UNOFFICIAL SOLUTIONS BY TheLongCat

B3: QUANTUM, ATOMIC AND MOLECULAR PHYSICS

TRINITY TERM 2018

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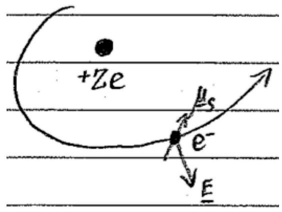
Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. (DRAFT)

(a) As the e$^-$ orbits around the nucleus with angular momentum $\mathbf{l} \neq 0$, it experiences magnetic field in its rest trame due to the transformation of the electric field from the nucleus. The interaction between this magnefic field and the intrinsic magnetic moment of the e$^-$ leads to a splitting in energy level called fine structure.



In e$^-$ rest frame,

$$\mathbf{B} = -\frac{\mathbf{v} \times \mathbf{B}}{c^2} \quad \text{and}$$
$$\mathbf{v} = \frac{\mathbf{p}}{m}, \; -e\mathbf{E} = -\frac{\partial U}{\partial r}$$

where $U(r)$ is Coulomb potential of the nucleus.

$$\Rightarrow \mathbf{B} = -\frac{1}{m_e c^2}\left(\frac{1}{er}\frac{\partial U}{\partial r}\right)\mathbf{p} \times \mathbf{r} \quad \text{since } \hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$$
$$= \frac{\hbar}{m_e c^2}\left(\frac{1}{er}\frac{\partial U}{\partial r}\right)\mathbf{l} \quad \text{for } \mathbf{r} \times \mathbf{p} = \hbar \mathbf{l}$$

Also magnetic moment of e$^-$:

$$\boldsymbol{\mu}_s = -g_s \mu_{\mathrm{B}} \mathbf{s}$$

with $g_s \simeq 2$ the gyromagnetic ratio, $\mu_{\mathrm{B}} = \dfrac{e\hbar}{2m_e}$.

Hamiltonian of magnetic interaction:

$$\Delta \hat{H}_{\mathrm{SO}} = -\boldsymbol{\mu}_s \cdot \mathbf{B}$$
$$= g_s \mu_{\mathrm{B}} \cdot \frac{\hbar}{m_e c^2}\left(\frac{1}{er}\frac{\partial U}{\partial r}\right)\mathbf{s} \cdot \mathbf{l}$$
$$\to (g_s - 1)\frac{e\hbar}{2m_e} \cdot \frac{\hbar}{m_e c^2}\left(\frac{1}{er}\frac{Ze^2}{4\pi\epsilon_0 r^2}\right)\mathbf{s} \cdot \mathbf{l}$$

by Thomas precession and ${}^{\partial U}\!/_{\partial r} = {}^{\partial}\!/_{\partial r}\left(-\dfrac{Ze^2}{4\pi\epsilon_0 r}\right)$.

$$\Delta E_{\mathrm{SO}} = \frac{\hbar^2}{2m_e^2 c^2}\left\langle \frac{1}{r^3} \right\rangle \frac{Ze^2}{4\pi\epsilon_0} \underbrace{\langle \mathbf{s} \cdot \mathbf{l} \rangle}_{\frac{1}{2}\langle \hat{\mathbf{j}}^2 - \hat{\mathbf{l}}^2 - \hat{\mathbf{s}}^2 \rangle}$$
$$= \frac{\beta}{2}\left[j(j+1) - l(l+1) - s(s+1)\right]$$

where

$$\begin{aligned}\beta &= \frac{\hbar}{2m_e^2c^2}\left\langle\frac{1}{r^3}\right\rangle\cdot\frac{Ze^2}{4\pi\epsilon_0}\\ &= \frac{\hbar}{2m_e^2c^2}\cdot\left(\frac{Z}{na_0}\right)^3\cdot\frac{1}{l(l+\frac{1}{2})(l+1)}\cdot\frac{Ze^2}{4\pi\epsilon_0}\\ &= \frac{\hbar}{2m_e^2c^2}\cdot\frac{e^2}{4\pi\epsilon_0}\cdot\frac{Z^4}{l(l+\frac{1}{2})(l+1)(na_0)^3}\end{aligned}$$

(b) $n=2$ shells: $2s$, $2p$

$n=3$ shells: $3s$, $3p$, $3d$

Config $2s \to$ Term ${}^2S \to$ Level ${}^2S_{1/2}$

Config $2p \to$ Term ${}^2P \to$ Level ${}^2P_{1/2}$, ${}^2P_{3/2}$

Config $3s \to$ Term ${}^2S \to$ Level ${}^2S_{1/2}$

Config $3p \to$ Term ${}^2P \to$ Level ${}^2P_{1/2}$, ${}^2P_{3/2}$

Config $3d \to$ Term ${}^2D \to$ Level ${}^2D_{3/2}$, ${}^2D_{5/2}$

Energy shifts:

$$\begin{aligned}2p\,{}^2P_{1/2}:\ \beta &= \frac{\hbar^2}{2m_e^2c^2}\cdot\alpha 2\pi hc\cdot\frac{2^4}{1\left(\frac{3}{2}\right)(2)(2a_0)^3}\\ &= \frac{(197.33\,\text{MeV fm})^3\cdot 16\pi}{(0.511\,\text{MeV})^2\cdot 24(5.292\times10^4\,\text{fm})^3}\\ &= 4.158\times10^{-7}\,\text{MeV} = 0.4158\,\text{eV}\\ \Rightarrow \Delta E_{\text{SO}}\left(j=\frac{1}{2}\right) &= \frac{\beta}{2}\left[\frac{1}{2}\left(\frac{3}{2}\right)-1(2)-\frac{1}{2}\left(\frac{3}{2}\right)\right]\\ &= -\beta = -0.4158\,\text{eV}\end{aligned}$$

$$\begin{aligned}2p\,{}^2P_{3/2}:\ \Delta E_{\text{SO}} &= \frac{\beta}{2}\left[\frac{3}{2}\left(\frac{5}{2}\right)-1(2)-\frac{1}{2\left(\frac{3}{2}\right)}\right]\\ &= \frac{\beta}{2} = 0.2079\,\text{eV}\end{aligned}$$

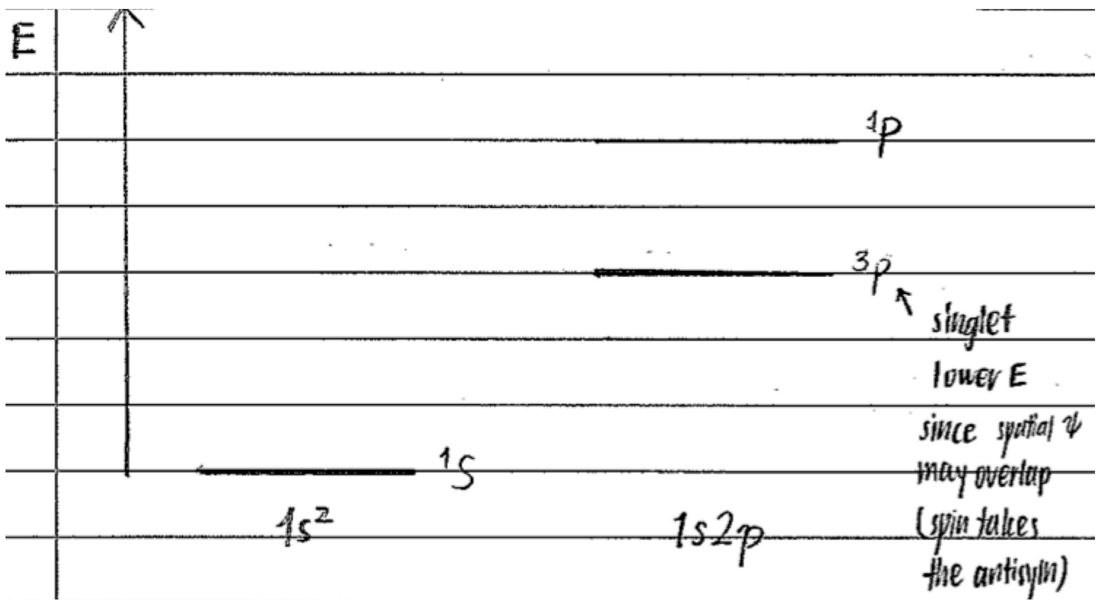
$$
\begin{aligned}
3p\ {}^2P_{1/2}:\ \beta &= \frac{(197.33\,\mathrm{MeV\,fm})^3 \cdot 2^4\pi}{(0.511\,\mathrm{MeV})^2 \cdot 3 \cdot 3^3 (5.292 \times 10^4\,\mathrm{fm})^3} \quad \text{similarly} \\
&= 1.232 \times 10^{-7}\,\mathrm{MeV} = 0.1232\,\mathrm{eV} \\
\Delta E_{\mathrm{SO}}\left(j = \frac{1}{2}\right) &= \frac{\beta}{2} \cdot \left[\frac{1}{2}\left(\frac{3}{2}\right) \dots\right] \\
&= -\beta \\
&= -0.1232\,\mathrm{eV} \quad \text{similarly}
\end{aligned}
$$

$$
\begin{aligned}
{}^2P_{3/2}:\ \Delta E\left(j = \frac{3}{2}\right) &= \frac{\beta}{2} \\
&= 0.0616\,\mathrm{eV}
\end{aligned}
$$

$$
\begin{aligned}
3d\ {}^2D_{3/2}:\ \beta &= \frac{(197.33\,\mathrm{MeV\,fm})^3 \cdot 16\pi}{(0.511\,\mathrm{MeV})^2 \cdot 2\left(\frac{5}{2}\right)(3) \cdot 3^3 (5.292 \times 10^4\,\mathrm{fm})^3} \\
&= 2.464 \times 10^{-8}\,\mathrm{MeV} \\
&= 0.024\,64\,\mathrm{eV} \\
\Delta E\left(j = \frac{3}{2}\right) &= \frac{\beta}{2}\left[\frac{3}{2}\left(\frac{5}{2}\right) - 2(3) - \frac{1}{2}\left(\frac{3}{2}\right)\right] \\
&= -\frac{3\beta}{2} \\
&= -0.036\,96\,\mathrm{eV}
\end{aligned}
$$

$$
\begin{aligned}
\Delta E\left(j = \frac{5}{2}\right) &= \frac{\beta}{2}\left[\frac{5}{2}\left(\frac{9}{2}\right) - 2(3) - \frac{1}{2}\left(\frac{3}{2}\right)\right] \\
&= \beta \\
&= 0.024\,64\,\mathrm{eV}
\end{aligned}
$$

