UNOFFICIAL SOLUTIONS BY TheLongCat

B3: QUANTUM, ATOMIC AND MOLECULAR PHYSICS

TRINITY TERM 2019

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Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. **(DRAFT)**

(a) For a many-e$^-$ system, its Hamiltonian is

$$\hat{H} = \sum_i \left[\frac{\hat{\mathbf{p}}_i^2}{2m_e} - \frac{Ze^2}{4\pi\epsilon_0 \hat{r}_i} + \sum_{j>i} \frac{e^2}{4\pi\epsilon_0 \hat{r}_{ij}} \right]$$

where the first term is kinetic energy, second term is potential due to nucleus, and the last term is the inter-e$^-$ repulsion.

Introduce a central field such that $\hat{H} = \hat{H}_{\text{CF}} + \Delta\hat{H}_{\text{RE}}$, where

$$\hat{H}_{\text{CF}} = \sum_i \left[\frac{\hat{\mathbf{p}}_i^2}{2m_e} - \frac{Ze^2}{4\pi\epsilon_0 \hat{r}_i} + S(r_i) \right] \quad \text{and}$$

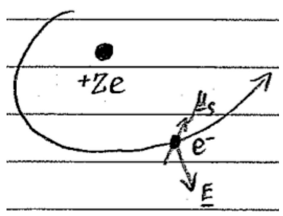
$$\Delta\hat{H}_{\text{RE}} = \sum_i \left[\sum_{j>i} \frac{e^2}{4\pi\epsilon_0 \hat{r}_{ij}} - S(r_i) \right]$$

and the system may be treated as a central field with non-central perturbation.

As the strength of $\Delta\hat{H}_{\text{RE}} \ll \hat{H}_{\text{CF}}$, the LS coupling scheme applies where the energy eigenstates may be labelled as $|nlm_l sm_s\rangle$ for each e$^-$ $\to |nLM_L SM_S\rangle$ for the whole system.

As $\Delta\hat{H}_{\text{RE}}$ is an internal interaction, the total orbital angular momentum is then conserved, rendering L a good quantum number by Ehrenfest's Theorem. Similarly S is also a good quantum number.

This leads to the idea of terms, where the eigenstates are labelled with ${}^{2S+1}L$. L takes the form of S, P, D . . .



(b) In e$^-$ rest frame,

$$\begin{aligned} \mathbf{B} &= -\frac{1}{c^2}\mathbf{v} \times \mathbf{E} \quad \text{by Lorentz transformation} \\ &= -\frac{1}{c^2}\frac{\mathbf{p}}{m_e} \times \frac{Ze^2}{4\pi\epsilon_0 r^3}\mathbf{r} \end{aligned}$$

Recall orbital angular momentum $\hbar\hat{\mathbf{l}} = \mathbf{r} \times \mathbf{p}$, so:

$$\mathbf{B} = \frac{\hbar}{m_e c^2} \cdot \frac{Ze^2}{4\pi\epsilon_0 r^2}\hat{\mathbf{l}}$$

Magnetic dipole energy:

$$\begin{aligned} \langle E_{\text{SO}} \rangle &= \langle -\boldsymbol{\mu}_s \cdot \mathbf{B} \rangle \\ &= \left\langle -g_s \mu_{\text{B}} \hat{\mathbf{s}} \cdot \hat{\mathbf{l}} \cdot \frac{\hbar}{m_e c^2} \cdot \frac{Ze^2}{4\pi\epsilon_0 r^2} \right\rangle \\ &= \beta_{\text{SO}} \hat{\mathbf{l}} \cdot \hat{\mathbf{s}} \end{aligned}$$

