UNOFFICIAL SOLUTIONS BY TheLongCat

B6: CONDENSED-MATTER PHYSICS

TRINITY TERM 2019

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Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. (DRAFT)

(a) Lattice: a mathematical set of points related by an infinite number of primitive lattice vectors (which are linearly independent of one another).

Basis: a construct of repeated motifs with respect to a lattice point.

(b) Reciprocal PLV:

$$\mathbf{b}_i = \frac{2\pi \mathbf{a}_j \times \mathbf{a}_k}{\mathbf{a}_i \cdot (\mathbf{a}_j \times \mathbf{a}_k)}$$

For $\mathbf{R} = h\mathbf{a}_1 + k\mathbf{a}_2 + l\mathbf{a}_3$,

$$\begin{aligned} \mathbf{R} \cdot \mathbf{G} &= 2\pi (h + k + l) \\ \Rightarrow e^{i\mathbf{G}\cdot\mathbf{R}} &= 1 \end{aligned}$$

for any integer h, k, l.

So $\mathbf{G}$ is a primitive reciprocal lattice vector.

(c)

$$\begin{aligned} V_q &= \int e^{i\mathbf{q}\cdot\mathbf{x}} V(\mathbf{x}) \, \mathrm{d}\mathbf{x} \\ &= \sum_{\mathbf{R}} \int_{\mathrm{UC}} e^{i\mathbf{q}\cdot(\mathbf{R}+\mathbf{x}')} V(\mathbf{R}+\mathbf{x}') \, \mathrm{d}\mathbf{x}' \\ &= \underbrace{\sum_{\mathbf{R}} e^{i\mathbf{q}\cdot\mathbf{R}}}_{\substack{\text{This term} \\ \text{vanishes} \\ \text{unless } \mathbf{q}=\mathbf{G} \\ \text{so it becomes} \\ \text{an infinite sum}}} \int_{\mathrm{UC}} e^{i\mathbf{q}\cdot\mathbf{x}'} \underbrace{V(\mathbf{x}')}_{V(\mathbf{x})=V(\mathbf{x}+\mathbf{R})} \, \mathrm{d}\mathbf{x}' \end{aligned}$$

(d) Matrix element:

$$\begin{aligned} \langle \mathbf{k}' | V(\mathbf{x}) | \mathbf{k} \rangle &= \int e^{i\mathbf{q}\cdot\mathbf{x}} V(\mathbf{x}) \, \mathrm{d}^3\mathbf{x} \quad \text{where } \mathbf{q} = (\mathbf{k}' - \mathbf{k}) \\ &= \underbrace{\sum_{\mathbf{R}} e^{i\mathbf{q}\cdot\mathbf{R}}}_{\text{over system}} \int_{\mathrm{UC}} e^{i\mathbf{q}\cdot\mathbf{x}} V(\mathbf{x}) \, \mathrm{d}^3\mathbf{x} \end{aligned}$$

similar to part c.

Structure factor:

$$\begin{aligned}
b_p(\mathbf{G}) &= \int_{\mathrm{UC}} e^{i\mathbf{q}\cdot\mathbf{x}} V(\mathbf{x})\,\mathrm{d}^3\mathbf{x} \\
&= \int_{\mathrm{UC}} e^{i\mathbf{q}\cdot\mathbf{x}} \left[\sum_j u_j(\mathbf{x}-\mathbf{x}_j)\right] \mathrm{d}^3\mathbf{x} \quad \text{by Laue condition} \\
&= \sum_j \int_{\mathrm{UC}} e^{i\mathbf{G}\cdot\mathbf{x}} u_j(\mathbf{x}-\mathbf{x}_j)\,\mathrm{d}^3\mathbf{x} \\
&= \underbrace{\sum_j}_{\text{in UC only}} \int_{\mathrm{UC}} e^{i\mathbf{G}\cdot(\mathbf{x}-\mathbf{x}_j)} u_j(\mathbf{x})\,\mathrm{d}^3\mathbf{x}
\end{aligned}$$

So by relabelling $\mathbf{R} \to \mathbf{r}_p$,

$$S(\mathbf{G}) = \sum_p b_p(\mathbf{G}) e^{i\mathbf{G}\cdot\mathbf{r}_p}$$

$$\Rightarrow I(\mathbf{G}) \propto \Gamma \propto |S(\mathbf{G})|^2$$

(e) For cubic lattice, $d = \dfrac{a}{\sqrt{h^2+k^2+l^2}}$ where d is reciprocal lattice plane spacing.

