Part II
Technical background

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10:40 – 12:30

(1h50m)

Plan

- Formal grammars (30m)
- Semantics for formal grammars (30m)
- Bayesian inference (30m)

9:00-10:20	Introduction: On the very idea of an LoT
10:40-12:30	**Technical background**
12:30-13:30	Lunch
13:30-15:00	Bayesian program induction (LOTlib3)
15:20-16:30	Case studies
16:30-17:00	Summary

Formal grammars

What’s a *grammar*

Start with a *language* (Typically, a natural language)

A notion of **well-formedness** independent of meaning

- E.g., ‘3 is more curious than the past table’
- We call this *grammatical* well-formedness.

We can build an **abstract device** to encode grammaticality

- Two types of such devices are popular: automata and formal grammars.
- There is a correspondence between automata and grammars!
- In the rest of the course we’ll just use grammars.

Formal grammars

Grammars: Infinite use of finite means

Four ingredients:

1. A finite set N of *nonterminal symbols*
2. A finite set Σ of *terminal symbols*
3. A finite set P of *production rules*
4. A symbol S in N: the *start symbol*

Formal grammars

1. *N* (nonterminal symbols)
2. Σ (terminal symbols)
3. *P* (production rules)

4. *S* (start symbol)

1. {S, x, y}
2. {a, b}
3. 1. S → x
   2. x → xy
   3. y → a
   4. x → b

Let's derive a sentence in this grammar!

Formal grammars

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   3. by → ba
   4. x → b
   5. b → a **WRONG!** Why?

Let's derive a sentence in this grammar!

CFG - Context Free Grammars

Context-free grammar (CFGs) are grammars with rules of the form:

$$A \rightarrow \alpha$$

Where:

- A: *single* nonterminal symbol
- α: (possibly empty) string of terminals and/or nonterminals

PCFG - Probabilistic CFG

G = (*N*, Σ, *P*, *S*, Π)

1. *N:* nonterminal symbols
2. Σ: terminal symbols
3. *P:* production rules
4. *S:* start symbol
5. Π: probabilities on production rules

New: a function $\Pi: P \rightarrow \mathbb{R}$

- *Conditional* probability of applying rule $\alpha \rightarrow \beta$
- *Conditional* on the left side being α.

Every derivation has a probability of being derived

- The product of the probabilities of the applied rules.
- Higher probability to smaller trees

PCFG - Probabilistic CFG

1. N (nonterminal symbols)
2. Σ (terminal symbols)
3. P (production rules)
4. S (start symbol)
5. Π (probabilities on production rules)

1. {S}
2. {a, b}
3.

		Π
1.	S → aSa	0.3
2.	S → bSb	0.3
3.	S → e	0.2
4.	S → a	0.1
5.	S → b	0.1

Interpreting a grammar

Writing down functions

- 1, 2, 3, …, +, %, 'every', 'some', … **vs** x, y, z, …
  - Unsaturated vs saturated 'x + 1' vs '1 + 1'

- Unsaturated to function
  - f(x) = x+1
  - f(x) notation is *inconvenient*: forces us to name

- Solution: *lambda* expressions
  - Start with expression $x + 2$
  - Make λ expression w/ variable $\lambda x. x + 2$
  - *Function* from bound variable to the evaluated expression

Multiple λs and languages

- We can nest lambda expressions!
  - For instance: $\lambda y. \lambda x. x + y$
- Any expression can go inside
  - E.g., English: $\lambda x. x$ is a bird
- We'll mostly use variations of predicate logic.

The notation of λ-calculus

Notation for applying an argument to a function

- $\lambda x. P(x)$
- Apply argument N to function $\lambda x. P(x)$: $\big(\lambda x. P(x)\big)N$
- β-reduction:
  - $\big(\lambda x. P(x)\big)N$
  - Replace N in every occurrence of x and remove the lambda
  - $P(N)$

- We can also rename variables w/ alpha conversion (Let's ignore this)

Example of β reduction

- $\left(\left((\lambda x.\lambda y.x(y))\lambda z.P(z)\right)a\right)b$

Substitute $\lambda z.P(z)$ for x into $\lambda y.x(y)$:

- $\left((\lambda y.\lambda z.P(z)(y))a\right)b$

Substitute a for y in $\lambda z.P(z)(y)$

- $(\lambda z.P(z)(a))b$

Substitute b for z in $P(z)(a)$:

- $P(b)(a)$

Compositional interpretation

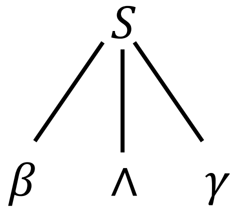
Let's build an *interpretation function* for a grammar!
- Associate each sentence with a meaning!

E.g., propositional logic
- S → p | q | (S ∧ S) | (S ∨ S) | ¬S

Interpretation of basic symbols:
- $I_@(p)$ = True
- $I_@$ (q) = False
- $I_@ (\wedge) = \lambda x. \lambda y. \mathrm{x} = 1 \; and \; y = 0$
- $I_@ (\vee) = \lambda x. \lambda y. x = 1 \; or \; y = 0$
- $I_@ (\neg) = \lambda x. x = 0$

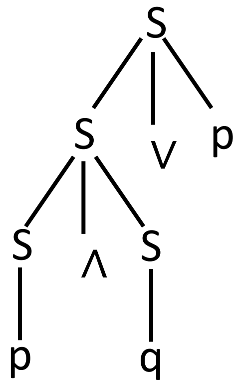
Compositional interpretation

Meaning of complex sentences:

If α has the form 

then $I(\alpha) = (I(\wedge)(I(\beta))\ I(\gamma))$

Can you tell what the rules are for the other entries?

Example: 

Bayesian inference

A motivating example

- Question: “Is my new haircut better than previous one?”

- You are completely unsure: 50/50
  - Mother of a friend says “Yes” –> new belief?
  - Your worst enemy says “Yes” –> new belief?

- You are pretty sure it’s worse than it was: 90/10
  - Mother of a friend says “Yes” –> new belief?
  - Your worst enemy says “Yes” –> new belief?

What is at play?

- Prior
- Likelihood

The components of Bayes theorem

$$\overbrace{P(H \mid D)}^{\text{Posterior}} = \frac{\overbrace{P(D \mid H)}^{\text{Likelihood}}\,\overbrace{P(H)}^{\text{Prior}}}{\underbrace{P(D)}_{\text{Evidence}}}$$

Four ingredients in Bayes theorem:

1.	**Posterior**	Probability of hypothesis given data
2.	**Likelihood**	Probability of the data *given* the hypothesis
3.	**Prior**	Probability of the hypothesis, NOT conditioned on data
4.	**Evidence**	Probability of the data, NOT conditioned on H

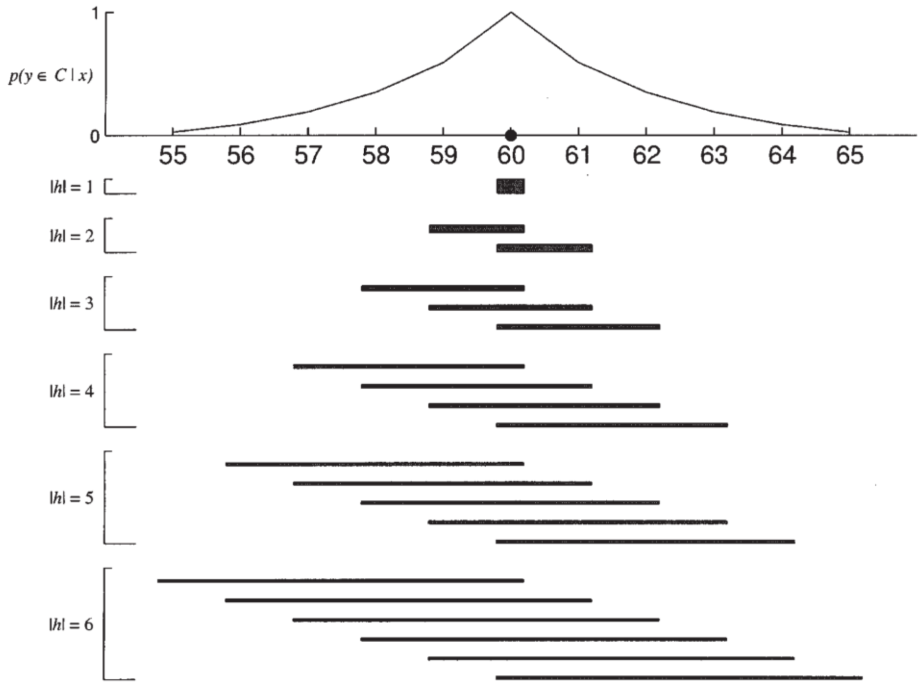
Case study: Simple category learning

Task: learn a category from examples.

- The space is simply the integers from 1 to 50
- The examples are numbers from the category
- The category is a *convex* region
- We get examples from inside the category

Bayesian category learning

- What's the space of hypotheses?
- What's the posterior, likelihood, and prior?
- What happens if we get more observations?

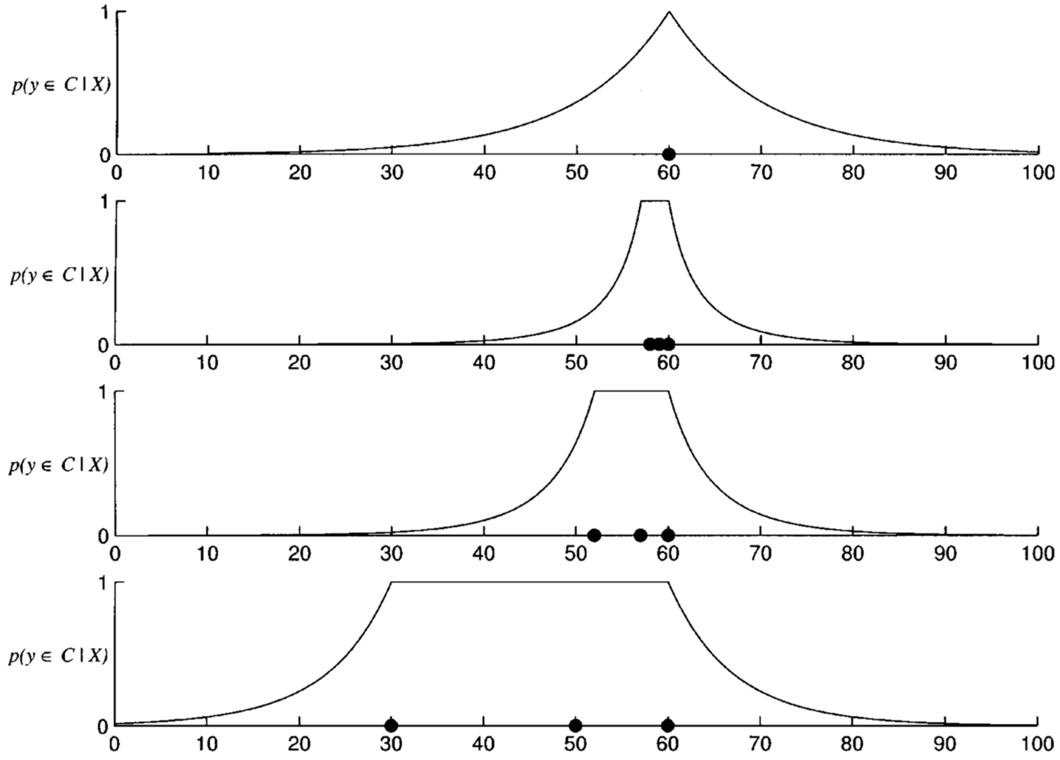


Tenenbaum & Griffiths (2001)

Case study: Simple category learning

Strong sampling –> *size effect*

- Can you see why intuitively?
- Can you see why formally?



Tenenbaum & Griffiths (2001)

Conclusions

We've learned about

- PCFGs
- their interpretation
- Bayesian category learning

We can use this as a model of the pLoT

- Can you see how?

Next session:

- Combine into computational model!

Lunch!

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If there’s time left…

Language to grammar

- We’ve seen grammar –> language
- We can also go language –> grammar
- Let’s try to write a grammar that produces all the palindromes in {a, b}*

1. *N* (nonterminal symbols)	1. {S}
2. Σ (terminal symbols)	2. {a, b}
3. *P* (production rules)	3. 1. S → aSa 2. S → bSb 3. S → *e* 4. S → a 5. S → b
4. *S* (start symbol)	

Language to grammar

- All strings with the form: $a^n b^n$

1. *N* (nonterminal symbols)	1. {S}
2. Σ (terminal symbols)	2. {a, b}
3. *P* (production rules)	3. 1. S → aSb 2. S → *e*
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Type theory

- For reasons that will become clear soon, we associate each of our expressions with a *type*.
- Let's define the set of types:
  - e and t are types
  - If σ and τ are types, then $< \sigma, \tau >$ is a type
  - Nothing else is a type
- And how to interpret them:
  - e refers to the set of individuals
  - t refers to the set of truth values
  - $< \sigma, \tau >$ refers to the set of functions from objects of type σ to objects of type τ

Type theory

Let's consider some expressions and what type they are:

- $\lambda x. P(x)$, where P is a predicate and x an individual.
    - $< e,t >$
- $\lambda x. \lambda y. Q(x, y)$, where Q is a predicate with two arguments and x and y individuals
    - $< e, < e,t \gg$
- $\lambda X. X(a)$, where a is an individual and X is a predicate
    - $\ll e,t >, t >$
- $\lambda X. \lambda Y. X(a) \wedge Y(a)$, where a is an individual and X and Y predicates
    - $\ll e,t >, \ll e,t >, t \gg$
- Basically, it can get as complicated as you want!

Type theory

In order to keep things tidy, we can put domain restrictions after a colon.

Therefore, we can write the type of each argument of a lambda function as follows:

- $\lambda x: x \in e. P(x)$, where P is a predicate
- $\lambda x: x \in e. \lambda y: y \in e. Q(x, y)$, where Q is a predicate with two arguments
- $\lambda X: X \in < e, t >. X(a)$, where a is an individual
- $\lambda X: X \in < e, t >. \lambda Y: Y \in < e, t >. X(a) \wedge Y(a)$

Sometimes, you'll also see the type written as a subfix:

- $\lambda x e. P(x)$