UNOFFICIAL SOLUTIONS BY TheLongCat

B3. ATOMIC AND LASER PHYSICS

TRINITY TERM 2022

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Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. (DRAFT)

(a) In many-e$^-$ systems, by separating the atomic Hamiltonian $\hat{H}$ into a central field $\hat{H}_{\mathrm{CF}}$ and a residual electrostatic $\Delta\hat{H}_{\mathrm{RE}}$ parts, the systems may be treated as an H-like system with perturbation from non-central components.

Further including spin-orbit interaction $\Delta\hat{H}_{\mathrm{SO}}$ then gives rise to the LS coupling scheme, provided that $\Delta\hat{H}_{\mathrm{SO}} \ll \Delta\hat{H}_{\mathrm{RE}}$ so spin-orbit may be treated as a secondary perturbation after $\Delta\hat{H}_{\mathrm{RE}}$.

In this scheme, $\hat{\mathbf{L}}$ the total angular momentum is conserved as the inter-e$^-$ repulsion is an internal interaction. Then by the conservation of total angular momentum $\hat{\mathbf{J}}$, the spin-orbit interaction should only involve the precession of $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ about $\hat{\mathbf{J}}$, making $\hat{\mathbf{L}}$, $\hat{\mathbf{S}}$, $\hat{\mathbf{J}}$ good operators.

E-dipole selection rules:

General	LS
$\Delta n =$any	$\Delta L = 0, \pm 1$ $(0 \nrightarrow 0)$
$\Delta l = \pm 1$	$\Delta S = 0$
	$\Delta J = 0, \pm 1$ $(0 \nrightarrow 0)$
	$\Delta M_J = 0, \pm 1$ $(0 \nrightarrow 0$ iff $\Delta J = 0)$

For configuration $5s^2$, $L = 0$, $S = 0 \rightarrow$ term ${}^1S \rightarrow$ level ${}^1S_0$.

$5s5p$:

$$L = 1, \begin{matrix} S = 0 \\ S = 1 \end{matrix} \rightarrow \text{terms} \begin{matrix} {}^1P \\ {}^3P \end{matrix} \rightarrow \text{levels} \begin{matrix} {}^1P_1 \\ {}^3P_0,\ {}^3P_1,\ {}^3P_2 \end{matrix}$$

$5s6s$:

$$L = 0, \begin{matrix} S = 0 \\ S = 1 \end{matrix} \rightarrow \text{terms} \begin{matrix} {}^1S \\ {}^3S \end{matrix} \rightarrow \text{levels} \begin{matrix} {}^1S_0 \\ {}^3S_1 \end{matrix}$$

(b) Assuming e-dipole transitions, $\lambda_0 = 461\,\mathrm{nm}$ in absorption $\Rightarrow$ transitions from ground state since $k_{\mathrm{B}}T \sim 0.1\,\mathrm{eV}$.

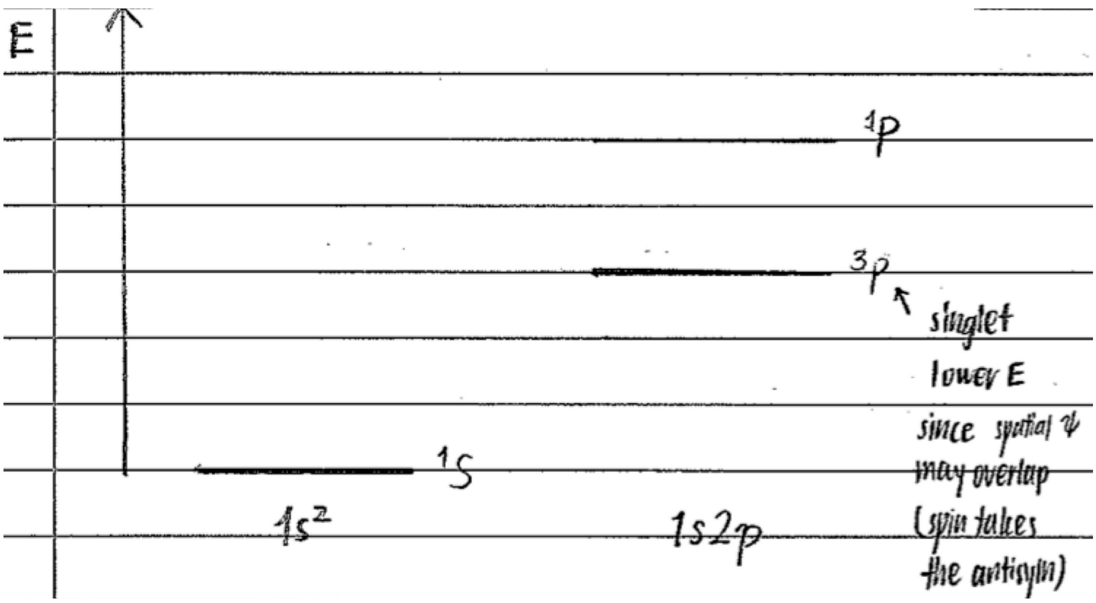
The only possible transition is $5s^2\ {}^1S_0 \rightarrow 5s5p\ {}^1P_1$.