Energy level diagram:



Electric dipole selection rule:

$\Delta n =$ any $\quad \Delta l = \pm 1$

$\Delta S = 0 \quad \Delta L = 0, \pm 1\ (0 \nrightarrow 0) \quad \Delta J = 0, \pm 1 (0 \nrightarrow 0) \quad \Delta M_J = 0, \pm 1\ (0 \nrightarrow 0$ iff $\Delta J = 0)$

So possible transitions:

$2s/3s\ \ ^2S_{1/2}$ — $2p\ \ ^2P_{1/2}$

$2s/3s\ \ ^2S_{1/2}$ — $2p\ \ ^2P_{3/2}$

$2s/3s\ \ ^2S_{1/2}$ — $3p\ \ ^2P_{1/2}$

$2s/3s\ \ ^2S_{1/2}$ — $3p\ \ ^2P_{3/2}$

(c) Ω^- baryon belongs to the $J^P = \frac{3}{2}^+$ group so it possesses $s = \frac{3}{2}$.

For the exotic hydrogenic ion Pb^{81+}, replace $m_e \to m_\Omega$ and $s = \frac{1}{2} \to \frac{3}{2}$.

So:

$$\begin{aligned}
\beta &= \frac{\hbar^2 c^2}{2m_\Omega^2 c^4} \cdot 2\pi\alpha\hbar c \cdot \frac{82^4}{9\left(\frac{19}{2}\right)(10)(10)^3(a_0)^3} \\
&= \frac{(197.33\,\mathrm{MeV\,fm})^3 \cdot 82^4\pi}{(1672\,\mathrm{MeV})^2 \cdot 855\,000(5.292\times 10^4\,\mathrm{fm})^3} \\
&= 3.081\times 10^{-12}\,\mathrm{MeV} \\
&= 3.081\times 10^{-6}\,\mathrm{eV}
\end{aligned}$$

Config $10l \rightarrow$ Term ${}^4L \rightarrow$ Level ${}^4L_{15/2}$, ${}^4L_{17/2}$, ${}^4L_{19/2}$, ${}^4L_{21/2}$

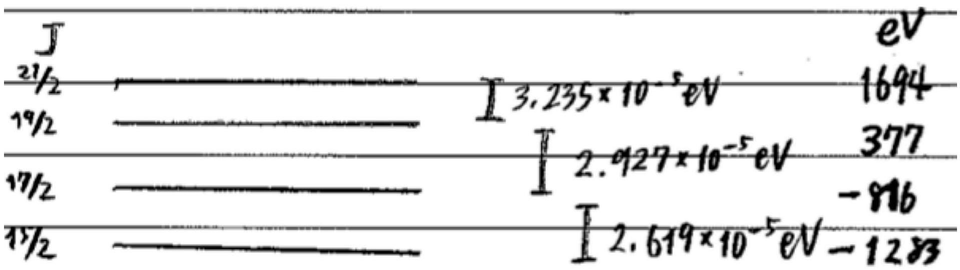
So:

$$\begin{aligned}
\Delta E\left(j=\frac{15}{2}\right) &= \frac{\beta}{2}\left[\frac{15}{2}\left(\frac{17}{2}\right) - 9(10) - \frac{3}{2}\left(\frac{5}{2}\right)\right] \\
&= -15\beta = -4.621\times 10^{-5}\,\mathrm{eV}
\end{aligned}$$

$$\begin{aligned}
\Delta E\left(j=\frac{17}{2}\right) &= \frac{\beta}{2}\left[\frac{17}{2}\left(\frac{19}{2}\right) - 9(10) - \frac{3}{2}\left(\frac{5}{2}\right)\right] \\
&= -\frac{13}{2}\beta = -2.003\times 10^{-5}\,\mathrm{eV}
\end{aligned}$$

$$\begin{aligned}
\Delta E\left(j=\frac{19}{2}\right) &= \frac{\beta}{2}\left[\frac{19}{2}\left(\frac{21}{2}\right) - 9(10) - \frac{3}{2}\left(\frac{5}{2}\right)\right] \\
&= 3\beta = 9.243\times 10^{-6}\,\mathrm{eV}
\end{aligned}$$

$$\begin{aligned}
\Delta E\left(j=\frac{21}{2}\right) &= \frac{\beta}{2}\left[\frac{21}{2}\left(\frac{23}{2}\right) - 9(10) - \frac{3}{2}\left(\frac{5}{2}\right)\right] \\
&= \frac{27}{2}\beta = 4.159\times 10^{-5}\,\mathrm{eV}
\end{aligned}$$



(d) Possible spins:

$$\begin{aligned}
S &= \left|\frac{3}{2} - \frac{3}{2}\right| \cdots \left(\frac{3}{2} + \frac{3}{2}\right) \\
&= 0, 1, 2, 3
\end{aligned}$$

For excited state $1s2s \rightarrow$

Term	${}^1S$	${}^3S$	${}^5S$, ${}^7S$	
	$\vert$	$\vert$	$\vert$	$\vert$
Level	${}^1S_0$	${}^3S_1$	${}^5S_2$	${}^7S_3$

For $1s2p$,

$$\begin{array}{lcccc} \text{Term} & {}^1P & {}^3P & {}^5P & {}^7P \\ & & \downarrow & & \\ \text{Level} & {}^1P_1 & {}^3P_0,\ {}^3P_1,\ {}^3P_2 & {}^5P_1,\ {}^5P_2,\ {}^5P_3 & {}^7P_2,\ {}^7P_3,\ {}^7P_4 \end{array}$$

For ground state $1s^2$, Pauli exclusion principle states that no 2 fermions may occupy the same state, so the 2 Ω^-'s must not have aligned spins. So the available spins are $S = 0, 1, 2$.

