UNOFFICIAL SOLUTIONS BY TheLongCat

C2: LASER SCIENCE AND QUANTUM INFORMATION PROCESSING

TRINITY TERM 2016

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Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

5. Qubit gates and their fidelities.

(a) Density operator: $\hat{\rho} = \sum_n P_n \left|\psi_n\right\rangle \left\langle\psi_n\right|$

Therefore we have density matrix:

$$\begin{aligned}\rho &= \left|\psi\right\rangle \left\langle\psi\right| \\ &= \begin{pmatrix}\alpha \\ \beta\end{pmatrix} \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} \qquad \text{for } \left|\psi\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle \\ &= \begin{pmatrix}|\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2\end{pmatrix}\end{aligned}$$

Conjugate of ρ:

$$\begin{aligned}\rho^\dagger &= \left(\left|\psi\right\rangle\left\langle\psi\right|\right)^\dagger \\ &= \left(\left\langle\psi\right|\right)^\dagger \left(\left|\psi\right\rangle\right)^\dagger \\ &= \left|\psi\right\rangle\left\langle\psi\right| = \rho\end{aligned}$$

Thus ρ is Hermitian.

Also $\rho^2 = \left|\psi\right\rangle \left\langle\psi|\psi\right\rangle \left\langle\psi\right| = \rho \Rightarrow \mathrm{tr}(\rho^2) = \mathrm{tr}(\rho) = 1$.

In this case ρ is unitary, however for a mixed state $\rho^2 \neq \rho$ hence this is not general.

(b) We make definition `TRANSPOSE` $\equiv T$ for ease of writing.

Then we have:

$$\begin{aligned}T\rho &= \rho^T \\ &= \begin{pmatrix}|\alpha|^2 & \alpha^*\beta \\ \alpha\beta^* & |\beta|^2\end{pmatrix} \\ &= \begin{pmatrix}\alpha^* \\ \beta^*\end{pmatrix} \begin{pmatrix}\alpha & \beta\end{pmatrix} \\ &= \left|\psi^T\right\rangle \left\langle\psi^T\right|\end{aligned}$$

So $\left|\psi^T\right\rangle$ has its amplitudes in computational basis conjugated from $\left|\psi\right\rangle$.

For state on z-axis, we pick $\left|\psi\right\rangle = \left|0\right\rangle$.

For y-axis, we pick $\left|\psi\right\rangle = \left|+\right\rangle = (\left|0\right\rangle + \left|1\right\rangle)/\sqrt{2}$.

For x-axis, we pick $\left|\psi\right\rangle = \left|R\right\rangle = (\left|0\right\rangle + \mathrm{e}^{i\pi/2}\left|1\right\rangle)/\sqrt{2}$.

$\left\|\psi\right\rangle$	$\left\|\psi^T\right\rangle$
$\begin{pmatrix}1 \\ 0\end{pmatrix}$	$\begin{pmatrix}1 \\ 0\end{pmatrix}$
$\begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{pmatrix}$
$\begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}}\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}}\end{pmatrix}$

For $T \sim \mathbb{1}$, we have $\mathbb{1}\,|\psi\rangle = |\psi\rangle$ for all basis, thus:

$$\mathcal{F} = \begin{cases} 1 & \text{for z, x} \\ \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \right|^2 = 0 \end{cases}$$

So average $\mathcal{F} = {}^2\!/_3 + 0 = {}^2\!/_3$.

