UNOFFICIAL SOLUTIONS BY TheLongCat

C2: LASER SCIENCE AND QUANTUM INFORMATION PROCESSING

TRINITY TERM 2023

Last updated: 30th May 2025

Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

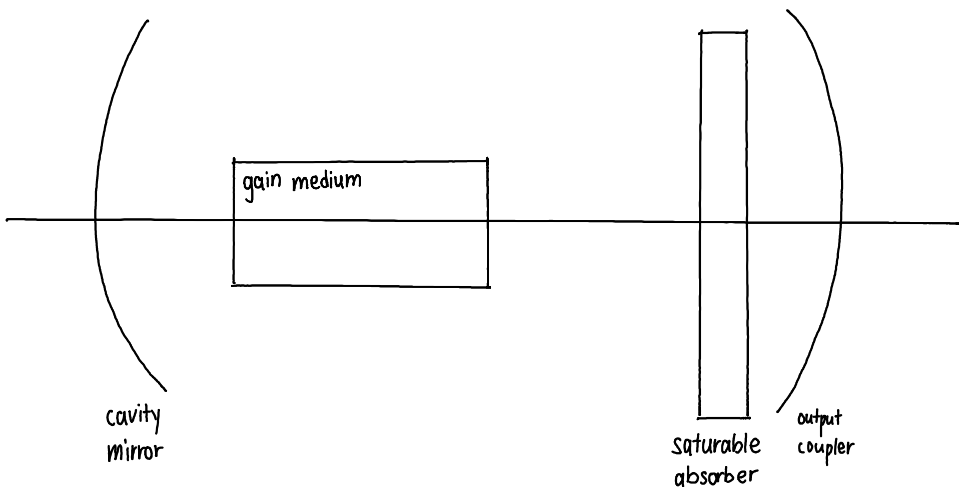
1. Classic question on transient behaviour of a laser.

(a) Relaxation oscillation: variation in laser output as a result of disturbance to the system, e.g. change in cavity length, this is more regularly spaced and gentler than laser spiking.

Laser spiking: sharp and rather irregular spikes in laser output when it is first turned on.

The reason why both conditions occur is due to the underdamp in the cavity mode and may be summarised as $\tau_2 \ll \tau_c$ where τ_2 is the lifetime of the upper level, τ_c is the cavity lifetime.

(b) N_s^*: threshold population inversion – this is the boundary of population inversion beyond which photon population begins to grow.



In the absence of gain, we have:

$$\begin{aligned}
\frac{\mathrm{d}n}{\mathrm{d}t} &= -\underbrace{\left(\frac{1}{2}nc\right)}_{\text{flux in each direction}} \cdot \underbrace{A_{\text{cavity}}}_{\text{beam cross-sectional area}} \cdot \frac{(1-R_1)+(1-R_2)}{\underbrace{V_{\text{cavity}}}_{\text{cavity volume}}} \\
&= -\frac{1}{2}nc \cdot \frac{1}{L} \cdot (0+T) \\
&= -\frac{cT}{2L}n \equiv -\frac{n}{\tau_c} \\
\Rightarrow \tau_c &= \frac{2L}{cT}
\end{aligned}$$

Cavity round-trip time is given as:

$$t_{\text{rt}} = \frac{2L}{c} \Rightarrow \tau_c = t_{\text{rt}}/T$$

At steady state, $\dfrac{\mathrm{d}N^*}{\mathrm{d}t} = \dfrac{\mathrm{d}n}{\mathrm{d}t} = 0$:

$$\begin{aligned}
\frac{1}{\eta}\frac{N^*}{N_s^*}\frac{n}{\tau_c} + \frac{N^*}{\tau_2} &= S_2 \\
\xRightarrow{N^*/N_s^* - 1 = 0 \Rightarrow N^* = N_s^*} \frac{1}{\eta}\cdot\frac{n}{\tau_c} &= S_2 - \frac{N_s^*}{\tau_2} \\
\Rightarrow n_{ss} &= \eta\tau_c\left[S_2 - \frac{N_s^*}{\tau_s}\right]
\end{aligned}$$

In the absence of stimulated emission:

$$\begin{aligned}
\frac{\mathrm{d}N^*}{\mathrm{d}t} &= S_2 - \frac{N^*}{\tau_2} \\
\Rightarrow N^*(0) &= S_2\tau_2 \quad \text{at steady state}
\end{aligned}$$

Hence overpumping ratio $r = \dfrac{S_2\tau_2}{N_s^*} = \dfrac{N^*(0)}{N_s^*}$.

(c) For small perturbation we have:

$$\begin{aligned}
N^* &= N^*_{\mathrm{ss}} + \Delta N^* \\
n &= n_{\mathrm{ss}} + \Delta n_{ss}
\end{aligned}$$

where X_{ss} is steady state value of X.

Plugging them into the rate equations gives:

$$\begin{aligned}
\frac{\mathrm{d}N^*}{\mathrm{d}t} &= \frac{\mathrm{d}(\Delta N^*)}{\mathrm{d}t} \\
&= S_2 - \frac{1}{\eta}\frac{(N_s^* + \Delta N^*)}{N_s^*}\cdot\frac{(n_{\mathrm{ss}} + \Delta n)}{\tau_c} - \frac{N_s^* + \Delta N^*}{\tau_2} \\
&= S_2 - \frac{1}{\eta\tau_c}\left(1 + \frac{\Delta N^*}{N_s^*}\right)(n_{\mathrm{ss}} + \Delta n) - \frac{N_s^* + \Delta N^*}{\tau_2} \\
&\simeq S_2 - \frac{1}{\eta\tau_c}\left(n_{\mathrm{ss}} + \Delta n + \frac{n_{\mathrm{ss}}}{N_s^*}\Delta N^*\right) - \frac{N_s^* + \Delta N^*}{\tau_2} \quad \text{keeping 1st order terms} \\
&= S_2 - S_2 + \frac{N_s^*}{\tau_2} - \frac{\Delta n}{\eta\tau_c} - \left(S_2 - \frac{N_s^*}{\tau_2}\right)\cdot\frac{\Delta N^*}{N_s^*} - \frac{N_s^*}{\tau_2} - \frac{\Delta N^*}{\tau_2} \\
&= -\Delta N^*\left[\frac{S_2}{N_s^*}\right] - \frac{\Delta n}{\eta\tau_c} \\
&= -\frac{r}{\tau_2}\Delta N^* - \frac{\Delta n}{\eta\tau_c}
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}n}{\mathrm{d}t} = \frac{\mathrm{d}(\Delta n)}{\mathrm{d}t} &= \underbrace{\left[\frac{N_s^* + \Delta N^*}{N_s^*} - 1\right]}_{\Delta N^*} \frac{n_{\text{ss}} + \Delta n}{\tau_c} \\
&\simeq \frac{n_{\text{ss}}}{\tau_c}\frac{\Delta N^*}{N_s^*} \quad \text{by linearisation} \\
&= \eta \left(S_2 - \frac{N_s^*}{\tau_2}\right) \Delta N^* \\
&= \frac{\eta}{\tau_2} \underbrace{\left(\frac{S_2\tau_2 - N_s^*}{N_s^*}\right)}_{r-1} \Delta N^*
\end{aligned}$$

Now insert ansatz $\Delta N^* = a\mathrm{e}^{mt}$, $\Delta n = b\mathrm{e}^{mt}$:

$$\begin{aligned}
\frac{\mathrm{d}n}{\mathrm{d}t} &= bm\mathrm{e}^{mt} = \frac{(r-1)\eta}{\tau_2} a\mathrm{e}^{mt} \\
\Rightarrow m &= \frac{(r-1)\eta}{\tau_2}\frac{a}{b} \Rightarrow \frac{b}{a} = \frac{(r-1)\eta}{\tau_2 m} \\
\frac{\mathrm{d}\Delta N^*}{\mathrm{d}t} &= -\frac{r}{\tau_2} a\mathrm{e}^{mt} - \frac{1}{\eta\tau_c} b\mathrm{e}^{mt} \\
\Rightarrow m &= -\frac{r}{\tau_2} - \frac{1}{\eta\tau_c}\frac{b}{a}
\end{aligned}$$

Hence:

$$\begin{aligned}
m + \frac{1}{\eta\tau_c} \cdot \frac{(r-1)\eta}{\tau_2 m} &= -\frac{r}{\tau_2} \\
\Rightarrow m^2 + \frac{r}{\tau_2} m + \frac{r-1}{\tau_c\tau_2} &= 0 \\
m = \frac{-r/\tau_2 \pm \sqrt{(r/\tau_2)^2 - 4(r-1)/\tau_c\tau_2}}{2}
\end{aligned}$$

We want m to be complex for oscillations:

$$\begin{aligned}
\left(\frac{r}{\tau_2}\right)^2 - \frac{4(r-1)}{\tau_c\tau_2} &< 0 \\
\frac{r^2}{\tau_2} &< \frac{4(r-1)}{\tau_c}
\end{aligned}$$

Note that the scenario $\tau_2 \ll \tau_c$ satisfies this inequality.

Thus we have the angular frequency (ω_{ro}) and frequency (f_{ro}) of relaxation oscillations:

$$\begin{aligned}
\omega_{ro}^2 &= \frac{1}{4}\left[\frac{4(r-1)}{\tau_2\tau_c} - \left(\frac{r}{\tau_2}\right)^2\right] \\
f_{ro} &= \frac{\omega_{ro}}{2\pi}
\end{aligned}$$

In the limit $\tau_2 \ll \tau_c$, $\omega_{ro}^2 \to (r-1)/\tau_2\tau_c$, so:

$$\begin{aligned}
r - 1 &= (2\pi f_{ro})^2\,\tau_2\tau_c \\
r &= (2\pi f_{ro})^2\,\tau_2\tau_c + 1 \\
\frac{N^*(0)}{N_s^*} &= (2\pi f_{ro})^2\,\frac{\tau_2 t_{\mathrm{rt}}}{T} + 1 \\
\eta N^*(0)\sigma_{21}\underbrace{c\tau_c}_{2L/T} &= \dots \\
\Rightarrow \delta &= (2\pi f_{ro})^2\,\tau_2 t_{\mathrm{rt}} + T
\end{aligned}$$

(d) From above, we have $f_{ro} \propto \sqrt{r} \propto \sqrt{S_2}$ the pump rate, hence the graph exhibits a square root dependency.

In addition, as the oscillations in laser power are of frequency f_{ro}, the frequency spectrum should exhibit a peak at f_{ro}.

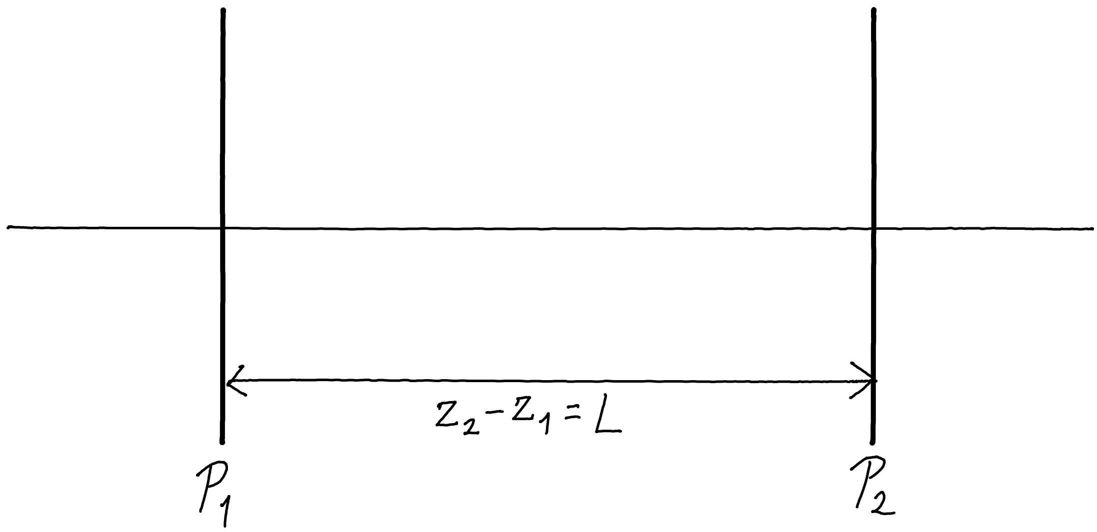
From the graph, we infer that $f_{ro} \simeq 190\,\mathrm{kHz}$ @ $1000\,\mathrm{mW}$ pump power.

With $t_{\mathrm{rt}} = 4\,\mathrm{ns}$, $\tau_2 = 450\,\mu\mathrm{s}$, $T = 9.5\%$, we have $\delta = 2.660$.

The advantage of this method is that the measurement error in cavity dimensions do not get into the calculation, enhancing the accuracy of gain measurement.

2. **(DRAFT)** Question on ray transfer and application on controlling dispersion.

(a) **(TO EXPAND)** Sketch of the setup:



At $\mathcal{P}_1$, $E = 1/q_1 \mathrm{e}^{ikz_1} \mathrm{e}^{ik\rho_1^2/2q_1}$ where $\rho_1^2 = x_1^2 + y_1^2$.

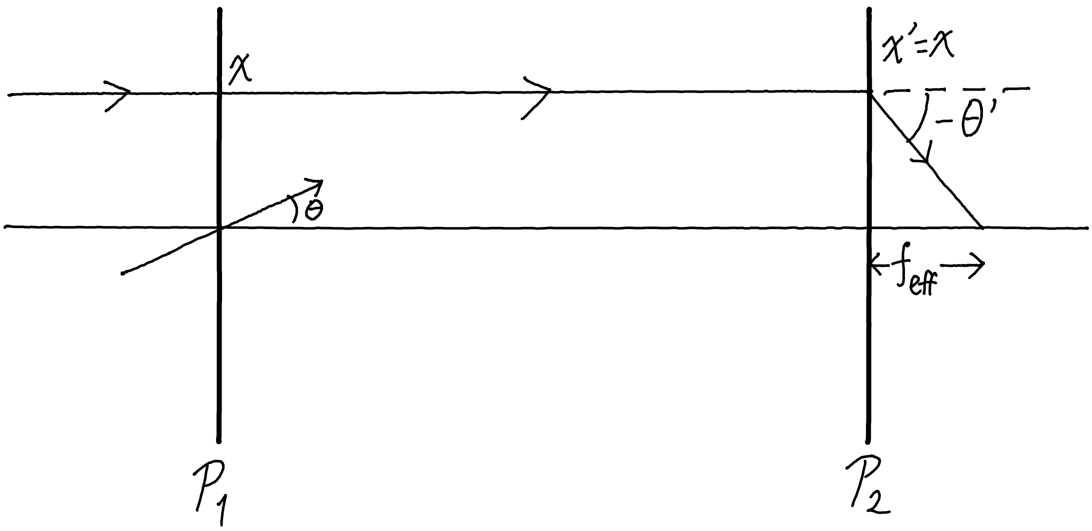
By Collins integral, we have:

$$E(\mathcal{P}_2) = \frac{1}{i\lambda B} \int E\mathrm{ikl}\, \mathrm{d}x_1\, \mathrm{d}y_1$$
$$= \frac{1}{i\lambda B} \frac{\mathrm{e}^{ikz_1}}{q_1} \mathrm{e}^{ikl_0} \mathrm{e}^{ik/2BD(x_2^2+y_2^2)} \int \mathrm{e}^{ik\rho_1^2/2q_1} \mathrm{e}^{ik/2B\left[A(x_1^2+y_1^2)-2(x_1x_2+y_1y_2)\right]}\, \mathrm{d}x_1\, \mathrm{d}y_1$$

We further evaluate the integral: **(TO CHECK)**

$$\int \exp\left\{ik\left[x_1^2\left(\frac{1}{2q_1}+\frac{A}{2B}\right) - x_1 \cdot \frac{2x_2}{2B} + y_1^2\left(\frac{1}{2q_1}\right) - y_1 \cdot \frac{2y_2}{2B}\right] \dots\right\}$$

(b) **(TO EXPAND)** Sketch of the setup again:



Ray transfer matrix: ???

(c) Ray transfer matrix for propagation over $f \to$ lens $\to$ propagation over f:

$$\begin{pmatrix}1 & f\\0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\-\frac{1}{f} & 1\end{pmatrix}\begin{pmatrix}1 & f\\0 & 1\end{pmatrix} = \begin{pmatrix}1 & f\\0 & 1\end{pmatrix}\begin{pmatrix}1 & f\\-\frac{1}{f} & 0\end{pmatrix}$$
$$= \begin{pmatrix}0 & f\\-\frac{1}{f}0 & \end{pmatrix}$$

So $\mathcal{P}_1 \to \mathcal{P}_2$ has ray matrix:

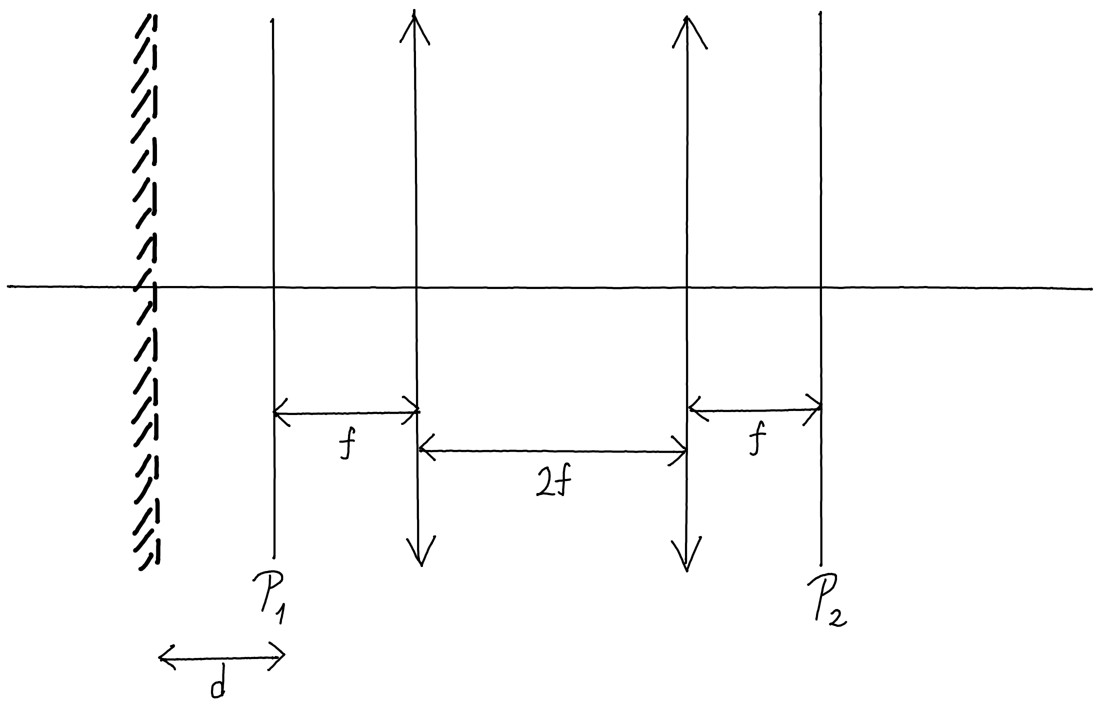
$$\begin{pmatrix}0 & f\\-\frac{1}{f} & 0\end{pmatrix}\begin{pmatrix}0 & f\\-\frac{1}{f}0 & \end{pmatrix}\begin{pmatrix}1 & d\\0 & 1\end{pmatrix} = \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & d\\0 & 1\end{pmatrix}$$
$$= \begin{pmatrix}-1 & -d\\0 & -1\end{pmatrix}$$

So we have l:

$$l = (d + 4f) + \frac{1}{-2d}\left[-(x_1^2 + y_1^2) - 2(x_1x_2 + y_1y_2) - (x_2^2 + y_2^2)\right]$$

Note that compared to just propagating through d, A, B, and D have all their signs flipped.

For negative d, consider the system below:



We have propagation by $|d| \to$ reflect $\to$ (f lens f):

$$\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}1 & |d|\\0 & 1\end{pmatrix} = ???$$

So B flips $\Rightarrow l$ increases then decreases.

(d) Noting that l may be adjusted to be positive or negative, thus this may be used to control dispersion in a CPA chain.

5. (DRAFT) Nasty NMR as usual.

(a) NMR Hamiltonian:

$$\mathcal{H} = \hbar\gamma B \frac{\sigma_z}{2}$$
$$\Rightarrow \Delta E = \hbar\gamma B = \hbar\omega$$
$$\Rightarrow B = \frac{\omega}{\gamma}$$

Usually we would have a B field of around 10 T, thus the level splitting would involve frequency around $\omega = B\gamma \simeq 425.8\,\mathrm{MHz}$, thus we are in the RF regime.

(b) Single-qubit gate is implemented by applying an RF pulse with specific frequency for qubit addressing since each species in the ensemble possesses different resonance frequencies due to either them being distinguishable nuclei (heteronuclear system) or chemical screening (homonuclear system).