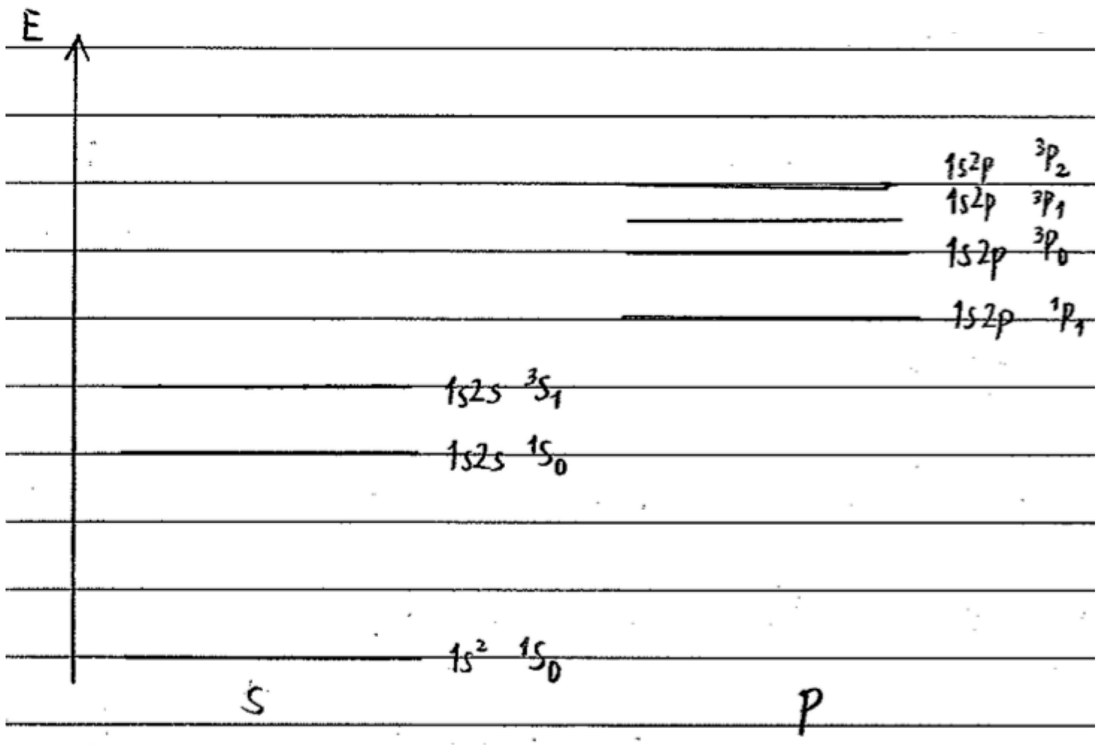
$$\Rightarrow \begin{array}{llll} \text{Ground term} & {}^1S & {}^3S & {}^5S \\ \text{Level} & {}^1S_0 & \cancel{{}^3S_1} & {}^5S_2 \end{array}$$

This differs from helium whereby only ${}^1S_0$ is available as e^- has spin $\dfrac{1}{2}$ instead of $\dfrac{3}{2}$.

2. (DRAFT)

(a) First 2 excited configs of helium:

Config	$1s2p$					$1s2s$
Terms	${}^1P$			${}^3P$	${}^1S$	${}^3S$
Levels	${}^1P_1$	${}^3P_0$	${}^3P_1$	${}^3P_2$	${}^1S_0$	${}^3S_1$



So allowed transitions:

$1s^2\ {}^1S_0$ — $1s2p\ {}^1P_1$

$1s2s\ {}^1S_0$ — $1s2p\ {}^1P_1$

$1s2s\ {}^3S_1$ — $1s2p\ {}^3P_2$

$1s2s\ {}^3S_1$ — $1s2p\ {}^3P_1$

$1s2s\ {}^3S_1$ — $1s2p\ {}^3P_0$

Note that the $1s^2$–$1s2p$ transition is likely to be observed in absorption as most, if not all, helium atoms reside at ground state under room temperature.

As helium has significant inter-e$^-$ interaction, it is likely to observe intercombination lines, e.g. $1s2s\ {}^1S_0$–$1s^2\ {}^1S_0$ which violates $\Delta l = \pm 1$.

(b) LS coupling assumes that $\Delta\hat{H}_{\mathrm{RE}} \ll \hat{H}_{\mathrm{CF}}$, where $\Delta\hat{H}_{\mathrm{RE}}$ is residual electrostatic interaction and $\hat{H}_{\mathrm{CF}}$ the central field Hamiltonian.

For weak field, $\Delta\hat{H}_z = -\boldsymbol{\mu}_s \cdot \mathbf{B}$ where $\boldsymbol{\mu}$ is atomic magnetic moment must be much smaller than $\Delta\hat{H}_{\mathrm{RE}}$.

