UNOFFICIAL SOLUTIONS BY TheLongCat

C2: LASER SCIENCE AND QUANTUM INFORMATION PROCESSING

TRINITY TERM 2019

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Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. Classic Q-switching question.

(a) Q-switching is a technique by which the quality factor of a laser cavity is periodically varied to intentionally build up population inversion N^* beyond the equilibrium value, thus producing a laser output with *large peak intensity* and *short pulse length.*

Q-switching may be achieved by employing a saturable absorber at the output coupler. As the absorption of the absorber varies with the intensity I, it is able to suppress lasing until saturation, thereby achieving Q-switching without active clock source.

(b) i. N^*_{th} represents the threshold population inversion – this is the value of N^* when photon population in the cavity begins to grow.

ii. *Bookwork from Simon's notes. Essential points:*

1. *Integrate* $\dfrac{\mathrm{d}n}{\mathrm{d}t}$ *to get relationship between* $\dfrac{n}{\tau_c}$*'s*
2. *Write output power as function of* $\dfrac{n}{\tau_c}$ *by considering the net photon output rate*
3. *???* *You know what to do, integration!*
4. *Profit!*

In any case, I shall demonstrate a (hopefully!) standard answer below.

We integrate $\dfrac{\mathrm{d}n}{\mathrm{d}t}$ as given in the question:

$$\int_{n(t\to-\infty)}^{n(t\to+\infty)} \mathrm{d}n = \int_{-\infty}^{+\infty} \mathrm{d}t \left(\frac{N^*}{N^*_{\text{th}}} - 1\right)\frac{n}{\tau_c}$$

$$\Rightarrow \quad \underbrace{n(t\to+\infty) - n(t\to-\infty)}_{\text{0 since we have null intensity in either limit!}} = \ldots$$

$$\Rightarrow \int_{-\infty}^{+\infty} \mathrm{d}t\,\frac{n}{\tau_c} = \int_{-\infty}^{+\infty} \mathrm{d}t\,\frac{N^*}{N^*_{\text{th}}}\frac{n}{\tau_c}$$

Then note that the power output is just (net photon output rate) $\times\,\hbar\omega$:

$$P = \left(\frac{nV_c}{\tau_c}\right)\hbar\omega \tag{1}$$

Note that the net output photon *(not density!)* rate involves the product with the cavity volume!

Integrating (1) then gives (also noting the given $\dfrac{\mathrm{d}N^*}{\mathrm{d}t}$):

$$\begin{aligned}
E &= \int_{-\infty}^{+\infty} P\,\mathrm{d}t \\
&= \int_{-\infty}^{+\infty} \mathrm{d}t\,\frac{n}{\tau_c}V_c\hbar\omega \\
&= \int_{-\infty}^{+\infty} \mathrm{d}t\,\frac{N^*}{N^*_{\mathrm{th}}}\,\frac{n}{\tau_c}V_c\hbar\omega \\
&= \int_{N^*(t\to-\infty)}^{N^*(t\to+\infty)} \mathrm{d}N^*\,\left(-\frac{f_c}{\beta}V_c\hbar\omega\right) \\
&= \frac{N^*_{\mathrm{i}}-N^*_{\mathrm{f}}}{\beta}V_g\hbar\omega \\
&= \eta N^*_{\mathrm{i}}V_g\hbar\omega
\end{aligned}$$

iii. From above, $\eta = \dfrac{N^*_{\mathrm{i}}-N^*_{\mathrm{f}}}{\beta N^*_{\mathrm{i}}}$ is the energy utilisation factor. This encodes how efficient the laser system is in extracting energy from population inversion.

(c) i. To find $n(t)$, we divide $\dfrac{\mathrm{d}n}{\mathrm{d}t}$ by $\dfrac{\mathrm{d}N^*}{\mathrm{d}t}$ to get:

$$\begin{aligned}
\frac{\mathrm{d}n}{\mathrm{d}N^*} &= -\frac{f_c}{\beta}\left(1-\frac{N^*_{\mathrm{th}}}{N^*}\right) \\
\Rightarrow \int_{N^*(0)}^{N^*(t)}\left(1-\frac{N^*_{\mathrm{th}}}{N^*}\right)\mathrm{d}N^* &= \int_{n(0)}^{n(t)} -\frac{\beta}{f_c}\,\mathrm{d}n \\
-\frac{\beta}{f_c}\left(n(t)-n(0)\right) &= \left(N^*(t)-N^*(0)\right) - N^*_{\mathrm{th}}\ln\left(\frac{N^*(t)}{N^*(0)}\right)
\end{aligned}$$

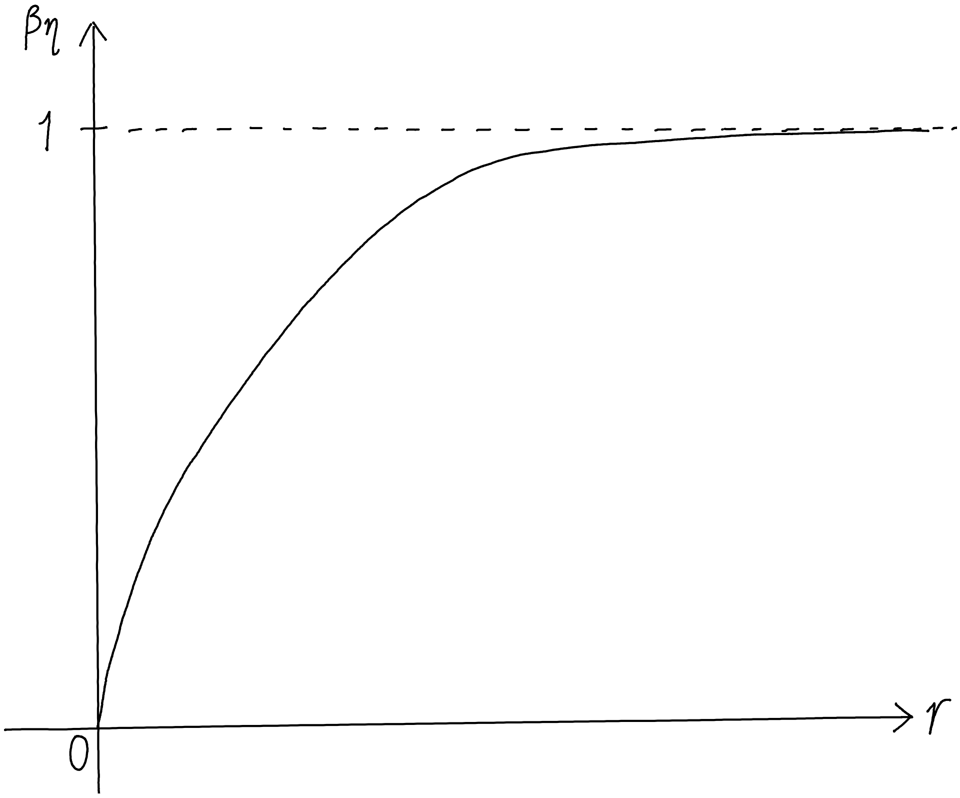
Equating $n(0)=0$ and $N^*(0)=N^*_{\mathrm{i}}$ gives:

$$n(t) = \frac{f_c}{\beta}\left[N^*_{\mathrm{th}}\ln\left(\frac{N^*(t)}{N^*_{\mathrm{i}}}\right) + \left(N^*_{\mathrm{i}}-N^*(t)\right)\right]$$

ii. Now putting in limits of $n(t\to+\infty)=0$, $N^*(t\to+\infty)=N^*_{\mathrm{f}}$:

$$\begin{aligned}
N^*_{\mathrm{f}}-N^*_{\mathrm{i}} &= N^*_{\mathrm{th}}\ln\left(\frac{N^*_{\mathrm{f}}}{N^*_{\mathrm{i}}}\right) \\
\Rightarrow \beta\eta N^*_{\mathrm{i}} &= N^*_{\mathrm{th}}\ln\left(\frac{N^*_{\mathrm{i}}}{N^*_{\mathrm{i}}\left(1-\beta\eta\right)}\right) \\
\beta\eta &= \frac{1}{r}\ln\left(\frac{1}{1-\beta\eta}\right) \\
r &= -\frac{\ln\left(1-\beta\eta\right)}{\beta\eta}
\end{aligned}$$

iii. *Precise sketch may be found in Simon's notes:*



iv. For $E/E_{\max} = 95\,\%$, we have:

$$\begin{aligned} \beta\eta &= 0.95 \\ \Rightarrow r &= -\frac{\ln 0.05}{0.95} \\ &= 3.15 \end{aligned}$$

