UNOFFICIAL SOLUTIONS BY TheLongCat

B3: ATOMIC AND LASER PHYSICS

TRINITY TERM 2020

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Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. **(DRAFT)**

(a) By considering a quantity called central field $S(r)$, the Hamiltonian may be written as

$$\hat{H} = \hat{H}_{\text{CF}} + \Delta\hat{H}_{\text{RE}}$$

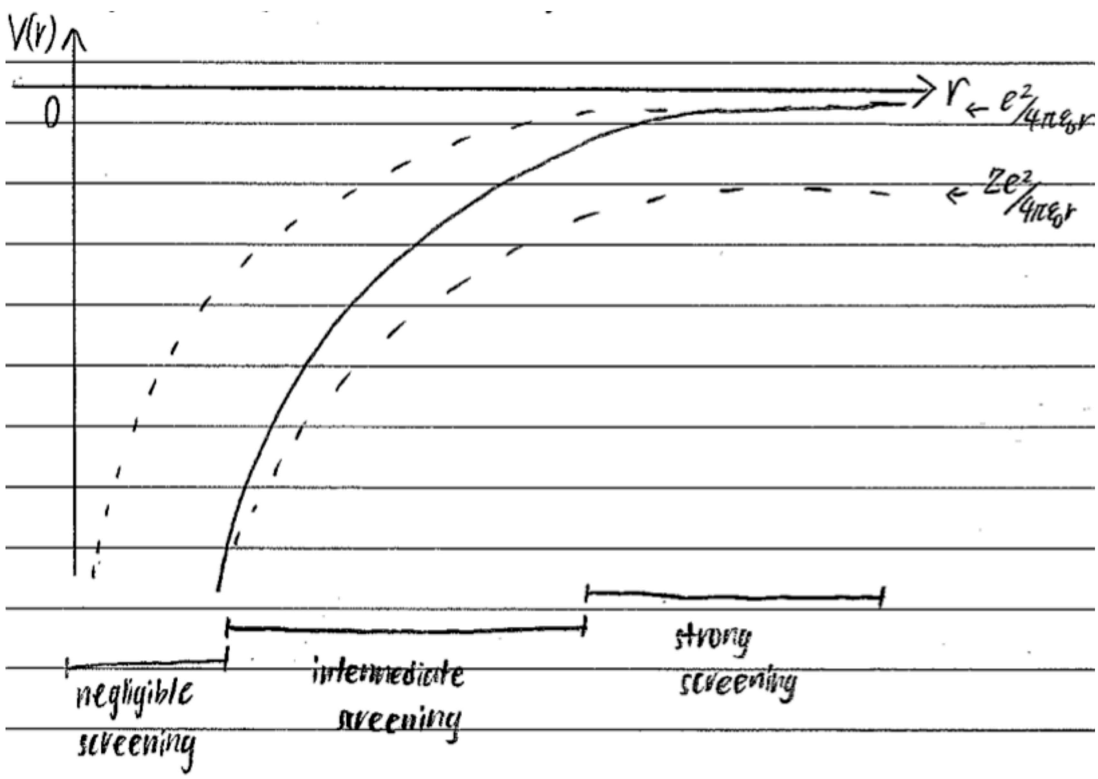
where $\hat{H}_{\text{CF}} = \sum_i \left[\frac{\hat{\mathbf{p}}_i^2}{2m} - \frac{Ze^2}{4\pi\epsilon_0 \hat{r}_i} + S(r_i)\right]$ is central field, $\Delta\hat{H}_{\text{RE}} = \sum_i \left[-S(r_i) + \sum_{j>i} \frac{e^2}{4\pi\epsilon_0 \hat{r}_{ij}}\right]$ is residual electrostatic.

This approximation is called central field approximation where a suitable choice of $S(r)$ minimises $\Delta\hat{H}_{\text{RE}}$. The Hamiltonian is then separable into different equations for e$^-$, giving electronic configuration with quantum numbers: n the principal quantum number, and l the orbital angular momentum.

For $\Delta\hat{H}_{\text{RE}}$, it couples $\mathbf{l}$'s of different e$^-$, however since it is an internal interaction, the total orbital angular momentum $\mathbf{L}$ is conserved, making L a good quantum number. Likewise, sinee $\Delta\hat{H}_{\text{RE}}$ does not act on total e$^-$ spin $\mathbf{S}$, S is also a good quantum number. The labelling of eigenstates as $|LM_LSM_S\rangle$ or $|LSJM_J\rangle$ is called LS coupling.

(b) At small distances, the potential is simply $-\dfrac{Ze^2}{4\pi\epsilon_0 r}$ as there is no screening effect.

At large distances, however, the potential is $-\dfrac{e^2}{4\pi\epsilon_0 r}$ as the e$^-$'s screen the nuclear charge, making an effective charge of $+e$.



