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# B2. SYMMETRY AND RELATIVITY

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**TRINITY TERM 2017**

**Last updated: 30th May 2025**

*Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.*

**Turn over as you please – we are NOT under exam conditions here.**

**1. (DRAFT)**

(a) For a worldline $\mathsf{X}^\mu = (ct, x, y, z)$, we have 4-velocity:

$$\begin{aligned} \mathsf{U}^\mu &= \frac{\mathrm{d}\mathsf{X}^\mu}{\mathrm{d}\tau} \\ &= \gamma \frac{\mathrm{d}\mathsf{X}^\mu}{\mathrm{d}t} \\ &= \gamma(c, \mathbf{v}) \end{aligned}$$

where $\mathbf{v}$ is the 3-velocity.

4-momentum is given as $\mathsf{P}^\mu = m\mathsf{U}^\mu$, so 4-force shall be:

$$\begin{aligned} \mathsf{F}^\mu &= \frac{\mathrm{d}\mathsf{P}^\mu}{\mathrm{d}\tau} \\ &= \gamma \frac{\mathrm{d}\mathsf{P}^\mu}{\mathrm{d}t} \\ &= \gamma(\dot{\gamma}mc + \gamma\dot{m}c, \dot{\gamma}m\mathbf{v} + \gamma m\mathbf{a} + \gamma\dot{m}\mathbf{v}) \end{aligned}$$

where $\mathbf{a} = {}^{\mathrm{d}\mathbf{v}}/_{\mathrm{d}t}$ is the 3-acceleration.

**(TO EXPAND)** For $\mathsf{D}^\mu \neq 0$ and $\mathsf{G}^\mu \neq 0$, if $\mathsf{D}_\mu\mathsf{G}^\mu = 0$, it implies that $\mathsf{D}_0\mathsf{G}^0 = \mathsf{D}_i\mathsf{G}^i$.

Choosing $\mathsf{D}^\mu = (1, \mathbf{0})$, we then have

$$\begin{aligned} &\mathsf{G}^0 = 0 \\ \Rightarrow\ &\mathsf{G}_\mu\mathsf{G}^\mu \leq 0 \end{aligned}$$

But since $\mathsf{G}^\mu \neq 0$, $\mathsf{G}^\mu$ is space-like.

(b) Choose a frame where the particle is at rest, then $\mathsf{P}^\mu = (mc, \mathbf{0})$:

$$\begin{aligned} \mathsf{P}_\mu\mathsf{P}^\mu &= -m^2c^2 = 0 \\ &\Rightarrow m = 0 \end{aligned}$$

so the particle must be massless. One such example is the photon.

(c) Proper time is defined as the spacetime interval between 2 events at which the space interval is null (i.e. the clock does not move).

Pure force is a force such that the mass of a body remains constant, ie. $\dot{m} = 0$.

If $\mathsf{F}^\mu$ is pure then:

$$\begin{aligned} \mathsf{F}^\mu &= \gamma(\dot{\gamma}mc, \dot{\gamma}m\mathbf{v} + \gamma m\mathbf{a}) \\ \Rightarrow \mathsf{U}_\mu\mathsf{F}^\mu &= -\dot{\gamma}mc^2\gamma^2 + \gamma^2\left(\dot{\gamma}v^2 + \gamma\mathbf{a}\cdot\mathbf{v}\right) \\ &= 0 \quad \text{in the particle rest frame where } \mathbf{v} = 0,\ \dot{\gamma} = 0 \end{aligned}$$

The 4-acceleration is then:

$$\begin{aligned} \mathsf{A}^\mu &= \frac{\mathrm{d}\mathsf{U}^\mu}{\mathrm{d}\tau} \\ &= \gamma(\dot{\gamma}c, \dot{\gamma}\mathbf{v} + \gamma\mathbf{a}) \end{aligned}$$

Noting that ${}^{\mathrm{d}\gamma}\!/_{\mathrm{d}t} = {}^{\mathrm{d}}\!/_{\mathrm{d}t}\left[(1-\beta^2)^{-1/2}\right]$, we then have:

$$\begin{aligned} \mathsf{A}_\mu \mathsf{A}^\mu &= -\gamma^2\dot{\gamma}^2 c^2 + \left(\gamma\dot{\gamma}\mathbf{v} + \gamma^2\mathbf{a}\right)^2 \\ &= \gamma^4 a^2 \end{aligned}$$

Assuming a constant 3-force,

$$\begin{aligned} \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma m\mathbf{v}\right) = f_x \\ \Rightarrow f_x &= \dot{\gamma} m\mathbf{v} + \gamma m\mathbf{a} \\ &= \beta^2\gamma^3\frac{a}{v}m\mathbf{v} + \gamma m\mathbf{a} \quad \text{for } \mathbf{v} \parallel \mathbf{a} \\ &= \gamma m a\left[1 + \beta^2\gamma^2\frac{v}{v}\right] \quad \text{in } x \text{ direction} \\ &= \gamma^3 m a\left[\frac{1}{\gamma^2} + \beta^2\right] = \gamma^3 m a \end{aligned}$$