(d) Setting $\dfrac{\mathrm{d}n}{\mathrm{d}t} = 0$ gives $N^* = N^*_{\mathrm{th}}$. substituting this into $n(t)$ then gives:

$$\begin{aligned} n_{\mathrm{peak}} &= \frac{f_c}{\beta}\left[N^*_{\mathrm{th}} \ln\left(\frac{N^*_{\mathrm{th}}}{N^*_{\mathrm{i}}}\right) + (N^*_{\mathrm{i}} - N^*_{\mathrm{th}}))\right] \\ &= \frac{f_c}{\beta} N^*_{\mathrm{i}} \left[\frac{1}{r}\ln\left(\frac{1}{r}\right) + \left(1 - \frac{1}{r}\right)\right] \end{aligned}$$

Substituting this into (1) then gives the peak power:

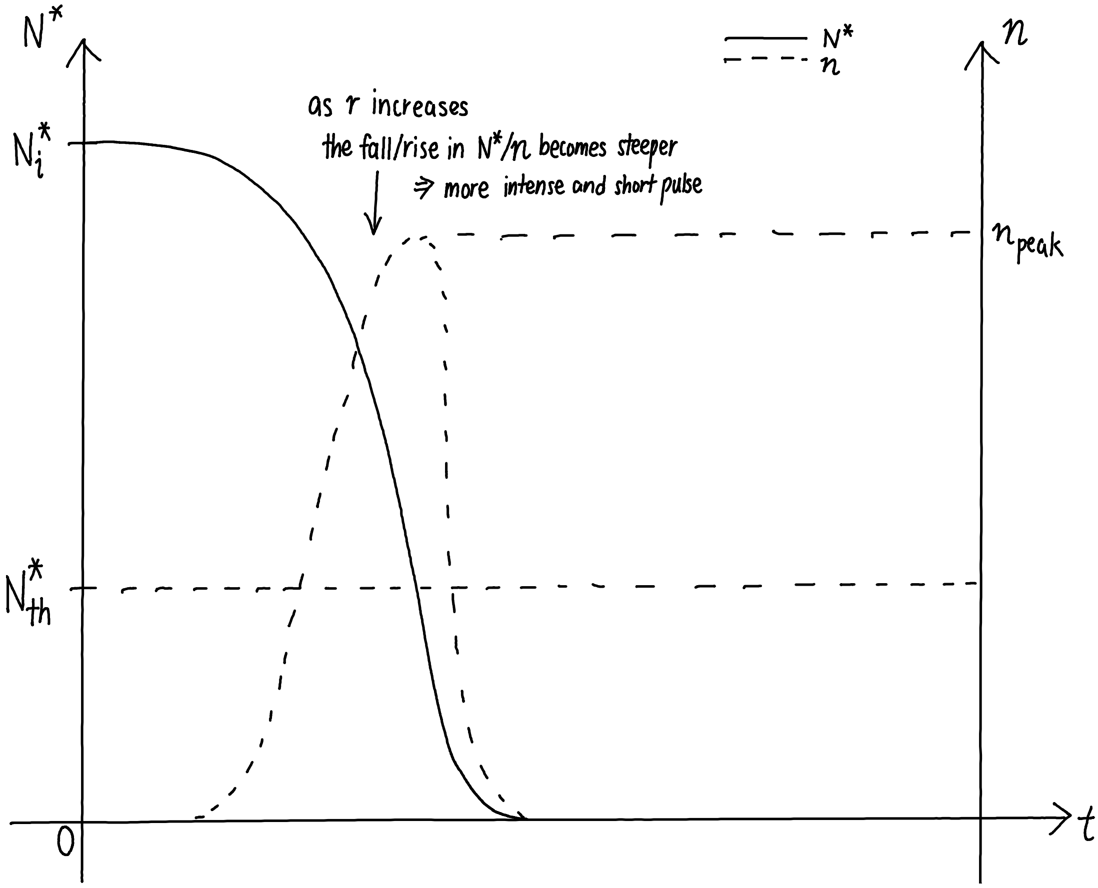
$$\begin{aligned} P_{\mathrm{peak}} &= \frac{N^*_{\mathrm{i}} f_c V_c \hbar\omega}{\beta\tau_c}\left[\frac{-\ln r + r - 1}{r}\right] \\ &= \frac{N^*_{\mathrm{i}} V_g \hbar\omega}{\beta\tau_c}\left[\frac{r - 1 - \ln r}{r}\right] \end{aligned} \tag{2}$$

Rewriting (2) in terms of N^*_{th}:

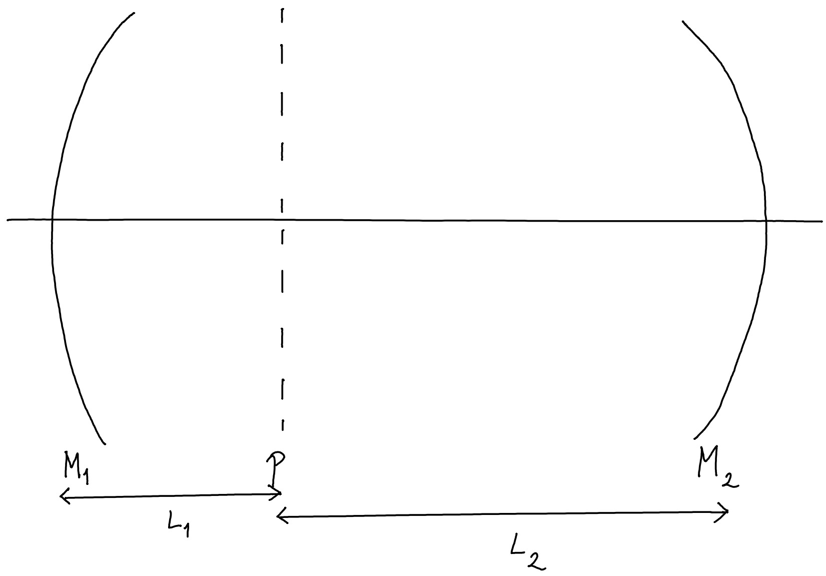
$$P_{\text{peak}} = \frac{N^*_{\text{i}} f_c V_c \hbar\omega}{\beta\tau_c}\left[\frac{-\ln r + r - 1}{r}\right]$$
$$= \frac{N^*_{\text{th}} V_g \hbar\omega}{\beta\tau_c}\left[r - 1 - \ln r\right]$$

At large r, the peak output power grows towards ∞ as the laser system is able to burn almost all of the population inversion down to generate an intense pulse.