Preparation of NMR qubit is difficult as trying to initialise a zeor state requires cooling the liquid NMR down, which can be physically infeasible. In addition, while a pseudo-pure state may achieve initialisation for a liquid NMR system, it is difficult to scale up due to the shrinking effective purity.

Measurement of a NMR computer is rather challenging as the pulse absorption happens across the ensemble rather than an individual qubit $\Rightarrow$ readout is a statistical average than the direct result of a qubit.

(c) **(TO EXPAND)** Recall that in NMR we need a background B field at $\hat{\mathbf{z}}$, but applying another B field at other directions would lead to non-unitarity as different directions of σ do not commute.

However, we can rewrite $\sigma_x = (\sigma_+ + \sigma_-)/2$, $\sigma_y = (\sigma_+ - \sigma_-)/2i$ so in effect we are projecting the state onto the z-axis.

Gradient strength???

(d) **(TO EXPAND)** As a quantum system evolves unitarily in time, by applying a sufficiently frequent measurement, one may retain the original constant in time with high probability. In an NMR, the issue is that the response time of a qubit can take a while before decoherence kicks in?

(e) Consider logical qubit $|0_L\rangle \equiv |01\rangle + |10\rangle$, $|1_L\rangle \equiv |00\rangle + |11\rangle$ ignoring normalisation.

A simultaneous X flip protects the logical qubits from errors $\Rightarrow$ decoherence-free subspace (DFS).

In a heteronuclear system, the resonance frequency for each qubit is different $\Rightarrow$ inherent protection against simultaneous gate.

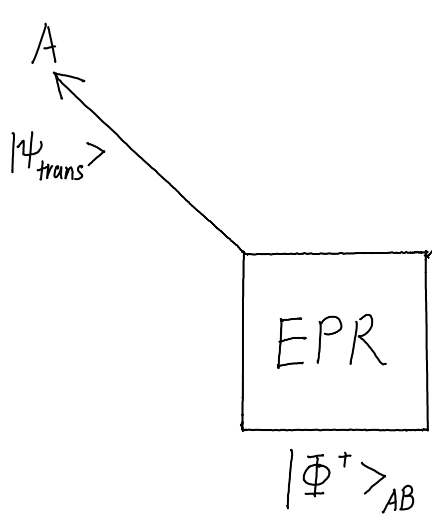
(f) **(TO EXPAND)** Spin-echo: suppresses unitary evolution via sequences of Y gates.

So to act on a single nuclear type, ???

6. Quantum information with a twist on multiple Hilbert spaces.

(a) In general for two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ to be entangled, their composite structure may not be separable.

e.g. for joint density matrix ρ_{AB}, it cannot decomposed into $\rho_{\mathrm{A}} \otimes \rho_{\mathrm{B}} = \sum_{i,j} p_i p_j \left|\psi_i\right\rangle \left\langle\psi_i\right| \otimes \left|\psi_j\right\rangle \left\langle\psi_j\right|$.



(b) To facilitate quantum teleportation with a Bell state, first the qubit pair has to be distributed between two parties A and B. A then performs a Bell state measurement between her qubit with the qubit to be transported $\left|\psi_{\text{trans}}\right\rangle = \alpha\left|0\right\rangle + \beta\left|1\right\rangle$.

By linearity of Hilbert space, we may rewrite the system state as the superposition of 4 Bell basis *between* $\left|\psi_{\text{trans}}\right\rangle$ and A, and now B has $\left|\psi_{\text{trans}}\right\rangle$ with $\mathbb{1}$, X, Y or Z flip. Hence if A now communicates the measurement result with B, B may reconstruct $\left|\psi_{\text{trans}}\right\rangle$ without knowing $\left|\psi_{\text{trans}}\right\rangle$ itself!

(c) *Here I present a brute-force approach, though a smarter way around this is in my attempt of the 2024 Finals. (Hint: partial trace is your friend)*

$$a\left|00\right\rangle + b\left|11\right\rangle = \begin{pmatrix} a \\ 0 \\ 0 \\ b \end{pmatrix}$$

For separable states,

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix} = \begin{pmatrix} \alpha\gamma \\ \alpha\delta \\ \beta\gamma \\ \beta\delta \end{pmatrix}$$

However note that $\alpha\delta = \beta\gamma = 0$ so one of the coefficients in the pairs (α, β) and (γ, δ) must be 0, which translates to the condition $a = 0$ or $b = 0$.

(d) Suppose instead of $|\Phi^+\rangle$, we have $|\xi\rangle = a\,|00\rangle + b\,|11\rangle$ with $a > 0$ and $b > 0$ and both real. The system state may then be described by the following ket vector:

$$\begin{aligned}
|\text{sys}\rangle &= \frac{1}{\sqrt{2}}\left[|0\rangle_{\text{trans}} + |1\rangle_{\text{trans}}\right] \otimes \left[a\,|0_{\text{A}}0_{\text{B}}\rangle + b\,|1_{\text{A}}1_{\text{B}}\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[a\,|0_{\text{trans}}0_{\text{A}}0_{\text{B}}\rangle + b\,|0_{\text{trans}}1_{\text{A}}1_{\text{B}}\rangle + a\,|1_{\text{trans}}0_{\text{A}}0_{\text{B}}\rangle + b\,|1_{\text{trans}}1_{\text{A}}1_{\text{B}}\rangle\right]
\end{aligned}$$

We now rewrite:

$$|00\rangle = \frac{|\Phi^+\rangle + |\Phi^-\rangle}{\sqrt{2}} \qquad |01\rangle = \frac{|\Psi^+\rangle + |\Psi^-\rangle}{\sqrt{2}}$$

$$|11\rangle = \frac{|\Phi^+\rangle - |\Phi^-\rangle}{\sqrt{2}} \qquad |10\rangle = \frac{|\Psi^+\rangle - |\Psi^-\rangle}{\sqrt{2}}$$

Thus:

$$\begin{aligned}
|\text{sys}\rangle = \frac{1}{\sqrt{2}}\Bigg[&\frac{a}{\sqrt{2}}\left(|\Phi^+\rangle_{\text{trans, A}} + |\Phi^-\rangle_{\text{trans, A}}\right)|0_{\text{B}}\rangle + \frac{b}{\sqrt{2}}\left(|\Psi^+\rangle_{\text{trans, A}} + |\Psi^-\rangle_{\text{trans, A}}\right)|1_{\text{B}}\rangle + \\
&\frac{a}{\sqrt{2}}\left(|\Psi^+\rangle_{\text{trans, A}} - |\Psi^-\rangle_{\text{trans, A}}\right)|0_{\text{B}}\rangle + \frac{b}{\sqrt{2}}\left(|\Phi^+\rangle_{\text{trans, A}} - |\Phi^-\rangle_{\text{trans, A}}\right)|1_{\text{B}}\rangle\Bigg] \\
= \frac{1}{2}\Big[&|\Phi^+\rangle_{\text{trans, A}}\left(a\,|0_{\text{B}}\rangle + b\,|1_{\text{B}}\rangle\right) + |\Phi^-\rangle_{\text{trans, A}}\left(a\,|0_{\text{B}}\rangle - b\,|1_{\text{B}}\rangle\right) + \\
&|\Psi^+\rangle_{\text{trans, A}}\left(a\,|0_{\text{B}}\rangle + b\,|1_{\text{B}}\rangle\right) + |\Psi^-\rangle_{\text{trans, A}}\left(-a\,|0_{\text{B}}\rangle + b\,|1_{\text{B}}\rangle\right)\Big]
\end{aligned}$$

Note that the amplitudes a and b are carried over due to the linearity of Hilbert space, thus the fidelity of the output state is:

$$\begin{aligned}
\mathcal{F} &= \left|\left(a\,\langle 0| + b\,\langle 1|\right)\left(\frac{1}{\sqrt{2}}\,|0\rangle + \frac{1}{\sqrt{2}}\,|1\rangle\right)\right|^2 \\
&= \left|\frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}}\right|^2 \\
&= \frac{a^2 + b^2}{2} + ab \\
&= \frac{1}{2} + ab
\end{aligned}$$

(e) Now suppose $|\psi_{\text{trans}}\rangle$ is entangled to some other system, so $|\psi_{\text{trans}}\rangle = \alpha\,|0\rangle \otimes \rho_0 + \beta\,|1\rangle \otimes \rho_1$ where $\rho_{0,1}$ are the entangled external entities.

Repeating the scheme above and it is clear that ρ_0 always follow the coefficient α and ρ_1 with β, so after the transportation, we realise that $|\psi_{\text{B}}\rangle = |\psi_{\text{trans}}\rangle$ *precisely*, hence the entanglement is also transported.