For multi-e$^-$,

$$
\begin{aligned}
\langle E_{\mathrm{SO}} \rangle &= \sum_i \beta_{\mathrm{SO}_i} \hat{\mathbf{l}}_i \cdot \hat{\mathbf{s}}_i \\
&= \beta_{\mathrm{SO}} \hat{\mathbf{L}} \cdot \hat{\mathbf{S}} \quad \text{by Wigner-Eckart where } \hat{\mathbf{l}}_i \to \frac{\hat{\mathbf{l}}_i \cdot \hat{\mathbf{L}}}{\hat{L}^2},\ \hat{\mathbf{s}}_i \to \frac{\hat{\mathbf{s}}_i \cdot \hat{\mathbf{S}}}{\hat{S}^2} \\
&= \frac{1}{2}\beta_{\mathrm{SO}} \left(J(J+1) - L(L+1) - S(S+1) \right)
\end{aligned}
$$

since $\hat{\mathbf{J}}^2 = \left(\mathbf{L} \hat{+} \hat{\mathbf{S}}\right)^2$.

So for different combination of J, there exists different energy levels.

Consider:

$$
\begin{aligned}
\Delta E_{J,J-1} &= \frac{1}{2}\beta_{\mathrm{SO}} \left[\begin{aligned} &J(J+1) - \cancel{L(L+1)} - \cancel{S(S+1)} \\ &\quad - (J-1)J + \cancel{L(L+1)} + \cancel{S(S+1)} \end{aligned} \right] \\
&= \beta_{\mathrm{SO}} J
\end{aligned}
$$

$$
\begin{aligned}
\Delta E_{J-1,J-2} &= \frac{1}{2}\beta_{\mathrm{SO}} \left[\begin{aligned} &(J-1)J - \cancel{L(L+1)} - \cancel{S(S+1)} \\ &\quad - (J-2)(J-1) + \cancel{L(L+1)} + \cancel{S(S+1)} \end{aligned} \right] \\
&= \beta_{\mathrm{SO}} (J-1)
\end{aligned}
$$

$$
\Rightarrow \frac{\Delta E_{J,J-1}}{\Delta E_{J-1,J-2}} = \frac{J}{J-1} \Rightarrow \text{Interval Rule}
$$

(c) Assuming electric dipole transition, then:

1 e$^-$ moves Δn =any $\Delta l = \pm 1$	$\Delta L = 0, \pm 1$ $(0 \nrightarrow 0)$ $\Delta S = 0$	$\Delta J = 0, \pm 1$ $(0 \nrightarrow 0)$ $\Delta M_J = 0, \pm 1$ $(0 \nrightarrow 0$ iff $\Delta J = 0)$
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Valence e$^-$ configuration of Mg: $3s^2$ $(2+8+2=12)$ at ground.

$$
\begin{matrix} L = 0 \\ S = 0 \end{matrix} \to {}^1S \text{ term only} \to {}^1S_0 \text{ level}
$$

1st excited state:

$$
3s3p \to L = 1 \begin{matrix} S = 0 \\ S = 1 \end{matrix} \to \begin{matrix} {}^1P \\ {}^3P \end{matrix} \to \begin{matrix} {}^1P_1 \\ {}^3P_0, {}^3P_1, {}^3P_2 \end{matrix}
$$

2nd excited state:

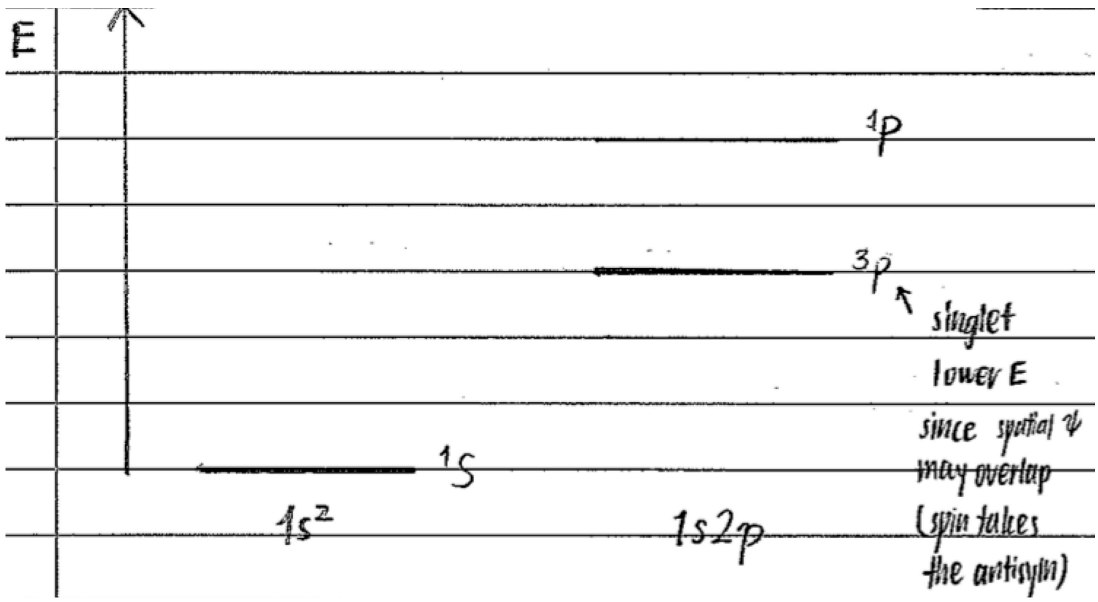
$$
3s4s \to L = 0 \begin{matrix} S = 0 \\ S = 1 \end{matrix} \to \begin{matrix} {}^1S \\ {}^3S \end{matrix} \to \begin{matrix} {}^1S_0 \\ {}^3S_1 \end{matrix}
$$

285.21 nm absorption $\Rightarrow$ ground level excitation $\Rightarrow$ only transition $3s^2 \quad {}^1S_0 \to 3s3p^1P_1$ possible.

Other possible transitions:

- $3s3p\ {}^1P_1 \to 3s4s\ {}^1S_0$
- $3s3p\ {}^3P_0 \to 3s4s\ {}^3S_1$
- $3s3p\ {}^3P_1 \to 3s4s\ {}^3S_1$
- $3s3p\ {}^3P_2 \to 3s4s\ {}^3S_1$

For the closely grouped lines, they belong to the ${}^3P \to {}^3S$ family.



Energy $E = \dfrac{hc}{\lambda} = \dfrac{1}{\lambda}$ in natural units.

${}^1P_1$: $\dfrac{1}{285.21\,\mathrm{nm}} = 3\,506\,200\,\mathrm{m}^{-1}$

${}^1S_0$: $E_{{}^1P_1} + \dfrac{1}{1182.8\,\mathrm{nm}} = 4\,351\,600\,\mathrm{m}^{-1}$

${}^3S_1$: $E_{{}^1S_0} - \Delta = 4\,121\,000\,\mathrm{m}^{-1}$

${}^3P_2$: $E_{{}^3S_1} - \dfrac{1}{518.36\,\mathrm{nm}} = 2\,191\,000\,\mathrm{m}^{-1}$

${}^3P_1$: $E_{{}^3S_1} - \dfrac{1}{517.27\,\mathrm{nm}} = 2\,187\,000\,\mathrm{m}^{-1}$

${}^3P_0$: $E_{{}^3S_1} - \dfrac{1}{516.73\,\mathrm{nm}} = 2\,185\,800\,\mathrm{m}^{-1}$

$$\begin{aligned}\Delta E_{2,1} &= E_{{}^3P_2} - E_{{}^3P_1} \\ &= 4065.2\,\mathrm{m}^{-1}\end{aligned}$$

$$\begin{aligned}\Delta E_{1,0} &= E_{{}^3P_1} - E_{{}^3P_0} \\ &= 2020.3\,\mathrm{m}^{-1}\end{aligned}$$

$\dfrac{\Delta E_{2,1}}{\Delta E_{1,0}} = 2.012 \simeq 2$ so Interval Rule is obeyed well.

