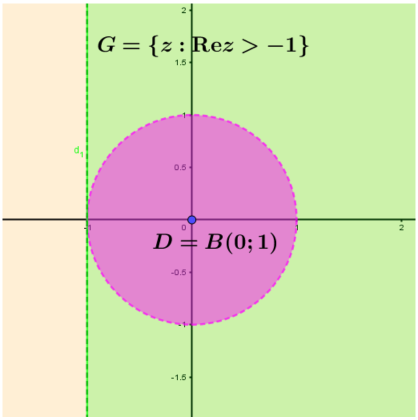

Figure 3.11:

Nevertheless, it is desirable to recognise the relationship between f and g. This leads, therefore, to the concept of a germ of analytic functions.

Definition 3.4.1. A **function element** is a pair (f, G) where G is a region and f is an analytic function on G. For a given function element (f, G) define the **germ of** f **at** a to be the collection of all function elements $(g,\, D)$ such that $a \in D$ and $f(z) = g(z)$ for all z in a neighborhood of a. Denote the germ by $[f]_a$.

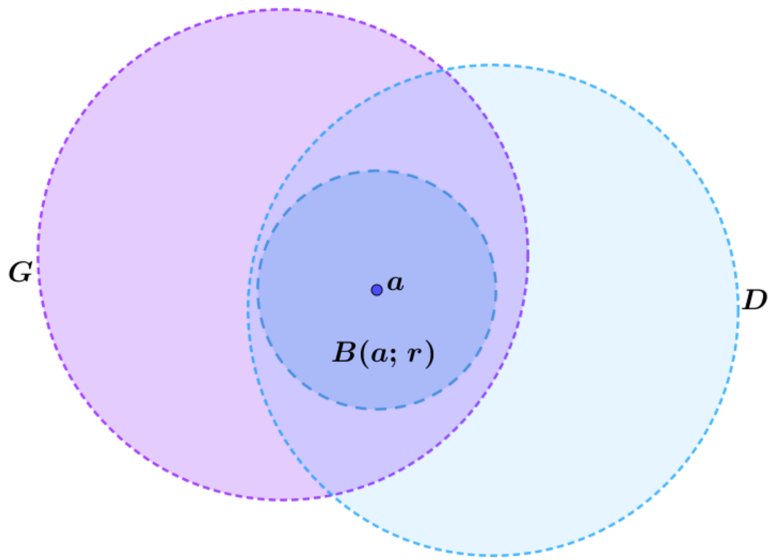

Figure 3.12: If f is analytic on G and g is analytic on D, and if $f(z) = g(z)$ for all z in a neighborhood $B(a;\, r)$, then g is an element in the germ of f at a.

Notice that $[f]_a$ is a collection of function elements and it is not a function element itself. Also $(g,\, D) \in [f]_a$ if and only if $(f,\, G) \in [g]_a$ (Verify!)[23]. It should also be emphasized that

[23] $(g,\, D) \in [f]_a$ if and only if $(f,\, G) \in [g]_a$: $(g,\, D) \in [f]_a$ if and only if $a \in D$ and $g(z) = f(z)$ for all z in a neighborhood of a. This is if and only if $a \in D$ and $f(z) = g(z)$ for all z in a neighborhood of a; if and only if $(f,\, G) \in [g]_a$.

it makes no sense to talk of the equality of two germs $[f]_a$ and $[g]_b$ unless the points a and b are the same. For example, if (f, G) is a function element then it makes no sense to say that $[f]_a = [f]_b$ for two distinct points a and b in G.

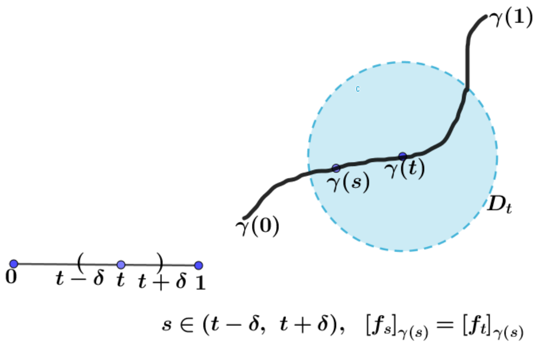

Figure 3.13:

Definition 3.4.2. Let $\gamma : [0, 1] \to \mathbb{C}$ be a path and suppose that for each t in $[0, 1]$ there is a function element (f_t, D_t) such that:

(a) $\gamma(t) \in D_t$;

(b) for each t in $[0, 1]$ there is $\delta > 0$ such that $|s - t| < \delta$ implies $\gamma(s) \in D_t$ and

$$[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}. \tag{3.25}$$

Then (f_1, D_1) is **the analytic continuation of** (f_0, D_0) **along**

the path γ; **or,** (f_1, D_1) **is obtained from** (f_0, D_0) **by analytic continuation along** γ.

Before proceeding examine Part(b) of this definition. Since γ is a continuous function and $\gamma(t)$ is in the open set D_t, it follows that there is a $\delta > 0$ such that $\gamma(s) \in D_t$ for $|s-t| < \delta$. The important content of part (b) is that (3.25) is satisfied whenever $|s-t| < \delta$. That is

$$f_s(z) = f_t(z), \quad z \in D_s \cap D_t \text{ whenever } |s-t| < \delta.$$

Whether for a given curve and given function element there is an analytic continuation along the curve can be a difficult question. Since no degree of generality can be achieved which justify the effort, no existence theorem for analytic continuation will be proved. Each individual case will be considered by itself. Instead uniqueness theorem for continuation are proved. One such theorem is Monodromy Theorem of the next section. This theorem gives a criterion by which one can tell *when a continuation along two different curves connecting the same points result in the same function element.* The next preposition fixes a curve and shows that two different continuations along this curve of the same function element result in the same function element. Alternately this result can be considered as an affirmative answer to the following question: is it possible to define the concept of "the continuation of germ along a curve?"

Proposition 3.4.3. *Let $\gamma : [0,\, 1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t,\, D_t) : 0 \leq t \leq 1\}$ and $\{(g_t,\, B_t) : 0 \leq t \leq 1\}$ be analytic continuations along*[24] *γ such that $[f_0]_a = [g_0]_a$. Then $[f_1]_b = [g_1]_b$.*

Proof. This proposition will be proved by showing that the set

$$T = \{t \in [0,\, 1] : [f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}\} \tag{3.26}$$

is both open and closed in $[0,\, 1]$; since T is non empty (because $0 \in T$) it will follow that[25] $T = [0,\, 1]$ so that, in particular $1 \in T$ and hence $[f_1]_b = [g_1]_b$. The easiest part of the proof is to show that T is open. So fix t in T and assume $t \neq 0$ or 1. (If $t = 1$ the proof is complete; if $t = 0$ the argument about to be given will also show that $[a,\, a + \delta) \subset T$ for some $\delta > 0$.) By the definition of analytic continuation there is a $\delta > 0$ such that for $|s - t| < \delta$, $\gamma(s) \in D_t \cap B_t$ and

$$\left.\begin{aligned} [f_s]_{\gamma(s)} &= [f_t]_{\gamma(s)}, \\ [g_s]_{\gamma(s)} &= [g_t]_{\gamma(s)}. \end{aligned}\right\} \tag{3.27}$$

But since $t \in T$, $f_t(z) = g_t(z)$ for all z in $D_t \cap B_t$. Hence

[24] continuations along the same curve γ

[25] The only nonempty open and closed subset of the connected set $[0,\, 1]$ is $[0,\, 1]$.

$[f_t]_{\gamma(s)} = [g_t]_{\gamma(s)}$ for all $\gamma(s)$ in $D_t \cap B_t$. So it follows from (3.27) that $[f_s]_{\gamma(s)} = [g_s]_{\gamma(s)}$ whenever $|s-t| < \delta$. That is, $(t-\delta,\, t+\delta) \subset T$ and so T is open.

To show that T is closed[26] let t be a limit point of T, and again choose $\delta > 0$ so that $\gamma(s) \in D_t \cap B_t$ and (3.27) is satisfied whenever $|s-t| < \delta$. Since t is a limit point of T there is a point s in T with $|s-t| < \delta$; so $G = D_t \cap B_t \cap D_s \cap B_s$ contains $\gamma(s)$ and, therefore, is a nonempty open set. Thus, $f_s(z) = g_s(z)$ for all z in G by the definition of T. But according to (3.27), $f_s(z) = f_t(z)$ and $g_s(z) = g_t(z)$ for all z in G. So $f_t(z) = g_t(z)$ for all z in G and because G has a limit point in $D_t \cap B_t$, this gives that $[f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}$. That is, $t \in T$ and so T is closed. $\square$

Definition 3.4.4. If $\gamma : [0,\, 1] \to \mathbb{C}$ is a path from a to b and $\{f_t,\, D_t) : 0 \le t \le 1\}$ is an analytic continuation along γ then **the germ $[f_1]_b$ is the analytic continuation of $[f_0]_a$ along** γ.

The preceding preposition implies that Definition 3.4.4 is unambiguous. As stated this definition *seems to depend on the choice of the continuation* $\{(f_t,\, D_t)\}$. However Proposition 3.4.3 says that if $\{(g_t,\, B_t)\}$ is another continuation along γ with $[f_0]_a = [g_0]_a$ then $[f_1]_b = [g_1]_b$. So in fact the definition does not depend on the choice of continuation.

[26] To show that T is closed it is enough to show that it contains all its limit points.

Definition 3.4.5. If (f, G) is a function element then the **complete analytic function obtained from** (f, G) is the collection $\mathcal{F}$ of all germs $[g]_b$ for which there is a point a in G and a path γ from a to b such that $[g]_b$ is the analytic continuation of $[f]_a$ along γ.

Definition 3.4.6. A collection of germs $\mathcal{F}$ is called a **complete analytic function** if there is a function element (f, G) such that $\mathcal{F}$ is the complete analytic function obtained from (f, G).

Notice that the point a in the definition is immaterial; any point in G can be chosen since G is an open connected subset of[27] $\mathbb{C}$. Also, if $\mathcal{F}$ is the complete analytic function associated with (f, G) then $[f]_z \in \mathcal{F}$ for all z in G.

Although there is no ambiguity in the definition of a complete analytic function there is an incompleteness about it. Is it a function? We should refrain from calling an object a function unless it is indeed a function. To make $\mathcal{F}$ a function one must manufacture a domain (the range will be $\mathbb{C}$) and show that $\mathcal{F}$ gives a "rule". This is easy. In a sense we let $\mathcal{F}$ be its own domain; more precisely,let

$$\mathcal{R} = \{(z, [f]_z) : [f]_z \in \mathcal{F}\}.$$

[27] An open set $G \subset \mathbb{C}$ is connected if and only if for any two points a, b in G there is a polygon from a to b lying entirely inside G.