As suggested by the rate equations above, large value of r will also cause a rapid population burning, hence the pulse will possess a short pulse length as shown in the sketch below:



2. Low-loss transverse mode question. As always drawing out a schematic of the cavity helps:



(a) ABCD rule gives:

$$q' = \frac{Aq + B}{Cq + D}$$

And for a low-loss mode we have $q' = q$:

$$Cq^2 + Dq = Aq + B$$
$$Cq^2 + (D - A)q - B = 0$$
$$\Rightarrow q = \frac{(A - D) \pm \sqrt{(A - D)^2 + 4BC}}{2C} \tag{3}$$

We want q to remain complex so the square-root term in (3) must be negative:

$$(A - D)^2 + 4 \underbrace{BC}_{\substack{AD - BC = 1 \\ \Rightarrow BC = 1 + AD}} < 0$$
$$\Rightarrow (A + D)^2 < 4$$
$$\Rightarrow -2 < A + D < 2 \tag{4}$$

(b) Ray transfer matrix for $\mathcal{P} \to M_1 \to \mathcal{P}$:

$$\begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\dfrac{2}{R_1} & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ -\dfrac{2}{R_1} & -\dfrac{2L_1}{R_1} + 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \dfrac{2L_1}{R_1} & 2L_1 - \dfrac{2L_1^2}{R_1} \\ -\dfrac{2}{R_1} & -\dfrac{2L_1}{R_1} + 1 \end{pmatrix}$$

By symmetry, ray transfer matrix for $\mathcal{P} \to M_2 \to \mathcal{P}$:

$$\begin{pmatrix} 1 - \dfrac{2L_2}{R_2} & 2L_2 - \dfrac{2L_2^2}{R_2} \\ -\dfrac{2}{R_2} & -\dfrac{2L_2}{R_2} + 1 \end{pmatrix}$$

Set $\mathcal{P}$ at M_2 gives $L_1 = L$, $L_2 = 0$, thus the resultant matrix is:

$$\begin{pmatrix} 1 & 0 \\ -\dfrac{2}{R_2} & 1 \end{pmatrix} \begin{pmatrix} 1 - \dfrac{2L}{R_1} & 2L - \dfrac{2L^2}{R_1} \\ -\dfrac{2}{R_1} & 1 - \dfrac{2L}{R_1} \end{pmatrix} = \begin{pmatrix} 1 - \dfrac{2L}{R_1} & \dots \\ \dots & -\dfrac{4L}{R_2} + \dfrac{4L^2}{R_1 R_2} + 1 - \dfrac{2L}{R_1} \end{pmatrix}$$

We then have $A + D$:

$$\begin{aligned} A + D &= 2 - \frac{4L}{R_1} - \frac{4L}{R_2} + \frac{4L^2}{R_1 R_2} \\ &= 4\left(\frac{1}{2} - \frac{L}{R_1} - \frac{L}{R_2}\left(1 - \frac{L}{R_1}\right)\right) \\ &= 4\left[\left(1 - \frac{L}{R_1}\right)\left(1 - \frac{L}{R_2}\right) - \frac{1}{2}\right] \end{aligned}$$

Plugging this into (4) then gives:

$$\begin{gathered} -\frac{1}{2} < \left(1 - \frac{L}{R_1}\right)\left(1 - \frac{L}{R_2}\right) - \frac{1}{2} < \frac{1}{2} \\ 0 < \underbrace{\left(1 - \frac{L}{R_1}\right)}_{g_1}\underbrace{\left(1 - \frac{L}{R_2}\right)}_{g_2} < 1 \end{gathered} \tag{5}$$

(c) For a symmetric confocal cavity, $R_1 = R_2 = R$ and $L = R$, we then set $\mathcal{P}$ in the middle so $L_1 = L_2 = {}^{L}\!/_{2}$.

$\mathcal{P} \to M_1 \to \mathcal{P}$ gives:

$$\begin{pmatrix} q' \\ 1 \end{pmatrix} = \mathcal{U}\mathcal{F} \underbrace{\begin{pmatrix} 1 - \dfrac{L}{L} & L - \dfrac{L^2}{2L} \\ -\dfrac{2}{L} & 1 - \dfrac{L}{L} \end{pmatrix}}_{\begin{pmatrix} 0 & {}^{L}\!/_{2} \\ -{}^{2}\!/_{L} & 0 \end{pmatrix}} \begin{pmatrix} q \\ 1 \end{pmatrix}$$

$$\Rightarrow q' \equiv z_0' - iz_R' = \frac{{}^{L}\!/_{2}}{-{}^{2}\!/_{L} \times q} \quad \text{as per the definition of Gaussian beam}$$

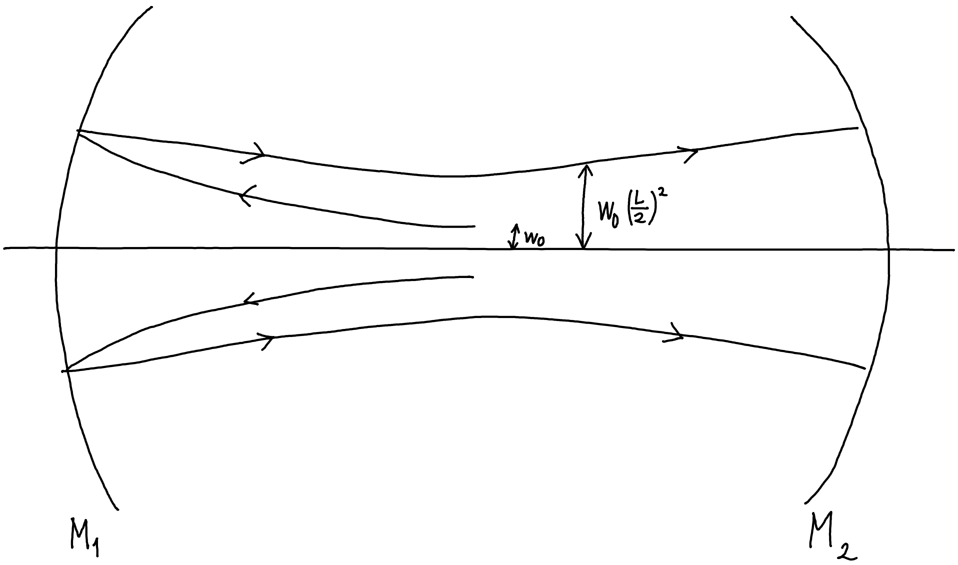
$$\Rightarrow z_0' - iz_R' = -\left(\frac{L}{2}\right)^2 \times \frac{1}{-iz_R}$$

$$\Rightarrow z_0' = 0 \quad \text{and} \quad z_R z_R' = \left(\frac{L}{2}\right)^2$$

Next we note that $\exp\left(\frac{ik\rho^2}{2q}\right) = \exp\left[\frac{ik\rho^2}{2}\left(\frac{i}{z_R}\right)\right] = \exp\left[-\frac{k\rho^2}{2z_R}\right]$ and thus the spot size $w \propto z_R$ ($\rho^2 = x^2 + y^2$ are the in-plane coordinates).

Also we have $z'_R = \frac{1}{z_R}\left(\frac{L}{2}\right)^2$ and that $z''_R = z_R$ by the low-loss condition.