(d) Under LS coupling scheme,

$$6s5d \to L = 2 \begin{array}{l} S=0 \\ S=1 \end{array} \to \begin{array}{c} {}^1D_2 \\ {}^3D_1, {}^3D_2, {}^3D_3 \text{ -- expect 3 closely grouped lines} \end{array}$$

$$6s6p \to L = 1 \begin{array}{l} S=0 \\ S=1 \end{array} \to \begin{array}{c} {}^1P_1 \\ {}^3P_0, {}^3P_1, {}^3P_2 \text{ -- expect 3 closely grouped lines} \end{array}$$

$$6s^2 \to L = 0 \; S = 0 \to {}^1S_0 \text{ only}$$

The levels appear fine so far.

Next examine the Interval Rule: <u>$6s6p$</u>:

$$\begin{array}{l} \Delta E_{2,1} = 38\,103\,\mathrm{m}^{-1} \\ \Delta E_{1,0} = 18\,153\,\mathrm{m}^{-1} \end{array} \Rightarrow \frac{\Delta E_{2,1}}{\Delta E_{1,0}} = 2.099 \simeq 2$$

<u>$6s5d$</u>:

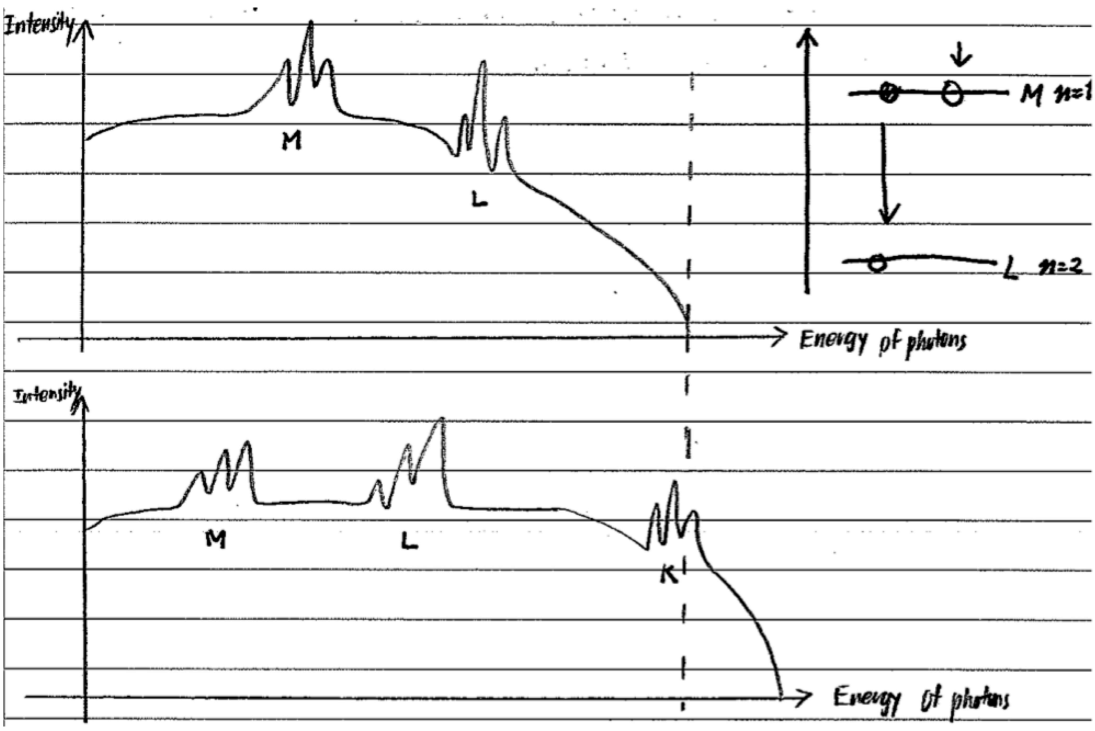
$$\begin{array}{l} \Delta E_{3,2} = 87\,812\,\mathrm{m}^{-1} \\ \Delta E_{2,1} = 37\,060\,\mathrm{m}^{-1} \end{array} \Rightarrow \frac{\Delta E_{3,2}}{\Delta E_{2,1}} = 2.369 \not\approx \frac{3}{2} = 1.5$$

So LS coupling is only good for $6s^2$ and $6s6p$ configs. For $6s5d$ the Interval Rule begins to break down.