Define $\mathcal{F} : \mathcal{R} \to \mathbb{C}$ by

$$\mathcal{F}(z, [f]_z) = f(z).$$

In this way $\mathcal{F}$ becomes an "honest" function. Nevertheless there is still a dissatisfaction. To have a satisfying solution a structure will be imposed on $\mathcal{R}$ which will make it possible to discuss the concept of analyticity for functions defined on $\mathcal{R}$. In this setting, the function $\mathcal{F}$ defined above becomes analytic; moreover it reflects the behaviour of each function element belonging to a germ that is in $\mathcal{F}$. The introduction of this structure is beyond the scope of the syllabus.

3.5 Monodromy Theorem

Let a and b be two complex numbers and suppose γ and σ are two paths from a to b. Suppose $\{(f_t, D_t)\}$ and $\{(g_t, B_t)\}$ are analytic continuations along[28] γ and σ respectively, and also suppose that $[f_0]_a = [g_0]_a$. Does it follow that $[f_1]_b = [g_1]_b$? If γ and σ are the same path then Proposition 3.4.3 gives an affirmative answer. However, if γ and σ are *distinct* then the answer can be no. There are examples that illustrate the possibility that $[f_1]_b \neq [g_1]_b$.

If (f, D) is a function element and $a \in D$, then f has a

[28] Here analytic continuations along paths γ and σ from a to b (here paths may be same or different).

power series expansion at $z = a$. The first step in proving the Monodromy Theorem is to investigate the behaviour of the radius of convergence for an analytic continuation along a curve.

Lemma 3.5.1. *Let $\gamma : [0,\ 1] \to \mathbb{C}$ be a path and let $\{(f_t,\ D_t) : 0 \le t \le 1\}$ be an analytic continuation along γ. For $0 \le t \le 1$ let*[29] *$R(t)$ be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$. Then*[30] *either $R(t) \equiv \infty$ or $R : [0,\ 1] \to (0,\ \infty)$ is continuous.*

Proof. If $R(t) = \infty$ *for some value* of t then it is possible to extend f_t to an entire function. It follows that $f_s(z) = f_t(z)$ for all z in D_s so that $R(s) = \infty$ for each s in $[0,\ 1]$; that is $R(s) \equiv \infty$. So suppose that $R(t) < \infty$ for all t. Fix t in $[0,\ 1]$ and let $\tau = \gamma(t)$; let

$$f_t(z) = \sum_{n=0}^{\infty} \tau_n (z - \tau)^n$$

be the power series expansion[31] of f_t about τ. Now let $\delta_1 > 0$ be such that $|s - t| < \delta_1$ implies that[32] $\gamma(s) \in D_t \cap B(\tau;\ R(t))$ and $[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$. Fix s with $|s - t| < \delta_1$ and let $\sigma = \gamma(s)$. Now f_t can be extended to an analytic function on $B(\tau;\ R(t))$.

[29] $R(t) = 0$ is not possible because f_t is analytic in a disk centered at $\gamma(t)$.
[30] either $R(t) \equiv \infty$ or the positive real valued function $R : [0,\ 1] \to (0,\ \infty)$ is continuous.
[31] The power series expansion is valid. D_t of f_t.
[32] D_t is the domain of f_t.

Since f_s agrees with f_t on a neighborhood of σ, f_s can be extended so that it is also analytic on $B(\tau;\ R(t)) \cup D_s$. If f_s has power series expansion

$$f_s(z) = \sum_{n=0}^{\infty} \sigma_n (z - \sigma)^n$$

about $z = \sigma$, then the radius of convergence $R(s)$ must be at least as big as the distance from σ to the circle $|z - \tau| = R(t)$: that is, $R(s) \geq d(\sigma, \{z : |z - \tau| = R(t)\} \geq R(t) - |\tau - \sigma|$. But this gives that $R(t) - R(s) \leq |\gamma(t) - \gamma(s)|$. A similar argument gives that $R(s) - R(t) \leq |\gamma(t) - \gamma(s)|$; hence

$$|R(s) - R(t)| \leq |\gamma(t) - \gamma(s)| \tag{3.28}$$

for $|s - t| < \delta_1$. Since $\gamma : [0,\ 1] \to \mathbb{C}$ is continuous[33] it follows that R must be continuous at t. $\square$

Lemma 3.5.2. *Let $\gamma : [0,\ 1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t,\ D_t) : 0 \leq t \leq 1\}$ be an analytic continuation along γ. There is a number $\varepsilon > 0$ such that if $\sigma : [0,\ 1] \to \mathbb{C}$ is any path from a to b with $|\gamma(t) - \sigma(t)| < \varepsilon$ for all t, and*

[33] Since $\gamma : [0,\ 1] \to \mathbb{C}$ is continuous, for a given $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|\gamma(t) - \gamma(s)| < \varepsilon$$

for $|s - t| < \delta_2$. Now let $\delta = \min\{\delta_1,\ \delta_2\}$. With this (3.28) gives that

$$|R(s) - R(t)| < \varepsilon$$

for $|s - t| < \delta$ showing that R is continuous at t.

if $\{(g_t, B_t) : 0 \le t \le 1\}$ *is any continuation along* σ *with* $[g_0]_a = [f_0]_a$*; then* $[g_1]_b = [f_1]_b$.

Proof. For $0 \le t \le 1$ let $R(t)$ be the radius of convergence of the power series expansion of f_t about $z = \gamma(t)$. It can be shown that if $R(t) \equiv \infty$ then any value of ε will suffice. So suppose $R(t) < \infty$ for all t. Since, by the preceding lemma, R is a continuous function and since $R(t) > 0$ for all t, R has a positive minimum value. Let

$$0 < \varepsilon < \frac{1}{2} \min\{R(t) : 0 \le t \le 1\} \tag{3.29}$$

and suppose that σ and $\{(g_t, B_t)\}$ are as in the statement of this lemma. Furthermore, suppose that D_t is a disk of radius $R(t)$ about $\gamma(t)$. The truth of the conclusion will not be affected by this assumption[34], and the exposition will be greatly simplified by it.

Since[35]

$$|\sigma(t) - \gamma(t)| < \varepsilon < R(t),$$

$\sigma(t) \in B_t \cap D_t$ for all t.

Thus, it makes sense to ask whether $g_t(z) = f_t(z)$ for all z in $B_t \cap D_t$. Indeed, to complete the proof we must show that

[34] Why?

[35] B_t is the domain of g_t and D_t is the domain of f_t.

this is precisely the case for $t = 1$. Define the set

$$T = \{t \in [0,\, 1] : f_t(z) = g_t(z) \text{ for } z \text{ in } B_t \cap D_t\};$$

and show that $1 \in T$. This is done by showing that T is non empty open and closed subset of $[0,\, 1]$.

From the hypothesis of the lemma, $0 \in T$ so that T is nonempty. To show T is open fix t in T and choose $\delta > 0$ such that

$$\left.\begin{array}{l} |\gamma(s) - \gamma(t)| < \varepsilon, \quad [f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}, \\ |\sigma(s) - \sigma(t)| < \varepsilon, \quad [g_s]_{\sigma(s)} = [g_t]_{\sigma(s)}, \quad \text{and} \\ \quad \sigma(s) \in B_t \end{array}\right\} \tag{3.30}$$

whenever $|s - t| < \delta$. We will now show that

$$B_s \cap B_t \cap D_s \cap D_t \neq \emptyset \text{ for } |s - t| < \delta;$$

in fact, we will show that $\sigma(s)$ is in this intersection. If $|s-t| < \delta$ then

$$|\sigma(s) - \gamma(s)| < \varepsilon < R(s)$$

so that $\sigma(s) \in D_s$. Also

$$|\sigma(s) - \gamma(t)| \leq |\sigma(s) - \gamma(s)| + |\gamma(s) - \gamma(t)| < 2\varepsilon < R(t)$$

by (3.29); so $\sigma(s) \in D_t$. Since we already have that $\sigma(s) \in$

$B_s \cap B_t$ by (3.30)

$$\sigma(s) \in B_s \cap B_t \cap D_s \cap D_t = G.$$

Since $t \in T$ it follows that $f_t(z) = g_t(z)$ for all z in G. Also, from (3.30) $f_s(z) = f_t(z)$ and $g_s(z) = g_t(z)$ for z in G; but since G has a limit point in $B_s \cap D_s$ it must be that $s \in T$. That is, $(t - \delta,\, t + \delta) \subset T$ and T is open. The proof that T is closed is similar and is left as an exercise. □

Definition 3.5.3. Let $(f,\, D)$ be a function element and let G be a region which contains D; then $(f,\, D)$ **admits unrestricted analytic continuation in** G if for any path γ in G with initial point in D there is an analytic continuation of $(f,\, D)$ along γ.

If $D = \{z : |z - 1| < 1\}$ and f is the principal branch of $\sqrt{z}$ or $\log z$ then $(f,\, D)$ admits unrestricted continuation in the punctured plane but not in the whole plane.

It has been stated before that an existence theorem for analytic continuations will not be proved. In particular, if $(f,\, D)$ is a function element and G is a region containing D, no criterion will be given which implies that $(f,\, D)$ admits unrestricted continuation in G. The Monodromy Theorem assumes that G has this property and states a uniqueness criterion.

Theorem 3.5.4. Monodromy Theorem *Let $(f,\ D)$ be a function element and let G be a region containing D such that $(f,\ D)$ admits unrestricted continuation in G. Let $a \in D$, $b \in G$ and let γ_0 and γ_1 be paths in G from a to b; Let $\{(f_t,\ D_t) : 0 \leq t \leq 1\}$ and $\{(g_t,\ D_t) : 0 \leq t \leq 1\}$ be analytic continuations of $(f,\ D)$ along γ_0 and γ_1 respectively. If γ_0 and γ_1 are FEP homotopic in G, then*

$$[f_1]_b = [g_1]_b.$$

Proof. Since γ_0 and γ_1 are FEP homotopic in G there is a continuous function $\Gamma : [0,\ 1] \times [0,\ 1] \to G$ such that

$$\begin{array}{ll} \Gamma(t,\ 0) = \gamma_0(t) & \qquad \Gamma(t,\ 1) = \gamma_1(t) \\ \Gamma(0,\ u) = a & \qquad \Gamma(1,\ u) = b \end{array}$$

for all t and u in $[0, 1]$. Fix u, $0 \leq u \leq 1$ and consider the path γ_u, defined by $\gamma_u(t) = \Gamma(t,\ u)$, from a to b. By hypothesis there is an analytic continuation

$$\{(h_{t,\,u}, D_{t,\,u}) : 0 \leq t \leq 1\}$$

of $(f,\ D)$ along γ_u. It follows from Proposition 3.4.3 that $[g_1]_b = [h_{1,\,1}]_b$ and $[f_1]_b = [h_{1,\,0}]_b$. So it suffices to show that

$$[h_{1,\,0}]_b = [h_{1,\,1}]_b.$$

To do this introduce the set

$$U = \{u \in [0,\, 1] : [h_{1,\,u}]_b = [h_{1,\,0}]_b\},$$

and show that U is a non-empty open and closed subset of $[0,\, 1]$. Since $0 \in U$, $U \neq \emptyset$. To show that U is both open and closed we will establish the following.