We then have the sketch below:



(d) For the reflected beams to have the same spot size, we then need $z_R z'_R = z_R^2 = (L/2)^2$:

$$\begin{aligned} z_R &= \frac{L}{2} \\ \Rightarrow w^2 &= \frac{2z_R}{k} \quad \text{at cavity centre} \\ &= \frac{2z_R}{\frac{2\pi}{\lambda}} \\ &= \frac{L\lambda}{2\pi} \end{aligned}$$

At the mirror, we have:

$$\begin{aligned} q &= \frac{L}{2} - iz_R = \frac{L}{2}(1-i) \\ \Rightarrow \exp\left[\frac{ik\rho^2}{2q}\right] &= \exp\left[\frac{ik\rho^2}{2} \times \frac{2}{L} \times \frac{1+i}{1^2+1^2}\right] \\ &= \exp\left[-\frac{k\rho^2}{2L} + i\ldots\right] \end{aligned}$$

So spot size $w^2 = \frac{2L}{k} = \frac{L\lambda}{\pi}$.

Note that the $g_1 g_2$ factor in (5) for a symmetric confocal cavity is borderline unstable, therefore in practice it can be undesirable to employ such cavity as slight disturbance can lead to high loss.

3. Classic Jaynes-Cummings question.

(a) Jaynes-Cummings model quantises the electric dipole interaction as follows:

$$\begin{aligned}\mathcal{H} &= \mathbf{p}\cdot\mathbf{E} \\ &\to (\sigma_+ + \sigma_-)\left(a + a^\dagger\right)\end{aligned}$$

where $\sigma_\pm$ are the raising/lowering operators for the atomic energy levels, a and $a^\dagger$ are annihilation/creation operators.

In the rotating frame, we may write the interacting Hamiltonian as:

$$\begin{aligned}\mathcal{H}_{\text{int}} &= \frac{\hbar\Omega}{2}\left[\sigma_+ a e^{-i(\omega_0-\omega)t} + \sigma_+ a^\dagger e^{-i(\omega_0+\omega)t} + \sigma_- a e^{i(\omega_0-\omega)t} + \sigma_- a^\dagger e^{i(\omega_0+\omega)t}\right] \\ &\approx \frac{\hbar\Omega}{2}\left[\sigma_+ a e^{-i(\omega_0-\omega)t} + \sigma_- a^\dagger e^{i(\omega_0-\omega)t}\right] \quad \text{by RWA}\end{aligned}$$

where ω_0 is the energy difference between ground and excited states of the atom, ω is the frequency of the photon, Ω is the Rabi frequency of the system.

Thus the total Hamiltonian on resonance is:

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_{\text{atom}} + \mathcal{H}_{\text{light}} + \mathcal{H}_{\text{int}} \\ &= \hbar\omega\frac{\sigma_z}{2} + \hbar\omega a^\dagger a + \frac{\hbar\Omega}{2}\left(\sigma_+ a^\dagger + \sigma_- a\right)\end{aligned}$$

(b) TDSE: $i\hbar\frac{\partial}{\partial t}|\psi\rangle = \mathcal{H}|\psi\rangle$.

In matrix notation the Hamiltonian reads:

$$\mathcal{H} = \begin{array}{c} \begin{matrix} \langle \text{e, } 0| & \quad\langle \text{g, } 1| \end{matrix} \\ \begin{pmatrix} \dfrac{\hbar\omega}{2} & \dfrac{\hbar\Omega}{2} \\ \dfrac{\hbar\Omega}{2} & -\dfrac{\hbar\omega}{2} + \hbar\omega \end{pmatrix} \end{array}$$

Diagonalising $\mathcal{H}$ then gives:

$$\begin{aligned}&\begin{vmatrix} \dfrac{\hbar\omega}{2} - E & \dfrac{\hbar\Omega}{2} \\ \dfrac{\hbar\Omega}{2} & \dfrac{\hbar\omega}{2} - E \end{vmatrix} = 0 \\ &\Rightarrow E_\pm = \frac{\hbar\omega}{2} \pm \frac{\hbar\Omega}{2}\end{aligned}$$

where $E_\pm$ are the energies of the dressed states: $|\pm\rangle = {}^1\!/\!_{\sqrt{2}}\left[|\text{e, } 0\rangle \pm |\text{g, } 1\rangle\right]$.

Thus the system evolves in the Schrödinger picture as:

$$|\psi(t)\rangle = \alpha|+\rangle \exp\left(-\frac{iE_+ t}{\hbar}\right) + \beta|-\rangle \exp\left(-\frac{iE_- t}{\hbar}\right)$$

We have initial condition $|\psi(0)\rangle = |\mathrm{e},\, 0\rangle$ so:

$$\frac{\alpha}{\sqrt{2}} + \frac{\beta}{\sqrt{2}} = 1 \qquad\qquad \frac{\alpha}{\sqrt{2}} - \frac{\beta}{\sqrt{2}} = 0$$

$$\Rightarrow \alpha = \beta = \frac{1}{\sqrt{2}}$$

Thus

$$\begin{aligned}|\psi(t)\rangle &= \frac{1}{2}\left\{\exp\left[-i\left(\frac{\omega+\Omega}{2}\right)t\right]|+\rangle \;+\; \exp\left[-i\left(\frac{\omega-\Omega}{2}\right)t\right]|-\rangle\right\} \\ &= \exp\left(-\frac{i\omega}{2}t\right)\left[\cos\left(\frac{\Omega}{2}t\right)|\mathrm{e},\, 0\rangle + i\sin\left(\frac{\Omega}{2}t\right)|\mathrm{g},\, 1\rangle\right]\end{aligned}$$

(c) From above, we have probability of excited state:

$$P_{|\mathrm{e},\, 0\rangle}(t) = \cos^2\left(\frac{\Omega}{2}t\right)$$

For $P > 0.9$, we have:

$$\begin{aligned}\cos^2\left(\frac{\Omega}{2}t\right) &> 0.9 \\ \frac{\Omega}{2}t &< \arccos\sqrt{0.9} = 0.322 \\ t &< \frac{0.644}{\Omega}\end{aligned}$$

(d) To measure 1 photon, we need state $|\mathrm{g},\, 1\rangle$:

$$P_{|\mathrm{g},\, 1\rangle}(t) = \sin^2\left(\frac{\Omega}{2}t\right)$$

After such measurement, the dressed states shall collapse into $|\mathrm{g},\, 1\rangle$ immediately.

(e) Without the measurement result, we have a probabilistic description given by the density matrix below:

$$\rho = P_{|\mathrm{e},\, 0\rangle}(t)\,|\mathrm{e},\, 0\rangle\langle\mathrm{e},\, 0| + P_{|\mathrm{g},\, 1\rangle}(t)\,|\mathrm{g},\, 1\rangle\langle\mathrm{g},\, 1|$$

$$\begin{array}{cc} \langle\mathrm{e}| & \langle\mathrm{g}| \end{array}$$

$$= \begin{pmatrix} \cos^2\dfrac{\Omega t}{2} & 0 \\ 0 & \sin^2\dfrac{\Omega t}{2} \end{pmatrix}$$

From Boltzmann distribution we have:

$$\begin{aligned}
\frac{N_\mathrm{e}}{N_\mathrm{g}} &= \exp\left(-\frac{\hbar\omega}{k_\mathrm{B}T}\right) \\
\Rightarrow p_\mathrm{e} &= \frac{N_\mathrm{e}}{N_\mathrm{e}+N_\mathrm{g}} \\
&= \frac{N_e}{N_e\left[1+\exp\left(\frac{\hbar\omega}{k_\mathrm{B}T}\right)\right]} \\
&= \frac{1}{1+\exp\left(\frac{\hbar\omega}{k_\mathrm{B}T}\right)} \\
\Rightarrow p_\mathrm{g} &= 1-p_\mathrm{e} \\
&= \frac{1}{1+\exp\left(-\frac{\hbar\omega}{k_\mathrm{B}T}\right)}
\end{aligned}$$