$679\,\mathrm{nm}$, $688\,\mathrm{nm}$ and $707\,\mathrm{nm}$ are very close to one another, therefore the transitions must involve the $5s5p\ {}^3P$ terms.

The allowed transitions are:

$$\begin{aligned} 6s5p\ {}^3P_0 &\leftrightarrow 5s6s\ {}^3S_1 \\ {}^3P_1 &\leftrightarrow {}^3S_1 \\ {}^3P_2 &\leftrightarrow {}^3S_1 \end{aligned}$$

So $1.12\,\mathrm{\mu m}$ corresponds to the transition $5s5p\ {}^1P_1 \leftrightarrow 5s6s\ {}^1S_0$.



Weak 689 nm $\Rightarrow$ non e-dipole transition so $^1S_0 \to {}^3P_1$.

(c) $5s5d$

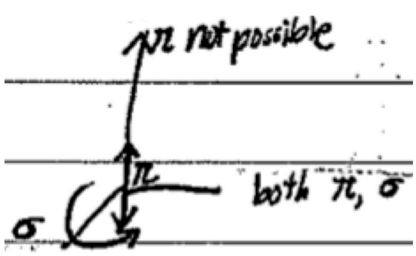
$$L = 2 \begin{matrix} S=0 \to \text{ term } ^1D \\ S=1 \to \text{ term } ^3D \end{matrix} \to \text{ levels} \begin{matrix} ^1D_2 \\ ^3D_1, {}^3D_2, {}^3D_3 \end{matrix}$$

For e-dipole transitions, possible transitions:

$$\begin{array}{rl} 5s5p\ \ {}^1P_1 \leftrightarrow 5s5d\ \ {}^1D_2 & 3 \text{ (Normal)} \\ {}^3P_0 \leftrightarrow {}^3D_1 & 3 \\ {}^3P_1 \leftrightarrow {}^3D_1 & 6 \\ {}^3P_1 \leftrightarrow {}^3D_2 & 9 \\ {}^3P_2 \leftrightarrow {}^3D_1 & 9 \\ {}^3P_2 \leftrightarrow {}^3D_2 & \vdots \\ {}^3P_2 \leftrightarrow {}^3D_3 & \vdots \end{array}$$

For photons perpendicular to B field, transition with $\Delta M_J = 0, \pm 1$ is expected by conservation of angular momentum.

So the number of splittings observed should correspond to the degeneracies of the level $5s5p\ \ {}^3P_1 \leftrightarrow 5s6s\ \ {}^3D_1$ (together with anomalous effect so $\times 2$).