Also:

$$\begin{aligned} \gamma &= \cosh\rho \\ v^2 &= c^2\left(1 - \operatorname{sech}^2\rho\right) \\ v &= c\tanh\rho \\ \frac{\mathrm{d}v}{\mathrm{d}\rho} &= c\operatorname{sech}^2\rho \end{aligned}$$

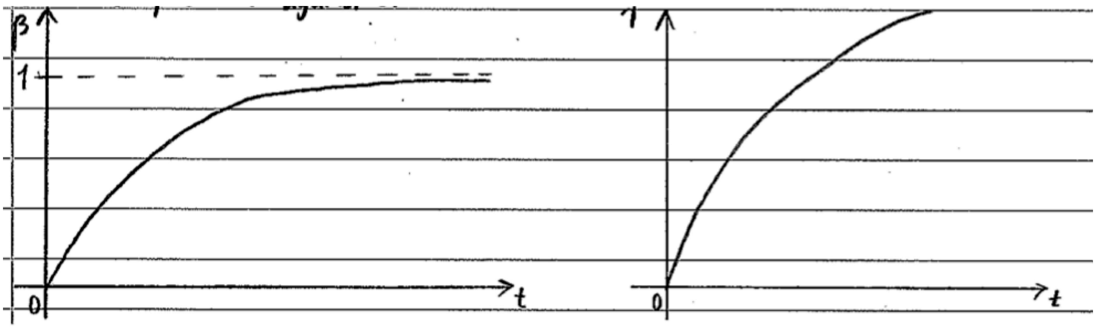
We then have:

$$\begin{aligned} f_x &= \gamma^3 m a = \gamma^3 m \underbrace{\frac{\mathrm{d}v}{\mathrm{d}t}}_{\frac{\mathrm{d}v}{\mathrm{d}\rho}\cdot\frac{\mathrm{d}\rho}{\mathrm{d}t}} \\ \Rightarrow f_x &= \cosh^3\rho\cdot m\cdot c\operatorname{sech}^2\rho\frac{\mathrm{d}\rho}{\mathrm{d}t} \\ \Rightarrow \int_0^\rho \cosh\rho\,\mathrm{d}\rho &= \int_0^t \frac{f_x}{mc}\,\mathrm{d}t \\ \sinh\rho = \beta\gamma &= \frac{f_x t}{mc} \end{aligned}$$

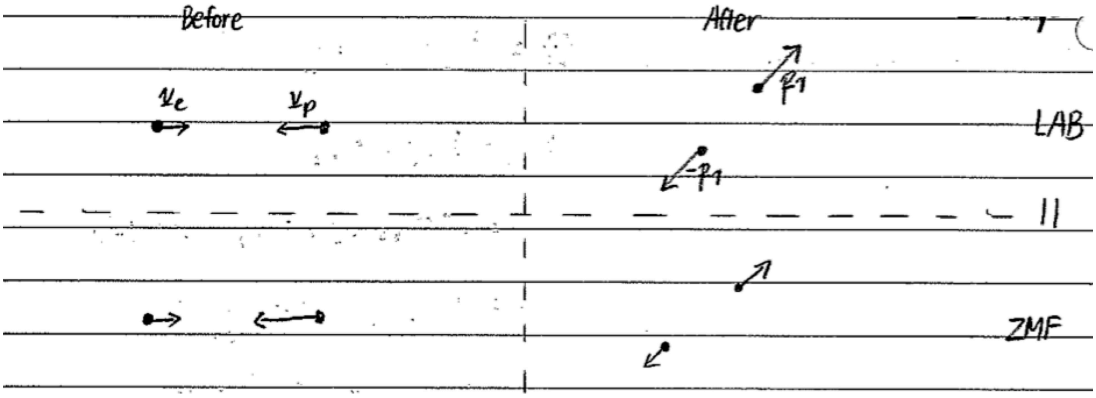
Then

$$\begin{aligned} \gamma &= \sqrt{1 + \sinh^2\rho} \\ &= \sqrt{1 + \left[\frac{f_x t}{mc}\right]^2} \\ \Rightarrow \beta &= \frac{f_x t/mc}{\sqrt{1 + \left(f_x t/mc\right)^2}} \\ \Rightarrow \mathsf{U}^\mu &= \gamma\left(c, \beta c, 0, 0\right) \end{aligned}$$

Sketch of $\beta$ and $\gamma$ against $t$:




(d) Diagram of the collision:



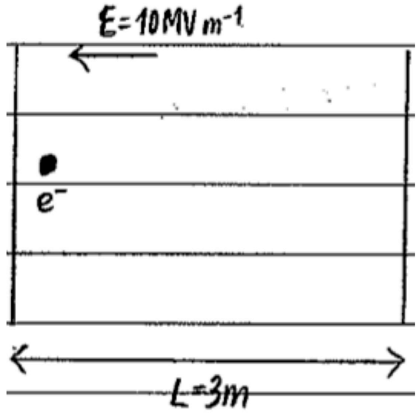

System 4-momentum is given by $\mathsf{P}^\mu = (2E/c, \mathbf{0})$ where $E^2 = (m_e c^2)^2 + (p_e c)^2$ is the energy of the electron/positron.

Suppose 1 photon is generated, the final 4-momentum would also be $\mathsf{P}^\mu$, but since $\mathsf{P}_\mu \mathsf{P}^\mu \neq 0$, this implies that the photon must possess finite mass, which is unphysical! Therefore a minimum of 2 photons must be produced.

Conservation of 3-momentum means that each of the platons must possess equal but opposite $p_\gamma \Rightarrow$ equal energy so $E_\gamma = E$ by symmetry.

$$\mathsf{P}_\mu \mathsf{P}^\mu = -\frac{4E^2}{c^2}$$
$$\Rightarrow E = \sqrt{(0.511\,\mathrm{MeV})^2 + (1\,\mathrm{GeV})^2}$$
$$= 1\,\mathrm{GeV}$$

(e) Sketch of the plates:




Energy gained $E$:

$$E = e\epsilon L = \gamma m_{\mathrm{e}} c^2$$
$$\Rightarrow \gamma = \frac{e\epsilon L}{m_{\mathrm{e}} c^2} = 58.7$$

We finally have:

$$\beta = \left[1 - \gamma^{-2}\right]^{1/2}$$
$$= \sqrt{1 - \left(\frac{m_{\mathrm{e}} c^2}{e\epsilon L}\right)^2} = 0.9999$$

**2. (DRAFT)**

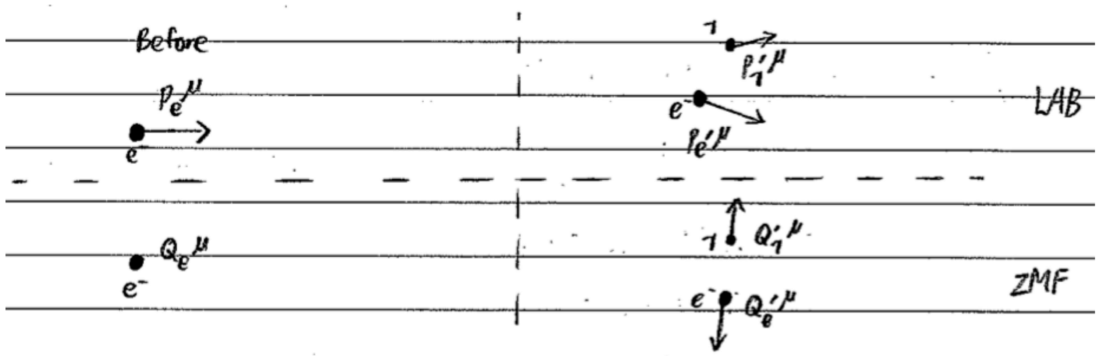
(a) A 4-wavevector is defined as $\mathsf{K}^\mu = \partial^\mu \phi = (\omega/c, \mathbf{k})$ where $\phi(t, \mathbf{x})$ is the phase of a wave, $\omega$ is the frequency of the wave, and $\mathbf{k}$ is the 3-wavevector.

The phase may be given by (with $\mathsf{X}^\mu = (ct, \mathbf{x})$):

$$\begin{aligned}\phi &= \mathsf{K}_\mu \mathsf{X}^\mu \\ &= \omega t - \mathbf{k}\cdot\mathbf{x}\end{aligned}$$

However as $\mathsf{K}_\mu \mathsf{X}^\mu$ is a contraction of two 4-vectors, and Lorentz transformation is unitary, $\phi$ shall be Lorentz invariant.

