DOCTORAL THESIS

The Fundamental Theorem of Calculus

A case study into the didactic transposition of proof

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To Marek

Empty room

Empty heart

Since you've been gone

I must move on

ABSTRACT

The relationship between academic mathematics as practiced by researchers at universities and classroom mathematics (the mathematical practices in classrooms in primary, lower and upper secondary education as well as in undergraduate university education) is a fundamental question in mathematics education. The focus of the study presented here is on how this relationship is seen from the perspective of mathematics education and by researching mathematicians, with a focus on proof. The Fundamental Theorem of Calculus (FTC) and its proof provide an illuminating but also curious example. The propositional content of the statements, which are connected to this name, varies. Consequently, also the proofs differ. The formulations of different versions of "the" FTC cannot be understood in isolation from its historical and institutional context.

The study comprises a historical account of the invention of the FTC and its proof, including its appearance in calculus textbooks. Interviews with researching mathematicians from different sub-fields provide a picture of what meaning and relevance they attribute to the FTC. The outcomes of the historical account and the data about the mathematicians' views are discussed from the perspective of the theory of didactic transposition.

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CONTENTS

> The fact is that symbolism is useful because it makes things difficult. (This is not true of the advanced parts of mathematics, but only of the beginnings.) What we wish to know is, what can be deduced from what. Now, in the beginnings, everything is self-evident; and it is very hard to see whether one self-evident proposition follows from another or not. Obviousness is always the enemy to correctness. Hence we invent some new and difficult symbolism, in which nothing seems ob-vious. Then we set up certain rules for operating on the symbols, and the whole thing becomes mechanical. In this way we find out what must be taken as premiss and what can be demonstrated or defined. (Bertrand Russell, *Mysticism and logic and other essays*, p. 77)

1. INTRODUCTION

BACKGROUND AND OUTLINE

Russell (1917), in the course of his characterisation of the functioning of modern mathematics as an axiomatic-deductive system, admits that "the proof of self-evident propositions may seem, to the uninitiated, a somewhat frivolous occupation", but continues that: "One of the merits of a proof is that it instills a certain doubt as to the result proved; and when what is obvious can be proved in some cases, but not in others, it becomes possible to suppose that in these other cases it is false" (ibid., p.77). To Russell, in mathematics "familiarity with truth eliminates the vulgar prejudices of common sense" (ibid., p.81). Such a formalistic view of mathematics has never been seriously advertised among mathematics educators. A purely axiomatic-deductive presentation of mathematics is considered inadequate because it does not amount to understanding the essence of the propositions to be proved. But this does not mean that mathematical proof is avoided. The teaching and learning of mathematical proof, and of particular proofs, has been a controversial subject in mathematics education. Proof is not equally valued in school curricula in different countries. Bell (1976, p. 23) sees this divergence in practice due to the "tension between the awareness that deduction is essential to mathematics, and the fact that generally only the ablest school pupils have achieved understanding of it." Also for undergraduate university courses, the status of proof varies. Not always are students asked to show their mathematical competence by producing proofs or constructing fragments of original proofs. Undergraduate university students do not necessarily develop views of proof that are consistent with mathematicians' perspectives (Harel & Sowder, 1998; Martin & Harel, 1989). Hazzan and Leron (1996) found that undergraduate students sometimes use theorems, especially those with names, as vague slogans, especially when they tried to solve problems to which these theorems only seemingly applied.

The relationship between academic mathematics as practiced by researchers at universities and classroom mathematics (the mathematical practices in classrooms in primary, lower and upper secondary education as well as in undergraduate university education) is a fundamental question in mathematics education. The focus of the study presented here is on how this relationship is seen from the perspective of mathematics education and by researching mathematicians, with a focus on proof. The Fundamental Theorem of Calculus (FTC) and its proof provide an illuminating but also curious example. The propositional content of the statements, which are connected to this name, varies. Consequently, also the proofs differ. The formulations of different versions of "the" FTC cannot be understood in isolation from its historical and institutional context.

The teaching of the FTC is situated in different institutions, depending on the mathematics curriculum for upper secondary education. It is "in between" school and university mathematics. Consequently, it can be expected to resemble more of the practice of academic mathematics than proofs that are a topic exclusively in school mathematics.

The study comprises a historical account of the invention of the FTC and its proof, including its appearance in calculus textbooks. Interviews with researching mathematicians from different sub-fields provide a picture of what meaning and relevance they attribute to the FTC. The outcomes of the historical account and the data about the mathematicians' views are discussed from the perspective of the theory of didactic transposition (Chevallard, 1985; 1991).

THE NOTION OF MATHEMATICAL PROOF

The notion of proof signifies a range of concepts in different disciplines and has different connotations in different contexts. The English noun "proof" refers to both the process as well as to the protocol of this process. The word (cf. its origin from the Latin "probare") signifies the activity of *checking* and also its result. But the obligation to prove something is with the one who puts forward a proposition and not with the one who checks for the purpose of opposing. In other languages, the focus is more on *showing* the result. For example, the Swedish word “bevis” and the German “Beweis”, are derived from the verb “visa”/ “weisen” with the older meaning of “making someone to know something”. The group of related words (“veta”/ “wissen”, “vis”/ “weise”; also the Russian “videt” and the Polish “dowodzić”) originally mean “to see” or “having seen” (cf. the Latin “vidare”). The word “Beweis” was used in the context of law and has been introduced into mathematics in the second half of the18th century as a substitute for “demonstratio”. In Romanian languages, the words “démontrer” (French), “demonstrar” (Spanish), “dimostrare” (Italian), which all mean to “prove” but also “to show” or “make visible” are used for mathematical proof, but also “prouver”, “probar”, “provare” are common. So in these languages both connotations, “showing” as well as “checking”, are associated with proof.

In the following there are some examples of characterisations of proof that specify mathematical proof; these point to different features and aspects:

> **(1) Proof** (noun), **prove** (verb): the Latin adjective probus meant "upright, honest," from the Indo-European root per- "forward, through” with many other meanings. The derived verb probare meant "to try, to test, to judge". One meaning of the verb then came to include the successful result of testing something, so to prove meant "to test and find valid". Similarly, if you approve of something, you test it and find it acceptable. In a deductive system like mathematics, a proof tests a hypothesis only in the sense of validating it once and for all. In early 19th century American textbooks, prove was used in the etymological sense of "check, verify"; for example, multiplication was "proved" by casting out nines. (*The Words of Mathematics*, S. Schwartzman, 1994)
>
> **(2) Proof** n. a sequence of statements, each of which is either validly derived from those preceding it or is an axiom or assumption, and the final member of which, the conclusion, is the statement of which the truth is thereby established. (*The Harper Collins Dictionary of Mathematics*, E.J. Borowski, 1991)
>
> **(3) Proof** a chain of reasoning using rules of inference, ultimately based on a set of axioms, that lead to a conclusion. (*The Penguin Dictionary of Mathematics*, J. Daintith, 1989)

(4) Proof – a reasoning conducted according to certain rules in order to demonstrate some proposition (statement, theorem); it is based on initial statements (axioms). In practice, however, it may also be based on previously demonstrated propositions. Any proof is relative, since it is based on certain unprovable assumptions. Rules of conducting a reasoning and methods of proof form a main topic in logic. (*Encyclopaedia of Mathematics,* M. Hazewinkel**,** 1991)

(5) In mathematics, a **proof** is a convincing demonstration that some mathematical statement is necessarily true, within the accepted standards of the field. A proof is a logical argument, not an empirical one. That is, the proof must demonstrate that a proposition is true in all cases to which it applies, without a single exception. An unproven proposition believed or strongly suspected to be true is known as a conjecture.

Proofs employ logic but usually include some amount of natural language which usually admits some ambiguity. In fact, the vast majority of proofs in written mathematics can be considered as applications of informal logic. Purely formal proofs are considered in proof theory. The distinction between formal and informal proofs has led to much examination of current and historical mathematical practice, quasi-empiricism in mathematics, and so-called folk mathematics (in both senses of that term). The philosophy of mathematics is concerned with the role of language and logic in proofs, and mathematics as a language.

Regardless of one's attitude to formalism, the result that is proved to be true is a theorem; in a completely formal proof it would be the final line, and the complete proof shows how it follows from the axioms alone by the application of the rules of inference. Once a theorem is proved, it can be used as the basis to prove further statements. A theorem may also be referred to as a lemma if it is used as a stepping stone in the proof of a theorem. The axioms are those statements one cannot, or need not, prove. These were once the primary study of philosophers of mathematics. Today focus is more on practice, i.e. acceptable techniques. (*Wikipedia online a.* November 15, 2008)

> **(6)** Mathematical proof consists, of course, of explicit chains of inference following agreed rules of deduction, and is often characterised by use of formal notation, syntax and rules of manipulation. Yet clearly, for mathematicians proof is much more than a sequence of correct steps; it is also and, perhaps most importantly, a sequence of ideas and insights with the goal of mathematical understanding – specifically, understanding why a claim is true. Thus, the challenge for educators is to foster the use of mathematical proof as a method to certify not only that something is true but also why it is true. (*ICMI Study 19 Discussion Document,* 2009, p. 2)
> Understanding the relationship between argumentation (a reasoned discourse that is not necessarily deductive but uses arguments of plausibility) and mathematical proof (a chain of well-organised deductive inferences that uses arguments of necessity) may be essential for designing learning tasks and curricula that aim at teaching proof and proving. Some researchers see mathematical proof as distinct from argumentation, whereas others see argumentation and proof as parts of a continuum rather than as a dichotomy. (ibid., p. 4)

> **(7)** By **proof** we understand not just the formal process of constructing logically consistent arguments based on axioms, definitions, and theorems traditionally found in school geometry courses but all the activity that leads to discovering a mathematical fact, establishing a conjecture and constructing a justification, including exploring, generalizing, reasoning, arguing and validating. (*Assessment of reasoning and proof,* 2007)

These characterisations specify different aspects of mathematical proof. Description (1) can be taken as referring to the establishment of an “eternal truth” through a mathematical proof. The characterisations (2), (3) and (4) refer to the structure of a proof in terms of the status of the components as axioms, assumptions, proved statements and the conclusion. In (2) and (3) it is stated that these components have to be validly derived from each other, but what rules of inference or types of reasoning are used is not specified. In (4) it is pointed out that the rules and methods used in mathematical proofs are an object of study in logic. In addition this characterisation points to the fact that full formalisation cannot be achieved. The metaphor of a "series" or of a "chain" of arguments from axioms/ accepted statements to the conclusion does not immediately suggest including indirect proofs.

Description (5) introduces an audience to which the proof is addressed and stresses that standards of proof are an object of change throughout history. It also points out that there remains an unavoidable rest of ambiguity in every mathematical proof because of the embeddedness in natural language. This claim might be challenged by the existence of computerised proofs. The characterisation also draws attention to the fact that the level of formalisation differs in different mathematical practices. The point stressed in respect to the form of argument used in mathematical proof, is its deductive nature in contrast to empirical reasoning. In (6) different levels of reading a proof are mentioned: not only as a sequence of inferences but also as conveying an idea and an insight, though this seems to be restricted to mathematicians. It includes a comparison of proofing with other forms of reasoning in mathematics, though there is no agreement about whether there is a continuum between argumentation and formal proof, or whether these are to be seen as two qualitatively different forms of reasoning. Characterisation (7) intends to expand the notion of mathematical proof by including exploration, generalising, reasoning, arguing and validating (in addition to justifying through formal reasoning) by stressing the role of those in the context of discovery.

Some of these characterisations can also be seen as an expression of different philosophies of mathematics: a platonistic view in (1), a formalistic in (6) and a sociological in (5).

2. CONCEPTUAL FRAMEWORK

INTRODUCTION

The relationship between academic mathematics as practiced at universities and classroom mathematics, in particular in lower and upper secondary education, has been discussed by professional mathematics educators from the outset since mathematics education has been established as a research field. This discourse focussed on epistemological issues concerning the nature of mathematical knowledge in relation to the knowledge to be aimed at. The often quoted statement of René Thom (1973, p. 2004) that "In fact, whether one wishes or not, all mathematical pedagogy, even scarcely coherent, rests on a philosophy of mathematics" indicates this focus.

Agreement about what philosophy of mathematics would best suit the purpose of serving as a basis for mathematics education cannot be reached. But as a purely formalistic view of mathematics is too limiting, there is a tradition of studying the meaning of mathematical objects, which draws on a range of conceptual frameworks. Very prominent became the distinction between “concept definition” (the formal definition within a mathematical theory) and “concept image” (the concept representation and properties attached to it by an individual) (Niss, 2006, p. 51;Tall & Vinner, 1981), or the “duality” between operational and structural meanings (e.g. Sfard, 1991, p. 7-9). These frameworks focus on the construction of meaning by individual learners and make it possible to identify difficulties they face, for example, when condensing mathematical processes into objects (cf. Tall, 1991a). In research about students’ understanding of the concepts of calculus, especially the concept of “limit”, these approaches are central (e.g. Bingolbali & Monaghan, 2008; Cottrill, Dubinsky, Nichols, Schwingendorf, Thomas, & Vidakovic, 1996; Przenioslo, 2004). Cognitively oriented studies of such concepts have also focused on the distinction between intuitive and formal thinking (see Harel & Trgalova, 1996) and on embodied metaphors (Núñez, Edwards, & Matos, 1999). Such a cognitive psychological framework is not adopted here because the focus of the study is on how the relationship between research mathematics and classroom mathematics developed in the history of the FTC and how it is seen by mathematicians in terms of knowledge structure and its symbolisations, and not in terms of individual cognition.

The challenge of the study is to interpret the historical development of the FTC and its proof, which forms a main part of the analysis, in the light of developing rationales for its presentation in teaching and to conceptualise parallels or differences between institutionalised forms of knowledge. In the following, the notion of “epistemological obstacle” is briefly discussed as an alternative for a conceptual framework. The other option is to employ the notion of “didactic transposition”. While the first draws on similarities between the historical development of research mathematics and the development of knowledge in classrooms, the second notion focuses on differences between academic mathematics and classroom mathematics.

EPISTEMOLOGICAL OBSTACLES

One important goal for carrying out historical and epistemological analyses of mathematical concepts has been the search for parallels in knowledge development throughout history and in individuals' acquirement of this knowledge. What has become to be known as the "genetic principle" for teaching, relies on taking the historical development, including its pitfalls and problems, as a blueprint for didactical decisions. It emerged as an alternative to an axiomatic-deductive style of teaching mathematics (Mosvold, 2003). The "genetic principle", in essence, demands that the presentation of a topic should resemble the principles that have determined its development into its present form. There could be a positive parallelism (overcoming insufficient conceptions in the face of new problems), as well as a negative parallelism (repeating the pitfalls) (Thomaidis & Tzanakis, 2007). A negative parallelism includes a correspondence between the students' views that act as an obstacle to further developing their concepts and similar views held by mathematicians in the past. This suggests that identification of "epistemological obstacles" in the history of mathematics could be useful.

The notion of "epistemological obstacle" was introduced by Gaston Bachelard (1938) as a concept to be applied to the history of physics and later was used as a concept in mathematics education (Brousseau, 1997, pp. 85-114). In this view scientific knowledge provides solutions to problems, which are independent of individuals who try to solve them, but are worked out within given constraints. The fundamental principle says that building of scientific knowledge is not a continuous process but involves rejection of previous forms of knowledge; these forms of obsolete knowledge are called "epistemological obstacles". The learners' misunderstandings are not random, but linked to the set of constraints in the sphere of mathematical knowledge within which an optimal solution can be conceived. The epistemological obstacles live in this sphere. The discussion of obstacles includes also others, such as cultural and individual obstacles (with ontogenetic origin) as well as didactical ones that are connected with the way of teaching (Brousseau, 1997, pp. 86-87). Epistemological obstacles are described as follows:

> Obstacles of really epistemological origin are those from which one neither can or should escape, because of their formative role in the knowledge being sought. They can be found in the history of the concepts themselves. This doesn't mean that we must amplify their effect or reproduce in the school context the histo rical conditions under which they were vanquished (ibid., p. 87).

Epistemological obstacles are culturally shared beliefs and ways of thinking, which prevent alternative ways of conceptualising ideas and experience. These beliefs are implicit in a practice (e.g. of scientists or of the students in a classroom) and can have dogmatic effects. In this sense epistemological obstacles are seen as playing a negative role, but in the process of learning their role can be positive as they initiate understanding of the relevance and meaning of the new knowledge that is constructed through teaching. The metaphor of overcoming an obstacle is also reflected in the "magical moments" or "sudden flashes" of understanding that have been described by many mathematicians. Polya (1965), for example, describes "a sudden clarification that brings light, order, connection and purpose to details which before appeared obscure, confused, scattered and elusive" (p. 54).

The history of the development of the FTC could indeed be interpreted from a perspective of overcoming epistemological obstacles. The physical or geometrical interpretation of the integral as the area of a region (as a concrete number suggesting a static image) and the derivative as a slope (as changes of magnitude suggesting a dynamic view) refer to different phenomena and it is difficult to see the inverse character of the concomitant operations. From this point of view integration and differentiation do not appear as "operations" and the FTC is harder to "see". One could interpret the historical development such as that seeing the integral as an area provided an epistemological obstacle, a "geometrical obstacle". As a means of overcoming this obstacle the FTC can be seen as meaningful and as a significant result in the development of mathematics.

On the other hand, if the integral is conceptualised as antiderivative then the FTC appears as trivial, but it allows avoiding the "geometrical obstacle". For the mathematicians who were trying to reorganize and systematize analysis in the 18th century and were concerned with a more rigorous formulation of the foundation of calculus, an algebraic formal approach without references to geometric ideas, but with seeing differentiation and integration as "operations", was suitable. The inverse relationship between operations is no longer a problem or difficulty. This change from a geometric to an algebraic approach is seen in the works of Euler and Johann Bernoulli, when the methods of calculus were already proven to work for a number of different problems and it seemed important to settle the formal, logical grounds for the subject.

While this approach leads to formalization, the first approach allows further generalizations to extended applications. The notion of the area of a geometric region is generalized to the notion of a measure in terms of a function defined on an abstract space. (Also the notion of area or volume itself was questioned; cf. the famous Banach-Tarsky-paradoxon). Generalising led to the definition of the Lebesgue integral and measure theory.

In view of the above interpretation the following alternatives would arise with respect to the teaching of the FTC: Avoiding the "geometrical obstacle" or deliberately introduce it together with the ways of overcoming it because pointing out the relationship between integration and differentiation is more meaningful if it is not obvious. The view of overcoming obstacles that are intrinsic to the knowledge development suggests to construct teaching sequences (in the format of problems to be solved) that make the "epistemological obstacles" as visible as possible. While the epistemological obstacles should not be avoided because they are essential for the meaning of the new knowledge, the teaching should not resemble the historical situations that led mathematicians to overcome these obstacles (Brousseau 1983, p. 178).

Sierpinska (1994) sees the obstacles not as necessary steps in knowledge development, but as obstacles that can make individuals' reorganisations and modifications of their theories about mathematical concepts difficult. A parallelism in history and students' understanding is not to be seen in the supposed recapitulation of phylogenesis in ontogenesis, but rather in a commonality of mechanisms of these developments. In drawing on Skarga (1989), Sierpinska (1994) argues that these include "the preservation, in linguistic tradition and the metaphorical use of words, of the past senses" (ibid., p. 122).

Identifying obstacles to further development in the history of mathematics might help to hypothesise about the obstacles students are likely to face. This does not necessarily suggest modeling a teaching approach according to the historical development, including the obstacles. It is not common practice, for example, to introduce calculus without a preceding definition of a function, but instead use the terms quantities, their rates of change and differences connected with given curves modelled in the way Newton and Leibniz worked, in order to motivate a proper definition of function.

It remains difficult to identify qualitatively different stages in the development of mathematics and the obstacles that accounted for a change of perspective without an underlying theoretical framework for such a historical analysis. Damerow (2007) shows that establishing a connection between historical developments and individuals' cognition depends on how much cultural relativism is attributed to the development of mathematical concepts. He develops some principles of a historical epistemology of logico-mathematical thought. Similarly, Radford, Boero and Vasco (2003) point out that interpreting the history of mathematics for the purpose of understanding individual learners' difficulties relies on implicit assumptions about the epistemology of mathematical knowledge.

Fried (2008) draws attention to the goal of mathematics education as an introduction into *modern* mathematics:

> For mathematics education - at least as it is usually conceived (and this qualification is not trivial) - aims towards modern mathematics, but treats mathematics as it is conceived today as if it were mathematics *tout court*; thus, in the classroom, although one might refer to modern algebra to mean the theory of algebraic structures, one would generally see the addition of 'modern', if one considered it at all, as supererogatory (ibid., p.3).

He also points to a fundamental problem in writing a history of mathematics and compares it with other historical studies:

> Historians of mathematics are like anthropologists who study mathematical cultures very different from our own; at work, historians must consider mathematics as ever changing and having no eternal, fixed, reference. […] For this reason, historians of mathematics assiduously try to keep modern notions of mathematics away from the mathematics of the past, that is, to keep anachronism at bay. More generally, even though it is interested in understanding how the present has come to be as it is, the practice of history studiously avoids measuring the past according to modern conceptions of what mathematics is and modern standards of what is mathematically significant. Failure to do this leads one to what is known in historiography as "Whiggism," after the tendency of certain British historians to see history as marching ever towards the liberal values and aspirations of the Whig party (ibid., pp. 3-4).

In the face of the differentiation and segmentation of mathematical knowledge due to its high level of specialisation, it is not easy to see towards which shared values and accepted forms mathematical knowledge would be marching.

Artigue (1992) draws attention to the problem that what might have been identified by researchers as an epistemological obstacle, is often closely related to the teaching and learning process, that is, to the choices and characteristics of the educational system. The obstacle is then of a didactical and not of an epistemological nature (Artigue 1992, p. 110). This points to a framework that sets out to conceptualise forms of institutionalised mathematical knowledge and helps to identify the (systematic) differences between academic mathematical knowledge and knowledge developed in the course of teaching, rather than searching for parallels between those two mathematical practices.

DIDACTIC TRANSPOSITION AND MATHEMATICAL PRAXEOLOGIES

Didactic transposition

Kitcher (1984), in accordance with others, conceive of the history of a scientific field as a history of changing practices. Such a practice consists of different components, such as a language, a set of statements that are accepted, questions that are seen as relevant, and accepted standards of methods, including standards of definitions and proof (cf. Kitcher, 1984, pp. 163 sqq.). Distinct views about the scope of mathematics are included, which are dependant on the rationalities of the culture to which the practice belongs. Distinguishing between different mathematical practices (or "cultures"), which are linked to different institutions is not only reasonable for historiographic purposes.

The meaning and the relevance of the FTC and of its proof is dependant on the mathematical practice in which it is embedded. We find different versions not only in the history but also in modern calculus textbooks with different standards for notation and rigour. The history of the theorem also shows its changing status and forms. Consequently, the question of what version of this outcome of research mathematics is chosen and how it is transformed into a piece for teaching as well as what is changed throughout this process and for what reasons, is a an important question in relation to this theorem and its proof as well as in relation to the teaching of proof in general.

The “Antropological Theory of Didactics” proposes a framework that can be used for analysing mathematical practices in different institutions. Mathematical organisations are called *praxeologies*. A distinct praxeology includes a set of tasks that are to be solved (and might be considered as problematic), techniques that are available to solve them, and a technological or theoretical discourse that describes, explains and justifies the techniques used (see the section on mathematical praxeologies below). A key concept of the theory is the notion of *didactic transposition* of mathematical knowledge.

The first comprehensive outline of the Didactic Transposition Theory was developed in Chevallard (1985), with an extended second edition in Chevallard (1991), though the term ‘transposition didactique’ was used earlier, for example in Chevallard (1978). The theory is aimed at producing a scientific analysis of didactic systems and is based on the assumption that the (mathematical) knowledge set up as a teaching object (‘savoir enseigné’), in an institutionalised educational system, normally has a pre-existence, which is called *scholarly knowledge* (‘savoir savant’). By this is meant “a *body* of knowledge, not knowledge in itself” (Chevallard, 1992, p. 11), ranging from “genuinely scholarly bodies of knowledge to scholarly-like or even pseudo-scholarly ones” (ibid., p. 12).

So in the didactic transposition what is being taught at school is something generated outside school that is moved, ‘transposed’ to school. There is a distinction among the ‘original’ mathematical knowledge as it is produced by mathematicians (or others) and knowledge ‘to be taught’ as it is designed by curricula. Mathematical knowledge is most often produced outside teaching institutions and it needs a series of adaptations before being accepted for teaching.

Chevallard claims that there often is an immense difference between these objects of knowledge (Chevallard, 1991, p. 43), and defines the didactic transposition as the work done during the transformation from *scholarly knowledge* via *knowledge to be taught* and the actual *knowledge taught* to *learnt knowledge* (see e.g. Bosch & Gascon, 2006, for an overview of these four levels of bodies of knowledge). Along with the notion of didactic transposition there are other notions introduced: the bodies of knowledge, the noosphere, proto- and para-mathematical knowledge, the illusion of transparency.

The process of didactic transposition has its beginning far away from school. Then follows a process of rebuilding different elements of the knowledge with the aim of making them teachable. The first step in this sequence of transformations of knowledge is taking place in the *noosphere*, i.e. a non-structured set of experts, educators, politicians, curriculum developers, recommendations to teachers, textbooks etc. Analysing the ‘knowledge to be taught’ through the elements from the noosphere reveals the conditions and constraints under which the ‘knowledge to be taught’ is constituted.

That there really is a ‘transposition’ is evidenced by the objects, the reconstruction of the objects or dysfunctions it creates (i.e. the outcome of the didactic transposition), existing in the didactic arena of school mathematics (ibid., p. 42). The related analysis of the didactician aims at making visible the difference between the transposed (taught) object and the scholarly object, a difference not spontaneously perceived by the teacher. In addition, while ruled by norms and values attached to the educational institution, the teacher does not always take responsibility of the epistemological consequences of this difference. This is related to the *illusion of transparency*, i.e. a feeling that the knowledge taught is not to be questioned, which may lead to an “epistemological rupture” of the knowledge objects (ibid., p. 42-43).

Some objects of scholarly mathematical knowledge are defined as direct teaching objects and constructed in the didactic system (by definition or construction), i.e. *mathematical notions*, such as for example addition, the circle, or second order differential equations with constant coefficients. However, there are other knowledge objects, termed *para-mathematical notions*, useful in mathematical activities but often not set up as teaching objects per se but pre-constructed, such as the notions of parameter, equation, or proof. At a deeper didactic level, also *proto-mathematical notions* are used in school mathematics activities, such as using geometric patterns to highlight structures in symbolic expressions (e.g. $2\Delta + 5 = 11 + \Delta$) or the idea of ‘simplicity’. By this distinction, some objects become objects for teaching while others serve as auxiliary objects, and only the former are directly evaluated (ibid., pp. 49-53). However, in the didactic analysis their interrelation must be taken into account:

> “*Leur prise en compte différentielle est necessaire à l’analyse didactique*: c’est ainsi que l’analyse de la transposition didactique de telle notion mathématique (par exemple ‘identité $a^2 - b^2 = (a+b)(a-b)$) suppose la considération de notions paramathématiques (par exemple, les notions *de factorisation et*

de simplification), qui à leur tour doivent être vues à la lumière de certaines notions protomathématique (la notion de 'pattern', de 'simplicté', etc.)" (ibid., p. 55).

[*Providing a differentiated account of them* [the auxiliary objects] *is necessary in a didactical analysis*: and so the analysis of such a mathematical concept (for examle the 'identity' $a^2 - b^2 = (a+b)(a-b)$) requires the consideration of para-mathematical concepts (for example of the concepts of factorisation and simplification), which on their part must be seen in the light of certain proto-mathematical concepts (the concept of 'pattern', of 'simplicity' etc.)]

The level of "teachability" of a body of knowledge is then seen to depend on three factors, related to social organisation and cultural values: epistemological relevance, cultural relevance, and degree of exposure to society (Chevallard, 1991, p. 58). Though claimed by the noosphere to have strong cultural relevance, by expressions like the society "needs" strong mathematical knowledge in its citizens (though in fact it "demands" this, according to Chevallard), mathematics lacks this and relies on its epistemological relevance (Chevallard, 1992, p. 225). The public exposure (openness) makes possible a social control of the learners by developing systems for testing. The didactic transposition implies a *textualisation* of knowledge, as well as a *depersonalisation*, thus producing an objectification possible to be made public and to form a basis for social control (Chevallard, 1991, pp. 61-62). The text of knowledge produced thus serves as a norm for knowledge and for what it means to know, as well as for the progression of knowledge authorising didactical choices. As a consequence, a view of the learning process as isomorphic to the text, metaphorically speaking, is implied. But the order of acquisition is not isomorph to the order of exposition of knowledge. Since a text by necessity is linear, in contrast to the knowledge it sets out to describe, the textualisation process strongly constrains the didactic system: "*l'apprentissage du savoir n'est pas le décalque du texte du savoir*" (ibid., p. 63). [The acquirement of knowledge is not the pause in the text of knowledge].

A given text of knowledge to teach sets up a *didactic time*: "La processus didactique existe comme *interaction d'un texte et d'une durée*" (ibid., p. 65). [The didactical process exists as interaction between a text and a time span]. The objects of mathematical knowledge have a history and are by the didactic transposition reconstructed, thus becoming new objects. In both cases a process of 'erosion' is taking place, which produces, in the teaching situations, a contradiction or dialectic between the old and the new. The objects of teaching are thus 'victims' of didactic time, forced into an erosion-renewal didactic process. Another key difference between the scholarly and the didactically transposed knowledge is the role of the problem, which in the scholarly field is the driving force of the construction of knowledge, while in the teaching process the progression is run by the contrast between the old and the new objects (ibid.).

The didactical triangle, that is the relation between teachers, students and mathematical knowledge, is set up as a didactical machine, served by the teacher but run by the contradiction between the old and the new. The teacher is the one who (already) knows, and therefore is in charge of running didactic time: "la *chronogénèse* du savoir", realising the fiction of a unique didactic time, progressive, cumulative, and irreversible. The "chronogénèse" has to be interpreted as the allocated time and pace that organises the timely order of the learning for the student. The learner can master the past, only the teacher can also master the future (of the learner's learning). The didactic transposition tends to organise qualitative differences, institutionalising two ways of knowing, i.e. that of the teacher (knowing on a holistic theoretical level) and the learner (learning only what is given to learn within the didactic system), as well as how to know it. This assumes a dichotomisation of knowledge with one version for the teacher and one for the learner: "L'objet d'enseignement a été noté comme *objet transactionelle entre passé et avenir* (dans la chronogénèse); il apparaît maintenant comme *objet transactionelle entre les deux régimes didactiques du savoir* (dans la topogénèse)" (ibid., p. 76). [The teaching object is recorded as an object of a transaction between the past and the future (in the *chronogénèse*, see above); it becomes now palpable as an object of a transaction between the two didactical regimes of knowledge (in the *topogénèse*)]. The "topogénèse" refers to the distribution of responsibility in the relationship with the knowledge, that is, to the places of the teacher and the student in a classroom. A common outcome of this system is a fragmentation of knowledge and a tendency to algorithmisation.

Instead of looking at cause-effect issues in education the didactic transposition theory suggests a focus on the *ecological problematic*, by studying the conditions and constraints within didactic systems (Chevallard, 1992, p. 7).