Equating p_g with $P_{|\mathrm{g},\,1\rangle}$ gives:

$$\begin{aligned}
\sin^2\frac{\omega t}{2} &= \frac{1}{1+\exp\left(-\frac{\hbar\omega}{k_\mathrm{B}T}\right)} \\
\Rightarrow 1+\exp\left(-\frac{\hbar\omega}{k_\mathrm{B}T}\right) &= \frac{1}{\sin^2\frac{\Omega t}{2}} \\
\frac{\hbar\omega}{k_\mathrm{B}T} &= -\ln\left(\frac{1}{\sin^2\frac{\Omega t}{2}}-1\right) \\
&= \ln\left(\frac{\sin^2\frac{\Omega t}{2}}{1-\sin^2\frac{\Omega t}{2}}\right) \\
\Rightarrow T &= \frac{\hbar\omega}{k_\mathrm{B}\ln\left(\tan^2\frac{\Omega t}{2}\right)}
\end{aligned}$$

where T is the critical temperature at which both probabilities match.

However note that with Boltzmann distribution, this equality only holds for t where $0 \leq P_{|\mathrm{g},\,1\rangle} \leq 1/2$ for all T.

4. Quantum information with a twist at quantum money.

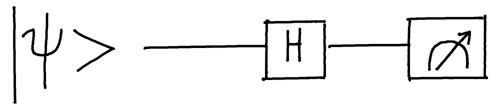
(a) Description of quantum states after measurement:

i. After measurement, Alice will get one of the 2 basis states, thus ρ_{A} is either $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$.

ii. However Bob has no information on the measurement result, therefore the best he can do is to encode the decoherence with $\rho_{\mathrm{B}} = |\alpha|^2 |0\rangle\langle 0| + |\beta|^2 |1\rangle\langle 1|$.

The difference is that while Alice has the deterministic result of her measurement, Bob can only observe the decoherence and thus has the results probabilistically mixed in ρ_{B} – similar to what one does to account for unintentional measurement in a quantum system.

(b) i. We know that Hadamard gate transforms between $|0\rangle \leftrightarrow |+\rangle$ and $|1\rangle \leftrightarrow |-\rangle$, thus we have the following network for X basis measurement:

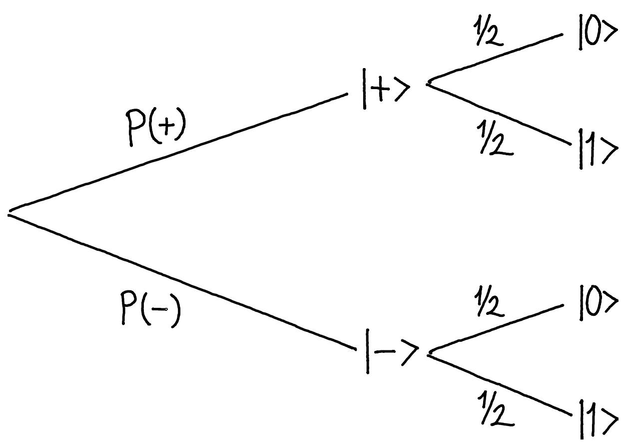


$$
\begin{aligned}
|\psi\rangle &= \alpha\,|0\rangle + \beta\,|1\rangle \\
&= \alpha\left(\frac{|+\rangle + |-\rangle}{\sqrt{2}}\right) + \beta\left(\frac{|+\rangle - |-\rangle}{\sqrt{2}}\right) \\
&= \frac{\alpha+\beta}{\sqrt{2}}\,|+\rangle \; + \; \frac{\alpha-\beta}{\sqrt{2}}\,|-\rangle
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow P(+) &= \left|\frac{\alpha+\beta}{\sqrt{2}}\right|^2 = \frac{1}{2}\,|\alpha+\beta|^2 \\
P(-) &= \frac{1}{2}\,|\alpha-\beta|^2
\end{aligned}
$$

ii. Realise that after measurement, Alice would have a pure state of $|+\rangle$ or $|-\rangle$, either containing $(|0\rangle \pm |1\rangle)/\sqrt{2} \Rightarrow P(0) = P(1) = {}^{1}\!/_{2}$ after X measurement.

From Bob's perspective, we then have the tree diagram:



Thus ρ_B is now:

$$\begin{aligned}
&\frac{P(+)+P(-)}{2}\left|0\right\rangle\left\langle 0\right| + \frac{P(+)+P(-)}{2}\left|1\right\rangle\left\langle 1\right| \\
&\quad= \frac{1}{4}\Big[|\alpha|^2+|\beta|^2+\alpha^*\beta+\beta^*\alpha+|\alpha|^2+|\beta|^2-\alpha^*\beta-\beta^*\alpha\Big]\big[\left|0\right\rangle\left\langle 0\right|+\left|1\right\rangle\left\langle 1\right|\big] \\
&\quad= \frac{1}{2}\big[\left|0\right\rangle\left\langle 0\right|+\left|1\right\rangle\left\langle 1\right|\big]
\end{aligned}$$

(c) i. As Calvin has no information on $|\psi\rangle$, he can only construct $\rho_\mathrm{C} = \mathrm{diag}(|\alpha|^2, |\beta|^2)$ in Z basis. Hence fidelity:

$$\begin{aligned}
\mathcal{F} &= \langle\psi|\,\rho_\mathrm{C}\,|\psi\rangle \\
&= |\alpha|^4+|\beta|^4 \\
&= 2|\alpha|^4-2|\alpha|^2+1
\end{aligned}$$

ii. Note if $|\psi\rangle$ is either $|0\rangle$ or $|1\rangle$, we have $\mathcal{F}=1 \Leftarrow$ maximum fidelity states.

iii. But if $|\alpha| = |\beta| = 1/\sqrt{2}$, i.e. $|\psi\rangle$ lies on the equator of the Bloch sphere, we have $|\alpha|^2 = 1/2 \Rightarrow \mathcal{F} = 1/2$.

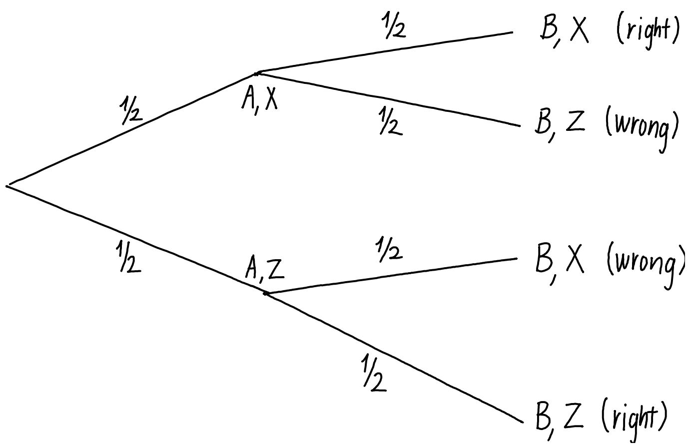
iv. Thus average fidelity is $\langle\mathcal{F}\rangle = 1/3\,\mathcal{F}_z + 1/3\,\mathcal{F}_x + 1/3\,\mathcal{F}_y = 1/3 \times 1 + 2/3 \times \frac{1}{2} = 2/3$.