(b) Consider the process $\mathrm{e}^- \to \mathrm{e}^-\gamma$:




Conservation of 4-momentum tells us:

$$\begin{aligned}\mathsf{P}_\mathrm{e}^\mu &= \mathsf{P}_\mathrm{e}'^\mu + \mathsf{P}_\gamma'^\mu \\ \Rightarrow \left(\mathsf{Q}_\mathrm{e}^\mu - \mathsf{Q}_\gamma'^\mu\right)^2 &= \left(\mathsf{Q}_\mathrm{e}'^\mu\right)^2 \\ -m_\mathrm{e}^2c^2 + 0 + 2m_\mathrm{e}c \cdot p_\gamma &= -m_\mathrm{e}^2c^2 \\ \Rightarrow m_\mathrm{e}cp_\gamma &= 0 \\ \Rightarrow m_\mathrm{e} &= 0\end{aligned}$$

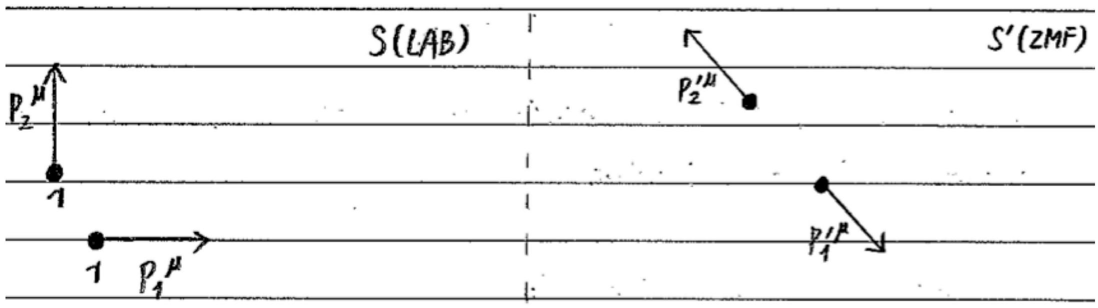
which is unphysical, therefore the process is kinematically forbidden.

(c) Space-like interval:

$$\begin{aligned}\left(\mathsf{D}^\mu \mathsf{G}^\mu\right)^2 &> 0 \\ \Rightarrow c^2\left(t_d - t_g\right)^2 &< \left(\mathbf{x}_d - \mathbf{x}_g\right)^2\end{aligned}$$

For space-like interval, since the two events cannot be connected by a signal path, it is possible to find a frame where they are simultaneous since causality is not possible between them.

Sketch of the collision:




(d) System 4-momentum:

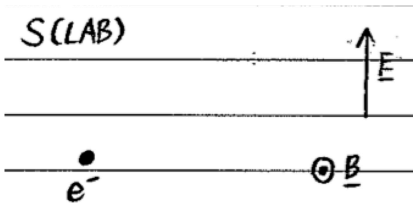
$$\begin{aligned} \mathsf{P}^\mu &= \mathsf{P}_1^\mu + \mathsf{P}_2^\mu \\ &= (p+p, p, p, 0) \quad \text{where } p = \hbar\omega/c \text{ is the photon momentum} \\ \Rightarrow \mathsf{P}_\mu \mathsf{P}^\mu &= -(2p)^2 + 2p^2 \\ -m_\gamma^2 c^2 &= -2p^2 \\ m_\gamma &= \frac{p}{c} = \frac{\hbar\omega}{c} \end{aligned}$$

(e) Rotate the axis such that the new $x$-axis points at $^1/_{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{y}})$, now:

$$\mathsf{P}^\mu = (2p, 2p\sin 45°, 0, 0)$$

Lorentz transform and impose the condition that $\mathsf{P}' = 0$:

$$\begin{aligned} 0 &= -\beta\gamma(2p) + \gamma(2p\sin 45°) \\ 2\beta\gamma &= \frac{1}{\sqrt{2}} \cdot 2p \\ \beta &= \frac{1}{\sqrt{2}} \\ \Rightarrow \mathbf{v} &= \frac{1}{\sqrt{2}} c \left[ \frac{1}{\sqrt{2}} (\hat{\mathbf{x}} + \hat{\mathbf{y}}) \right] \end{aligned}$$




(f) Lorentz transformation of EM fields:

$$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp - \mathbf{v} \times \mathbf{B}) \qquad \mathbf{E}'_\parallel = \mathbf{E}_\parallel$$

$$\mathbf{B}'_\perp = \gamma\left(\mathbf{B}_\perp - \frac{\mathbf{v} \times \mathbf{E}}{c^2}\right) \qquad \mathbf{B}'_\parallel = \mathbf{B}_\parallel$$

(g) $E^2/c^2 > B^2 \Rightarrow$ No frame with $E = 0$ since $E^2/c^2 - B^2$ is Lorentz invariant.