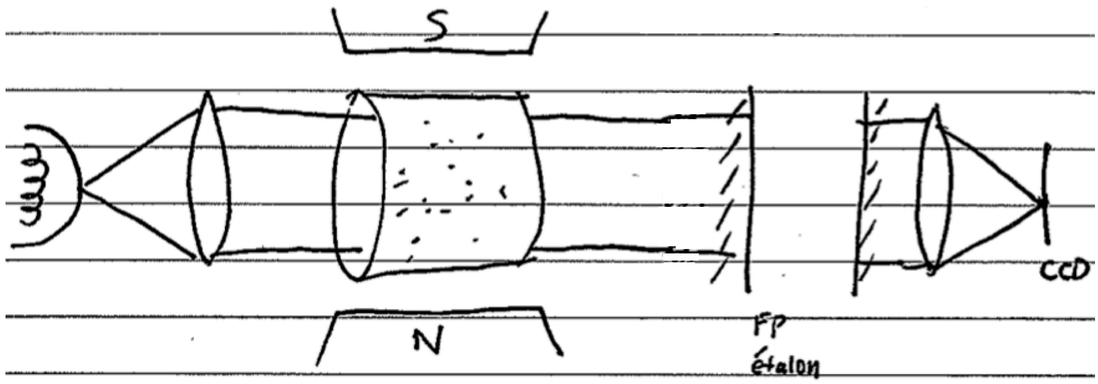


$$\Delta M_J = \underbrace{0}_{\pi}, \underbrace{\pm 1}_{\sigma}$$

Can argue g_J diff since S same, L diff.

(d) **(TO EXPAND)** Note that Zeeman splitting is extremely small compared to the level splitting, so a high precision optical apparatus such as a Fabry-Pérot étalon should be used to distinguish between the fine splitting.

Sketch of experimental setup:



2. (DRAFT)

(a) Propose an operator $\hat{\mathbf{F}} = \hat{\mathbf{I}} + \hat{\mathbf{J}}$ and by squaring we get:

$$\hat{\mathbf{F}}^2 = \hat{\mathbf{I}}^2 + \hat{\mathbf{J}}^2 + 2\hat{\mathbf{I}} \cdot \hat{\mathbf{J}}$$
$$\Rightarrow \hat{\mathbf{I}} \cdot \hat{\mathbf{J}} = \frac{1}{2}\left[\hat{\mathbf{F}}^2 - \hat{\mathbf{I}}^2 - \hat{\mathbf{J}}^2\right]$$

The addition of angular momenta also gives us $F = |I - J|, |I - J + 1|, \ldots, (I + J)$ as good quantum numbers.

Hence:

$$\begin{aligned}\left\langle \hat{H} \right\rangle &= A \left\langle \hat{\mathbf{I}} \cdot \hat{\mathbf{J}} \right\rangle \\ &= \frac{1}{2} A\left[F(F+1) - I(I+1) - J(J+1)\right]\end{aligned}$$

Now the step between adjacent energy levels:

$$\begin{aligned}\Delta E &= \frac{1}{2}A\left[(F+1)(F+2) - I(I+1) - J(J+1)\right] \\ &\quad - \frac{1}{2}A\left[F(F+1) - I(I+1) - J(J+1)\right] \\ &= \frac{1}{2}A\left[F^2 + 3F + 2 - F^2 - F\right] \\ &= A(F+1) = E_{F+1} - E_F\end{aligned}$$

Hence the interval rule.

In atoms there are 2 interactions of this sort: spin-orbit interaction between the electron magnetic moment and nuclear magnetic field in its rest frame, and hyperfine interaction between nuclear magnetic moment and electron magnetic moment.

For spin-orbit interactions, $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ couple so that the set of good quantum numbers is $|nl_i LSJM_J\rangle$.

For hyperfine interaction, $\hat{\mathbf{I}}$ and $\hat{\mathbf{J}}$, the set is $|nl_i IJFM_F\rangle$, where L is total orbital angular momentum, S is total spin, I is nuclear spin, J is total e$^-$ angular momentum, F is total atomic angular momentum.

