UNOFFICIAL SOLUTIONS BY TheLongCat

C2: LASER SCIENCE AND QUANTUM INFORMATION PROCESSING

TRINITY TERM 2022

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Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. Laser modelocking – Simon's self study problem set is a good starting point.

(a) Q-switching: the overpumping ratio $r = \dfrac{N_i^*}{N_{\text{th}}^*}$, where N_i^* is the initial population inversion before the pulse and N_{th}^* is the threshold population inversion, governs the Q pulse profile.

N_{th}^* is dependent on the size of cavity and the population kinematics of the gain medium (i.e. the β factor).

Modelocking: multiple modes are synchronised in phase to produce an intense, short pulse. The bandwidth of the laser system, i.e. the number of modes in the cavity, governs the laser pulse length (strictly for inhomogeneous medium).

Modelocking tends to produce short pulse length as it is easier to excite more modes than shortening the cavity length down to the µm scale.

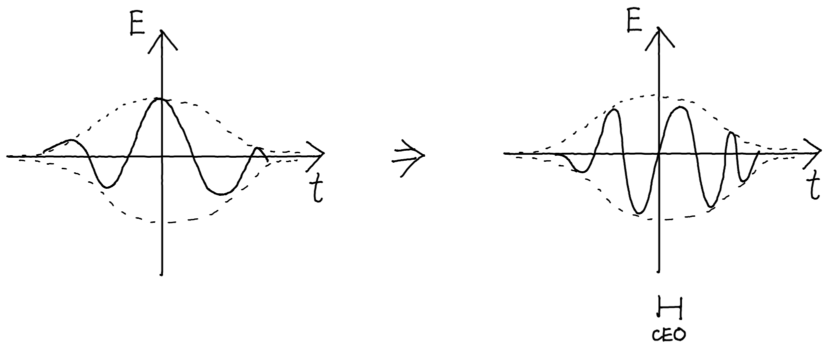
(b) Each mode have field component $E_p(t) = a(\omega_p)\mathrm{e}^{i(kz-\omega_p t+\phi_0)}$, setting the origin at the output coupler then gives the total field:

$$\begin{aligned} E(t) &= \sum_p a(\omega_p)\mathrm{e}^{-i(\omega_{ce}+p\Delta\omega)t}\mathrm{e}^{i\phi_0} \\ &= \underbrace{\mathrm{e}^{-i(\omega_{ce}-\phi_0)}}_{\text{carrier}}\underbrace{\sum_p a(\omega_p)\mathrm{e}^{-ip\Delta\omega t}}_{\text{envelope}} \end{aligned}$$

The intensity is then:

$$\begin{aligned} I(t) &\propto |E(t)|^2 \\ &= \sum_p\sum_q a(\omega_p)a^*(\omega_q)\mathrm{e}^{-i(p-q)\Delta\omega t} \end{aligned}$$

Note that we have a maximum when $\Delta\omega t = 2\pi m$ where m is an integer. Hence $t_m = \frac{2\pi}{\Delta\omega}m$.



CEO offset is the phase difference between the carrier wave and the envelope after 1 optical cycle, this is important when the envelope contains few carrier peaks (i.e. $\omega_{ce} \simeq \Delta\omega$).

Substitute t_m into $E(t)$ then gives:

$$\begin{aligned} E(t_m) &= \mathrm{e}^{-i(\frac{2\pi\omega_{ce}}{\Delta\omega}m-\phi_0)}\sum_p a(\omega_p)\mathrm{e}^{-ip\cdot 2\pi m} \\ &= \mathrm{e}^{\phi_{\text{slip}}\cdot m-\phi_0}\sum_p a(\omega_p) \quad \text{where } \phi_{\text{slip}} = \frac{2\pi\omega_{ce}}{\Delta\omega} \end{aligned}$$

(c) From the given spectral component we have:

$$a(\omega_p) = A \exp\left[-\left(\frac{p\Delta\omega + \omega_{ce} - \omega_0}{\Delta\omega_b}\right)^2\right]$$