Can also compare $\dfrac{\Delta E_{\mathrm{RE}}}{\Delta E_{\mathrm{SO}}} \sim \dfrac{1}{0.17}$.

2. **(DRAFT)**

(a) The emission of X-rays is due to the introduction of holes in the inner shell via e$^-$ bombardment, which knocks off the inner e$^-$. In addition, X-rays may also be emitted via bremsstrahlung when the incoming e$^-$ accelerates under the atomic potential.

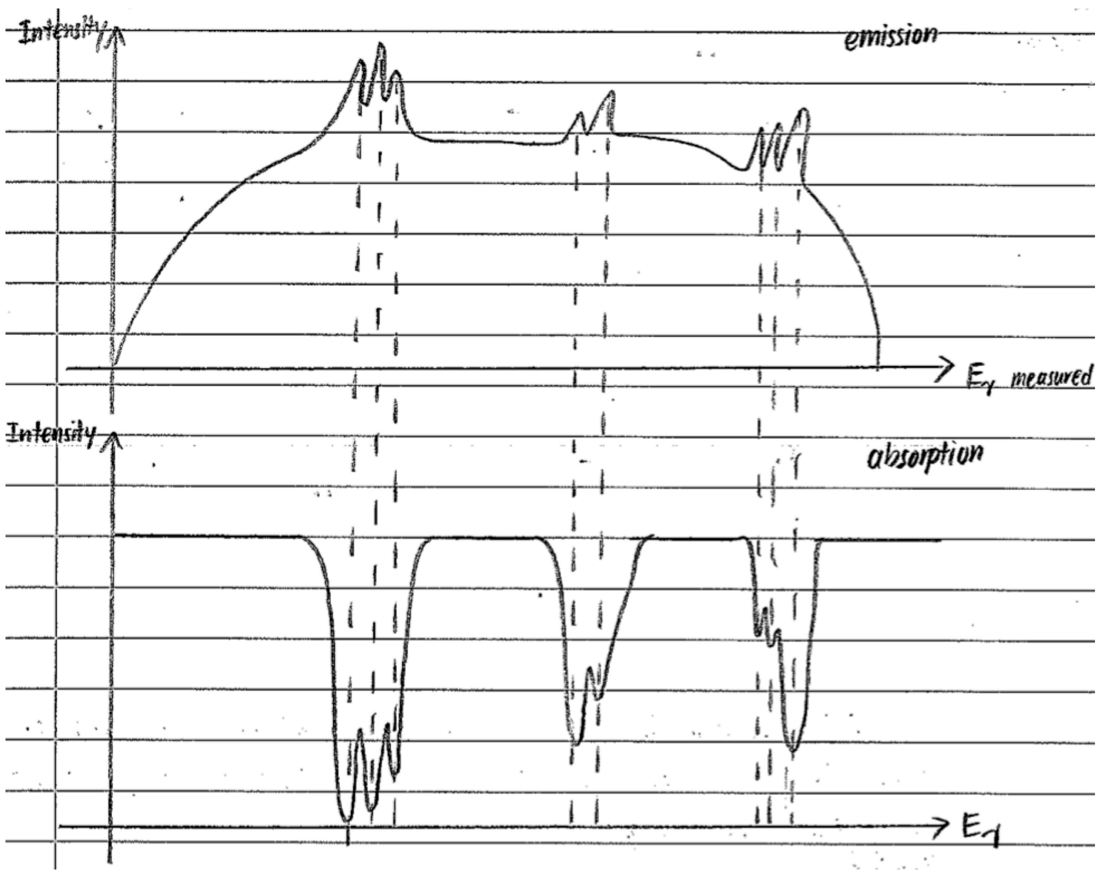


The emission spectrum is a curve with several resonance peaks due to the background bremsstrahlung, while the peaks are simply the emission of X-rays from the hole transitioning away from the inner shells.

Furthermore, the peaks will not appear until the probed energy (E_e) exceeds the energy level of the inner shells (see dashed line above). And the cut off point of the bremsstrahlung curve is simply the energy of the e$^-$ beam.