So choose $B = 0$ and boost by $\gamma_u$ along $+\hat{\mathbf{x}}$:

$$0 = B - \frac{uE}{c^2}$$
$$\Rightarrow u = \frac{Bc^2}{E}$$

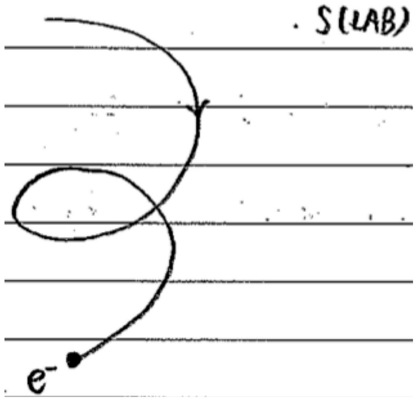
So $E'^2 = E^2 - B^2c^2$ from the invariant:

$$\Rightarrow E'_y = \gamma_u (E - uB) = \sqrt{E^2 - B^2c^2}$$

**(TO EXPAND)** So in $S'$, the $e^-$ gains velocity in $\hat{\mathbf{y}}$, and the Lorentz force is:

$$\mathbf{f} = -e\mathbf{E}'$$
$$f_y = -e\sqrt{E^2 - B^2c^2}$$

In $S$ the $e^-$ would appear to be travelling in $-\hat{\mathbf{y}}$ before turning under the B field.




(h) Conversely if $E^2/c^2 < B^2$, we can choose $E' = 0$ in $S'$.

Boosting by $\gamma_u$ along $+\hat{\mathbf{x}}$:

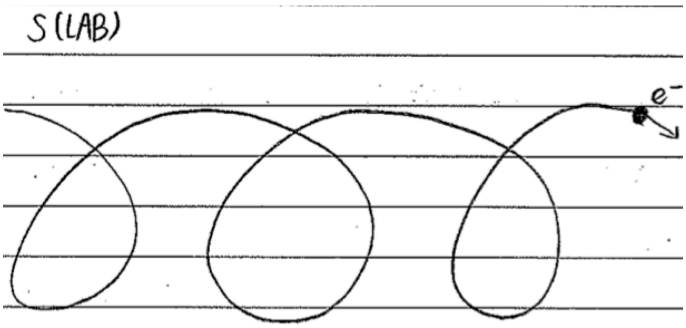
$$0 = E - uB \Rightarrow u = \frac{E}{B}$$
$$B' = \gamma_u \left( B - \frac{uE}{c^2} \right)$$
$$\Rightarrow -B'^2 = \frac{E^2}{c^2} - B^2$$
$$B' = \sqrt{B^2 - \frac{E^2}{c^2}}$$
$$\Rightarrow f' = -evB'$$

So the particle would undergo a circular motion in $S'$ with $f$ as follows:

$$\begin{aligned} |f'| &= \gamma_u^3 m_{\mathrm{e}} a \\ \Rightarrow evB' &= \gamma_u^3 m_{\mathrm{e}} \frac{u^2}{r} \\ \Rightarrow r &= \frac{\gamma_u^3 m_{\mathrm{e}} u^2}{evB'} \end{aligned}$$

where $r$ is the radius.

In $S$ this would appear as a helical-like path similar to the trajectory a type makes when rolling:




Consider the 4-momentum of the system:

$$\begin{aligned} \mathsf{P}^\mu &= \left(\sum_i W_i, \sum_i \mathbf{p}_i\right) \\ \Rightarrow \mathsf{P}_\mu \mathsf{P}^\mu &= -\left(\sum_i W_i\right)^2 + \left(\sum_i \mathbf{p}_i\right)^2 = -S^2 \quad \text{is invariant} \end{aligned}$$

For photon gas, $W_i = p_i c = p_i$, and for a non-interacting gas, $\sum_i \mathbf{p}_i = 0$, thus:

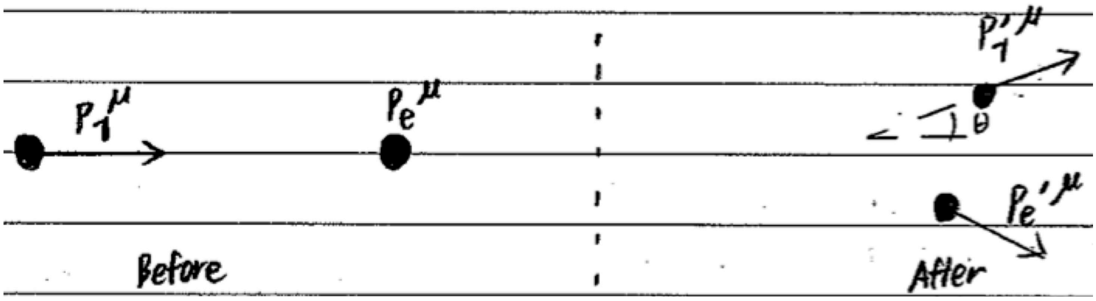
$$S^2 = \left(\sum_i W_i\right)^2 = \left(\sum_i p_i\right)^2$$

**3. (DRAFT)**

(a) 1. Headlight effect – the distribution of photons emitted by a body is skewed towards its motion, a consequence of the Lorentz transformation.