Claim.

For u in $[0,\, 1]$ there is a $\delta > 0$ such that if $|u - v| < \delta$

$$\text{then } [h_{1,\,u}]_b = [h_{1,\,v}]_b. \tag{3.31}$$

For a fixed u in $[0,\, 1]$ apply Lemma 3.5.2 to find an $\varepsilon > 0$ such that if σ is any path from a to b with

$$|\gamma_u(t) - \sigma(t)| < \varepsilon \text{ for all } t, \text{ and}$$

if $\{(k_t,\, E_t)\}$ is *any* continuation of $(f,\, D)$ along σ, then

$$[h_{1,\,u}]_b = [k_1]_b. \tag{3.32}$$

Now Γ is a uniformly continuous function, so there is a $\delta > 0$ such that if $|u - v| < \delta$ then

$$|\gamma_u(t) - \gamma_v(t)| = |\Gamma(t,\ u) - \Gamma(t,\ v)| < \varepsilon \ \text{ for all } \ t\,.$$

Claim 3.31 now follows by applying (3.32). Suppose $u \in U$ and let $\delta > 0$ be the number given by Claim 3.31. By the definition of U, $(u-\delta,\, u+\delta) \subset U$; so U is open. If $u \in \overline{U}$ and δ is again chosen as in (3.31) then there is a u in U such that $|u-v| < \delta$. But by (3.31) $[h_{1,u}]_b = [h_{1,v}]_b$; and since $v \in U$, $[h_{1,v}]_b = [h_{1,0}]_b$. Therefore $[h_{1,u}]_b = [h_{1,0}]_b$ so that $u \in U$; that is, U is closed. □

The following corollary is the most important consequence of the Monodromy Theorem.

Corollary 3.5.5. *Let $(f,\, D)$ be a function element which admits unrestricted continuation in the simply connected region G. Then there is an analytic function $F : G \to \mathbb{C}$ such that*

$$F(z) = f(z) \text{ for all } z \text{ in } D.$$

Proof. Fix a in D and let z be any point in G. If γ is a path in G from a to z and $\{(f_t,\, D_t) : 0 \leq t \leq 1\}$ is an analytic continuation of $(f,\, D)$ along γ then let

$$F(z,\, \gamma) = f_1(z).$$

Since G is simply connected $F(z,\, \gamma) = F(z,\, \sigma)$ for any two paths γ and σ in G from a to z. [36] Thus, $F(z) = F(z,\, \gamma)$ gives

[36] Why? Since G is simply connected, the paths γ and σ in G from a to z are FEP homotopic. Now apply Monodromy Theorem.

a well defined function $F : G \to \mathbb{C}$. To show that F is analytic let $z \in G$ and let γ and $\{(f_t, D_t)\}$ be as above. A simple argument gives that $F(w) = f_1(w)$ for w in a neighborhood of z; so F must be analytic. □

3.6 Entire Functions: Introduction

Let us recall the Weierstrass Factorization Theorem for entire functions. Let f be an entire function with a zero of multiplicity $m \geq 0$ at $z = 0$; let $\{a_n\}$ be the zeros of f, $a_n \neq 0$, arranged so that a zero of multiplicity k is repeated in this sequence k times. Also assume that $|a_1| \leq |a_2| \leq \cdots$ If $\{p_n\}$ is a sequence of integers such that

$$\sum_{n=1}^{\infty} \left(\frac{r}{|a_n|}\right)^{p_n+1} < \infty \quad \text{for every } r > 0 \tag{3.33}$$

then

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right) \tag{3.34}$$

converges uniformly on compact subsets of the plane, where

$$E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right) \quad \text{for} \quad p \geq 1 \tag{3.35}$$

and

$$E_0(z) = 1 - z. \tag{3.36}$$

Consequently

$$f(z) = z^m e^{g(z)} P(z) \tag{3.37}$$

where g is an entire function. An interesting line of investigation begins if we ask the questions: What properties of f can be deduced if g and P are assumed to have certain "nice" properties? Can restrictions be imposed on f which will imply that g and P have particular properties? The plan that will be adopted in answering these questions is to assume that g and P have certain characteristics, deduce the implied properties of f, and then try to prove the converse of this implication.

How to begin? Clearly the first restriction on g in this program is to suppose that it is a polynomial. It is equally clear that such an assumption must impose a growth condition on $e^{g(z)}$. A convenient assumption on P is that all the integers p_n are equal. From equation (3.33), we see that this is to assume that there is an integer $p \geq 1$ such that

$$\sum_{n=1}^{\infty} |a_n|^{-p} < \infty$$

that is, it is an assumption on the growth rate of the zeros of f. In the next section of this chapter Jensen's Formula is deduced. Jensen's Formula says that there is a relation between the growth rate of the zeros of f and the growth of

$M(r) = \sup\left\{\left|f(re^{i\theta})\right| : 0 \le \theta \le 2\pi\right\}$ as r increases.

3.7 Jensen's Formula

Let f be analytic in an open set containing

$$\overline{B(0;\ r)}.$$

Case 1. Suppose f doesn't vanish in $\overline{B(0;\ r)}$. Then $\log|f|$ is harmonic there.[37] Hence it has the Mean Value Property[38] ; that is

$$\underbrace{\log|f(0)|}_{(\log|f|)(0)} = \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\log\left|f(re^{i\theta})\right|}_{(\log|f|)(re^{i\theta})} d\theta. \tag{3.38}$$

Case 2. Suppose f has exactly one zero $a = re^{i\theta}$ on the circle $|z| = r$. If

$$g(z) = \frac{f(z)}{z-a}$$

[37] Here log function is the natural logarithmic function $\log : (0,\ \infty) \to R$ in Calculus.

[38]

Definition 3.7.1. A continuous function $u : G \to \mathbb{R}$ has the **Mean Value Property (MVP)** if whenever $\overline{B(a;\ r)} \subset G$

$$u(a) = \frac{1}{2\pi}\int_0^{2\pi} u(a + re^{i\theta})d\theta.$$

then g doesn't vanish in $\overline{B(0;\ r)}$ so Case 1 and hence (3.38) can be applied to g to obtain

$$\begin{aligned}
\underbrace{\log|g(0)|}_{(\log|g|)(0)} &= \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\log\left|g(re^{i\theta})\right|}_{(\log|g|)(re^{i\theta})} d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} \underbrace{\log\left|\frac{f(re^{i\theta})}{re^{i\theta}-re^{i\alpha}}\right|}_{\text{since } \ g(z)=\frac{f(z)}{z-a}} d\theta
\end{aligned}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left[\log\left|f(re^{i\theta})\right| - \log\left|re^{i\theta}-re^{i\alpha}\right|\right] d\theta$$

Since

$$\begin{aligned}
\log|g(0)| &= \log\left|\frac{f(0)}{0-a}\right| \\
&= \log|f(0)| - \log \underbrace{|-a|}_{\substack{=r,\text{ since} \\ a=re^{i\alpha}}} \\
&= \log|f(0)| - \log r
\end{aligned}$$

it will follow that (3.38) remains valid where f has a single zero on $|z| = r$, if it can be shown that

$$\frac{1}{2\pi}\int_0^{2\pi} \log\left|re^{i\,\theta} - re^{i\,\alpha}\right| d\theta = \log r$$

or that

$$\underbrace{\frac{1}{2\pi}\int_0^{2\pi} \left[log\left|re^{i\,\theta} - re^{i\,\alpha}\right| - \log r\right] d\theta = 0}_{\text{since } \frac{1}{2\pi}\int_0^{2\pi} d\theta = 1}$$

or that

$$\int_0^{2\pi} \left[\log\left|re^{i\,\theta} - re^{i\,\alpha}\right| - \log r\right] d\theta = 0$$

or that

$$\int_0^{2\pi} \log\frac{\left|re^{i\,\theta} - re^{i\,\alpha}\right|}{r} d\theta = 0$$

or that

$$\int_0^{2\pi} \log\left|e^{i\,\theta} - e^{i\,\alpha}\right| d\theta = 0$$

or that

$$\int_0^{2\pi} \log \underbrace{\left|e^{i\,\theta}\right|}_{1} \left|1 - e^{i\,(\alpha-\theta)}\right| d\theta = 0$$

or that

$$\underbrace{-\int_0^{2\pi} \log \left|1 - e^{i\eta}\right| d\eta = 0}_{\text{by taking } \eta=\alpha-\theta,\ d\eta=-d\theta}$$

alternately, if it can be shown that

$$\int_0^{2\pi} \log \left|1 - e^{i\theta}\right| d\theta = 0.$$

But this follows from the fact that

$$\int_0^{2\pi} \log(\sin^2 2\theta) d\theta = -4\pi \log 2.$$

39

So (3.38) remains valid if f has a single zero on $|z| = r$; by

[39]Details:

$$\begin{aligned}
\log\left|1 - e^{i\,\theta}\right| &= \log|1 - (\cos\theta + i\,\sin\theta)| \\
&= \log|(1-\cos\theta) - i\,\sin\theta| \\
&= \log\sqrt{(1-\cos\theta)^2 + (-\sin\theta)^2} \\
&= \frac{1}{2}\log\left(2 - 2\cos\theta\right) \\
&= \frac{1}{2}\log\left((2)(1-\cos\theta)\right) \\
&= \frac{1}{2}\log\left(4\sin^2(\theta/2)\right) \\
&= \frac{1}{2}\log 4 + \frac{1}{2}\log\left(\sin^2(\theta/2)\right)
\end{aligned}$$

Thus,

$$\begin{aligned}
\int_0^{2\pi}\log\left|1 - e^{i\,\theta}\right| d\theta &= \underbrace{\frac{1}{2}\log 4}_{\log 4^{1/2} = \log 2}\underbrace{\int_0^{2\pi} d\theta}_{2\pi} + \frac{1}{2}\int_0^{2\pi}\log\left(\sin^2(\theta/2)\right) d\theta \\
&= 2\pi\log 2 + \frac{1}{2}\int_0^{2\pi}\log\left(\sin^2(\theta/2)\right) d\theta \qquad (3.39)
\end{aligned}$$

Now to evaluate $\int_0^{2\pi}\log\left(\sin^2(\theta/2)\right) d\theta$:

$$\begin{aligned}
\int_0^{2\pi}\log\left(\sin^2(\theta/2)\right) d\theta &= \underbrace{\int_0^{\pi/2}\log\left(\sin^2 2u\right)(4)du}_{2u=\theta/2,\ 2du=(1/2)d\theta} \\
&= \underbrace{4\cdot\frac{1}{4}\cdot\int_0^{2\pi}\log\left(\sin^2 2u\right) du}_{\text{as } \sin^2 2u \text{ is } \pi/2 \text{ periodic}} \\
&= -4\pi\log 2
\end{aligned}$$

induction, (3.38) is valid as long as f has no zeros in $B(0;\, r)$.
Case 3. The next step is to examine what happens if f has zeros inside $B(0;\, r)$. In this case, $\log|f(z)|$ is no longer harmonic so that MVP is not present.