Even though the theory of didactic transposition focuses on school mathematical knowledge, it can be expanded to include the teaching of mathematics in undergraduate university programmes. The conceptualisation of knowledge in the theory is linked to locating it in institutions, that is in particular, the knowledge institutionalised in the practice of research mathematics on the one hand, and the mathematical knowledge institutionalised at school. The practice of undergraduate mathematics teaching can be seen as a distinct institution (for example at colleges). Undergraduate or collegiate education serves different goals than specialised mathematics education in post-graduate programmes. It includes courses for students who will not specialise and do not aim at becoming research mathematicians (e.g. engineering students, teacher students). These usually include introductory calculus in which one version of the FTC is included. On the other hand, pre-calculus is a subject at school in many places. In some, the school subject includes calculus. A question that can be analysed from the perspective of didactic transposition is, what different institutions (research institutions, teaching institutions) define as legitimate knowledge.

An essential difference between research mathematics and school mathematics is seen in the fundamental principles that govern the growth of knowledge. While research is problem-driven, school mathematics develops by a dialectic between the ‘old’ and the ‘new’ material. Research-type mathematical behaviours and attitudes are difficult, if not impossible, to obtain in the mathematical practice of classrooms and this becomes in particular clear when proving theorems is at issue. A fact, which Balacheff (1990) describes as follow:

> We have to realise that most of the time students do not act as theoreticians but as practical men. Their task is to give a solution to the problem the teacher has given to them, a solution that will be acceptable with respect to the classroom situation. In such a context the most important thing is to be efficient, not to be rigorous. It is to produce a solution, not to produce knowledge (ibid., p. 3).

The theory of didactic transposition relies on the assumption that there is an object (‘savoir savant mathématique’) which can be identified as a part of the body of scholarly knowledge. Only if this is possible, the knowledge for teaching can be compared with the scholarly knowledge in order to judge a transposition as more or less ‘legitimate’. Chevallard used the term ‘scholar’ in quite an ironical way to characterise knowledge that guarantees and legitimates the teaching process. The “savants” from yesterday are today called “chercheurs” (researchers), and as an adjective for signifying a specific research field, the term “scientifique“ is used and not “savante“ (Freudenthal, 1986, p. 324). Kang and Kilpatrick (1992, p. 2) interpret the notion of the ‘savoir savant’ (the scholarly body of knowledge) as “knowledge used both to produce new knowledge and to organize the knowledge newly produced into a coherent theoretical assemblage.” Chevallard (1989) himself recognises the difficulty that might be involved in identifying such an object:

> In most cases a given body of knowledge will appear only in fragments. […] The first step in establishing some body of knowledge as teachable knowledge therefore consists in making it into a body of knowledge, i.e., into an organized and more or less integrated whole (ibid., p. 57).

Mathematical praxeologies

The scope of the theory of didactic transposition was in the mid 1990s widened into *the anthropological theory of didactics* by studies of the ecology of mathematical knowledge within institutions. The unit of analysis used for such studies was set up by the notion of a *mathematical organisation* or *mathematical praxeology*, based on a model of any human activity as comprised by a practical component, the know-how, and a discursive component, the know-why (Bosch & Gascon, 2006). In order to solve some *type of problem* within an institution, appropriate *techniques* are developed. However, this know-how generally does not exist isolated from a discourse about why the chosen techniques apply, giving rise to a level of *technology*, which in turn is put into a wider context of meanings by reflections in terms of a *theory*:

> En toute institution, l’activité des personnes occupant une position donnée se décline en différents *types de tâches T*, accomplis au moyen d’une certaine *manière de faire*, ou *technique*, t. Le couple [*T*/t] constitue, par définition, un *savoir-faire*. Mais un tel savoir-faire ne saurait vivre à l’état isolé : il appelle un *environnement technologico-théorique* [q/Q], ou *savoir* (au sens restreint), formé d’une *technologie*, q, « discours » rationnel (*logos*) censé justifier et rendre intelligible la

> technique (*tekhnê*), et à son tour justifié et éclairé par une *théorie*, Q, généralement évanouissante (Chevallard, 1997, p. 14).
>
> [In every institution the activity of persons in a given position takes the shape of different *types of tasks* T, accomplished by means of a specific *way of doing*, or *technique*, t. The couple [*T*/t] constitutes by definition a *know-how*. But such a know-how cannot live in an isolated state: it calls for a *technological-theoretical environment* [q/Q], or *know-why* (in a restricted sense), shaped by a *technology*, q, a rational « discourse » (*logos*) supposed to make the technique (*tekhnê*) understandable, and on the next level also to be justified and clarified by a *theory*, Q, mostly without affecting it.]

This description sets up the structure of an institutional body of knowledge, such as a sub-area of mathematical work (e.g. calculus) or a specific part of a sub-area (e.g. the FTC), suggesting a dynamic and a division of labour within the institution:

> Le système de ces quatre composantes, noté [*T*/t/q/Q], constitue alors une *organization praxéologique* ou *praxéologie*, dénomination qui a le mérite de rappeler la structure bifide d'une telle organisation, avec sa partie pratico-technique [*T*/t] (savoir-faire), de l'ordre de la *praxis*, et sa partie technologico-théorique [q/Q] (savoir), de l'ordre du *logos* (ibid.).
>
> [The system of these four components, written [*T*/t/q/Q], constitutes a *praxeological organisation* or a *praxeology*, a naming which manages to allude to the two-part structure of such an organisation, with its practical-technical part [*T*/t] (know-how), at the level of praxis, and its technological-theoretical part [q/Q] (know-why), at the level of *logos*.]

The praxeology thus links these two double-faced dimensions of the activity, acknowledging the necessity of always taking both into account, even one of them being more or less undeveloped:

> The word 'praxeology' indicates that practice (*praxis*) and the discourse about practice (*logos*) always go together, even if it is sometimes possible to find local know-how which is (still) not described and systematised, or knowledge 'in a vacuum' because one does not know (or one has forgotten) what kinds of problems

it can help to solve (Barbé, Bosch, Espinoza, & Gascon, 2005, p. 237).

This may happen when, for example, a task is ill-defined or a technique only sketched, a technology is vague or a theory does not yet exist (Chevallard, 1997).

According to Chevallard (1998), the types of problems to solve, i.e. the task, type of task, or genre of task – do not exist per se but are artefacts constructed within an institution. The level of task in a praxeology is often indicated by the presence of verbs such as solve, find, or construct. Techniques applied to solve the task define the praxeology that is used or in development. Within an institution a 'canonical' technique need not be justified by a technology. The role of the technology may be twofold, i.e. to justify *that* a technique works and to explain *why* it works. In mathematics the *justification* function dominates over the *explanation* function through its proof claims (ibid., p. 94). The same discourse may have a double function of carrying both the technique and the technology, as for example when justifying the technique of using a solution formula to solve a second degree equation by doing a completion of squares. A third function of a technology is the *production* of (new) techniques. Theory plays the same role to technology as technology does to technique, i.e. as justification, explanation, or production. In terms of the 'actors' of the institution employing the praxeology, the technicians are doing it, the technologists are designing how to do it to work, and the theorists are reflecting on the others without participating in the activities (Chevallard, 1998).

In order to accomplish an analysis of all steps of the process of a didactic transposition, Chevallard (ibid.) classifies praxeologies as point ("ponctuelle"), local, regional, or global. A given specific type of task defines a triplet of technique, technology, and theory, setting up a *point* praxeology. Such a situation is, however, not a desired goal in a didactical system as a set of unconnected point praxeologies sets the stage for a fragmentarisation of knowledge. A common technology for an aggregate of techniques for a set of types of tasks defines a *local* praxeology, while a set of technologies covered by one theory will specify a *regional* praxeology. When putting together, within an institution, a complex of regional praxeologies covered by different theories, a *global* praxeology is established. In a structural sense, the practical component of a praxeology (the *praxis*) may be viewed as an application of the discursive component (the *logos*). The following example is given by Chevallard (1998, p. 96):

> Dans l’enseignement des mathématiques, un *thème d’étude* („Pythagore“, „Thalès“, etc.) est souvent identifié à une *technologie* q déterminée (théorème de Pythagore, théorème de Thalès), ou plutôt, implicitement, au bloc de savoir [q,Q] correspondant, cette technologie permettant de produire et de justifier, à titre d’applications, des techniques relatives à divers types de tâches. On notera cependant que d’autres thèmes d’étude („factorisation“, „développement“, „résolution d’équations“, etc.) s’expriment, très classiquement, en termes de types de tâches.
>
> [In the teaching of mathematics, a theme of study („Pythagoras”, „Thales”, etc.) is often identified as a specific technology q (the theorem of Pythagoras, the theorem of Thales), or rather, implicitly as the know-why [q,Q] corresponding to this technology, allowing to produce and justify, as applications, the techniques in relation to different types of tasks. In this context one can also note that other themes of study („factorisation”, „expansion”, „solving equations”, etc.) are classically expressed in terms of types of tasks.]

A teaching institution can generally not be described in terms of a distinct established praxeology, but rather by its praxeological dynamics; for example, the techniques promoted for dealing with proportional problems, in mathematics teaching, are changing over time (Chevallard, 1998, p. 96). These changes have to be seen as related to the dynamics between the noosphere and the teaching institutions in the course of the didactic transposition process.

3. RESEARCH QUESTIONS AND METHODOLOGY

GENERAL RESEARCH GOAL AND SPECIFIC RESEARCH QUESTIONS

The relationship between academic mathematics as practiced by researchers at universities and classroom mathematics (the mathematical practices in classrooms in primary, lower and upper secondary education as well as in undergraduate university education) is a fundamental question in mathematics education. The general focus of the study presented here is on how this relationship is seen from the perspective of mathematics education and by researching mathematicians. The Fundamental Theorem of Calculus (short: FTC) and its proof serve as an example. The exemplarity is threefold. Firstly, the analysis of the development of this theorem provides an example for the changing relationship between research mathematics (the scholarly body of knowledge) and mathematics for teaching. Secondly, what has become called the FTC is situated in different institutions, depending on the mathematics curriculum for upper secondary education. It is "in between" school and university mathematics. Consequently, it can be expected to resemble more of the practice of academic mathematics than such proofs that are exclusively in school mathematics curricula. Thirdly, the FTC is an influential theorem for the systematisation of the calculus as a body of scholarly knowledge and consequently high value is attributed to it, hence its name "fundamental".

The specific research questions of the study are:

1. How is the relation between scholarly knowledge and knowledge to be taught seen in mathematics education with respect to the status and role of proof?

2. What is the propositional content of "the" Fundamental Theorem of Calculus (FTC) as part of a body of scholarly knowledge?

A. How did the statements connected with the FTC develop to become a fundamental theorem for the (new) sub-area of mathematics called calculus? How did it evolve in relation to basic notions and to the systematisation of the calculus? How was the process of this development reflected in different formulations and names?

B. How is this particular scholarly knowledge seen by researching mathematicians from different fields of expertise?

3. How can the FTC be identified and described as an object for teaching?

A. How is it presented in contemporary and subsequent textbooks?

B. How do researching mathematicians remember the FTC from their experience as students? How do they evaluate different versions presented in textbooks? What didactic transposition do they prefer or suggest for teaching?

4. With reference to the two questions above, which insights can be derived from the case study of the FTC, about the relationship between scholarly knowledge and knowledge for teaching in terms of praxeologies and of the theory of didactic transposition?

The study sets in with a literature review that forms the basis for a discussion of functions and forms proof in mathematics and in mathematics education. It comprises a historical account of the invention of the FTC and its proof, including its appearance in calculus textbooks. Interviews with researching mathematicians from different sub-fields provide a picture of what meaning and relevance they attribute to the FTC. The outcomes of the historical account and the data about the participants' views (mathematicians, students) are discussed from the perspective of the theory of didactic transposition (Chevallard, 1985; 1991).

The development of the FTC and its proof as part of an institutionalised body of scholarly knowledge is approached through a historical analysis. A study of contemporary textbooks is to reveal how the FTC and its proof have been transposed for the purpose of teaching. How this particular scholarly knowledge is seen today, is reconstructed through the eyes of researching mathematicians from different fields of expertise.

RESEARCH STRATEGIES AND METHODS

Approaches for investigating the research questions

The first research question is approached by a literature review about the forms, roles and functions of proof in mathematics as a research domain in comparison to mathematics education. The main basis is literature from within mathematics education, and not from philosophy of mathematics. This restriction is reasonable because it is the rationales for and the proposed ways of including proof in mathematics curricula at different levels, which are at issue. There is an accumulated body of literature about proof, which includes theorems, although this is not referred explicitly because the notion of proof includes that there is a proposition, which might have reached the status of a theorem, that has to be or has been proved. The outcomes of this literature review will be discussed in relation to the theoretical framework of the study. The FTC will be used as an example.

The second research question will be investigated through a historical analysis as well as through an empirical study of the meanings and relevance attributed to the FTC by researching mathematicians. This approach is based on the fact that the theorem has a long history and its meaning and proof changed with respect to the theoretical development of the area of calculus with concomitant changes in the standards of formalisation and proof. In addition, it can be assumed that there is no uniform meaning and relevance attributed to the FTC by mathematicians, as can be seen from different approaches taken in undergraduate courses and also from informal discussions with staff at mathematics departments. It is the didactic transposition, which is the focus of the study. Consequently, the knowledge, that is assumed to be there before it undergoes such a transposition, is at issue. This knowledge can be interpreted as the personal knowledge held by researching mathematicians, in addition to its manifestation in a standardised version after, at some point, it was not further developed (at least not under the name of the "FTC") in the course of its history.

The third research question is approached by two different strategies. Firstly, by an analysis of historical and more recent textbooks with a focus on the relationship between the scholarly knowledge and the knowledge for teaching. Thus, it is not the different formulations with different approaches to the proof that are at issue; these could have been studied, for example, by an analysis of wide-spread textbooks for undergraduate calculus courses. Secondly, the views of researching mathematicians on the didactic transpositions are investigated. This investigation comprises also their memories of how they have been experiencing the theorem (in a transposed version) as students.

The fourth research question is of a theoretical nature. Implications with respect to the basic assumptions and key concepts of the theory of didactic transposition and of mathematical praxeologies will be drawn from the outcomes of the first three questions.

The historical study

The development of the statements connected with the FTC is studied with reference to original works of prominent researchers. Classical works about the history of calculus were used as secondary sources. Identifying a point in time, in which calculus became a delineated body of scholarly knowledge, is not as easy as it would seem at first sight. This is because studies in the history of mathematics primarily aim at tracing the innovations in the field of research rather than their institutionalisation as a delineated and named sub-area. The reference to a sub-area or to a proposition with a common name can be taken as an indication for whether a piece of scholarly knowledge has become institutionalised. Consequently the historical study includes an investigation of the emergence of names for basic concepts and theorems as well as for the sub-area. The dissemination through textbooks or handbooks is taken as another indicator that a piece of knowledge has become institutionalised. The study of the propositions related to the FTC and of the names used for basic concepts in calculus includes all well-known early textbooks including a mathematical handbook. As “textbooks” are taken publications that are written by researchers in the field intended for an audience with less specialised knowledge in the area of knowledge to which the sub-area under consideration belongs. The use of a shared name (in some variation) for the theorem is a criterion for the selection of later textbooks. For the more recent textbooks, the choice resembles a “longitudinal cut” with some examples from different decades and from different universities. The focus is on Sweden textbooks. In many places it is also common that researching mathematicians who teach undergraduate calculus courses produce their own “textbooks” in the form of more local publications and do not choose to use commercial and common texts. If available, examples are included. Only the formulation of the FTC and its proof are investigated.

The case study: interviews with mathematicians

Yin defines the case study research method as an empirical inquiry that investigates a contemporary phenomenon within its real-life context, when the boundaries between phenomenon and context are not clearly evident (Yin, 1994, p. 13). A more operational description of a case study is the following, quoted in Mertens (2005, p. 237):

> A case study is a method for learning about a complex instance, based on a comprehensive understanding of that instance obtained by extensive descriptions and analysis of that instance taken as a whole and in its context.

Unlike quantitative methods of research, like a survey, which focus on the questions of who, what, where, how much, and how many, and archival analysis, which often situates the participant in some form of historical context, case studies are the preferred strategy chosen by researchers when how or why questions are asked. In addition, unlike more specifically directed experiments, case studies require a problem that seeks a holistic understanding of the event or situation in question, using inductive logic reasoning from specific to more general terms.

As a distinct approach to research, use of the case study originated in the early 20th century. One of the areas in which case studies have been gaining popularity is education and particular education evaluation (Stake, 1995). The popularity of case studies has developed in recent decades in the area of mathematics education. In the 1960s, researchers were becoming concerned about the limitations of quantitative methods (Hamel, Dufour, & Fortin, 1993). Strauss and Glaser's (1967) concept of "grounded theory" became very influential in the discussion of methodologies.

Researchers from many disciplines use the case study method to build upon theory, to produce new theory, to dispute or challenge theory, to explain a situation, to explore or to describe an object or phenomenon. Case studies tend to be selective, focusing on one or two issues that are fundamental to understanding the system being examined.

A criticism of case study methodology is about its dependence on a single case that is no sufficient for generalizing conclusion. Yin (1994) argued that the size of the sample (whether it is 2 or 100 cases) does not transform a multiple case into a macroscopic study. Case studies do not need to have a minimum number of cases.

Stake (1995) and Yin (1994) describe some sources of evidence in case studies: documents (could be letters, administrative documents, newspapers articles), archival records (lists of names, survey data), interviews (one of the most important sources of case of study information), direct observation (data collection activities, formal protocols to measure and record behaviours), participant-observation (occurs in studies of groups) and physical artefacts (tools, instruments, physical evidence that may be collected during the study).

Altogether, these characterisations suggest that a case study is appropriate for the research outlined above. As the question is about personal views, interviews are the most appropriate source for generating data because; the views which are of interest in this study are not documented elsewhere. There is no conceptual framework that could have been used as a basis for the construction of a questionnaire for a big sample, which would capture researching mathematicians' views of a particular theorem or of didactic transposition.

Selection of the participants

For selecting the cases in a case study, researchers often use information-oriented sampling, as opposed to random sampling. This is because the typical or average case is often not the richest in information. Extreme or atypical cases reveal more information (Flyvbjerg, 2006). This principle has been used for the selection of the mathematicians in the study.

The mathematicians have diverse backgrounds in education in different countries and are working in different, mostly unrelated, areas of mathematics. They also differ in their teaching experiences. 11 mathematicians (9 from a university in Canada; 2 from a Swedish university) are chosen. They represent researchers in different areas of mathematics:

Interviewee 1 (Int1): works as an active researcher in Mathematical Physics with his university studies in a developing country. He does not have an experience of teaching at the elementary level, but only in multivariate calculus.

Interviewee 2 (Int2): is not an active researcher but his area of interest is within Homological Algebra and Category Theory. He finished his university study in Canada. He works mostly as an instructor and has a long experience in teaching different courses at university level.

Interviewee 3 (Int3): is former researcher within Ergodic Theory but currently working mostly as instructor and administrator. He has quite a long teaching experience in different university courses. He studied at a university in Canada.

Interviewee 4 (Int4): is an active researcher in Dynamical Systems and Philosophy of Mathematics, but has also a broad experience in teaching different courses at university level. He never taught calculus courses but analysis courses in many years. He finished his university study in North America.

Interviewee 5 (Int5): has his background in one Eastern European country where he finished his university study. He is a researcher within Mathematical Physics (Integrable Systems, Classical and Quantum Gravity) but he also has experience in teaching some advanced mathematical courses.

Interviewee 6 (Int6): has his mathematical background in Logic but is working mostly as instructor and has a very long (more then 30 years) teaching experience at university level both in Canada as well as in Europe. He finished his university study in North America.

Interviewee 7 (Int7): is a researcher within Harmonic Analysis, Partial Differential Equations and Several Complex Variables with the university study in USA. She taught calculus courses more then 10 years ago, but has a long and ongoing teaching experience in courses within Measure Theory.

Interviewee 8 (Int8): is an active researcher in Number Theory with a university background in South America and a teaching experience in calculus courses.

Interviewee 9 (Int9): is an active researcher in Mathematics and Computer Science (Computational Group Theory, Sporadic Groups, Computing Galois Groups) without any teaching experience and has his university background in North America.

Interviewee 10 (Int10): is researcher in mathematics with the biggest interest in Functional Analysis and Inequalities and a long experience of teaching mathematical courses (calculus, analysis and others) on different levels. He finished his university study in an Eastern European country and worked at universities in Europe, North and South America.

Interviewee 11 (Int11): his field of research is variational analysis and viscosity solutions of nonlinear partial differential equations. He has a broad experience in teaching mathematical courses at different levels as well as administrating them. He has his university background in Europe and North America.

The interview questions

In the following, a justification of each question is given in relation to its specific formulation, intentions and to the theoretical framework.

Question 1: What is the "Fundamental Theorem of Calculus", as you understand it? Do you know several versions of it? In what version have you first heard about it as a student?

The first question (Q1) was asked to reveal the mathematicians' most spontaneous conceptions of the FTC and their most deeply rooted formulations of the theorem. Thus, Q1 focused on the identification of the FTC as scholarly knowledge. What notions and conceptions are connected to the theorem? What are the differences and similarities in the mathematicians' formulations of the FTC? Which formulation is most common? What counts as scholarly knowledge when it comes to the FTC?

Question 2: Which, if any, of the following statements would you consider as the closest to the FTC as you understand it?

1. Differentiation and integration are inverse operations

2. $d\int y dx = y$

3. If $\mathrm{F(x)} = \int_a^x \mathrm{f(t)dt}$, then $\mathrm{F}'(x) = \mathrm{f(x)}$

4. The rate of change of the area under a curve, as a function of the abscissa of the curve, is the same as the function which describes the curve. So, to calculate the area under a given curve, it is enough to find the antiderivative of the function which describes the curve.

5. To find the antiderivative of a function, it is enough to find the area under the curve as a function of the abscissa.

6. Area under a curve is to the abscissa as the ordinate of the curve is to the abscissa.

7. If *f* is continuous on [a,b] and if F is an antiderivative of *f* on [a,b] then $\int_a^b f(x)dx = F(b) - F(a)$

8. Let *f* be a continuous function on an interval I, and let *a* be a point in I. If F is defined by $F(x) = \int_a^x f(t)dt$, then $F'(x) = f(x)$ at each point x in the interval I.

9. Let f be a continuous function on a closed interval $a \le x \le b$. Then by "the definite integral of f(x) from a to b" denoted $\int_a^b f(x)dx$, we understand the number F (b) – F (a), where F is any primitive function of f in the interval (a, b), and $F'_+(a) = f(a)$ and $F'_-(b) = f(b)$.

10. The flux of a vector fields **F** = M**i** + N**j** + P**k** across a closed oriented surface S in the direction of the surface's outward unit normal field **n** equals the integral of $\nabla \cdot F$ over the region D enclosed by the surface

$$\iint_S F \cdot ndo = \iiint_D \nabla \cdot FdV$$

11. The integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region.

The second question (Q2) is concerned with the identification of the scholarly knowledge of the FTC in relation to its historical development. The question presents several textualisations of knowledge (i.e. different versions of the FTC) for the mathematicians, to be confronted with their existing views on the FTC. Would the mathematicians revise their first formulations?

The rationales behind the choices of these eleven formulations in Q2 refer to the following:

1. this can be described as folklore, concise, informal; a popular formulation of the FTC that is even sometimes used in textbooks;

2. is a literal rendering (representation) of the first one by using symbols for integration and derivation ($\int$ and d);

3. represents the formulation of the first part of the FTC in textbooks used in calculus courses but without assumptions; how important are actually assumptions for mathematicians – as researchers and teachers?

The following three formulations are used for revealing the existence of problems for mathematicians with a formulation of the FTC when using old geometrical terms instead of derivative and integral:

4. can be seen as a formulation of the FTC similar to Newton's approach as presented in Boyer (1949, p. 191);

5. in this formulation the geometrical concepts can be seen as attributed to Barrow and Leibniz; it can be used as the first part of the FTC;

6. this one can be seen as the second part of the FTC, close to the formulation used by Leibniz;

In some books (e.g. Adams, 2006) the FTC is formulated in two parts: the first one is represented in example 8 and the second part in example 7.

7. is the usual formulation of one of the parts of the FTC that we can find in many textbooks (Greenwell, Ritchey & Lial, 2003; in that book it is actually the only part);

8. is the other part of the FTC (Part I) in textbooks (R. A. Adams "Calculus: A Complete Course, 2003, Addison Wesley, Toronto);

9. a slightly different approach for the FTC with "right" and "left" primitive function (Kuratowski, 1977);

10. an example of a multidimensional formulation of FTC as the divergence theorem in Thomas and Finney (1996);

11. found in Thomas and Finney (1996), at the end of the chapter as an attempt to present a popular, informal formulation of the same theorem (as in formulation 1 above).

Question 3: When a theorem is formulated, it is normally a result of mathematicians' work on some problem. According to you, what was this problem, in case of the FTC?

The third question (Q3) was designed to investigate how important the connections between the theoretical level of mathematics and specific types of problems/ applications are for mathematicians. That is, in terms of praxeology, the question was set up to inform on the structure of an institutional body of knowledge of which the FTC is a part. In particular, the aim was to identify the type of task to be solved by the FTC, as seen by these mathematicians. The question relates to the production of knowledge at the level of technology.

Question 4: What is the significance of the FTC in the present day mathematics, pure and applied?

The fourth question (Q4) is about the importance attributed to the FTC in present day mathematics scholarly knowledge. Is the name ‘the fundamental theorem’ still adequate or is the FTC not seen any more as a fundamental theorem in mathematics? Significance can be related to types of tasks/ problems (i.e. applications) to solve by way of the FTC, or to its role in theory development (i.e. ‘down’ or ‘up’ within a praxeology). The question also opens up a possibility to identify different praxeologies in different institutions.

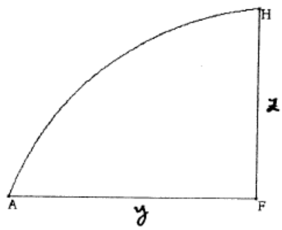
Question 5: Did understanding of the FTC change the way you thought about variables, functions and mathematics in general?

Quite often students’ difficulties with understanding the FTC are described as connected to insufficient understanding of function or limit concepts. In the fifth question (Q5) the aim is to find out how it works for mathematicians. Can they see these connections between understanding different concepts involved in the formulation of the FTC? Newton and Leibniz did not think about functions at all because that concept was not defined at this time.

The next two questions (Q 6 and Q7) are related to the identification of proof: what is needed in the reasoning so that it can be accepted as a mathematical proof?

Question 6: What do you think about the following reasoning?

We show that the general problem of finding areas under curves can be reduced to the finding of a line that has a given law of tangency.



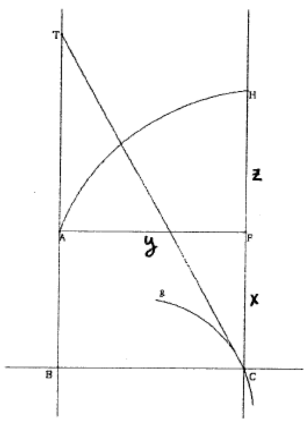
Suppose we want to find the area under the curve AH, described by the ordinate FH = z.
Let AF = y be the abscissa of the curve.

Suppose the line g is such that the tangent TC at point C of g satisfies

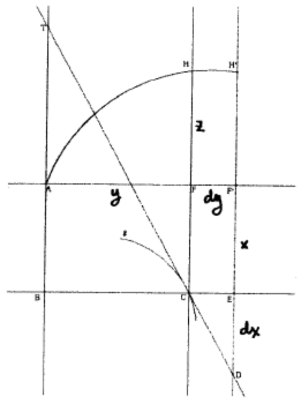
(1) $$\frac{TB}{BC} = \frac{FH}{a},$$ where a is a constant.

This means that the law of tangency of g is the same as the ordinate z, which describes the curve AH.

Denote: FC = x.



We will show that the sum of all lines z (i.e. the area under the curve AH) is equal to ax, and hence it is determined by the curve g.



We do it by way of a "motion": we extend the curve AH to H'.

By similarity of triangles TBC and CED, $ED : EC = TB : BC$.

By (1), $TB : BC = FH : a$.

And $EC = FF'$.

So $a\,ED = FH \times FF'$

Now, $FF' = dy$ and $ED = dx$ are increments of y and x, resp.

Then $a\,dx = z\,dy$.

Therefore the sum of all rectangles a by dx is equal to the sum of all rectangles x by dy

$\int adx = \int zdy$

Since the sum of all rectangles a by dx is equal to ax, we get

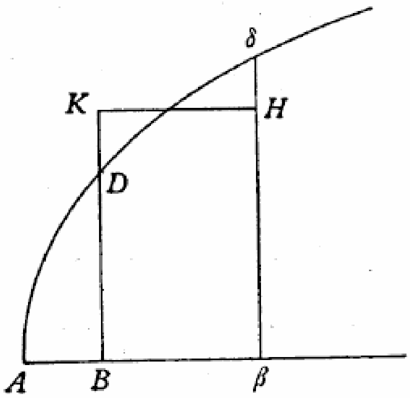
$\int zdy = ax$

i.e. once we know a curve whose law of tangency is equal to z, we know the area under the curve with ordinate z.

The reasoning in Q6 is taken from the book "Mathematical Expeditions. Chronicles by the Explorers" (pp. 133-134) as Leibniz's formulation and proof of FTC. We were interested in how the mathematicians would react to such a completely strange and different formulation of the proof as compared to the present standard versions. How foreign is this reasoning for our interviewees? As a comment of curiosity, the example illustrates how difficult it was to notice the relationship between tangent and area.

The version of the FTC displayed in Q7 (next page) shows Newton's reasoning as described in "The History of Mathematics" by J. Fauvel & J. Gray (p. 384). Do mathematicians consider this reasoning as correct proof?

Question 7: What do you think about the following reasoning, attributed to Newton? How does it differ from the present day reasoning?



Let any curve $AD\delta$ have base $AB = x$, perpendicular ordinate $BD = y$ and area $ABD = z$. Take $B\beta = o$, $BK = v$ and the rectangle $B\beta HK(ov)$ equal to the space $B\beta\delta D$. It is, therefore, $A\beta = x + o$ and $A\delta\beta = z + ov$. With these premises, from any arbitrarily assumed relationship between x and z I seek y in the way you see following.

Take at will $\frac{2}{3}x^{\frac{3}{2}} = z$ or $\frac{4}{9}x^3 = z^2$. Then, when $x + o(A\beta)$ is substituted for x and $z + ov(A\delta\beta)$ for z, there arises (by the nature of the curve) $\frac{4}{9}(x^3 + 3x^2o + 3xo^2 + o^3) = z^2 + 2zov + o^2v^2$. On taking away equal quantities ($\frac{4}{9}x^3$ and z^2) and dividing the rest by o, there remains $\frac{4}{9}(3x^2 + 3xo + o^2) = 2zv + ov^2$. If we now suppose $B\beta$ to be infinitely small, that is, o to be zero, v and y will be equal and terms multiplied by o will vanish and there will consequently remain $\frac{4}{9} \times 3x^2 = 2zv$ or $\frac{2}{3}x^2(= zy) = \frac{2}{3}x^{\frac{3}{2}}y$, that is, $x^{\frac{1}{2}}(= x^2/x^{\frac{3}{2}}) = y$. Conversely therefore if $x^{\frac{1}{2}} = y$, then will $\frac{2}{3}x^{\frac{3}{2}} = z$.