Also:

$$\begin{aligned}
|\mathbf{G}| = G &= \frac{2\pi}{d} \\
&= \frac{2\pi\sqrt{h^2+k^2+l^2}}{a}
\end{aligned}$$

Bragg's law:

$$2d\sin\theta = \lambda$$

$$\sin\theta \propto \frac{1}{d} \propto \sqrt{h^2+k^2+l^2}$$

Assuming all the silicons have similar $b_p(\mathbf{G})$, i.e. ignoring the asymmetry in nearest neighbour distances:

$$\begin{aligned}
S(\mathbf{G}) &= b_{\mathrm{Si}}\left[e^{i\pi 0} + e^{2i\pi\left(\frac{h+k+l}{4}\right)} + e^{2i\pi\left(\frac{h+k}{2}\right)} + e^{2i\pi\left(\frac{k+l}{2}\right)} + e^{2i\left(\frac{h+l}{2}\right)} + \dots\right] \\
&= b_{\mathrm{Si}}\left[1 + e^{i\pi(h+k)} + e^{i\pi(k+l)} + e^{i\pi(h+l)}\right]\left[1 + e^{\frac{i\pi}{2}(h+k+l)}\right]
\end{aligned}$$

So FCC selection rules apply, albeit with some absences where $(h+k+l)\cdot\frac{1}{2}$ is odd:

- $\{111\}$ ✓
- $\{200\}$ ✗ – $\dfrac{2}{2} = 1$ so absence due to above condition
- $\{220\}$ ✓

So the 2 lowest angle of scattering corresponds to:

$$G = \frac{2\pi\sqrt{3}}{a}$$

and

$$G = \frac{2\pi\sqrt{8}}{a}$$

2. (DRAFT)

(a) Einstein model assumptions:

- The atoms do not interact with one another;
- Each atom constitutes a harmonic oscillator $\Rightarrow$ energy $= \hbar\omega\left(n+\dfrac{1}{2}\right)$ where ω is Einstein frequency (a parameter).

Single particle partition function:

$$\begin{aligned}
Z &= \sum_{n_x,n_y,n_z} e^{-\beta\hbar\omega\left(n_x+n_y+n_z+\frac{3}{2}\right)} \\
&= \sum_n \left(e^{-\beta\hbar\omega(n+\frac{1}{2})}\right)^3 \quad \text{by isotropy} \\
&= \frac{e^{-3\beta\hbar\omega\cdot\frac{1}{2}}}{1-e^{-3\beta\hbar\omega}} \\
&= \frac{1}{e^{\frac{3}{2}\beta\hbar\omega}-e^{-\frac{3}{2}\beta\hbar\omega}} \\
&= \frac{2}{\sinh\left(\frac{3}{2}\beta\hbar\omega\right)}
\end{aligned}$$

So expected energy:

$$\begin{aligned}
\langle E\rangle &= -\frac{1}{Z}\frac{\partial Z}{\partial\beta} \\
&= -\frac{1}{2}\sinh\left(\frac{3}{2}\beta\hbar\omega\right)\cdot\left(-\frac{2\cosh\left(\frac{3}{2}\beta\hbar\omega\right)}{\sinh^2\left(\frac{3}{2}\beta\hbar\omega\right)}\right)\cdot\frac{3}{2}\hbar\omega \\
&= \frac{3}{2}\hbar\omega\coth\left(\frac{3}{2}\beta\hbar\omega\right)
\end{aligned}$$