2. Transverse/Oblique Doppler effect – Galilean transformation fails to explain frequency shift at high velocities.

3. Abberation – direction of emitted photons seemingly "bent" in the direction of boost.




Conservation of 4-momentum gives:

$$\mathsf{P}_\gamma^\mu + \mathsf{P}_\mathrm{e}^\mu = \mathsf{P}_\gamma'^\mu + \mathsf{P}_\mathrm{e}'^\mu$$
$$\Rightarrow \left(\mathsf{P}_\gamma^\mu + \mathsf{P}_\mathrm{e}^\mu - \mathsf{P}_\gamma'^\mu\right)^2 = \left(\mathsf{P}_\mathrm{e}'^\mu\right)^2$$

$$\cancelto{0}{(\mathsf{P}_\gamma)_\mu (\mathsf{P}_\gamma)^\mu} + \underbrace{(\mathsf{P}_\mathrm{e})_\mu (\mathsf{P}_\mathrm{e})^\mu}_{-m_\mathrm{e}^2c^2} + \cancelto{0}{(\mathsf{P}_\gamma')_\mu (\mathsf{P}_\gamma')^\mu}$$
$$+\,2\left[(\mathsf{P}_\gamma)_\mu (\mathsf{P}_\mathrm{e})^\mu - (\mathsf{P}_\gamma)_\mu (\mathsf{P}_\gamma')^\mu - (\mathsf{P}_\mathrm{e})_\mu (\mathsf{P}_\gamma')^\mu\right] = \underbrace{(\mathsf{P}_\mathrm{e}')_\mu (\mathsf{P}_\mathrm{e}')^\mu}_{-m_\mathrm{e}^2c^2}$$
$$\Rightarrow 2\left[-\frac{E_\gamma}{c}\cdot m_\mathrm{e}c + 0 + \frac{E_\gamma E_\gamma'}{c^2} - p_\gamma p_\gamma' \cos\theta + \frac{E_\gamma'}{c}\cdot m_\mathrm{e}c - 0\right] = 0$$
$$\left(p_\gamma' - p_\gamma\right) m_\mathrm{e}c + p_\gamma p_\gamma' - p_\gamma p_\gamma' \cos\theta = 0$$
$$p_\gamma' \left[m_\mathrm{e}c + p_\gamma\left(1-\cos\theta\right)\right] = p_\gamma m_\mathrm{e} c$$
$$p_\gamma' = \frac{p_\gamma m_\mathrm{e} c}{m_\mathrm{e}c + p_\gamma\left(1-\cos\theta\right)}$$
$$\Rightarrow \frac{1}{\lambda'} = \frac{1}{\lambda}\left(\frac{m_\mathrm{e}c}{m_\mathrm{e}c + {}^{h}\!/_{\lambda}\left(1-\cos\theta\right)}\right)$$
$$= \frac{1}{\lambda - {}^{h}\!/_{m_\mathrm{e}c}\left(1-\cos\theta\right)}$$

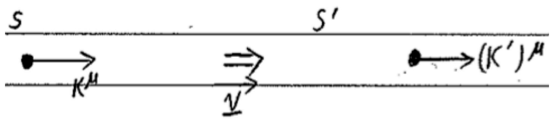
Rotate in $xy$-plane such that the 4-wavevector $\mathsf{K}^\mu = (\omega/c, k\cos\theta, k\sin\theta, 0)$ with $k = k_\pi$.

Boosting along the $x$-axis (by say $+u\hat{\mathbf{x}}$) gives:

$$
\begin{aligned}
(\mathsf{K}')^\mu &= \Lambda^\mu_\nu \mathsf{K}^\nu \\
\begin{pmatrix} \dfrac{\omega'}{c} \\ k'\cos\theta' \\ k'\sin\theta' \\ 0 \end{pmatrix} &= \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dfrac{\omega}{c} \\ k\cos\theta \\ k\sin\theta \\ 0 \end{pmatrix} \quad \text{where } \beta = +\frac{u}{c},\ \gamma = (1-\beta)^{-1/2} \\
\Rightarrow \frac{\omega'}{c} &= \gamma\frac{\omega}{c} - \beta\gamma k\cos\theta \\
\omega' &= \gamma\omega - \beta\gamma \underbrace{kc}_{\omega \text{ for EM wave}} \cos\theta \\
\omega' &= \gamma\left(1 - \beta\cos\theta\right)\omega
\end{aligned}
$$