Atomic magnetic moment $\boldsymbol{\mu} = -\mu_{\mathrm{B}}\mathbf{L} - g_s\mu_{\mathrm{B}}\mathbf{S}$ where $\mathbf{L} = \sum_i \mathbf{l}_i$ is total orbital angular momentum, $\mathbf{S} = \sum_i \mathbf{s}_i$ is total spin (e^-).

$$\Rightarrow \Delta\hat{H}_z = +\mu_{\mathrm{B}}\frac{L^2 + \mathbf{L}\cdot\mathbf{S} + g_sS^2 + g_s\mathbf{S}\cdot\mathbf{L}}{J^2}\mathbf{J}\cdot\mathbf{B}$$

$$\Rightarrow \Delta E_z = \mu_{\mathrm{B}}M_JB\frac{\begin{array}{c}L(L+1) + \frac{1}{2}\left[J(J+1) - L(L+1) - S(S+1)\right]\\ + g_sS(S+1) + \frac{g_s}{2}\left[J(J+1) - L(L+1) - S(S+1)\right]\end{array}}{J(J+1)}$$

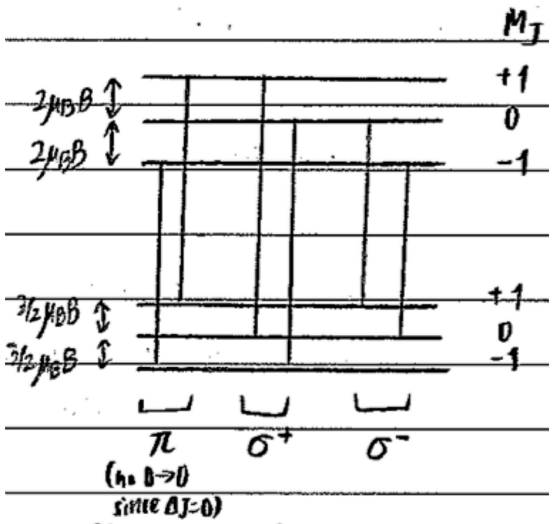
Approximating $g_s \simeq 2$,

$$\Delta E_z \simeq \mu_{\mathrm{B}}M_JB\frac{\begin{array}{c}\cancel{2L(L+1)} + J(J+1) - L(L+1) - S(S+1)\\ + \cancel{2S(S+1)} + 2J(J+1) - \cancel{2L(L+1)} - \cancel{2S(S+1)}\end{array}}{2J(J+1)}$$

$$= g_J\mu_{\mathrm{B}}M_JB$$

where $g_J = \dfrac{3}{2} - \dfrac{L(L+1) - S(S+1)}{2J(J+1)}$.

(c) $(1s3s)\ ^3S_1 \to (1s2p)\ ^3P_1$

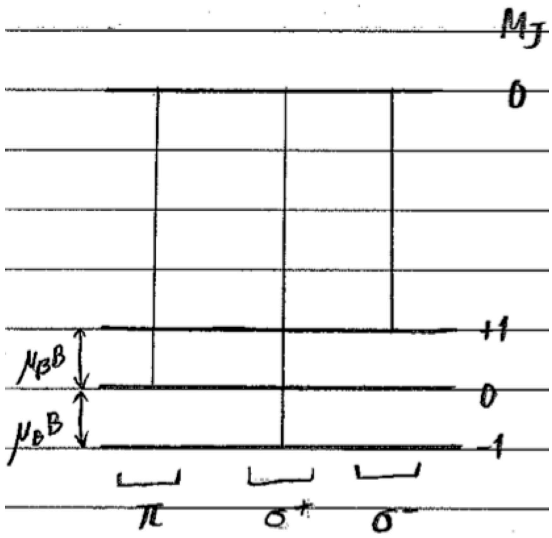


Upper level $g_J = \dfrac{3}{2} - \dfrac{0 - 1(2)}{2\cdot 1(2)} = 2$

Lower level $g_J = \dfrac{3}{2} - \dfrac{1(2) - 1(2)}{2\cdot 1(2)} = \dfrac{3}{2}$

Since g_J differs, we expect anomalous Zeeman splitting.

$(1s3s)\ ^1S_0 \to (1s2p)\ ^1P_1$



Lower level $g_J = \frac{3}{2} - \frac{1(2) - 0}{2 \cdot 1 \cdot 2} = 1$

Since the energy gaps do not differ within each polarisation, we expect normal Zeeman splitting.

(d) For a total of 6 lines, we can have 4 σ's from $(1s2p)\ {}^3P_1$; 2 σ's from $(1s2p)\ {}^1P_1$.

So the observer is observing parallel to the magnetic field.

Collisional mean time:

$$\frac{1}{\tau_e} = n\sigma v$$

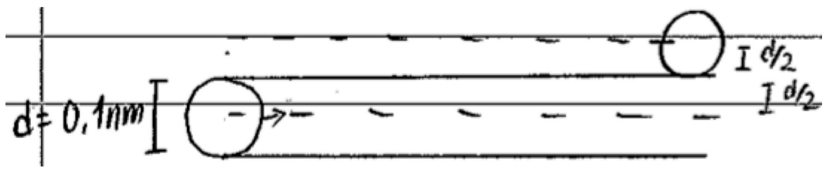
where n is number density of atoms, σ is collisional cross section, v is speed of atoms.