For $T \sim \mathrm{X}$, we have:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \to \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \to \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Hence we have:

$$\begin{aligned} \mathcal{F}\left(\langle\psi^T|\,,\, \mathrm{X}\,|\psi\rangle\right) &= \frac{1}{3}\left[\left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2 + \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2 + \left| \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \right|^2 \right] \\ &= \frac{2}{3} \end{aligned}$$

For $T \sim \texttt{MEASUREMENT}$, we have:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \to \begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix}$$
$$\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{pmatrix} \to \begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix}$$

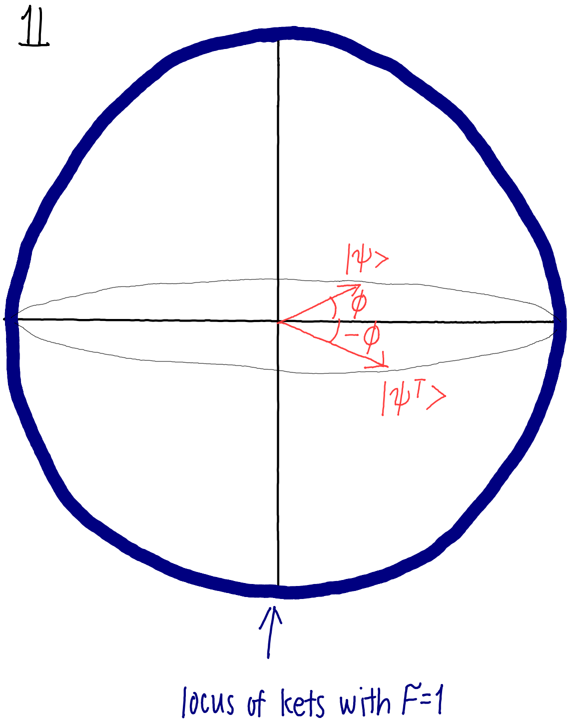
So we have $\mathcal{F}_z = 1$, $\mathcal{F}_x = \mathcal{F}_y = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \\ & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$.

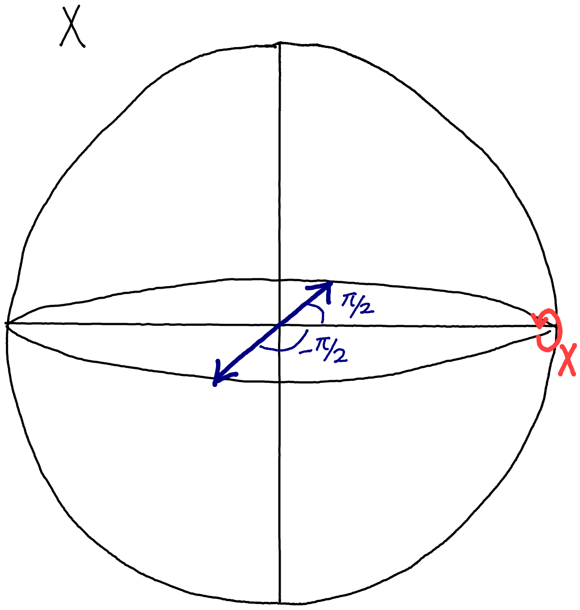
Thus average $\mathcal{F} = {}^1\!/_3 + {}^2\!/_3 \times {}^1\!/_2 = {}^2\!/_3$.

For $\mathbb{1}$, since conjugation relates to the azimuthal inversion, the set of kets that satisfies $\mathcal{F} = 1$ would be the ones wiht azimuthal angle $\phi = 0$.

For X, it would be the ones with $\phi = \pm\pi/2$.

For `MEASUREMENT`, it would be the basis $|0\rangle$ and $|1\rangle$.

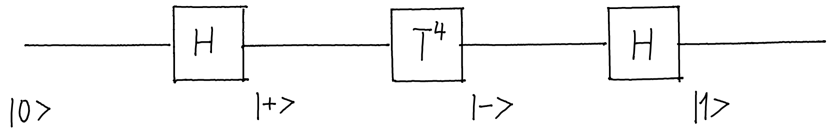




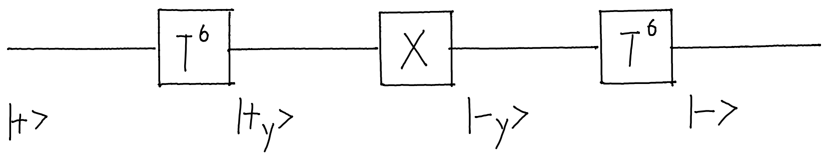
6. Grover algorithm and some nasty NMR computing at the end!