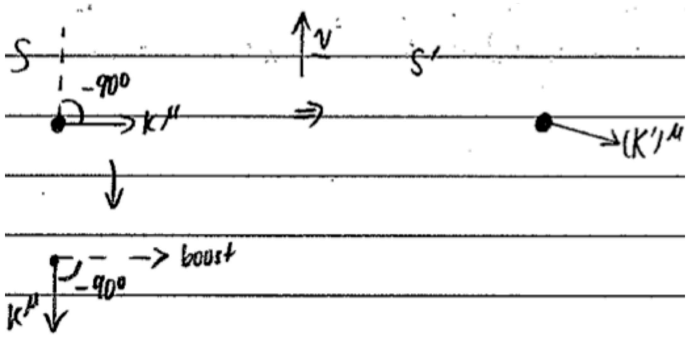
(b) For frame $S'$ moving in $\mathbf{v} = (v_x, 0, 0)$, $\theta = 0$ so:

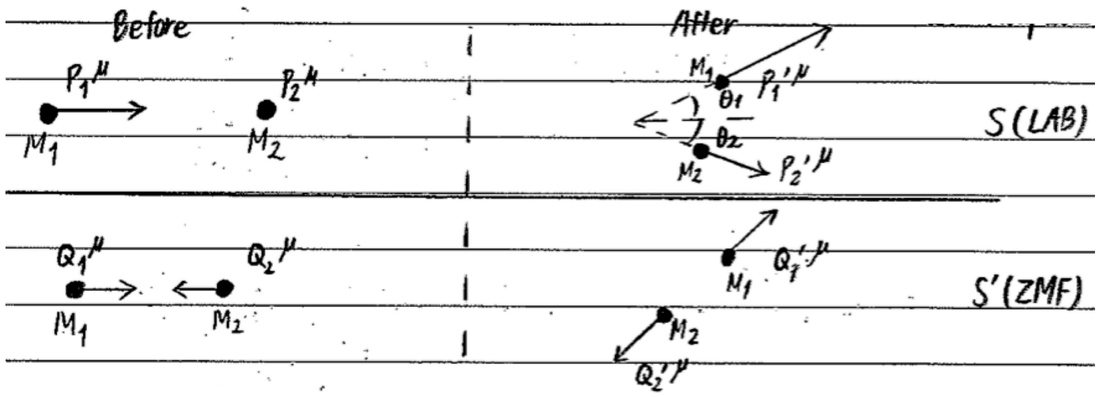
$$
\begin{aligned}
\omega' &= \gamma\left(1-\beta\right)\omega \\
&= \frac{1 - v_x/c}{\sqrt{1 - \left(v_x/c\right)^2}}\omega
\end{aligned}
$$




(c) For $\mathbf{v} = (0, v_y, 0)$, $\theta = -90°$, so:

$$
\begin{aligned}
\omega' &= \gamma\omega \\
&= \frac{1}{\sqrt{1 - \left(v_y/c\right)^2}}\omega
\end{aligned}
$$






Conservation of 4-momentum gives (in $S$):

$$\mathsf{P}_1^\mu + \mathsf{P}_2^\mu = (\mathsf{P}_1')^\mu + (\mathsf{P}_2')^\mu$$
$$\Rightarrow \left(\mathsf{P}_1^\mu + \mathsf{P}_2^\mu - \mathsf{P}_1'^\mu\right)^2 = \left(\mathsf{P}_2'^\mu\right)^2$$
$$(\mathsf{P}_1)_\mu (\mathsf{P}_1)^\mu + (\mathsf{P}_2)_\mu (\mathsf{P}_2)^\mu + (\mathsf{P}_1')_\mu (\mathsf{P}_1')^\mu + 2\left[(\mathsf{P}_1)_\mu (\mathsf{P}_2)^\mu - (\mathsf{P}_1)_\mu (\mathsf{P}_1')^\mu - (\mathsf{P}_2)_\mu (\mathsf{P}_1')^\mu\right] = (\mathsf{P}_2')_\mu (\mathsf{P}_2')^\mu$$
$$-M_1^2c^2 - \cancel{M_2^2c^2} - -M_1^2c^2 + 2\left[-\frac{E_1}{c}M_2c + 0 + \frac{E_1E_1'}{c^2} - p_1p_1'\cos\theta_1 + M_2c\frac{E_1'}{c} - 0\right] = -\cancel{M_2^2c^2}$$
$$M_2c\left[\frac{E_1' - E_1}{c}\right] + \frac{E_1E_1'}{c^2} - p_1p_1'\cos\theta_1 = M_1^2c^2 \tag{1}$$

$$\text{Similarly, } M_2c\left[\frac{E_2' - E_1}{c}\right] + \frac{E_1E_2'}{c^2} - p_1p_2'\cos\theta_2 = M_1^2c^2 \tag{2}$$

Conservation of 3-momentum in $S$:

$$p_1 = p_1'\cos\theta_1 + p_2'\cos\theta_2 \qquad \Rightarrow p_1'\cos\theta_1 = p_1 - p_2'\cos\theta_2$$
$$0 = p_1'\sin\theta_1 - p_2'\sin\theta_2 \qquad \Rightarrow p_1'\sin\theta_1 = p_2'\sin\theta_2$$