Ideal gas gives:

$$pV = Nk_BT$$
$$\Rightarrow n = \frac{N}{V} = \frac{p}{k_BT}$$

Also equipartition theorem gives:

$$\frac{1}{2}mv^2 = \frac{3}{2}k_BT$$
$$\Rightarrow v = \sqrt{\frac{6k_BT}{m}}$$



Collisional cross section:

$$\sigma = \pi\left(\frac{d}{2}\right)^2 = \frac{\pi d^2}{4}$$

So:

$$
\begin{aligned}
\frac{1}{\tau_c} &= \frac{p}{k_{\mathrm{B}}T} \cdot \sqrt{\frac{6k_{\mathrm{B}}T}{m}} \cdot \frac{\pi d^2}{4} \\
&= 2009\,\mathrm{s}^{-1} \\
\Rightarrow \Delta\omega &\sim \frac{1}{\tau_c} \\
\Rightarrow \Delta E &\sim \frac{\hbar}{\tau_c} = 2.12 \times 10^{-31}\,\mathrm{J}
\end{aligned}
$$

Zeeman $\Delta E_z \sim \mu_{\mathrm{B}} B$ so equating both gives:

$$
\begin{aligned}
B &\sim \frac{\Delta E}{\mu_{\mathrm{B}}} \\
&= 2.28 \times 10^{-8}\,\mathrm{T}
\end{aligned}
$$

(e) Positron and electron are distinguishable and thus Pauli exclusion principle does not apply.

Total possible spins:

$$
\begin{aligned}
S &= \left|\frac{1}{2} - \frac{1}{2}\right|, \left(\frac{1}{2} + \frac{1}{2}\right) \\
&= 0, 1
\end{aligned}
$$

Zeeman Hamiltonian:

$$
\begin{aligned}
\Delta \hat{H}_z &= -\boldsymbol{\mu} \cdot \mathbf{B} \\
&= -g_s \mu_{\mathrm{B}} \left(\mathbf{S}_+ \cdot \mathbf{B} - \mathbf{S}_- \cdot \mathbf{B}\right)
\end{aligned}
$$

with $\boldsymbol{\mu} = +g_s\mu_{\mathrm{B}}\mathbf{S}_+ - g_s\mu_{\mathrm{B}}\mathbf{S}_-$.

By Wigner-Eckart,

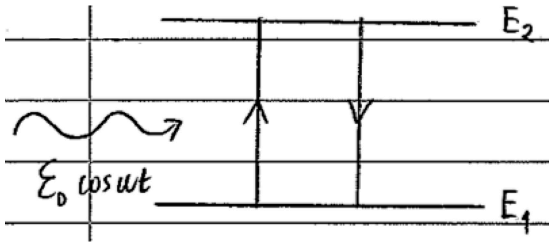
$$
\mathbf{S}_+ \to \frac{\mathbf{S}_+ \cdot \mathbf{S}}{S^2}\mathbf{S} \qquad \mathbf{S}_- \to \frac{\mathbf{S}_- \cdot \mathbf{S}}{S^2}\mathbf{S}
$$

$$
\begin{aligned}
\Rightarrow \Delta \hat{H}_z &= -g_s \mu_{\mathrm{B}} \left[\frac{S_+^2 + \mathbf{S}_+ \cdot \mathbf{S}_- - S_-^2 - \mathbf{S}_- \cdot \mathbf{S}_+}{S^2}\right] \\
&= 0
\end{aligned}
$$

since $S_+^2 = S_-^2 = \frac{1}{2}\left(\frac{3}{2}\right)$.

So no matter which term the “atom” is in, there will be no Zeeman splitting.

3. (DRAFT)



(a) In semi-classical treatment of light-atom interaction, we obtain the given differential equations by TDSE.

Rabi frequency is defined such that:

$$\begin{aligned}\hbar\Omega_R &= \langle 2|\, e\boldsymbol{\mathcal{E}}_0 \cdot \mathbf{r}\,|1\rangle \\ &= \langle 2|\, e\mathcal{E}_0 x\,|1\rangle\end{aligned}$$

is the matrix element of transition due to the electric field.

At $\omega = \Omega_R$, Rabi oscillation may occur for purely radiative processes.

(b) Weak light: $\to c_1(t) \simeq 1$, $c_2(t) \simeq 0$:

$$\Rightarrow \begin{cases} \dot{c}_1 &= 0 \\ \dot{c}_2 &= 0 \end{cases}$$

$$\begin{aligned}\Rightarrow c_2(t) &= \int_0^t -i\Omega_R \cdot \left[\frac{e^{i(\omega_0+\omega)t} + \omega_0 - \omega}{2}\right] \mathrm{d}t \\ &= -\frac{i\Omega_R}{2}\left[\frac{e^{i(\omega_0+\omega)}}{i(\omega_0+\omega)} + \frac{e^{i(\omega_0-\omega)t}}{i(\omega_0-\omega)}\right]_{t=0}^{t}\end{aligned}$$

Invoke rotating wave approximation, $\omega_0 + \omega \gg |\omega_0 - \omega|$:

$$\begin{aligned}c_2(t) &\simeq -\frac{i\Omega_R}{2} \cdot \frac{e^{i(\omega_0-\omega)t} - 1}{i(\omega_0-\omega)} \\ &= -\frac{i\Omega_R}{2} \cdot e^{i\frac{(\omega_0-\omega)}{2}t} \frac{e^{i\frac{(\omega_0-\omega)}{2}t} - e^{-i(\frac{(\omega_0-\omega)}{2})t}}{2i(\frac{\omega_0-\omega}{2}t)} t \\ &= -\frac{i\Omega_R}{2} e^{i\frac{\delta}{2}t} \operatorname{sinc}\left(\frac{\delta}{2}t\right) t\end{aligned}$$

where $\delta = \omega_0 - \omega$.