(c) e^- density $n(p) = \dfrac{8\pi}{3h^3}p^3$

$$\frac{p^2}{2m} = eV$$
$$\Rightarrow \rho = -\frac{8\pi}{3h^3}\left(2meV\right)^{3/2}$$

Maxwell: $\nabla^2 V = \dfrac{8\pi}{3\epsilon_0 h^3}\left(2meV\right)^{3/2}$

Ansatz $V(r) = \dfrac{\chi(r)Ze}{4\pi\epsilon_0 r}$:

$$\begin{aligned}\Rightarrow \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(\dot{\chi}\frac{Zer}{4\pi\epsilon_0} - \chi\frac{Ze}{4\pi\epsilon_0}\right) \\ &= \ddot{\chi}\frac{Ze}{4\pi\epsilon_0 r} + \cancel{\dot{\chi}\frac{Ze}{4\pi\epsilon_0 r^2}} - \cancel{\dot{\chi}\frac{Ze}{4\pi\epsilon_0 r^2}} \\ &= \ddot{\chi}\frac{Ze}{4\pi\epsilon_0 r}\end{aligned}$$

RHS reads:

$$\frac{8\pi e}{3\epsilon_0 h^3}\left(2me\frac{\chi Ze}{4\pi\epsilon_0 r}\right)^{3/2} = \chi^{3/2}\cdot\frac{16\sqrt{2}\pi e^{5/2}Z^{3/2}}{23\epsilon_0^{5/2}\pi^{3/2}h^3 r^{3/2}}$$

So:

$$\begin{aligned}\Rightarrow \ddot{\chi} &= \chi^{3/2}\frac{2\sqrt{2}e^{5/2}Z^{1/2}}{\frac{3}{4}\pi^{-1/2}\epsilon_0^{3/2}h^3 r^{1/2}} \\ &= \chi^{3/2}\frac{8\sqrt{2}\pi^{1/2}e^{5/2}Z^{1/2}}{3\epsilon_0^{3/2}h^3 r^{1/2}} \\ &= \chi^{3/2}\left(\frac{b}{r}\right)^{1/2}\end{aligned}$$

where $b = \dfrac{128\pi e^5 Z}{9\epsilon_0^3 h^6}$.

(d) We have assumed that the e^- behaves like a Fermi gas, but the shell structure means that the distribution of e^- is not uniform, so this model would only work for massive atoms where the distribution is approximately uniform. ERRATA: assumed uniform ρ so need large number of e^-.

(e) Lower angular momentum states are favoured more as they reside deeper in the energy level for multiple reasons: lower angular momentum tends to penetrate the core e^- more often, thereby experiencing lower screening and is more tightly bound to nucleus. Furthermore, the existence of angular momentum barrier as $\dfrac{L^2}{2mr^2}$ means that lower angular momentum would contribute to having lower energy for similar penetration.

ADDENDUM:

$4s$ before $3d$: minimise ang mtm to minimise centrifugal barrier, lower l lower potential.

2. (DRAFT)

(a) $A\mathbf{I}\cdot\mathbf{J}$: the hyperfine interaction where the intrinsic magnetic moments of nucleus and e^- interact via magnetic dipole interaction.

$g_J\mu_\mathrm{B}\mathbf{J}\cdot\mathbf{B}$: external magnetic interaction between magnetic moments of e^- and the external magnetic field. $\mathbf{J}$ is used in place of $\sum_i \mathbf{l}_i = \mathbf{L}$ and $\sum_i \mathbf{s}_i = \mathbf{S}$ by Wigher-Eckart theorem.

$g_I\mu_\mathrm{B}\mathbf{I}\cdot\mathbf{B}$: similar to above but it is the nuclear magnetic moments which interact instead.

(b) In the weak field limit, $A\mathbf{I}\cdot\mathbf{J} \gg g_I\mu_\mathrm{B}\mathbf{I}\cdot\mathbf{B}$ (nuclear term negligible).

Hyperfine interaction has typical energies in microwave range, e.g. $\lambda = 21\,\mathrm{cm}$.

So:

$$\begin{aligned} \mu_\mathrm{B}B &\sim \frac{hc}{\lambda} \\ \Rightarrow B &\sim \frac{hc}{\lambda\mu_\mathrm{B}} = 0.10\,\mathrm{T} \end{aligned}$$

within this the weak field limit applies.