(d) For each qubit, $P(\text{Alice having Z}) = 1/2$ and note that even for the wrong basis, we have a probability of $1/2$ of getting the right value.

i. Thus we have:

$$P(\text{correct value}|\text{measure Z constantly}) = \underbrace{\frac{1}{2}}_{\substack{\text{right}\\\text{basis}}} + \underbrace{\frac{1}{2}}_{\substack{\text{wrong}\\\text{basis}}} \times \underbrace{\frac{1}{2}}_{\substack{\text{but still get}\\\text{the correct value}}} = \frac{3}{4}$$

ii. With reference to the tree diagram below, we have:



$$P(\text{correct value}|\text{X, Z at random}) = 2\underbrace{\left(\frac{1}{2}\times\frac{1}{2}\right)}_{\text{right basis}} + 2\underbrace{\left(\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\right)}_{\substack{\text{wrong basis}\\\text{but right value}}}$$

$$= \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$$

(e) i. For Alice to detect the forgery, she needs to obtain the wrong value so we have for every bit:

$$P(\text{wrong value} \cap \text{wrong basis}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

For $n = 24$ we have:

$$\begin{aligned} P(\text{detection}) &= 1 - P(\text{no wrong value}) \\ &= 1 - \left(\frac{3}{4}\right)^n \\ &= 0.999 \end{aligned}$$

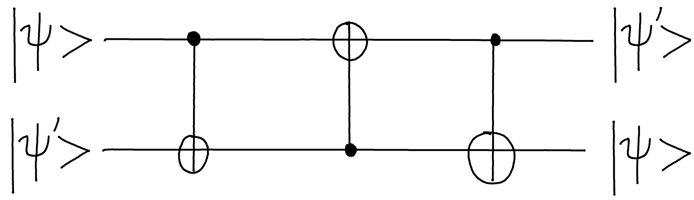
ii. If Daisy has access to the original banknote, the best she could do is to gain knowledge of some of her basis being incorrect as per the $^1\!/_4$ probability above.

By no-cloning theorem, she is unable to copy the note either – measurement would also collapse the state into her measurement basis and render the note useless for further probing.

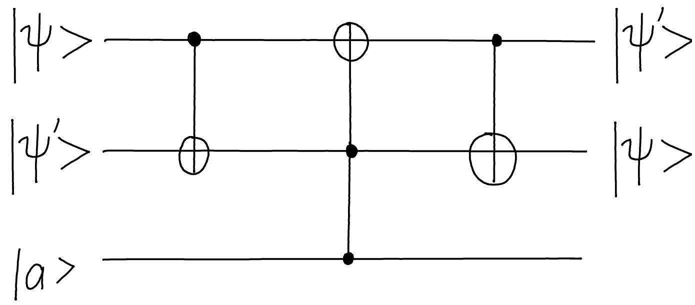
5. Usual quantum information with the usual hindrance of practical implementation.

(a) `NOT`, `CNOT`, and `TOFFOLI` are universal in that they are able to replicate a general multi-qubit gate to an arbitrary accuracy.

To construct a `FREDKIN` gate, we want to have a controlled sequence of `XOR` gates (i.e. `CNOT`):



Now realise that `XOR` is an inverse of itself, we may then replace the central `CNOT` with `TOFFOLI`:



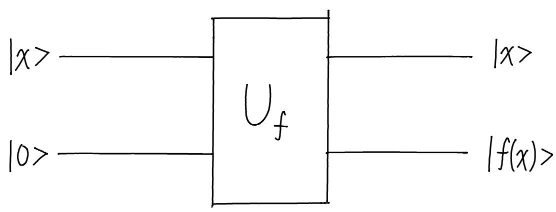
Note that if $|a\rangle = |0\rangle$ we have the circuit as `IDENTITY` and `SWAP` if $|a\rangle = |1\rangle$.

(b) $f : \{0, 1\} \to \{0, 1\}$

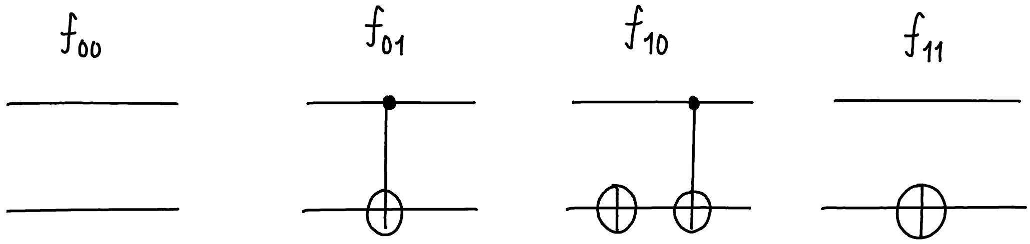
i. There are only 4 such functions:

Input	f_{00}	f_{01}	f_{10}	f_{11}
0	0	0	1	1
1	0	1	0	1
	constant	**balanced**	**balanced**	**constant**

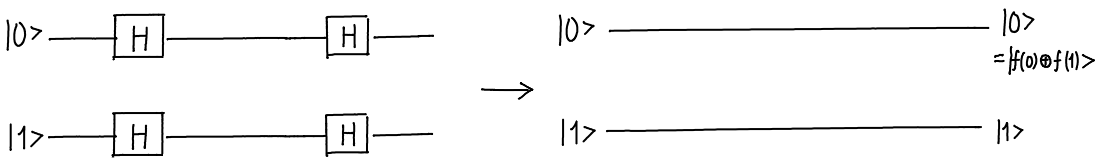
ii. Classically to determine whether f is balanced/constant, we need 2 calls with inputs $|x\rangle = |0\rangle$ and $|x\rangle = |1\rangle$ to obtain the results and compare them against the table above.



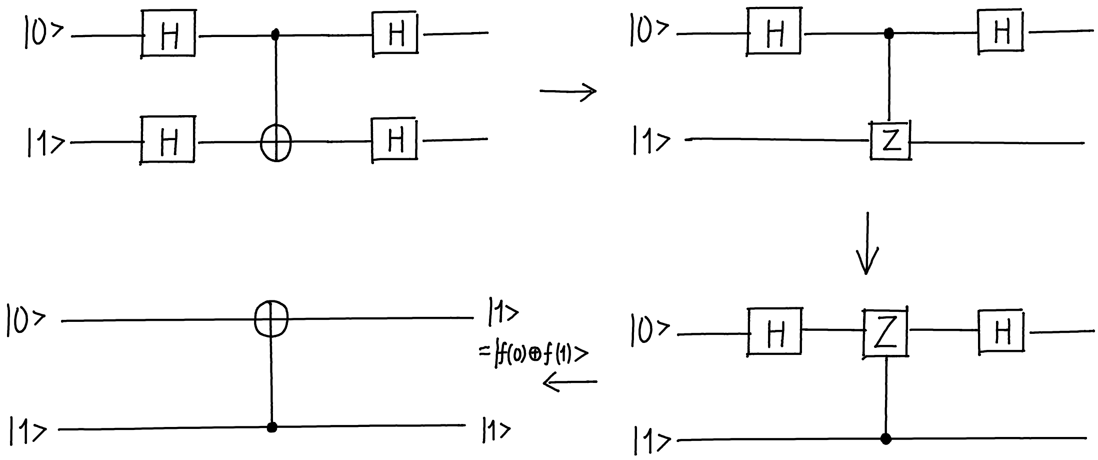
Explicit propagator for each f:



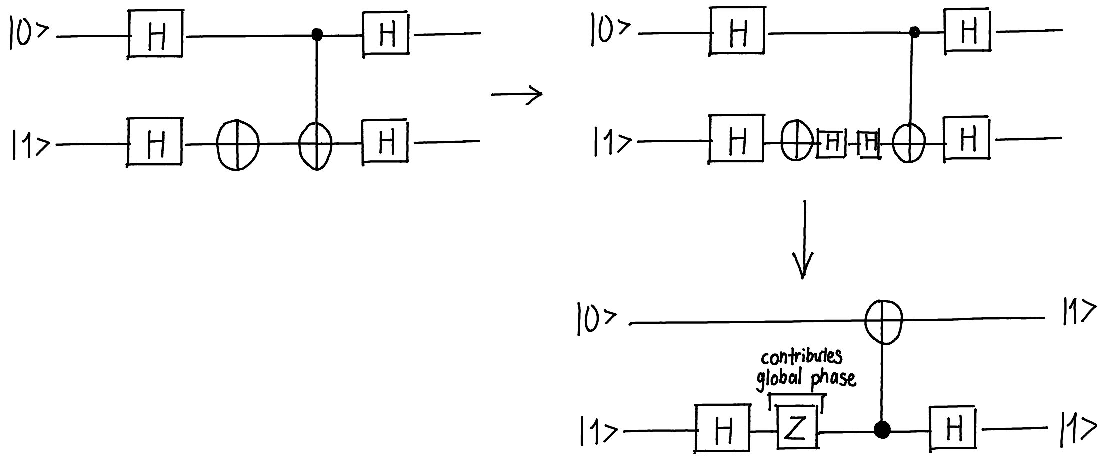
(c) Case f_{00}:



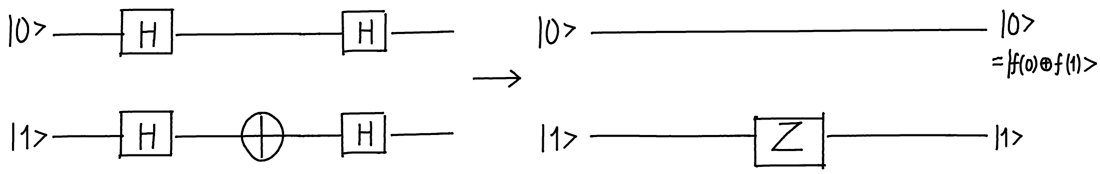
Case f_{01}:



Case f_{10}:



Case f_{11}:



(d) **(DRAFT)** In ion trap computing,

Single-qubit gates: implemented via Rabi flopping – the pulse time corresponds to the applied phase.

Two-qubit gates: we exploit the cavity condition to achieve 2-qubit manipulation via the long range Coulomb interaction.

(e) **(DRAFT)** For large number of qubits, the Coulomb interaction would inadvertently affect other qubits along the chain of ions, thereby introducing errors to the system.

6. Classic GHZ question.

(a) Want eigenstates of X and Y:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \quad \text{Seek} \quad \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm 1$$

$$\Rightarrow \quad \text{Eigenstates} \quad |\mathrm{H}'\rangle = \frac{|\mathrm{H}\rangle + |\mathrm{V}\rangle}{\sqrt{2}}$$
$$|\mathrm{V}'\rangle = \frac{|\mathrm{H}\rangle - |\mathrm{V}\rangle}{\sqrt{2}}$$

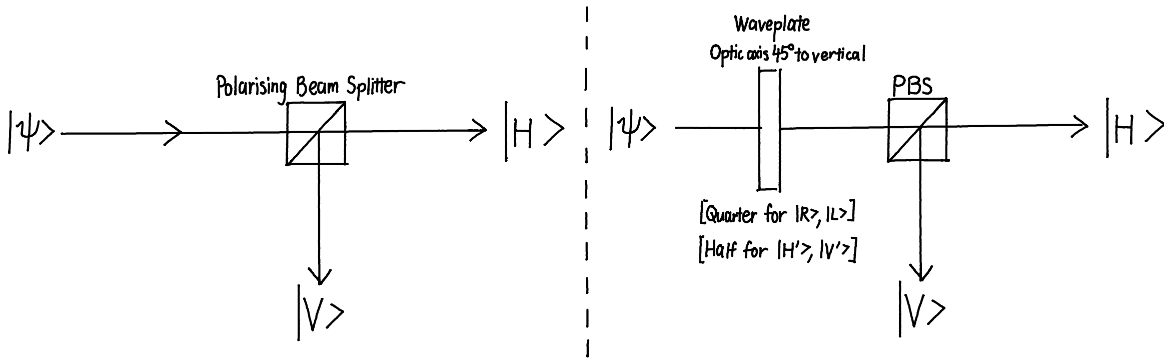
Similarly,

$$\sigma_y = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \Rightarrow \quad \text{Seek} \quad \begin{vmatrix} -\lambda & i \\ -i & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \pm i$$

$$\Rightarrow \quad \text{Eigenstates} \quad |\mathrm{R}\rangle = \frac{|\mathrm{H}\rangle + i\,|\mathrm{V}\rangle}{\sqrt{2}}$$
$$|\mathrm{L}\rangle = \frac{|\mathrm{H}\rangle - i\,|\mathrm{V}\rangle}{\sqrt{2}}$$

Note that in optics the correspondence are $\{|\mathrm{H}'\rangle, |\mathrm{V}'\rangle\}$ $\equiv$ linear polarisation 45° to the $\{|\mathrm{H}\rangle, |\mathrm{V}\rangle\}$ axes, and $\{|\mathrm{R}\rangle, |\mathrm{L}\rangle\}$ $\equiv$ right/left-handed polarisation.

To measure these polarisation, consider the following setup:



(b) $|\mathrm{GHZ}\rangle = {}^{1}\!/\!_{\sqrt{2}}\,[|\mathrm{HHH}\rangle - i\,|\mathrm{VVV}\rangle]$

Noting orthogonality between the basis, we have:

$$P(\mathrm{HHH}) = P(\mathrm{VVV}) = \left|\frac{1}{\sqrt{2}}\right|^2 = \left|-\frac{i}{\sqrt{2}}\right|^2 = \frac{1}{2}$$
$$P(\mathrm{HHV}) = P(\mathrm{HVH}) = \ldots = 0$$

From above, we have for X basis:

$$|\mathrm{H}\rangle = \frac{|\mathrm{H}'\rangle + |\mathrm{V}'\rangle}{\sqrt{2}}$$
$$|\mathrm{V}\rangle = \frac{|\mathrm{H}'\rangle - |\mathrm{V}'\rangle}{\sqrt{2}}$$

For Y basis:

$$|\mathrm{H}\rangle = \frac{|\mathrm{R}\rangle + |\mathrm{L}\rangle}{\sqrt{2}}$$
$$|\mathrm{V}\rangle = -i\frac{|\mathrm{R}\rangle - |\mathrm{L}\rangle}{\sqrt{2}}$$

$$\begin{aligned}
\text{So } |\mathrm{GHZ}\rangle &= \frac{1}{\sqrt{2}}\left[\left(\frac{|\mathrm{H'}\rangle + |\mathrm{V'}\rangle}{\sqrt{2}}\right)^{\otimes 2}\left(\frac{|\mathrm{R}\rangle + |\mathrm{L}\rangle}{\sqrt{2}}\right) - i\left(\frac{|\mathrm{H'}\rangle - |\mathrm{V'}\rangle}{\sqrt{2}}\right)^{\otimes 2}\left(-i\frac{|\mathrm{R}\rangle - |\mathrm{L}\rangle}{\sqrt{2}}\right)\right] \\
&= \frac{1}{4}\Big[\big(|\mathrm{H'}\rangle + |\mathrm{V'}\rangle\big)\big(|\mathrm{H'}\rangle + |\mathrm{V'}\rangle\big)\big(|\mathrm{R}\rangle + |\mathrm{L}\rangle\big) \\
&\quad - \big(|\mathrm{H'}\rangle - |\mathrm{V'}\rangle\big)\big(|\mathrm{H'}\rangle - |\mathrm{V'}\rangle\big)\big(|\mathrm{R}\rangle - |\mathrm{L}\rangle\big)\Big]
\end{aligned}$$