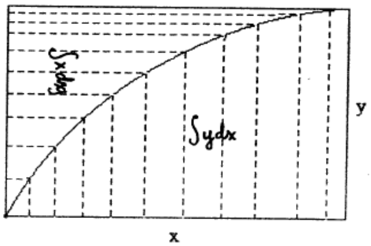
Or in general if $[n/(m+n)]ax^{(m+n)/n} = z$, that is, by setting $na/(m+n) = c$ and $m + n = p$, if $cx^{p/n} = z$ or $c^n x^p = z^n$, then when $x + o$ is substituted for x and $z + ov$ (or, what is its equivalent, $z + oy$) for z there arises $c^n(x^p + pox^{p-1} \dots) = z^n + noyz^{n-1} \dots$, omitting the other terms, to be precise, which would ultimately vanish. Now, on taking away the equal terms $c^n x^p$ and z^n and dividing the rest by o, there remains $c^n px^{p-1} = nyz^{n-1}(= nyz^n/z) = nyc^n x^p/cx^{p/n}$. That is, on dividing by $c^n x^p$, there will be $px^{-1} = ny/cx^{p/n}$ or $pcx^{(p-n)/n} = y$; in other words, by restoring $na/(m+n)$ for o and $m + n$ for p, that is, m for $p - n$ and na for pc, there will come $ax^{m/n} = y$. Conversely therefore if $ax^{m/n} = y$, then will $[n/(m+n)]ax^{(m+n)/n} = z$. As was to be proved.

Here in passing may be noted a method by which as many curves as you please whose areas are known may be found: namely, by assuming any equation at will for the relationship between the area z and from it in consequence seeking the ordinate y. So if you should suppose $\sqrt{[a^2 + x^2]} = z$, by computation you will find $x/\sqrt{[a^2 + x^2]} = y$. And similarly in other cases.

Question 8: Suppose a student has plotted the graph of the velocity of a car against time from observation, not from an algebraic formula. Suppose the student does not know the FTC. How would you explain to him that the distance of this car from the initial point can be measured by the area under the graph?

In this question (Q8) a situation is presented where a student does not know the connection between the area under the graph and the distance. If a student understands this, would he/she understand the FTC? The question was asked to examine how the mathematicians explain that connection – do they see difficulties in using approximation for an explanation purpose? How do they describe the relation between the type of problem studied and the technique developed to solve it, and the relation to the theoretical levels of the specific praxeology.

Question 9: What is wrong with the following reasoning?



Look at the figure above.

The area of the rectangle with sides x and y is equal, on the one hand, xy, and, on the other, the sum of two curvilinear triangles.

Thus $xy = \int xdy + \int ydx$

Therefore, $d(xy) = d\int xdy + d\int ydx = xdy + ydx$

The reasoning in the ninth question (Q9) is closely connected to the first formulation of the FTC in Q2. That formulation is quite popular and teachers may not be aware of problems it can create. The purpose of this question was thus to look at mathematicians' reactions and explanations for such a possible misconception problem.

Question 10: How do you teach the FTC? How would you teach it if you hade a choice?

This question (Q10) is related to the identification of 'taught knowledge' as a level in the didactic transposition process, relating this level to the level of scholarly knowledge (as described by the mathematicians). What is the relation between textualised and personalised scholarly knowledge to the 'knowledge taught'? What is the influence of traditions (in used textbooks and local institutions) on the didactic transposition? How strong is the 'pressure' or constraints of the noosphere and the textbooks on didactical choices? What didactical decisions would be based on the mathematicians' awareness of students' difficulties and misconceptions connected to the FTC.

Conduction of the interviews

The individual interviews with mathematicians, which took place in April-May 2004, lasted from 45 minutes to two hours and were tape-recorded and later transcribed. The interviews took place in each mathematician's office at their mathematics departments.

All participants have explicitly volunteered to be an interviewee. They have been informed 2-3 days earlier about the purpose, methods and goals of the study in face-to-face meetings. The set of questions were given to the participants in printed form at the beginning of the interviews. They were invited to read all of the questions before answering, if they liked, or responding directly after reading each question.

Ethical considerations

The conduction of the interviews followed the ethical guidelines set up by the Swedish Research Council concerning information, agreement, confidentiality and usage. All participants have explicitly volunteered to be an interviewee or a respondent, they have been informed about the purpose of the study, the reporting and storage of the data is anonymous. The aims of the study have been explained in personal meetings to the interviewees and they were informed how their contributions were going to be dealt with.

4. FORMS AND FUNCTIONS OF PROOF IN MATHEMATICS AND IN MATHEMATICS EDUCATION: AN ISSUE OF DIDACTIC TRANSPOSITION

INTRODUCTION

Characteristic structural or functional features of proof can be described from different perspectives, such as philosophy (of mathematics), history (of mathematics), mathematics or mathematics education with different agendas in mind. The main basis for the discussion in this chapter is literature from within mathematics education. This restriction is reasonable because it is the rationales for and the proposed ways of including proof in mathematics curricula at different levels, which are at issue.

Much of the discussion of the role of formal and other, more informal, types of "proof" in mathematics education draws on a comparison between forms and functions of proof in research mathematics and features and roles of proof suggested for different levels of mathematics education. There is agreement that introduction into academic mathematics includes an introduction into mathematical proof. School mathematical ways of proving, such as "visual proof", "inductive proof" and "proof by example", have been suggested to be included into the teaching for students at earlier grades.

In this chapter literature about the role of proof in mathematics and mathematics education is reviewed with the intention to discuss issues of didactic transposition of proof. In the first section, some forms and functions of proof in mathematics are discussed. In the second section the focus is on mathematics education.

PROOF IN MATHEMATICS

Types of mathematical proof

A main distinction of mathematical proof from a logical point of view has been made between direct and indirect proof. A *direct proof* is a way of showing the truth or falsehood of a given proposition (statement) by a straightforward combination of established facts (such as axioms, lemmas, theorems) without introducing further assumptions, by a logical deduction from the established facts to the conclusion. The protocol of such a proof is a number of statements that follow logically from each other in a chain. The last one of these statements is the proposition that was to be proved.

In contrast, an *indirect proof* (also called “reductio ad absurdum” or proof by contradiction) begins with the assumption that what is to be proved is false (that is, the logical negation of the proposition), and then shows that something contradictory happens. Such a proof does not show directly that the conclusion to be proved is true, but instead that all of the alternatives are false. The contradiction is with a proposition that is implied by the negation of the original proposition. Proof by contradiction is often used when proving the impossibility of something. One of the first proofs by contradiction is attributed to Euclid for the theorem that there are infinitely many prime numbers. Another famous example is the proof of the irrationality of $\sqrt{2}$.

This distinction between direct and indirect mathematical proof is based on a reflection of different ways of proving, and thus refers to the level of theory in a praxeology. In order to get the status of a “proof” — in contrast to a specific solution of a mathematical problem, the protocol of a mathematical reasoning has to refer to a proposition with a certain level of generality and relatedness within a theory so that there are theoretical consequences. Such a proposition might become called a theorem afterwards. The label “proof” is also used for the solution of an exemplar of a range of problems that lead to a new concept development, as for example, the proofs of the irrationality of $\sqrt{2}$ or of the transcendentality of π.

There are some common labels for distinct methods of proving, such as proof by exhaustion, proof by mathematical induction. Indirect proof can also be seen as a proving method and not only as a logical classification for a type of proof. In terms of a mathematical praxeology, distinct proving methods are technologies or techniques for solving tasks that have theoretical consequences so that their solutions gain the status of a proof.

It is not always clear where to draw the line between a proving technology and a technique. For example, a proof for the proposition that the sum of two odd integers is always even could be accomplished by exhaustion (see below), that is, by exploring the consequences of adding all possible pairs of numbers (in decimal notation) with an odd number as their last digit. In this example, proof by exhaustion is a technique. In a proof for the same proposition that employs algebra (and uses the definition of even integers as well as the distributive law) the method operates at the level of technology.

Proof by mathematical induction as a technique is typically used to establish that a given statement is true for all natural numbers or sometimes for a bounded subset of natural numbers. Within mathematics the term *proof by induction* often is used as shorthand for a proof by mathematical induction. However, the term proof by induction also refers to forms of inductive reasoning, which are not part of the accepted standards of proof in mathematics.

Proof by exhaustion, or proof by cases, perfect induction, or "the brute force method" (The Language of Mathematics, retrieved April 15, 2009**)**. In such a proof the conclusion is established by dividing it into a number of cases and proving each one separately. As to the standards of mathematical proof, there is no upper limit to the number of cases accepted in a proof by exhaustion. For example, the first proof of the four colour theorem explored 1936 cases. Proofs by exhaustion with a large number of cases are generally avoided because of the lack of insight they convey and are often considered as "inelegant" by researching mathematicians (see below for a referenced discussion of an example). A proof with a large number of cases may leave an impression that the theorem is only true by coincidence, and not because of some underlying principle or connection.

A *constructive proof*, or proof by example, is a proof that demonstrates the 'existence' of a mathematical entity with distinct properties by showing that it is possible to construct a specific example of such an object. A constructive proof thus establishes that a particular object does exist by providing a method for finding it. In Euclidean geometry, for example, the existence of a tangent to a circle was proved by showing how to construct it. A constructive proof in geometry is at the level of technique when analysing geometry as a mathematical praxeology.

A *non-constructive proof* (also known as an *existence proof* for an *existence theorem*) proves the existence of a mathematical object with certain properties, without explaining how such an object can be found. For example, the theorem that there exist irrational numbers a and b such that a^b is rational can be established via a constructive proof, or via a non-constructive proof. Often the formulation of the FTC includes a proposition about the existence of a primitive function; the concomitant part of the proof then could be seen as an existence proof.

Until the 20th century it was assumed that any proof could be checked by a competent mathematician to confirm its validity, but a number of developments in some parts of mathematics that involve the growing use of computers led to challenges of the accepted standards of proof in relation to the criteria and methods for validity check. The challenges not only concern computer-aided proofs of theorems, but also the validity of computational algorithms as well as specific calculations. *Computer-assisted proofs* include numerical methods carried out by computerised algorithms for a large number of cases, and are often preceded by reliance on computer experiments in the context of discovery (cf. Hanna & Jahnke, 1996, p. 880-881). The development includes new types of proof that do not follow the deductive argumentation of a "typical" proof. Goldwasser, Micali & Rackoff (1985) describe a *zero-knowledge proof* metaphorically as an interactive procedure between a prover and a verifier that could be used to prove that a mathematical proposition is true without revealing anything about the proof itself. As a result the verifier is convinced that the proposition is true but at the same time has "zero" knowledge of the proof itself. A zero-knowledge proof must satisfy three properties: completeness (the verifier, representing the testing procedure, has to be convinced), soundness (the chance of cheating the verifier is very small), and zero knowledge. So-called *holographic proofs* (also *transparent proof*, *instantly checkable proof*, *probabilistically checkable proof*) is a form in which every proof or record of a computation can be presented in a way that the presence of any errors is instantly seen after checking a small, part of the proof or computation. The introduction of these new types of proving changes the mathematical praxeology in which they are employed by converting large parts of a proof into a set of techniques through computerisation. Even if these types of proof are new, they are still substitutes for traditional analytic proofs based on deductive reasoning and do not challenge the value attributed to it. However, some mathematicians work with confirming mathematical properties experimentally (Borwein & Bailey, 2004).

Functions of proof in mathematics

Many authors who discuss functions of proof in research mathematics from the point of view of mathematics education, refer to Bell's (1976) distinction of the three functions (i) verification, (ii) illumination and (iii) systematization. Bell argues that the main function attributed to mathematical proof by many teachers is its role as a means for convincing the students of the correctness of a mathematical proposition. He points out that conviction in mathematics can be obtained "by quite other means than that of following a logical proof" (ibid., p. 24). In mathematics, he argues, the role of proof is not restricted to verification or justification, which only depends on the formal validity of the proof. A good proof, that is, one which is aesthetically pleasing, also illuminates why a proposition is true. The third "sense of proof is the most characteristically mathematical", the organisation of results into a deductive system (ibid., p. 24).

De Villiers (1990) presents the following functions of proof in mathematics, which include, in addition to Bell's (1976) list, also "discovery" and "communication":

- verification (concerned with the truth of a statement),
- explanation (providing insight into why it is true),
- systematisation (the organization of various results into a deductive system of axioms, major concepts and theorems),
- discovery (the discovery or invention of new results),
- communication (the transmission of mathematical knowledge) (ibid., p. 18).

Later he added the function of "intellectual challenge" (see below) as the self-realization/fulfilment derived from constructing a proof (de Villiers, 2003). Hanna (2000) sees three additional functions that are not subsumed under the ones previously mentioned:

- construction of empirical theory,
- exploration of the meaning of a definition or of the consequences of an assumption,
- incorporation of a well-known fact into a new framework and thus viewing it from a fresh perspective (ibid., p. 8).

Hemmi (2006) describes the function of "transfer" that adds to the relevance of teaching and learning mathematical proof as follows:

> The function of transfer refers to two basically different things. Firstly, working with proofs can be useful in other contexts than in mathematics. Secondly, some proofs can provide methods or techniques useful in other mathematical contexts (ibid., p. 223).

The different functions of proof can be distinguished analytically, and a given proof can be described according to its main functions. All the characterisations describe proofs and their meaning in mathematical practice. There is agreement that the knowledge about them can have an influence on students' understanding of the role of proof in mathematics, which is considered as central. In the following, the different functions attributed to mathematical proof will be briefly discussed, if appropriate with a view on the FTC.

Verification, justification and conviction

It is obvious that a proof in mathematics serves to verify or justify a statement about a conjecture or a result that has been found otherwise. This "validation function" is seen as the most important function of mathematical proofs. Ernest (1999) sees two aspects of the justification function, which include demonstrating the existence of mathematical objects, that is, an ontological function, and persuading mathematicians of the validity of knowledge claims, that is, an epistemological function. Hanna and Jahnke (1996) describe the standard view of proofs:

> The standard view of proof, stemming from the Euclidean paradigm, is that it transfers truth to a new theorem from axioms which are intuitively true and from theorems which have already been proven. A new insight is reduced to insights already established. This view is not only an epistemological claim, but also a true reflection of the subjective feelings of a mathematician producing a proof or of a person who is learning a mathematical theory but is already quite at home with it. It does not reflect the feelings of beginners, however. And in new areas of mathematics, in applications and at the borderlines between disciplines, this view of proof is not always adequate, as will now be shown (ibid., p. 894).

Proof is not always seen as necessary for conviction; being convinced may precede the attempt of proving. Polya (1954) sees confidence in a proposition as a condition which leads to constructing a proof, rather than as an outcome of proof:

> […] having verified the theorem in several particular cases, we gathered strong inductive evidence for it. The inductive phase overcame our initial suspicion and gave us a strong confidence in the theorem. Without such confidence we would have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is true, you start proving it (Polya, 1954, pp. 83-84).

The role of intuition and quasi-empirical verification (in contrast to formal explicit methods) for mathematical discovery and conviction has been stressed by many authors. Wilder (1984, p. 43) states that "without intuition, there is no creativity in mathematics [but] the intuitive component is dependent for its growth on the knowledge component." Stewart (1995, p.13) talks about an "instinct" rather than intuition: "The mathematician's instinct is to structure that process of understanding by seeking generalities that cut across the obvious sub-divisions." Some of the mathematicians in Burton's study (2004) were uncomfortable about saying that they use intuition in their work. Intuition, in their eyes, was the result of experience and knowledge, often unreliable and in the need of checking. But nevertheless they saw it as an important ingredient of mathematical thinking and research.

According to Hanna (1994, p. 58) there exist a number of factors that allow mathematicians to accept new theorems and their proofs:

- They understand the theorem, the concepts embodied in it, its logical antecedents, and its implications. There is nothing to suggest it is not true.
- The theorem is significant enough to have implications in one or more branches of mathematics (and is thus important and useful enough to warrant detailed study and analysis).
- The theorem is consistent with the body of accepted mathematical results;
- The author has an unimpeachable reputation as an expert in the subject matter of the theorem.
- There is a convincing mathematical argument for it (rigorous or otherwise), of a type they have encountered before.

A new proof needs to be accepted by other mathematicians. Then it can also be refined and improved. From a huge number of theorems published every year only a few are judged as significant and their proofs can be corrected and refined. Others remain unexamined. When an error is discovered in the proof of one of these important theorems, usually the proof is changed and not the theorem (Hanna, 1994).

By pointing to the influence of different research environments, Hodgkin (1976) suggests that a view of mathematical truth as based on objective criteria for "rightness" obtained through shared standards for proof is only an outsider's image. Mathematicians do not see it as being fixed and certain.

> We have important ways in which scientific knowledge changes character, even locally, with changing culture. And this determining character of social institutions on the knowledge acquired is even *well-known* among scientists: X was at Princeton, at Grenoble, at Warwick *and so* he/she sees a differential equation in a particular way, even has a particular notion of 'proof'. But this knowledge is not spread outside the restricted community of scientists, so that the outsiders still believe in a universal agreement on an abstract 'rightness'. Which is something they themselves have been *taught* to believe in – at school. And in mathematics in particular. (Hodgkin, 1976, p. 39, original emphasis)

The changing standards of proof are also visible in the history of the FTC. In addition, a separate development of the calculus in two communities of mathematicians shows the conditioning influence of the institutions in two different cultural environments (see chapter 5).

Explanation and illumination

A proof is not only to verify a result but also to provide some insight into why it is true. In many situations, where there is a high degree of conviction of the validity of a conjecture, its proof does not provide satisfactory explanation why the conjecture may be true. Davis and Hersh (1981) stress the explanatory function attributed to proof in the context of a discussion of the Riemann Hypothesis:

> It is interesting to ask, in a context such as this, why we still feel the need for a proof, or what additional conviction would be carried if a proof should be forthcoming which was, say, 200 or 300 pages long, full of arduous calculations where even the most persistent may sometimes lose their way. It seems clear that we want a proof because we are convinced that all the properties of the natural numbers *can* be deduced from a single set of axioms, and if something is true and we *can't* deduce it in this way, this is a sign of a lack of understanding on our part. We believe, in other words, that a proof would be a way of understanding *why* the Riemann conjecture is true; which is something more than just knowing from convincing heuristic reasoning that it *is* true. (ibid., p. 368)

For many mathematicians the explanatory aspect of a proof is more important than the aspect of verification. The criticism of the proof of the four colour theorem provides a famous example. Paul Halmos, for example, complained that one does not learn anything from the proof because the proof gives no indications as to why one only needs four colours to colour maps. He compares the insight gained with that an oracle would give about the special status of the number four (Hersh, 1998). Computer aided experimental proofs are an example of the dominance of the verification aspect over the explanatory power of proofs. Zero-knowledge proofs (see above) challenge the explanation (and the systematisation function of proof.

However, classical proofs by mathematical induction also do not carry much explanatory power. Kitcher (1984, p. 183) discusses the example of proof by induction of sums of finite series. Mathematical induction helps to prove a specific conjecture, but it does not solve the general problem because it does not provide a methodological insight. Indirect proof also does not carry much explanatory power. Hanna and Jahnke (1996) claim that for practising mathematicians, better understanding of a theorem is the main value of a proof: “rigour is secondary in importance to understanding and significance” (ibid., p. 878).

Sierpinska (1994) stresses the differences between proving and explaining something. Both activities aim at answering the question ‘why?’, but explanations draw on examples and visualizations to describe something, while in a proof it is only seen as a valuable addition to use extra explanations to highlight the central idea of the proof.

For Dreyfus (1999) there exists a route from explanation via argument and justification to proof, but the distinctions between them are not always clear. Hanna (2007, p. 7) describes *mathematical experimentation* as experiments to examine properties of mathematical conjectures and notices a shift of paradigm: “the use of computers gives mathematicians another view of reality and another tool for investigating the correctness of a piece of mathematics through investigating examples.” In such experiments, explanation and justification tend to approach each other, while the role of argumentation remains more implicit and the need of justification may become less strong for some, while others (the critics) argue that this practice increases the need for a formal deductive justification. In the mathematics classroom this calls for an awareness of these roles and for a didactical choice.

As to the FTC, in Sweden, its proof achieves exclusively an explanatory function in several textbooks for upper secondary mathematics, by only providing a graph where the change of the area function is illustrated in a way that illuminates its relation to the derivative. This approach does not count as a proper mathematical justification.

Discovery and production

Many mathematical propositions are formulated and believed to be true prior to verification by proof, but there are also examples of new results that were discovered by means of deduction, as, for example, non-Euclidean geometry. Thus, a mathematical proof can also serve the function of exploring and inventing new results (de Villiers, 2003). There are also cases where the most interesting result in a mathematical publication is not the content of the statement that is proved, but the method of the proof itself, which can be transferred to other problems. In this case the verification function is not the main point of a proof. It is rather its methodological function for the creation of new results. Such a method might be transferable to other contexts. Hemmi's (2006) function of *transfer* refers to the situation when proofs offer new techniques or insights that can be used for solving other problems that are different from the original context:

> For example, Galois' result that the fifth degree equation cannot be solved by radicals has had much less importance to mathematics than his proof for the theorem, which opened a possibility to develop a new theory (ibid., p. 61).

Some proving methods that have been developed in the context of an isolated problem, have become part of the standardised methodological repertoire in mathematics. Ernest (1999) refers to the function of proof in helping working mathematicians to develop and extend knowledge as 'methodological'. If experimental verification gains the status of an official means of justification, then such a "proof" lacks any transfer function for the development of new theory, but the methodological function for expanding the body of existing knowledge is obvious.

Systematisation

As a mathematical (deductive) proof creates a connection between statement(s) that it proofs and statements by which it proofs these statement(s), it amounts to a systematisation that aims at theory building. This is the meaning of deduction, which systematises already known results into a logical system of axioms, definitions and theorems by exposing the relationship between statements. De Villiers (2003) draws attention to a number of important functions of a deductive systematisation of known results. Organising statements into a coherent whole (that might become a theory):

- helps identify inconsistencies, circular arguments and hidden or not explicitly stated assumptions;
- unifies and simplifies mathematical theories by integrating unrelated statements, theorems, and concepts with one another;
- provides a useful global perspective of a topic by exposing the underlying axiomatic structure of that topic from which all the other properties may be derived;
- helps for applications both within and outside mathematics, since it makes it possible to check the applicability of a whole complex structure or theory by simply evaluating the suitability of its axioms and definitions;
- leads to alternative deductive systems that provide new perspectives.

Connecting statements in a proof is only possible if the statements are recognised to be about objects that can be described with the same means of symbolic techniques. In the case of the FTC the establishment of a connection between two areas that had been linked to two different objects of study is seen as one of its major achievements. That is, the theorem is attributed an important systematisation function.

Communication and institutionalisation of mathematical results

According to Bell (1976, p. 24) "proof is essentially public activity which follows the reaching of conviction, though it may be conducted internally, against an imaginary doubter." Proof offers a unique way of communicating mathematical results between professional mathematicians but also between mathematicians or teachers and students. Formal proof is part of the official written mathematical discourse. Certain standards have to be met in publications. A result together with its proof becomes an element of the official body of mathematical knowledge by this type of communication in officially sanctioned journals, conference reports and books. As is well known, this communication does not include the history of the discovery of a result. Davis and Hersh (1986) stress the "incompleteness" of formal proof and point to the contextuality of mathematical argumentation:

> We recognize that mathematical argument is addressed to a human audience, which possesses a background knowledge enabling it to understand the intentions of the speaker or author. In stating that mathematical argument is not mechanical or formal, we have also stated implicitly what it is (…) namely, a human interchange based on shared meanings, not all of which are verbal or formulaic (ibid., p. 73).

The history of the FTC (see chapter 5) offers some insights into the changing standards for communicating mathematical results. The development and institutionalisation of shared notational systems play an important role in the process of conveying mathematical results.

Construction of an empirical theory

Hanna and Jahnke (1996, pp. 894-896) discuss Newton's proof of Kepler's laws, that is his reduction of Kepler's laws to the law of gravity, as an example of "a mathematical empirical theory". Newton's version of the law of mass attraction states that the attraction between two masses is inversely proportional to the square of their distance. The law has been derived from measurements, but Newton's version relied on an uncertain hypotheses. His generalisation is based on introducing concepts of another order, that is, of mass and force. This could be conceived as a theoretical explanation for Kepler's law. This theoretical explanation, the proof, has the function of deriving an empirically well-established fact from something less comprehensible and less certain. The proof rather gives credibility to the assumptions (the law of gravity) behind it than to the conclusions. After application to sufficiently many cases (in the different areas that the law connects), the law became accepted, and also the consequences. The suggestion of Hanna and Jahnke is to understand a mathematical empirical theory in a dynamic way, rather than as static. A static view would see it as a network of theorems (laws) and measurements, the laws connected by deductively established relationships, that is, by proofs. A more dynamic view is to see the laws as statements about measurement, the credibility of which is enhanced by proofs. By the proof an empirical law becomes part of a theory. It is then established not only by those measurements, which directly refer to it, but also by measurements that confirm other laws of the same theory. The whole theory is confirmed rather than individual laws by individual tests.

Hanna and Jahnke apply the same argument to Euclidean geometry, which is viewed as an empirical theory for describing spatial relations of objects.

Measuring angles of triangles establishes an empirical law. The sum turns out to be always near to 180 degrees. The result is then proven as a mathematical theorem. The function of that proof is to enhance the conviction of the measurement result, "not because mathematics has a mysterious power to make statements about triangles, a power that goes beyond that of measurement, but because it connects the measurements of angles in triangles with a wealth of other measurements which taken together confirm that Euclidean geometry viewed as an empirical theory is one of the most well-established theories of all" (ibid., p. 896).

Intellectual challenge

Proof can become an intellectual challenge for mathematicians. It can serve the function of self-realization and fulfilment, which can be appealing for researchers as an intellectual challenge for them. It is then not the existence of the result which is important but whether it is possible to prove it:

> Perhaps, though, there is another purpose to proof – as a testing ground for the stamina and ingenuity of the mathematician. We admire the conqueror of Everest, not because the top of Everest is a place we want to be, but just because it is so hard to get there (Davis & Hersh, 1981, p. 369).

Many mathematical theorems have been proven by different proofs, which is visible, for example, in the growing collection of proofs for the Pythagorean Theorem. The task in adding a new proof cannot be seen as the establishment of a theorem, but as a challenge of finding a new re-description of the proposition and its proof.

PROOF IN MATHEMATICS EDUCATION

Functions of proof in mathematics education

Introduction

Depending on theoretical frameworks and research goals, there is a diversity of studies in mathematics education on the role of proof that take different directions, such as analyses of the role of proof in mathematics curricula, students' conceptions of proof, teaching experiments to teach students to prove, and the use of new technologies for the teaching and learning of proof (Holton, 2001). The interest for questions and problems connected with notions of theorems and proof changed over the last two decades. A comparison of the references to proof in *The International*

Handbook of Mathematics Education (Bishop et al., 1996) and in the *Second International Handbook of Mathematics Education* (2003) reveals a decline in interest.

In the first handbook a whole chapter is devoted to the issue (Hanna & Jahnke, 1996). The introduction to the section *Perspectives & interdisciplinary contexts*, under which the chapter appears, announces that the chapter deals with the main question whether the idea and notion of proof still remains at the heart of mathematics and explores the consequences of conceptions of proof for the mathematics curriculum and for teaching practices. In several other chapters there are references to proof. These concern the relationships of proofs with computers in teaching mathematics, questions connected with the traditional geometry curriculum, questions in the philosophy of mathematics education (Hilbert's programme and Lakatos' notion of proof). In the second handbook proof is only mentioned at three occasions, a main reference occurs in connection to dynamic geometry systems in which students' attention can be changed from investigating the constructions of figures to explanation, verification and even proof (Bishop, Clements, Keitel, Kilpatrick, & Leung, 2003, p. 335).

For many mathematics educators too much stress on formal proof is feared to become an impeding factor for fostering productivity and creativity of the students, while others are critical about a disappearance of proof in the curricula of school mathematics and also in introductory university courses (Hanna & Jahnke, 1996; Hemmi, 2006; Holton, 2001). Theories about the learning of mathematics within a developmental psychological perspective, even suggest that mathematical proof might be inaccessible to students at lower levels. Tall (1991b), for example, describes a process of how mathematical objects are constructed by learners by moving through three levels: perceived objects, "procepts" (consisting of a process for constructing a mathematical object and a symbol representing either process or object) and axiomatic objects. The last are according to the theory only common in advanced mathematics where axioms and definitions are used for proving that the objects posses some distinct properties.

On the other hand there are also mathematics educators who are concerned about curricula that do not comprise mathematical proof but focus on students' argumentation and justification processes during their problem solving experiences. The situation is considered to be based on a misconception of the nature of mathematical proof; the roles of proof in mathematical practice are not well understood (Greeno, 1994, p.270).

Goals associated with the teaching of proofs of theorems include the development of students' awareness of the role of assumptions and of the interrelationship between concepts. Students are hoped to learn how to deduce information from given facts and also to use visual representations of mathematical objects. To what extent and in which form proofs can or should be included naturally depends on the assumed mathematical background of the students. Some forms of "proof" suggested for young students may not even be considered as a proof by expert mathematicians (see below the section on forms of didactic transpositions of mathematical proof). The degree of formalisation of school mathematics differs with the curriculum traditions in different cultural contexts.

Explanation/ illumination and justification/ verification

Students are hoped to build powerful and lasting images of mathematical ideas if they are investigating problems in situations, which are interesting for them and when they ask whether and why relationships they discover are true (Hanna, 1995).

> [proof] deserves a prominent place in the curriculum because it continues to be a central feature of mathematics itself, as the preferred method of verification, and because it is a valuable tool for promoting mathematical understanding (ibid., p. 43).

However, it has been questioned, whether formal proof as such promotes mathematical understanding. McCallum (2000) starts from the observation that students in calculus courses usually are asked to memorize but not to understand the definitions and theorems. The traditional courses that include the learning and repetition of proofs of relevant theorems are inefficient and do not satisfy the students' and teachers' expectations and needs. Teachers may ask students to prove statements but these are generally not produced by the students but suggested by the teacher and this is often reserved to the top level students. Even more seldom students are asked to produce conjunctures themselves. In relation to the FTC, McCallum reports:

> When I first started teaching calculus in 1979, as a graduate teaching assistant at Harvard University, I was thrilled with the prospect of explaining the concept of limit, derivative, and integral, and looked forward to the climax of the story, the

> Fundamental Theorem of Calculus. However, I gradually discovered that teaching calculus was as far removed from my own mathematical experience as teaching neat handwriting would be to student of English Literature. I discovered that most of my students were attempting to navigate the course entirely by memorizing rules and procedures, and that they were largely succeeding because the homework and exam questions were all cut from a limited set of templates defined by examples in the book (McCallum, 2000, p.12).

The so-called calculus reform movement in the 1980ies in the United States started with the dissatisfaction of many mathematicians with courses in that subject, which were labelled as 'cookbook courses', which comprise a number of algorithms in the form of 'recipes', of which the students cannot see how and why they should be used (see Ganter, 2000). The debate focussed on the incorporation of technology (hoped to make up for the students' weak computational skills) and the role of applications, but not so much on the relevance of proof for enhancing the students' understanding.

Dreyfus (1999) notes that the ability to prove a theorem depends on forms of knowledge to which most students are rarely if ever exposed. Most high school and college students do neither know what a proof is nor what role it has. Even by the time they graduate from high school, most students have not been introduced into the practice of justifying and proving their mathematical work and knowledge:

> College students do not usually read mathematics research papers or see research mathematicians in action. But they do listen to lectures and participate in exercise sessions; they see and experience the talk and actions by their teachers; they read textbooks; they hand in assignments and tests, and they consider the grader's remarks when they receive them back; their mathematical behaviour is shaped, consciously or subconsciously, by these influences (ibid., p. 96).

A number of research studies reviewed by Dreyfus (1999) confirm that very few students ever learn to appreciate formal proofs of mathematical theorems and learn to construct such arguments themselves. He sees evidence that understanding and constructing formal proofs is extremely difficult even for those with reasonable proficiency and understanding. This observation suggests that there is not so much explanatory and illuminatory value in formal proofs in the context of teaching and learning mathematics.

Raman (2002) introduces a private and a public aspect of proof, which are linked to the functions of proof as a means of explanation and justification:

> The private being that which engenders understanding and provides a sense of why a claim is true. The public aspect is the formal argument with sufficient rigor for a particular mathematical setting which gives a sense that the claim is true (ibid., p.2-3).

For teachers both aspects are connected through the key idea of the proof, that is, "the essence of the proof which gives a sense of why a claim is true" (ibid., p. 3), but for students the aspects appear disconnected because they do not realize the key idea of the proof. So it is possible that clarifying the role of the key ideas can be important for helping students develop a correct view of mathematical proof (Raman, 2002).

Argumentation

Proving in mathematics is a very complex process that involves logical and deductive arguments, visual or empirical evidence and mathematical results and facts, as well as intuition, beliefs and social norms.

To prove a statement includes establishing its validity by an argument in a convincing way. Davis (1986) claims that introducing a proof can be helpful in that it puts a focus on the process of justification, which can open a debate. Hanna (2007) observes that it is already known that students have a lot of difficulties with constructing and understanding logical argumentation and consequently also with mathematical proof. However, when students are asked to show their work or to justify their answers to mathematical questions, some form of reasoning is involved and this should be made clear and visible for the students. For Dreyfus (1999), the distinction between argumentation, explanation and proof presents a didactical dilemma:

> As didacticians, we must sharpen our awareness of the distinctions between explanation, argument and proof, and we must reflect on what we can and what we should expect from students in different age groups, levels and courses. And as teachers, we must attempt the difficult task of helping students to understand what we expect from them (p. 103).

Consequently, there are some suggestions for stressing types of argumentation and 'proof' that can open up a road to formal proof, which is not accessible to students before having developed adequate pre-knowledge.

Developing an adequate view of mathematical proof and of mathematics

Szendrei-Radnai and Töröks (2007, p. 118) state a number of reasons why the teaching of proofs is important in mathematics education:

- it is a major characteristic and inherent part of mathematics as a discipline;
- classical theorems and their proofs are part of the human culture,
- proofs provide tools and methods for reasoning;
- a proof can work as a generic example of a logical structure;
- the teaching of proving raises the level of clarity and helps to avoid ambiguity.

The first two reasons are linked to the goals for the teaching of mathematics as a cultural heritage, and not so much because of its usefulness. As deductive proof is seen as the core of mathematical reasoning, this aspect should be stressed. The issue is then not so much the teaching of how to proof, but rather to show examples of proofs and also to point to characteristic features of mathematical proofs. This function of proof in a mathematics curriculum is connected with teaching *about* mathematics rather than with the teaching of specific mathematical topics.

Developing general reasoning competences and critical thinking

Learning how to establish a proposition by contesting and justifying conjectures and hidden assumptions is seen as important for developing general skills in reasoning. Consequently, mathematics educators have started out to propose ways of teaching that support this goal. An early example can be found in Harold Fawcett's works (1938, reprinted 1995). The goals that he promoted for geometry teaching include the development of critical thinking:

> [It is] … to find a way not only to teach the important facts of geometry but also to acquaint the pupil with the kinds of thinking one needs in life situations which can best be learned by study of geometry (Fawcett, 1995, editor's preface, p. v).

The purpose of his study was to find and describe some classroom activities, which allowed using geometric proofs to influence pupils' general thinking and reasoning capacities as a contribution of geometry to students' general education:

> Teachers of mathematics agree, at last verbally, that the most important reason for teaching demonstrative geometry is to acquaint the pupils with certain ideas related to the nature of deductive proof and to make them familiar with postulational thinking as a general method of thought (Fawcett, 1995, p. 117)

For Fawcett (ibid., p.120) it was clear that the study of proof cannot be composed as a special course but it should be seen as a part of the students' general education because proof "is a concept which not only pervades his [the pupil's] work in mathematics but is also involved in all situations where conclusions are to be reached and decisions to be made". He saw the concept of "proof" not restricted to mathematics:

> The ability to express ideas concisely and accurately, the ability to abstract from a situation those qualities which make it different from other situations, the ability to define and the ability to generalize are all recognized as educational values which are common to many areas of learning." (ibid., p.121)

Some research carried out in the 1930ies indeed showed that the study of geometry did not automatically improve students' reasoning and understanding in non-geometric situations (ibid., pp. 7 - 8). But Fawcett insisted that by using some ideas and methods from geometry, which serve to illustrate the nature of proof rather than the factual content of the subject matter it is possible to help students with developing critical thinking about new ideas and information in non-mathematical situations before accepting them (ibid., p. 12). But he also acknowledged that this competence does not follow directly from dealing with geometrical proofs without consideration during the teaching-learning process:

> No transfer [of the knowledge] will occur unless the material is learned in connection with the field to which transfer is desired. Isolated ideas and subjects do not integrate (ibid., p. 13).

Fawcett proposed to overcome this lack of transfer by developing classroom activities that involve the students in developing careful definitions, questioning conclusions presented by others, analysing evidence in order to distinguish facts from assumptions, uncovering hidden assumptions and evaluating assumptions and arguments (ibid., pp. 11-12). This is to develop a critical attitude also towards their own beliefs that guide the students' actions. The importance attributed to this formative goal of teaching mathematics can be seen from the following early publication:

> The good teacher of mathematics nowadays knows, perhaps as do few others, that to have searched and found, leaves a pupil a different person from what he would be if he merely understands and accepts the results of others' search and formulation (Kilpatrick, 1922, p.23).

FORMS OF DIDACTIC TRANSPOSITIONS OF MATHEMATICAL PROOF

Blum and Kirsch (1991, p. 184) see three levels of proving. These are (i) experimental 'proofs', (ii) intuitional proofs (a translation of "inhaltlich-anschauliche Beweise") and (iii) formal (scientific) proofs. For them the "border line" between "proofs", that are not proofs for mathematicians, and (real) proofs runs between the first and the second level and not between the second and third. They also believe that not only in school but also in teacher training, students should have the possibility to work with intuitional proofs by using a correct but intuitive argumentation. In their opinion it is necessary that students get involved in "doing mathematics on a preformal level" (ibid., p. 186). But this preformal level does not mean that the mathematics becomes easier or simpler but maybe more obvious and natural. In their opinion preformal proofs can be a challenge even (or especially?) for experienced mathematicians.

The authors are explicit about using the expression "preformal" instead of "non-formal" or "informal", as the prefixes 'non' and 'in' would represent negations and therefore suggest something not of full value, which they see in sharp contrast to their intentions. For them preformal proofs must be "valid, rigorous [but not formal] proofs" (ibid., p.201). Their definition of the proof is connected to Semadeni's (1984) concept of "action proofs". These are intended for primary classrooms, in which, according to Piaget's developmental psychological account, the pupils are not able to engage in hypothetical deductive reasoning. Blum and Kirsch (1991, p.187) use the following definition: "we mean by a preformal proof a chain of correct, but not formally represented conclusions which refer to valid, non-formal premises" with the following examples of such premises: concretely given real objects, geometric-intuitive facts, reality-oriented basic ideas, intuitively evident and psychologically obvious statements. Depending on the forms of representations of the premises and conclusions, the authors present three types of preformal proofs: (i) Action proofs, which contain some concrete actions which correspond to mathematical arguments, (ii) geometric-intuitive proofs using basic geometric concepts and "intuitively evident [geometric] facts" as well as (ii) reality-oriented proofs with basic ideas that have clear connection to (empirical or experienced) "reality". All conclusions derived in preformal proofs, when formalized, should become correct, formal mathematical arguments. However, "It requires a competent mathematician to judge whether a given preformal proof is acceptable" (ibid., p.189).

The reference to concepts like 'intuitive', 'obvious', 'psychologically obvious' or 'self-evident' does not necessarily allow to distinguish preformal from formal proofs, because total formalisation of mathematical proofs is out of reach. What counts as obvious, and what not, means different things for different people and can be changed with experience and with changing standards of proof. If neither teachers or students are in the position to judge the validity of a preformal proof, it does not seem to be easy to construct these proofs.

Another, related transposition of formal mathematical proof are "generic proofs". These proofs draw on arguments that are generalised from one example and can be used in the teaching of number theory. Both, explanation and conviction are hoped to be achieved (Rowland, 2001, p. 160). A generic proof, is given in terms of a set of particular numbers but nowhere relies on any specific properties of these numbers (Mason & Pimm, 1984). Some might involve images of figurative numbers. Division by nine, sums of consecutive odd numbers, or the sum of consecutive integers provide some well-known examples.

Lakatos (1976) proposes that the definition of mathematical proof as a sequence of propositions that follows from preceding propositions or axioms, refers to narrow class of proofs only. Mathematicians use and accept also other kinds of proof which he calls informal or pre-formal proofs:

> I shall begin with a rough classification of mathematical proofs; I classify all proofs accepted as such by working mathematicians or logicians under three heads: (1) pre-formal proofs, (2) formal proofs, (3) post-formal proofs. Of these (1) and (3) are kinds of informal proofs (Lakatos, 1978, p.61).

From this perspective, different versions of informal proofs are not only didactic transpositions or replacements of formal proofs, but belong to the accepted repertoire of scholarly mathematical practice.

Depending on the repertoire of basic notions and symbolising techniques available to the students, in teaching institutions formal proofs are often substituted by other forms of argumentation and proofs. Many proofs of theorems that have already been proved are invented for didactical purposes. This is the case also for the FTC.

The role of proof for developing an adequate picture of mathematics and general reasoning competencies that are transferable to other contexts is a role only attributed to it in general education. Courses that aim into the introduction of a delineated body of scholarly knowledge, such as calculus at the undergraduate level, do generally not explicitly include the goals of developing an adequate view of mathematics and of enhancing the students' general reasoning capacities.

5. THE HISTORICAL DEVELOPMENT OF THE FTC AND ITS PROOF

INTRODUCTION

What is called “calculus” nowadays includes the study of limits, derivatives, integrals and infinite series. It constitutes a major part of modern undergraduate university mathematics education. Pre-calculus, and in some countries also calculus, is included in upper secondary education. The level of formalisation and the order of topics in modern calculus courses show that the teaching does not resemble the historical development. However, some of the earlier methods are still part of the school curriculum (e.g. a didactic transposition of Cavalieri’s Principle). Calculus courses usually start with differentiation and later introduce integration, while historically the ideas of the integral calculus were developed before the differential calculus. The idea of integration first arose in connection with finding length, areas, volumes and arc lengths. Differentiation was created in connection with tangents to curves and questions about maxima and minima (Artique, 1991). These problems became important only with the development of mathematical physics.

In order to understand and discuss the difference between “knowledge that is used” and “knowledge for teaching”, a difference that is the basic assumption in the theory of didactic transposition, it must be possible to identify a point in time, when the piece of knowledge under consideration becomes an institutionalised part of the body of scholarly mathematical knowledge. That means, it has to become a shared “rule” to use a particular version of concepts and methods of a distinct sub-area. The establishment of internal coherence of such a “sub-area” is a prerequisite for distinguishing it from other areas. Coherence of an area can be achieved by a set of distinctive basic notions, specific methods and objects of study, a notational system, a methodology for relating the propositional statements to each other (e.g. by proofs), or its common intellectual roots. The development of these parts might take place independently or in a dialectic process of mutual inducement. Not all of these parts need to be exclusively related to one area, but may overlap with developments in other areas. Calculus is clearly a sub-area of mathematics where this is the case. Consequently, identifying a point in time, in which it became a delineated body of scholarly knowledge, turns out to be not as easy as it would seem at first sight. This is because studies in the history of mathematics primarily aim at tracing the innovations in the field of knowledge production rather than their institutionalisation as a delineated and named sub-area. As long as new methods and results were transmitted verbally, in personal correspondence and through privately circulated manuscripts, identification of a point of institutionalisation remains generally problematic. As to the historical development, the invention of the calculus takes place in a period, in which mathematics itself as a "body" of scholarly knowledge only became established as a specialised academic discipline.

The following historical account aims at providing a basis for the discussion of the didactic transposition of the calculus, in particular of the FTC and of its proof. The long history of the calculus shows that logical deduction from accepted mathematical propositions, that is proof, is not a prerequisite for the acceptance of mathematical methods and results. Many of its methods were known to work long before justification of their functioning by given standards of proof was at issue, and many of its theorems were established long before the results required to set the calculus on solid conceptual foundations were integrated into a body of scholarly knowledge (Boyer, 1959).

One indication for whether a piece of scholarly knowledge has become institutionalised is the possibility to refer to it with a common name. Consequently the study of the emergence of names for basic concepts and theorems as well as for the sub-area is of interest. The dissemination through textbooks or handbooks is another indicator that a piece of knowledge has become institutionalised.

INVENTION OF THE CALCULUS AND INSTITUTIONALISATION OF THE FTC

Predecessors

The invention of infinitesimal calculus (referred to as the "Calculus" since the 17th century), ranges under one of the most important achievements in European mathematics. Most writings about the history of calculus begin with ancient Greece and the works of Greek mathematicians, in particular of Antiphon, Eudoxus, Euclid and Archimedes (for a challenge of this focus on European roots, see Joseph, 1991; for Arabian contributions to the development of calculus see Juschkewitsch & Rosenfeld, 1963). In the 16th century, through translations and commentaries, the works of Greek mathematicians became available. Some of the typical problems that had been posed were taken up in the 17^{th} and 18^{th} century by Oresme, Galilei and Kepler (Becker, 1975). Eudoxus of Cnidus (410/408-355/347 B.C.), a student at Plato's Academy in Athens, is known for the development and skilful use of the famous Greek method of exhaustion (Boyer, 1959). Archimedes of Syracuse (287?-212 B.C.) applied that method to plane curves (e.g. parabolas) and solid figures (e.g. spheres and cones). By using the method he was, for example, able to show the volume of a cone and of a pyramid to be 1/3 that of the cylinder and prism respectively, of the same height and the same base. He also proved a number of specific results by exhaustion, but did not show or indicate how he found them (Katz, 1998). The area and volume calculation had to be worked out for each individual case. But the idea of using inscribed and circumscribed rectangles together with an estimation of the decrease of the error when increasing the number of rectangular stripes, can be seen as the basic idea of integral calculus. Whiteside (1964) even claims that the technique is equivalent to the definite Cauchy-Riemann integral on a convex set of points in the plane. The further development shows, that over a long period some prerequisites had to be invented, before the calculus could be generalised and algorithmisised. These comprise the invention of formulas and variables to allow full symbolic representation of mathematical statements without everyday language, analytical geometry and a beginning theory of functions and their differentiation and integration that allow to see commonalities in different phenomena (such as velocity, centre of mass/ gravity, slope of a curve, slope of its tangent etc.).

Prerequisites

As to some ideas of a beginning theory of functions developed in the Late Middle Ages, in particular of graphical representations, Nicole Oresme (ca. 1323-1382) probably for the first time used the area under a velocity-time curve for representing the covered distance, but without an explanation why this is possible. He invented some general rules for representing the quantity of a given “quality” (e.g. velocity). He formulated the proposition that the distance traversed by a body starting from rest and moving with uniform acceleration is the same as that which the body would cover if it were to move for the same interval of time with a uniform velocity which is one-half of the final velocity. Oresme even worked with infinite series - another concept, which was essential in the development of calculus. For example, he considered a body moving with uniform velocity for half of a period of time, with double of this velocity for the next quarter of the time, three times this velocity for the next eighth and so on. He used a geometrical method and by comparison of areas corresponding to the distances, he found that the total distance would be four times that covered in the first half of the time (Boyer, 1959).

Another aspect that made the development of the calculus possible was the generalization of numbers, that is, the introduction of symbols for the quantities involved in algebraic relations. As early as in the 13th century, letters had been used as symbols for quantities by Jordanus Nemorarius, but there had been no possibility to distinguish quantities assumed to be known from those unknown, which are to be found, that is variables from parameters (Boyer, 1968). The French mathematician François Viéte (1540-1603) started to use consonants to represent known quantities and vowels for the unknown (Edwards, 1979). This symbolism was essential to the progress of analytic geometry and the calculus in the following centuries because it made possible to use the concepts of variability and functionality. The convention to use letters near the beginning of the alphabet to represent known quantities, while letters near the end to represent unknown quantities, was introduced later by Descartes in *La Gèometrie* (Edwards, 1979).

Various types of calculus problems studied in the 17th century were inherited from Greek mathematics: problems of quadratures and cubatures (finding areas and volumes), questions concerning centres of gravity, problems of determining tangents and problems about extreme values. The list of the 17th century calculus problems is longer, but those were the most influential in terms of the process that led to the generalisation and algorithmisation of the calculus. Many mathematicians made their contributions to the development of calculus, some of which took up inherited ideas about producing figures and solids through "flowing" elements ("forma fluens", "fluxus formae"; cf. Becker, 1975, p.144). In the following there are some examples of methods for finding areas and volumes.

Bonaventura Cavalieri's (a student's of Galilei) theory of indivisibles, presented in his *Geometria indivisibilibus continuorum nova quadam ratione promota* (1635) was a development of a method for representing continuous "wholes" (lines, areas and volumes) by indivisibles of a lower dimension (Edwards, 1979). Cavalieri thought of an area as being made up of components which were intersecting lines parallel to a tangent; the "flowing" of one characteristic line (the "regula") produces the area (Becker, 1975). He stated that if two solids have the same height then their volumes will be proportional to the areas of their bases ("the theorem of Cavalieri"). Roberval (1602-1675) considered problems of the same type; for him a curve was a path of a moving point and then a tangent line at a given point showed the direction of the motion at that point; he looked at the area between a curve and a line as being made up of an infinite number of infinitely narrow rectangular strips (Edwards, 1979). Blaise Pascal (1632-1662) worked with infinitesimally small rectangles and triangles instead of simple indivisibles and derived some results equivalent to trigonometric integrals. This idea was taken up by Leibniz (Baron, 1987; Becker, 1975). Pierre de Fermat (1601-1665) invented a general and systematic method for the quadrature of higher parabolas and hyperbolas (Boyer, 1959). The techniques that were used during the 17^{th} century for finding areas and volumes were designed for specific geometrical forms while in modern calculus the methods are general and algorithmic.

The "FTC" and the calculus

Isaac Barrow's (1630-1677) *Lectiones Opticae et Geometricae* (1669) contain methods of drawing tangents to curves and of determining areas bounded by curves. Barrow is generally credited to be the first who saw differentiation and integration as inverse operations. What became to be called the fundamental theorem of calculus is formulated and proved, in a geometrical disguise, in his *Lectiones* (Lectio X, Prop. 11, pp. 30-32) (Edwards, 1979). Barrow probably was not aware of the fundamental nature of the two theorems presented in his text. The lack of an analytic representation of the operations restricted the effective use of the inverse relationship (Jahnke, 2003). He reduced inverse-tangent problems to quadratures but not the opposite. The geometrical argumentation makes the text hard to understand:

> Let *ZGE* (Fig.1) be any curve of which the axis is *VD* and let there be perpendicular ordinates to the axis (*VZ, PG, DE*) continually increasing from the initial ordinate *VZ*; also let *VIF* be a line such that, if any straight line *EDF* is drawn perpendicular to *VD*, cutting the curves in the points *E, F,* and *VD* in *D*, the rectangle contained by *DF* and a given length *R* is equal to the intercepted space *VDEZ*; also let *DE:DF* = *R:DT*,

and join (T and F). Then TF will touch the curve VIF. (Struik, 1986, p. 255)

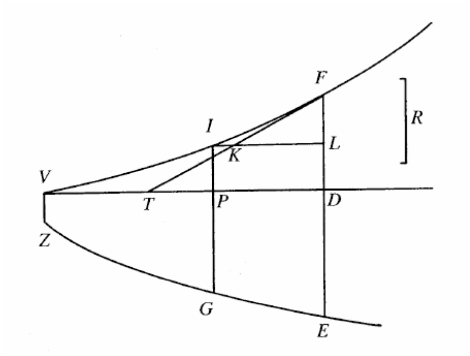


Fig. 1

Figure 1: Reconstruction of Barrow's approach (Struik, 1986, p.225)

In modern symbolism $y = f(x)$ stands for the curve ZGE and $z = g(x)$ for the curve VIF. Then Barrow's theorem reads that TF is the tangent at F. One can understand $\frac{DF}{DT}$ as the slope of the second curve at the point F, which is equivalent to the formulation that $y : z = R : z\frac{dx}{dz}$ and $y = \frac{d}{dx}\int_a^x ydx$ if $Rz = \int_a^x ydx$.

The use of the classical definition of the tangent may have caused a rather static interpretation that did not suggest to handle problems about rates of change.

Similar ideas are attributed to James Gregory (1638-1675), a mathematician of Scottish origin who worked for a period in Italy. Prag (1939) classifies Gregory's *Geometriae pars universalis* (1668) as an early, if not the first, attempt to write a systematic textbook on what became to be called calculus. The work includes a proof that the method of tangents was inverse to the method of quadratures.

The existing techniques were not yet integrated into a closed system, which would allow algorithmic treatment of different types of problems, such as the study of motion, the problem of determining the speed and acceleration of moving bodies and the converse problem of finding the speed and distance travelled by the body at any instant of time if the acceleration is unknown. Some techniques remained unpublished and hence were less influential. Quadrature had become recognized as the geometric model for all integration processes, while differential processes were connected to tangent methods and problems of finding extreme values.

Isaac Newton and Gottfied Wilhelm Leibniz have been accorded a central role in the invention of the calculus. Newton was the first to apply calculus to more general problems and Leibniz developed much of the notation that is still used in calculus today. Leibniz started with integration and Newton with differentiation. But the calculus developed by Newton and Leibniz had not the form that students see today: the main object of their study was not functions but curves.

When Newton and Leibniz first published their results there emerged, as it is well known, a controversy over who deserved the credit. Newton derived his results first, but Leibniz published earlier. At his death Newton left around 5000 pages of unpublished mathematical manuscripts that first appeared in the Cambridge edition of *The Mathematical Papers of Isaac Newton* edited by D. T. Whiteside in 1964. Both Newton and Leibniz realized that a whole variety of problems follow from two basic problems and that these two problems were the inverse of each other, that is, the fundamental theorem of calculus.

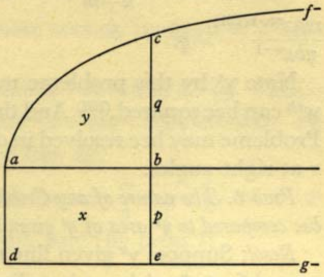
The controversy on the priority of the invention of the calculus eventually caused an isolation of British mathematics from further developments on the continent. British mathematicians did rather explain and apply Newton's work than attempt to develop it, while the Leibnizian calculus continued to develop. After some time it could not anymore be easily translated into the Newtonian style, terminology and notation. Leibniz used the name "Calculus differentialis et calculus integrali" (Kaiser & Nöbauer, 1984, p. 47). Newton called his calculus "The science of fluxions" (Boyer, 1959).

Newton wrote a tract on fluxions in October 1666, not published at the time, but seen by many mathematicians as the work that had a major influence on the direction calculus was to take (Edwards, 1979).

Soe if y^e curve bee $x^3 = byy$. Then is $\mathcal{X} = 3x^3$. $\mathcal{X} = -2byy$. $\mathcal{X} = 6x^3$. $\mathcal{X} = -2[b]yy$. $\mathcal{X} = 0$. And therefore ck(95) $= 3y + \frac{4xx}{3y}$, which hath no least nor the curve any least crookednesse.(96)

Prob 5^t. To find y^e nature of y^e crooked line whose area is expressed by any given equation.

That is, y^e nature of y^e area being given to find y^e nature of y^e crooked line whose area it is.



Resol. If y^e relation of $ab = x$, & $\triangle abc = y$ bee given & y^e relation of $ab = x$, & $bc = q$ bee required (bc being ordinately applyed at right angles to ab). Make $de \parallel ab \perp ad \parallel be = 1$. & y^n is $\square abed = x$. Now supposing y^e line cbe by parallel motion from ad to describe y^e two superficies $ae = x$, & $abc = y$; The velocity w^th w^ch they increase will bee, as be to bc: y^t is, y^e motion by w^ch x increaseth being $be = p = 1$, y^e motion by w^ch y increaseth will bee $bc = q$. which therefore may bee found by prop: 7^th. viz: $\frac{-\mathcal{X}y}{\mathcal{X}x} = q = bc$.(97)

(95) Or $\frac{[(3x^3)^2y^2 + (-2by^2)^2x^2].-2by^2}{-(3x^3)^2(-2y^2b)y - (-2by^2)^2(6x^3)y}$.

(96) Newton entered this in cancellation of 'which is least when $x = -\frac{4b}{27}$'. Where

$$ck = 3y + 4x^2/3y = 3y + \tfrac{4}{3}b^{\frac{2}{3}}y^{\frac{1}{3}}, \quad \text{then} \quad d/dx(ck) = 3 + 4b/9x$$

which increases uniformly as x decreases and is zero for $x = -4b/27$. However Newton wishes here to consider only real points on the curve and so restricts x to the interval $[0, \infty]$. Hence this 'minimal' value for curvature defined by $x = -4b/27$ is not admissible in his scheme. More directly, we may calculate the radius of curvature to be

$$cm = \frac{(9x^2+4bx)^{\frac{3}{2}}}{6bx}, \quad \text{so that} \quad \frac{d(cm)}{dx} = \frac{\sqrt{[9x^2+4bx]}}{2bx^2}(9x^2+6bx)$$

and cm has therefore an extreme value for x in $[0, \infty]$ at $x = \infty$ (which, in fact, defines the inflexion point at infinity on the semicubic parabola $by^2 = x^3$). We may also show that cm increases with x in the region of $x = 0$, so that the curvature at a real point takes on an apparent maximum at $x = 0$, $y = 0$. (Compare the difficulties which Newton had in the winter of 1664/5 in considering the extreme values of curvature in the case of a conic. See 4, §2.5 above.)

(97) Since, where $\mathcal{X} \equiv f(x, y) = 0$, we have $\mathcal{X} = xf_x$, $\mathcal{X} = yf_y$ and $f_x + f_y\frac{dy}{dx} = 0$, therefore $q = -\frac{\mathcal{X}y}{\mathcal{X}x} = \frac{dy}{dx}$. Hence the area '$\triangle abc$' $= \int q.dx = y$, a statement of the fundamental theorem of the calculus that $\int\left(\frac{dy}{dx}\right).dx = y$.

Figure 2: Newton's fundamental theorem (Whitheside, 1964, Vol. I, p. 427)

Newton's first manuscript notes date from 1665, in which he used "pricked'' letters such as $\dot{x}$, which he had started to use it consequently in late 1691. In 1710 William Jones made a transcript of the 1671 work on fluxions and inserted the dot notation. This transcript was subsequently copied in all published editions. Newton acknowledges that he had been led to his first discoveries in analysis and fluxions by Wallis' works. Studying Wallis's *Arithmetica Infinitorum* (1655) he had also discovered the binomial series. Newton stated that he was in possession of his fluxionary calculus in 1665-1666 but the first notice of his calculus was given in 1669 in *De Analysi Per Aequationes Numero Terminorum Infinitas* (Boyer, 1959; Scott, 1960). In this work he did not use the fluxionary idea but worked with the idea of an indefinitely small rectangle or "moment" of area to find the quadratures of numerous curves. Newton began this short summary of his discoveries with three rules (Jahnke, 2003):

Rule 1: If $y = ax^{\frac{m}{n}}$, then the area under y is $\frac{an}{(n+m)}x^{\frac{m}{n}+1}$

Rule 2: If y is given by the sum of more terms (also infinite number of terms) $y = y_1 + y_2 + \ldots$, then the area under y is given by the sum of the areas of the corresponding terms.

Rule 3: In order to calculate the area under a curve $f(x, y) = 0$ one must expand y as a sum of terms of the form $ax^{\frac{m}{n}}$ and apply Rule 1 and Rule 2.

Later in this work he gave "a general" procedure for finding the relation between the quadrature of a curve and its ordinate in which it is possible to see that Newton recognized the inverse relationship of integration and differentiation. The procedure is the following (Whiteside, 1964, Vol. II, pp.242-245):

Let area $ABD = z, BD = y, AB = x, B\beta = o, BK = v$ chosen in such way that area $BD\delta\beta =$ area $BKH\beta = ov$.

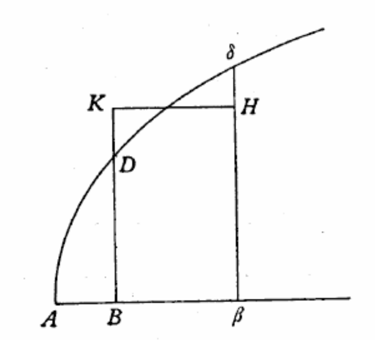


Fig. 3

Figure 3: Reconstruction of Newton's approach (Whiteside, 1964, Vol. II, p. 242)

And then follows an example with the curve $z=\frac{2}{3}x^{\frac{3}{2}}$. If one substitutes $x+o$ for x and $z+ov$ for z one gets:

$$(z+ov)^2=\frac{4}{9}(x+o)^3$$

$$z^2+2zov+o^2v^2=\frac{4}{9}\left(x^3+3x^2o+3xo^2+o^3\right).$$

Removing terms without o (which are equal) and dividing both sides by o, results in

$$2zv+ov^2=\frac{4}{9}\left(3x^2+3xo+o^2\right).$$

Now, Newton takes $B\beta$ as "infinitely small", so $v=y$ and terms where o vanishes:

$$2zy=\frac{4}{3}x^2$$

Using the first expression for z results in $y=x^{\frac{1}{2}}$.

After the example Newton used the same method to show that if $\frac{n}{m+n}ax^{\frac{m+n}{n}}=z$ then $ax^{\frac{m}{n}}=y$, and he finished by saying "And similarly in other cases".

The essential element in the procedure is the substitution of the "small" increments o and ov for x and z in the equation. Newton made use of this method also in the determination of maxima and minima, tangents and curvature. Later he reformulated these algorithms in terms of “fluents” and “fluxions”. So if one considers the point D as moving along the curve then the corresponding ordinate y, abscissa x, quadrature z (or any other variable quantity connected with the curve) would increase or decrease, that is, “flow”. Newton called such flowing quantities “fluents” and he called their rate of change with respect to time “fluxion”. In earlier studies he used different letters and by 1671 (in *Methodus fluxionum et sericum infinitarum*) he introduced the dot-notation, where the fluxions of the fluents x,y,z are $\dot{x},\dot{y},\dot{z}$ respectively. Newton often made the additional assumption that one of the variables, say x, moves uniformly so $\dot{x}=1$. It is possible to do so because he was not interested in the values of the fluxions but in their ratio $\frac{\dot{y}}{\dot{x}}$, which is equal to the slope of the tangent. He explains that the ratio of the fluxions $\frac{\dot{y}}{\dot{x}}$ is equal to the “prime” (when x and y come into existence) or “ultimate” ratio (when they cease to exist) of augments or decrements of y and x (Struik, 1986). From these concepts emerges one of the foundational questions in the debate and critique of the calculus: *Do prime or ultimate ratios exist*?

For Newton, time was the universal independent variable. He thought of variables as moving quantities and focused attention on their velocities. He regarded o as a small time interval and used p for the velocity of the variable x (later $\dot{x}$), so the change in x over the time interval o was op. In the opening chapter of the *Fluxions* (1671) Newton states two basic problems (Jahnke, 2003):

> 1. Given the length of the space continuously (that is, at every time) to find the speed of motion at any time proposed [modern explanation: given an expression for distance in terms of time $s=f(t)$, compute the velocity $v(t)=\frac{ds}{dt}=f'(t)$].
>
> 2. Given the speed of motion continuously, to find the length of the space described at any time proposed [modern explanation: given an expression for velocity in terms of time $v(t)=\frac{ds}{dt}=\Phi(t)$, compute the distance $s=\int_0^t \Phi(x)dx$].

It is easier to understand some of Newton's statements in the notation of modern calculus. Newton did not use a single notation for the area under a curve. Usually he used phrases as “the area of” and sometimes a symbol of a rectangle with the term inside it (Jahnke, 2003). He also used a small vertical bar above *x* for notating the integral of *x* (Cajori, 1923).

Altogether, Newton's method of fluxions was based on change:

- a fluent as a changing quantity,
- the fluxion of the fluent: its rate of change,
- the moment of a fluent: the infinitely small amount of change experienced by a fluent in an infinitely small time interval.

Leibniz’s conception of the foundations was different. While Newton considered variables changing with time, Leibniz thought of variables *x, y* as ranging over sequences of infinitely close values. He introduced *dx* and *dy* as differences between successive values of these sequences. Leibniz recognised that the search for a good notation was of a fundamental importance. Newton, on the other hand, was more informal in his notations.

The following basic ideas can be said to underlie Leibniz’s invention of the calculus (Baron, 1987):

- Leibniz’s interest in symbolism and notation (in connection with his idea of a general symbolic language),
- the insight that summing of sequences and taking their differences are inverse operations and so determining quadratures and tangents are inverse operations,
- the use of the characteristic triangle in deriving general transformations of quadratures.

Leibniz’s idea about integrals, derivatives and calculus were derived from close analogies with finite sums and differences. He combined these ideas in a series of studies on the analytic treatment of infinitesimal problems.

His notation appeared to be more clear than Newton’s (Boyer, 1959). He is responsible for introducing the integral and differential sign *dx*. The symbols $dx, dy, \frac{dx}{dy}$ were introduced in a manuscript from November 11, 1675 (Cajori, 1923). Before introducing the integral symbol, Leibniz wrote *omn.* in front of the term to be integrated. The manuscript from 29th of October contains the following result, which he comments to be “a very fine theorem, and one that is not at all obvious” (Edwards, 1979, p.252):

$$\frac{\overline{omn.l}^{2}}{2} = omn.\overline{\overline{omn.l}\,\frac{l}{a}}$$

where overbars are used in place of parentheses, which means in modern version:

$$\frac{1}{2}\left(\int dy\right)^2 = \int \left(\int dy\right) dy.$$

The integral symbol was first used in this unpublished manuscript. Later in 1675, he proposed the use of the integral symbol in a letter to H. Oldenburg (secretary of the Royal Society). The first appearance of the integral symbol in print was in an article in the *Acta Eruditorum* from 1686 but it did not look exactly as the one used today. The symbol missed the lower part and was similar to the letter "f" (Cajori, 1923). The modern definite integral symbol was first introduced in 1822 by Jean Baptiste Fourier in his work *The Analytical Theory of Heat* (Cajori, 1923). Fourier was the one who extended the integral symbol with the lower and upper interval notation (Cajori, 1923, p. 35).

Leibniz and Newton are considered as the inventors of the calculus, but they did not invent the same calculus. The differences between Leibniz's and Newton's versions of the calculus can be summarised as follows (Baron, 1987):

- The conception of variable: Newton considered variables as changing in time (flowing quantities), Leibniz considered them as ranging over sequences of infinitely close values.
- The fundamental concepts: Newton's fluxion is a kinematic concept that draws on the velocity or rate of change; Leibniz's differential is the difference of two successive values in the sequence that could not be finite and had to be infinitely small; both are not directly related to problems about quadratures and tangents.
- While Leibniz worked with infinitely small quantities, Newton's fluxion was a finite velocity.
- The conception of the integral and the fundamental theorem: for Newton integration meant finding fluent quantities for given fluxions and so the fundamental theorem is implied in the definition of integration; Leibniz conceived integration as summation and so the inverse relation between differentiation and integration is not implied in the definition of integration.

- Notation: Leibniz used separate symbols d, $\int$ which are more easy to use in complicated formulae; Newton worked with dots for fluxions but did not have a special symbol for integration and his notation requires that all variables are considered as functions of time.

Calculus changed a lot before developing the form that still is presented in introductory courses, especially in terms of formalisation and changing standards of proof.

Arithmetisation of the calculus

One conception for describing and explaining calculus is by infinitesimals, which are objects that can be treated like infinitely small numbers, which on a number line have zero distance from zero. Any multiple of an infinitesimal is still infinitely small. Then calculus is a collection of techniques for manipulating infinitesimals. In the course of exactification, in the 19th century infinitesimals were replaced by limits. Calculus becomes a collection of techniques for manipulating certain limits. Limits are used as the standard approach to calculus. The development of the limit concept was thus crucial for the institutionalisation of the calculus and linked to the unresolved discussion on the ontological status of the infinitesimals, initiated by Berkeley's critic in 1734 (see e.g. Katz, 1998, pp. 628-632). Some details of this historical development are therefore worth mentioning.

An early description of a limit notion similar to present use is Jean Baptiste d'Alembert's in 1754 in the French Encyclopedia, though not very influential. He calculated the slope of the tangent as the limit of the slope of the secants exactly as it is done today without using a formal definition of limit (Struik, 1986, p. 343).

Also with minor influence was Simon l'Huilier, who introduced the notation "lim." for limit in his *Exposition élémentaire des principes des calculs supérieures* from 1786 and used the notion of limit to define the derivative: Lim. $\frac{\Delta y}{\Delta x}$ (see Cajori, 1993, Vol. II, p. 257). More influential on the development was Joseph Louis Lagrange, even if his first outline of the differential calculus from 1772 (*Sur une nouvelle espèce de calcul relatif á la differentiation et á l'intégration des quantités variables*) was not used though much praised. However, his book from 1797 (*Théorie des functions analytiques, contenant les principes du calcul ….*), where he further developed the derivatives ("fonctions dérivées"), was the most important attempt at the time to form a basis for the calculus (Boyer, 1959, p. 260; Katz, 1998, p. 633). Then, through the work of Sylvestre François Lacroix, building on Lagrange, limits became familiar by his shorter version from 1802 (*Traité élémentaire*) of his earlier text from 1797 (*Traité du calcul différentiel et du calcul integral*). This shorter version was the most famous and ambitious textbook of the time, translated into many languages (Boyer, 1959, p. 264-265). Both Lagrange and Lacroix expressed serious concern for the role of notations used in the calculus (Cajori, 1993, Vol. II, p. 213-215). Another popular text using limits was Lazare Carnot's *Oeuvres mathématiques du Citoyen Carnot* from 1797, though more focused on practical use than on foundation issues, and not building on the function concept as Augustin Louis Cauchy did later. The work of Bernhard Bolzano (1817) was not much recognized by his time, and it is unclear if his non-geometric definition of continuity by limits and of the derivative as a limit of a quotient, similar to l'Huilier and almost identical to Cauchy, did influence Cauchy (Katz, 1998, p. 770). An important idea was his use of $\frac{dy}{dx}$ as a symbol for a single function and not as a quotient of infinitesimals. With Cauchy (1821; 1823) the non-geometrical arithmetic definition of limit and the derivative, for which he used the notation $f'(x)$ for the derivative of y with respect to x, the development moves beyond the problematic ontological interpretation of the infinitesimals, to be completed by the present rigorous ε-δ-definitions of Karl Theodor Weierstrass in 1854, where also the notation $\underset{n=\infty}{\mathrm{Lim.}}\, p_n$ introduced. The arrow notation for limits was not introduced until 1905 by John Gaston Leathem in his *Volume and surface integrals used in physics*, and then promoted by Godfrey Harold Hardy in 1908 (see Cajori, 1993, Vol. II, p. 257).

For the most of the 18th century, mathematicians were mainly interested in the results and the methods of the calculus, especially in its applications in physics, in particular mechanics, and astronomy. The foundations of the basic concepts of the calculus became a major issue for d'Alembert and Lagrange. In the 19th century a major task for mathematicians like Cauchy, Bolzano and Weierstrass was the rigorisation of the foundations. Some contradictions that derived from calculating with infinite series drew attention to the concept of convergence. However, standardisation for the purpose of teaching was another motivation for rigorisation. Universities became the centres of mathematical training and research, which led to the development of pure mathematics as an independent field (Jahnke, 2003).

In the same period the ideas of calculus were generalized to Euclidean space and to the complex plane. Lebesgue generalized the notion of the integral so that any function has an integral, while Schwarz extended differentiation in the same way (Boyer, 1959).

Cauchy is regarded as the founder of the differential calculus in its modern version (Boyer, 1959). The critical mathematics of the 19th century aimed at developing foundations that comprised the objects to work with without relying on informal understanding and sensory perception (Becker, 1975). Cauchy started to search for ways of finding formal, logical grounds for the calculus. The view that if concepts exist and are understood there is no need for definitions - simple ideas sometimes have complicated formal definitions, turned out to be unsatisfactory (Boyer, 1959). Cauchy's work can be described in the following way:

> Cauchy was able to see – where nobody else had been able to see – how these ideas could be used to built a new rigorous calculus. We do not insist that an architect make every brick he uses with his own hands; instead, we marvel that the beauty of his creations can come from such commonplace materials. Augustin-Louis Cauchy neither began nor completed the rigorization of analysis. But more than any other mathematician, he was responsible for the first great revolution in mathematical rigor since the time of the ancient Greeks (Grabiner 1981, p. 166).

The major ideas of modern calculus – derivative, continuity, integral, convergence/divergence of sequences and series, became defined in terms of limits. The notion of limit was included in works of some mathematicians before Cauchy but it missed a formal definition, probably because it was based on geometrical representations (Boyer, 1959). Limit is to be seen as the fundamental concept of calculus, which distinguishes it from other branches of mathematics.

In 1821 Cauchy was searching for a rigorous development of calculus to be presented to his engineering students at the École Polytechnique in Paris. He started his calculus course with a definition of the limit. In his writings Cauchy used limits as the basis for definitions of continuity and convergence, the derivative and the integral:

> When the values successively attributed to a particular variable approach indefinitely a fixed value so as to differ from it by as little as one wishes, this latter value is called the limit of the others (Cauchy, 1821, series 2, vol. 3, p.19).

Cauchy’s work established a new way of looking at the concepts of the calculus and transformed it from a set of usefully applicable methods into a mathematical discipline, integrated by definitions and proofs. Equipped with the definition for the integral of any continuous function he gives a proof of the fundamental theorem of calculus.

Riemann, in 1854 in his paper on trigonometric series, generalized Cauchy’s work to include the integrals of bounded functions. He defines the oscillation of the function in an interval as the difference between the greatest and least value of the function on that interval and with this approach he could integrate functions with an infinite number of discontinuities (Kline, 1972). However, the modern notion of Riemann sums was not actually completed by Riemann himself but by Gaston Darboux, who defined the upper and lower Riemann sums and showed that a function is integrable only when upper and lower Riemann sums approach the same value as the maximum subinterval approaches zero in any partition over the interval (Kline, 1972).

Even after Riemann generalized Cauchy’s concept of the integral, mathematicians discovered functions that could not be integrated, for example the Dirichlet-function that takes the value 1 on the rational numbers, and 0 on the irrationals. Riemann integration could not handle such functions because of the large number of discontinuities in any interval one may choose for integration.

The FTC in 20th century mathematics

For some mathematicians who were trying to organize and systematize calculus in the 18^{th} century, the approach of seeing the integral as antiderivative was preferable, because it is algebraic and formal with no references to “intuitive” geometric ideas. When seeing differentiation and integration as operations, the inverse relationship between the operations is no longer a problem or difficulty. A shift from a geometric to an algebraic approach is for example found in the works of Euler and Johann Bernoulli.

While the approach of conceptualising the integral as antiderivative leads to formalization, conceptualisation as area goes in the direction of generalization to wider applications. The notion of area of geometric regions is generalized to the notion of measure as a function defined on an abstract space. This approach led to the definition of the Lebesgue integral and measure theory. The Lebesgue integral is considered a quite difficult mathematical concept but also a very important tool that can be used in a number of various applications in modern mathematics. As Norbert Wiener formulated it (1956):

> The Lebesgue integral is not easy conception for the layman to grasp (…) It is easy enough to measure the length of an interval along a line or the area inside a circle or other smooth, closed curve. Yet, when one tries to measure sets of points which are scattered over an infinity of segments or curve-bound areas, or sets of points so irregularly distributed that even this complicated description is not adequate for them, the very simplest notions of area and volume demand high-grade thinking for their definition. The Lebesgue integral is a tool for measuring such complex phenomena. The measurement of highly irregular regions is indispensable to the theories of probability and statistics; and these two closely related theories seemed to me, even in those remote days before the war, to be on the point of taking over large areas of physics (ibid., pp. 22-23).

The development of the function concept in the 19^{th} century with the expansion of arbitrary functions into Fourier series led to Riemann's definition of the integral in which he used a new class of functions: integrable functions. But still Riemann's integration and differentiation were found not to be completely reversible because the process of differentiation of a function f leads to a bounded derivative f' which is not necessarily Riemann integrable (Jahnke 2003, p. 271). This problem led to another conceptualisation. Henry Lebesgue introduced his theory of integration in 1902 (in his doctoral thesis *Intégrale, longueur, aire*) and expended it in 1904 in his book *Leçons sur l'integration et la recherché des functions primitives*.

Presently, the standard integral in advanced mathematics is the the Lebesgue integral which generalizes the Riemann integral in the sense that any function that is Riemann-integrable is also Lebesgue-integrable and integrates to the same value, but the class of integrable functions is much larger; there are functions whose improper Riemann integral exists but which are not Lebesgue-intagrable. Lebesgue started from the problem of measurability of sets, Lebesgue's integral uses a generalization of length, which is called the measure of a set. A function is measurable if the inverse image of any open interval is a measurable set. Lebesgue's theory inverts the roles played by inputs and function values in defining the integral by looking on the range of a function instead for its domain. In his definition he was interested in some properties for the integral (*Intégrale, longueur, aire*, p.253):

- Riemann's definition should be a special case.
- The definition can be used in the cases of one and of several variables.
- It should guarantee the solution in the fundamental problem of calculus.

Lebesgue proved that a bounded function on a closed interval is Riemann-integrable if and only if its set of discontinuities has measure zero (almost everywhere) but every bounded measurable function on a closed interval is Lebesgue-integrable. Lebesgue's integral can be extended to the case of several variables with the proper measure on *R*, but even with the Lebesgue-integral it cannot be proved that $\int_a^b f' = f(b) - f(a)$ without some additional assumptions for *f*. In the introduction of his thesis he wrote (p. 203):

> It is known that there are derivatives which are not integrable, if one accepts Riemann's definition of the integral; the kind of integration as defined by Riemann does not allow in all cases to solve the fundamental problem of calculus: find a function with a given derivative. It thus seems to be natural to search for a definition of the integral which makes integration the inverse operation of differentiation in as large a range as possible.

Lebesgue's Fundamental Theorem of Calculus:

The integral of *f* over a set *E* is written as $\int_E f(x)dm$. In one dimension it is equivalent to $\int_\alpha^\beta f(t)dt$ when $E = (\alpha, \beta)$. A non-negative function *f* is called integrable if it is Lebesgue measurable and $\int_E f(x)dm < \infty$. Then, if the function $f:[a,b] \to R$ is Lebesgue integrable and $F(x) = \int_a^x f(t)dt$ is its indefinite integral then for almost every $x \in [a,b]$, $F'(x)$ exists and is equal to *f(x)*.

That does not characterize indefinite integrals. If a function *G* has a derivative almost everywhere and its derivative is an integrable function *f*, this does not imply that *G* differs from the indefinite integral of *f* by a constant. This is satisfied for another type of function: absolutely continuous.

The following theorems state that differentiation and Lebesgue integration are inverse operations on very large classes of functions:

Theorem: Let $f:[a,b]\to R$ be integrable.

The indefinite integral F of f is absolutely continuous. For almost every x, $F^{'}(x)$ exists and is equal $f(x)$. If G is an absolutely continuous function and $G^{'}(x)=f(x)$ for almost every x, then G differs from F by a constant.

In the next theorem he proved that integration and differentiation are reversible if only the derivative is bounded.

Theorem: If a function f on $[a,b]$ has a bounded derivative $f^{'}$, then $f^{'}$ is Lebesgue integrable and $\int_a^b f^{'}(x)dx=f(b)-f(a)$.

By the 19th century mathematics had moved far beyond of being simply a tool for science and it became possible to define something as abstract as the Lebesgue measure and integration.

NAME GIVING AND TEXTBOOKS

Textbooks

The short outline of the history of the FTC above did not focus on the issue of terminology and textbooks. The following section concentrates on the emergence of shared names and on the appearance of the FTC in textbooks.

Before the official publication of Leibniz's work, there existed several textbooks in calculus. Naturally, these cannot be expected to contain what we now call the FTC, but are traces of the first didactic transposition of the new knowledge about tangents and areas.

The first printed textbook in differential calculus appeared in Paris in 1696 *Analyse des infiniment petits pour l'intelligence des lignes courbes*. This book was edited anonymously but was written by Marquis de l'Hospital with the help of Johann Bernoulli (Kaiser & Nöbauer, 1984, p. 49). Maligranda (2003) states that some parts (about 30 pages) of it were indeed written in 1691-1692 by Bernoulli as the material for the private lectures given by him to de l'Hospital. It was only in the second editions (which exist from 1715 and from 1716) that l'Hospital was named as the author.

The book consists of ten chapters about rules of differential calculus, applications of differentials for determinations of tangents, maximum and minimum problems and other problems and also contains a formula known later as l'Hospital's Rule, but is missing any formulation of the fundamental theorem of calculus. The preface of the second edition contains historical comments as well as an overview of the contents of the work:

> The type of analysis we shall describe in this work presupposes an acquaintance with ordinary analysis, but is very different from it. Ordinary analysis deals only with finite quantities whereas we shall be concerned with infinite ones. We shall compare infinitely small differences with finite quantities; we shall consider the ratios of these differences and deduce those of the finite quantities, which, by comparison with the infinitely small quantities are like so many infinities. We could never say that our analysis takes us beyond infinity because we shall consider not only these infinitely small differences but also the ratios of the differences of these differences, and those of the third differences and the fourth differences and so on, without encountering any obstacle to our progress. So we shall not only deal with infinity but with an infinity of infinity or an infinity of infinities [De forte qu'elle n embrasse pas seulement l'infini ; mais l'infini de l'infini ou une infinité d'infinis]." (l'Hospital, 1716, p. iii-iv).

The new methods here are introduced as calculations with ratios of infinitely small differences. In terms of modern notions, the textbook is about differential calculus. The introduction continues:

> All this is only the first part of M Leibniz's work on calculus, which consists of working down from integral quantities to consider the infinitely small differences between them and comparing these infinitely small differences with each other, whatever their type: this part is called Differential Calculus. The other part of M Leibniz's work is called the Integral Calculus, and consists of working up from these infinitely small quantities to the quantities of totals of which they are the differences: that is, it consists of finding their sums. I had intended to describe this also. But M Leibniz wrote to me to say that he himself was engaged upon describing the integral calculus in a treatise he

> calls De Scientia infiniti, and I did not wish to deprive the public of such a work, … (l'Hospital, 1716, p. xii).

However, the second part of the planned textbook about the integral calculus never appeared, probably because Leibniz died in 1716. From the introduction (above) it becomes clear that the name "Integral Calculus" ["Calcul integral"] was already in use. Thus, by having a specific name it had gained an "official" status as a connected part of knowledge to which one could easily refer. However, it is not clear whether the name initially referred to Leibniz's method. With reference to Boyer (1939), Domingues (2008, p. 140) points out that the name "Integral" was used by the Bernoullis (in a printed work of Jakob), not for denoting Leibniz's definition as the sum of infinitesimally narrow rectangles, but as indicating the definition of the integral as antiderivative. According to Gerhardt (1860) the name "Calculus integralis" was officially sanctioned by Leibniz, and eventually superseded the "Calculus summatorius". In the *Elementa universae: commentationem de methodo mathematica…* (Wolf, 1732), a mathematics handbook, there is a section on "Calculo integrali seu summatorio". But this section still does only refer to the integral as antiderivative (see Fig. 4).

The first English translation of l'Hospital's book by John Colson appeared in London in 1736 as *The method of fluxions and infinite series with its applications to the geometry of curve-lines*. The re-naming obviously reflects a cultural identity that links the works in calculus to Newton. In another English translation of the book from 1930 it has the title *The method of fluxions both direct and inverse*. The title was supplied by the translator E. Stone. Leibniz's differentials were still replaced by Newton's fluxions (Maligranda, 2003).

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qui eſt, in ſemicirculo, ſit itidem rectus, (§. 317 *Geom.*); erit △ AMP ∽ △ ANB (§. 267 *Geom.*) &

$$PM : AM = AN : AB$$
$$y : x = a-x : a$$
$$ay = ax - x^2$$
$$ady = adx - 2xdx = 0$$
$$a - 2x = 0$$
$$a = 2x$$
$$\tfrac{1}{2}a = x$$

Hinc porro $y = x - \frac{x^2}{a} = \frac{1}{2}a - \frac{1}{4}a = \frac{1}{4}a$

Eſt igitur in caſu applicatæ maximæ AM $= \frac{1}{2}a$: unde reperitur AP $= \frac{1}{4}\sqrt{3a^2}$ (§. 417. *Geom.*)

SECTIO SECUNDA.

DE CALCULO INTEGRALI SEU SUMMATORIO.

CAPUT I.

De natura Calculi integralis.

DEFINITIO V.

91. *Calculus Integralis* ſeu *Summatorius* eſt Methodus quantitates differentiales ſummandi, hoc eſt, ex quantitate differentiali data inveniendi eam, ex cujus differentiatione reſultat differentiale datum.

COROLLARIUM.

92. Integrationis itaque ſeu ſummationis rite peractæ indicium eſt, ſi quantitas inventa juxta regulas Cap. 1. Sect. I. traditas differentiata eam producit, quæ ad ſummandum proponebatur.

SCHOLION.

93. *Quoniam Angli differentialia quantitatum fluxiones vocant* (§ 6); *Calculum, quem nos differentialem dicimus*, Methodum fluxionum; *quem vero integralem vocamus & qui a differentiis ad ſummas, ſeu, ut cum Anglis loquar, a fluxionibus ad quantitates fluentes* (*ita nimirum variabiles dicunt*) *aſcendit*, Methodum fluxionum inverſam *appellant.*

HYPOTHESIS.

94. *Signum ſummæ aut quantitatis integralis ſit* ∫, *ita ut* ∫ydx *denotet ſummam ſeu integrale differentialis* ydx.

PROBLEMA XXIV.

95. *Quantitatem differentialem integrare ſeu ſummare.*

RESOLUTIO.

Ex ſuperioribus manifeſtum eſt, quod ſit

I. $\int dx = x$ (§. 8).

II. $\int (dx \mp dy) = x \mp y$ (§. 11).

III. $\int (xdy + ydx) = xy$ (§. 12).

IV. $\int mx^{m-1}\,dx = x^m$ (§. 13).

V. $\int (n:m)\,x^{(n-m):m}\,dx = x^{n:m}$ (§. 17).

VI. $\int (ydx - xdy):y^2 = x:y$ (§. 19).

Ex his caſus quartus & quintus frequentius occurrunt, in quibus quantitas differentialis ſummatur, ſi exponenti variabilis unitas additur, & ea, quæ prodit, dividitur per novum exponentem ductum in differentiale radicis e. gr. in caſu quarto per $(m - 1 + 1)\,dx$, hoc eſt, per mdx.

Quodſi quantitas differentialis ad ſummandum

Figure 4: Page 440 of Wolf’s Handbook

Leibniz's work from 1684 was printed in Acta Eruditorum with the title Nova methodus pro maximis et minimis, itemque tangentibus quae nec fractas nec irrationals quantitates moratur, et singulare pro illis calculi genus. It cannot be called a textbook but it contains a definition of the differential together with the rules for computing differentials of powers, products and quotients. Consequently the year 1684 is officially accepted as the beginning of the differential calculus. In 1671 Newton wrote in Latin a book about the "method of fluxions" that was first published only nine years after his death as The Method of Fluxions and Infinite Series with its Applications to the Geometry of Curve-lines (1736) in an English translation by John Colson. Consequently it was not influential for the early development of the calculus.

Euler's *Introductio in analysin infinitorum* from 1748 is the first textbook in calculus that can easily be read even today because of the modern notation and terminology and of the function concept playing a central role; for the first time functions instead of curves were the principal objects of study (Edwards, 1979).

Cauchy's Course of analysis of the École Polytechnique (Cours d'analyse de l'École Polytechnique, 1821) and the Summary of the lectures given at the École Polytechnique on the infinitesimal calculus (Résumé des leçons données a l'École Polytechnique sur le Calcul Infinitesimal, 1823) were the first textbooks in which calculus appeared in a general character as an integrated body of knowledge:

> In the integral calculus, it has appeared to me necessary to demonstrate generally the existence of the integrals or primitive functions before making known their diverse properties. In order to attain this object, it was found necessary to establish at the outset the notion of integrals taken between given limits or definite integrals (Cauchy in Oeuvres (2), IV, pp. ii-iii).

Cauchy is also viewed as being responsible for changing the attitude towards the value of dealing with the foundations of calculus. In his books he stressed the definitions of the basic concepts and included many examples of a new style of reasoning. He demonstrated the necessity for rigour not only in defining basic concepts but also in proving theorems (Grabiner, 1981).

CALCUL INTÉGRAL. 151

VINGT-SIXIÈME LEÇON.

INTÉGRALES INDÉFINIES.

Si, dans l'intégrale définie $\int_{x_0}^{X} f(x)\,dx$, on fait varier l'une des deux limites, par exemple la quantité X, l'intégrale variera elle-même avec cette quantité; et, si l'on remplace la limite X devenue variable par x, on obtiendra pour résultat une nouvelle fonction de x, qui sera ce qu'on appelle une intégrale prise à partir de l'*origine* $x = x_0$. Soit

$$(1) \qquad \mathscr{F}(x) = \int_{x_0}^{x} f(x)\,dx$$

cette fonction nouvelle. On tirera de la formule (19) (vingt-deuxième Leçon)

$$(2) \qquad \mathscr{F}(x) = (x - x_0)\,f[x_0 + \theta(x - x_0)], \qquad \mathscr{F}(x_0) = 0,$$

θ étant un nombre inférieur à l'unité, et de la formule (7) (vingt-troisième Leçon)

$$\int_{x_0}^{x+\alpha} f(x)\,dx - \int_{x_0}^{x} f(x)\,dx = \int_{x}^{x+\alpha} f(x)\,dx = \alpha\,f(x + \theta\alpha)$$

ou

$$(3) \qquad \mathscr{F}(x + \alpha) - \mathscr{F}(x) = \alpha\,f(x + \theta\alpha).$$

Il suit des équations (2) et (3) que, si la fonction $f(x)$ est finie et continue dans le voisinage d'une valeur particulière attribuée à la variable x, la nouvelle fonction $\mathscr{F}(x)$ sera non seulement finie, mais encore continue dans le voisinage de cette valeur, puisqu'à un accrois-

Figure 5: Cauchy in Oeuvres (2), IV, p. 151

With two operations (differentiation and integration) defined independently of each other, Cauchy established the inverse relationship between them without relying on an informal concept of area. In his "Twenty sixth lesson" (ibid., pp.151-155, see fig. 4, 5), Cauchy presents the formulation of the fundamental theorem of calculus.

He showed that if *f(x)* is a continuous function, the function defined as the definite integral $F(x)=\int_{x_0}^{x} f(x)dx$ has as its derivative the function *f(x)*. This was perhaps the first rigorous demonstration of the proposition known as the fundamental theorem of calculus (Boyer, 1959, pp.279-280).

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sement infiniment petit de x correspondra un accroissement infiniment petit de $\mathcal{F}(x)$. Donc, si la fonction $f(x)$ reste finie et continue depuis $x=x_0$ jusqu'à $x=X$, il en sera de même de la fonction $\mathcal{F}(x)$. Ajoutons que, si l'on divise par α les deux membres de la formule (3), on en conclura, en passant aux limites,

$$\mathcal{F}'(x)=f(x). \tag{4}$$

Donc l'intégrale (1), considérée comme fonction de x, a pour dérivée la fonction $f(x)$ renfermée sous le signe $\int$ dans cette intégrale. On prouverait de la même manière que l'intégrale

$$\int_x^X f(x)\,dx=-\int_X^x f(x)\,dx,$$

considérée comme fonction de x, a pour dérivée $-f(x)$. On aura donc

$$\frac{d}{dx}\int_{x_0}^x f(x)\,dx=f(x) \quad \text{et} \quad \frac{d}{dx}\int_x^X f(x)\,dx=-f(x). \tag{5}$$

Si aux diverses formules qui précèdent on réunit l'équation (6) de la septième Leçon, il deviendra facile de résoudre les questions suivantes.

Problème I. — *On demande une fonction $\varpi(x)$ dont la dérivée $\varpi'(x)$ soit constamment nulle. En d'autres termes, on propose de résoudre l'équation*

$$\varpi'(x)=0. \tag{6}$$

Solution. — Si l'on veut que la fonction $\varpi(x)$ reste finie et continue depuis $x=-\infty$ jusqu'à $x=+\infty$, alors, en désignant par x_0 une valeur particulière de la variable x, on tirera de la formule (6) (septième Leçon)

$$\varpi(x)-\varpi(x_0)=(x-x_0)\varpi'[x_0+\theta(x-x_0)]=0$$

et, par suite,

$$\varpi\, x=\varpi(x_0), \tag{7}$$

Figure 6: Cauchy in Oeuvres (2), IV, p. 152

But still he did not use the name the “Fundamental Theorem of Calculus”. In addition, Cauchy proves on page 155 in this *Résumé des leçons données a l’École Polytechnique sur le Calcul Infinitesimal* from 1923 the computational formula (in modern terms), and he also introduces the term 'indefinite integral' with present day notation (page 154). He does not give any particular name to these results.

Emergence of a shared name for the “FTC”

In these early textbooks on calculus the propositions similar to the theorem presently called FTC are not named. The origin may well be from the French tradition, from the French word “fondamentale” for something basic.

The Course d’analyse mathématiques from 1902 by the French mathematician Eduard Goursat, which was translated into English already in 1904 (A course in mathematical analysis, applications to geometry, expansion in series, definite integrals, derivatives and differentials) and widely spread, was based on his university lectures (Osgood, 1903) and can thus be considered a textbook. In volume II of this book the text “the fundamental formula of the integral calculus” (p. 63) refers to volume I, where the expression “fundamental theorem” is used for the fact that “every continuous function f(x) is the derivative of some other function” (p. 140) and “the fundamental formula becomes $\int_a^b f(x)dx = F(b) - F(a)$” (p. 155). Later, in volume II, this name of the theorem is also used for complex analysis: “the fundamental formula of the integral calculus can be extended to the case of complex variables: $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$” (p. 72). In the same vein, Charles Jean de la Vallée Poussin in Cours d’analyse infinitésimale from 1921, uses the term “relation fondamentale pour le calcul des integrals définies” (p. 211) and writes

> $\int_a^b f(x)dx = F(b) - F(a)$ C’est la formule fondamentale pour le calcul des integrals définies.

Possibly, also Ernest Hobson was influenced by the well known book by Goursat in *The theory of functions of a real variable & the theory of Fourier series*, published in 1907. In this book a whole chapter has the title “The fundamental theorem of the integral calculus”. Later on in the same book there is one chapter with the title “The fundamental theorem of the integral calculus for the Lebesgue integral” and another chapter called “The fundamental theorem of the integral calculus for the Denjoy integral". That the name of the theorem is extended to a more general application indicates strong institutionalization of the theorem.

In the textbook *An introduction to the summation of differences of a function* by Benjamin Feland Groat, printed in 1902, the expression "the fundamental theorem of the integral calculus" is used, as well as more short "fundamental theorem":

> to find the limits of sums of the form $\sum \phi(x)\Delta x$, it was necessary to have an identity of the form:
>
> $\phi(x)\Delta x = \psi(x) - \psi(x+\Delta x) + F(x,\Delta x)\Delta x^2$. The fundamental theorem of the integral calculus puts into mathematical language a rule for finding the limit of any sum, of the kind considered, provided an identity of the right form can be found; and the rules an formulae of the integral calculus afford a method for the discovery of the essential form of the identity when it exists.
>
> 33. **Fundamental theorem.** $\int_a^b f'(x)dx = f(b) - f(a)$. Or, more explicitly,
>
> $\underset{\Delta x=0}{\mathrm{Lt.}} \sum_a^b \psi'(x)\Delta x = \psi(b) - \psi(a)$, where $\psi'(x)$ is any function of x and $\psi(x)$ any function whose differential coefficient with regard to x is $\psi'(x)$ (pp. 40-41).

In addition to this quote, the fundamental theorem is also expressed without employing mathematical formulae. This indicates the effort of the author to address an audience that is not fluent to read specialized technical language, that is, an attempt of a didactic transposition.

In his textbook, *A course of pure mathematics* from 1908, Hardy employs the same name of the theorem. In a paragraph called "areas of plane curves" (derivatives and integral) the FTC is proved but not named. Then later in a paragraph on "Definite Integrals. Areas of curves" (in the chapter: "Additional theorems in the differential and integral calculus") it is restated as a formal named theorem ("the fundamental theorem of the integral calculus") with reference to the proof (p. 293):

(10) **The Fundamental Theorem of the Integral Calculus**.

The function

$$F(x) = \int_a^x f(t)dt$$

has a derivative equal to f(x).

This has been proved already in § 145, but it is convenient to restate the result here as a formal theorem. It follows as a corollary, as was pointed out in § 157, that F(x) is a continuous function of x.

In this case, the name seems to be linked to the statement rather than to the statement along with its proof. The listing of “additional” theorems without proofs (but with names), of which the proofs are given earlier in the book, points to a distinct didactical rationale for this way of structuring the topic.

A reference to another version of the name that seemed to be common as well as the formulation of the FTC is found in *Differential and Integral Calculus* (1934) by Richard Courant (pp.113-114):

The question about the group of all primitive functions is answered by the following theorem, sometimes referred to as the fundamental theorem of the differential and integral calculus:

The difference of two primitives $F_1(x)$ *and* $F_1(x)$ *of the same function* f(x) *is always a constant:*

$$F_1(x) - F_2(x) = c.$$

Thus, from any one primitive function F(x) *we can obtain all the others in the form*

$$F(x) + c$$

by suitable choice of the constant c. Conversely, from every value of the constant c the expression $F_1(x) = F(x) + c$ *represents a primitive function of* f(x).

From this theorem Courant derives the formula $\int_a^b f(u)du = F(b) - F(a)$, also stated as the *important rule:*

> If F(x) is any primitive of the function f(x) whatsoever, the definite integral of f(x) between the limits a and b is equal to the difference $\mathrm{F}(b)-\mathrm{F}(a)$ (p. 117).

Norbert Wiener refers several times to “the fundamental theorem of the calculus” in *Fourier Transforms in the Complex Domain* from 1934. That this name became standardised is evident from the classical book *What is mathematics?* where Courant and Robbins (1941) use the chapter title “The fundamental theorem of the calculus”, and write:

> There is no separate differential calculus and integral calculus, but only one *calculus*. It was the great achievement of Leibniz and Newton to have first clearly recognized and exploited this *fundamental theorem of the calculus* (p. 436).

The reason for skipping the appendix “of the integral calculus” is evident from the quotation above. This stresses the role of the FTC and its proof as a means of integrating two related but still distinct sub-areas of <u>the</u> calculus. In later textbooks, the definite article has been left out, which can be taken as a reference to a general field and not to a particular method.

In commercial textbook production, different traditions of formulating and proving the FTC seem to have developed for different markets. In the U.S. a division of the theorem into two parts is common. An early example is to be found in Morrey (1962), where the “first form” of the FTC refers to the formula for computing the definite integral by the primitive function and the “second form” to the derivation of the integral. In modern U.S. calculus textbooks it seems to be “standard” to refer to this second form as *the first part of the FTC* and the computational formula as *the second part of the FTC* (e.g. Adams, 2006). In other countries this division is less common.

In many places it is also common that researching mathematicians who teach undergraduate calculus courses produce their own “textbooks” in the form of lecture notes or local publications and do not choose to use commercial and common texts. Some examples of these texts (selected as a convenience sample), show that these are less standardised in their approach and display unusual versions of the FTC.

The early Swedish textbook by Björling (1877) has an outline of the definite integral similar to that of Cauchy. Björling gives the name “Grundsats” to the computational formula, $\int_a^b f(x)dx = F(b)-F(a)$, i.e. the ‘second part’ of the FTC. In another early Swedish textbook, which is one example of how notes from a lecture series have been compiled to a printed book, the FTC is not named though described by the words (Malmquist, 1923, p. 266):

> differentiation och integration äro fullständigt motsatta processer, vadan följande likheter gälla:
>
> $d\int f(x)dx = f(x)dx$, $\int df(x) = f(x) + C$"
>
> [Differentiation and integration are completely opposite processes, from which the following equalities hold:]

The same author is later involved in a calculus textbook (Malmquist, Stenström, & Danielsson, 1951), where a theorem named "Integralkalkylens fundamentalsats" [fundamental theorem of the integral calculus] refers to the following proposition: "If the derivative of a function equals zero on an interval, then the function is constant on that interval." The formula, $\int_a^b f(x)dx = F(b) - F(a)$, usually included in the FTC, is contained in the same section of the book, but not named. In more modern Swedish textbooks the name "integralkalkylens huvudsats" [main theorem of the integral calculus] is commonly used for the computational formula $\int_a^b f(x)dx = F(b) - F(a)$, with the assumption that *f* is continuous (e.g. Hyltén-Cavallius & Sandgren, 1968; Domar, Haliste, Wallin, & Wik, 1969; Ullemar, 1972; Hellström, Molander, & Tengstrand, 1991). It can be noted that in an earlier edition of Hyltén-Cavallius and Sandgren (1968), i.e. Hyltén-Cavallius and Sandgren (1956), the name "integralkalkylens huvudsats" did not appear. When the name "analysens huvudsats" is used in Persson and Böiers (1990) and in Forsling and Neymark (2004), it refers only to the 'first part' of the FTC, i.e. the proposition that $\frac{d}{dx}\int_a^x f(t)dt = f(x)$ for any continuous function *f*, though the name indicates the key role of the theorem for the coherence of the calculus (in Swedish 'analys') more than does the name "integralkalkylens huvudsats". However, in this textbook the second part is also named, using the term "insättningsformeln" the insert formula]. A completely different approach is used in Eriksson, Larsson and Wahde (1975), where the definite integral from *a* to *b* of a continuous function *f* is *defined* as $F(b) - F(a)$, where *F* is any primitive function to *f*. It is then proved, from a previous definition of area measure, that this formula computes the area under the curve *f*. No name is given to this proposition.

In a German textbook from (von Mangoldt & Knopp, 1932; 13th edition from 1967) only this computational part is called the "Hauptsatz der Differential- und Integralrechnung" [main theorem for the differential and integral calculus], whereas in a textbook from the German Democratic Republic (Belkner & Bremer, 1984) this computational part is called the "Umkehrung des Hauptsatz der Differential- und Integralrechnung" [converse of the main theorem for the differential and integral calculus].

In a Polish textbook from 1929, *The differential and integral calculus*, written by Stefan Banach there is a chapter (§ 9) about definite integrals and primitive functions, in which a formulation of what can be recognised as the FTC is presented without naming it; it is divided into three parts. It is also interesting to look on the proof Banach proposes for these theorems. The first part is formulated as the following theorem:

If a function $f(x)$ is integrable on (a,b), $a \le \alpha \le b$ and $a \le x \le b$, then the derivative of the function $\int_{\alpha}^{x} f(t)dt$ exists and it is equal to a function under the integral in every point at which the function f is continuous.

Proof. Let $f(x)$ be a continuous function at x_0 in an interval (a,b). Taking an arbitrary ε>0 we can find $\eta > 0$ such that every point *x* in (a,b) is satisfying the inequality

$|x - x_0| \le \eta$; (1)

it follows that

$$|f(x) - f(x_0)| \le \varepsilon,$$

that is

$$f(x_0) - \varepsilon \le f(x) \le f(x_0) + \varepsilon . \quad (2)$$

Putting $F(x) = \int_{\alpha}^{x} f(t)dt$ we see that

$$\frac{F(x) - F(x_0)}{x - x_0} = \frac{1}{x - x_0} \int_{x_0}^{x} f(t)dt ,$$

thus if *x* satisfies inequality (1), then by inequality (2) and the theorem from § 6 we get:

$$f(x_0) - \varepsilon \le \frac{F(x) - F(x_0)}{x - x_0} \le f(x_0) + \varepsilon.$$

Since ε was arbitrary, it follows that

$$\lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0),$$

i.e. $F'(x_0)$ exists and $F'(x_0) = f(x_0)$.

From this theorem there follows directly another one:
A continuous function $f(x)$ on an interval (a,b) has primitive function on this interval. The primitive function is $F(x) = \int_a^x f(t)dt + c.$

And the third theorem is formulated and proved as follows:
If $F(x)$ is a primitive function of $f(x)$, which is a continuous function on the interval (a,b), then $\int_\alpha^x f(t)dt = F(x) - F(a), a \le \alpha \le b, a \le x \le b.$

Proof. Because both $F(x)$ and $\int_\alpha^x f(t)dt$ are primitive functions of *f(x)* they differ only by a constant. Hence

$$\int_\alpha^x f(t)dt = F(x) + c.$$

Taking $x = \alpha$ we obtain $0 = F(\alpha) + c$. There it follows that $c = -F(\alpha)$. Hence

$$\int_\alpha^x f(t)dt = F(x) - F(\alpha).$$

Remark: If we put $x = b, \alpha = a$ in the last equality, then we obtain

$$\int_a^b f(t)dt = F(b) - F(a).$$

The last formula allows to compute a definite integral if we know a primitive function, that is, the indefinite integral.

DISCUSSION

The notion of the integral can be defined in two different ways, (i) as an area or (ii) as the antiderivative. The historical development of calculus shows that mathematicians used both of these. From the beginning the calculus had been based on geometrical representations. Later these "intuitive" notions have been replaced by precise and formal definitions of the concepts of function, limit, continuity, derivative and integral. Before Newton and Leibniz there existed different methods for different curves. Leibniz understood the integral as a sum of infinitely small rectangles (of differentials) and Newton as the inverse of fluxion, called fluent, the problem being to find the fluent quantity of a given fluxion. Johann Bernoulli gave up the definition of the integral as a sum and used the approach of the inverse of the differential and Euler started with the sum concept but later followed Bernoulli's approach. With this approach there is a limited number of functions that can be integrated. The class of functions was more and more widening, with many exceptional cases requiring a change of interpretation of the integral. In the work of Cauchy and then also of Riemann, there is a change in the approach to the integral that separates integration from the derivative by using the conception of area under a curve for the definition of the integral.

Each approach has different consequences for conceptualising the relationship between tangents, quadratures, derivatives, antiderivatives and definite integrals. In the first approach the relationship between integration and differentiation is more meaningful because it is not obvious. On the other hand, if the integral is defined as antiderivative, then the FTC becomes a form of a definition. In modern textbooks, we usually find a presentation of the calculus that resembles Cauchy's overall approach:

> With Cauchy, it may safely be said, the fundamental concepts of the calculus received a rigorous formulation. Cauchy has for this reason commonly been regarded as the founder of the exact differential calculus in the modern sense. Upon a precise definition of the notion of limit, he built the theory of continuity and infinite series, of the derivative, the differential, and the integral. Through the popularity of his lectures and textbooks, his exposition of the calculus became that generally adopted and the one which has been accepted down to the present time (Boyer, 1959, p. 282-283).

The two main features of this exposition include (i) the fundamental concept of the differential calculus is the derivative defined as a limit and (ii) the concept of integral is that of a limit sum under refining partitions and not the inverse of differentiation. The FTC has become a theorem needing proof rather than a corollary of the definition of the integral.

As to the FTC, another possible choice could have been avoiding a geometrical interpretation and thus make it less linked to a language that draws on specific geometrical conceptions, which makes it easier to conceptualise, but also less significant. Traces of such an approach can actually be found in calculus textbooks, in which a chapter called "antiderivatives" often precedes a chapter about the definition of the integral as a sum. The common choice in overall approach comprises to deliberately choose interpretations that make the FTC more significant, but harder to conceptualise, an approach that resembles indeed some lines of the historical development.

The early printed textbook in differential calculus *Analyse des infiniment petits pour l'intelligence des lignes courbes* from 1696 (edited anonymously but written by Marquis de l'Hospital with the help of Johann Bernoulli) presents many techniques. It includes rules of differential calculus, applications of differentials for determinations of tangents, maximum and minimum problems and other problems as well as the formula known later as l'Hospital's Rule (or Bernoulli-Hospital Rule), but no proposition associated with the FTC. In a geometrical disguise, Barrow's *Lectiones opticae et geometricae* from 1669 contains a proposition and a proof that links tangents to quadratures. Gregory's *Geometriae pars universalis* from 1668, an early attempt to write a systematic textbook on what became to be called calculus, also includes a proof that the method of tangents is inverse to the method of quadratures.

As mentioned above, Cauchy's lecture notes from 1823 include what today is associated with the FTC. In the textbooks from the 20th century, there are different names used for the same propositions, but also the name fundamental theorem is used for different statements. The widely-spread *Course d'analyse mathématiques* from 1902 by Eduard Goursat (translated into English in 1904), based on university lectures, includes a "fundamental theorem" stating that "every continuous function f(x) is the derivative of some other function" (Vol. I, p. 140) and "the fundamental formula of the integral calculus" referring to the computation of definite integrals (Vol. I, p. 155). In the second volume of the same work, this name of the theorem is also used for complex analysis: “the fundamental formula of the integral calculus can be extended to the case of complex variables” (Vol. II, p.72). In the textbook *An introduction to the summation of differences of a function* by Benjamin Feland Groat, printed in 1902, the expression “the fundamental theorem of the integral calculus”, or just the Fundamental Theorem is used for the computational part. Another well known book by Goursat, *The theory of functions of a real variable & the theory of Fourier series*, published in 1907, includes a chapter titled “The fundamental theorem of the integral calculus”. Later on in the same book there is one chapter with the title “The fundamental theorem of the integral calculus for the Lebesgue integral” and another chapter called “The fundamental theorem of the integral calculus for the Denjoy integral”. Hardy’s textbook *A course of pure mathematics* from 1908, includes a proof of what in many modern textbooks is the first part of FTC (derivative of the integral) in a section called “areas of plane curves”, but without naming it. In a chapter containing “additional theorems in the differential and integral calculus” it is restated as “the fundamental theorem of the integral calculus“ with reference to the proof in the other section. In the *Differential and Integral Calculus* (1934), Courant states that the “fundamental theorem of the differential and integral calculus” is often used for the fact that the difference of two primitive functions is a constant. The computational formula is mentioned as “the important rule” in that book.

Björling (1877) in a Swedish textbook with an outline of the definite integral similar to that of Cauchy, the author gives the name “Fundamental theorem“ [Grundsats] to the computational formula. Another early Swedish textbook, a compilation of lecture notes into a printed book, states that “differentiation and integration are completely opposite processes” (Malmquist, 1923, p. 266). A non-standard use of the name “Fundamental theorem of the integral calculus” [Integralkalkylens fundamentalsats] can be found in a later calculus textbook (Malmquist, Stenström, & Danielsson, 1951). It refers to the proposition: “If the derivative of a function equals zero on an interval, then the function is constant on that interval.” In more modern Swedish textbooks the name “Main theorem of the integral calculus” [Integralkalkylens huvudsats] is commonly used for the computational formula, with the assumption that the function is continuous. A similar use of the name can be found in older German textbooks. In some books, the name “Main theorem of the calculus” [Analysens huvudsats] is used for the the part of the FTC referring to the derivative of the integral (e.g. in Persson and Böiers, 1990; Forsling and Neymark, 2004). In commercial textbook production, different traditions of formulating and proving the FTC seem to have developed for different markets. In the U.S. a division of the theorem into two parts is common, with the first referring to the derivation of the integral and the second to the formula for computing the definite integral by the primitive function. In other places this is less common.

The development of the names used for denoting the sub-area of the calculus and for the FTC reflects the complex process of its institutionalisation. The acceptance of a shared name can be taken as an expression of attributing value. This is, for example, witnessed by the discussion between Bernoulli and Leibniz about the preference of "calculus integralis" over "calculus summatorius". Thus, name giving can be taken as an important evidence for an institutionalisation in a community of practice. The development from rather long names denoting two sub-areas (calculus differentialis et calculus integralis) into one (the calculus, or more general without the definite article) mirrors the development of the area. The same pattern has been observed for the naming of the "FTC".

These examples from textbooks show that the FTC long after its invention, in its basic form at the elementary undergraduate level at universities, has been and is still being given different or no names (also within the same country), different formulations, and different proofs, in commercial textbook production as well as in local texts produced directly by mathematicians teaching courses in calculus. From the examples listed here the overall development can be seen to move from a name referring to integration only to a name emphasising the role of the FTC as a fundamental link between differentiation and integration, thus defining calculus as a strongly classified sub-area of mathematics. However, still today there is not one theorem with one name and one proof that has become an internationally standardised version of the FTC, neither in the educational or the academic field.

6. THE MATHEMATICIANS' VIEWS: OUTCOMES OF THE INTERVIEWS

INTRODUCTION

This chapter reports the outcomes from the interviews with the sample of eleven mathematicians as described in chapter 4. The compilation includes some typical statements. As the interviewees answered at different levels, more general or more with reference to an example, and interpreted the interview prompts in different ways, a systematic coding procedure could not be applied. In order to provide an institutionally based picture of the scholarly knowledge of the FTC and the transposed knowledge for its teaching at the undergraduate level, as viewed through the answers of the mathematicians to the interview questions, the results will be presented by question, and not by interviewee. Then a summary and an analysis and discussion of this report will follow, in terms of the relevant research questions and the theoretical framework outlined in chapter 3.

RESPONSES TO THE INTERVIEW QUESTIONS

Question 1 (Q1)

When answering Q1, six of the interviewees used the word 'inverse' in some sense when describing the FTC, saying that differentiation and integration are 'inverse processes', 'inverse operations', or simply 'inverses'. It is in most cases unclear if they thought about it in a trivial sense (seeing the integral only as the antiderivative; cf. chapter 5). In only one interview (Int8) it was pointed out that the integral should be defined as a limit of a sum, and not simply as an antiderivative, to make the FTC interesting.

Usually the interviewees had a quite informal approach to the formulation of the theorem without providing assumptions or a remark about possible parts of the theorem. However, Int3 gave a precise formulation with assumptions:

> If you have a function $f(x)$ which is continuous almost everywhere then f is Riemann integrable (first of all) and the derivative of $F(x)=\int f(x)dx$ is differentiable at every point x where f is continuous and $F(x)=f(x)$. That is essentially the FTC. (Int3)

He also mentioned that there are two parts of the FTC and commented that the computational part of the theorem should be called the first part rather than the common textbook practice of naming it the second part:

> the most important part [of FTC] from the computational point of view is $\int_a^b f(t)dt = F(b) - F(a)$ (Int3)

Those mathematicians having some experience of teaching calculus courses for students usually gave a more formal formulation of the FTC as composed of two parts, valid for continuous functions. Only one person (Int5) formulated the FTC with the assumption that f is a piecewise continuous function. Three persons (Int1, Int2, Int11) mentioned that there exist other versions of the FTC, for multivariable functions or with weaker assumptions.

For one person (Int2) the FTC is seen as a way of finding the area under the curve, thus emphasising the technical aspect of the theorem, while Int11 more generally talks about the FTC as a connection between area and function.

Int4 presents a different epistemological approach to Q1 than all the other interviewees, possibly due to his background in the philosophy of mathematics. He connected the FTC with the concept of *ambiguity*:

> Mathematical situations in which there are two separate contexts and those contexts are in conflict with one another on some level. So those contexts themselves are well defined and consistent, everything makes sense but there is a tension between the two ways of looking at it. (Int4)

For him the FTC represents an example of such ambiguity when the two contexts are differential calculus and integral calculus. Then the FTC is seen to connect these two contexts:

> So the fundamental theorem of calculus says that there is one calculus but there are two different ways of looking at that calculus. One way of looking at it is what we call differential calculus and the other way of looking at it is what we call integral calculus but you are really talking about the same thing. (Int4)

But he also clearly points out that it does not mean that differential and integral calculus are identical things. For him the FTC allows a dual view of calculus and even if the FTC is “a very accessible theorem” it is also

> one of those deep theorems that you keep learning more about as you have more familiarity with mathematics. (Int4)

Also Int2 points to this function of the FTC to connect “two different areas”.

Question 2 (Q2)

Even if the Q2 asked specifically about which formulation(s) of the FTC, among the eleven different ones that were presented (referred to in the following by their number in the question sheet as Q2.1, Q2.2, etc.), the interviewee found to be closest to his/ her own understanding of the FTC, most of the interviewed mathematicians commented on all or several of these formulations. Generally, all formulations were “accepted” apart from sometimes Q2.1, Q2.9, Q2.10 and Q2.11, and in one case Q2.2 and Q2.6. Two of the mathematicians (Int7 and Int9) explicitly accepted all formulations. Most preferred were Q2.3, Q2.7 and Q2.8 (given in about half of the answers). Reasons given for these choices were more or less elaborated, due to the open format of the question. For Int4 the formulation Q2.3 was selected as the closest to his version of the FTC, with the following comments:

> I think of it as basically number three…with a picture so to speak [ ] where I look at a graph and $f(x)$ is the area so far under the graph and we look at how quickly that area is changing. I like to think about things geometrically. So I mean that has a nice geometric feeling for me.

The choice of the words “a nice geometric feeling for me” indicates a personalised way of looking at the FTC. Int4 related the character of Q2.3 to his spontaneous preference of Q2.1:

> Number two you differentiate and number three you integrate and then you differentiate. So they are inverse processes. (Int4)

When chosing Q2.3, two interviewees (Int8 and Int10) added the assumption that the function must be continuous, while Int6 here was less concerned with the assumptions than in his response to Q1.

Among all formulations in Q2 Int3 chose without any doubts Q2.7 and Q2.8, with the reason that he used them in his teaching. Also Int4 refers to students in his choice of the same formulations of the FTC:

> Number seven is very often the way you see it but really it's as I said before, it's part of it only…This is the way the student would study first. (Int4)

One person also referred to his own time as a student:

> When I first studied calculus the way in which I was taught and the way in which I understood it was [number] seven. (Int1)

The same choices, with some additions, are made by Int7, who however is more precise in her arguments, referring to the two parts of the FTC:

> So there are two parts. One part is three and eight, and the other part is seven and nine, and again the continuity [assumptions in seven, eight and nine] is not necessary if you use Riemann integrable. The others of course are just applications in various contexts.

When selecting Q2.3, Q2.7 and Q2.8, in most cases also Q2.1 was accepted. Only two persons (Int4, Int10) explicitly opposed to Q2.1 and they said it clearly:

> To say that they [integration and derivation] are inverse operations, I think, misses the point. (Int4)

> The first one not, because if you take not continuous functions, this is completely not good. (Int10)

One person (Int6) accepted the correctness of formulation Q2.1, however not as the theorem but as a definition.

Persons with experience in teaching calculus courses usually choose Q2.7 and Q2.8 (or sometimes Q2.3) as the formulations closest to their understanding of the FTC. Most of them knew that usually the FTC is formulated in two parts, most explicitly expressed by Int7. Three persons (Int4, Int9, Int11) talked about the first part and one person (Int1) only about the second part of the FTC.

Reasons for not accepting the formulations Q2.9, Q2.10 and Q2.11 varied among the interviewees. Int2 does not like Q2.9 because it is a bit "pedantic", Int6 says that "number nine misses the whole point as far as I am concerned" while Int8 simpy states that "that [Q2.9] is one that I do not like". Some comments on Q2.10 and Q2.11 are rather vague. Int11 says that "This is an abstract version or something", and Int3 talks about a generalisation:

> Well, I guess ten and eleven are little bit generalised, a little bit too much [ ] but I would not necessarily say that this is the way I understand it. I do not think that way. (Int3)

Nobody really accepted formulations Q2.10 and Q2.11 as the FTC. One mathematician was very clear about his view on Q2.11:

> What is a differential operator? What is a field? This is complete nonsense. They publish books with a lot of nonsense. (Int5)

A slightly positive reaction to Q2.10 was given by Int9, who said:

> I am quite happy with Q2.1. But you could do something like Q2.10, I suppose. (Int9)

Also Int1 can accept Q2.10, seeing it in terms of different dimensions:

> Number ten and number seven I would say…they are the same formulations of the same thing, in different dimensions. (Int1)

In contrast, Int10 said that "this is not the FTC, this is something else".

The formulations by Newton and Leibniz (Q2.4, Q2.5, and Q2.6) were not accepted as formal versions of the FTC. They were not completely rejected as wrong but accepted only as useful for explanations or as applications of the FTC.

> It is not quite good this number five…When the area under the curve, the function is negative you have problems. I would not like number five. (Int2)

> Four, five, six are useful in terms of explaining or motivating. But I would not necessarily use these as formulations of FTC. (Int3)

However, Int6 accepts Q2.4 as his way to present the FTC for teaching:

> Number four, I think, is how I try to present the FTC, how to show where it comes from. I have a curve and I imagine the area under the curve being gradually exposed at a rate of change of area as it comes out. If you go from x to $x+\Delta x$ you have roughly a little rectangle of base Δx and height $f(x)$. So that's how I try to make it meaningful to them. Whether I get anywhere or not, that is a different question. But I think, certainly for students heading

to the science, they should at least be exposed to what the FTC really says.

There were not many comments on Q2.5 but it was accepted by Int11 with some correction:

> Finding the area is more or less the other way around. First you find the antiderivative and then you use it to find the area rather than the other way. (Int11)

Question 3 (Q3)

There were different approaches to answer this question. Most of the mathematicians accepted the statement in the first sentence and answered the question, one person elaborated on the statement in the first sentence, and some interpreted the question as asking about the significance of FTC and answered by addressing its historical significance.

On Q3 about a half of all the interviewees pointed to important role of a problem for the development of a theory. Quite many did not provide a specific problem which could have been involved in the genesis of the FTC, and also did not find it necessary. The importance of a theorem is then seen either in an application to 'reality' or as an element in theory development. References to history were made, emphasising the key role of the invention of the FTC for the development of mathematics, even one of the most important moments in its history:

> I don't have a very simple problem in my mind. I mean, it seems to me that, I consider the fundamental theorem of calculus as the culmination of, I don't know, 2000 years of mathematics. [ ] Because it is at this exact moment where, I think, in a way modern mathematics is born, a lot of things that come out of that. And let face it, before this happens no one knows what a function is, I mean never mind derivative, never mind a continuous function, none of these fundamental mathematical ideas. (Int4)

Only about half of the interviewed mathematicians gave quadrature, i.e. the calculation of area, as the problem related to the FTC: "presumably this is in connection with some sort of quadrature problem" (Int9), or "I guess, the problem is the calculation of area" (Int8). For them the concept of integral is equivalent to the area and the problem is equivalent to the task of calculating areas by using antiderivatives. However, two of these persons did not find this 'simple' problem very important. For example, in his answer to Q3 Int3 could not formulate any concrete problem:

> I think every theorem is a solution to a problem. But certainly not necessarily applied problem. (Int3)

He completely agrees, though, that using some real life problems from physics, biology, or economics can be useful in teaching, providing better explanation and understanding, not only for the FTC but also for the whole calculus. It can help to motivate students. It is not enough with the simple explanation that by using the FTC they can find the area under the curve. It has to be a type of problem in which it is possible to see the connection between different concepts like area, slope, integral, and derivative but in terms of this particular problem:

> So this is, sort of way, I bring up idea that you need to find area under the curve. So it is not just area under the curve because, you know, that seems trivial. Why would people want to find the area under the curve, why would people want to find a slope at a point? So what? Who cares? But you have to explain why this has important application and I try to give a real problem, where it is not finding the slope but what it means in terms of this particular problem.

Thus, Int3 seems to be talking more about teaching for understanding than about the genesis of the FTC through attempts to solve a specific type of problem. Other interviewees pointed more directly to quadrature as the problem for the FTC. Int2 agrees with the statement that to formulate the theorem one first needs a problem to work with:

> You need a problem…it could be like a real life problem…or any problem. (Int2)

For him the only mathematical problem connected with the FTC is the area under the curve. In his answer to Q3, Int9 mentioned that it was related to "some sort of quadrature problem", but he also thinks that it is not clear that mathematical results are closely connected with concrete problems:

> Maybe one gets motivated by some problems but I don't think the problem has necessarily anything directly to do with the mathematics. (Int9)

Similar ideas were expressed by other interviewees. For example, Int1 could not see any specific application of the FTC. He was aware about the long history behind it, and the long history of the calculus but could not formulate any specific problem that led to the discovery of FTC:

> The fundamental theorem of calculus has a much earlier history or a longer history, ok, the fact the differentiation and integration are inverse processes, I mean, this was known for a long time from the early days of calculus, but I don't really know what exactly, I mean historically, which problems really led to it. (Int1)

For him the only real application of the FTC is the physical interpretation with Stoke's theorem. Also Int10 could not give an example of a problem that was solved by the FTC but this is not an important issue for him:

> I don't care if this has applications in Physics or somewhere. [ ] For me, I take the real analysis course and this is a beautiful theorem because instead of this crazy definition everything I have is a very easy description. One in the terms of complete abstract and a second in terms of area.

Only one person suggested another problem as the root of the FTC:

> I guess the problem is how do you integrate, and this is what the result is, because they knew how to differentiate. They wanted to integrate. (Int7)

This answer to Q3 is connected to the first formulation of the FTC [as the inverse operations] by Int7. One person (Int5) elaborated more on the first statement in Q3 (that a theorem is formulated out of work on some problem), trying to explain what is the driving force of development of mathematical theory, in particular when formulating a certain theorem in a theory. His answer was connected to mathematics as a research domain in general and not only to the problem in the case of the FTC. He meant that building the theory is like art for art's sake and not just the tool needed to solve some real life or mathematical problems. But still for him, who is a mathematical physicist, a theory is a model of some reality and the significance of a theory can be "measured" or judged by assessing how well it describes the reality:

> When I try to look on a problem for myself it should have some relationship to some reality. Then the question is how real the model is? What is important is how close you can get to the real. (Int5)

One person (Int10) commented on the possibility of introducing the FTC as a solution to a problem of quadrature in teaching. When students come to the first course in calculus at the university level they already have some experience from high school with working with integrals by using antiderivatives. And then it is very difficult for them to understand the significance of the FTC. The situation is different when it comes to higher courses like in real analysis. Then one can use the Riemann integral for bounded functions and a number of examples for finding the area by the partition method and then "suddenly…it's a very nice tool which is a fundamental theorem of calculus". However, he does not see the need to motivate the FTC by a problem that it solves.

Question 4 (Q4)

The answers to Q4 differ depending on how the expression "significance" is interpreted, either as a mathematical significance or as an educational significance. Mathematically the FTC is seen to have (or had) importance for applications (in other subjects such as physics as well as in other areas of mathematics), for theory, and for the historical development of mathematics. Some interviewees talk about one of these aspects, some of two of them or all three.

Int1 refers to the important use of the FTC in applications of differential geometry in physics by Stoke's and Green's theorems: "these have been used all the time in electromagnetism and hydrodynamics". Also Int5 mentions these theorems and adds:

> It's just the basic true. Even what Einstein did follows from it. (Int5)

For him, then, the FTC is a very important result in mathematics. Similarly, Int8 says that it is "the most important theorem in mathematics" and that "everybody use it", though without being more specific about some application.

In addition to its applications, Int5 and Int8 point to the significance of the FTC for mathematics itself. This is done also by Int11 and emphasised by Int7, who sees the role of the FTC for pure mathematics as essential:

> And in my area of analysis, Harmonic Analysis, this theorem is of essential importance. [ ] The Lebesgue version of that theorem is really of fundamental importance. (Int7)

Several of the interviewed mathematicians talk about the historical significance of the FTC, which according to Int6 is outstanding:

> It's a non-trivial thing [FTC]. It really changed science, changed the history of science. So we should be aware that this is something. (Int6)

Also for Int4 the formulation of the FTC is one of the most important moments in the history of mathematics (see quote from Int4 above on Q3). He also relates this historical perspective to education and the students' difficulties to understand some mathematical concepts:

> The past is the student's present in some sense. The difficulties in the past point to often what the student's difficulties are in the present, so in a certain sense, they are recapitulating the past at least in some level. (Int4)

He attributes a major significance of the FTC to its potential to create a moment of insight, due to the 'ambiguities' within mathematics that the FTC resolves. Also Int10 relates the significance of the FTC to what it can offer students, if treating the partitions involved in the Riemann integral in a 'clever' way, "not written anywhere in a book":

> And then it's a lot of work. You know there are three times two hours teaching and suddenly you put the theorem which is the fundamental theorem of calculus. This is for me significance. Unbelievable, I cannot believe. (Int10)

On the other hand not all were convinced about the importance of the FTC in present mathematics because they do not use it themselves, though not opposing to the importance of the concepts of differentiation and integration (Int9, Int11). One of interviewees strongly opposed to the name "the fundamental theorem":

> So you see, you have a bunch of so called fundamental theorems: in algebra, in all parts of mathematics [ ] But you see, you must be very careful before you put this label fundamental theorem. Why it is fundamental theorem? (Int2)

He admits that it was a fundamental theorem when it was formulated in the 17^{th} century but not any longer:

> When Newton discovered this theorem, this was like post doc mathematics, then it went down to graduate mathematics and then it went to undergraduate mathematics. Now it is like junior

> college or even it is high school. [ ] What seemed to be very, very deep, like this, in Newton's day, today isn't so deep. (Int2)

However, the same person added (later in the interview) that "You can't do anything without that theorem", thus expressing its pragmatic significance. When it comes to the name of the theorem Int10 accepts it as the fundamental theorem of calculus but not for the whole of mathematics. Then it is just one of its many theorems.

Question 5 (Q5)

Only five persons agreed to some extent with the fact that the FTC changed their understanding about some concepts in mathematics (Int1, Int4, Int6, Int7, and Int10). However, some of them at the same disagreed with this, depending on content or in which role you use the FTC. Int1 acknowledged influence on conceptual knowledge about integration and differentiation but not about variables and functions, while Int4 completely agreed that the FTC changed his understanding of mathematical concepts,

> Because I went through all of my basic mathematical concepts and I understood that this form of duality or ambiguity is present in everything (Int4),

but did not connect his answer to the concept of variables or functions. Two persons (Int7 and Int10) acknowledged this influence only in connection to teaching calculus, not to learning it as a student. For students, they claim, the FTC is mainly providing a technique for computing integrals and students are not aware of, or interested in, why the FTC is true. Along this line, Int2 says that when students see the FTC for the first time it is just a nice way of finding the solution to the area problem by using the anti-derivative:

> For any student, the first time you see the FTC this isn't mathematics. It is the quick way to find the solution. (Int2)

Int10 admits that his understanding of the theorem as a student was the same:

> But at this time this was just the instruction how to calculate the integrals. By the way, the first year I even hated this. Because it's calculate this integral. It's crazy. Without understanding what you are doing at all. (Int10)

His perception of the FTC started to change first when he had to teach students for the first time. But even then he did not see anything special with the FTC. The change came with teaching the real analysis course for the second and third year student:

> Then I really observed how nice this is, how beautiful and what difficulties they had, in history, to prove. [ ] Therefore it is difficult to believe that you can find such very general theorems, because this is a very general theorem. Okay, the starting point was really teaching the students and the course where you must prove all the theorems. (Int10)

That the act of teaching the FTC has an influence on your own conceptual understanding of the basic mathematical concepts related to it was most explicitly expressed by Int6:

> I would say just about every time I taught calculus [ ] every time I go through setting it up at the board, I would get what I feel is deeper understanding of what limits are about, what Riemann sums and its generalizations really are. (Int6)

One of the interviewees could not give a clear answer:

> I don't know. I first learned this theorem and then learned my precise definition [of function]. (Int8)

Most of the mathematicians did not see, as students, the influence of the FTC on their understanding of mathematical concepts, probably because they did not have enough knowledge about these notions:

> It did not change. At that time, when the FTC was introduced I knew not enough. (Int5)

> I never had really sort of these ideas what is a function, really, and that never bothered me. (Int3)

Answers indicate, however, that these issues are relevant for mathematicians in teaching situations.

Questions 6 and 7 (Q6, Q7)

The outcome from Q6 and Q7 are presented together, as the interviewees often compared the different ways of reasoning of Leibniz and Newton. Leibniz's reasoning in Q6 caused quite a lot of confusion during the interviews but also appreciation. It seems that the proof as presented in Q6 is completely removed from the way mathematicians work and think today: “I am not familiar with it” (Int11), or, after some time spent reading it, “Well, very strange” (Int5). Int10 showed a quite strong reaction: “Wow, this is crazy. This one is nothing for me. I don’t understand.” Some interviewees asked about expressions used in the example, especially ‘the law of tangency’. The most common point of view was that the proof is very geometric and that it was difficult to understand it: “I never thought of it (FTC) geometrically, in this way” (Int1), and “This is complicated, but it’s kind of geometrical” (Int7). One person did not want to read the proof with the explanation that “I never understand Leibniz before” (Int4) but some of the interviewees also had a positive view on that kind of reasoning. One person liked it:

> It would take me a little bit longer to absorb it but it looks like a good argument. [ ] It’s a nice geometric way to prove it. (Int7)

Int3 also appreciated the way of reasoning in Q6 and said that “First of all the proof is clever”, and somehow put himself in the time of Leibniz:

> Yes, it is strange. Doesn’t it? But if you don’t see what we call FTC and look for the way to find the area that is related to the derivative, this is certainly a clever way to do it. (Int3)

However, the notation used by Leibniz in this proof seemed to be one of the possible causes for difficulties to understand it:

> I think one of the things about mathematics that is probably not appreciated sufficiently is getting good notation. When things are related to each other if you chose good notation it makes it a lot easier to follow. (Int9)

One of them mentioned that usually opinions about other mathematical work are based upon one’s own experience and because the proof in Q6 is very far from what they have seen before it is difficult to accept it (Int11). Mathematical traditions, originated in the different approaches by Leibniz and Newton during the time of the invention of the calculus, were referred to as causing difficulties:

> Is this the European approach? [ ] It is totally different, of course, from the way it’s usually presented in North American textbooks. [ ] Well, it seems to be pulled out of a hat. (Int6)

Comparing the texts by Leibniz and Newton also initiated reflections on what constitutes a proof. Everybody agreed that both texts in Q6 and Q7 are quite removed from today's reasoning but that the one in Q7 (Newton) is much easier to understand. One person compared it to physics:

> That's how it's done in physics books all the time. I am much familiar with this. [ ] This is very easy. [ ] You know, this is the way we do it. (Int1)

Another person found Newton's way of working more modern:

> It's similar to the modern day proof [ ] So that's sort of the analytical proof as opposed to the previous one which was geometric. (Int7).

That a geometric way of proving is seen to be more unusual today is expressed also by Int3:

> Well, it is interesting. [ ] So Newton was really much more on the right track than Leibniz. That is a very unusual technique [Leibniz's reasoning] which never went anywhere.

However, most of the interviewees found that the reasoning in Q7 is not exactly a proof as we understand it now and for some of them it would be easier to accept the previous one as the proof. Even if he admits that the "intuition is better in Newton's", Int2 does not see the reasoning as proof:

> This won't be accepted as a proof. It's reasoning. It's like building of intuition. [ ] To me this [Leibniz] is a proof. And Newton is not a proof. (Int2)

Nevertheless, the interviewees usually prefer Newton's way of thinking:

> I would put Newton before Leibniz. […] Intuition before the proof. (Int2).

> The Newton way of thinking I accept more than Leibniz. This is for sure. (Int10)

A historically based comment was made by Int6, who found the formulation and reasoning in Q7 funny:

> Because this was the whole thing with the calculus that you started out by clearly saying that something was not zero and then all of sudden you let it to be zero. (Int6)

Comments by Int4 illustrate, by discussing differences between what people think and what they write down, that these things about what is proof are not very clear. One has to choose what to write down in a proof and what to leave implicit, and he says that the difference between a mathematician and a student in this respect is that the mathematician can fill in all the details but the student usually not. About the reasoning in the text in Q7 he says:

> In Newton's case he may well have thought of it completely different than the way he wrote it down. I think in fact he wrote everything down geometrically and that was completely political on his part. (Int4)

Question 8 (Q8)

The most common explanation for this problem (by 8 persons) was to start with a constant, sometimes average velocity and in the next step, when velocity is changing, break up the interval into small subintervals in which we can assume the velocity to be constant ("more or less"). The last step would be to take the sum of all small rectangles that cover the area under the graph and 'in some sense' take a limit. The quote below is a typical example of an answer to this question.

> I would go with little rectangles. [ ] The average velocity in one hour or the average velocity in a minute and this is a minute velocity, so this is the distance. So the area of a rectangle would be the distance. The sum of distances would be what the distance is covered. [ ] This is the way we do it. (Int2)

That the answers were so similar may be explained by the comment by Int3 that "This is the thing that [ ] I do in a class". It is, however, usually not seen as very important by these mathematicians to explain, to the students, the 'leap' between the constant and non constant velocity, even if they are aware about its significance and the problems that can appear in that step. For Int1 this step seems to be quite natural because

> a lot of these mathematical formulae are derived in physics books exactly by this method that Δx is not differential, Δx is always considered to be a small interval and you add up these increments and in some sense take a limit. (Int1)

Four persons (Int4, Int6, Int9, Int11) formulated some reservation to this way of explanation, as for example this caution:

> There is in calculus, as I was saying before, something basic about nonlinearity. So if you start with a linear function you get a false idea of what's going on. (Int4)

Nevertheless, they still thought that this is the best way of dealing with the problem and it should be sufficient when teaching the calculus. This also reflects a view of what is needed or possible in an introductory calculus course. For example, Int6 is completely aware that such approach is not correct but he also thinks that for most students such reasoning is sufficient:

> I think you have to adapt it [teaching] to the kind of students you are working with. If you have an honours class [ ] you can go a bit further. But really, if they had as good understanding of the basic principles as Newton and Leibniz did, that would be extraordinary. And Euler, if they learn how to manipulate as well as Euler did, that would be incredible. (Int6)

An uncertainty about how to teach this topic may be hidden behind the following explanation, where a student is given a graph:

> Since he's got all information in that graph, you might persuade him that what he is doing is finding the area underneath, point by point. I mean, you can do that slowly. (Int9)

Question 9 (Q9)

The interviewees did not offer any detailed answers to Q9, and some were quite confused. Most of the 'explanations' for the possible misconception referred to the use of vaguely or ill defined terms, in particular the letter *d*. Almost all the mathematicians were aware that the formulation of the FTC stating that the integration and derivation are inverse operations (similar to Q2.1 and Q2.2 in Q2) does not implicate the possibility of cancelling out the symbols $\int$ and d in the equality $d(xy) = d\int xdy + d\int ydx = xdy + ydx$. They say that it is important to define the meaning of *d*, but not of $\int$ so the meaning of the integral symbol is supposed to be clear from the beginning. One person (Int3) referred to a general problem with students' understanding of mathematical notations, and in calculus he pointed to the use of the same symbol for both the indefinite and the definite integral.

> When the notation is ambiguous then this kind of mistakes are easy to make. (Int3)

For Int2 it was important to define the symbol *d*. If it is a derivative then it is important to provide a clear explanation what is the variable for such operation. In his opinion we do not have a problem today with wrong interpretations of derivative or integral:

> With the notation we have today, we don't make that mistake. (Int2)

This view seems to be shared by Int4, who elaborates on the idea of meaning:

> These *d* was just a bad notation, so then you have this development of calculus without the *d* [ ] but now I think that things have gone full circle. These *d* is actually an act of genius, it is a very brilliant notation. [ ] So I think Leibniz had a meaning. We lost it, which means we didn't understand the context within which he was thinking and then we created later on a new context which made it meaningful again. (Int4)

He is certain that we still do not understand some of Newton's and Leibniz's ideas that they had themselves during the work of formulating calculus.

Also the meaning of x and y in this example was discussed. Int7 missed rigorous definitions: "on one hand x and y are fixed sides of the rectangle and on the other hand variables". She expressed a belief that this problem could be 'fixed', which is in line with her choice of formulation Q2.2 of the FTC in Q2. Also Int8 had problems with x and y:

> Lots of things are wrong. Here x and y are constants and you are integrating over x and y. So you cannot compare. (Int8)

He added, however: "It is not totally wrong. So it is interesting." Also Int5 found the example intriguing, seeing a problem to interpret the figure only when the function is negative:

> But otherwise it's great. I don't even see what's wrong with that. (Int5)

In contrast, for Int11 it is very important that all expressions should be defined rigorously, with all the required assumptions.

Question 10 (Q10)

Very different views on teaching the FTC were expressed in the interviews. For some the FTC should be the most important theorem in the calculus course and not taught as "just another theorem in succession of theorems" (Int4). It is very important that students are aware of the value of that theorem:

> You could claim that maybe we should only teach two theorems of calculus, you know, the chain rule and the fundamental theorem. (Int4)

Int7 points to the importance of the FTC first in the real analysis course, in connection to Riemann and Lebesgue integration. For several other interviewees this distinction between the introductory calculus course and the more advanced real analysis was crucial in regard to the way it can be taught. For example, Int10 divides his answer to Q10 in two parts because, as he said:

> I am saying that it is different how I am teaching at the first year calculus and it is different when I have a real analysis, where I am proving the theorems. Very big difference. Completely different kind and way of thinking about the theorem.

In the first course the FTC is presented as "a recipe" for how to find the function that satisfies some requirements, followed by a number of almost the same exercises. Then, "nothing is beautiful with fundamental theorems of calculus".

There is, for Int10, another situation in the real analysis course that requires knowledge about continuity, uniform continuity, and the Riemann integral. He starts his teaching with the partition and the upper and lower sum for some easy functions, showing that it takes a lot of work to find the integral with this method. Then it is time to formulate the FTC which is so useful in similar situations. He does not see it as necessary to change that way of teaching the FTC:

> Why we should explain for first year students that this theorem is beautiful? Maybe we should not? Maybe it is a recipe and that's all. And why we should, why we should force people outside the university to understand that some theorems are beautiful? Should we? I don't believe so.

A similar point of view is expressed by Int 6, as evidenced in his answers both to Q8 and Q10:

> I think you have to adapt it [teaching] to the kind of students you are working with. If you have an honors class [ ] you can go a bit further. But really, if they had as good understanding of the basic principles as Newton and Leibniz did, that would be extraordinary. And Euler, if they learn how to manipulate as well as Euler did, that would be incredible. (Int6)

He believes that students should first get the procedural knowledge of calculus before the conceptual understanding:

> The mechanical stuff becomes more or less automatic so that later on they can turn their minds towards understanding at a deeper level without getting bogged down. (Int6)

Others express the view that the teacher should explain how the FTC works (by showing a number of examples) but that it is not so important to explain why it works. It is not necessary to include the proof because the proof does not increase the understanding of the theorem (Int2). Some feel that a graphical approach to the FTC can be helpful: the picture with the graph of the function, the area and the explanation of the connection between derivative, tangent, and antiderivative can be useful for explanation (Int3, Int5). In other answers there are statements that it is important in teaching to point to generalisations of the FTC, such as Green's and Stoke's theorems:

> it's one thing which people can get some understanding, even at the elementary level [ ] It's an unifying concept which comes in elementary calculus, then in differential geometry [ ] it's a concept which carries through for a long time. (Int1)

Another person (Int11) was satisfied with the 'traditional way' of teaching the FTC with some short theoretical introduction about the area and the antiderivative concepts, then formulation of the theorem followed by a number of examples that should "make students aware that differentiation and integration are two separate things that finally meet". However, he says that

> we usually don't spend much time on the theoretical parts of the foundation of the subject, not enough time to make the students really become aware of the differences. (Int11)

He also believes that it is necessary to teach students such theorems as the FTC because the education at the university level should be "different from what is done at high school", with some abstract and theoretical parts of the foundation of mathematics.

DISCUSSION AND ANALYSIS

What is the FTC, really?

Research question 2B asked for an identification of scholarly knowledge of the FTC as expressed by mathematicians from different fields of experience. Summarizing the outcomes of interview questions 1 and 2, which focused on this issue, the following picture emerges.

About half of the interviewed mathematicians saw some inverse relationship between differentiation and integration as the propositional content of the FTC, sometimes described in a very informal way. Those having some experience of teaching calculus courses for students usually gave a more formal formulation of the FTC as composed of two parts. Variations included statements about multivariable functions or with weaker assumptions, for example formulating the FTC for a piecewise continuous function. A few mathematicians saw it as a way of finding the area under the curve, one as reconciling two different ways of looking at "the" calculus.

As to the different formulations in interview question 2, all or several of these were "accepted". The most common first preferences were, "without any doubts" as one mathematician expressed it, Q2.7 (the computational formula in symbolic terms, with assumptions), Q2.8 (differentiating the integral gives 'back' the function, in symbolic terms with assumptions), or Q2.3 (same as Q2.8 but without assumptions). However, these choices were expressed only with relation to teaching or to their memories of how they encountered the FTC as students. Versions that were not accepted by some mathematicians include Q2.9 (the definite integral defined by the computational formula in Q2.7, with assumptions), which was labelled a bit pedantic, as missing the whole point, or being abstract. Nobody really accepted formulations Q2.10 (a detailed vector field formulation, in words and symbolic terms) and Q2.11 (a vector field formulation without using mathematical symbols), which were qualified as a little bit generalised, complete nonsense or something else than the FTC. The formulations by

Newton and Leibniz (Q2.4, Q2.5, and Q2.6) were not accepted as formal versions of the FTC, but also not completely rejected as wrong. Additional data on these versions can be found in the outcome from interview questions 6 and 7, where Leibniz's reasoning, of which one thought it is "crazy", caused quite a lot of confusion but also appreciation. Newton's was more accessible and thus preferred, but Leibniz's was seen as resembling more of what can be called a proof.

In the responses from some mathematicians to the questions about their own views of the FTC, one could trace an influence from their teaching experience. For example, even the persons who accepted formulation Q2.1 ("Differentiation and integration are inverse operations") as the FTC were not so positive to Q2.2, which can be seen as the symbolic translation of the first one. One interpretation of this is that they are aware that the formulation in Q2.1 should not be used in teaching situations without additional information and explanation for students about derivation and integration concepts, and problems with using them in such informal way. In addition, even if most of the interviewees chose a formal version of the FTC in interview question 2, they still were not very strict with assumptions. They quite often chose Q2.3 instead of Q2.8. Only three persons paid attention for assumptions in both interview questions 1 and 2. Again, this could possibly be related to their long experience of teaching calculus to undergraduate students.

The big variation of the answers to interview question 2 show that there are substantial differences in how the mathematicians evaluate the given formulations of the FTC as close to their own views. That they were expressing their personal views rather than what they thought a standardised version would be, is also evidenced in formulations such as *for me, as far as I am concerned, a version that I do not like,* or *this one is nothing for me*.

In terms of mathematical praxeologies, the different answers given in the interviews relate to the three levels technique, technology and theory:

Technique: the FTC is formulated with an emphasis on the computational formula, thus representing the level of technique;

Technology: the FTC is described in general terms with an emphasis on how two mathematical operations or processes are related, without specifying how these process are to be carried out in a technical sense, sometimes with reference given to necessary assumptions or definitions;

Theory: the FTC is represented by different formulations (for example also in a multivariable version) or put into a wider theoretical/ philosophical context.

Taken together, the answers in the interviews show that the ‘scholarly knowledge’ of the FTC among mathematicians is not well defined and connected to their sub-institutions (areas of research), as well as being personalised by for example teaching experiences. Thus, the interviews point to two aspects of scholarly knowledge: an official (institutionalised) version of the FTC and a personalised version.

Significance and the problem that led to the FTC

Another element of the scholarly knowledge of the FTC concerns its significance for mathematics or applications in other subjects, as well as for individuals’ understanding of mathematical notions that are involved. Related to this is its genesis, that is an identification of those types of problems that triggered the development of mathematics towards a formulation of the FTC.

The type of problem or task from which the FTC developed most commonly identified by the mathematicians is quadrature, i.e. the calculation of area. One person saw finding out how to integrate as the problem related to the FTC. Others pointed more generally to the use of the FTC in real world applications, rather than being generated by them. Comments about the need of a problem to build on when teaching the FTC were also made, to make it more accessible to students. However, some interviewees could not formulate any specific problem that led to the development of the FTC but conceived of it as a theoretical achievement. References to history were made, emphasising the key role of the invention of the FTC for the development of mathematics.

Some of the interviewed mathematicians questioned the ‘status’ today of the FTC as ‘fundamental’, as compared to when it played a more prominent role in the development of mathematics. Mathematical significance of the theorem was attributed to its importance for applications outside mathematics (a ‘pragmatic significance’), and use inside mathematics, including theory development (a ‘horizontal significance’). Also a ‘historical significance’ of the FTC was pointed at by the interviewees, by its unique role for the development of mathematics. At another level several mathematicians described an important ‘educational significance’ of the FTC, both in its potential to provide deep insights and to demonstrate the power of mathematics.

These different views expressed on the question about which problem led to the FTC, and its significance, relate to mathematical praxelogies with different emphases on the practical and theoretical levels, reflecting the interviewees' different ontological and epistemological orientations towards mathematics.

The outcomes on the question about a problem behind the emergence of the FTC also relate to the issue of different functions of proof. When starting from a task and developing a technique to solve the task, and the constructing of the related technology and proof as the result, the functions of proof refer to explanation and justification. In contrast, when starting from an established theory and developing techniques to solve a type of task (application), it is the validation as function of proof.

Suggested didactic transpositions and their constraints

The issue of teaching the FTC was touched upon during several of the interview questions, most directly in questions 8 and 10 but also for example in questions 5 and 9. The views expressed on these questions by the mathematicians suggest didactically transposed version of their knowledge related to the FTC.

For several interviewees a distinction between the introductory calculus course and the more advanced real analysis was crucial in regard to the way it can be taught. For undergraduate students it was not by all seen necessary to include the proof because the proof does not increase the understanding of the theorem. In addition, beginning students were seen to be interested mainly in the computational aspects of the FTC, as well as generally in their mathematics studies showing a procedural approach rather than a conceptual. This was also acknowledged by the interviewed mathematicians to be their own way of working during their undergraduate studies. One of the interviewees put forward the argument that it necessary to teach students such theorems as the FTC because the education at the university level should be “different from what is done at high school”. Some felt that a graphical approach to the FTC can be helpful, and one thought that it is important in teaching to point to generalisations of the FTC.

A common didactical technique in mathematics education is to use different kinds of representational means to better ‘explain’ some mathematical issue that is regarded as ‘formal’ or ‘difficult’. Such a situation is set up in interview question 9, where the FTC is ‘shown’ by way of a diagram. In the interviews this initiated mainly a discussion about mathematical notations and the need for clear definitions in teaching. This relates, in terms of praxeologies, to the need to elaborate on the theoretical level when an ‘explanation’ to the FTC is given, and the different functions of proof.

In interview question 8 another situation (the velocity graph) is set up which relates to interview question 3, that is how work on a type of problem may develop levels of technique and technology of a mathematical praxeology. The uniformity of the answers suggest that this is a ‘standard’ task for the interviewees, linked to an established way of setting up a praxeology with an informal technological level, relying on visual impression, intuition, and uncontrolled approximations.

Since the mathematicians express an awareness that the procedure described to solve the task is not fully conclusive, there seems to be an agreement that the students imagined here need not be invited to take part in an institutionalized activity of formal mathematics. One interviewee outlined an insider-outsider perspective, where only a small community were seen to be able to take part in and get something out of the theoretical work. For the ‘outsiders’, only technical skills need to be taught.

In terms of praxeologies, some mathematicians expressed a view that for introductory calculus courses, the FTC should be seen mainly as a technology to supply a technique for solving tasks on integration, without any need for justification by proof. Only more advanced courses in real analysis called for a more elaborated theoretical part of the praxeology.

As the different views put forward about the teaching of the FTC relate to the different personal backgrounds of the mathematicians, such as area of expertise and experience of teaching at the undergraduate or more advanced levels, they indicate the impact of the ‘personal’ scholarly knowledge on the didactic transposition process. It is therefore necessary, when discussing the didactic transposition of the FTC, to discuss the two aspects of scholarly knowledge identified: an official (institutionalised) version of the FTC and a personalised version. Didactic transposition in this case means standardisation. This is a process of mass education which also is visible from the historical textbooks or the ones later that are written by more exposed scholars in the field, such as Banach or Hardy.

7. OUTCOMES AND CONCLUSIONS

A starting point for this thesis has been the nature of the relationship between academic mathematics as practiced by researchers at universities and what has been called educational mathematics (Bergsten & Grevholm, 2008), encompassing institutionalised teaching and learning mathematics at different levels within educational systems. This study set its focus on how this relationship is seen from the perspective of mathematics education and by researching mathematicians. The Fundamental Theorem of Calculus (FTC) and its proof here served as an example. The results presented and discussed in chapters 4, 5 and 6 will here be summarised and elaborated, in relation to the specific research questions defined in chapter 3. The implications and conclusions of the study are discussed within these sections.

RESEARCH QUESTION 1

How is the relation between scholarly knowledge and knowledge to be taught seen in mathematics education with respect to the status and role of proof?

Transformations of functions of proof in teaching institutions

The notion of proof in mathematics is embedded in the notion of theorem. A theorem is a proposition about relationships between properties of mathematical objects that has become a piece of accepted knowledge within an institution by way of its proof and is seen as important. Not every proved proposition reaches the status of a theorem. The status and function of different kinds of proof have been studied within mathematics and the philosophy of mathematics as well as in mathematics education. As discussed in chapter 4, the functions of proof in mathematics as a research discipline and in mathematics as an educational task vary considerably in focus and scope. The proof functions to develop general reasoning skills and meta-knowledge of mathematics belong only to teaching institutions. While the justification/ verification function is a major focus in research mathematics, education puts more emphasis on explanation/ illustration. This may be related to the role of proof for communicating new results: this function resides only outside teaching institutions, as the communication takes place between scholars. A piece of knowledge (such as a proved proposition) is usually converted into knowledge for teaching only after official sanction, and the need for justification of this knowledge, such as the FTC, is then strongly reduced. Students see no reason to discuss the propositional content of statements in mathematics already established by an expert community: they trust what is in the book, or what their teacher says. Facts and procedures that seem obvious or common sense to the students is another category of knowledge that does not call for justification in the eyes of the students. What may be seen as obvious and what not, is sometimes related to didactical order and time, the chronogénèse, as will be further discussed below.

The systematisation function of theorems and proofs

The systematisation function is already implied in the notion of proof. By definition, a mathematical proof links the proposition under consideration to other, already accepted, propositions. But it can also be interpreted as related to a unifying agenda, in general achieved by the propositional content of a theorem, which in this respect is then more or less important. Such an agenda also can carry a didactical purpose to better understand how the different parts being systematised relate. Both these aspects are present in research mathematics, evidenced for example by the many monographs that are being published within different sub-areas of mathematics. In education, notably at upper secondary and introductory tertiary level, the didactical aspect is clearly a major focus (cf. Ernest, 1999). A systematisation of a body of knowledge can be done by way of hierarchy, for example embedding calculus in analysis and then in functional analysis, or by technologies, as when classifying differential equations by types of equations with different techniques to solve tasks. In the case of the FTC, its systematisation function contributes to the value attached to it. It is achieved by linking two previously separated sets of technologies. Systematisation may be done at different praxeological levels (for example to compare techniques or technologies), while a hierarchical systematisation leads to establishing a hierarchy in terms of punctual, local and regional praxeologies. But the systematisation function of a theorem and its proof is hard to be made visible in undergraduate courses or at school. The well known problem to make meta-discussions ‘get through’ in classrooms, in a spirit of systematisation for the purpose of deepened understanding, may be related to the need of knowing the parts before one can systematise them. This point was, at least implicitly, evidenced in the interviews with the mathematicians.

Para-mathematical aspects and standards of proof

Applied proving techniques differ also in levels of what is counted as proof within the different types of institutions. As the border line between what is 'really' a proof and what is not is fuzzy already within the scholarly knowledge of mathematics, the status and forms of didactically transposed proofs for theorems such as the FTC are strongly institutionally biased. While experimental and formal proof, in the distinction made by Blum and Kirsch (1991), are clearly separated as to their status as 'real' proof, the intermediate level of institutional proofs may include a wide range of types of justifications, verifications or explanations counted as preformal within the institution. The interviews with the mathematicians show that also at the introductory tertiary level it is common to 'stay' at a preformal proof level with the FTC, or even to fully leave the proof out. Proof can itself be an explicit object of teaching, i.e. didactically treated as a mathematical notion to be taught, but is rarely the case. Usually it is just treated as a para-mathematical notion, i.e. used but not explicitly trained or assessed. This second alternative is especially common in so called service courses in educational programmes where mathematics is used mainly as a tool.

Mathematics education literature comprises terms such as informal proof, proof by example, visual proof (i.e. institutional proof) in contrast to rigid proof or formal proof. Obviously the proofs themselves undergo a transposition. Often the level of generality is reduced and in the school curriculum a special case of a more general statement is included and proven. Often new 'proofs' are invented only in the teaching institutions by didactical purposes, with an aimed explaining function. However a transposition is not specific to didactical purposes. A variety of mathematical proofs often exist of the same statement depending within which theory it is formulated and upon the degree of formalisation. A well known example is Pythagoras' theorem with its several hundreds of published proofs, but also 'new' proofs of more advanced theorems are published in mathematical journals. Such a transposition would by definition be identified as a didactic transposition when the theorem is selected as an object for teaching and purpose to adapt to the didactical constraints in the specific teaching institution. However, one could also argue that when the function of the proof is to systematise or to explain, also within a scholarly institution, the transposition would be didactical in essence.

The FTC and its proof within different didactical praxeologies

The modern standardised textbook version of the FTC has two parts, the first one about the existence of a mathematical object with certain properties (i.e. a function which has a given function as its derivative), and a computational part (i.e. providing a technique to compute definite integrals). This version of the FTC is a standard formulation only in the teaching institutions. By this formulation it is not possible to construct the mathematical objects in the first part but only in the second part by way of the first part. This gives the FTC an asymmetry with didactical consequences. Setting up the mathematical praxeology in teaching can be done in two ways. In the 'standard' DTP format of teaching at the university, i.e. following the order definition – theorem – proof – examples (Weber, 2004), the propositional content of the theorem moves 'down' in the praxeology to a generalised technique/ technology which can then be applied without knowing the relationships in the theorem. The student can apply a method of validation for each single task, in which a technique is used, by just testing the specific case without reference to the discursive level (the propositional content of the theorem); in the case of the FTC this would come down to differentiating the function to be used in the computational formula of the integral. In fact, this is exactly what good theorems do in mathematics, freeing the user from the need to validate each step. This is a general phenomenon in relation to 'canonical' techniques of praxeologies within an institution (Chevallard, 1998).