(c) Ignoring nuclear term since it is negligible ($\mu_\mathrm{N} = \dfrac{m_e}{m_\mathrm{p}} \ll \mu_\mathrm{B}$):

$$\Delta\hat{H} = A\mathbf{I}\cdot\mathbf{J} + g_J\mu_\mathrm{B}\mathbf{J}\cdot\mathbf{B}$$

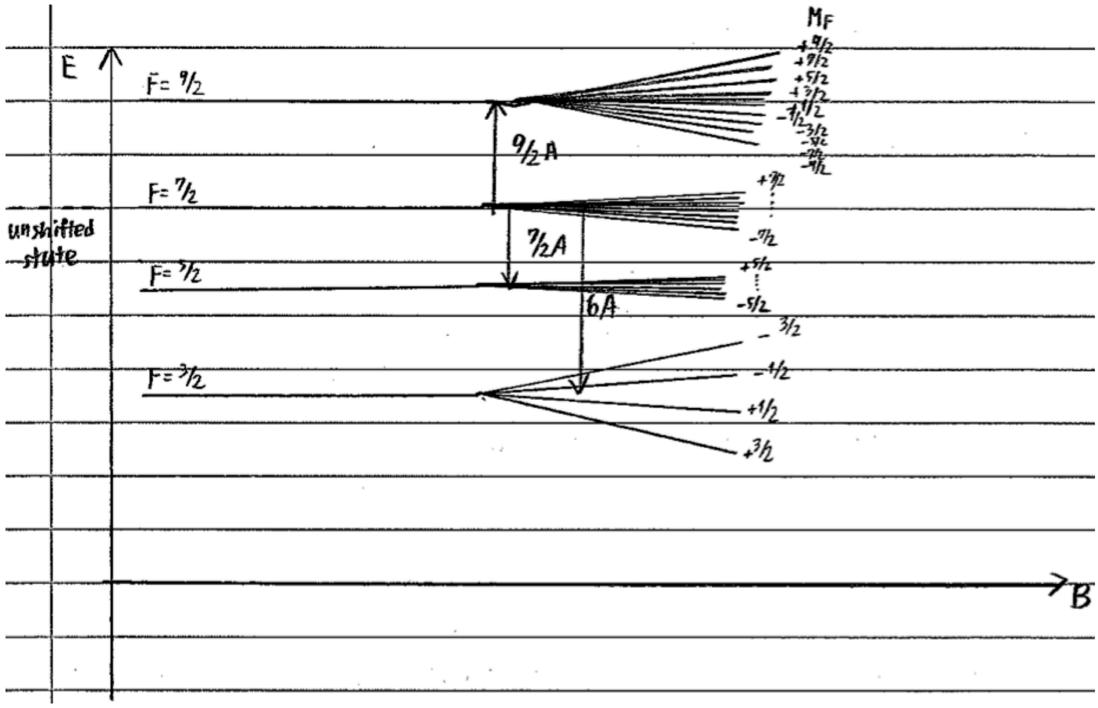
Under weak field, the dominant interaction is the hyperfine interaction, which means that $\mathbf{F} = \mathbf{I} + \mathbf{J}$ is conserved and so F is a good quantum number.

By Wigner-Eckart, $\mathbf{J} \to \dfrac{\mathbf{J}\cdot\mathbf{F}}{F^2}\mathbf{F}$ so:

$$\begin{aligned} \Delta\hat{H} &= A\cdot\frac{1}{2}\left[\hat{\mathbf{F}}^2 - \hat{\mathbf{I}}^2 - \hat{\mathbf{J}}^2\right] + g_J\mu_\mathrm{B}\frac{\hat{J}^2 + \hat{\mathbf{I}}\cdot\hat{\mathbf{J}}}{F^2}\mathbf{F}\cdot\mathbf{B} \\ \Rightarrow \Delta E &= \frac{A}{2}\underbrace{[F(F+1) - I(I+1) - J(J+1)]}_{K} \\ &\quad + \underbrace{g_J\frac{J(J+1) + \frac{1}{2}[F(F+1) - I(I+1) - J(J+1)]}{F(F+1)}}_{g_F}\mu_\mathrm{B}M_F B \end{aligned}$$

(d) $J = 3/2$, $I = 3$. So the possible values of F are:

$$\begin{aligned} \frac{3}{2} &\longrightarrow K = -12 & \frac{g_F}{g_J} &= -\frac{3}{5} \\ \frac{5}{2} &\longrightarrow K = -7 & \frac{g_F}{g_J} &= \frac{1}{35} \\ \frac{7}{2} &\longrightarrow K = 0 & \frac{g_F}{g_J} &= \frac{5}{21} \\ \frac{9}{2} &\longrightarrow K = 9 & \frac{g_F}{g_J} &= \frac{1}{3} \end{aligned}$$



(e) In the strong field case, the dominant interaction is the external Hamiltonian, making M_I and M_J good quantum numbers instead.

$$\Rightarrow \Delta E = AM_IM_J + g_J\mu_\mathrm{B}M_JB - g_I\mu_\mathrm{N}M_IB$$