For N atoms, total partition function:

$$\begin{aligned}
Z_N &= (Z)^N \\
\Rightarrow \langle E_N\rangle &= \frac{3}{2}N\hbar\omega\coth\left(\frac{3N}{2}\beta\hbar\omega\right)
\end{aligned}$$

So:

$$\begin{aligned}
C &= \frac{\partial\langle E_N\rangle}{\partial T} \\
&= \frac{\partial\langle E_N\rangle}{\partial\beta}\cdot\frac{\partial\beta}{\partial T} \\
&= \frac{3}{2}N\hbar\omega\left[\frac{\sinh(\dots)}{\sinh(\dots)}-\frac{\cosh^2(\dots)}{\sinh^3(\dots)}\right]\cdot\left(-k_\mathrm{B}\beta^2\right) \\
&= \left(\frac{3N}{2}\hbar\omega\right)^2 k_\mathrm{B}\beta^2\left(1-\coth^2(\dots)\right) \\
&= \left(\frac{3N}{2}\beta\hbar\omega\right)^2 k_\mathrm{B}\left[1-\coth^2\left(\frac{3N}{2}\beta\hbar\omega\right)\right]
\end{aligned}$$

(b) In high temperature limit, $\beta \ll \hbar\omega$.

Also:

$$\begin{aligned}\coth x &= \frac{e^x + e^{-x}}{e^x - e^{-x}} \\ &= \frac{e^{2x}+1}{e^{2x}-1} \\ &= \frac{e^{2x}-1}{e^{2x}-1} + \frac{2}{e^{2x}-1} \\ &= 1 + 2n_\mathrm{B}(2x)\end{aligned}$$

where $n_\mathrm{B}(x) = \dfrac{1}{e^x - 1}$.

$$\begin{aligned}\langle E \rangle &= 3N\hbar\omega \left[n_\mathrm{B}(3N\beta\hbar\omega) + \frac{1}{2} \right] \\ &= 3N\hbar\omega \left[\frac{1}{3N\beta\hbar\omega} - \cancel{\frac{1}{2}} + \frac{3N\beta\hbar\omega}{12} + \cancel{\frac{1}{2}} + \dots \right] \\ &= \frac{1}{\beta} + \frac{(3N\hbar\omega)^2 \beta}{4} + \dots \\ &= k_\mathrm{B}T + \frac{(3N\hbar\omega)^2}{4k_\mathrm{B}T} + \dots\end{aligned}$$

So:

$$C = k_\mathrm{B} - \frac{(3N\hbar\omega)^2}{4k_\mathrm{B}T^2}$$

$$\begin{aligned}\Rightarrow A_0 &= k_\mathrm{B} \\ A_1 &= 0 \\ A_2 &= -\frac{(3N\hbar\omega)^2}{4k_\mathrm{B}}\end{aligned}$$

(c) Einstein temperature:

$$\begin{aligned}\hbar\omega &= k_\mathrm{B}T_E \\ \Rightarrow A_2 &= -\frac{(3N)^2 k_\mathrm{B}^2 T_E^2}{4k_\mathrm{B}} \\ &= -\frac{(3N)^2 k_\mathrm{B}}{4} T_E^2\end{aligned}$$

Next note that $A_0 \to NA_0$, $A_2 \to NA_2$:

$$\begin{aligned}\Rightarrow N &= \frac{A_0}{k_\mathrm{B}} \\ \Rightarrow T_E^2 &= -\frac{4}{9}\frac{k_\mathrm{B}^2}{A_0^2}\frac{A_2}{k_\mathrm{B}} \\ T_E &= 11.54\,\mathrm{K}\end{aligned}$$

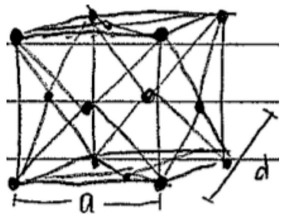
(d) Argon atom weighs $40u$.