(a) Universal single qubit gates are the set of gates that form the basis of constructing a general qubit operation to an arbitrary accuracy, in this case H and T gates may replicate other single qubit operations to F 1.

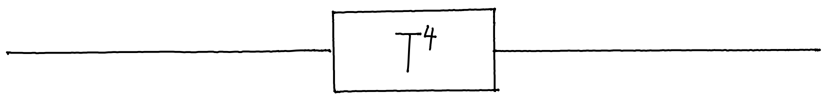
X:



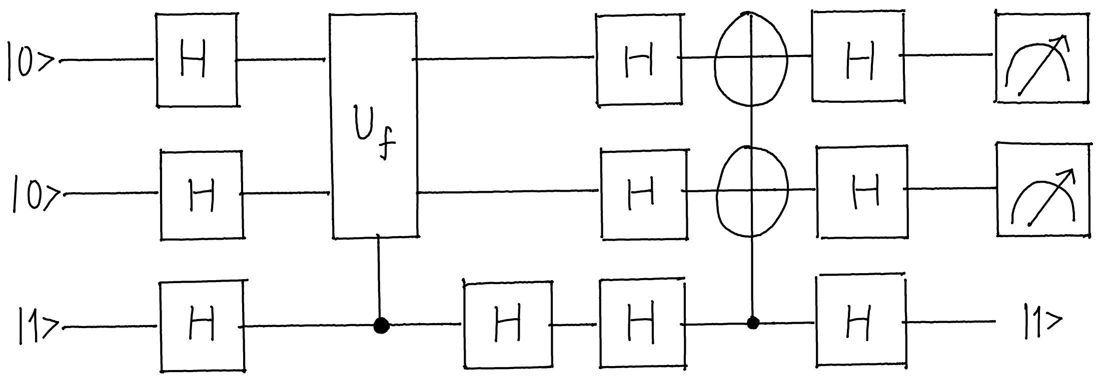
Y:



Z:



(b) Grover for $n = 2$: $f : \{0,\, 1\}^2 \to \{0,\, 1\}$ only one x gives $f(x) = 1$.

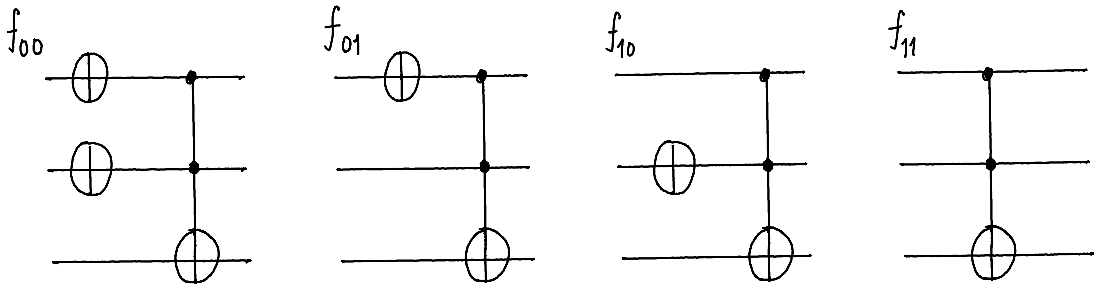


Note that the operation $|x\rangle\,|y\rangle \to |x\rangle\,|y \oplus f(x)\rangle$ with $x \equiv$ the input, $y \equiv$ the ancilla becomes $|x\rangle\,(|0\rangle - |1\rangle) \to |x\rangle\,(|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle) = (-1)^{f(x)}\,|x\rangle\,(|0\rangle - |1\rangle)$ ignoring normalisation factor.

Due to the linearity of Hilbert space, the phase $(-1)^{f(x)}$ may be thought of as being applied onto the superposition of inputs while leaving the ancilla unchanged, hence we may largely ignore ancilla in this algorithm for simplicity.

A promise about f is a specific fact that we know about f, without knowing the function itself. In Grover, the promise is that only one of the inputs returns $f(x) = 1$.

We may then write f_{jk} as the satisfying input being $|jk\rangle$.



After the oracle, we have states (in subspace of input qubits):

$$\begin{aligned}|x\rangle &= |00\rangle + |01\rangle - |10\rangle + |11\rangle \\ &= \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix}\end{aligned}$$

We then have propagator:

$$\begin{aligned}\mathrm{H}^{\otimes 2} \cdot \texttt{TOFFOLI} \cdot \mathrm{H}^{\otimes 2} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}\end{aligned}$$

Acting this on $|x\rangle$:

$$\begin{aligned}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \\ 1 \end{pmatrix} &= \begin{pmatrix} 0 \\ 4 \\ 0 \\ 0 \end{pmatrix} \\ &\propto |01\rangle\end{aligned}$$

Hence we obtain a unique output here, though a further $\texttt{NOT}^{\otimes 2}$ is required to get the correct answer.

(c) NMR Hamiltonian:

$$\mathcal{H} = \hbar\omega_1 \frac{\sigma_{1z}}{2} + \hbar\omega_2 \frac{\sigma_{2z}}{2} + \hbar\omega_{12} \frac{\sigma_{1z} \cdot \sigma_{2z}}{4}$$

To implement f_{11}, we exploit the J coupling between qubits to apply the Toffoli gate via a π pulse at ω_{12}.

As shown in the diagram above, applying a π pulse at ω_1 and ω_2 to flip qubits 1 and 2 before the π pulse at ω_{12} should achieve the function f_{00}.

7. **(DRAFT)** Classic GHZ question.

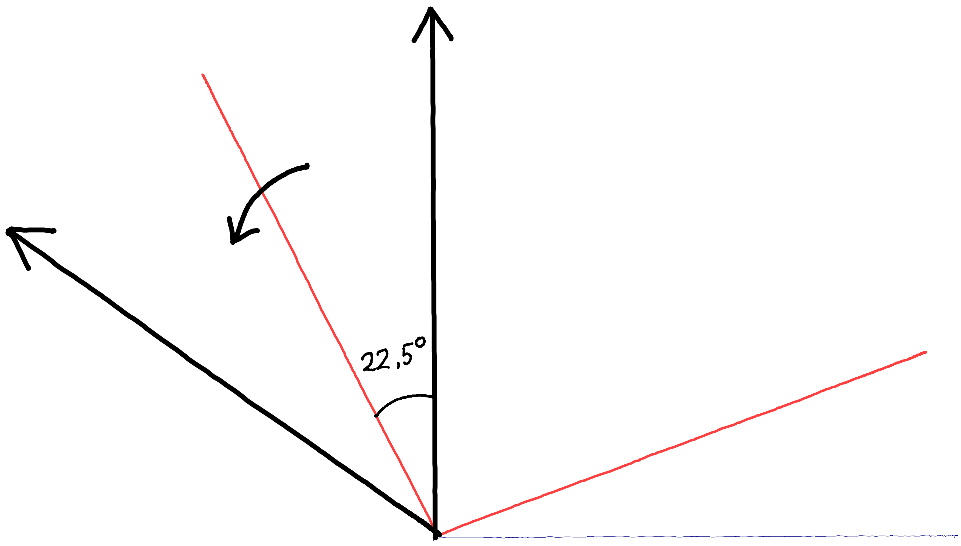
(a) **(TO EXPAND)** Noting that vertical and horizontal polarisations are orthogonal, we make identification:

$$|0\rangle \equiv |\mathrm{H}\rangle \qquad |1\rangle \equiv |\mathrm{V}\rangle$$

The states $|\pm\rangle$ are then $|\mathrm{H}\rangle \pm |\mathrm{V}\rangle$, i.e. polarisation 45° to horizontal/vertical.

Similarly, $|R\rangle$ and $|L\rangle$ are $|\mathrm{H}\rangle \pm i\,|\mathrm{V}\rangle$, i.e. right/left handed polarisation.

A Hadamard gate may be achieved by placing a π waveplate 22.5° to horizontal as shown below:



A controlled phase gate may be achieved by adjusting the angle and phase of a waveplate by Pockels effect. The control should come from?

The main reasons why a 2-photon gate in such basis is difficult to implement is due to its sensitivity to polarisation angle, a slight error could render the gate unusable.

(b) Ignoring normalisation:

$$\begin{aligned}
|000\rangle &\xrightarrow{\mathrm{H}^{\otimes 3}} |{+}{+}{+}\rangle = |000\rangle + |001\rangle + \ldots + |111\rangle \\
&\xrightarrow{\mathrm{CZ}_{13}} |000\rangle + \ldots + |100\rangle - |101\rangle + |110\rangle - |111\rangle \\
&\xrightarrow{\mathrm{CZ}_{12}} |000\rangle + \ldots + |100\rangle - |101\rangle - |110\rangle + |111\rangle = |0\rangle\,|{+}{+}\rangle + |1\rangle\,|{-}{-}\rangle \\
&\xrightarrow{\mathbb{1}\otimes\mathrm{H}^{\otimes 2}} |000\rangle + |111\rangle
\end{aligned}$$

So we have generated a GHZ state in the ZZZ basis.

To rewrite the GHZ state in other bases, we first note the following transformations:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}} \quad \text{in X basis} \qquad = \frac{|R\rangle + |L\rangle}{\sqrt{2}} \quad \text{in Y basis}$$

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}} \quad \text{in X basis} \qquad = \frac{|R\rangle - |L\rangle}{\sqrt{2}} \quad \text{in Y basis up to a global phase of } i$$

So we may rewrite the GHZ state as:

$$\begin{aligned}|\mathrm{GHZ}\rangle &= (|+\rangle + |-\rangle)(|R\rangle + |L\rangle)(|R\rangle + |L\rangle) + (|+\rangle - |-\rangle)(|R\rangle - |L\rangle)(|R\rangle - |L\rangle) \\ &= |{+RR}\rangle + |{+RL}\rangle + |{+LR}\rangle + |{+LL}\rangle + |{-RR}\rangle + \ldots + \\ &\quad |{+RR}\rangle - |{+RL}\rangle - \ldots + |{+LL}\rangle - |{-RR}\rangle + \text{cross terms} - |{-LL}\rangle \\ &= |+\rangle\left[|RR\rangle + |LL\rangle\right] - |-\rangle\left[|LR\rangle + |RL\rangle\right] \quad \text{in XYY basis}\end{aligned}$$

Similarly we have:

$$\begin{aligned}|\mathrm{GHZ}\rangle &= |R+R\rangle + |L+L\rangle - |L-R\rangle - |R-L\rangle \quad \text{in YXY} \\ &= \left[|RR\rangle + |LL\rangle\right]|+\rangle - \left[|LR\rangle + |RL\rangle\right]|-\rangle \quad \text{in YYX}\end{aligned}$$

Note that there is an entanglement between the X and Y pairs – if Y is measured to be the same $\Rightarrow$ X must be $|+\rangle$ and vice versa.

In XXX basis:

$$\begin{aligned}|\mathrm{GHZ}\rangle &= (|+\rangle + |-\rangle)^{\otimes 3} + (|+\rangle - |-\rangle)^{\otimes 3} \\ &= |{+}{+}{+}\rangle + |{+}{-}{-}\rangle + |{-}{-}{+}\rangle + |{-}{+}{-}\rangle \quad \text{noting that odd powers of } |-\rangle \text{ cancels} \\ &= |+\rangle|\Phi^+\rangle + |-\rangle|\Psi^+\rangle\end{aligned}$$

Hence a measurement in X basis reveals a maximally entangled state, which is known for violating the CHSH inequality, i.e. $B = 2\sqrt{2} > 2$ as predicted by local realism.

However the slight difference is that the Bell state is encapsulated in the GHZ state when measured in X basis, as opposed to being an inherent feature in computational basis.