Theorem 3.7.2. ***[Jensen's Formula]****Let f be an analytic function on a region containing $\overline{B(0;\ r)}$ and suppose that*

$$a_1,\ \ldots,\ a_n$$

are the zeros of f in $B(0;\, r)$ repeated according to multiplicity, if $f(0) \neq 0$ then

$$\log|f(0)| = -\sum_{k=1}^{n} \log\left(\frac{r}{|a_k|}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log\left|f(re^{i\,\theta})\right| d\theta.$$

Proof. If $|b| < 1$ then the map

$$\frac{z-b}{1-\overline{b}z}$$

takes the disk $B(0,\, 1)$ onto itself and maps the boundary onto

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Hence from (3.39), we obtain

$$\int_0^{2\pi} \log\left|1-e^{i\,\theta}\right| d\theta = 2\pi\log 2 + \frac{1}{2}(-4\pi\log 2) = 0$$

as desired.

itself. Hence

$$\frac{r^2(z-a_k)}{r^2-\bar{a}_k z}$$

maps $B(0;\, r)$ onto itself and takes the boundary to the boundary. Therefore

$$F(z) = f(z)\prod_{k=1}^{n}\frac{r^2-\bar{a}_k z}{r(z-a_k)}$$

is analytic in an open set containing $\overline{B(0;\, r)}$, has no zeros in $B(0;\, r)$, and

$$|F(z)| = |f(z)|$$

for $|z| = r$.[40]

So (3.38) applies to F to give

$$\log|F(0)| = \underbrace{\frac{1}{2\pi}\int_0^{2\pi}\log\left|F(re^{i\theta})\right|d\theta = \frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|d\theta}_{|F(z)|=|f(z)| \text{ on the boundary of the disk, so } F(re^{i\theta})=f(re^{i\theta})}$$

[40]The function $\frac{r^2(z-a_k)}{r^2-\bar{a}_k z}$ maps the circle $|z| = r$ (the boundary $B(0;\, r)$) onto itself. So, for a point z on $|z| = r$, $\frac{r^2(z-a_k)}{r^2-\bar{a}_k z}$ is also a point on $|z| = r$, so its modulus is r. Hence

$$\left|\frac{r^2-\bar{a}_k z}{r(z-a_k)}\right| = r\left|\frac{r^2-\bar{a}_k z}{r^2(z-a_k)}\right| = \frac{r}{r} = 1.$$

Thus, $|F(z)| = |f(z)|$ for $|z| = r$.

However

$$F(0) = f(0)\prod_{k=1}^{n}\frac{r^2 - \bar{a}_k(0)}{r(0-a_k)} = f(0)\prod_{k=1}^{n}\left(-\frac{r}{a_k}\right)$$

so that

$$|F(0)| = |f(0)| \underbrace{\prod_{k=1}^{n}\frac{r}{|a_k|}}_{\text{as }\left|\prod\limits_{k=1}^{n}\left(-\frac{r}{a_k}\right)\right| = \prod\limits_{k=1}^{n}\frac{r}{|a_k|}}$$

so that

$$\log|F(0)| = \log|f(0)| + \log\prod_{k=1}^{n}\frac{r}{|a_k|}$$

so that

$$\frac{1}{2\pi}\int_0^{2\pi}\log\left|f(re^{i\theta})\right|d\theta = \log|f(0)| + \sum_{k=1}^{n}\log\left(\frac{r}{|a_k|}\right)$$

and hence Jensen's Formula results. □

If the same methods are used but the MVP is replaced by Corollary in the footnote[41], $\log|f(z)|$ can be found for

[41]

Corollary 3.7.3. *If $u : \bar{D} \to R$ is a continuous function that is harmonic in D then*

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)u(e^{it})dt$$

$z \neq a_k$, $1 \leq k \leq n$.

Theorem 3.7.4. ***[Poisson Jensen Formula]*** *Let f be analytic in a region which contains $\overline{B(0;\ r)}$ and let $a_1, \ldots, a_n$ be the zeros of f in $B(0;\ r)$ repeated according to multiplicity. If $|z| < r$ and $f(z) \neq 0$ then*

$$\log|f(z)|$$

$$= -\sum_{k=1}^{n} \log\left|\frac{r^2 - \bar{a}_k z}{r(z - a_k)}\right| + \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Re}\left(\frac{re^{i\theta} + z}{re^{i\theta} - z}\right) \log\left|f(re^{i\theta})\right| d\theta$$

Exercise: Prove Theorem 3.7.4.

3.8 The Genus and Order of an Entire Function

Definition 3.8.1. Let f be an entire function with zeros $\{a_1, a_2, \ldots\}$, repeated according to multiplicity and arranged such that $|a_1| \leq |a_2| \leq \cdots$. Then f is of **finite rank** if there

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for $0 \leq r < 1$ and all θ. Moreover, u is the real part of the analytic function

$$f(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(e^{it}) dt.$$

is an integer p such that

$$\sum_{n=1}^{\infty} |a_n|^{-\ (p+1)} < \infty. \tag{3.40}$$

If p is the smallest integer such that this occurs, then f is said to be of **rank** p; a function with only a finite number of zeros has rank 0. A function is of **infinite rank** if it is not of finite rank.

From equation (3.33) it is seen that if f has finite rank p then the canonical product P in (3.37) can be taken to be

$$P(z) = \prod_{n=1}^{\infty} E_p\left(\frac{z}{a_n}\right) \tag{3.41}$$

Notice that if f is of finite rank and p is any integer larger than the rank of f, then (3.40) remains valid. So there is a second canonical product (3.41), and this shows that the factorization (3.37) of f is not unique. However, if the product P is defined by (3.41) where p is the rank of f then the factorization (3.37) is unique except that g may be replaced by $g + 2\pi mi$ for any integer m.

Definition 3.8.2. Let f be an entire function of rank p with zeros $\{a_1, a_2, \ldots\}$. Then the product defined in (3.41) is said to be in **standard form for** f. If f is understood then it will he said to be in **standard form.**

Definition 3.8.3. An entire function f has **finite genus** if f has finite rank and if

$$f(z) = z^m e^{g(z)} P(z),$$

where P is in standard form, and g is a polynomial. If p is the rank of f and q is the degree of the polynomial g, then $\mu = \max(p,\, q)$ is called the **genus of f.**

Notice that the genus of f is well defined integer because once P is in standard form, then g is uniquely determined up to adding a multiple of $2\pi i$. In particular, the degree of g is determined.

Theorem 3.8.4. *Let f be an entire function of genus μ. For each positive number α there is a number r_0 such that for $|z| > r_0$*

$$|f(z)| < \exp(\alpha\, |z|^{\mu+1}).$$

Proof. Since f is an entire function of genus μ

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} E_\mu\left(\frac{z}{a_n}\right),$$

where g is a polynomial of degree $\leq \mu$. Notice that if $|z| < \frac{1}{2}$

then

$$\begin{aligned}
\log|E_\mu(z)| &= \operatorname{Re}\left\{\log(1-z)+z+\cdots+\frac{z^\mu}{\mu}\right\} \qquad (3.42)\\
&= \operatorname{Re}\left\{-\frac{1}{\mu+1}z^{\mu+1}-\frac{1}{\mu+2}z^{\mu+2}-\cdots\right\}\\
&\leq |z|^{\mu+1}\left\{\frac{1}{\mu+1}+\frac{|z|}{\mu+2}+\cdots\right\}\\
&\leq |z|^{\mu+1}\left\{1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\cdots\right\}\\
&= 2\,|z|^{\mu+1}.
\end{aligned}$$

Also

$$|E_\mu(z)| \leq (1+|z|)\exp\left(|z|+\cdots+\frac{|z|^\mu}{\mu}\right)$$

so that

$$\log|E_\mu(z)| \leq \log(1+|z|)+\log\exp\left(|z|+\cdots+\frac{|z|^\mu}{\mu}\right)$$

and hence

$$\log|E_\mu(z)| \leq \log(1+|z|)+|z|+\cdots+\frac{|z|^\mu}{\mu}.$$

Hence,

$$\lim_{z\to\infty}\frac{\log|E_\mu(z)|}{|z|^{\mu+1}}=0.$$

So if $A > 0$ then there is a number $R > 0$ such that

$$\log |E_\mu(z)| \le A\, |z|^{\mu+1}\,, \quad |z| > R. \tag{3.43}$$

But on $\{z : \frac{1}{2} \le |z| \le R\}$ the function $|z|^{-(\mu+1)} \log |E_\mu(z)|$ is continuous except at $z = +1$, where it tends to $-\infty$. Hence there is a constant $B > 0$ such that

$$\log |E_\mu(z)| \le B\, |z|^{\mu+1}\,, \quad \frac{1}{2} \le |z| \le R. \tag{3.44}$$

Combining (3.42), (3.43), and (3.44) gives that

$$\log |E_\mu(z)| \le M\, |z|^{\mu+1} \tag{3.45}$$

for all z in $\mathbb{C}$, where $M = \max\{2,\, A,\, B\}$.

Since $\sum |a_n|^{-(\mu+1)} < \infty$, an integer N can be chosen so that

$$\sum_{n=N+1}^{\infty} |a_n|^{-(\mu+1)} < \frac{\alpha}{4M}.$$

But, using (3.45),

$$\sum_{n=N+1}^{\infty} \log \left| E_\mu\left(\frac{z}{a_n}\right)\right| \le M \sum_{n=N+1}^{\infty} \left|\frac{z}{a_n}\right|^{\mu+1} \le \frac{\alpha}{4}\, |z|^{\mu+1}. \tag{3.46}$$

Now notice that in the derivation of (3.43), A could be chosen as small as desired by taking R sufficiently large. So choose

$r_1 > 0$ such that

$$\log|E_\mu(z)| \leq \frac{\alpha}{4N}|z|^{\mu+1} \quad \text{for} \quad |z| > r_1.$$

If

$$r_2 = \max\{|a_1|\, r_1,\ |a_2|\, r_2,\ \ldots\ ,\ |a_N|\, r_1\}$$

then

$$\sum_{n=1}^{N} \log\left|E_\mu\left(\frac{z}{a_n}\right)\right| \leq \frac{\alpha}{4}|z|^{\mu+1} \quad \text{for} \quad |z| > r_2.$$

Combining this with (3.46) gives that

$$\log\,|P(z)| = \sum_{n=1}^{\infty} \log\left|E_\mu\left(\frac{z}{a_n}\right)\right| < \frac{\alpha}{2}|z|^{\mu+1} \quad \text{for} \quad |z| > r_2. \tag{3.47}$$

Since g is a polynomial of degree $\leq \mu$,

$$\lim_{z\to\infty} \frac{m\log|z| + |g(z)|}{|z|^{\mu+1}} = 0.$$

So there is an $r_3 > 0$ such that

$$m\log|z| + |g(z)| < \frac{1}{2}\alpha\,|z|^{\mu+1}.$$

Together with (3.47) this yields

$$\log|f(z)| < \alpha\,|z|^{\mu+1}$$

for $|z| > r_0 = \max\{r_2, r_3\}$. By taking the exponential of both sides, the desired inequality is obtained. $\square$

The preceding theorem says that by restricting the rate of growth of the zeros of the entire function $f(z) = z^m \exp g(z) P(z)$ and by requiring that g be a polynomial, then the growth of $M(r) = \max\left\{\left|f(re^{i\theta})\right| : 0 \leq \theta \leq 2\pi\right\}$ is dominated by

$$\exp\left(\alpha\, |z|^{\mu+1}\right)$$

for some μ and any $\alpha > 0$.

We wish to prove the converse to this result.

Definition 3.8.5. An entire function f is of **finite order** if there is a positive constant a and an $r_0 > 0$ such that $|f(z)| < \exp\left(|z|^a\right)$ for $|z| > r_0$. If f is not of finite order then f is of **infinite order.**

If f is of finite order then the number

$$\lambda = \inf\left\{a : |f(z)| < \exp(|z|^a) \text{ for } |z| \text{ sufficiently large}\right\}$$

is called the **order of** f.

Notice that if

$$|f(z)| < \exp\left(|z|^a\right) \text{ for } |z| > r_a > 1 \text{ and } b > a$$

then

$$|f(z)| < \exp\left(|z|^b\right).$$

The next proposition is an immediate consequence of this observation.

Proposition 3.8.6. *Let f be an entire function of finite order λ. If $\varepsilon > 0$ then*

$$|f(z)| < \exp\left(|z|^{\lambda+\varepsilon}\right)$$

for all z with $|z|$ sufficiently large; and a z can be found, with $|z|$ as large as desired, such that

$$|f(z)| \geq \exp\left(|z|^{\lambda-\varepsilon}\right).$$

Although the definition of order seems a *priori* weaker than the conclusion of Theorem 3.8.4, they are, in fact, equivalent.

So it is desirable to know if every fu:nction of finite order has finite genus (a converse of Theorem 3.8.4). That this is in fact the case is a result of Hadamard's Factorization Theorem, proved in the next section.

The proof of the next proposition is left to the reader.

Proposition 3.8.7. *Let f be an entire function of order λ and let $M(r) = \max\{|f(z)| : |z| = r\}$; then*

$$\lambda = \limsup_{r\to\infty} \frac{\log\log M(r)}{\log r}.$$

Consider the function $f(z) = \exp(e^z)$; then

$$|f(z)| = \exp(\text{Re } e^z) = \exp(e^r \cos\theta) \text{ if } \ z = re^{i\theta}.$$

Hence $M(r) = \exp(e^r)$ and

$$\frac{\log\log M(r)}{\log r} = \frac{r}{\log r};$$

thus, f is of infinite order. On the other hand if $g(z) = \exp(z^n)$, $n \geq 1$, then

$$|g(z)| = \exp(\text{Re} z^n) = \exp(r^n \cos m\theta)\,.$$

Hence $M(r) = \exp(r^n)$ and so

$$\frac{\log\log M(r)}{\log r} = n;$$

thus g is of order n. Using this terminology , Theorem 3.8.4 can be rephrased as follows.

Corollary 3.8.8. *If f is an entire function of finite genus μ then f is of finite order $\lambda \leq \mu + 1$.*

3.9 Hadamard Factorization Theorem

In this section the converse of Corollary 3.8.8 is proved; that is each function of finite order has finite genus. Since a function of finite genus can be factored in a particularly pleasing way this gives a factorization theorem.

Lemma 3.9.1. *Let f be a non-constant entire function of order λ with $f(0) = 1$, and let $\{a_1, a_2, \ldots\}$ be the zeros of f counted according to multiplicity and arranged so that $|a_1| \le |a_2| \le \ldots$. If an integer $p > \lambda - 1$ then*

$$\frac{d^p}{dz^p}\left[\frac{f'(z)}{f(z)}\right] = -p!\sum_{n=1}^{\infty}\frac{1}{(a_n - z)^{p+1}}$$

for $z \ne a_1, a_2, \ldots$.

Proof. Let $n = n(r) =$ the number of zeros of f in $B(0; r)$; according to the Poisson-Jensen formula

$$\log|f(z)| = -\sum_{k=1}^{\infty}\log\left|\frac{r^2 - \bar{a}_k z}{r(z - a_k)}\right| + \frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\left(\frac{re^{i\theta} + z}{re^{i\theta} - z}\right)\log\left|f(re^{i\theta})\right| d\theta$$

for $|z| < r$. Using Leibniz's rule for differentiating under an integral sign this gives

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \sum_{k=1}^{n}\frac{1}{z - a_k} + \sum_{k=1}^{n}\frac{\bar{a}_k}{r^2 - \bar{a}_k z} \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}\frac{2re^{i\theta}}{(re^{i\theta} - z)^2}\log\left|f(re^{i\theta})\right| d\theta\end{aligned}$$

for $|z| < r$ and $z \ne a_1, \ldots, a_n$. Differentiating p times yields:

$$\frac{d^p}{dz^p}\left[\frac{f'(z)}{f(z)}\right] = -p!\sum_{k=1}^{\infty}\frac{1}{(a_k-z)^{p+1}} + p!\sum_{k=1}^{\infty}\frac{\bar{a}_k^{p+1}}{(r^2-\bar{a}_k z)^{p+1}}$$
$$+(p+1)!\frac{1}{2\pi}\int_0^{2\pi}\frac{2re^{i\theta}}{(re^{i\theta}-z)^{p+2}}\log\left|f(re^{i\theta})\right|\,d\theta \tag{3.48}$$

Now as $r \to \infty$, $n(r) \to \infty$ so that the result will follow if it can be shown that the last two summands in (3.48) tend to zero as $r \to \infty$.

To see that the second sum converges to zero let $r > 2|z|$; then $|a_k| \le r$ gives $|r^2 - \bar{a}_k z| \ge \frac{1}{2}r^2$ so that

$$\left(\frac{|\bar{a}_k|}{|r^2-\bar{a}_k z|}\right)^{p+1} \le \left(\frac{2}{r}\right)^{p+1}.$$

Hence the second summand is dominated by $n(r)\left(\frac{2}{r}\right)^{p+1}$. But it is an easy consequence of Jensen's Formula that $\log 2n(r) \le \log M(r)$. Since f is or order λ, for any $\varepsilon > 0$ and r sufficiently large

$$\frac{\log 2n(r)}{r^{p+1}} \le \frac{\log[M(r)]}{r^{p+1}} \le r^{(\lambda+\varepsilon)-(p+1)}$$

But $p+1 > \lambda$ so that ε may be chosen with $(\lambda+\varepsilon)-(p+1) < 0$. Hence $n(r)\left(\frac{2}{r}\right)^{p+1} \to 0$ as $r \to \infty$; that is, the second summand in (3.48) converges to zero.

To show that the integral in (3.48) converges to zero notice that

$$\int_0^{2\pi} \frac{re^{i\theta}}{(re^{i\theta}-z)^{p+2}} d\theta = 0$$

since this integral is a multiple of the integral of $\frac{1}{(w-z)^{p+2}}$ around the circle $|w| = r$ and this function has a primitive. So the value of the integral in (3.48) remains unchanged if we substitute $\log|f| - \log M(r)$ for $\log|f|$.

So for $2|z| < r$, the absolute value of the integral in (3.48) is dominated by

$$(p+1)!\frac{2^{p+3}}{r^{p+1}} \cdot \frac{1}{2\pi} \int_0^{2\pi} \left[\log M(r) - \log\left|f(re^{i\theta})\right|\right] d\theta\,. \qquad (3.49)$$

But according to Jensen's formula,

$$\frac{1}{2\pi} \int_0^{2\pi} \log\left|f(re^{i\theta})\right| \, d\theta \ \geq 0$$

since $f(0) = 1$. Also $\log M(r) \leq r^{\lambda+\varepsilon}$ for sufficiently large r so that (3.49) is dominated by

$$(p+1)! 2^{p+3} r^{\lambda+\varepsilon-(p+1)}$$

As before, ε can be chosen so that $r^{\lambda+\varepsilon-(p+1)} \to 0$ as $r \to$

∞. □

Note that the preceding lemma implicitly assumes that f has infinitely many zeros. However, if f has only a finite number of zeros then the sum in Lemma 3.9.1 becomes a finite sum and the lemma remains valid.

Theorem 3.9.2. Hadamard's Factorization Theorem *If f is an entire function of finite order λ then f has finite genus $\mu \leq \lambda$.*

Proof. Let p be the largest integer less than or equal to λ; so $p \leq \lambda \leq p + 1$. The first step in the proof is to show that f has finite rank and that the rank is not larger than p. So let $\{a_1, a_2, \ldots\}$ be the zeros of f counted according to multiplicity and arranged such that $|a_1| \leq |a_2| \leq \ldots$. It must be shown that

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^{p+1}} < \infty. \tag{3.50}$$

There is no loss in generality in assuming that $f(0) = 1$. Indeed, if f has a zero at the origin of multiplicity m and $M(r) = \max\{|f(z)| : |z| = r\}$ then for any $\varepsilon > 0$ and $|z| = r$

$$\log\left|\frac{f(z)}{z^m}\right| \leq \log\frac{M(r)}{r^m} \leq r^{\lambda+\varepsilon} - m\log r \leq r^{\lambda+2\varepsilon}$$

if r is sufficiently large. So $f(z)z^{-m}$ is an entire function of order λ with no zero at the origin. Since multiplication by a scalar docs not affect the order, the assumption that $f(0) = 1$ is justified.

Let $n(r) =$ the number of zeros of f in $B(0; r)$. It follows that $[\log 2]n(r) \leq \log M(r)$. Since f has order λ, $\log M(r) \leq r^{\lambda+\frac{\varepsilon}{2}}$ for any $\varepsilon > 0$ so that

$$\lim_{n\to\infty} \frac{n(r)}{r^{\lambda+\varepsilon}} = 0.$$

Hence

$$n(r) \leq r^{\lambda+\varepsilon}$$

for sufficiently large r. Since $|a_1| \leq |a_2| \leq \ldots,$

$$k \leq n(|a_k|) \leq |a_k|^{\lambda+\varepsilon}$$

for all k larger than some integer k_0. Hence

$$\frac{1}{|a_k|^{p+1}} \leq \frac{1}{k^{p+\frac{1}{\lambda+\varepsilon}}}$$

for $k > k_0$. So if ε is chosen with $\lambda + \varepsilon < p + 1$ (recall that $\lambda < p+1$) then $\sum \frac{1}{|a_k|^{p+1}}$ is dominated by a convergent series; (3.50) now follows.

Let $f(z) = P(z) \exp g(z)$ where P is a canonical product in standard form. Hence for $z \neq a_k$

$$\frac{f'(z)}{f(z)} = g'(z) + \frac{P'(z)}{P(z)}.$$

Using Lemma 3.9.1 gives that

$$-p!\sum_{n=1}^{\infty}\frac{1}{(a_n-z)^{p+1}} = g^{(p+1)}(z) + \frac{d^p}{dz^p}\left[\frac{P'(z)}{P(z)}\right].$$

However it is easy to show that

$$\frac{d^p}{dz^p}\left[\frac{P'(z)}{P(z)}\right] = -p!\sum_{n=1}^{\infty}\frac{1}{(a_n-z)^{p+1}}$$

for $z \neq a_1, a_2, \ldots$. Hence $g^{(p+1)} \equiv 0$ and g must be a polynomial of degree $\leq p$. So the genus of $f \leq p \leq \lambda$. □

As an application of Hadamard's Theorem, a special case of Picard's can be proved.

Theorem 3.9.3. *Let f be an entire function of finite order, then f assumes each complex number with one possible exception.*

Proof. Suppose there are complex numbers α and β, $\alpha \neq \beta$, such that $f(z) \neq \alpha$ and $f(z) \neq \beta$ for all z in $\mathbb{C}$. So $f-\alpha$ is an entire function that never vanishes; hence there is an entire function g such that $f(z) - \alpha = \exp g(z)$. Since f has finite order, so does $f - \alpha$; by Hadamard's Theorem g must be a polynomial. But $\exp g(z)$ never assumes the value $\beta - \alpha$ and this means that $g(z)$ never assumes the value $\log(\beta - \alpha)$, a contradiction to the Fundamental Theorem of Algebra. □

One might ask how many limes f assumes a given value α. If g is a polynomial of degree $n \geq 1$, then every α is assumed exactly n times. However $f = e^g$ assumes each value (with the exception of zero) an infinite number times. Since the order of e^g is n the next result lends some confusion to this problem.

Theorem 3.9.4. *Let f be an entire function of finite order λ where λ is not an integer; then f has infinitely many zeros.*

Proof. Suppose f has only a finite number of zeros $\{a_1, a_2, \ldots, a_n\}$ counted according to multiplicity. Then

$$f(z) = e^{g(z)}(z - a_1) \cdots (z - a_n)$$

for an entire function g. By Hadamard's Theorem, g is a polynomial of degree $\leq \lambda$. But it is easy to see that f and e^g have the same order. Since the order of e^g is the degree of g, λ must be an integer. This completes the proof. $\square$

Corollary 3.9.5. *If f is an entire function of order λ and λ is not an integer then f assumes each complex value an infinite number of times.*

Proof. If $\alpha \in f(\mathbb{C})$, apply the preceding theorem to $f - \alpha$. $\square$

Appendices

Appendix A

The Extended Plane and its Spherical Representation

A.1 Stereographic Projection

While discussing functions in complex analysis we will be concerned with functions that become infinite as the variable approaches a given point. To discuss this situation we introduce the ***extended plane*** which is

$$\mathbb{C} \cup \{\infty\} \equiv \mathbb{C}_\infty.$$

We also wish to introduce a ***distance function*** on $\mathbb{C}_\infty$ in order to discuss continuity properties of functions assuming the value infinity. To accomplish this and to give a concrete picture of $\mathbb{C}_\infty$ **we represent $\mathbb{C}_\infty$ as the unit sphere in $\mathbb{R}^3$** given by

$$S = \left\{(x_1,\ x_2,\ x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\right\}.$$

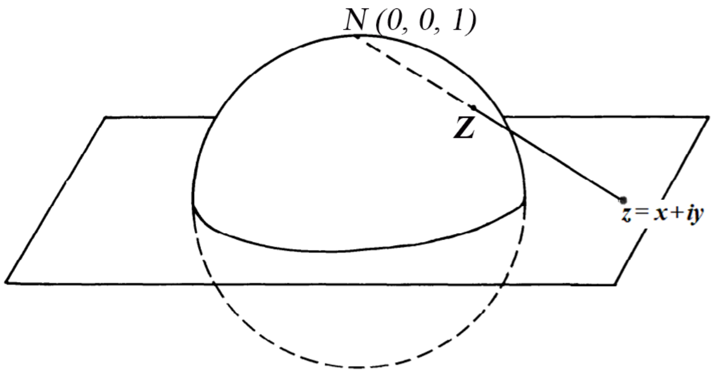

Figure A.1: The complex plane $\mathbb{C}$ shown in the figure cuts the sphere along the equator. The north pole is N with coordinates $(0,\ 0,\ 1)$. For a point $z = x + i\,y$ on $\mathbb{C}$, straight line through z and N intersects the sphere in exactly one point $Z \neq N$.

Let $N = (0,\ 0,\ 1)$; that is, N is the **north pole** on S (Fig. A.1). Also, **identify $\mathbb{C}$ with** $\{(x_1,\ x_2,\ 0) : x_1,\ x_2 \in \mathbb{R}\}$ **so that $\mathbb{C}$ cuts S along the equator.** Now for each point z in $\mathbb{C}$ consider the straight line in $\mathbb{R}^3$ through z and N. This intersects the sphere in exactly one point $Z \neq N$.

- If $|z| > 1$ (i.e., if the point z lie outside the unit circle on $\mathbb{C}$) then Z is in the northern hemisphere, because the straight line in $\mathbb{R}^3$ through z and N intersects the sphere in exactly one point on the northern hemisphere;

- if $|z| < 1$ then Z is in the southern hemisphere; and

- for $|z| = 1$ (i.e., the point z lies on the unit circle), then $Z = z$ (i.e., the point Z is z itself because the straight line in $\mathbb{R}^3$ through z and N intersects the sphere at the point z itself).

What happens to Z as $|z| \to \infty$? Clearly Z approaches N; **hence, we identify N and the point ∞ in $\mathbb{C}_\infty$. Thus $\mathbb{C}_\infty$ is represented as the sphere S.**

Let us explore this representation. Put $z = x + iy$ and let $Z = (x_1, x_2, x_3)$ be the corresponding point on S. **We will find equations expressing x_1, x_2, and x_3 in terms of x and y.** The parametrization of the line in $\mathbb{R}^3$ through $z = x + i\,y = (x, y, 0)$ and $N = (1, 0, 0)$ is obtained as follows: $N - z = [1-x, 0-y, 0-0] = [1-x, -y, 0]$ is a vector parallel to the required line.

Vector zP is parallel to the vector $N - z$; and hence zP is a scalar multiple of $N - z$, so that

$$zP = t(N - z) \text{ for some scalar } t$$

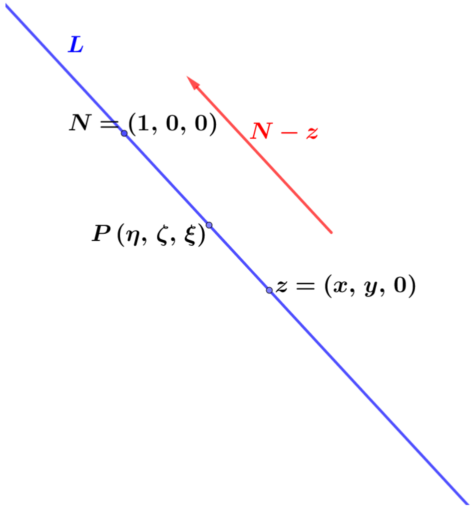

Figure A.2: Vector zP is parallel to the vector $N-z$; and hence zP is a scalar multiple of $N-z$.

With $O\,(0,\,0,\,0)$ as the origin,

$$\overrightarrow{Oz}+\overrightarrow{zP}=\overrightarrow{OP}$$

i.e.,

$$\overrightarrow{OP}=\overrightarrow{Oz}+\overrightarrow{zP}$$

i.e.,

$$\overrightarrow{OP}=z+t(N-z)$$

i.e.,

$$\overrightarrow{OP}=(1-t)z+tN.$$

As the point P is an arbitrary point on the straight line, we have a point on the straight line is $(1-t)z+tN$ for some scalar t. Hence the line is the set of points of the form $(1-t)z+tN$; i.e., the line is given by

$$L = \{(1-t)z + tN : -\infty < t < \infty\}.$$

As $N = (0,\, 0,\, 1)$ and $z = (x,\, y,\, 0)$, we have

$$L = \{(1-t)(x,\, y,\, 0) + t(0,\, 0,\, 1) : -\infty < t < \infty\}.$$

i.e.,

$$L = \{((1-t)x,\, (1-t)y,\, t) : -\infty < t < \infty\}. \tag{A.1}$$

Hence, **we can find the coordinates** $(x_1,\, x_2,\, x_3)$ **of the point** Z **of intersection of the line with the unit sphere** $\alpha^2+\beta^2+\gamma^2 = 1$ **if we can find the value of** t **at which this line intersects** S. If t is this value then substituting $\alpha = (1-t)x$, $\beta = (1-t)y$, $\gamma = t$ on the unit sphere $\alpha^2+\beta^2+\gamma^2 = 1$ we obtain

$$\begin{aligned} 1 &= (1-t)^2x^2 + (1-t)^2y^2 + t^2 \\ &= (1-t)^2\,|z|^2 + t^2, \text{ since } |z|^2 = x^2 + y^2, \end{aligned}$$

from which we get

$$1 - t^2 = (1-t)^2 \, |z|^2 \, .$$

Since $t \neq 1 \, (z \neq \infty)$[1] we can divide $1-t$ from both sides and get

$$1 + t = (1-t) \, |z|^2 \, .$$

which yields

$$t = \frac{|z|^2 - 1}{|z|^2 + 1}.$$

Thus, using (A.1),

$$\begin{aligned} x_1 &= (1-t)x = \left(1 - \frac{|z|^2 - 1}{|z|^2 + 1}\right) x \\ &= \frac{2x}{|z|^2 + 1}. \end{aligned}$$

Similarly,

$$x_2 = \frac{2y}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}$$

Thus,

$$x_1 = \frac{2x}{|z|^2 + 1}, \quad x_2 = \frac{2y}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1} \tag{A.2}$$

[1] Why $t \neq 1$? $t = 1$ gives $((1-t)x, \, (1-t)y, \, t) = (0, \, 0, \, 1)$ which is the north pole N; and this is a contradiction because we have observed above that $Z \neq N$.

This together with

$$x = \text{Re}z = \frac{z+\bar{z}}{2} \quad \text{and} \quad y = \text{Im}z = \frac{z-\bar{z}}{2i}$$

gives

$$x_1 = \frac{z+\bar{z}}{|z|^2+1}, \quad x_2 = \frac{-i(z-\bar{z})}{|z|^2+1}, \quad x_3 = \frac{|z|^2-1}{|z|^2+1} \tag{A.3}$$

Writing z in terms of $Z = (x_1, x_2, x_3)$:

If the point $Z = (x_1, x_2, x_3)$ is given $(Z \neq N)$ we find z as follows: Using (A.1), we have

$$((1-t)x, (1-t)y, t) = (x_1, x_2, x_3)$$

and hence $t = x_3$, so that

$$x_1 = (1-t)x = (1-x_3)x$$

which gives

$$x = \frac{x_1}{1-x_3}.$$

Similarly,

$$y = \frac{x_2}{1-x_3}.$$

Thus,

$$z = x + iy = \frac{x_1 + ix_2}{1-x_3} \tag{A.4}$$

A.1.1 Defining a distance function between points in the extended plane:

Now let us define a *distance function* between points in the extended plane in the following manner:

Definition A.1.1. For z, z' in $\mathbb{C}_\infty$ define the distance from z to z', $d(z,\ z')$, to be the distance between the corresponding points Z and Z' in $\mathbb{R}^3$. If $Z = (x_1,\ x_2,\ x_3)$ and $Z' = (x'_1,\ x'_2,\ x'_3)$ then

$$d(z,\ z') = \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2}. \qquad \text{(A.5)}$$

The above gives

$$\begin{aligned} [d(z,\ z')]^2 &= \underbrace{x_1^2 + x_2^2 + x_3^2} + \underbrace{{x'}_1^2 + {x'}_2^2 + {x'}_3^2} \\ &\quad -2\left(x_1{x'}_1 + x_2{x'}_2 + x_3{x'}_3\right) \end{aligned}$$

Using the fact that Z and Z' are on the unit sphere S, it follows that $x_1^2 + x_2^2 + x_3^2 = 1$ and ${x'}_1^2 + {x'}_2^2 + {x'}_3^2 = 1$, so (A.6) gives

$$[d(z,\ z')]^2 = 2 - 2(x_1{x'}_1 + x_2{x'}_2 + x_3{x'}_3) \qquad \text{(A.6)}$$

By using equation (A.3) we get

$$\begin{aligned}
[d(z,\ z')]^2 &= 2-2\left\{\frac{z+\bar{z}}{|z|^2+1}\cdot\frac{z'+\overline{z'}}{|z'|^2+1}-\frac{z-\bar{z}}{|z|^2+1}\cdot\frac{z'-\overline{z'}}{|z'|^2+1}\right. \\
&\qquad\left.+\frac{|z|^2-1}{|z|^2+1}\cdot\frac{|z'|^2-1}{|z'|^2+1}\right\} \\
&= \frac{4\left\{|z|^2+|z'|^2-z\overline{z'}-\bar{z}z'\right\}}{\left(|z|^2+1\right)\left(|z'|^2+1\right)} \\
&= \frac{4(z-z')\overline{(z-z')}}{\left(|z|^2+1\right)\left(|z'|^2+1\right)}
\end{aligned}$$

$$= \frac{4\left|z-z'\right|^2}{\left(|z|^2+1\right)\left(|z'|^2+1\right)}$$

which gives

$$d(z,\ z') = \frac{2\left|z-z'\right|}{\sqrt{(1+|z|^2)(1+|z'|^2)}},\quad (z,\ z'\in\mathbb{C}\ ) \tag{A.7}$$

In a similar manner for the complex number z and ∞ in $\mathbb{C}_\infty$ the distance from z to ∞, denoted by $d(z,\ \infty)$, is the distance between the corresponding points Z and $N=(0,\ 0,\ 1)$ in $\mathbb{R}^3$. If $Z=(x_1,\ x_2,\ x_3)$ then

$$d(z,\ \infty)=\sqrt{(x_1-0)^2+(x_2-0)^2+(x_3-1)^2}. \tag{A.8}$$

The above gives

$$[d(z,\ \infty)]^2 = \underbrace{x_1^2 + x_2^2 + x_3^2} + 1 - 2x_3 \tag{A.9}$$

Using the fact that Z is on the unit sphere S, it follows that $x_1^2 + x_2^2 + x_3^2 = 1$ so (A.9) gives

$$[d(z,\ \infty)]^2 = 2 - 2(x_3) \tag{A.10}$$

By using equation (A.3) we get

$$\begin{aligned} [d(z,\ \infty)]^2 &= 2 - 2\left(\frac{|z|^2 - 1}{|z|^2 + 1}\right) \\ &= \frac{2\left\{|z|^2 + 1 - \left(|z|^2 - 1\right)\right\}}{\left(|z|^2 + 1\right)} \\ &= \frac{4}{|z|^2 + 1} \end{aligned}$$

which gives for z in $\mathbb{C}$

$$d(z,\ \infty) = \frac{2}{\sqrt{|z|^2 + 1}} \tag{A.11}$$

This correspondence between points of S and $\mathbb{C}_\infty$ is called the **stereographic projection.**

Appendix B

Uniform Convergence of Improper Integrals

We recall Cauchy criterion for convergence of series of real numbers and Cauchy criterion for **uniform** convergence that of **series of functions.**

B.1 Cauchy Criterion

Theorem A (Cauchy Criterion for Series of Real Numbers) The series of real numbers

$$\sum_{n=1}^{\infty} x_n = x_1 + x_2 + \cdots + x_n + \cdots = \lim_{m\to\infty} \underbrace{\sum_{n=1}^{m} x_n}_{s_m}$$

converges if and only if for every $\varepsilon > 0$ there exists an $M(\varepsilon) \in \mathbb{N}$such that if $m > n \geq M(\varepsilon)$, then

$$\left|\underbrace{x_{n+1} + x_{n+2} + \cdots + x_m}_{s_m - s_n}\right| < \varepsilon.$$

Theorem B (Cauchy Criterion for Series of Real Valued Functions of Real Numbers) Let (f_n) be a sequence of functions on $D \subseteq \mathbb{R}$ to $\mathbb{R}$. The series of functions $\sum_{n=1}^{\infty} f_n = \lim_{m\to\infty} \underbrace{\sum_{n=1}^{m} f_n}_{S_m}$ is uniformly convergent on D if and only if for every $\varepsilon > 0$ there exists an $M(\varepsilon) \in \mathbb{N}$such that if $m > n \geq M(\varepsilon)$, then

$$\left|\underbrace{f_{n+1}(x) + \cdots + f_m(x)}_{S_m(x) - S_n(x)}\right| < \varepsilon \quad \text{for} \quad \text{all} \quad x \in D.$$

i.e., for a given $\varepsilon > 0$, the same $M(\varepsilon)$ works for all $x \in D$.

We use the above results only to compare with the Cauchy criterion for **uniform** convergence of improper integrals of functions of two variables where the integration is with respect to one variable only. Before that we recall improper integrals of two kinds:

Improper Integral of First Kind

The definition or evaluation of the integral

$$\int_a^\infty f(x)dx$$

does not follow from the discussion on Riemann integration since the interval $[a, \infty)$ is not bounded. Such an integral is called an **improper integral of first kind**. The theory of this type of integral resembles to a great extent the theory of infinite series.

We define

$$\int_a^\infty f(x)dx$$

as follows:

Definition (Improper Integrals of the First Kind) If $f \in R\,[a, s]$ for every $s > a$, *then*

$$\int_a^\infty f(x)dx$$

is defined to be the ordered pair $\langle f,\ F \rangle$ where

$$F(s) = \int_a^s f(x)\,dx \qquad (a \le s < \infty).$$

Integrals of the type discussed in this section are sometimes called **improper integrals of the first kind.**

Analogy with the Definition of the Series

The analogy with definition of the series is strong. The func-

tion f
corresponds to the sequence $\{a_k\}_{k=1}^{\infty}$ while the "partial integral"

$$F(s) = \int_a^s f$$

corresponds to the Nth partial sum$S_N = \sum_{n=1}^{N} a_n$.

Convergence of Improper Integral

Definition We say that $\int_a^{\infty} f$ is **convergent** to A if $\lim_{s\to\infty} F(s) = A$. In this case we write $\int_a^{\infty} f = A$. If $\int_a^{\infty} f$ does not converge, we say that $\int_a^{\infty} f$ is **divergent.**

Example B.1.1. We show that the integral $\int_1^{\infty} \frac{1}{x^2}\, dx$is convergent. With the usual notation, we have $F(s) = \int_1^s \frac{1}{x^2}\, dx$, and then $F(s) = 1 - \frac{1}{s}$and hence $\lim_{s\to\infty} F(s) = 1$.Thus

$$\int_1^{\infty} \frac{1}{x^2}\, dx = 1.$$

Example B.1.2. We show that the integral $\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$ diverges. The integral

$$\int_1^{\infty} \frac{1}{\sqrt{x}}\, dx$$

diverges since

$$F(s) = \int_1^s \frac{1}{\sqrt{x}}\, dx = 2(\sqrt{s} - 1)$$

and $\lim_{s\to\infty} F(s)$ does not exist.

Theorem B.1.3. ***Direct Comparison Test*** *Let* f and g *be continuous on* $[a,\infty)$ *and suppose that* $0 \leq f(x) \leq g(x)$ *for all* $x \geq a$*. Then*

(i) $\int_a^\infty f(x)dx$ *converges if* $\int_a^\infty g(x)dx$ *converges*

(ii) $\int_a^\infty g(x)dx$ *diverges if* $\int_a^\infty f(x)dx$ *diverges.*

Definition B.1.4. Improper Integral of the Second Kind If $f \in \mathcal{R}\ [a+\varepsilon, b]$for all εsuch that $0 < \varepsilon < b-a$,but $f \notin \mathcal{R}[a,b]$, we define $\int_a^b f(x)\,dx$as the ordered pair $\langle f,\ F\rangle$where

$$F(\varepsilon)\ = \int_{a+\varepsilon}^{b} f(x)\,dx(0 < \varepsilon < b-a).$$

We say that$\int_a^b f$**converges** to A if $\lim\limits_{\varepsilon\to 0+} F(\varepsilon) = A$.We say that $\int_a^b f$**diverges** if $\int_a^b f$ does not converge. The integral $\int_a^b f$ is called an **improper integral of the second kind.**

Example B.1.5. We examine the convergence of $\int_0^1 \frac{dx}{x^2}$. The integrand $\frac{1}{x^2}$ is unbounded at $x = 0$. Hence the given is an improper integral of second kind.
Hence $\int_0^1 \frac{1}{x^2}\,dx = \lim\limits_{\varepsilon\to 0+} \int_\varepsilon^1 \frac{1}{x^2}\,dx = \lim\limits_{\varepsilon\to 0+} \left[-\frac{1}{x}\right]_\varepsilon^1 = \lim\limits_{\varepsilon\to 0} \left(\frac{1}{\varepsilon} - 1\right) = \infty$.
Hence $\int_0^1 \frac{dx}{x^2}$ is divergent.

Theorem B.1.6. ***Direct Comparison Test*** *Let* f *and* g *be two positive functions, both are bounded at* $x = a$*and such that* $f(x) \leq g(x)$ *for* $a < x \leq b$*. Then* *(i)* $\int_a^b f(x)\,dx$ *converges if* $\int_a^b g(x)\,dx$ *converges; and*

(ii) $\int_a^b g(x)\,dx$ *diverges if* $\int_a^b f(x)\,dx$ *diverges.*

Example B.1.7. We show that

$$\int_a^b \frac{dx}{(x-a)^p}$$

converges if $p < 1$ and diverges if $p \geq 1$. The given integral is a proper integral if $p \leq 0$ and hence converges. So let $p > 0$. Then the integrand $\frac{1}{(x-a)^p}$ is unbounded at $x = a$.

Hence, $\int_a^b \frac{1}{(x-a)^p}\,dx = \lim\limits_{\varepsilon\to 0+} \int_{a+\varepsilon}^b \frac{1}{(x-a)^p}\,dx$

$= \lim\limits_{\varepsilon\to 0+} \left[\frac{(x-a)^{-p+1}}{-p+1}\right]_{a+\varepsilon}^b$, if $p \neq 1$

$$= \lim_{\varepsilon\to 0+} \frac{1}{1-p}\left[(b-a)^{-p+1} - \varepsilon^{-p+1}\right], \text{ if } p \neq 1$$

$$= \frac{1}{1-p}(b-a)^{-p+1} - \frac{1}{1-p}\lim_{\varepsilon\to 0+} \varepsilon^{-p+1}$$

But $\lim\limits_{\varepsilon\to 0+} \varepsilon^{-p+1} = 0$, if $p < 1$

$$= \infty, \text{ if } p > 1.$$

Hence, $\int_a^b \frac{1}{(x-a)^p}\,dx$ is convergent if $p < 1$ and divergent if $p > 1$.

When $p = 1$,

$\int_a^b \frac{1}{(x-a)^p}\ dx = \int_a^b \frac{1}{x-a}\, dx$, improper integral of second kind

$$= \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^{b} \frac{1}{x-a}\ dx$$

$$= \lim_{\varepsilon\to 0+} \left[\ln(x-a)\right]_{a+\varepsilon}^{b}$$

$$= \lim_{\varepsilon\to 0+} \left[\ln(b-a) - \ln\varepsilon\right]$$

$$= \ln(b-a) - \lim_{\varepsilon\to 0+} \ln\ \varepsilon.$$

Since the limit on the right does not exist $\int_a^b \frac{1}{(x-a)^p}\, dx$is not convergent at $p = 1$.

B.2 Uniform convergence of improper integrals of functions of two variables where the integration is with respect to one variable only.

Henceforth we deal with functions $f = f(x,\, y)$ with domains $I \times S$, where S is an interval or a union of intervals and I is of one of the following forms:

- $[a,\, b)$ with $-\infty < a < b \leq \infty$;
- $(a,\, b]$ with $-\infty \leq a < b < \infty$;
- $(a,\, b)$ with $-\infty < a < b < \infty$.

In all cases it is to be understood that f is locally integrable with respect to x on I . When we say that the improper integral $\int\limits_a^b f(x,\ y)\,dx$ **has a stated property on S** we mean that it has the property for every $y \in S$.

Definition B.2.1.

If the improper integral

$$\int\limits_a^b f(x,\ y)\,dx = \lim_{r \to b^-} \int\limits_a^r f(x,\ y)\,dx$$

converges on S, it is said to **converge uniformly** (or be **uniformly convergent)** on S if , for each $\varepsilon > 0$, there is an $r_0 \in [a,\,b)$ such that

$$\left| \int\limits_a^b f(x,\ y)\,dx - \int\limits_a^r f(x,\ y)\,dx \right| < \varepsilon,\ \ y \in S,\ \ r_0 \le r < b,$$

or,equivalently,

$$\left| \underbrace{\int\limits_r^b f(x,\ y)\,dx}_{\int\limits_a^b f(x,\ y)\,dx - \int\limits_a^r f(x,\ y)\,dx} \right| < \varepsilon,\ \ y \in S,\ \ r_0 \le r < b.$$

The crucial difference between pointwise and uniform convergence is that $r_0(y)$ may depend upon the particular value of y, while the r_0 in the last two equations does not: one choice must work for all $y \in S$. Thus,uniform convergence implies pointwise convergence, but pointwise convergence does not imply uniform convergence.

B.3 Cauchy criterion for uniform convergence of improper integrals of functions of two variables where the integration is with respect to one variable only.

Cauchy Criterion for Uniform Convergence: The improper integral in

$$\int_a^b f(x,\ y)\,dx = \lim_{r \to b^-} \int_a^r f(x,\ y)\,dx$$

converges uniformly on S if and only if, for each $\varepsilon > 0$, there is an $r_0 \in [a,\, b)$ such that

$$\left| \int_r^{r_1} f(x,\ y)\,dx \right| < \varepsilon, \quad y \in S, \quad r_0 \leq r, \ r_1 < b. \tag{B.1}$$

Suppose $\int_a^b f(x,\ y)\,dx$ converges uniformly on S and $\varepsilon > 0$. From Definition, there is an $r_0 \in [a,\ b)$ such that

$$\left|\int_r^b f(x,\ y)\,dx\right| < \frac{\varepsilon}{2} \qquad \text{and} \quad \left|\int_{r_1}^b f(x,\ y)\,dx\right| < \frac{\varepsilon}{2},$$
$$y \in S,\ \ r_0 \le r,\ r_1 < b.$$

Since

$$\int_r^{r_1} f(x,\ y)\,dx = \int_r^b f(x,\ y)\,dx - \int_{r_1}^b f(x,\ y)\,dx,$$

(B.2) and the triangle inequality imply (B.1). For the converse, denote

$$F(y) = \int_a^r f(x,\ y)\,dx.$$

Since (B.1) implies that

$$|F(r,\ y) - F(r_1,\ y)| < \varepsilon, \quad y \in S,\ \ r_0 \le r,\ r_1 < b.$$

Syllabus

SEMESTER 4

MTH4E05: ADVANCED COMPLEX ANALYSIS
No. of Credits: 3
No. of Hours of Lectures/week: 5

TEXT: **JOHN B. CONWAY**, FUNCTIONS OF ONE COMPLEX VARIABLE (2nd Edn.); Springer International Student Edition; 1992

Module 1

The Space of continuous functions $C(G, \Omega)$, Spaces of Analytic functions, Spaces of meromorphic functions, The Riemann Mapping theorem, Weierstrass Factorization Theorem [Chapter. VII Sections 1, 2, 3, 4, and 5]

Module 2

Factorization of the sine function, Gamma function, The Riemann Zeta function, Runge's theorem, Simple connectedness [Chapter. VII Sections 6, 7, and 8, Chapter. VII Sections 1 and 2]

Module 3

Mittage-Leffler's Theorem, Schwarz reflection principle, Analytic
continuation along a path, Monodromy theorem, Jensen's formula, The Genus and order of an entire function, Statement of Hadamard factorization theorem [Chapt. VII: Section 3, Chapter IX Sections 1, 2, and 3, Chapter XI Sections 1, 2, and Section 3 Statement of Hadamard factorization theorem only]

References

1. H. Cartan:, Elementary Theory of analytic functions of one or several variables; Addison - Wesley Pub. Co.; 1973.

2. T.O. Moore and E.H. Hadlock:Complex Analysis, Series in Pure Mathematics Vol. 9; World Scientific; 1991.

3. L. Pennisi: Elements of Complex Variables(2nd Edn.); Holf,
Rinehart and Winston; 1976.

4. W Rudin: Real and Complex Analysis(3rd Edn.); Mc Graw - Hill International Editions; 1987

5. H. Sliverman: Complex Variables; Houghton Mifflin Co. Boston; 1975

6. R. Remmert:, Theory of Complex Functions, UTM, Springer-Verlag, NY Inc.; 1991.