So for hydrogen ground state $1s\,^2S_{1/2}$:

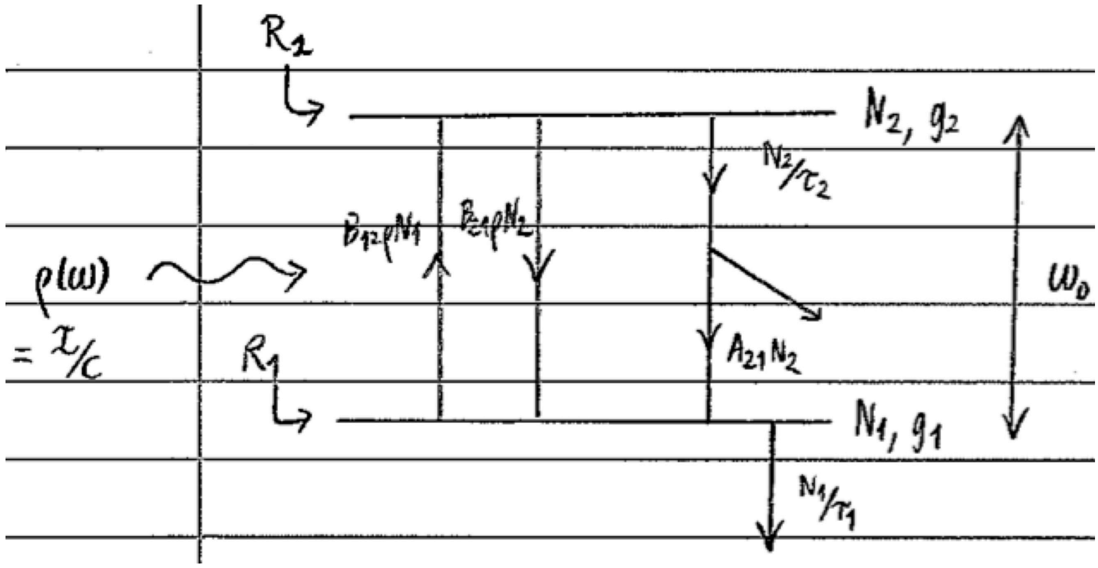
$$\Delta E\left(J_z = \frac{1}{2}, I_z = -\frac{1}{2}\right) = -\frac{3}{4}A + \frac{B}{2}\left[g_J\mu_\mathrm{B} + g_I\mu_\mathrm{N}\right]$$
$$\Delta E\left(J_z = -\frac{1}{2}, I_z = -\frac{1}{2}\right) = \frac{3}{4}A + \frac{B}{2}\left[-g_J\mu_\mathrm{B} + g_I\mu_\mathrm{N}\right]$$
$$\Delta E\left(J_z = \frac{1}{2}, I_z = \frac{1}{2}\right) = \frac{3}{4}A + \frac{B}{2}\left[g_J\mu_\mathrm{B} - g_I\mu_\mathrm{N}\right]$$

So:

$$\frac{\Delta E\left(J_z = \frac{1}{2}, I_z = -\frac{1}{2}\right) + \Delta E\left(J_z = -\frac{1}{2}, I_z = -\frac{1}{2}\right)}{\Delta E\left(J_z = \frac{1}{2}, I_z = -\frac{1}{2}\right) + \Delta E\left(J_z = \frac{1}{2}, I_z = \frac{1}{2}\right)} = \frac{\frac{B}{2}\left[2g_I\mu_\mathrm{N}\right]}{\frac{B}{2}\left[2g_J\mu_\mathrm{B}\right]} = \frac{g_p\mu_\mathrm{N}}{g_s\mu_\mathrm{B}}$$

for hydrogen (^{1}H) specifically.

3. (DRAFT)

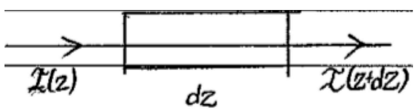


(a) When a light passes through the system, it interacts with it by inducing stimulated emission/absorption, which for an incoherent light may be described by Einstein's B coefficient.

Otherwise Rabi oscillations may occur and we may proceed via time-dependent perturbation theory.

Rate equations (for a single frequency):

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = R_2 + B_{12}\rho N_1 - B_{21}\rho N_2 - \frac{N_2}{\tau_2}$$
$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = R_1 - B_{12}\rho N_1 + B_{21}\rho N_2 + A_{21}N_2 - \frac{N_1}{\tau_1}$$



Energy change within a volume element $A\,\mathrm{d}z$: (replace $\rho \to \rho g\delta\omega$ for finite ω width)

$$\mathrm{d}\mathcal{I}\cancel{A\delta\omega} = \left[-B_{12}\rho g\cancel{\delta\omega}N_1 + B_{21}\rho g\cancel{\delta\omega}N_2\right]\hbar\omega_0\cancel{A}\,\mathrm{d}z$$
$$\frac{\mathrm{d}\mathcal{I}}{\mathrm{d}z} = \underbrace{\left[N_2 - \frac{g_2}{g_1}N_1\right]}_{N^*} B_{21}\rho g\hbar\omega_0 \quad \text{since } g_1B_{12} = g_2B_{21}$$
$$= N^*\,\underbrace{B_{21}(\omega-\omega_0)\frac{\hbar\omega_0}{c}}_{\sigma_{21}(\omega-\omega_0)}\mathcal{I}$$

Now rewrite the rate equations with N^*, σ_{21} and $\int_0^\infty \mathrm{d}\omega$ to get $\int_0^\infty \mathcal{I}\,\mathrm{d}\omega$ total intensity:

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = R_2 - N^*\sigma_{21}\frac{I}{\hbar\omega} - \frac{N_2}{\tau_2}$$
$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = R_1 + N^*\sigma_{21}\frac{I}{\hbar\omega} + A_{21}N_2 - \frac{N_1}{\tau_1}$$

At steady state,

$$\begin{aligned} N_2 &= R_2\tau_2 - N^*\sigma_{21}\frac{I}{\hbar\omega}\tau_2 \\ N_1 &= R_1\tau_1 + N^*\sigma_{21}\frac{I}{\hbar\omega}\tau_1 + A_{21}N_2\tau_1 \end{aligned}$$

Hence:

$$\begin{aligned} N^* &= N_2 - \frac{g_2}{g_1}N_1 \\ &= R_2\tau_2 - N^*\sigma_{21}\frac{I}{\hbar\omega}\left(\tau_2 - \frac{g_2}{g_1}\tau_1\right) - \frac{g_2}{g_1}R_1\tau_1 - \frac{g_2}{g_1}A_{21}\tau_1\left[R_2\tau_2 - N^*\sigma_{21}\frac{I}{\hbar\omega}\tau_2\right] \\ \Rightarrow N^*(I) &= \frac{N^*(0)}{1+\frac{I}{I_s}} \end{aligned}$$

where

$$\begin{aligned} N^*(0) &= R_2\tau_2 - \frac{g_2}{g_1}R_1\tau_1 - \frac{g_2}{g_1}A_{21}\tau_1R_2\tau_2 \\ &= R_2\tau_2\left[1 - \frac{g_2}{g_1}A_{21}\tau_1\right] - \frac{g_2}{g_1}R_1\tau_1 \end{aligned}$$

$$\begin{aligned} I_s &= \left[\frac{\sigma_{21}}{\hbar\omega}\underbrace{\tau_2 - \frac{g_2}{g_1}\tau_1 + \frac{g_2}{g_1}A_{21}\tau_1\tau_2}_{\tau_R}\right]^{-1} \\ &= \frac{\hbar\omega}{\sigma_{21}\tau_R} \end{aligned}$$

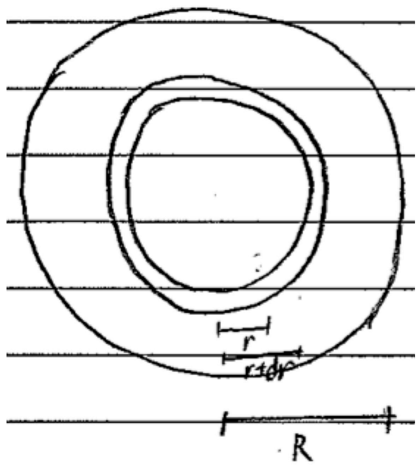
So gain coefficient:

$$\begin{aligned} \alpha(I) &= N^*(I)\sigma_{21} \\ &= \frac{\alpha(0)}{1+\frac{I}{I_s}} \end{aligned}$$

with $\alpha(0) = N^*(0)\sigma_{21}$.

So ${}^{\mathrm{d}I}\!/_{\mathrm{d}z} = \alpha(I)I$ from before, but we integrated over all ω to get I.

(b) Sketch of the spherical laser:



Uniform population density $\Rightarrow N_i(r) = N_i(R)\left(\frac{r}{R}\right)^3$

Spherical symmetry means that we can reuse ${}^{\mathrm{d}I}\!/_{\mathrm{d}z}$ from before:

$$\frac{\mathrm{d}I}{\mathrm{d}r} = \underbrace{N^*(I,r)\sigma_{21}}_{\alpha(I)\left(\frac{r}{R}\right)^3} I$$

with $N^*(I,r) = N^*(I)\left(\frac{r}{R}\right)^3$.

(c) So now we have:

$$\frac{1}{I} + \frac{1}{I_s}\,\mathrm{d}I = \alpha(0)\left(\frac{r}{R}\right)^3 \mathrm{d}r$$
$$\ln\left(\frac{I}{I_0}\right) + \frac{I - I_0}{I_s} = \frac{\alpha(0)}{4R^3}r^4$$

Since $I(R) = 0.1I_s \ll I_s$, we may ignore the linear term:

$$I = I_0 e^{\frac{\alpha(0)}{4R^3}r^4}$$

Hence power $\mathcal{P}(r) = I(r)\cdot 4\pi r^2$ increases exponentially in the sphere.

(d) Differentiating:

$$\frac{\mathrm{d}I}{\mathrm{d}r} = \frac{4I_0\alpha(0)}{4R^3}r^3 e^{\frac{\alpha(0)}{4R^3}r^4}$$

It is clear that when $r = 0$, ${}^{\mathrm{d}I}\!/_{\mathrm{d}r} = 0$ and it constitutes a minimum since exp is a monotonically increasing function.

(e) When $r = 0$, $I = I_0$ which should be non-zero for lasing to occur. This also implies that for a spherical laser medium, a spontaneous emission is required to initiate population inversion – which makes it difficult to achieve under normal conditions.

However, astronomical lasers tend to have extreme conditions, e.g. high pressure or temperature that make such lasing possible.

4. **(DRAFT)**

(a) Hamiltonian $\hat{H} = \hat{H}_0 + \hat{V}(t)$ where $\hat{H}_0$ is the unperturbed Hamiltonian.

TDSE gives $\hat{H}\left|\psi\right\rangle = i\hbar {}^{\partial}/_{\partial t}\left|\psi\right\rangle$.

Substituting ansatz

$$\psi(t, \mathbf{r}) = c_1(t)\psi_1(\mathbf{r})e^{-\frac{iE_1 t}{\hbar}} + c_2(t)\psi_2(\mathbf{r})e^{-\frac{iE_2 t}{\hbar}}$$

then gives: (since ψ_1 and ψ_2 are the stationary states with energies E_1 and E_2, they come with a phase factor $e^{-\frac{iEt}{\hbar}}$ as per TDSE, this is simply a superposition of the two states)

$$i\hbar\begin{bmatrix} \dot{c}_1\psi_1 e^{-\frac{iE_1 t}{\hbar}} - \cancel{\frac{iE_1}{\hbar}c_1\psi_1 e^{-\frac{iE_1 t}{\hbar}}} \\ + \dot{c}_2\psi_2 e^{-\frac{iE_2 t}{\hbar}} - \cancel{\frac{iE_2}{\hbar}c_2\psi_2 e^{-\frac{iE_2 t}{\hbar}}} \end{bmatrix} = \begin{matrix} \cancel{E_1 c_1\psi_1 e^{-\frac{iE_1 t}{\hbar}}} + \cancel{E_2 c_2\psi_2 e^{-\frac{iE_2 t}{\hbar}}} \\ + \hat{V}c_1\psi_1 e^{-\frac{iE_1 t}{\hbar}} + \hat{V}c_2\psi_2 e^{-\frac{iE_2 t}{\hbar}} \end{matrix}$$

$$i\hbar\dot{c}_1\psi_1 e^{-\frac{iE_1 t}{\hbar}} + i\hbar\dot{c}_2\psi_2 e^{-\frac{iE_2 t}{\hbar}} = \hat{V}c_1\psi_1 e^{-\frac{iE_1 t}{\hbar}} + \hat{V}c_2\psi_2 e^{-\frac{iE_2 t}{\hbar}}$$

$\times\psi_1^*$ then gives:

$$\begin{aligned} i\hbar\dot{c}_1 e^{-\frac{iE_1 t}{\hbar}} &= \left\langle 1\right|\hat{V}\left|1\right\rangle c_1 e^{-\frac{iE_1 t}{\hbar}} + \left\langle 1\right|\hat{V}\left|2\right\rangle c_2 e^{-\frac{iE_2 t}{\hbar}} \\ \Rightarrow \dot{c}_1 &= -\frac{i}{\hbar}\left\langle 1\right|\hat{V}\left|2\right\rangle e^{-i\omega_0 t}c_2 \quad \text{for } \hbar\omega_0 = E_2 - E_1 \text{ and } \hat{V} \text{ prohibits null transition} \\ &= \frac{i}{\hbar}\mathcal{E}_0\chi_{12}\cos(\omega t)e^{-i\omega_0 t}c_2 \quad \text{since } \hat{V}(t) = ex\mathcal{E}_0\cos(\omega t) \\ &= \frac{i\mathcal{E}_0\chi_{12}}{2\hbar}\left[e^{i(\omega-\omega_0)t} + e^{-i(\omega+\omega_0)t}\right]c_2 \end{aligned}$$

Similarly for $\dot{c}_2$, we replace $e^{-i\omega_0 t} \to e^{i\omega_0 t}$:

$$\Rightarrow \dot{c}_2 = \frac{i\mathcal{E}_0\chi_{12}}{2\hbar}\left[e^{+i(\omega+\omega_0)t} + e^{-i(\omega-\omega_0)t}\right]c_1$$

(b) For weak perturbation, $c_2 \simeq 0$, $c_1 \simeq 1$:

$$\begin{aligned} \Rightarrow \dot{c}_2 &= i\frac{\mathcal{E}_0\chi_{12}}{2\hbar}\left[e^{-i(\omega-\omega_0)t} + e^{i(\omega+\omega_0)t}\right] \\ c_2(t) &= i\frac{\mathcal{E}_0\chi_{12}}{2\hbar}\left[\frac{e^{-i(\omega-\omega_0)t}}{-i(\omega-\omega_0)} + \frac{e^{i(\omega+\omega_0)t}}{i(\omega+\omega_0)}\right]_{t=0}^{t} \end{aligned}$$

Since $\left|\frac{\delta\omega}{\omega_0}\right| \ll 1$, we may invoke rotating wave approximation where $|\omega - \omega_0| \ll \omega + \omega_0$:

$$\begin{aligned} \Rightarrow c_2(t) &\simeq i\frac{\mathcal{E}_0\chi_{12}}{2\hbar}\left[\frac{e^{-i(\omega-\omega_0)t} - 1}{-i(\omega-\omega_0)}\right] \\ &= i\frac{\mathcal{E}_0\chi_{12}}{\hbar}\cdot\frac{e^{-i\frac{\delta\omega}{2}t}}{\delta\omega}\left[\frac{e^{-i\frac{\delta\omega}{2}t} - e^{i\frac{\delta\omega}{2}t}}{-2i}\right] \quad \text{for } \delta\omega = \omega - \omega_0 \\ &= \frac{ie^{-i\frac{\delta\omega}{2}t}}{\delta\omega}\frac{\mathcal{E}_0\chi_{12}}{\hbar}\sin\left(\frac{\delta\omega}{2}t\right) \end{aligned}$$

So:

$$|c_2(t)|^2 = \left(\frac{1}{\delta\omega}\right)^2 \left(\frac{\mathcal{E}_0\chi_{12}}{\hbar}\right)^2 \sin^2\left(\frac{\delta\omega}{2}t\right)$$

(c) Write $\int_{\omega_0}^{\omega_0+\Delta\omega} u(\omega)\,\mathrm{d}\omega = \rho(\omega) \Rightarrow \mathcal{E}_0^2 = \dfrac{2\rho}{\epsilon_0}$.

So $|c_2(t)|^2 = \left(\dfrac{1}{\delta\omega}\right)^2 \dfrac{2\rho}{\epsilon_0} \sin^2\left(\dfrac{\delta\omega}{2}t\right)$.

Note that Einstein's B coefficient is defined as $u(\omega)B_{21} = \dfrac{|c_2(t)|^2}{t}$ for each $\mathrm{d}\omega$ interval.

Also by realising that the sinc function has a sharp resonance at ω_0 so $u(\omega) \simeq u(\omega - \omega_0)$ since we assume it varies slowly, we write:

$$\begin{aligned}
u(\omega_0)B_{21} &= \frac{1}{t}\int_0^\infty \frac{2u(\omega_0)}{4\epsilon_0}\left(\frac{\chi_{12}}{\hbar}\right)^2 \cdot \left(\frac{1}{\delta\omega}\right)^2 \sin^2\left(\frac{\delta\omega}{2}t\right)\mathrm{d}\omega \\
u(\omega_0)B_{21} &= \frac{2\pi t}{t}\cdot\frac{2u(\omega_0)}{4\epsilon_0}\left(\frac{\chi_{12}}{\hbar}\right)^2 \\
B_{21} &= \frac{\pi\chi_{12}^2}{\epsilon_0\hbar^2} \\
&\to \frac{\pi\chi_{12}^2}{3\epsilon_0\hbar^2} \quad \text{to account for the fact that we have defined } B \text{ for unpolarised light}
\end{aligned}$$

(d) Since this interaction does not life the degeneracy in m_l, we choose $m_l = 1$ for convenience.

$$\begin{aligned}
\chi_{12} &= -e\int \psi_{1s}^* x \psi_{2p}\,\mathrm{d}^3\mathbf{r} \\
&= -e\int_0^{2\pi}\mathrm{d}\phi\int_0^\pi \mathrm{d}\theta\int_0^\infty \frac{e^{-\frac{r}{a_0}}}{\sqrt{\pi}a_0^{3/2}}\cdot r\sin\theta\cos\phi\cdot\frac{e^{-\frac{r}{2a_0}}}{\sqrt{2\pi}a_0^{3/2}}\cdot\frac{r}{a_0}\sin\theta r^2\sin\theta e^{+i\phi}\,\mathrm{d}r \\
&= -e\frac{1}{\sqrt{2}\pi a_0^3 a_0}\int_0^{2\pi} e^{i\phi}\cos\phi\,\mathrm{d}\phi\int_0^\pi \sin^2\theta\,\mathrm{d}\theta\int_0^\infty e^{-\frac{3r}{2a_0}}r^3\,\mathrm{d}r \\
&= -\frac{e}{\sqrt{2}\pi a_0^4}\int_0^{2\pi}\frac{e^{2i\phi}+1}{2}\mathrm{d}\phi\int_0^\pi\left(1-\cos^2\theta\right)\sin\theta\,\mathrm{d}\theta\int_0^\infty r^3 e^{-\frac{3r}{2a_0}}\,\mathrm{d}r
\end{aligned}$$

Evaluate radial integral:

$$F = r^3 \qquad \Rightarrow \qquad \mathrm{d}F = 3r^2\,\mathrm{d}r$$
$$\mathrm{d}G = e^{-\frac{3r}{2a_0}}\,\mathrm{d}r \qquad \Rightarrow \qquad G = -\frac{2a_0}{3}e^{-\frac{3r}{2a_0}}$$

$$\int_0^\infty r^3 e^{-\frac{3r}{2a_0}}\,\mathrm{d}r = \int_0^\infty 2a_0 r^2 e^{-\frac{3r}{2a_0}}\,\mathrm{d}r \quad \text{since } F\cdot G \text{ vanishes at the boundaries}$$

$$F = r^2 \qquad \Rightarrow \qquad \mathrm{d}F = 2r\,\mathrm{d}r$$
$$\mathrm{d}G = e^{-\frac{3r}{2a_0}}\,\mathrm{d}r \qquad \Rightarrow \qquad G = -\frac{2a_0}{3}e^{-\frac{3r}{2a_0}}$$

$$\to \int_0^\infty \frac{8}{3}a_0^2 r e^{-\frac{3r}{2a_0}}\,\mathrm{d}r$$

$$F = r \qquad \Rightarrow \qquad \mathrm{d}F = \mathrm{d}r$$
$$\mathrm{d}G = e^{-\frac{3r}{2a_0}}\,\mathrm{d}r \qquad \Rightarrow \qquad G = -\frac{2a_0}{3}e^{-\frac{3r}{2a_0}}$$

$$\begin{aligned}
&\to \int_0^\infty \frac{16}{9}a_0^3 e^{-\frac{3r}{2a_0}}\,\mathrm{d}r \\
&= \left[-\frac{32}{27}a_0^4 e^{-\frac{3r}{2a_0}}\right]_{r=0}^\infty \\
&= \frac{32}{27}a_0^4
\end{aligned}$$

θ integral:

$$\begin{aligned}
\int_0^\pi \sin\theta\,\mathrm{d}\theta + \int_0^\pi \cos^2\theta\,\mathrm{d}(\cos\theta) &= [-\cos\theta]_0^\pi + \left[\frac{\cos^3\theta}{3}\right]_0^\pi \\
&= 1 + 1 - \frac{1}{3} - \frac{1}{3} = \frac{4}{3}
\end{aligned}$$

ϕ integral:

$$\begin{aligned}
\int_0^{2\pi} \frac{e^{2-\phi}}{2}\,\mathrm{d}\phi &= \left[\frac{e^{2i\phi}}{4i} + \frac{\phi}{2}\right]_0^{2\pi} \\
&= \pi
\end{aligned}$$

So:

$$\begin{aligned}
\chi_{12} &= -\frac{e}{2\sqrt{2}\cancel{\pi}\cancel{a_0^4}}\cdot\cancel{\pi}\cdot\frac{4}{3}\cdot\frac{32}{27}\cancel{a_0^4} \\
&= -\frac{16}{81\sqrt{2}}
\end{aligned}$$

$$\Rightarrow B_{12} = \frac{\pi\chi_{12}^2}{3\epsilon_0\hbar^2} = 2.075\times10^{77}\,\mathrm{F^{-1}\,m\,J^2\,s^2}$$

ERRATA:

$$\chi_{12} = \frac{128\sqrt{2}}{243} e a_0$$

$$B_{12} = 4.2 \times 10^{20}\,\mathrm{m}^3\,\mathrm{rad}\,\mathrm{J}^{-1}\,\mathrm{s}^{-2}$$