A conversion of the 'chronogénèse' creates other tensions within the dynamics of the praxeology. When first, after doing differentiation (derivative), one does antidifferentiation (antiderivative; primitive functions) before defining the definite integral as a limit of a Riemann sum, the justification function of the proof of the first part of the FTC (in the standardised textbook version) is hard to achieve. This effect is also supported by a 'notational obstacle', created by the power of good notations in mathematics: introducing the standard integral symbol to denote primitive functions (without any reference to the integral as a limit of a Riemann sum) makes you resistant to the need of a proof for the first part of the FTC, as it is already inherited in the symbol used (now for definite integral). In terms of praxeologies, the (resulting) body of knowledge is anticipated by a chronogénèse that starts with techniques and employs symbolising tools that are taken from the body as a whole, and the justification function of proof becomes problematic. The FTC does not justify anything because of the start with the technique/technology that perfectly functions. However, a choice of chronogénèse starting with a problem to develop a technique for calculating for example an area, or computing distance by way of a non constant velocity over a specific time, may evoke a need for justification of how the sum (leading to an integral) can be evaluated by primitive functions, if the notion of primitive function and techniques for finding them has not yet been taught. Thus, if the techniques are introduced first, the technology and the motivation/ incentive for the development of a proof is created through a process of justification. This happens in situations of a breakdown or expansion of the praxeology. In this case the didactical praxeology would more resemble the historical development of the FTC in its proof.

RESEARCH QUESTION 2

What is the propositional content of "the" Fundamental Theorem of Calculus (FTC) as part of a body of scholarly knowledge?

RQ2 A. *How did the statements connected with the FTC develop to become a fundamental theorem for the (new) sub-area of mathematics called calculus? How did it evolve in relation to basic notions and to the systematisation of the calculus? How was the process of this development reflected in different formulations and names?*

Before a didactic transposition takes place, it must be possible to identify a point in time, when a piece of knowledge under consideration, such as calculus and the FTC, becomes an institutionalised part of the body of scholarly mathematical knowledge. Institutionalisation means that it has to become a shared "rule" to use a particular version of concepts and methods within an identifiable and clearly delineated distinct sub-area. Some internal coherence of such a sub-area is a prerequisite for distinguishing it from other areas.

Distinctiveness can be achieved by the use of a particular notational or representational system, shared basic notions, specific methods and objects of study, and a methodology for relating the propositional statements to each other and by its common intellectual roots. The history of the calculus and of the FTC shows that there are indeed developments of all of these parts, in parallel and sometimes overlapping with developments in other areas, until the sub-area of calculus became established. That the distinctiveness of a “body” of knowledge is also linked to the intellectual roots can be seen in much of the popular history writings about calculus with Newton and Leibniz as the central figures. Modern expansions of integral calculus are even linked to the name of the creators, that is, to Riemann and Lebesgue.

The forms of representation needed for establishing the calculus include graphical representations of curves (e.g. the velocity-time curve used by Oresme), and the introduction of symbols for the quantities involved in algebraic relations (e.g. by Nemorarius and Viéte). Newton’s symbolising techniques were less specialised than those of Leibniz, whose notation appeared to be more clear and more flexible for generalisations. The history of single pieces of the notational system shows the long time-span, within which these developed. For example from the introduction of the integral symbol (proposed by Leibniz 1675 and developed into the modern definite integral symbol in 1822 by Fourier). The development also included a set of typical problems for which the techniques and technologies were developed, such as the study of motion, tangents and quadratures. The achievement of an integration of existing techniques into a technology that allows algorithmic treatment of certain types of problems is attributed to Newton and Leibniz. This integration is one of the main values seen in the propositions that are associated with what became called the FTC.

The inverse relationship between the derivative and the integral has been called “the root idea of the whole of the differential and integral calculus” (Richard Courant, quoted in Boyer, 1959, p. 10-11). Newton’s version consists in “a general” procedure for finding the relation between the quadrature of a curve and its ordinate for a class of curves (see the interview questions, Q2.4 and Q7). His interest to find the “fluxions” of the “fluents” and conversely can be seen as the fundamental problem of the calculus (Boyer, 1959, p. 194). Leibniz’s version is geometrically disguised (see the interview questions, Q2.6 and Q6). However, Leibniz stated that ”the general problem of quadratures can be reduced to finding the curve that has a given law of tangency” (quoted in Katz, 2009, p. 572). Both versions of what can be interpreted as a form of the FTC are at the level of technology in terms of a mathematical praxeology, but are asserted theoretical status by many, because of the potential for merging two separated collections of technologies into “one calculus”. Before Newton and Leibniz, James Gregory and Isaac Barrow did also link the ideas of area and tangent, but Katz (2009, p. 539) argues that they did not invent the FTC.

Colin MacLaurin did, in his textbook *A treatise of fluxions* from 1742, among other things, calculate areas by integrals, and “probably the earliest analytic proof of part of the fundamental theorem of calculus, at least for the special case of power functions” (Katz, 2009, p. 615). Joseph-Louis Lagrange also proved one part (differentiation of the integral) of the FTC, though he used no formal definition of area (see Katz, 2009, p. 635)

The incentives for justifying the calculus technologies merged from ontological concerns about the existence of prime and ultimate ratios or infinitesimals. After arithmetisation, calculus developed from a collection of techniques for manipulating infinitesimals into a collection of techniques for manipulating certain limits. The concept of limit can be seen as the unifying basic concept for the body of scholarly knowledge called calculus.

In the approach of Cauchy and then also of Riemann, there is a clear separation of integration from the derivative. This can be taken as a revival of Leibniz’s approach of the integral as a sum.

Cauchy’s version (1823) of the proposition associated with the FTC includes what is called the first part of the FTC in modern textbooks and a proof by means of the mean value theorem for integrals based on a rigorous definition of the definite integral. He also proves the computational formula and introduces the term 'indefinite integral' with present day notation. However, he does not give any name to these results. These are part of the “Twenty sixth lesson” of a course at the teaching at the École Polytéchnique.

In the historical study of textbooks as well as in the literature about the history of mathematical notations and of the calculus, it was not possible to identify the author or the work, in which the name the Fundamental Theorem of Calculus first appeared.

The development of the function concept in the 19th century with the expansion of arbitrary functions into Fourier series eventually led to Riemann’s definition of the integral. Riemann proved Green’s theorem by using the FTC, Mikhail Ostragradsky the divergence theorem, and Hermann Hankel Stoke’s theorem (see Katz, 2009). These can be seen as generalisation of the propositions associated with the FTC.

The development of the calculus and of the FTC shows that the process of institutionalisation of a body of knowledge as an identifiable, delineated and named area has to be seen in relation to the publication practices and the ways of dissemination. As long as new methods and results are transmitted verbally, in personal correspondence and through privately circulated manuscripts, identification of a point of institutionalisation remains generally problematic. The borderline between a "textbook" and a publication addressed to an audience of other researchers who have less specialised knowledge in the sub-area under consideration, is not as clear as it is today throughout the history of the calculus. For example, reference to Cauchy's scholarly work is commonly done by drawing on his textbooks. This provides an example of a dynamic relationship between the development of "knowledge for teaching" and the intention of re-organising and re-describing a set of related outcomes of research in different sub-areas for the purpose of presenting it in a coherent way. This is clearly another indication that the 'systematisation function' of mathematical proof has a didactic component.

RQ2 B. *How is this particular scholarly knowledge seen by researching mathematicians from different fields of expertise?*

In the interviews the mathematicians were asked to describe their views on and understandings of the FTC. The interviewees were all research mathematicians, thus representing the producers of mathematical knowledge and here seen also as bearers of the scholarly knowledge of the (since a long time existing) theorem and its proof. Some had long experience of teaching calculus, including the FTC, while others were exclusively researchers. The portrait of the scholarly knowledge of the FTC painted by these mathematicians was varied to a much greater extent and in more aspects than perhaps could be expected. This variation was found in how the theorem was described in terms of focus and content, choice of language as more or less rich in mathematical symbolism, detail and possible variation of assumptions for and versions of the FTC, significance in terms of use, theory, and historical development, or for the understanding of mathematical concepts. The answers given could to some extent be related to the individual expertise and background in for example teaching experience, pointing to the need of considering a 'personalised' scholarly knowledge in addition to the institutionally standardised textualised scholarly knowledge of the FTC. The differences in answers are distributed over and within all levels in terms of mathematical praxeologies:

Type of task: the FTC is formulated and proved (invented) to solve the problem of quadrature, or how to integrate, or no specific problem;

Technique: the FTC is formulated with an emphasis on the computational formula, thus representing the level of technique;

Technology: the FTC is described in general terms with an emphasis on how two mathematical operations or processes are related (as inverses to each other), without specifying how these process are to be carried out in a technical sense, sometimes with reference given to necessary assumptions or definitions;

Theory: the FTC is represented by different formulations (for example also in a multivariable version) or put into a wider theoretical/ philosophical context.

Variations in significance of the FTC referred to a mathematical component as pragmatic (in applications), horizontal (in mathematical theory development), or historical (major role for the progression of mathematics), and an educational component to deepen understanding and demonstrate the strength of mathematics.

The variation of the answers also to the interview question where the interviewees were to make a preference among 10 different textbook formulations of the FTC, point to substantial differences in how the mathematicians see the given formulations as similar to their own conceptions. That they were expressing their personal views rather than referring to some standardised version is also evidenced in the choice of formulations in their answers, using expressions such as *for me*, *as far as I am concerned*, *a version that I do not like*, or *this one is nothing for me*.

A conclusion based on this outcome of the interviews is that the ‘scholarly knowledge’ of the Fundamental Theorem of Calculus among mathematicians is not well defined, it is connected to their sub-institutions (areas of research), and it is personalised (by for example experiences as teachers as well as learners, in addition to expertise). Thus, the interviews point out two aspects of scholarly knowledge: an official (institutionalised) version of the FTC and a personalised version.

As the interviews were performed in connection to work on a thesis in mathematics education, some interviewees may have seen themselves as representatives of university mathematics teachers rather than researchers. It was evident from the outcomes that mathematicians with extended experience as teachers brought in this experience into their ways of answering the questions. This is, however, a common situation among the scholar community of mathematicians, and the mix represented in the sample for the interviews account for this variation of personal backgrounds.

RESEARCH QUESTION 3

How can the ftc be identified and described as an object for teaching?

RQ3 A. *How is it presented in contemporary and subsequent textbooks?*

Cauchy's version of approaching calculus, represented at a level of formalisation stemming from the later 19th century (formalisation of the real numbers, the epsilon-delta definition of limit, and thus a different conception of continuity and a concept of function based on set-theory), is what can be found in current calculus courses. Usually a version of the FTC is included. In commercial textbook production, different traditions of formulating and proving the FTC seem to have developed for different markets. In the U.S. a division of the theorem into two parts is common, with first referring to the derivation of the integral and the second to the formula for computing the definite integral by the primitive function. In other places this is less common.

The development of the name used for the propositions associated with the FCT from no name, through rather long names including the role of the FCT as a link between two sub-areas to its short name as "The fundamental theorem of calculus" mirrors the development of the area. In some countries still the longer name "Fundamental (or main) theorem of the differential- and integral calculus" are used in textbooks, such as in Germany, Austria and in Poland. This long name indicates the systematisation function of the theorem and its proof.

The extent to which textbooks define the praxeologies can be seen in the variation of the formulations of the theorem, for example, in Sweden. These range from a more widely used version, which after defining the definite integral through sums establishing a link to differentiation, to singularities such as defining the definite integral as the difference of the primitive functions (see summary of the development included in the discussion section of chapter 5).

The propositional content of the statements connected with the FTC is in the textbooks presented at different levels, as a technique, as a technology as well as a theoretical proposition. Cauchy's lecture notes and textbook outline a theory with a stress on definitions of basic concepts. Many textbooks from the 20th century present the relationship between indefinite and definite integral as a technique for computing the definite integral. Goursat's *Course d'analyse mathématiques* from 1902 presents it as "the fundamental formula of the integral calculus" (while the "theorem" states that every continuous function is the derivative of some other function), Poussin's *Cours d'analyse infinitésimale* from 1921 refers to it as "the fundamental formula for the computation of definite integrals". Courant (1934) refers to it as the "important rule" (while the "theorem" states that the difference of two primitives of the same function is always a constant). In the modern textbooks for the U.S. market, this rule is usually presented as the "second part of the FTC" (the first part refers to the derivation of the integral). The German textbook cited in chapter 5 (Mangoldt & Knopp, 1932) refers to the computational part as the "main theorem of the differential- and integral calculus". Similarly, in Swedish textbooks produced during the second half of the 20th century, the "main theorem of the integral calculus" commonly refers to the computational formula.

Some textbooks stress the fact that the FTC gives a rationale to the technique, and thus present it as a technology, for example the early textbook *An introduction to the summation of differences of a function* by Groat from 1902. There are also some that stress the systematisation function and its theoretical value (Hardy, 1908; Courant and Robbins, 1941; Forsling and Neymark; 2004) and thus present it at the level of theory. This is also the case for the Polish textbook by Banach from 1929, although the formulations and the approach seems to be a singularity

Some of the less commercialised modern textbooks, often compiled original lecture notes, can be seen more as an expression of the scholars' personal views on the particular part of the knowledge. It is not only the perceived constraints of the anticipated or experienced teaching situation (e.g. the level of the students' knowledge, the target students in terms of programmes) that contribute to the structure of a calculus textbook. Sometimes exposed researchers in the field do engage in writing textbooks. This engagement can be seen as an expression of their own personalised mathematical knowledge and their ways of structuring it. That is, they might take a different approach than in standardised commercial textbooks, usually not written by eminent scholars. Banach's approach to the FCT in his textbook might be an example of such a personal approach. Cauchy's lecture notes provide another example, which is a rather influential one. The standardisation of the FTC and its proof in textbooks is due to its institutionalisation as a piece of knowledge for teaching in undergraduate calculus courses for large groups of students where commercial textbooks are used.

RQ3 B. *How do researching mathematicians remember the FTC from their experience as students? How do they evaluate different versions presented in textbooks? What didactic transposition do they prefer or suggest for teaching?*

Mathematicians at universities are often also engaged in teaching, to an extent that may vary over time and between individual persons from very little to the major part of their work. Since many students take the beginning calculus course most mathematicians have at least at some occasion taught the FTC. To investigate the didactic transposition of the FTC for teaching at beginning calculus courses, the interviewed mathematicians were asked about their evaluations of different textbook formulations of the FTC, their own memories of encountering the FTC as students, and their preferred way of teaching the FTC.

Evaluation of the FTC in textbooks

How the mathematicians evaluated different textbook versions of the FTC was investigated by the second interview question where 10 formulations of the FTC were presented, representing standardised modern versions in symbolic mathematical language with or without assumptions of both the first and second part of the theorem, as well as more informal formulations about "inverse operations", wordings with an older language, and generalised forms in several variables.

The outcome showed that most of these formulations of the FTC were seen by the interviewed mathematicians as possible versions of the theorem, but that preferences for the standard modern textbook versions were most common, i.e. the computational formula in symbolic terms with assumptions, the statement that differentiation of the (definite) integral gives 'back' the function, in symbolic terms with or without assumptions. These choices were related to teaching experiences or to how they remember the FTC from their time as students. The definite integral defined by the computational formula (with assumptions) was said to miss the whole point or being abstract, and a more generalised formulation in terms of vector fields, with no symbolic formulas, was really not accepted by anyone, qualified as complete nonsense or something else than the FTC. Another vector field version with a symbolic integral formula given was however chosen by two mathematicians to represent the FTC. The formulations by Newton and Leibniz (described only in words and no formulas) were not accepted but also not completely rejected as being wrong. In another interview question proofs of the FTC given by Leibniz (geometric) and Newton (analytic) could be compared. Newton's reasoning was seen as more similar to present approaches and was preferred, but Leibniz's text was seen to be more of a proof, even if his reasoning caused considerable confusion and was said to be difficult to follow. The texts by Leibniz and Newton cannot be seen to be results of didactic transpositions.

Persons who accepted the informal version "Differentiation and integration are inverse operations" as the FTC were not so positive to the formulation that can be seen as a concise symbolic translation of that version, both without assumptions. With mathematical formulas involved the need for assumptions may increase, while the former has an intuitive appeal without being precise. But even if most of the interviewees preferred a formal version of the FTC, they still were not strict with assumptions, preferring the shorter one without assumptions to the same formulation with assumptions. The three persons who paid attention to assumptions in both interview questions 1 and 2 had long experience of teaching calculus.

In conclusion, the most common preferences among these mathematicians were the standard modern textbook versions, though equally distributed between the first and the second part of the FTC and with or without assumptions. However, also other formulations were accepted to represent the FTC, showing that there are substantial differences in how the mathematicians evaluate the given formulations of the FTC as similar to their own views.

Memories from learning the FTC as a student

Most of the interviewed mathematicians reported a first conception of the FTC, from their time as students, with a strong procedural orientation. It was remembered mainly as a nice way to calculate integrals or an area. At that time, many said, they did not reflect so much about the deeper relations or the meaning of the FTC and its basic concepts, not seeing such things as a problem. One mathematician did not report the computational formula (i.e. the second part of the FTC) as his first encounter with the theorem but the first part, formulated very similar to the way Cauchy did it. Memories of a more conceptual content were put forward only by one of the interviewees, who understood from the FTC that one can see similar things in different ways in mathematics, such as differentiation and integration.

Almost all of these mathematicians pointed to a major difference in conceptualising the FTC between meeting it as student and teaching it: seeing it as a 'recipe' for technical work as novice learner of calculus to seeing it, as an experienced teacher, at a theoretical level as a technology with a propositional content linking different mathematical concepts.

Comparing these 'memories' of their first encounters with the FTC with the picture of the FTC that emerged from their answers to the first part of the first interview question (What is the FTC as you understand it?), it is clear that their views on the FTC have been influenced by their subsequent work in mathematics after this first encounter, now seeing the FTC more in general terms as inverse processes or operations, at a technological level. This, however, does not always determine how they teach the FTC in the introductory calculus course. There are also other constraints influencing the didactic transposition process.

Didactic transpositions of the FTC

The outcomes of this study show that the scholarly knowledge of the FTC is not well defined, neither in names, formulations and proofs, or reported significance. It was also observed that the scholarly knowledge as expressed by the individual mathematicians to a great extent was personalised, related to expertise and personal background such as experiences of teaching. One could therefore expect a rich variety of views on how to teach the FTC, due to the ill defined 'start' of the didactic transposition process.

The interviews with the mathematicians suggested, however, only two substantially different didactically transposed versions of teaching the FTC (and the calculus). Regarding the first version, several interviewees agreed that one cannot teach the introductory calculus course and the more advanced real analysis the same way with regard to the theoretical construction of the FTC, a view expressed by one mathematician with the words "completely different kind and way of thinking about the theorem". Arguments put forward concerned both the level of student understanding and their approach to study mathematics: the proof does not increase the understanding of the FTC, and beginning students tend to focus mainly on the computational part of the theorem. Also at a general level several interviewees expressed the view that students have a procedural approach rather than a conceptual in their mathematics studies, a view possibly related also to their experiences of their own undergraduate studies.

In contrast to this common view of the interviewed mathematicians, pointing to constraints in terms of the students to the didactic transposition of the FTC, a few emphasised the importance of also including a theoretical level of the praxeology. Arguments then focused on the need to make students aware of the FTC with regard to its unifying and generalising aspects, as well as more generally that university education should be "different from what is done at high school".

As discussed above, the extent to which different functions of proof are evoked, such as a need for justification or explanation, is constrained by the didactical praxeology chosen for teaching of the FTC. In particular, this amounts to the chronogénèse of techniques and technology. The uniformity of the answers of the mathematicians to interview question 8, where a situation was outlined with initial work on a problem (using a velocity graph) that would lead further to the technical and technological levels of the praxeology, suggests that a 'standardized' didactical praxeology has been established by a didactic transposition of this commonly used example. The answers show that it results in a mathematical praxeology with an informal technological level, relying on visual impression, intuition, and uncontrolled approximations. As the procedure described to solve the task is not fully justified, this didactic transposition of a mathematical problem related to the FTC can be seen as an expression of a view that the students imagined here need not be invited to take part in institutionalized formal mathematics. This represents an insider-outsider perspective, where only a small community is seen to be able to take part in and get something out of theoretical work in mathematics, while only technical skills need to be taught to the 'outsiders'.

In conclusion, a view suggested a didactic transposition where, for introductory calculus courses, the FTC has its main function as a technology to supply a technique for solving tasks on integration, without need for justification by formal proof. Some interviewees suggested preformal proofs with an explanation function, such as showing simple examples and explaining how things work rather than rigorous justification of why it works. According to this view, only more advanced courses in real analysis call for a more elaborated theoretical part of the praxeology. The didactical choice to even fully omit the proof can count as an extreme version of a didactic transposition of proof.

As an alternative didactic transposition, a few mathematicians favoured a praxeology with a more elaborated theoretical level, to make students aware of the value of the FTC and its systematisation function. One interviewee (Int 4, see chapter 6) expressed a strongly personalised view, suggesting that “you could claim that maybe we should only teach two theorems of calculus, you know, the chain rule and the fundamental theorem.”

In simple terms, when teaching the FTC and its proof, one possible didactical choice by the lecturer is to ‘write down’ what you know about it, thus staying close to your personalised scholarly knowledge, or to shift to doing something else, such as providing a ‘simple’ explanation using pictures or metaphors. The interviews point to the existence of both these options among the mathematicians for the introductory calculus course, with the most common preference for the latter option. A third more ‘extreme’ version of a didactic transposition of the FTC would be to simply leave out any kind of proof or explanation, which was also (implicitly) suggested by some. There was more agreement about the more advanced courses about the need of a strong theoretical level of the praxeology. Not many suggestions were made, however, regarding the chronogénèse of teaching calculus, that is the timely order of the different levels of praxis and logos and content related parts of the mathematical praxeology.

As the different views put forward about the teaching of the FTC relate to the different personal backgrounds of the mathematicians, such as area of expertise and experience of teaching at the undergraduate or more advanced levels, they indicate the impact of ‘personalised’ scholarly knowledge on the didactic transposition process. However, a picture that emerges is that the didactical praxeologies of teaching the FTC (and the calculus) are less varied than the mathematical praxeologies that could describe the scholarly knowledge of the FTC (and the calculus). This implies that the influence on the didactic transposition process of external constraints in relation to the scholarly knowledge are strong. Here some caution to the conclusions is needed, as the mathematicians who were interviewed in this study may represent shared locally developed traditions of teaching the introductory undergraduate mathematics courses.

RESEARCH QUESTION 4

With reference to the questions above, which insights can be derived from the case study of the FTC, about the relationship between scholarly knowledge and knowledge for teaching in terms of praxeologies and of the theory of didactic transposition?

In view of the outcomes of the study, some issues, which are discussed below, arise connected to a didactic transposition of the FTC as well as to the notion of praxeology and to didactic transposition itself.

Rigour and didactic time in terms of mathematical praxeologies

This study has initiated some reflections about the strengths and weaknesses of the conception of a mathematical praxeology as outlined in the anthropological theory of didactics (see chapter 2). In relation to this analytical tool, the notions of rigour and didactic time will be discussed here.

How can the theory conceptualise and explain the problems observed in this study to teach the FTC in a way that includes the theoretical level of the praxeology and preserves the need for justification and systematisation? Using a term commonly used among mathematicians: why is there a problem to teach the FTC with a rigorous proof at the undergraduate level? The notion of rigour in mathematics is most commonly associated with its foundations (cf. the quotation from Bertrand Russell at the beginning of the introductory chapter), but when discussing it in connection to teaching and learning mathematics, the notion is seldom defined but taken for granted. Rigour will here be viewed as including the following triplet of dimensions: (i)) well defined basic concepts, (ii) a functional specialised language and notational system, and (ii) deductive forms of argument. A sign of a lower degree of rigour would, for example, be the use of natural language for basic concepts (such as Newton writing about “the area of”), a wide use of para-mathematical concepts and heuristics or weakly defined forms of argument, possibly referring to proto-mathematical notions such as popular diagrams.

In the discussion above, the phenomenon of a notational obstacle was observed to explain why students come to not experience a need for justification of what is the first part of the FTC in a wide-spread textbook version. This was linked to the introduction of a notation (the integral sign) for a mathematical object (the antiderivative), which later was to be used for another object (the definite integral). To prove that the derivative of this new object then produces the function to be integrated was already inherited in the previously introduced notation, and thus trivial. This phenomenon is a consequence of didactical order and time, or chronogénèse, in the teaching of calculus, when moving 'up' to the theoretical levels of the praxeology, but anticipating a higher level of rigour at an earlier stage on part of the notation. The moving up does not take place with the same clock for didactic time at all the three dimensions. In parallel, but time-delayed, there is a necessary development of the basic notions involved in calculus and in the FTC, such as the concept of limit, derivative and integral. When looking at the praxeology at a specific time, synchronically, some elements involved to achieve rigour may have developed more than others, which means that there is an unbalance within the praxeology at that point in time. In teaching calculus there are many parts of the triplet of rigour that are unknown to the learner at different stages in didactic time, making the planning of the dynamic between synchronic and diachronic praxeologies for the teaching complex. This may explain the difficulties mentioned above in the teaching of the FTC and the calculus. Looking at another area of mathematics, elementary (Euclidean) geometry, the situation is different. That it is possible to teach proof in geometry already at the secondary school level can be explained by looking at the components of rigour needed there: basic concepts, notational systems, and forms of argumentation are more clearly delineated and accessible to the learner when (simple) proving tasks are introduced in teaching.

The notion of praxeology does not capture the difference between the teaching of calculus and geometry pointed to in the analysis above. It does not allow to differentiate between different dimensions of the mathematical repertoire that are involved at all levels. One can, for example, be at the level of theory (according to the praxeology), in a primary classroom where a generic proof is used for justifying a general proposition about numbers. The notion of rigour cannot be conceptualised in terms of the levels of a praxeology. This points to a challenge for the theory.

Which of the versions of the calculus that are, or have been, a delineated body of scholarly knowledge, could have been the starting point for a given didactic transposition?

As to the establishment of the calculus as a delineated body of scholarly knowledge, the history shows that it is hard to find a distinct point in time when it has become institutionalised. Consequently, the scholarly body of knowledge named "calculus", which could be the starting point for a didactic transposition, cannot easily be identified. For the FTC, the exemplary investigation of the different names used in research publications and textbooks shows that different names were used for denoting similar versions of "the" FTC, but also that the same name was used for different versions of it, or no specific name was used. There is no distinct transposition of a clearly identifiable piece of scholarly knowledge, but a series of re-descriptions. The standardisation seems to be happening within the "knowledge for teaching" rather than within the scholarly knowledge.

The distinction between a piece of knowledge as a part of a body of scholarly knowledge on the one hand, and a piece of knowledge for teaching, is hard to draw. However, it should be possible by distinguishing between the intended audiences of a publication. Textbooks, articles in handbooks, encyclopedic collections as well as popular science books that are written by researchers in the field, usually are intended for an audience with less specialised knowledge in the area of knowledge to which the sub-area under consideration belongs. Which part of the scholarly knowledge has been institutionalised independently of a didactic transposition, could then be identified by a detailed comparison of the sections in the didactically transposed versions with similar sections in other, that is, in research publications from the same sub-area.

What is the relationship between the institutionalised knowledge produced by research mathematicians and the knowledge intended for an introduction into that knowledge?

The development of calculus and of the FTC provides an example of the relationship under question. As the FTC and its proof, in the version that has become institutionalised, links two different fields of investigation, it can be attributed a systematising function. This systematisation has been achieved without reference to an intended presentation of the "calculus differentialis and integralis" that became "the calculus" as a delineated sub-area to an audience other than the researching mathematicians in the field. However, Cauchy's systematisation by means of introducing a set of basic concepts for an outline of the theory was developed in the context of the teaching at the École Polytéchnique. Common reference to his scholarly work is done by drawing on his textbooks. This provides an example of a dynamic relationship between the development of "knowledge for teaching" and the intention of re-organising and re-describing a set of related outcomes of research in different sub-areas for the purpose of presenting it in a coherent way. Felix Klein's *Elementarmathematik vom höheren Standpunkte aus* from 1908 is an example of a work that provided insights for both teachers of mathematics and researching mathematicians. The same year Godfrey Harold Hardy published his *A course of pure mathematics* (reprinted in many new editions), which "was intended to help reform mathematics teaching in the UK" and more specifically to prepare students to study mathematics at university (Wikipedia online b). It is not easy to locate such publications in relation to their role for a didactic transposition of scholarly knowledge. Transposition might include a 're-systematisation' of knowledge, for example, from a hierarchical structure of embedded specialised theories into one by shared techniques within different specialised areas.

The necessity for a didactic transposition assumes a separation between the institutions in which the producers of knowledge work and teaching institutions. If this separation includes a division of labour, the producers of knowledge are not the ones responsible for a transposition of the outcomes of research into knowledge for teaching. Several examples from the history of calculus (Bernoulli-l'Hospital, Lagrange, Cauchy) show that such a division of labour did not always exist. Prominent researchers in the area worked as 'transposers' of the knowledge produced by themselves for the purpose of the very introduction into that area. This is due to the relatively low degree of specialisation of the discipline, which reduces the gap between levels of mathematical knowledge in terms of a hierarchical knowledge organisation. The time-span, after which a piece of scholarly knowledge becomes an object for teaching, depends on the level of specialisation of the knowledge to be taught in relation to the level of the teaching institution. For example, Lebesgue integration soon entered advanced university course, but still is not the standard approach in introductory calculus courses. The calculus textbook by Hobson mentioned above, published in 1907, includes the Lebesgue integral that was published in 1904.

Specialisation of knowledge and the distance between the producers and distributors of mathematical knowledge

The relation and the 'distance' between producers of knowledge and distributors of knowledge are fundamental for the didactic transposition. The latter presupposes the existence external to the instutionalised didactic system of a 'knowledge' in the sense of a body of scholarly knowledge. In the early history of the calculus this distance was very short, as in some cases the same person who developed a body of knowledge also wrote a textbook where it was outlined for the purpose to be learned by others less knowledgeable. One example of this is Cauchy and his *Cours d'analyse* from 1821 and the follow up in the *Résumé des leçons données a l'École Polytechnique sur le Calcul Infinitesimal* from 1823, which was to meet the new demands arising from an institutionalisation of higher education after the French revolution, in the École Royale Polytechnique, even explicitly at "the urging of several of his colleagues" (Katz, 2004, p. 432). This tradition still prevails in present time and is not uncommon for example in the context of advanced textbooks in the U.S. university community. Such a practice establishes an institutionalisation of a body of knowledge through its didactic transposition. In this case there is a close relation as well as a short distance between producer and distributor of knowledge, which might even be the same person. The transposition is not achieved by another group of people, which is called the 'noosphere' in the theory of didactic transposition.

In the situation during the time of Cauchy the social base for the noosphere was different than today and the influence of market criteria still low. The body of knowledge which today is called calculus was not yet standardised; rather these early textbooks contributed to such standardisation in a dynamic relation between the producers and distributers of a developing body of 'new' knowledge. The historical sketch presented in chapter 5 illustrates how calculus as a body of knowledge was established by Cauchy through his use of the concept of limit as its key notion and his proof of (what later became called) the FTC as providing its inner coherence. Thus, the processes of institutionalisation and standardisation must be seen in their dynamic relationship to the didactic transposition.

As secondary school developed into a school not only for the preparation of a future elite, the number of students introduced into a particular mathematical topic is considerably high. The task of teaching large numbers of students' prompts the standardisation of curriculum with related specialised professions for developing, evaluating and producing transposed versions of scholarly knowledge. This is the noosphere. In such a context, the study of the relationship between institutionalised knowledge produced by research mathematicians and knowledge for teaching is likely to reveal a greater difference between these types of knowledge.

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