Now if $\theta_1 = \theta_2 = \phi$,

$$p_1' = p_2' = p'$$
$$\Rightarrow p'\cos\phi = p_1 - p'\cos\phi$$
$$\Rightarrow p'\cos\phi = \frac{1}{2}p_1$$

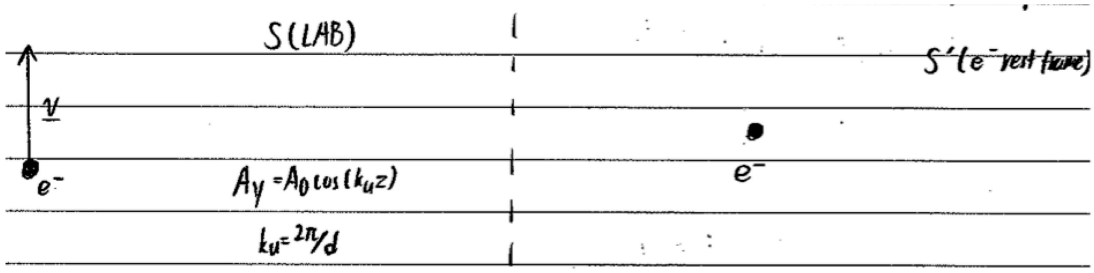
(1) $\div$ (2):

$$1 = \frac{M_2c\left[\frac{E_1'-E_1}{c}\right] + \frac{E_1E_1'}{c^2} - \overbrace{p_1p_1'\cos\theta}^{1/2\,p_1^2}}{M_2c\left[\frac{E_2'-E_1}{c}\right] + \frac{E_1E_2'}{c^2} - \underbrace{p_1p_2'\cos\theta}_{1/2\,p_1^2}}$$

Since the numerator and denominator must match, we must have $E'_1 = E'_2$:

$$\Rightarrow M_1^2 c^4 + \underbrace{p_1'^2}_{p'^2 c^2} = M_2^2 c^4 + \underbrace{p_2'^2}_{p'^2 c^2}$$
$$\Rightarrow M_1 = M_2$$

(d) Sketch of the system in both lab and $e^-$ rest frames:




4-potential $\mathsf{A}^\mu = (0, 0, A_0 \cos(k_u z), 0)$

$$\Rightarrow \mathsf{F}^{\mu\nu} = \partial^\mu \mathsf{A}^\nu - \partial^\nu \mathsf{A}^\mu = \begin{bmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ & 0 & B_z & -B_y \\ -\mathbf{E} & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{bmatrix}$$
$$\Rightarrow \mathbf{E} = 0 \text{ in } S$$

Non-zero component of $B$ involves $\frac{\partial \mathsf{A}^2}{\partial z}$:

$$\begin{aligned} B_x &= \partial^2 \mathsf{A}^3 - \partial^3 \mathsf{A}^2 \\ &= -\frac{\partial}{\partial z} A_0 \cos(k_u z) \\ &= A_0 k_u \sin(k_u z) \end{aligned}$$

Transformation of EM fields:

$$\mathbf{E}'_\parallel = \mathbf{E}_\parallel \qquad \mathbf{B}'_\parallel = \mathbf{B}_\parallel$$
$$\mathbf{E}'_\perp = \gamma\left(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}\right)$$
$$\mathbf{B}'_\perp = \gamma\left(\mathbf{B}_\perp - \frac{\mathbf{v} \times \mathbf{E}}{c^2}\right)$$

After boosting along $\hat{\mathbf{z}}$, we get:

$$\begin{aligned}
B'_z &= B_z = 0 \\
E'_z &= E_z = 0 \\
\mathbf{E}_\perp &= \gamma\left(0 + \mathbf{v} \times \mathbf{B}\right) \\
&= \gamma v \hat{\mathbf{z}} \times B_x \hat{\mathbf{x}} \\
&= \gamma v B_x \hat{\mathbf{y}} \\
\mathbf{B}_\perp &= \gamma\left(\mathbf{B} - \frac{\overbrace{\mathbf{v} \times \mathbf{E}}^{0}}{c^2}\right) \\
&= \gamma B_x \hat{\mathbf{x}}
\end{aligned}$$

So in $S'$,

$$\begin{aligned}
\mathbf{E}' &= \gamma v B_x \hat{\mathbf{y}} \\
\mathbf{B}' &= \gamma B_x \hat{\mathbf{x}}
\end{aligned}$$

where $\gamma = [1 - v^2/c^2]^{-1/2}$.

For EM wave, $B = E/c$, however $B'/E' = 1/v \sim 1/c$, so the observed field is not an EM wave, though it approaches one as $v \to c$.