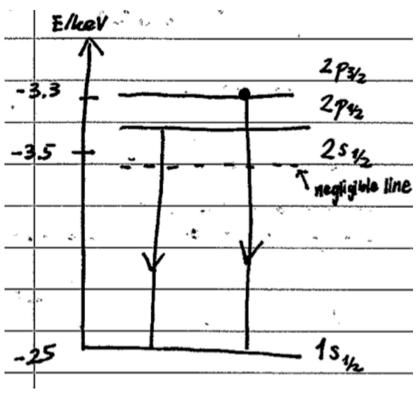
(b) The absorption and emission spectra will have the same peaks as they are inherent to the atomic structure.

However, the absorption spectrum lacks the bremsstrahlung features as a photon cannot emit another under EM field.

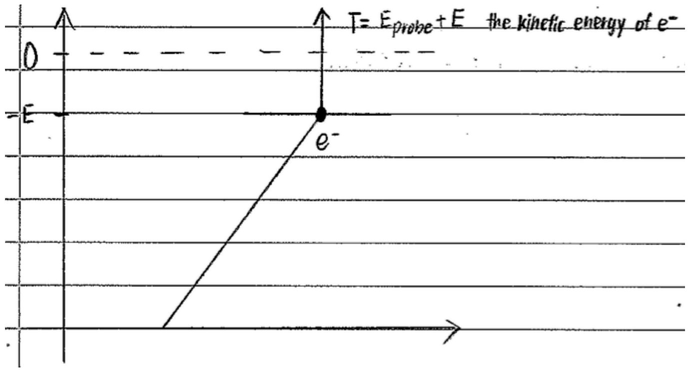


(c) Characteristic X-rays are emissions due to the introduction of holes in the inner shells. The final set of lines is due to the innermost (K) shell emission.

The transitions can be $K \to L$, $K \to M \ldots$

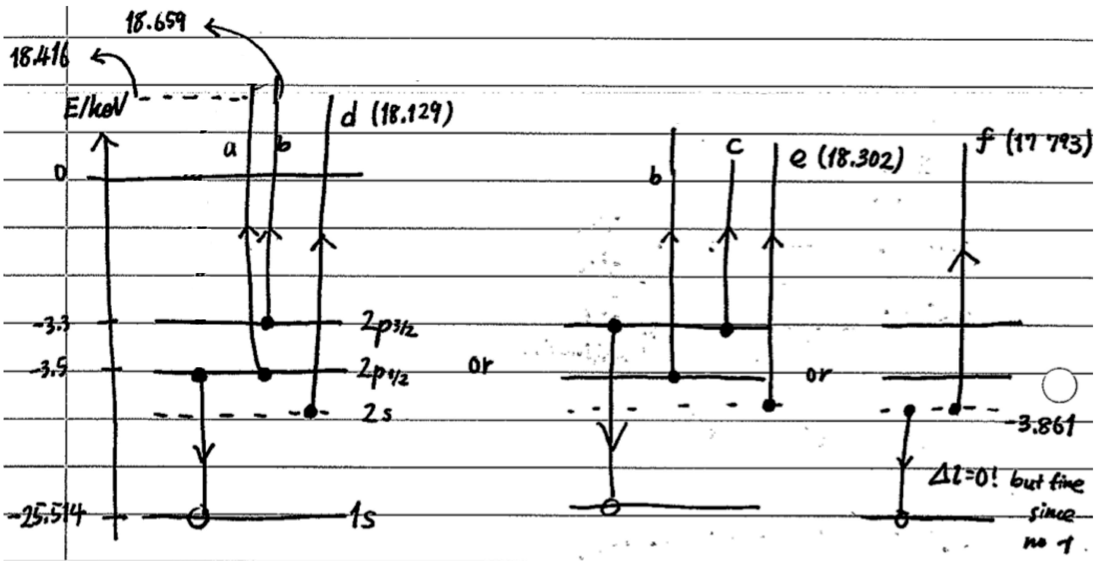


(d) The emission of e$^-$ is due to the Auger effect where energy of emission goes to ejecting e$^-$ instead of emitting a photon.