$\Rightarrow$ Mass of sample:

$$\begin{aligned} M &= 40Nu \\ &= \frac{40uA_0}{k_\mathrm{B}} && = 0.120\,\mathrm{kg} \end{aligned}$$

For lattice constant a, nearest neighbour distance d,

$$a = \sqrt{2}d$$



An FCC unit cell (conventional) contains 4 argon atoms.

Hence:

$$\begin{aligned} V &= \frac{N}{4}a^3 \\ \Rightarrow a^3 &= \frac{4V}{N} = \frac{4k_\mathrm{B}V}{A_0} \\ a &= 3.59 \times 10^{-8}\,\mathrm{cm} \\ &= 0.359\,\mathrm{nm} \\ \Rightarrow d &= \frac{a}{\sqrt{2}} \\ &= 0.254\,\mathrm{nm} \end{aligned}$$

(e) For a harmonic oscillator,

$$\begin{aligned} E &= \frac{1}{2}kx^2 + \frac{p^2}{2m} \\ \langle E \rangle &= \frac{1}{2}k\left\langle x^2 \right\rangle + \frac{\langle p^2 \rangle}{2m} \\ &= k\left\langle x^2 \right\rangle \quad \text{by equipartition theorem} \\ &= 2k_\mathrm{B}T \end{aligned}$$

$$\begin{aligned} \left\langle x^2 \right\rangle &= \int_{-\infty}^{\infty} P(x)x^2\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{e^{-\beta E}}{Z}x^2\,\mathrm{d}x \end{aligned}$$

where

$$Z = \int_{-\infty}^{\infty} \mathrm{d}\mathbf{p} \int e^{-\beta\left[\frac{\mathbf{p}^2}{2m} + \frac{k\mathbf{x}^2}{2}\right]}\,\mathrm{d}\mathbf{x}$$

since we are in the classical regime.

3. **(DRAFT)**

(a) 2-dimensional e$^-$ gas:

$$
\begin{aligned}
N &= 2\sum_{\mathbf{k}} n_\mathrm{F} \quad \text{where } n_\mathrm{F} = \frac{1}{e^{\beta[E(\mathbf{k})-\mu]}+1} \\
&= \frac{2V}{(2\pi)^2}\int_0^\infty 2\pi k n_\mathrm{F}\,\mathrm{d}k \\
n &= \int_0^\infty \frac{4\pi}{(2\pi)^2} k n_\mathrm{F}\,\mathrm{d}k
\end{aligned}
$$

From the dispersion relation,

$$
\begin{aligned}
E &= \frac{\hbar^2 k^2}{2m} \\
\Rightarrow\ \mathrm{d}E &= \frac{2\hbar^2 k}{2m}\,\mathrm{d}k \\
k\,\mathrm{d}k &= \frac{\cancel{2}m}{\cancel{2}\hbar^2}\,\mathrm{d}E
\end{aligned}
$$

So:

$$
\begin{aligned}
g(k) &= \frac{k}{\pi}\,\mathrm{d}k \\
&= \frac{m}{\pi\hbar^2}\,\mathrm{d}E = g(E)
\end{aligned}
$$

For e$^-$ in conduction band,

$$
n = \int_{\rceil_c}^\infty g(E-\rceil_c) n_\mathrm{F}\,\mathrm{d}E
$$

For $\mu \ll \rceil_c \Rightarrow E-\mu \gg 1 \Rightarrow n_\mathrm{F} \simeq e^{-\beta(E-\mu)}$:

$$
\begin{aligned}
n &= \int_{\rceil_c}^\infty \frac{m}{\pi\hbar^2} e^{-\beta(E-\mu)}\,\mathrm{d}E \\
&= \left[\frac{m}{\pi\hbar^2} e^{\beta\mu}\cdot\left(-\frac{1}{\beta}e^{-\beta E}\right)\right]_{E=\rceil_c}^\infty \\
&= \frac{m k_\mathrm{B} T}{\pi\hbar^2} e^{-\beta(\rceil_c-\mu)} \\
\Rightarrow C &= \frac{k_\mathrm{B}}{\pi\hbar^2},\ a=1,\ b=1
\end{aligned}
$$

(b) Similarly,

$$
\begin{aligned}
p(T) &= \frac{m_\mathrm{h}^* k_\mathrm{B} T}{\pi\hbar^2} e^{-\beta(\mu-\rceil_v)} \\
np &= \frac{m_\mathrm{e}^* m_\mathrm{h}^* (k_\mathrm{B}T)^2}{\pi^2\hbar^4} e^{-\beta(\rceil_c-\mu+\mu-\rceil_v)} \\
&= m_\mathrm{e}^* m_\mathrm{h}^* \left(\frac{k_\mathrm{B}T}{\pi\hbar^2}\right)^2 e^{-\beta E_g}
\end{aligned}
$$

where $E_g = \rceil_c - \rceil_v$ and $\rceil_v$ is the energy at the top of the valence band.

(c) Law of mass action states $np = n_i^2$ where n_i is intrinsic e^- concentration:

$$\begin{aligned}\Rightarrow n_i &= \sqrt{m_\mathrm{e}^* m_\mathrm{h}^*}\frac{k_\mathrm{B}T}{\pi\hbar^2}e^{-\frac{E_g}{2k_\mathrm{B}T}}\\ &= 1.06\times 10^{13}\,\mathrm{m}^{-2}\end{aligned}$$

for $t = 300\,\mathrm{K}$, $m_\mathrm{e}^* = 0.026m_\mathrm{e}$, $m_\mathrm{h}^* = 0.41m_\mathrm{e}$, $E_g = 0.36\,\mathrm{eV}$.

Transitioning from intrinsicn to extrinsic region requires:

$$\begin{aligned}n_i &\ll n_D\\ \Rightarrow \sqrt{m_\mathrm{e}^* m_\mathrm{h}^*}\frac{k_\mathrm{B}T}{\pi\hbar^2}e^{-\frac{E_g}{2k_\mathrm{B}T}} &= n_D = 10^{13}\,\mathrm{m}^{-2}\\ \Rightarrow T &\simeq 298\,\mathrm{K} \quad \text{by trial and error}\end{aligned}$$

(d) TISE: $\hat{H}\Psi = E\Psi$ where $\hat{H} = \frac{\hat{\mathbf{p}}^2}{2m} + V(r)$ with $V(r) = -\frac{e^2}{4\pi\epsilon_0\hat{r}}$:

$$\begin{aligned}-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}\Psi + V\Psi &= E\Psi\\ \Rightarrow -\frac{\hbar^2}{2m}\cdot Ab^2e^{-b|\mathbf{r}-\mathbf{r}_0|} + VAe^{-b|\mathbf{r}-\mathbf{r}_0|} &= EAe^{-b|\mathbf{r}-\mathbf{r}_0|}\\ \Rightarrow E &= V - \frac{\hbar^2b^2}{2m}\end{aligned}$$

For hydrogenic bound state,

$$\begin{aligned}E &= -R_\infty hc\left[\frac{m_\mathrm{e}^*}{m_\mathrm{e}}\frac{1}{\epsilon_r^2}\right]\\ &= -0.0016\,\mathrm{eV}\end{aligned}$$

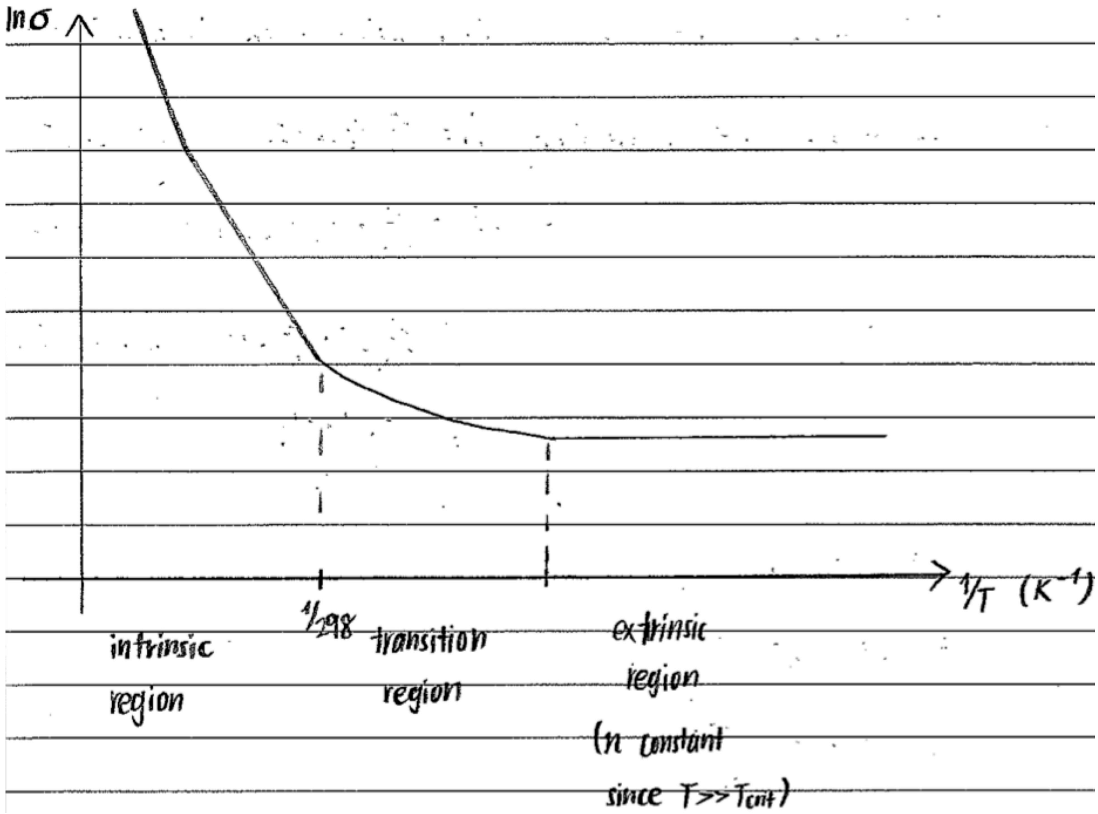
Also effective Bohr radius:

$$\begin{aligned}a_0^\mathrm{eff} &= a_0\left(\epsilon_r\frac{m_\mathrm{e}}{m_\mathrm{e}^*}\right)\\ &= 3.05\times 10^{-8}\,\mathrm{m}\\ \Rightarrow V &= -\frac{e^2}{e\pi\epsilon_0 a_0^\mathrm{eff}}\\ &= -0.047\,\mathrm{eV}\\ b^2 &= \frac{2m_\mathrm{e}^*(V-E)}{\hbar^2}\\ &= \frac{2m_\mathrm{e}^*c^2(V-E)}{\hbar^2c^2}\\ &= -1.19\times 10^{-12}\,\mathrm{fm}^{-2}\\ \Rightarrow b &= 1.09\times 10^{-6}i\,\mathrm{fm}^{-1}\end{aligned}$$

(e) Conductivity:

$$\begin{aligned}\sigma &= \frac{j}{E} \\ &= \frac{-nev}{E} \\ &= -n \underbrace{e\mu}_{\text{T independent}} \propto n\end{aligned}$$

So σ scales with n, which should exhibit both intrinsic and extrinsic behaviour over the temperature range of $30 - 600\text{K}$, freezing out is not shown as the critical temperature for so is $18.2\,\text{K}$.



4. (DRAFT)

(a) Paramagnet: material where $\chi > 0$ and $M = 0$ when $B = 0$.

Diamagnet: material where $\chi < 0$ uad $M = 0$ when $B = 0$.

Ferromagnet: usually $\chi > 0$, $M \neq 0$ even when $B = 0$.

χ is defined as $\lim_{H\to 0} \frac{\partial M}{\partial H}$ where M is magnetisation per unit volume, H is H field.

(b) For helium, its electronic state has $L = S = J = 0$ due to the filled shell, therefore Curie paramagnetism is impossible, rendering Larmor diamagnetism the significant interaction.