Noting that most cross terms vanish but those with matching parity, we have:

$$\begin{aligned}
|\mathrm{GHZ}\rangle &= \frac{1}{4}\Big[|\mathrm{H'H'L}\rangle + |\mathrm{V'V'L}\rangle + |\mathrm{H'V'R}\rangle + |\mathrm{V'H'R}\rangle\Big] \times 2 \\
&= \frac{1}{2}\Big[\dots\Big]
\end{aligned}$$

Noting the similarity in parity matching, we simply swap label for XYX and YXX:

$$|\mathrm{GHZ}\rangle = \begin{cases} {}^1\!/_2\big[|\mathrm{H'LH'}\rangle + |\mathrm{V'LV'}\rangle + |\mathrm{H'RV'}\rangle + |\mathrm{V'RH'}\rangle\big] & \mathrm{XYX} \\ {}^1\!/_2\big[|\mathrm{LH'H'}\rangle + |\mathrm{LV'V'}\rangle + |\mathrm{RH'V'}\rangle + |\mathrm{RV'H'}\rangle\big] & \mathrm{YXX} \end{cases}$$

The possible outcomes are then:

XXY: $|\mathrm{H'H'L}\rangle$, $|\mathrm{V'V'L}\rangle$, $|\mathrm{H'V'R}\rangle$, $|\mathrm{V'H'R}\rangle$ each with probability of $|{}^1\!/_2|^2 = {}^1\!/_4$.

Similar results follow for XYX and YXX.

(c) Following local realism, we observe that there is an element of anti-correlation between the first and second qubit when the third qubit is $|\mathrm{R}\rangle$, correlation when the third qubit is $|\mathrm{L}\rangle$. Hence in YYY basis we expect to have results $|\mathrm{LRR}\rangle$, $|\mathrm{RLR}\rangle$, $|\mathrm{RRL}\rangle$, $|\mathrm{LLL}\rangle$ each with probability ${}^1\!/_4$.

Now let's rewrite the state in YYY basis:

$$|\mathrm{GHZ}\rangle = \frac{1}{4}\Big[(|\mathrm{R}\rangle + |\mathrm{L}\rangle)(|\mathrm{R}\rangle + |\mathrm{L}\rangle)(|\mathrm{R}\rangle + |\mathrm{L}\rangle) - i(|\mathrm{R}\rangle - |\mathrm{L}\rangle)(|\mathrm{R}\rangle - |\mathrm{L}\rangle)(|\mathrm{R}\rangle - |\mathrm{L}\rangle)(-i)^3\Big]$$

Noting the opposite parity requirement, we have:

$$|\mathrm{GHZ}\rangle = \frac{1}{4}\big[|\mathrm{RRR}\rangle + |\mathrm{LLR}\rangle + |\mathrm{LRL}\rangle + |\mathrm{RLL}\rangle\big] \times 2$$

But here we yield a different set of outcomes: $|\mathrm{RRR}\rangle$, $|\mathrm{LLR}\rangle$, $|\mathrm{LRL}\rangle$, $|\mathrm{RLL}\rangle$ each with probability ${}^1\!/_4$!

The difference between a GHZ state and a Bell state lies in the nature of the violation of CHSH inequality – for GHZ state the violation comes from the entangled state itself, as opposed to being a statistical average of an ensemble of Bell states.

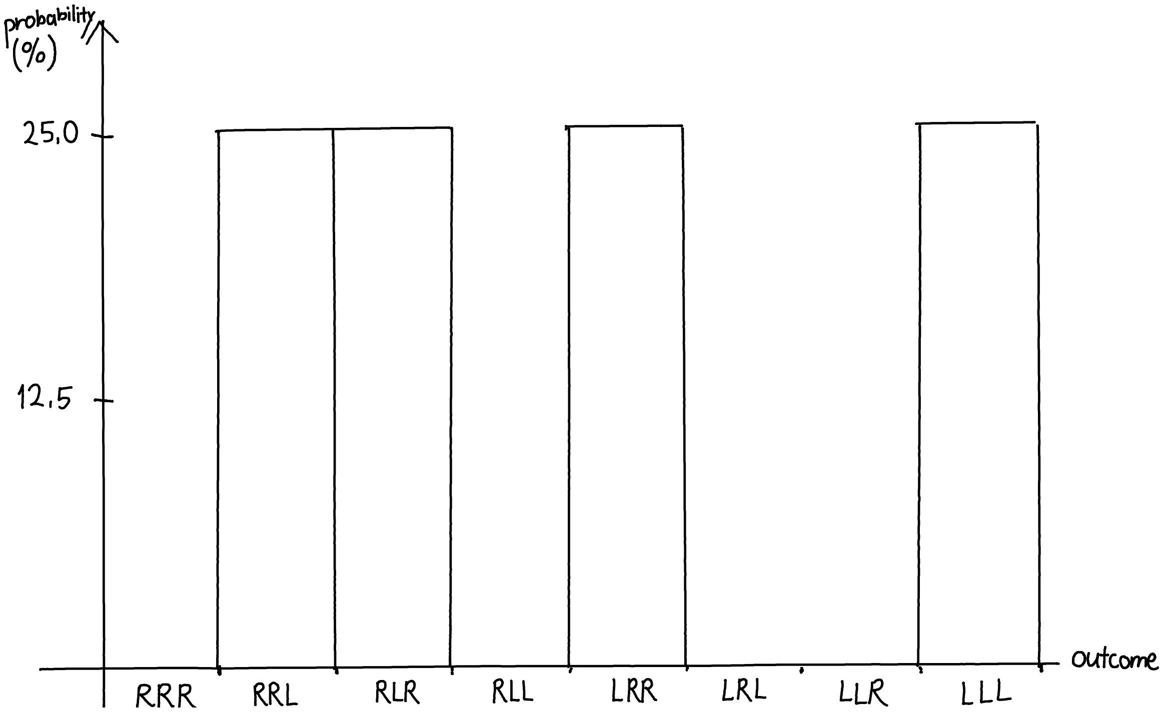
(d) In the event of a Z phase flip, we have $-i \to i$, so (apart from ZZZ which is protected) the XXY's bases would have the opposite parity requirement, while that for YYY remains the same. Hence we have the following outcome changes:

$$|\mathrm{H'H'L}\rangle \to |\mathrm{H'H'R}\rangle$$
$$|\mathrm{V'V'L}\rangle \to |\mathrm{V'V'R}\rangle$$
$$|\mathrm{H'V'R}\rangle \to |\mathrm{H'V'L}\rangle$$
$$|\mathrm{V'H'R}\rangle \to |\mathrm{V'H'L}\rangle$$

For YYY, we have $|\mathrm{GHZ_{flip}}\rangle = \frac{1}{2}\left[|\mathrm{RRL}\rangle + |\mathrm{LLL}\rangle + |\mathrm{LRR}\rangle + |\mathrm{RLR}\rangle\right]$, each having probability of $1/4 \times 8\,\% = 2\,\%$.

The original states would then each have probability $1/4 \times 92\,\% = 23\,\%$.

Histogram for local realism prediction:



Histogram for QM with phase flip error: