UNOFFICIAL SOLUTIONS BY TheLongCat

B2: SYMMETRY AND RELATIVITY

TRINITY TERM 2022

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Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. (DRAFT)

(a) Recall Lorentz transformation:

$$\begin{aligned} ct' &= \gamma ct - \gamma\beta x \\ x' &= -\gamma\beta ct + \gamma x \\ y' &= y \qquad z' = z \end{aligned}$$

Tensor transformation law also tells us:

$$\begin{aligned} \frac{\partial}{\partial x^\mu} &= \frac{\partial}{\partial x'^\nu} \cdot \frac{\partial x'^\nu}{\partial x^\mu} \\ \Rightarrow \frac{\partial}{\partial x'^\nu} &= \frac{\partial}{\partial x^\mu} \cdot \frac{\partial x_\nu}{\partial x'_\mu} \end{aligned}$$

$$\begin{aligned} \frac{\partial}{\partial x_\mu} &= \frac{\partial}{\partial x'_\nu} \cdot \frac{\partial x'_\nu}{\partial x_\mu} \\ \Rightarrow \frac{\partial}{\partial x'_\nu} &= \frac{\partial}{\partial x_\mu} \cdot \frac{\partial x^\nu}{\partial x'^\mu} \end{aligned}$$

Comparing against (1) and (2) gives:

$$\begin{aligned} \frac{\partial}{\partial x^\mu} &= \partial_\mu = \left(\frac{1}{c}\partial_t, \boldsymbol{\nabla}\right) \\ \frac{\partial}{\partial x_\mu} &= \partial^\mu = \left(-\frac{1}{c}\partial_t, \boldsymbol{\nabla}\right) \end{aligned}$$

(b)

$$\partial_\mu \partial^\mu = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2 \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

From above, we have transformed tensors:

$$\begin{aligned} (\partial')_\mu &= \frac{\partial x'_\mu}{\partial x_\nu} (\partial)_\nu \\ (\partial')^\mu &= \frac{\partial x'^\mu}{\partial x^\nu} (\partial)^\nu \end{aligned}$$

Hence:

$$\begin{aligned} (\partial')_\mu (\partial')^\mu &= \partial_\nu \partial^\nu \cdot \frac{\partial x'_\mu}{\partial x_\nu} \cdot \underbrace{\frac{\partial x'^\mu}{\partial x^\nu}}_{\left(\frac{\partial x_\mu}{\partial x'_\nu}\right)^{-1}} \\ &= \partial_\nu \partial^\nu \end{aligned}$$

So it is Lorentz invariant.

(c) Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{E}}{\partial t} \tag{3}$$
$$\nabla \times \mathbf{B} = +\mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{4}$$

$\div\cdot$(4) then gives:

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{B}) &= +\mu_0 \nabla \cdot \mathbf{J} + \mu_0 \epsilon_0 \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} \\ 0 &= +\mu_0 \nabla \cdot \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t} \underbrace{(\nabla \cdot \mathbf{E})}_{\frac{\rho}{\epsilon_0}} \end{aligned}$$
$$\Rightarrow \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

Note the relation may be simply written as $\partial_\mu \mathsf{J}^\mu = 0$ where $\mathsf{J}^\mu = (\rho c, \mathbf{J})$.

(d) $\mathsf{A}^\mu = (\phi/c, \mathbf{A})$ with $\partial_\mu \mathsf{A}^\mu = 0 \Rightarrow \phi$ independent of t and $\nabla \cdot \mathbf{A} = 0$.

$$\begin{aligned} \partial^\nu \partial_\nu \mathsf{A}^0 &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\phi}{c}\right) + \nabla^2 \left(\frac{\phi}{c}\right) \\ &= \nabla^2 \left(\frac{\phi}{c}\right) \end{aligned}$$

From (1),

$$\begin{aligned} \nabla \cdot \left(-\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}\right) &= \frac{\rho}{\epsilon_0} \\ \Rightarrow -\nabla^2 \phi &= \frac{\rho}{\epsilon_0} \\ \Rightarrow \partial^\nu \partial_\nu \mathsf{A}^0 &= -\frac{\rho}{\epsilon_0 c} = -\frac{c\rho}{\epsilon_0 c^2} = -\mu_0 \rho c \end{aligned}$$

Next we have:

$$\begin{aligned} \partial^\nu \partial_\nu \mathsf{A}^i &= -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\mathsf{A}^i\right) + \nabla^2 \left(\mathsf{A}^i\right) \\ \nabla \times \mathbf{B} &= \nabla \times (\nabla \times \mathbf{A}) \\ &= \cancelto{0}{\nabla (\nabla \cdot \mathbf{A})} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial}{\partial t}\left[-\nabla \phi - \frac{\partial \mathbf{A}}{\partial t}\right] \\ -\nabla^2 \mathbf{A} &= \mu_0 \mathbf{J} + \left(-\underbrace{\mu_0 \epsilon_0}_{1/c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) \end{aligned}$$
$$\Rightarrow -\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} + \nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} = \partial_\nu \partial^\nu \mathsf{A}^i$$

And so $\partial^\nu \partial_\nu \mathsf{A}^a = -\mu_0 \mathsf{J}^a$.

Since the contraction of $\partial^\nu \partial_\nu \mathsf{A}^a$ yields a 4-vector and that ∂_ν, ∂^ν are 4-components by quotient rule we must know that A^μ is a 4-vector.

(e)

$$\begin{aligned} \partial_\mu \mathsf{A}^\mu = 0 \Rightarrow \mathsf{K}_\mu \epsilon^\mu \sin\left(k^a x_a\right) = 0 \\ \Rightarrow \epsilon^0 = 0 \quad \text{for } \boldsymbol{\epsilon} \cdot \mathbf{k} = 0 \end{aligned}$$

Now with gauge transformation:

$$\begin{aligned} \mathsf{A}^\mu &= \left(\mathsf{A}'\right)^\mu + \partial^\mu x \\ &= \left(\epsilon'\right)^\mu \sin\left(k'^a x'_a\right) + \partial^\mu x \\ \Rightarrow \left(\epsilon'\right)^0 \sin\left(k'^a x'_a\right) + \partial^0 x &= \epsilon^0 = 0 \\ \Rightarrow x &= \int -\left(\epsilon'\right)^0 \sin\left(\ldots\right) \cdot c \, \mathrm{d}t \\ &= \cos\left(\ldots\right) \cdot \frac{\left(\epsilon'\right)^0 c}{\left(k'\right)^0} c \end{aligned}$$

So free to choose any ϵ such that $\mathbf{k} \cdot \boldsymbol{\epsilon} = 0$.

2. (DRAFT)

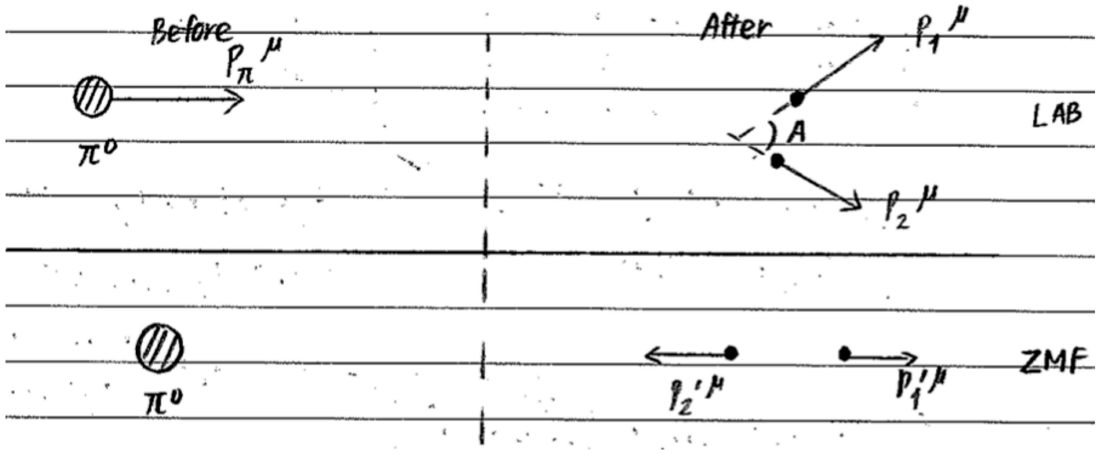
(a) Total momentum of the system:

$$\mathsf{P}^\mu = \left(\sum_i E_i, \sum_i \mathbf{p}_i\right)$$

Consider a transformation to the ZMF (align x-axis with $\sum_i \mathbf{p}_i$):

$$\begin{aligned}
(\mathsf{P}')^\mu &= \Lambda^\mu_\nu \mathsf{P}^\nu \\
\Rightarrow (\mathsf{P}')^1 &= 0 = -\beta\gamma\left(\sum_i E_i\right) + \gamma\left(\sum_i \mathbf{p}_i\right) \\
\Rightarrow \beta &= \frac{\sum_i \mathbf{p}_i}{\sum_i E_i}
\end{aligned}$$

(b) Sketch of the decay process:



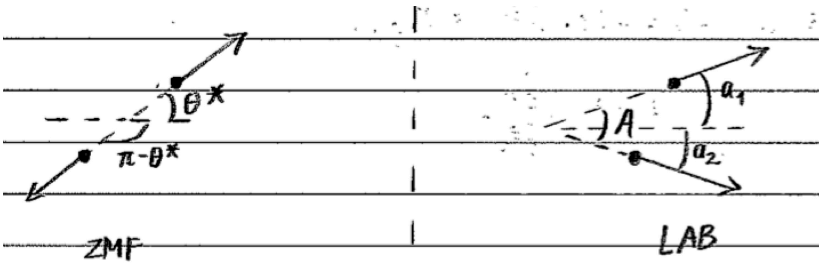
Conservation of 4-momentum gives:

$$\begin{aligned}
\mathsf{P}^\mu_\pi &= (\mathsf{P}^\mu_1 + \mathsf{P}^\mu_2) \\
\Rightarrow (\mathsf{P}^\mu_\pi)^2 &= (\mathsf{P}^\mu_1 + \mathsf{P}^\mu_2)^2 \\
-M_\pi^2 &= \cancel{(\mathsf{P}^\mu_1)^2}^{\,0} + \cancel{(\mathsf{P}^\mu_2)^2}^{\,0} + 2\,(\mathsf{P}_1)_\mu\,(\mathsf{P}_2)^\mu \\
&= 2\,(-E_1E_2 + p_1p_2\cos A) = 2p_1p_2\,(\cos A - 1) \qquad \text{since } E = pc \text{ for } \gamma
\end{aligned}$$

With $\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$, we then have:

$$\begin{aligned}
\Rightarrow M_\pi^2 &= -2E_1E_2\left(\cancel{1} - 2\sin^2\frac{A}{2} \cancel{-1}\right) \\
\Rightarrow \sin^2\frac{A}{2} &= \frac{M_\pi^2}{4E_1E_2}
\end{aligned}$$

(c) **(TO EXPAND)** Want ${}^{\mathrm{d}N}/_{\mathrm{d}A} = {}^{\mathrm{d}N}/_{\mathrm{d}\theta^*} \cdot {}^{\mathrm{d}\theta^*}/_{\mathrm{d}A}$.



Lorentz boosting from ZMF to LAB:

$$\begin{pmatrix} E_1 \\ \mathbf{p}_1 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} E' \\ \mathbf{p}' \end{pmatrix}$$

$$\Rightarrow E_1 = \frac{\gamma M_\pi}{2}\left(1 + \beta\cos\theta^*\right)$$

$$\begin{pmatrix} E_2 \\ \mathbf{p}_2 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} E' \\ \mathbf{p}' \end{pmatrix}$$

$$\Rightarrow E_2 = \frac{\gamma M_\pi}{2}\left(1 - \beta\cos\theta^*\right)$$

Substituting into ():

$$\begin{aligned}
\sin^2\frac{A}{2} &= \frac{M_\pi^2}{4}\cdot\frac{4}{\gamma^2 M_\pi^2}\cdot\frac{1}{1-\beta^2\cos^2\theta^*} \\
&= \frac{1}{\gamma\left(1-\beta^2\cos^2\theta^*\right)} \\
\Rightarrow \sin\frac{A}{2}\cos\frac{A}{2}\,\mathrm{d}A &= -\frac{1}{\gamma}\cdot\left[1-\beta^2\cos^2\theta^*\right]^{-2} 2\beta^2\cos\theta^*\,\mathrm{d}(\cos\theta^*) \\
\Rightarrow \frac{\mathrm{d}(\cos\theta^*)}{\mathrm{d}A} &= \frac{-\gamma\sin\frac{A}{2}\cos\frac{A}{2}}{2\beta^2\cos\theta^*}\left(1-\beta^2\cos^2\theta^*\right)^{-2} \\
&= a\frac{\sin\theta^*\,\mathrm{d}\theta^*}{\mathrm{d}A} = a\frac{\mathrm{d}(\cos\theta^*)}{\mathrm{d}A}
\end{aligned}$$

(d) **(TO EXPAND)**

$$A_{\max} = 2\arcsin\frac{1}{\underbrace{\gamma^2}_{\left(\frac{M_\pi}{E_\pi}\right)^2}}$$

${}^{\mathrm{d}N}/_{\mathrm{d}A} > 0$ so $A_{\min} = \pi$.

3. **(DRAFT)**

(a) Transforming time measuremnts:

$$\begin{pmatrix} c\Delta t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} c\Delta\tau \\ \mathbf{0} \end{pmatrix}$$

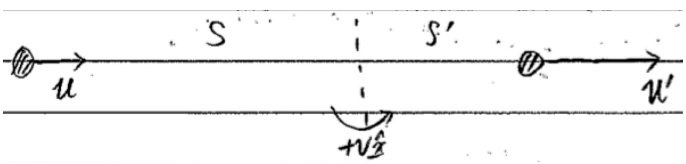
$$\Rightarrow c\Delta t = \gamma c \Delta\tau$$

$$\Rightarrow \mathrm{d}t = \gamma\,\mathrm{d}\tau$$

Definition of 4-velocity:

$$\begin{aligned} \mathsf{U}^\mu &= \frac{\mathrm{d}\mathsf{X}^\mu}{\mathrm{d}\tau} \\ &= \gamma\frac{\mathrm{d}\mathsf{X}^\mu}{\mathrm{d}t} \\ &= \gamma\,(c, \mathbf{u}) \quad \text{where } \mathbf{u} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \end{aligned}$$

(b) Sketch of the frames in question:



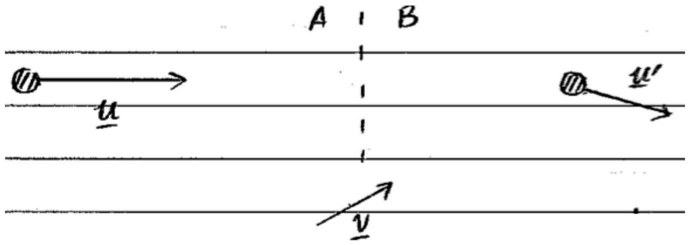
Transforming the 4-vectors:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad \text{with boost } +v\hat{\mathbf{x}}$$

$$\begin{aligned} \Rightarrow\ \mathrm{d}x' &= -\beta\gamma c\,\mathrm{d}t + \gamma\,\mathrm{d}x \\ c\,\mathrm{d}t' &= \gamma c\,\mathrm{d}t - \beta\gamma\,\mathrm{d}x \end{aligned}$$

$$\begin{aligned} \Rightarrow v' &= \frac{\mathrm{d}x'}{\mathrm{d}t'} \\ &= \frac{\gamma\,\mathrm{d}x - \beta\gamma c\,\mathrm{d}t}{\gamma\,\mathrm{d}t - \beta\gamma/c\,\mathrm{d}x} \\ &= \frac{{}^{\mathrm{d}x}\!/_{\mathrm{d}t} - \beta c}{1 - (\beta\,{}^{\mathrm{d}x}\!/_{\mathrm{d}t})/c} \\ &= \frac{u - v}{1 - uv/c^2} \quad \text{since } \beta = v/c \end{aligned}$$

(c) Sketch of the frames in question:



Rotate such that x-axis $\parallel$ $\mathbf{v}$ and so:

$$\mathbf{u} = \frac{\overbrace{(\mathbf{u}\cdot\mathbf{v})}^{\mathrm{d}x/\mathrm{d}t}}{v}\hat{\mathbf{x}} + \frac{\overbrace{(\mathbf{u}\times\mathbf{v})}^{\mathrm{d}y/\mathrm{d}t}}{v}$$

As with before, we have:

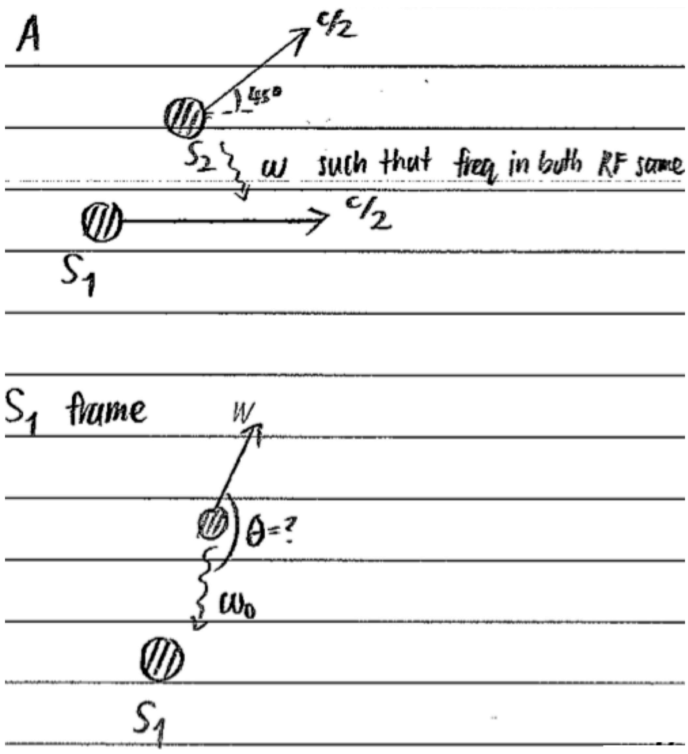
$$\begin{aligned} w_x &= \frac{\mathrm{d}x'}{\mathrm{d}t'} \\ &= \frac{(\mathbf{u}\cdot\mathbf{v})/v - v}{1 - \frac{\frac{(\mathbf{u}\cdot\mathbf{v})}{v} - v}{c^2}} \\ &= \frac{\frac{(\mathbf{u}\cdot\mathbf{v})}{v} - v}{1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \end{aligned}$$

$$\begin{aligned} w_y &= \frac{\mathrm{d}y'}{\mathrm{d}t'} \\ &= \frac{\mathrm{d}y}{\gamma\,\mathrm{d}t - \frac{\beta\gamma}{c}\,\mathrm{d}x} \\ &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\gamma - \frac{\beta\gamma}{c}\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{1}{\gamma_v}\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{1 - \frac{v}{c^2}\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{1}{\gamma_v}\frac{\frac{(\mathbf{u}\times\mathbf{v})}{v}}{1 - \frac{v}{c^2}\cdot\frac{(\mathbf{u}\cdot\mathbf{v})}{v}} \\ &= \frac{1}{\gamma_v}\frac{\frac{(\mathbf{u}\times\mathbf{v})}{v}}{1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \end{aligned}$$

Thus:

$$
\begin{aligned}
|\mathbf{w}| &= \sqrt{w_x^2 + w_y^2} \\
&= \frac{1}{1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \sqrt{\left[\frac{(\mathbf{u}\cdot\mathbf{v})}{v} - v\right]^2 + \left(\frac{(\mathbf{u}\times\mathbf{v})}{\gamma_v v}\right)^2} \\
&= \dots \left[\frac{(\mathbf{u}\cdot\mathbf{v})^2}{v^2} - 2\,(\mathbf{u}\cdot\mathbf{v}) + v^2 + \frac{(\mathbf{u}\times\mathbf{v})^2}{\gamma_v^2 v^2}\right]^{1/2} \\
&= \dots \left[\dots + \frac{(\mathbf{u}\times\mathbf{v})^2}{\gamma_v^2 v^2} \cdot \left(1 - \frac{v^2}{c^2}\right)\right]^{1/2} \\
&= \frac{1}{1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \left[\frac{\overbrace{(\mathbf{u}\cdot\mathbf{v})^2 + (\mathbf{u}\times\mathbf{v})}^{u^2v^2}}{v^2} - 2\,(\mathbf{u}\cdot\mathbf{v}) + v^2 - \frac{(\mathbf{u}\times\mathbf{v})^2}{c^2}\right]^{1/2} \\
&= \frac{\sqrt{(u^2 - 2\,(\mathbf{u}\cdot\mathbf{v}) + v^2) - \frac{1}{c^2}\,(\mathbf{u}\times\mathbf{v})^2}}{1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}
\end{aligned}
$$

(d) Sketch of the frames in question:



From c, S_2 should possess speed:

$$
w = \frac{\sqrt{(\mathbf{v}-\mathbf{u})^2 - \frac{1}{c^2}\,(\mathbf{u}\times\mathbf{v})^2}}{1 - \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}}
$$

with $\mathbf{u} = (c/2, 0, 0)$, $\mathbf{v} = (c/\sqrt{2}, c/\sqrt{2}, 0)$.

$$\begin{aligned}
w &= \frac{1}{1-2\sqrt{2}}\left[\left(\frac{\sqrt{2}-1}{2}c\right)^2 + \left(\frac{c}{\sqrt{2}}\right)^2 + \frac{1}{c^2}\left(\frac{c^2}{2\sqrt{2}}\right)\right]^{1/2} \\
&= \frac{1}{1-2\sqrt{2}}\left[\frac{2+1+2\sqrt{2}}{4} + \frac{1}{2} - \frac{1}{8}\right]^{1/2} c \\
&= \frac{\sqrt{\frac{9}{8}+\frac{\sqrt{2}}{2}}}{1-2\sqrt{2}}c
\end{aligned}$$

In S_2 rest frame, 4-wavevector K_0^μ is related by:

$$\begin{aligned}
\mathsf{K}^\mu &= \begin{pmatrix} \omega/c \\ k\cos\theta \\ k\sin\theta \\ 0 \end{pmatrix} \\
&= \begin{bmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{bmatrix} \begin{pmatrix} \omega_0/c \\ k_0\cos\theta_0 \\ k_0\sin\theta_0 \\ 0 \end{pmatrix}
\end{aligned}$$

Thus we have:

$$\Rightarrow \frac{\omega}{c} = \frac{\gamma\omega_0}{c} + \beta\gamma k_0\cos\theta_0 = \frac{\omega_0}{c} \tag{5}$$

$$\Rightarrow k\cos\theta = \frac{\beta\gamma\omega_0}{c} + \gamma k_0\cos\theta_0 = k_0\cos\theta \tag{6}$$

(6)÷(5):

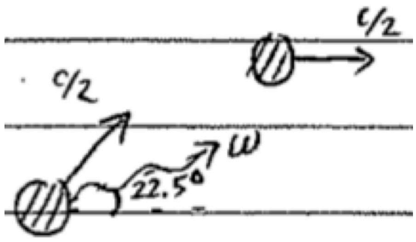
$$\begin{aligned}
\cos\theta &= \frac{\beta\gamma + \gamma\cos\theta_0}{\gamma + \beta\gamma\cos\theta_0} \\
&= \frac{\beta + \cos\theta_0}{1 + \beta\cos\theta_0}
\end{aligned}$$

where $\beta = w/c$.

Also from (5),

$$\begin{aligned}
\beta\gamma\cos\theta_0 &= 1-\gamma \\
\cos\theta_0 &= \frac{1-\gamma}{\beta\gamma} \\
\Rightarrow \cos\theta &= \frac{\beta + \frac{1-\gamma}{\beta\gamma}}{1+\beta\frac{1-\gamma}{\beta\gamma}} \quad \text{where } \gamma = \left[1-(w/c)^2\right]^{-1/2} \\
&= 0.255 \\
\Rightarrow \theta &= 75.2^\circ
\end{aligned}$$

For the Doppler shift to be negated, the photon must travel along the bisector between the path of the bodies in frame A (by symmetry so that Doppler shitts from $S_1 \to A \to S_2$ are cancelled precisely).



4. (DRAFT)

(a) Three-force:

$$
\begin{aligned}
\mathbf{F} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} \Rightarrow \mathbf{p}(t) &= \mathbf{F}t \\
\gamma m v &= \mathbf{F}t \\
\Rightarrow \frac{mv}{\sqrt{1-\frac{v^2}{c^2}}} &= Ft \\
m^2v^2 &= F^2t^2\left(1-\frac{v^2}{c^2}\right) \\
\left(m^2+\frac{F^2t^2}{c^2}\right)v^2 &= F^2t^2 \\
v &= \frac{Ft}{\sqrt{m^2+\frac{F^2t^2}{c^2}}}
\end{aligned}
$$

Note that as $t \to \infty$, $v \to \frac{Ft}{\frac{Ft}{c}} = c$.

(b) With 4-momentum $\mathsf{P}^\mu = (E/c, \mathbf{p})$, we have the 4-force:

$$
\begin{aligned}
\mathsf{F}^\mu = \frac{\mathrm{d}\mathsf{P}^\mu}{\mathrm{d}\tau} &= \gamma\frac{\mathrm{d}\mathsf{P}^\mu}{\mathrm{d}t} \\
&= \left(\frac{\gamma W}{c}, \gamma\mathbf{f}\right)
\end{aligned}
$$

where $W = \frac{\mathrm{d}E}{\mathrm{d}t}$, $\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$.

We also need the derivative of γ w.r.t. t:

$$
\begin{aligned}
\frac{\mathrm{d}\gamma}{\mathrm{d}t} &= -\frac{1}{2}\left(1-\frac{v^2}{c^2}\right)^{-3/2}\cdot\left(-\frac{2v\cdot a}{c^2}\right) \\
&= \gamma^3\frac{v\cdot a}{c^2}
\end{aligned}
$$

From energy $E = \gamma mc^2$,

$$
\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= \dot{\gamma}mc^2 \\
&= \gamma^3 mc^2\frac{\mathbf{v}\cdot\mathbf{a}}{c^2} = \gamma\frac{\mathbf{F}\cdot\mathbf{v}}{c}
\end{aligned}
$$

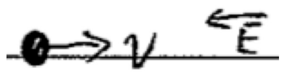
We also have:

$$
\begin{aligned}
\gamma \mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= \dot{\gamma} m \mathbf{v} + \gamma m \mathbf{a} \\
\Rightarrow \gamma^3 m \frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\mathbf{v} + \gamma m \mathbf{a} & \\
&= m\gamma^3 \left(\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\mathbf{v} + \left(1 - \frac{v^2}{c^2}\right)\mathbf{a} \right) \\
&= m\gamma^3 \left(\mathbf{a} + \frac{-v^2\mathbf{a} + (\mathbf{v}\cdot\mathbf{a})\,\mathbf{v}}{c^2} \right) \\
&= m\gamma^3 \left(\mathbf{a} + \frac{\mathbf{v}\times\mathbf{v}\times\mathbf{a}}{c^2} \right)
\end{aligned}
$$

(c) Larmor's formula:

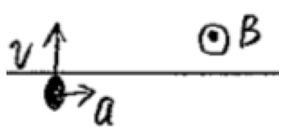
$$P = -\frac{1}{6\pi\epsilon_0}\frac{q^2}{m^2c^3}\mathsf{F}^\alpha \mathsf{F}_\alpha$$

Note that only the contraction plays a critical role in differentiating the power loss in both cases.



For the linac in LAB frame:

$$
\begin{aligned}
\mathsf{F}^\alpha \mathsf{F}_\alpha &= -\frac{\gamma^2 \left(\mathbf{F}\cdot\mathbf{v}\right)^2}{c^2} + \gamma^2 F^2 \\
&= \gamma^2 F^2 \underbrace{\left(1 - \frac{v^2}{c^2}\right)}_{\gamma^{-2}} \quad \text{since } \mathbf{F} \parallel \mathbf{v} \\
&= F^2 \\
&= \gamma^4 m^2 a^2
\end{aligned}
$$



Similarly, for a synchrotron in LAB frame:

$$
\begin{aligned}
\mathsf{F}^\alpha \mathsf{F}_\alpha &= -\frac{\gamma^2 \cancelto{0}{(\mathbf{F}\cdot\mathbf{v})^2}}{c^2} + \gamma^2 F^2 \\
&= \gamma^2 F^2 \\
&= \gamma^2 m^2 a^2
\end{aligned}
$$

where the difference arises from the fact that $\mathbf{F} \perp \mathbf{v}$ in a synchroton.

Thus for the same energy (i.e. same γ) and the same acceleration, a linac configuration would have higher power dissipation than a synchrotron.

(d) Energy of a particle $E = \gamma mc^2$. So with larger mass, one can pack more E without increasing γ. Hence muons are the better choice, provided that their decays are accounted for.

Additionally, since the synchroton has lower power dissipation as shown above, using it would be more economical.