Larmor diamagnetic Hamiltonian:

$$\begin{aligned}
\hat{H} &= \frac{e^2}{2m_\mathrm{e}} \frac{1}{4} |\mathbf{B} \times \mathbf{r}|^2 \quad \text{for an e}^- \\
\Rightarrow \langle E \rangle &= \frac{e^2}{2m_\mathrm{e}} \cdot \frac{1}{4} B^2 \langle x^2 + y^2 \rangle \quad \text{assuming } \mathbf{B} \parallel \hat{\mathbf{z}} \\
&= \frac{e^2}{8m_\mathrm{e}} B^2 \cdot \frac{2}{3} \langle r^2 \rangle \quad \text{assuming isotropy} \\
&= \frac{e^2 B^2 \langle r^2 \rangle}{12 m_\mathrm{e}}
\end{aligned}$$

Next we have:

$$\begin{aligned}
\mathrm{d}U &= T\,\mathrm{d}S - m\,\mathrm{d}B \\
\Rightarrow m &= -\frac{\partial U}{\partial B} \\
&= -\frac{e^2 B \langle r^2 \rangle}{6 m_\mathrm{e}}
\end{aligned}$$

Magnetisation:

$$M = \frac{Nm}{V} = -\frac{n e^2 B \langle r^2 \rangle}{6 m_\mathrm{e}}$$

where $n = 2\rho$ is electron density and ρ is helium density.

Hence susceptibility:

$$\begin{aligned}
\chi &= \lim_{H\to 0} \frac{\partial M}{\partial H} \\
&= \mu_0 \frac{\partial M}{\partial B} \\
&= -\frac{n e^2 \mu_0}{6 m_\mathrm{e}} \langle r^2 \rangle \\
&= -\frac{\rho e^2 \mu_0}{3 m_\mathrm{e}} \langle r^2 \rangle
\end{aligned}$$

(c) For ^{3}He nucleus, paramagnetic Hamiltonian:

$$\hat{H} = -\mu_{^3\mathrm{He}} \hat{\mathbf{s}} \cdot \mathbf{B}$$

Single particle partition function:

$$\begin{aligned} Z &= e^{-\beta\mu_{^3\mathrm{He}}B\cdot\frac{1}{2}} + e^{-\beta\mu_{^3\mathrm{He}}B\cdot\left(-\frac{1}{2}\right)} \\ &= 2\cosh\left(\frac{\beta\mu_{^3\mathrm{He}}B}{2}\right) \end{aligned}$$

Furthermore:

$$\begin{aligned} \mathrm{d}U &= T\,\mathrm{d}S - m\,\mathrm{d}B \\ \mathrm{d}F &= -S\,\mathrm{d}T - m\,\mathrm{d}B \\ \Rightarrow F &= -k_\mathrm{B}T\ln Z \\ \Rightarrow m &= -\frac{\partial F}{\partial B} \\ &= k_\mathrm{B}T\frac{\partial \ln Z}{\partial B} \\ &= k_\mathrm{B}T\cdot\frac{1}{Z}\cdot\frac{\partial Z}{\partial B} \\ &= \frac{\cancel{k_\mathrm{B}T}}{\cancel{2}\cosh(\dots)}\cdot\cancel{2}\sinh(\dots)\cdot\frac{\cancel{\beta}\mu_{^3\mathrm{He}}}{2} \\ &= \frac{\mu_{^3\mathrm{He}}}{2}\tanh\left(\frac{\beta\mu_{^3\mathrm{He}}B}{2}\right) \end{aligned}$$

Magnetisation:

$$\begin{aligned} M &= \frac{mN}{V} \\ &= \frac{n\mu_{^3\mathrm{He}}}{2}\tanh\left(\frac{\beta\mu_{^3\mathrm{He}}B}{2}\right) \end{aligned}$$

where n is the helium nucleus density.

Susceptibility:

$$\begin{aligned} \chi &= \lim_{H\to 0}\frac{\partial M}{\partial H} \\ &= \mu_0\lim_{H\to 0}\frac{\partial M}{\partial B} \\ &= \mu_0\frac{\partial}{\partial B}\left[\frac{n\mu_{^3\mathrm{He}}}{2}\cdot\frac{\beta\mu_{^3\mathrm{He}}B}{2}\right] \\ &= \frac{n\mu_0\mu_{^3\mathrm{He}}^2}{4k_\mathrm{B}T} \end{aligned}$$

(d) For ^{3}He, assuming negligible hyperfine interaction, total susceptibility:

$$\chi_\mathrm{tot} = \frac{n\mu_0\mu_{^3\mathrm{He}}}{4k_\mathrm{B}T} - \frac{ne^2\mu_0}{3m_\mathrm{e}}\langle r^2\rangle$$

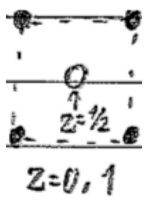
rewriting $\rho\to n$.

At critical temperature, $\chi_{\text{tot}} = 0$:

$$\begin{aligned}
\Rightarrow k_B T &= \frac{3m_e \cancel{n\mu_0}\mu_{^3\mathrm{He}}^2}{\cancel{n}e^2\cancel{\mu_0}\langle r^2\rangle} \\
T &= \frac{3m_e \mu_{^3\mathrm{He}}^2}{4k_B e^2 \langle r^2\rangle}
\end{aligned}$$

For helium, $\langle r^2\rangle \simeq a_0^2$:

$$T \simeq \frac{3m_e \mu_{^3\mathrm{He}}^2}{4k_B e^2 a_0^2} = 0.0797\,\mathrm{K}$$



(e) For a site i,

$$\hat{H}_i = -\frac{1}{2}\cdot 2\left[J_c\langle s_j\rangle + 3J_e\langle s_j\rangle\right]\cdot s_i$$

where $J_c = (1\,\mathrm{mK})k_B$, $J_e = (-0.5\,\mathrm{mK})k_B$.

Now compare with paramagnetic term $\hat{H}_i = -g_s\mu_B \mathbf{B}\cdot\mathbf{s}_i$ for e^-.

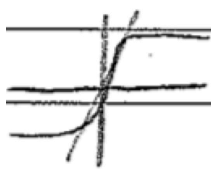
We get $g_s\mu_B\langle B_{\text{eff}}\rangle = (J_c + 3J_e)\langle s_j\rangle$.

And:

$$\begin{aligned}
Z &= 2\cosh\left(\beta\mu_B B_{\text{eff}}\right) \\
\Rightarrow m &= \mu_B \tanh\left(\beta\mu_B B_{\text{eff}}\right) = -g_s\mu_B\langle s_i\rangle \\
\Rightarrow \langle s_i\rangle &= -\frac{1}{2}\tanh\left(\beta\mu_B B_{\text{eff}}\right) \\
&= -\frac{1}{2}\tanh\left(\beta\cancel{\mu_B}\cdot\frac{(J_c+3J_e)\langle s_j\rangle}{g_s\cancel{\mu_B}}\right) \\
&= -\frac{1}{2}\tanh\left(\frac{1}{2}\beta(J_c+3J_e)\langle s_j\rangle\right)
\end{aligned}$$

Now invoke equivalence of means,

$$\langle s_i\rangle = \langle s_j\rangle = \langle s\rangle = -\frac{1}{2}\tanh\left(\frac{1}{2}\beta(J_c+3J_e)\langle s\rangle\right)$$



At critical temperature, $\dfrac{\partial}{\partial\langle s\rangle}\mathrm{RHS} = 1$:

$$\Rightarrow -\frac{1}{4}\,\mathrm{sech}\left(\frac{1}{2}\beta(J_c + 3J_e)\langle s\rangle\right)\cdot\beta(J_c + 3J_e) = 1$$

$$\Rightarrow T \simeq -\frac{J_c + 3J_e}{4k_\mathrm{B}} \quad \text{assuming small } T$$
$$= 0.125\,\mathrm{mK}$$