UNOFFICIAL SOLUTIONS BY TheLongCat

B4. SUB-ATOMIC PHYSICS

TRINITY TERM 2018

Last updated: 30th May 2025

Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. (DRAFT)

(a) Consider the SEMF:

$$B(Z,A) = \underbrace{a_V A - a_S A^{2/3}}_{\text{liquid-drop model}} - \underbrace{a_C \frac{Z^2}{A^{1/3}}}_{\text{Coulomb repulsion}} - a_A \underbrace{\frac{(A-2Z)^2}{A}}_{\text{asymmtery}} + \delta(A,Z)$$

where

$$\delta(A,Z) = \begin{cases} \frac{-12\,\text{MeV}}{\sqrt{A}} & \text{o-o} \\ 0 & \text{o-e/e-o} \\ \frac{12\,\text{MeV}}{\sqrt{A}} & \text{e-e} \end{cases}$$

We then seek the stationary point:

$$\left(\frac{\partial B}{\partial Z}\right)_A = 0$$
$$\Rightarrow Z = \frac{2a_A A}{a_C A^{2/3} + 4a_A}$$

where $a_C \sim 1\,\text{MeV}$, $a_A \sim 10\,\text{MeV}$.

Recall that the nucleons are fermions, which abide Pauli's exclusion principle. Therefore it is more energetically favourable to have approximately equal number of protons and neutrons than letting one type pile up on the energy level.

$$\lim_{A\to 0} Z = \frac{A}{2}$$

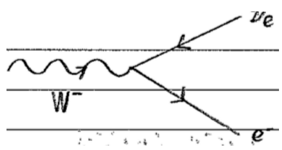
However, this scenario begins to break down for high Z nuclei as electromagnetic repulsion between protons begins to dominate over strong interaction, meaning that the balance will start to tip towards having more neutrons.

$$\lim_{A\to\infty} = \frac{2a_A}{a_C} A^{1/3}$$

In β^- decay, also for a fission free neutron, a neutron decays into a proton, an electron and an anti-electron neutrino.

$$\text{n} \to \text{p} + \text{e}^- + \bar{\nu}_\text{e}$$

The reason why an antineutrino is produced has to do with the conservation of lepton numbers, which is a consequence of $\text{W}^\pm$ coupling.

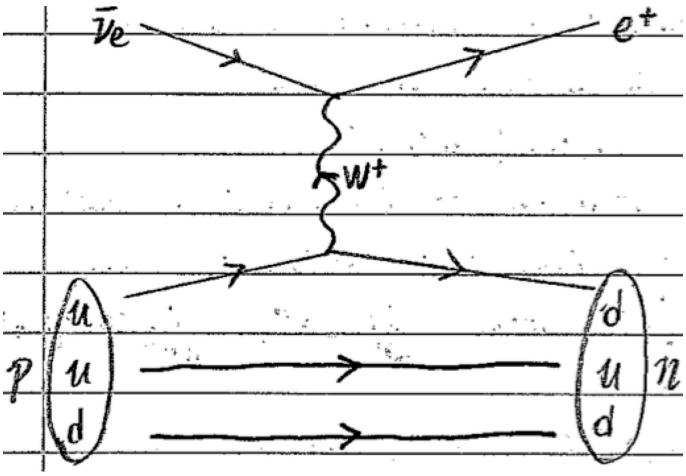


And so $0 = (+1) + (-1)$ since e^- has lepton number $+1$, $\bar{\nu}_\text{e}$ has -1.

As A decreases, we would have more n than p but need $Z \simeq \dfrac{A}{2}$ so β^- decay occurs.

(b) Inverse beta decay:

$$\bar{\nu}_{\mathrm{e}} + \mathrm{p} \to \mathrm{e}^{+} + \mathrm{n}$$



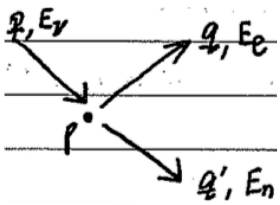
Recall that from Fermi's Golden Rule that

$$\Gamma = \frac{2\pi}{\hbar} \left|M_{if}\right|^2 \rho(E)$$

and matrix element M_{if} is dependent on the momentum transfer by the gauge boson. (how?)

Hence it is expected that the reaction cross-section, which is proportional to Γ the reaction rate, to increase with greater $\bar{\nu}_{\mathrm{e}}$ energy. (details needed)

Sargent's rule: $\Gamma \propto Q^5$



D.O.S.:

$$\begin{aligned}
\mathrm{d}^2 N &= \frac{V\,\mathrm{d}\Omega P^2\,\mathrm{d}P}{h^3}\frac{V\,\mathrm{d}\Omega Q^2\,\mathrm{d}Q}{h^3} \\
&= \frac{1}{4\pi^4}\frac{1}{h^6}p^2 q^2\,\mathrm{d}P\,\mathrm{d}Q \\
q &= \frac{(Q - E_\nu)}{c} \quad \text{for } Q \gg m_{\mathrm{e}}c^2
\end{aligned}$$

(c) Detected interactions in total:

$$\frac{\mathrm{d}I}{\mathrm{dE}_\nu} = \sigma_\nu \rho(E)$$

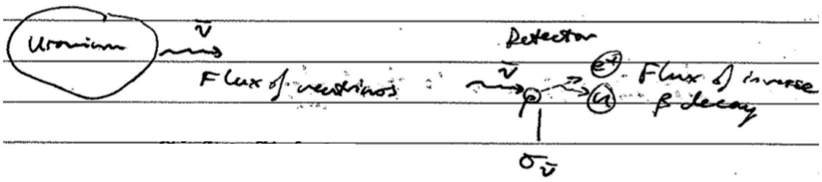
since reaction rate $\Gamma \propto \rho(E)$ and σ_ν.

Detected reaction rate $\propto \sigma_\nu$ since cross section dictates the total reaction rate (why?)

$\propto {}^{\mathrm{d}N_\nu}\!/_{\mathrm{d}E_\nu}\,\mathrm{d}E_\nu$ as a result of thermodynamical distribution (select population with specific energy for specific σ_ν).

$\sigma_{\bar{\nu}}\frac{\mathrm{d}N_{\bar{\nu}}}{\mathrm{d}E_{\bar{\nu}}}$ represents the cross section as a function of energy combined with the distribution of $\bar{\nu}_\mathrm{e}$ energies per fission reaction, i.e. total cross section of all $\bar{\nu}_\mathrm{e}$ produced at each energy per fission reaction. We have to integrate to get total cross section for all energies and particles.

$$\underbrace{\sigma\,\mathrm{d}E}_{\substack{\text{differential}\\ \text{cross section}\\ \text{for each } E}} \cdot \underbrace{\frac{\mathrm{d}N}{\mathrm{d}E}}_{\#\text{ of }\nu\text{ for each }E}$$



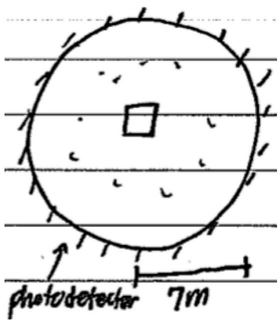
$$\sigma_{\bar{\nu}} = \frac{\text{No. events } t-1}{\text{Neutrino flux}}$$
$$\therefore\ \sigma_{\bar{\nu}}\,\mathrm{d}N_{\bar{\nu}} = \frac{\text{No. of events } t-1}{\text{Unit area}} \propto \text{Rate of observed } \bar{\nu} \text{ interactions}$$

(d) i. Generator produces 100 MW and each ^{235}U fission releases 210 MeV:

$$\therefore\ \text{rate} = \frac{\text{power}}{\text{Energy per fission}} = 2.97\times10^{18}\,\mathrm{s}^{-1}$$

ii. Approximate $\sigma{}^{\mathrm{d}N}\!/_{\mathrm{d}E}$ as a Gaussian, we can derive mean and s.d. from the data:

$$I = 0.669\times10^{-42}\,\mathrm{cm}^2/\text{fission}$$



iii. We have the rate given by:

$$\text{Rate} = \rho\sigma v$$

where ρ is the density of detector medium, σ is calculated in II per fission, $v = c$ for massless neutrinos.

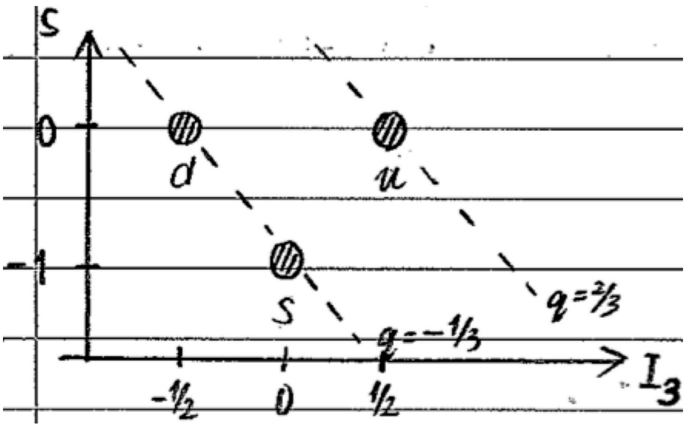
$$\begin{aligned}\text{Rate} &= 0.025\,\mathrm{s}^{-1}\\ &= \frac{\overbrace{N}^{\substack{\#\text{ of}\\ \text{scintillating particles}}}}{\underbrace{V}_{\frac{4}{3}\pi(7)^3}} \times 0.669 \times 10^{-42}\,\mathrm{cm}^2/\text{fission} \times 2.97 \times 10^{18}\,\text{fission/s} \times \underbrace{L}_{7\,\mathrm{m}}\end{aligned}$$

So

$$\begin{aligned}\rho &= N \times M_{\mathrm{C_9H_{10}}}\\ &= 6085\,\mathrm{kg}\end{aligned}$$

2. (DRAFT)

(a) In the light quark model, due to the invariance of strong force in the flavours of the quarks, we may propose a basis of representation as follows:



where q is electric charge, S is strangeness, I_3 is isospin.

Assembling the hadrons with these quarks then yields the Eightfold Way, explaining the mass, charge, isospin and strangeness.

For spin, recall that hadrons are composite particles of quarks, thus the quantum mechanical rule of spin addition holds, since each quark has spin $\frac{1}{2}$, a hadron may have spin $S = 0, \frac{1}{2}, 1, \frac{3}{2}$.

The parity of a particle is defined as $P = (-1)^L$ where L is total orbital angular momentum. At ground state, $L = 0 \Rightarrow P = \pm 1$.

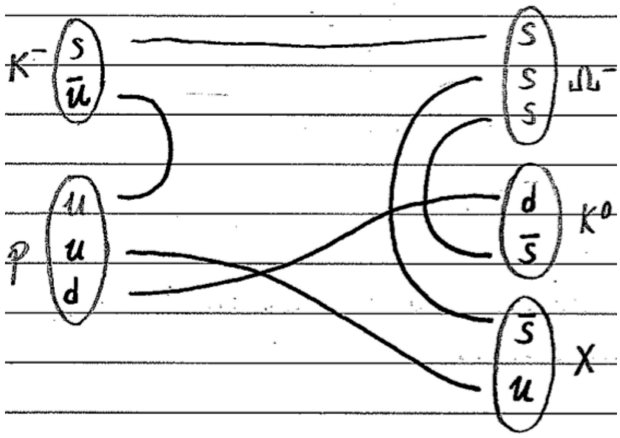
Give a more specific explanation for the multiplet given.

Why, for example, does the Σ^{*0} have a strangeness of -1? What values of q do the dashed lines represent?

(b)

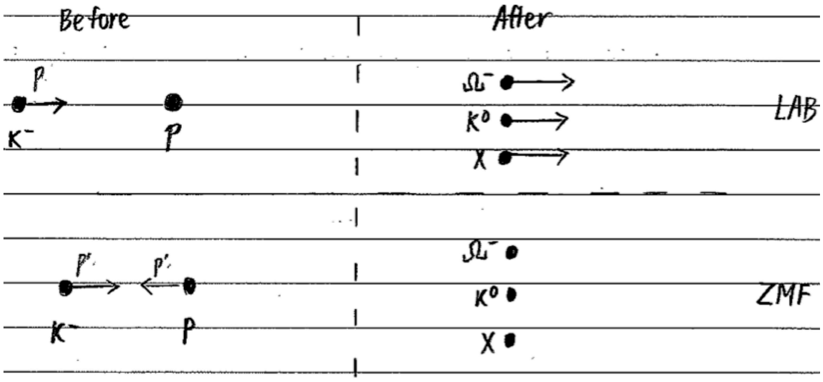
$$\mathrm{K}^- + \mathrm{p} \to \Omega^- + \mathrm{K}^0 + \mathrm{X}$$

Strong interaction preserves flavour so the quark contents should be preserved:



So X is K^+.

(c) For each strange quark added, the mass increases by $147\,\mathrm{MeV/c^2}$, so Ω^- should have mass $1532 + 147 = 1679\,(\mathrm{MeV/c^2})$.



At threshold energy, the products are at rest in ZMF.

Lorentz invariance of system 4-momentum:

$$
\begin{aligned}
\mathsf{P}_\mu \mathsf{P}^\mu &= \mathsf{P}'_\mu \mathsf{P}'^\mu \\
-\left(\frac{E}{c} + m_{\mathrm{p}} c\right)^2 + p^2 &= -\left(m_\Omega + m_{\mathrm{K}}^0 + m_{\mathrm{X}}\right)^2 c^2 \\
\underbrace{\left(\frac{E^2}{c^2} - p^2\right)}_{m_{\mathrm{K}}^2 c^2} + 2Em_{\mathrm{p}} + m_{\mathrm{p}}^2 c^2 &= \left(m_\Omega + m_{\mathrm{K}}^0 + m_{\mathrm{X}}\right)^2 c^2
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow E &= \frac{\left[\left(m_\Omega + m_{\mathrm{K}}^0 + m_{\mathrm{X}}\right)^2 - \left(m_{\mathrm{K}}^2 + m_{\mathrm{p}}^2\right)\right] c^2}{2m_{\mathrm{p}}} \\
&= 3200\,\mathrm{MeV}
\end{aligned}
$$

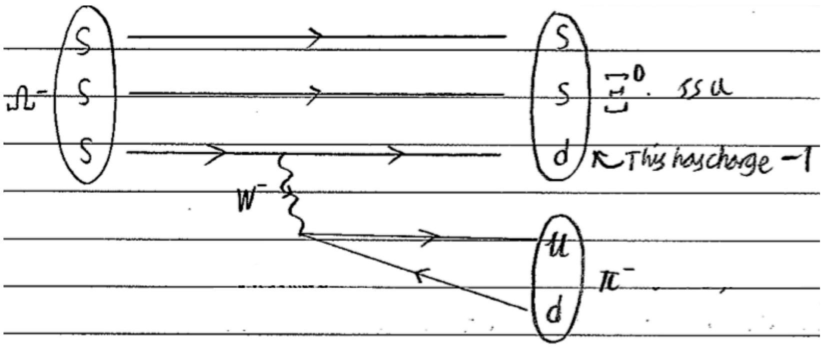
So threshold momentum:

$$
\begin{aligned}
pc &= \sqrt{E^2 - m_{\mathrm{K}}^2 c^4} \\
\Rightarrow p &= 3162\,\mathrm{MeV/c}
\end{aligned}
$$

(d)

$$
\begin{array}{ll}
\Omega^- \to \Xi^0 + & \pi^- \\
\mathrm{sss} \to \mathrm{ssu} & \mathrm{d\bar{u}}
\end{array}
$$

Conservation of flavour/quark number violated, try $W^\pm$ interaction:



(e) Momentum $p = \gamma m v = 2015\,\mathrm{MeV/c}$. Hence energy:

$$
\begin{aligned}
E &= \sqrt{m^2c^4 + p^2c^2} = \gamma m c^2 \\
\Rightarrow \gamma &= \frac{\sqrt{m^2c^4 + p^2c^2}}{mc^2} \\
\Rightarrow v &= c\left[1 - \gamma^{-2}\right]^{1/2} \\
&= c\left[1 - \frac{m^2c^4}{m^2c^4 + p^2c^2}\right]^{1/2}
\end{aligned}
$$

Measured lifetime:

$$
\begin{aligned}
t &= \frac{d}{v} \\
&= \frac{d}{c}\left(\frac{m^2c^4 + p^2c^2}{m^2c^4}\right)^{1/2}
\end{aligned}
$$

But time dilation means that $t = \gamma\tau$ where τ is proper lifetime:

$$
\begin{aligned}
\tau &= \frac{mc^2}{\sqrt{m^2c^4 + p^2c^2}} \cdot \frac{d}{c}\left[\frac{m^2c^4 + p^2c^2}{p^2c^2}\right]^{1/2} \\
&= \frac{mcd}{pc} \\
&= \frac{md}{p} \\
&= \frac{0.0208\,\mathrm{m}}{c} = 6.944 \times 10^{-11}\,\mathrm{s}
\end{aligned}
$$

3. (DRAFT)

(a) $\mathbb{B}$ is defined as $+\frac{1}{3}$ for quarks, $-\frac{1}{3}$ for anti-quarks, 0 for leptons. Therefore a baryon would have $\mathbb{B} = \pm 1$, where negative sign is for anti-baryons. Mesons only have $\mathbb{B} = 0$ since they contain a quark-antiquark pair. More detailed explanation for baryons and mesons would be good.

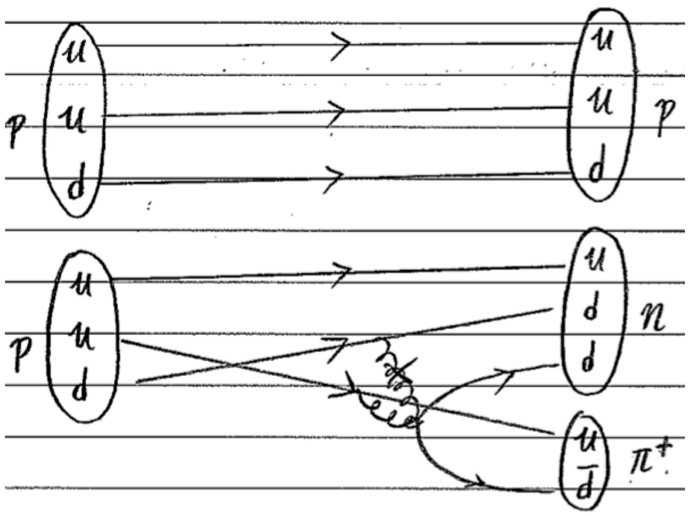
(b) i.

$$\mathrm{p} + \mathrm{p} \to \mathrm{n} + \mathrm{p} + \pi^+$$

$$\begin{array}{cc} \mathbb{B} = 2 & \mathbb{B} = 2 \\ \mathbb{Q} = +2 & \mathbb{Q} = +2 \\ \mathbb{L} = 0 & \mathbb{L} = 0 \end{array}$$

As $\mathbb{B}$, $\mathbb{Q}$, $\mathbb{L}$ are all conserved, the above process is allowed. (Try breakdown the calc.)

Feynman diagram:



ii.

$$\mathrm{p} + \mathrm{e}^- \to \rho^0$$

$$\begin{array}{cc} \mathbb{B} = 1 & \mathbb{B} = 0 \\ \mathbb{Q} = 0 & \mathbb{Q} = 0 \\ \mathbb{L} = +1 & \mathbb{L} = 0 \end{array}$$

Since lepton number and baryon number are not conserved, the process is forbidden.

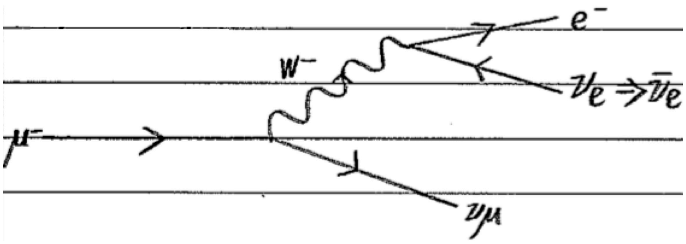
iii.

$$\mu^- \to \mathrm{e}^- + \nu_\mu + \bar{\nu}_\mathrm{e}$$

$$\begin{array}{cc} \mathbb{B} = 0 & \mathbb{B} = 0 \\ \mathbb{Q} = -1 & \mathbb{Q} = -1 \\ \mathbb{L} = 1 & \mathbb{L} = 1 \end{array}$$

Since $\mathbb{B}$, $\mathbb{Q}$, $\mathbb{L}$ are all conserved, the process is allowed.

Feynman diagram:



iv.

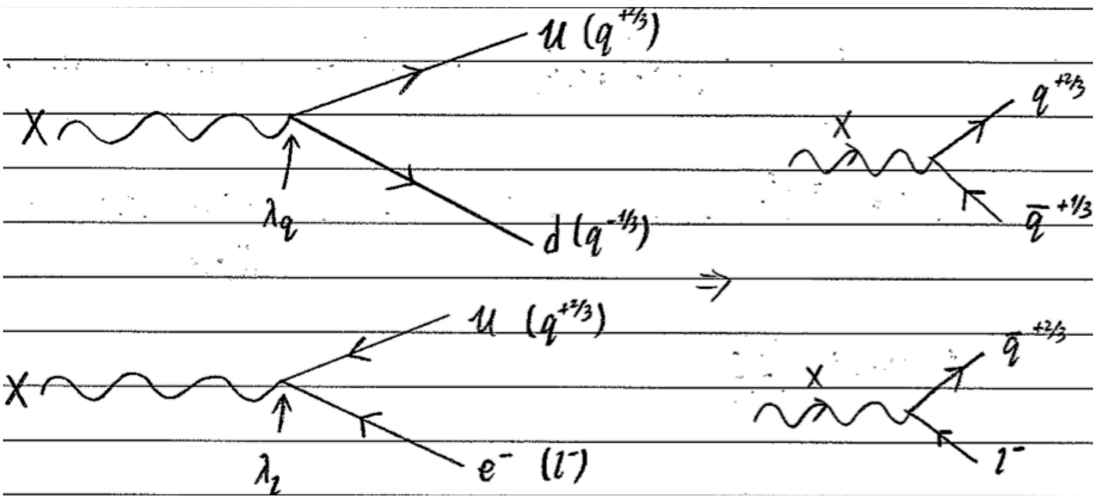
$$\nu_{\mathrm{e}} + \mathrm{e}^{+} \to \mathrm{d} + \bar{\mathrm{u}}$$

$$\begin{array}{ll} \mathbb{B} = 0 & \mathbb{B} = 0 \\ \mathbb{Q} = +1 & \mathbb{Q} = -1 \\ \mathbb{L} = 0 & \mathbb{L} = 0 \end{array}$$

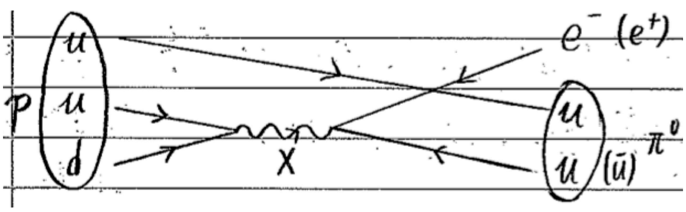
Since $\mathbb{Q}$ is not conserved, the process is forbidden.

(c) X boson: $\mathbb{Q} = +\frac{1}{3}$; $\mathbb{B}$, $\mathbb{L}$ not conserved; $m_{\mathrm{X}} \gg m_{\mathrm{p}}$, $\mathbb{Q}$ conserved.

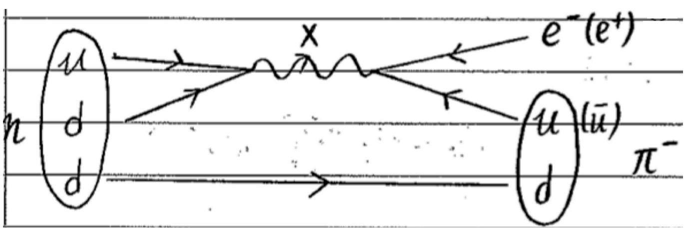
Fundamental vertices:



Now we try process $\mathrm{p} \to \mathrm{e}^{+} + \pi^{0}$:



$\mathrm{n} \to \mathrm{e}^{+} + \pi^{-}$:



(d) Recall matrix element: (for $\mathrm{p} \to \mathrm{e}^+ + \pi^0$)

$$\begin{aligned} |M_{if}| &= |\prod_i \text{vertex factor}| \times |\prod_i \text{propagator factor}| \\ &= \lambda_q \lambda_l \cdot \left| \frac{1}{\mathsf{P}^2 - m_\mathrm{X}^2 c^4} \right| \end{aligned}$$

with $\mathsf{P}^\mu = \left(\frac{E_\mathrm{u} + E_\mathrm{d}}{c}, \mathbf{p}_\mathrm{u} + \mathbf{p}_\mathrm{d} \right)$ the 4-momentum transferred.

But since $m_\mathrm{X} \gg m_\mathrm{p}$, $\mathsf{P}^2 \ll m_\mathrm{X}^2 c^2$ over most energy range in decay:

$$\Rightarrow |M_{if}| \simeq \frac{\lambda_q \lambda_l}{m_\mathrm{X}^2 c^4}$$

Fermi's Golden Rule gives reaction rate:

$$\Gamma = \frac{2\pi}{\hbar} |M_{if}|^2 \rho(E)$$

From Sargent's rule, density of states:

$$\rho(E) \propto E^5 = (m_\mathrm{p} c^2)^5$$

Thus:

$$\Gamma_{if} \propto \frac{2\pi}{\hbar} \frac{\lambda_l^2 \lambda_q^2}{m_\mathrm{X}^4} m_\mathrm{p}^5 c^2$$

So:

$$\begin{aligned} \tau_\mathrm{p} &\propto \frac{1}{\Gamma_{if}} \\ &= A \frac{m_\mathrm{X}^4 \hbar}{\lambda_l^2 \lambda_q^2 m_\mathrm{p}^5 c^2} \end{aligned}$$

4. (DRAFT)

(a) Differential cross section is the derivative of total cross section of some scattering w.r.t. the solid angle.

$$\sigma = \int \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\,\mathrm{d}\Omega$$
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{m}{2\pi\hbar^2}\right)^2 \left|\int V(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}\,\mathrm{d}^3\mathbf{x}\right|^2$$

where $\mathbf{q} = |\mathbf{k}' - \mathbf{k}|$ is the scattering wavevector, m is the rest mass of the particle.

(b) Potential:

$$V_\mu(r) = V_0\frac{e^{-\mu r}}{r}$$

$$\begin{aligned}\Rightarrow \int V(\mathbf{x})e^{i\mathbf{q}\cdot\mathbf{x}}\,\mathrm{d}^3\mathbf{x} &= 2\pi\int_0^\pi \mathrm{d}\theta \int_0^\infty V_0\frac{e^{-\mu r}}{r}e^{iqr\cos\theta}r^2\sin\theta\,\mathrm{d}r \\ &= 2\pi V_0\int_0^\pi \mathrm{d}\theta\int_0^\infty \sin\theta e^{r(-\mu+iq\cos\theta)}\cdot r\,\mathrm{d}r\end{aligned}$$

Integration by parts:

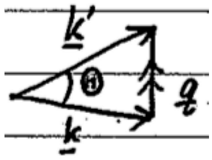
$$\begin{aligned} F &= r & &\Rightarrow & \mathrm{d}F &= \mathrm{d}r \\ \mathrm{d}G &= e^{r(-\mu+iq\cos\theta)}\,\mathrm{d}r & & & G &= \tfrac{e^{r(-\mu+iq\cos\theta)}}{-\mu+iq\cos\theta}\end{aligned}$$

$$\begin{aligned}&\Rightarrow 2\pi V_0\int_0^\pi \mathrm{d}\theta\left[\cancelto{0}{\left[r\frac{e^{r(-\mu+iq\cos\theta)}}{-\mu+iq\cos\theta}\right]} - \int_0^\infty \frac{e^{r(-\mu+iq\cos\theta)}}{-\mu+iq\cos\theta}\,\mathrm{d}r\right] \\ &= 2\pi V_0\int_0^\infty \mathrm{d}\theta\left[-\frac{e^{r(-\mu+iq\cos\theta)}}{(-\mu+iq\cos\theta)^2}\right]_{r=0}^{r=\infty} \\ &= 2\pi V_0\int_0^\pi \frac{1}{(-\mu+iq\cos\theta)^2}\,\mathrm{d}\theta \\ &= \frac{2\pi^2 V_0}{\mu^2 - q^2 - 2\mu q i}\end{aligned}$$

So:

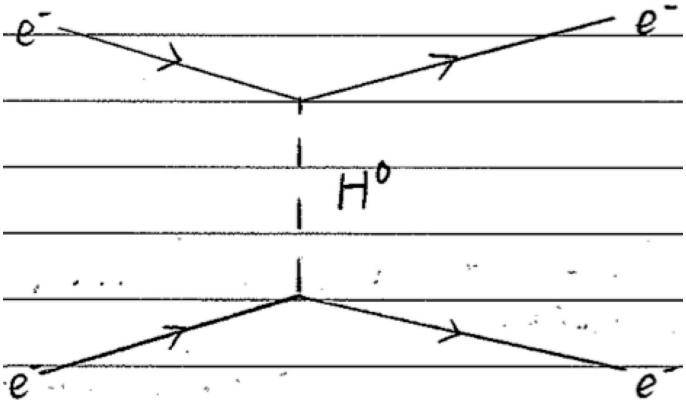
$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &= \left(\frac{m}{2\pi\hbar^2}\right)^2\left|\frac{2\pi^2V_0}{\mu^2-q^2-2\mu q i}\right|^2 \\ &= \left[\frac{m}{2\pi\hbar^2}\cdot 2\pi^2 V_0\right]^2\cdot\frac{1}{(\mu^2-q^2)+4\mu^2q^2} \\ &= \left(\frac{2mV_0}{\hbar^2}\right)^2\cdot\frac{1}{(q^2+\mu^2)^2}\end{aligned}$$

(TO EXPAND) For Coulomb scattering, $\mu = 0$, $q \to q\cos\theta$:



$$
\begin{aligned}
\Rightarrow \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} &\propto \frac{1}{q^4 \cos^4 \theta} \\
&\propto \frac{1}{\cos^4(\pi - \Theta)} \\
&= \frac{1}{(-\cos\Theta)^4} \\
&\propto \frac{1}{\cos^4 \Theta} \\
&= \sec^4 \Theta
\end{aligned}
$$

(c) Elastic e^--e^- scattering by H^0 bosons:

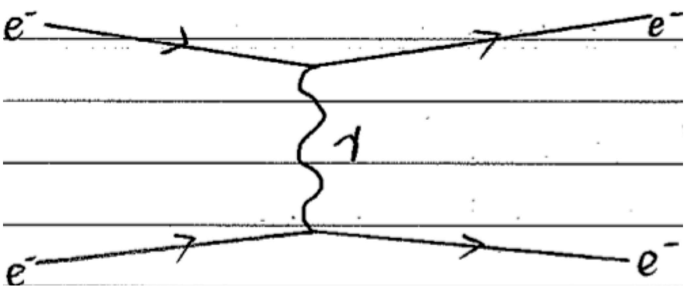


Matrix element:

$$M_{if} = \lambda_\mathrm{e}^2 \cdot \frac{1}{\mathsf{P}^2 - m_\mathrm{H}^2 c^2}$$

Since the scattering is elastic, no energy is transferred so $\mathsf{P}^2 = q^2$ where q is momentum of electrons.

Now for photon exchange:



Matrix element:

$$\begin{aligned} M'_{if} &= (\sqrt{4\pi\alpha})^2 \cdot \frac{1}{\mathsf{P}^2} \quad \text{since } \gamma \text{ is massless} \\ &= 4\pi\alpha \cdot \frac{1}{q^2} \quad \text{due to it being elastic} \end{aligned}$$

Hence:

$$\begin{aligned} R_H &= \frac{M_{if}}{M'_{if}} \\ &= \frac{\lambda_{\mathrm{e}}^2}{4\pi\alpha} \frac{q^2}{q^2 + \mu_{\mathrm{H}}^2} \end{aligned}$$

where $\mu_{\mathrm{H}} = i m_{\mathrm{H}} c$.

At ZMF energy of 200 GeV, $q \simeq 200\,\mathrm{GeV/c}$ so:

$$\begin{aligned} R_H &= \frac{\lambda_{\mathrm{e}}^2}{4\pi\alpha} \frac{q^2}{q^2 - m_{\mathrm{H}}^2 c^2} \\ &= \frac{(2.9 \times 10^{-6})^2}{\frac{4\pi}{137}} \cdot \frac{(200\,\mathrm{GeV/c})^2}{(200\,\mathrm{GeV/c})^2 - (125\,\mathrm{GeV/c})^2} \\ &= 1.5 \times 10^{-10} \lll 1 \end{aligned}$$

Since R_H is extremely small($\sim \alpha^5$ so much much smaller than EM interaction), measuring H^0 exchange via e^--e^- collision is not viable.