UNOFFICIAL SOLUTIONS BY TheLongCat

B3: ATOMIC AND LASER PHYSICS

TRINITY TERM 2021

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Turn over as you please – we are NOT under exam conditions here.

1. (DRAFT)

(a) Fine structure Hamiltonian:

$$\Delta\hat{H}_{\text{SO}} = \frac{\hbar^2}{2m_e^2c^2}\underbrace{\left\langle\frac{1}{r^3}\right\rangle}_{\frac{1}{l(l+\frac{1}{2})(l+1)}\left(\frac{Z}{na_0}\right)^3}\cdot\frac{Ze^2}{4\pi\epsilon_0}\cdot\langle\mathbf{s}\cdot\mathbf{l}\rangle \quad \text{for hydrogen}$$

$$\begin{aligned}\Rightarrow \Delta E_{\text{SO}} &= \frac{\hbar^2}{2m_e^2c^2}\cdot\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{l(l+\frac{1}{2})(l+1)}\left(\frac{1}{na_0}\right)^3\cdot\frac{1}{2}\left[j(j+1)-l(l+1)-s(s+1)\right]\\ &\sim \frac{\hbar^2}{2m_e^2c^2}\cdot\frac{e^2}{4\pi\epsilon_0}\cdot\frac{1}{a_0^3}\end{aligned}$$

Zeeman:

$$\begin{aligned}\Delta\hat{H}_z &= -\boldsymbol{\mu}\cdot\mathbf{B}\\ &= \mu_{\text{B}}\mathbf{L}\cdot\mathbf{B} + g_s\mu_{\text{B}}\mathbf{S}\cdot\mathbf{B}\\ &= g_J\mu_{\text{B}}\mathbf{J}\cdot\mathbf{B} \sim \mu_{\text{B}}B\end{aligned}$$

Equating both energies gives:

$$\begin{aligned}\mu_{\text{B}}B &= \frac{\hbar^2}{2m_e^2c^2}\cdot\underbrace{\frac{e^2}{4\pi\epsilon_0}}_{\alpha\hbar c}\cdot\frac{1}{a_0^3}\\ &= \frac{(\hbar c)^3}{2m_e^2c^4}\cdot\alpha\cdot\frac{1}{a_0^3}\\ &= \frac{(197.33\,\text{MeV fm})^3}{2(0.511\,\text{MeV})^2}\cdot\frac{1}{137.06}\cdot\frac{1}{(5.292\times10^4\,\text{fm})^3}\\ &= 7.24\times10^{-10}\,\text{MeV}\\ &= 7.24\times10^{-4}\,\text{eV} = 1.16\times10^{-22}\,\text{J}\\ \Rightarrow B &= 11.9\,\text{T}\end{aligned}$$

So within 12 T, Zeeman perturbation works on top of spin-orbit interaction.

(b) From above, $\Delta\hat{H}_z = \mu_{\text{B}}\left[\hat{\mathbf{L}}\cdot\hat{\mathbf{B}} + g_s\hat{\mathbf{S}}\cdot\hat{\mathbf{B}}\right]$.

By Wigner-Eckart theorem, we may substitute $\mathbf{L}$ and $\mathbf{S}$ with $\mathbf{J}$ with constant of proportionality akin to vector projection:

$$\begin{aligned}\Delta\hat{H}_z &= \mu_{\text{B}}\left[\frac{\hat{\mathbf{L}}\cdot\hat{\mathbf{J}}}{\hat{J}^2}\hat{\mathbf{J}}\cdot\mathbf{B} + g_s\frac{\hat{\mathbf{S}}\cdot\hat{\mathbf{J}}}{\hat{J}^2}\hat{\mathbf{J}}\cdot\mathbf{B}\right]\\ &= g_J\mu_{\text{B}}\hat{\mathbf{J}}\cdot\mathbf{B}\end{aligned}$$

with $g_J = \dfrac{3}{2} + \dfrac{S(S+1)-L(L+1)}{2J(J+1)}$ approximating $g_s \simeq 2$.

(c)

$$\begin{aligned}V_{\text{ext}} &= \alpha(x^2+y^2) + \beta z^2\\ &= \alpha r^2\cos^2\theta + \beta r^2\sin^2\theta\end{aligned}$$

Energy shift $\Delta E = \langle n| \hat{V}_{\text{ext}} |n\rangle$ for state $|n\rangle$.

For $1s$,

$$\begin{aligned}\Delta E_{1s} &= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \int_0^{\infty} \frac{1}{\pi a_0^3} e^{-\frac{2r}{a_0}} \left(\alpha r^2 \cos^2\theta + \beta r^2 \sin^2\theta\right) \times r^2 \sin\theta \,\mathrm{d}r \\ &= 2\pi \cdot \frac{1}{\pi a_0^3} \int_0^{\pi} \alpha \cos^2\theta + \beta \sin^2\theta \sin\theta \,\mathrm{d}\theta \int_0^{\infty} r^4 e^{-\frac{2r}{a_0}} \,\mathrm{d}r\end{aligned}$$

Radial integral:

$$\begin{aligned} F &= r^2 & \Rightarrow && \mathrm{d}F &= 2r\,\mathrm{d}r \\ \mathrm{d}G &= e^{-\frac{2r}{a_0}}\,\mathrm{d}r & \Rightarrow && G &= -\frac{a_0}{2} e^{-\frac{2r}{a_0}} \end{aligned}$$

$$\to \int_0^{\infty} a_0 r e^{-\frac{2r}{a_0}}\,\mathrm{d}r$$

$$\begin{aligned} F &= r & \Rightarrow && \mathrm{d}F &= \mathrm{d}r \\ \mathrm{d}G &= e^{-\frac{2r}{a_0}}\,\mathrm{d}r & \Rightarrow && G &= -\frac{a_0}{2} e^{-\frac{2r}{a_0}} \end{aligned}$$

$$\begin{aligned} &\to \int_0^{\infty} \frac{a_0^2}{2} e^{-\frac{2r}{a_0}}\,\mathrm{d}r \\ &= \left[-\frac{a_0^3}{4} e^{-\frac{2r}{a_0}}\right]_0^{\infty} \\ &= \frac{a_0^3}{4} \cdot \frac{12 a_0^2}{4} = \frac{3a_0^5}{4} \end{aligned}$$

θ integral:

$$\begin{aligned} &\int_0^{\pi} \alpha \cos^2\theta \sin\theta + \beta\left(1 - \cos^2\theta\right) \sin\theta \,\mathrm{d}\theta \\ &\qquad = \int_1^{-1} -\alpha \cos^2\theta - \beta\left(1 - \cos^2\theta\right) \mathrm{d}(\cos\theta) \\ &\qquad = \left[-\frac{\alpha\cos^3\theta}{3} - \beta\left(\cos\theta - \frac{\cos^3\theta}{3}\right)\right]_1^{-1} \\ &\qquad = \frac{2\alpha}{3} + \beta\left(2 - \frac{2}{3}\right) = \frac{2\alpha}{3} + \frac{4\beta}{3} \end{aligned}$$

So:

$$\begin{aligned}\Delta E_{1s} &= \frac{2}{a_0^3} \cdot \left(\frac{2\alpha}{3} + \frac{4\beta}{3}\right) \cdot \frac{3a_0^5}{4} \\ &= (\alpha + 2\beta)\, a_0^2 \\ &= 0 \quad \text{for } \beta = -\frac{\alpha}{2}\end{aligned}$$

For $2s$,

$$\Delta E_{2s} = \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \int_0^{\infty} \frac{1}{32\pi a_0^3} \left(2 - \frac{r}{a_0}\right)^2 e^{-\frac{r}{a_0}} r^2 \sin\theta \left(\alpha r^2 \cos^2\theta + \beta r^2 \sin^2\theta\right) \mathrm{d}r$$
$$= \ldots$$

Repeat for $2p$, $3s$, $3p$, $3d$. When $\beta = -\frac{\alpha}{2}$, shifting is null.

(d) **(TO EXPAND)** $\Delta\hat{H}_z = g_J \mu_\mathrm{B} \hat{\mathbf{J}} \cdot \mathbf{B}$

For $\mathbf{B} \parallel \hat{\mathbf{z}}$, $\hat{\mathbf{J}} \cdot \mathbf{B} \to M_J B$

For $\mathbf{B} \parallel \hat{\mathbf{x}}$, $\hat{\mathbf{J}} \cdot \mathbf{B} \to \hat{\mathbf{J}}_x \cdot B$ where $\hat{\mathbf{J}}_x = \frac{1}{2}\left[\hat{\mathbf{J}}_+ + \hat{\mathbf{J}}_-\right]$

Energy shift $\Delta E = \langle n| \Delta\hat{H}_z |n\rangle$

So $\Delta E = 0$ for $\mathbf{B} \parallel \hat{\mathbf{x}}$?

2. (DRAFT)

(a) The differential equations are the result of time-dependent perturbation theory of monochromatic light-matter interaction – c_1, c_2 are the amplitudes of each stationary state, $\hbar\Omega = \langle 2| e\mathcal{E}_0 x |1\rangle$ is Rabi frequency which characterises the transition frequeney in resonence. Rotating wave approximation is the dropping of term with $\omega + \omega_0$, since $|\omega - \omega_0| \ll (\omega + \omega_0)$ for optical transitions, causing the fast mode to be negligible.

(b)

$$
\begin{aligned}
c_2(t) &= \int_0^t -ie^{-i\delta t}\frac{\Omega}{2}\,\mathrm{d}t \quad \text{since } \tau_p \text{ is very short} \Rightarrow c_1 \simeq 1 \\
&= -\frac{i\Omega}{2}\left[\frac{e^{-i\delta t}}{-i\delta}\right]_0^t \\
&= -\frac{i\Omega}{2}\left(\frac{e^{-i\delta t}-1}{-i\delta}\right) \\
&= \frac{i\Omega}{\delta}e^{-i\frac{\delta}{2}t}\left(\frac{e^{-i\frac{\delta}{2}t}-e^{i\frac{\delta}{2}t}}{2i}\right) \\
&= -i\frac{i\Omega}{\delta}e^{-i\frac{\delta}{2}t}\sin\frac{\delta}{2}t \\
\Rightarrow |c_2(\tau_p)|^2 &= \frac{\Omega^2}{\delta^2}\sin^2\left(\frac{\delta\tau_p}{2}\right)
\end{aligned}
$$

In absence of radiation, $c_2 \to c_2 e^{i\delta t}$ since it is stationary (only relative phase matter so choose c_1 to have 0 phase).

After T, $c_2 \to c_2 e^{i\delta T} = c_2(T)$.

Repeating the same calculation above gives:

$$
\begin{aligned}
c_2(T+\tau_p) &= c_2(T) + \frac{-i\Omega}{\delta}\sin\left(\frac{\delta\tau_p}{2}\right)e^{-i\frac{\delta}{2}t} \\
&= -\frac{i\Omega}{\delta}\sin\left(\frac{\delta\tau_p}{2}\right)e^{i\frac{\delta}{2}\tau_p}\left[1+e^{i\delta T}\right] \\
\Rightarrow |c_2(t)|^2 &= \frac{\Omega^2}{\delta^2}\sin^2\left(\frac{\delta\tau_p}{2}\right)\cos^2\left(\frac{\delta T}{2}\right)
\end{aligned}
$$

(c) **(TO EXPAND)** Hyperfine $\Delta E = -g_I\mu_\mathrm{N}\mathbf{I}\cdot\mathbf{B}_e$

Assuming the Interval Rule holds, transition energy $A \propto g_I\mu_\mathrm{N}$

$$
\begin{aligned}
\Rightarrow \omega_0 &\simeq \frac{g_I\mu_\mathrm{N}}{\hbar} \\
&= 1.92\times 10^4\,\mathrm{T^{-1}\,s^{-1}}
\end{aligned}
$$