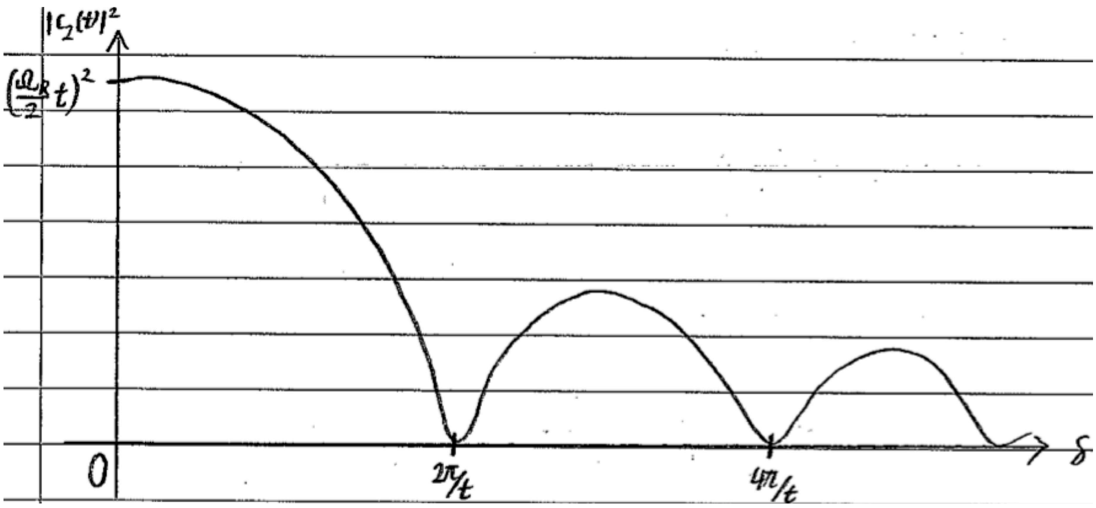
So:

$$\begin{aligned}|c_2(t)|^2 &= \left(\frac{\Omega_R}{2}\right)^2 \operatorname{sinc}^2\left(\frac{\delta t}{2}\right) t^2 \\ &= \frac{\Omega_R^2}{(2 \cdot \frac{\delta t}{2} \cdot \frac{1}{t})^2} \sin^2\left(\frac{\delta t}{2}\right) \\ &= \frac{\Omega_R^2}{\delta^2} \sin^2\left(\frac{\delta t}{2}\right)\end{aligned}$$

So $\alpha = \delta = \omega_0 - \omega$.

(c) For $\delta = 0$, $|c_2(t)|^2 = \left(\frac{\Omega_R}{2}\right)^2 t^2$.

Increasing detuning then causes $|c_2(t)|^2$ to oscillate with decreasing amplitude.



(d) Again for weak light, $c_1 \simeq 1$, $c_2 \simeq c_3 \simeq 0$:

$$\Rightarrow \begin{cases} \dot{c}_1 &= 0 \\ \dot{c}_2 &= -i\Omega_R \cos(\omega t)\, e^{i\omega_0 t} \\ \dot{c}_3 &= 0 \end{cases}$$

So similar to b:

$$c_2(t) = -\frac{i\Omega_R}{2} e^{i\frac{\delta t}{2}} \operatorname{sinc}\left(\frac{\delta t}{2}\right) t$$

(e) Plugging $c_2(t)$ into $\dot{c}_3(t)$:

$$\begin{aligned} \dot{c}_3 &= -i\Phi_R \cos(\omega t)\, e^{i\phi_0 t} \cdot \left(-\frac{i\Omega_R}{2}\right) \cdot \frac{e^{i\delta t} - 1}{i\delta} \\ &= \frac{i\Phi_R\Omega_R}{4} \left(e^{i\omega t} + e^{-i\omega t}\right) e^{i\phi_0 t} \left(e^{i\delta t} - 1\right) \end{aligned}$$

Know $\Delta\omega = \underbrace{\omega_0 - \omega}_{\delta} + \phi_0 - \omega$ negligible. So:

$$
\begin{aligned}
\dot{c}_3 &\simeq \frac{i\Phi_R\Omega_R}{4}\left[e^{i(\omega+\phi_0+\delta)t} \underbrace{- e^{-i(\omega+\phi_0)t} - e^{i(-\omega+\phi_0)t}}_{-2\cos(\omega+\phi_0)t}\right] \\
\Rightarrow c_3(t) &= \int_0^t \frac{i\Phi_R\Omega_R}{4}\left[\ldots\right]\mathrm{d}t \\
&= \frac{i\Phi_R\Omega_R}{4}\left[\frac{e^{i(\omega+\phi_0+\delta)t}}{i(\omega+\phi_0+\delta)} - \frac{2\sin(\omega+\phi_0)t}{\omega+\phi_0}\right]_{t=0}^{t} \\
&= \frac{i\Phi_R\Omega_R}{4}\left[\frac{e^{i(\omega+\phi_0)t}-1}{i(\omega+\phi_0)} - \underbrace{\frac{2\sin(\omega+\phi_0)t}{\omega+\phi_0}}_{\substack{\to 0 \\ \text{by RWA}}}\right] \\
&= \frac{\Phi_R\Omega_R}{4(\omega_0+\phi_0)} \cdot e^{i\left(\frac{\omega_0+\phi_0}{2}\right)t}\left[\frac{e^{i\frac{\omega_0+\phi_0}{2}t} - e^{-i\frac{\omega_0+\phi_0}{2}t}}{2i}\right]2i \\
&= \frac{i\Phi_R\Omega_R}{2\beta} \cdot e^{i\frac{\beta}{2}t}\sin\left(\frac{\beta t}{2}\right) \qquad \text{where } \beta = \omega_0 + \phi_0 \\
\Rightarrow |c_3(t)|^2 &= \frac{\Phi_R^2\Omega_R^2}{4\beta^2}\sin^2\left(\frac{\beta t}{2}\right)
\end{aligned}
$$

so $\Omega_{\text{eff}} = \dfrac{\Phi_R\Omega_R}{2}$.

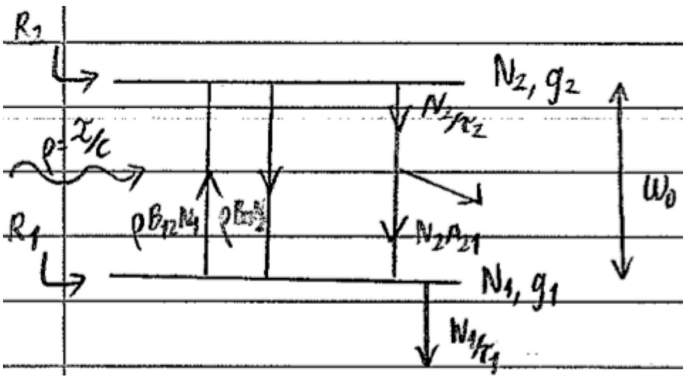
4. (DRAFT)

(a) Einstein coefficients of atomic transitions (level 2 higher than level 1):

A_{21}: constant of proportionality to spontaneous decay from level $2 \to 1$;

B_{21}: constant of proportionality to stimulated transition from level $1 \to 2$;

B_{12}: constant of proportionality to stimulated decay from level $2 \to 1$.



The net energy released by the system per volume element:

$$\begin{aligned}\Delta E &= \Delta N_\gamma \cdot \hbar\omega + A_{21} N_2 \hbar\omega_0 \\ &= (B_{21}\rho N_2 - B_{12}\rho N_1)\,\hbar\omega + A_{21} N_2 \hbar\omega_0\end{aligned}$$

(b) For purely radiative decay,

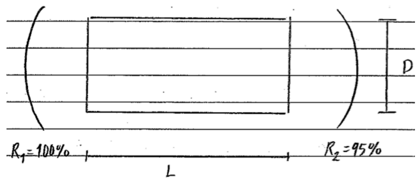
$$\begin{aligned}\Delta E &= (B_{21}\rho N_2 - B_{12}\rho N_1)\,\hbar\omega g_H \cancel{\delta\omega}\cancel{A}\,\mathrm{d}z = \mathrm{d}I \cdot \cancel{A\delta\omega} \\ \Rightarrow \frac{\mathrm{d}I}{\mathrm{d}z} &= B_{21} \underbrace{\rho}_{\frac{I}{c}} \left(N_2 - \frac{g_2}{g_1} N_1 \right) \hbar\omega g_H \\ &= \left(N_2 - \frac{g_2}{g_1} N_1 \right) \underbrace{\frac{\hbar\omega}{c} g_H B_{21}}_{\sigma_{21}} I\end{aligned}$$

Einstein relation:

$$\begin{aligned}\frac{A_{21}}{B_{21}} &= \frac{\hbar\omega^3}{\pi^2 c^3} \\ \Rightarrow B_{21} &= \frac{\pi^2 c^3}{\hbar\omega^3} A_{21} \\ \Rightarrow \sigma_{21} &\propto \frac{1}{\omega^2}\end{aligned}$$

So at $\omega = \nu_0$, $I_s \propto \dfrac{\nu_0}{\frac{1}{\nu_0^2}} = \nu_0^3$.

(c) **(TO EXPAND)**



Beam growth:

$$\frac{\mathrm{d}I}{\mathrm{d}z} = \alpha I$$

where $\alpha = N^*(I)\sigma_{21}$.

$$\begin{aligned} \Rightarrow \frac{1}{I} + \frac{1}{I_s}\,\mathrm{d}I &= \alpha(0)\,\mathrm{d}z \\ \ln\left(\frac{I}{I_0}\right) + \frac{I - I_0}{I_s} &= N^*(0)\sigma_{21} l = 1 \\ \Rightarrow \frac{I}{I_0} &= e^1 \\ I &= eI_0(?) \end{aligned}$$