(b) **(TO EXPAND)** The constant of proportionaity A is dependent on the nuclear g-factor g_I, which varies wildly with isotopes. It happens that europium has a large g_I that causes the hyperfine structure to be large. *s-e$^-$ penetrates nucleus more since $\psi(r = 0) \neq 0 \rightarrow$ more contribution.*

(c) **(TO EXPAND)** Interval Rule:

$$\Delta E_{F,F-1} \propto F$$
$$\Rightarrow \frac{\Delta E_{F,F-1}}{\Delta E_{F-1,F-2}} = \frac{F}{F-1}$$

$$\Delta E_{ab} = 2.82\,\text{GHz}$$
$$\Delta E_{bc} = 3.53\,\text{GHz}$$
$$\Delta E_{ce} = 4.24\,\text{GHz}$$

$$\frac{\Delta E_{ab}}{\Delta E_{bc}} = 0.80 \simeq \frac{4}{5}$$
$$\frac{\Delta E_{bc}}{\Delta E_{ce}} = 0.83 \simeq \frac{1}{1.2} = \frac{5}{6}$$

So the peaks between a, b, c, e obey the Interval Rule.

The next peak from e should then be $\Delta E_{e1} = \frac{7}{6}\Delta E_{ce} = 4.95\,\text{GHz}$ away.

So peak with position $10.60\,\text{GHz} - 4.95\,\text{GHz} = 5.65\,\text{GHz}$ is the next, i.e. peak i.

Similarly the next peak should be $\frac{8}{7}\cdot 4.95\,\text{GHz} = 5.66\,\text{GHz}$ away from i, and that corresponds to peak l.

Since the mass number is odd, we expect I to be half-integer, together with $J = \frac{11}{2}$, F should be an integer.

From the ratios:

$$F_{\max} = 8 \qquad F_{\min} = 4 - 1 = 3$$
$$\Rightarrow 2I = 8 - 3 = 5$$
$$\Rightarrow I = \frac{5}{2}$$

(d) Excluding the identified peaks, we should have the rest as the ones due to ^{153}Eu.

$$\begin{aligned} \Delta E_{df} &= 1.25 & \qquad \tfrac{\Delta E_{df}}{\Delta E_{fg}} &= 0.80 \simeq \tfrac{4}{5} \\ \Delta E_{fg} &= 1.57 & \tfrac{\Delta E_{fg}}{\Delta E_{gh}} &= 0.84 \simeq \tfrac{1}{1.2} = \tfrac{5}{6} \\ \Delta E_{gh} &= 1.88 \Rightarrow & \tfrac{\Delta E_{gh}}{\Delta E_{hj}} &= 0.85 \simeq \tfrac{1}{1.17} = \tfrac{6}{7} \\ \Delta E_{hj} &= 2.20 & \tfrac{\Delta E_{hj}}{\Delta E_{jk}} &= 0.88 \simeq \tfrac{1}{1.14} = \tfrac{7}{8} \\ \Delta E_{jk} &= 2.51 \end{aligned}$$

So similarly $F_{\max} = 8$, $F_{\min} = 3 \Rightarrow I = \frac{5}{2}$.

Recall $A \propto \boldsymbol{\mu}_I$, so pick ΔE of the same $F \rightarrow \frac{^157\mu_I}{^153\mu_I} = 2.2$.

3. (DRAFT)

(a) From TDSE:

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$$

$$\Rightarrow \begin{bmatrix} \dot{c}_1\psi_1 e^{-i\omega_1 t} + c_1\psi_1(-i\omega_1)e^{-i\omega_1 t} \\ + \dot{c}_2\psi_2 e^{-i\omega_2 t} + c_2\psi_2(-i\omega_2)e^{-i\omega_2 t} \end{bmatrix} = \begin{matrix} \left(\hat{H}_0 + \hat{D}_0\cos(\omega t)\right) \\ \left[c_1\psi_1 e^{-i\omega_1 t} + c_2\psi_2 e^{-i\omega_2 t}\right] \end{matrix} \quad \text{where } \omega_i = \frac{E_i}{\hbar}$$

$$\begin{matrix} i\hbar\dot{c}_1\psi_1 e^{-i\omega_1 t} + \cancel{\hbar\omega_1 c_1\psi_1 e^{-i\omega_1 t}} \\ i\hbar\dot{c}_2\psi_2 e^{-i\omega_2 t} + \cancel{\hbar\omega_2 c_2\psi_2 e^{-i\omega_2 t}} \end{matrix} = \begin{matrix} \cancel{E_1 c_1\psi_1 e^{-i\omega_1 t}} + \cancel{E_2 c_2\psi_2 e^{-i\omega_2 t}} \\ + \hat{D}_0\cos(\omega t)\left[c_1\psi_1 e^{-i\omega_1 t} + c_2\psi_2 e^{-i\omega_2 t}\right] \end{matrix}$$

$$\xRightarrow{\times\psi_1^*} i\hbar\dot{c}_1 \underbrace{|\psi_1|^2}_{\text{1 assuming normalised}} e^{-i\omega_1 t} = c_1 e^{-i\omega_1 t}\underbrace{\langle\psi_1|\hat{D}_0\cos(\omega t)|\psi_1\rangle}_{0} + c_2 e^{-i\omega_2 t}\langle\psi_1|\hat{D}_0\cos(\omega t)|\psi_2\rangle$$

$$\Rightarrow \dot{c}_1 = -\frac{i}{\hbar}c_2 e^{-i\omega_0 t}V_{12}\cos(\omega t) \quad \text{where } V_{ij} = \langle\psi_i|\hat{D}_0|\psi_j\rangle$$

Similarly:

$$\begin{aligned} \dot{c}_2 &= -\frac{i}{\hbar}c_1 e^{i\omega_0 t}V_{21}\cos(\omega t) \\ &= -\frac{i}{2\hbar}c_1 V_{21}\left(e^{i(\omega_0+\omega)t} + e^{i(\omega_0-\omega)t}\right) \end{aligned}$$

For $|\omega - \omega_0| \ll \omega_0$, we may invoke rotating wave approximation and drop the term with $\omega + \omega_0$:

$$\begin{aligned} \dot{c}_1 &= -\frac{i}{2\hbar}c_2 V_{12}e^{-i(\omega_0-\omega)t} \\ \dot{c}_2 &= -\frac{i}{2\hbar}c_1 V_{21}e^{i(\omega_0-\omega)t} \\ \Rightarrow \ddot{c}_2 &= -\frac{i}{2\hbar}\dot{c}_1 V_{21}e^{i(\omega_0-\omega)t} + i(\omega_0-\omega)\dot{c}_2 \\ 0 &= \ddot{c}_2 - i(\omega-\omega_0)\dot{c}_2 + \left(\frac{1}{2\hbar}\right)^2 c_2 \underbrace{V_{12}V_{21}}_{|V|^2} e^0 \end{aligned}$$

Try ansatz $c_2 = Ae^{i\alpha t}$:

$$\Rightarrow -\alpha^2 + \alpha(\omega-\omega_0) + \left(\frac{1}{2\hbar}\right)^2 |V|^2 = 0$$

$$\begin{aligned} \alpha &= \frac{(\omega-\omega_0) \pm \sqrt{(\omega-\omega_0)^2 + 4\left(\frac{V}{2\hbar}\right)^2}}{2} \\ &= \frac{\delta \pm \sqrt{\delta^2 + \Omega^2}}{2} \quad \text{where } \delta = \omega - \omega_0,\ \Omega = \frac{V}{\hbar} \\ &= \frac{\delta + \sqrt{\delta^2 + \Omega^2}}{2} \quad \text{since frequency} > 0 \end{aligned}$$

So $c_2 = Ae^{i\alpha t} + Be^{-i\alpha t}$, $c_2(0) = 0$:

$$\begin{aligned}
&\Rightarrow A = -B \\
\Rightarrow c_2(t) &= A\sin(\alpha t) \\
\Rightarrow \dot{c}_2(t) &= A\alpha\cos(\alpha t) \\
&= -\frac{i}{2\hbar}c_1 V e^{i\delta t}
\end{aligned}$$

$$\begin{aligned}
c_1(t) &= \frac{2i\hbar A\alpha}{V}\cos(\alpha t)e^{-i\delta t} \\
c_1(0) &= 1 \\
\Rightarrow A &= \frac{V}{2i\hbar\alpha} = \frac{\Omega}{2\alpha i}
\end{aligned}$$

So:

$$\begin{aligned}
|c_2(t)|^2 &= \left(\frac{\Omega}{2\alpha}\right)^2 \sin^2(\alpha t) \\
&= \left(\frac{\Omega}{\delta + \sqrt{\delta^2 + \Omega^2}}\right)^2 \sin^2\left(\frac{1}{2}(\delta + \sqrt{\delta^2 + \Omega^2})t\right)
\end{aligned}$$

Time average:

$$\begin{aligned}
\bar{c}_2 &= \frac{1}{\tau}\int_0^\tau \left(\frac{\Omega}{2\alpha}\right)^2 \sin^2\left(\frac{1}{2}\alpha t\right) \mathrm{d}t \\
&= \frac{1}{\tau}\left(\frac{\Omega}{2\alpha}\right)^2 \int_0^\tau \frac{1 - \cos(\alpha t)}{2} \mathrm{d}t \\
&= \frac{1}{2\tau}\left(\frac{\Omega}{2\alpha}\right)^2 \left[t - \frac{1}{\alpha}\sin(\alpha t)\right]_{t=0}^\tau \\
&= \frac{1}{2}\left(\frac{\Omega}{2\alpha}\right)^2 \left[1 - \frac{\sin(\alpha\tau)}{\alpha\tau}\right]
\end{aligned}$$

As $\tau \to \infty$, $\bar{c}_2 \to \dfrac{1}{2}\left(\dfrac{\Omega}{2\alpha}\right)^2$.

(b) **(TO EXPAND)** For Rabi oscillations to occur, $\Omega \gg \dfrac{1}{\tau}$

(c) **(TO EXPAND)** Decay term $e^{-\frac{\gamma t}{2}}$, where $\gamma = \dfrac{1}{\tau}$ is decay rate, should be added to $c_2(t)$ to account for natural broadening.

Fourier transforming the decay term should give broadening in frequency domain:

$$\begin{aligned}
\int_0^\infty \frac{1}{\pi} e^{i\omega t} e^{-\frac{\gamma t}{2}}\,\mathrm{d}t &= \frac{1}{\pi}\left[\frac{e^{(-\frac{\gamma}{2}+i\omega)t}}{i\omega - \frac{\gamma}{2}}\right]_{t=0}^{\infty} \\
&= \frac{1}{\pi}\frac{1}{\frac{\gamma}{2} - i\omega} \\
&= \frac{1}{\pi}\frac{\frac{\gamma}{2} + i\omega}{\left(\frac{\gamma}{2}\right)^2 + \omega^2} \\
&\to \frac{1}{\pi}\frac{\frac{\gamma}{2}}{\left(\frac{\gamma}{2}\right)^2 + \omega^2} \quad \text{by taking Re() component}
\end{aligned}$$

Hence natural broadening forms a Lorentzian in frequency domain.

Another example of Lorentzian broadening is phonon broadening.

(d) With the addition of $e^{-b|t|}$, the differentials now read:

$$\begin{aligned}
\dot{c}_1 &= -\frac{i}{2\hbar} c_2 V e^{-i(\omega_0-\omega)t} e^{-b|t|} \\
\dot{c}_2 &= -\frac{i}{2\hbar} c_1 V e^{i(\omega_0-\omega)t} e^{-b|t|}
\end{aligned}$$

Assuming weak radiation, then $c_1 \simeq 1$ and $c_2 \simeq 0$.

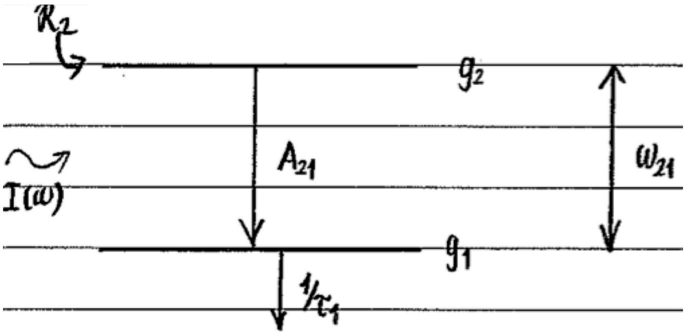
$$\begin{aligned}
\dot{c}_2 &= -\frac{i}{2\hbar} V e^{i(\omega_0-\omega)t} e^{-b|t|} \\
\Rightarrow c_2(t) &= \int_{-\infty}^{t} -\frac{i}{2\hbar} V e^{-b|t'|+i(\omega_0-\omega)t'}\,\mathrm{d}t' \\
&= -\frac{i}{2\hbar} V \frac{e^{(-b+i(\omega_0-\omega))t}}{-b+i(\omega_0-\omega)} \\
&= \underbrace{-\frac{i}{2}\Omega \frac{b+i(\omega_0-\omega)}{b^2+(\omega_0-\omega)^2}}_{-\frac{1}{2}\Omega\frac{ib-(\omega_0-\omega)}{b^2+(\omega_0-\omega)^2}} e^{-bt} e^{i(\omega_0-\omega)t}
\end{aligned}$$

Hence $|c_2(t)|^2 = \dfrac{1}{4}\Omega^2 \dfrac{1}{b^2+\delta^2} e^{-2bt}$.

As $t \to \infty$, $|c_2(t)|^2 \to \dfrac{1}{4}\Omega^2 \dfrac{1}{b^2+\delta^2}$.

Hence the excitation probability is a Lorentzian with FWHM $2b$.

4. (DRAFT)



(a) Rate equations:

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = \mathcal{R}_2 - A_{21}N_2$$
$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = A_{21}N_2 - \frac{N_1}{\tau_1}$$

Change in number of photons:

$$\Delta N_\gamma = A_{21}N_2 g_H(\omega-\omega_0)\rho(\omega,z)\delta\omega A\,\mathrm{d}z$$

where g_H is the broadening function, $\delta\omega$ is frequency width, A is the cross-sectional area of the amp, $\mathrm{d}z$ is a length element along propagation.

So energy contribution:

$$\Delta E = \Delta N_\gamma \cdot \hbar\omega = [I(z+\mathrm{d}z) - I(z)]\,A\delta\omega$$
$$\Rightarrow\ \mathrm{d}I = A_{21}N_2 g_H(\omega-\omega_0)\hbar\omega\,\mathrm{d}z\cdot\frac{I}{c} \quad \text{since spectral energy density } \rho = \frac{I}{c}$$
$$\frac{\mathrm{d}I}{\mathrm{d}z} = N_2\,\underbrace{A_{21}g_H(\omega-\omega_0)\frac{\hbar\omega}{c}}_{\sigma_{21}}\,I$$

So:

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = \mathcal{R}_2 - \frac{N_2\sigma_{21}I}{\hbar\omega}$$
$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = \frac{N_2\sigma_{21}I}{\hbar\omega} - \frac{N_1}{\tau_1}$$

At steady state,

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = 0$$
$$\Rightarrow N_2 = \frac{\hbar\omega\mathcal{R}_2}{\sigma_{21}I}$$

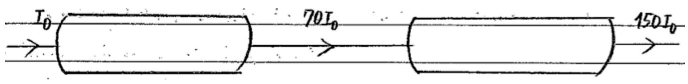
So gain coefficient:

$$\alpha = N_2\sigma_{21}$$
$$= \frac{\hbar\omega\mathcal{R}}{I}$$

(b)

$$
\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}z} &= \alpha I \\
\Rightarrow \frac{\mathrm{d}I}{I} &= \frac{\alpha_0}{1+\dfrac{I}{I_s}}\,\mathrm{d}z \\
\Rightarrow \frac{1}{I} + \frac{1}{I_s}\,\mathrm{d}I &= \alpha_0\,\mathrm{d}z \\
\ln\left(\frac{I_{\text{out}}}{I_{\text{in}}}\right) + \frac{I_{\text{out}} - I_{\text{in}}}{I_s} &= \alpha_0 z
\end{aligned}
$$

(c) Sketch of the setup:



$$
\begin{aligned}
\ln\left(\frac{70I_0}{I_0}\right) + \frac{70I_0 - I_0}{I_s} &= \alpha_0 L \\
\Rightarrow \ln 70 + 69\frac{I_0}{I_s} &= \alpha_0 L
\end{aligned}
$$

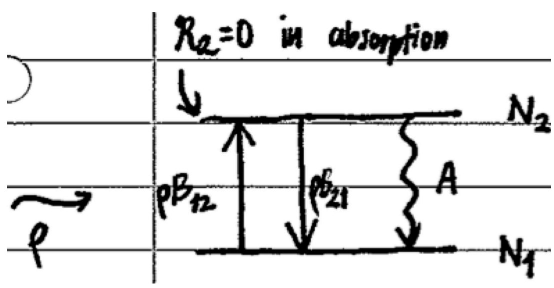
Also:

$$
\begin{aligned}
\ln\left(\frac{150I_0}{70I_0}\right) + \frac{150I_0 - 70I_0}{I_s} &= \alpha_0 L \\
\Rightarrow \ln\left(\frac{15}{7}\right) + 80\frac{I_0}{I_s} &= \alpha_0 L
\end{aligned}
$$

Combining the two expressions gives:

$$
\begin{aligned}
\ln 70 + 69\frac{I_0}{I_s} &= \ln\left(\frac{15}{7}\right) + 80\frac{I_0}{I_s} \\
11\frac{I_0}{I_s} &= \ln\left(\frac{70\cdot 7}{15}\right) \\
\Rightarrow \frac{I_0}{I_s} &= \frac{\ln\left(\dfrac{490}{15}\right)}{11} = 0.317
\end{aligned}
$$

(d) Sketch of the absorption dynamics:



At steady state,

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = \underbrace{\rho B_{12}N_1 - \rho B_{21}N_2}_{\rho B_{12}(N_1 - \frac{g_2}{g_1}N_2)} - A_{21}N_2 = 0$$

Also note that $N = N_1 + N_2$.

Next define $\tilde{N} = N_1 - \frac{g_1}{g_2}N_2$:

$$N - \tilde{N} = N_2\left(1 + \frac{g_1}{g_2}\right)$$
$$N_2 = \frac{N - \tilde{N}}{f}$$

where $f = 1 + \frac{g_1}{g_2}$.

Then:

$$A_{21}\left(\frac{N - \tilde{N}}{f}\right) = \rho B_{12}\tilde{N}$$
$$\tilde{N} = N\left[1 + \frac{\tilde{\sigma} I}{\hbar\omega}\frac{f}{A_{21}}\right]^{-1}$$

where $\rho B = \frac{\tilde{\sigma} I}{\hbar\omega}$.

Comparing with the form of N^* gives:

$$\tilde{I}_s = \frac{A_{21}\hbar\omega}{\tilde{\sigma} f}$$

The gain equation now reads:

$$\frac{\mathrm{d}I}{\mathrm{d}z} = -\alpha I = \left(\frac{\alpha_0}{1 + I/\tilde{I}_s}\right) I$$
$$\left[\ln I + \frac{I}{\tilde{I}_s}\right]_{I_0}^{I_z} = -\alpha_0 z$$
$$\Rightarrow \ln\frac{I_z}{I_0} + \frac{I_z - I_0}{\tilde{I}_s} = \ln\frac{I'_z}{I'_0} + \overbrace{\cancel{\frac{I'_z - I'_0}{\tilde{I}_s}}}^{\text{negligible}}$$

where LHS is the strong case, RHS is the weak case.

Solving for $\tilde{I}_s$ then gives:

$$\tilde{I}_s = 124\,\mathrm{W\,m^{-2}}$$
$$\tilde{\sigma} = 1.13 \times 10^{-13}\,\mathrm{m}^2$$