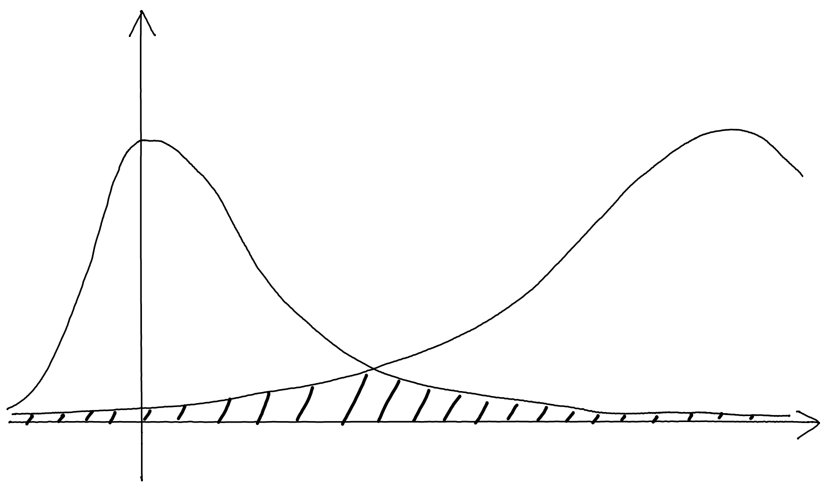
The resulting E field is then:

$$\begin{aligned}
E(t) &= \mathrm{e}^{-i(\omega_{ce}t - \phi_0)} \sum_p A\mathrm{e}^{-\left(\frac{p\Delta\omega + \omega_{ce} - \omega_0}{\Delta\omega_b}\right)^2} \mathrm{e}^{-ip\Delta\omega t} \\
&\simeq \mathrm{e}^{-i(\omega_{ce}t - \phi_0)} \frac{A}{\Delta\omega} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{\Omega^2}{\Delta\omega_b^2}} \mathrm{e}^{-i\Omega t} \mathrm{e}^{-i(\omega_{ce}+\omega_0)t}\, \mathrm{d}\Omega \quad \text{where } \Omega = p\Delta\omega + \omega_{ce} - \omega_0 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{d}\Omega = \mathrm{d}(p\Delta\omega) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{since } p \text{ is large and } a \text{ is} \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{bounded} \\
&= \mathrm{e}^{-i(\omega_{ce}t - \phi_0 - \omega_{ce}t + \omega_0 t)} \cdot \frac{A}{\Delta\omega} \int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{\Delta\omega_b^2}\Omega^2} \mathrm{e}^{-i\Omega t}\, \mathrm{d}\Omega \\
&= \frac{A}{\Delta\omega} \mathrm{e}^{-i(\omega_0 t - \phi_0)} \cdot \frac{\sqrt{\pi}}{1/\Delta\omega_b} \cdot \mathrm{e}^{-\left(\frac{t}{2/\Delta\omega_b}\right)^2} \\
&= \underbrace{\frac{A}{\Delta\omega}\sqrt{\pi}\Delta\omega_b \mathrm{e}^{-(\Delta\omega_b t/2)^2}}_{F(t)} \underbrace{\mathrm{e}^{-i(\omega_0 t - \phi_0)}}_{\psi = \phi_0}
\end{aligned}$$

Note that this form has no dependence on ω_{ce}, it is only the parameter $\Delta\omega_b$ and ω_0 that determines the temporal profile of the pulse. Also E is inversely proportional to $\Delta\omega$.

(d) Frequency doubling $\omega_p \to 2\omega_p$ so we have:

$$a(2\omega_p) = A \exp\left[-\left(\frac{2p\Delta\omega + 2\omega_{ce} - \omega_0}{\Delta\omega_b}\right)\right]$$

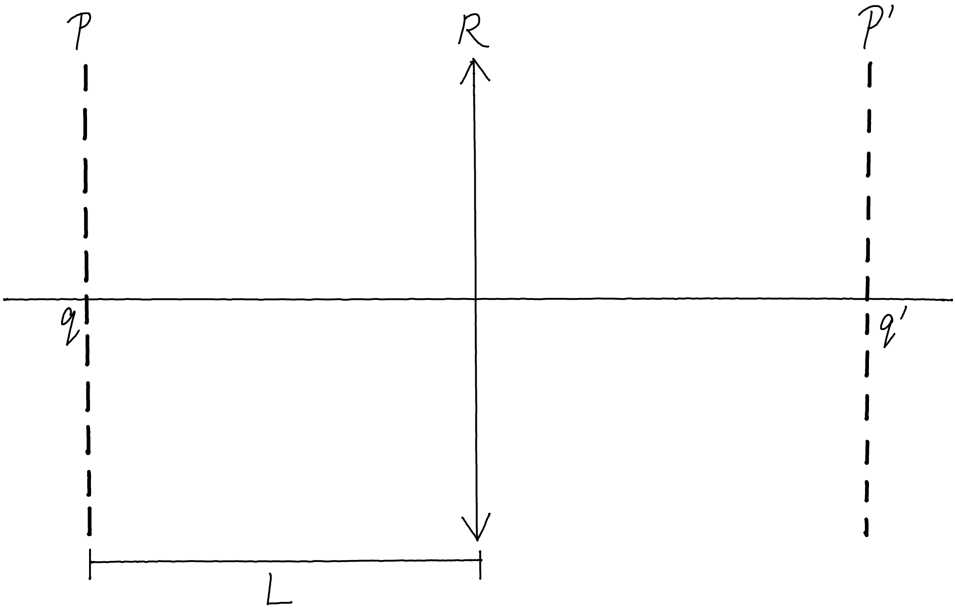


Noting the Gaussian form and letting $\Delta\omega_b$ goes large shows that there will be an overlap between the 2 spectrum, the minimum frequency difference is simply between like p's:

$$2\omega_{ce} - \omega_0 - \omega_{ce} + \omega_0 = \omega_{ce}$$

To achieve $\phi_{\text{slip}} = 0$, we may frequency double the output, measure the beating frequency between it and the original output, and adjust the cavity length such that there is no longer any beating, suggesting that $\omega_{ce} = 0 \Rightarrow \phi_{\text{slip}} = 0$.

2. Gaussian beam and low-loss mode conditions.



(a) Replacing the mirror with a lens gives:

$$
\begin{aligned}
\begin{pmatrix} q' \\ 0 \end{pmatrix} &= \mathcal{UF}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ -\dfrac{1}{f} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} q \\ 1 \end{pmatrix} \\
&= \mathcal{UF}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ -\frac{1}{f} & 1-\dfrac{L}{f} \end{pmatrix}\begin{pmatrix} q \\ 1 \end{pmatrix} \\
&= \mathcal{UF}\begin{pmatrix} 1-\dfrac{L}{f} & L+L-\dfrac{L^2}{f} \\ -\dfrac{1}{f} & 1-\dfrac{L}{f} \end{pmatrix}\begin{pmatrix} q \\ 1 \end{pmatrix} \\
\Rightarrow q' &= \frac{(1-L/f)q+2L-L^2/f}{-q/f+1-L/f} \\
&= \frac{(1-L/f)iz_R+(2L-L^2/f)}{(1-L/f)-iz_R/f} \\
&= \frac{[(2L-L^2/f)+iz_R(1-L/f)]\,[(1-L/f)+iz_R/f]}{(1-L/f)^2+(z_R/f)^2} \\
&= \frac{(2L-L^2/f)(1-L/f)-z_R^2(1-L/f)/f+i\,[z_R(1-L/f)^2+z_R/f(2L-L^2/f)]}{(1-L/f)^2+(z_R/f)^2}
\end{aligned}
$$

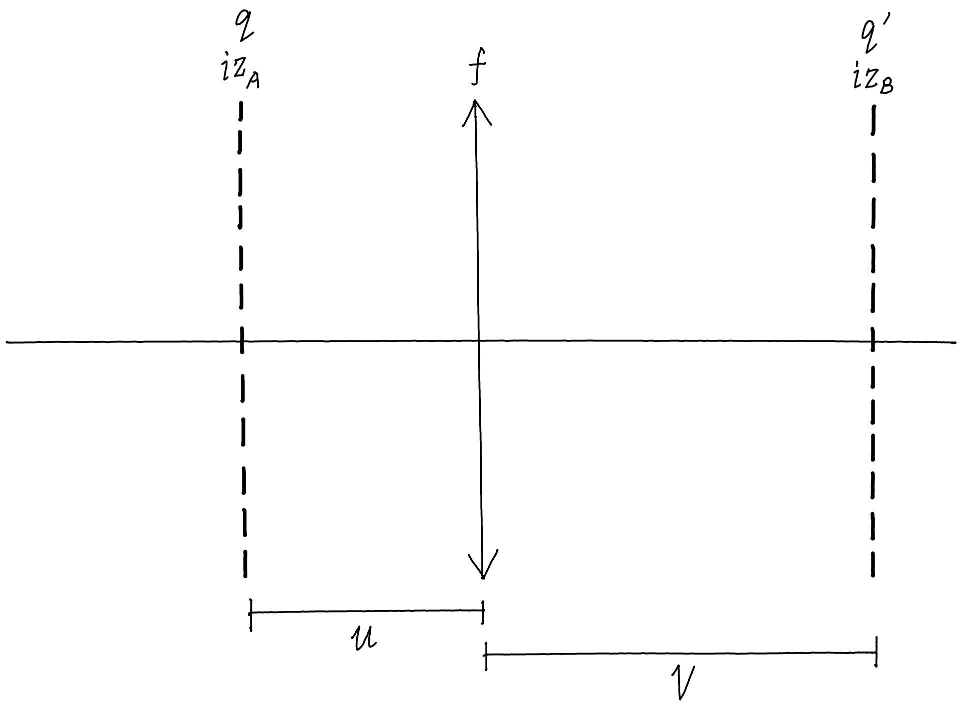
Note that the real part vanishes for $R = 2L = 2f$, hence the reflected beam has the waist on $\mathcal{P}$ (by reflecting $\mathcal{P}'$ to $\mathcal{P}$).

Now the imaginary part should be:

$$\begin{aligned} z'_R &= \frac{z_R/L(2L-L)}{0+(z_R/L)^2} \\ &= z_R \cdot \frac{L^2}{z_R^2} \\ &= \frac{L^2}{z_R} \end{aligned}$$

For $R \neq 2L$, we demand $q' = q \Rightarrow z'_R = z_R$ since the waist position is unchanged:

$$\begin{aligned} z_R &= \frac{(1-L/f)^2 z_R + z_R/f(2L-L^2/f)}{(1-L/f)^2+(z_R/f)^2} \\ &= \left(1-\frac{L}{f}\right)^2 + \left(\frac{z_R}{f}\right)^2 = \left(1-\frac{L}{f}\right)^2 + \frac{2L-L^2/f}{f} \\ \Rightarrow z_R^2 &= f^2\frac{2L-L^2/f}{f} \\ &= Lf\left(2-\frac{L}{f}\right) \\ &= LR\left(1-\frac{L}{R}\right) \\ \Rightarrow z_R &= \sqrt{LR\left(1-\frac{L}{R}\right)} \end{aligned}$$



(b) Similarly we have:

$$\begin{aligned}
\begin{pmatrix} q' \\ 1 \end{pmatrix} &= \mathcal{UF} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\dfrac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ 1 \end{pmatrix} \\
\Rightarrow \begin{pmatrix} q' \\ 1 \end{pmatrix} &= \mathcal{UF} \begin{pmatrix} 1 - \dfrac{v}{f} & v \\ -\dfrac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} q + u \\ 1 \end{pmatrix} \\
q' &= \frac{q + u - (q+u)/fv + v}{-(q+u)/f + 1} \\
&= \frac{f(u+v) - uv + iz_A(f - v)}{f - u - iz_A} = iz_B \\
\Rightarrow i(f-u)z_B + z_A z_B &= fu + fv - uv + iz_A(f - v)
\end{aligned}$$

Equating real part gives:

$$\begin{aligned}
z_A z_B &= uf + v(f - u) \\
\Rightarrow v(u - f) + z_A z_B &= uf \qquad (1)
\end{aligned}$$

Equating imaginary part gives:

$$\begin{aligned}
(f - u)z_B &= z_A(f - v) \\
\Rightarrow vz_A - z_B(u - f) &= z_A f \qquad (2)
\end{aligned}$$

From (2):

$$\begin{aligned}
z_A v &= z_A f + z_B \delta \quad \text{where } \delta = f - u \\
\overset{(1)}{\Longrightarrow} -\frac{\delta}{z_A}\left(z_A f + z_B \delta\right) + z_A z_B &= (f - \delta)\, f \\
-f\delta - \frac{z_B}{z_A}\delta^2 + z_A z_B &= f^2 - \delta f \\
\delta &= \sqrt{\frac{z_A}{z_B}\left(f^2 - z_A z_B\right)}
\end{aligned}$$

We want δ to be real, so $f^2 > z_A z_B$.

(c) Rayleigh range of the 1st cavity: $z_1 = \sqrt{L_1 R_1 (1 - L_1/R_1)} = 1000\,\text{mm}$.

Rayleigh range of the 2nd cavity: $z_2 = \sqrt{L_2 R_2 (1 - L_2/R_2)} = 866\,\text{mm}$.

From b, we have:

$$\begin{aligned}
\delta &= \sqrt{\frac{z_1}{z_2}\left(f^2 - z_1 z_2\right)} = u - f = 126.4\,\text{mm} \\
\Rightarrow u &= 1.626\,\text{m}
\end{aligned}$$

Also $v = f + z_2/z_1 \delta = 12.45\,\text{m}$.

(d) Compared to a symmetrical confocal cavity, this system may help us to focus the Gaussian beam down or defocusing it, thereby enabling us to control the laser beam size.