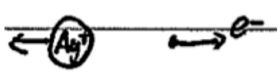


So electron energy:

$$
\begin{aligned}
T &= E_{\text{probe}} + E \\
\Rightarrow -E &= E_{\text{probe}} - T \\
&\Rightarrow 7.721\,\text{keV}, 7.426\,\text{keV}, 7.256\,\text{keV}, 7.183\,\text{keV}, 7.002\,\text{keV}, 6.833\,\text{keV}
\end{aligned}
$$



(e) We are having Ag^{++} instead of Ag^{+} so energy level not exactly the same. Conservation of momentum so measured lower than theoretical.



3. **(DRAFT)**

(a) Rabi frequency $\Omega_R = \langle 2| eE_0\hat{\mathbf{x}} |1\rangle$ where $\hat{\mathbf{x}}$ is the unit vector in the direction of $\mathbf{E}$, $|1\rangle$ and $|2\rangle$ are the lower and upper level states of the atom. This describes the strength of the interaction between the radiation and the atom, i.e. the hopping potential from one level to another.

(b) For optical transition, $\omega + \omega_0 \gg |\omega - \omega_0|$ since spread in frequency $\Delta\omega = |\omega - \omega_0| \ll \omega_0 \sim 10^7\,\mathrm{rad\,s^{-1}}$.

$\Rightarrow$ Rotating wave approximation suitable $\Rightarrow$ terms with $\omega + \omega_0$ may be dropped.

$$\begin{aligned}
\dot{c}_2 &= -i\Omega_R c_1 \frac{1}{2}\left[\cancelto{\text{RWA}}{e^{i(\omega_0+\omega)t}} + e^{i(\omega_0-\omega)t}\right] \\
\dot{c}_1 &= -i\Omega_R c_2 \frac{1}{2}\left[e^{i(-\omega_0+\omega)t} + \cancelto{\text{RWA}}{e^{i(-\omega_0-\omega)t}}\right]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \dot{c}_2 &= -\frac{i}{2}\Omega_R e^{i(\omega_0-\omega)t} c_1 \\
\Rightarrow \ddot{c}_2 &= \underbrace{\frac{\omega_0-\omega}{2}}_{i(\omega_0-\omega)\dot{c}_2} - \frac{i}{2}\Omega_R e^{i(\omega_0-\omega)t}\dot{c}_1 \\
\ddot{c}_2 &= -i\delta\omega\dot{c}_2 - \frac{i}{2}\Omega_R \cancel{e^{i(\omega_0-\omega)t}} \cdot \left(-i\Omega_R c_2 \frac{1}{2}\cancel{e^{-i(\omega_0-\omega)t}}\right) \qquad \text{where } \delta\omega = \omega - \omega_0 \\
\ddot{c}_2 &+ i\delta\omega\dot{c}_2 + \frac{1}{4}\Omega_R^2 c_2 = 0
\end{aligned}$$

Insert Ansatz $c_2 = Ae^{i\omega_A t}$ gives:

$$\begin{aligned}
-\omega_A^2 c_2 + i\delta\omega(i\omega_A)c_2 + \frac{\Omega_R^2}{4}c_2 &= 0 \\
\omega_A^2 + \delta\omega\omega_A - \frac{\Omega_R^2}{4} &= 0 \\
\Rightarrow \omega_A &= \frac{-\delta\omega \pm \sqrt{(\delta\omega)^2 + \Omega_R^2}}{2} \\
&= \frac{\sqrt{\Omega_R^2 + (\delta\omega)^2} - \delta\omega}{2} \qquad \text{since } \omega_A > 0
\end{aligned}$$

Boundary conditions:

$$\begin{aligned}
&c_2(0) = 0 \\
&\Rightarrow c_2 = Ae^{i\omega_A t} - Ae^{-i\omega_A t} \\
&\qquad = A(2i)\sin\omega_A t \to A\sin\omega_A t
\end{aligned}$$

Also $\dot{c}_2 = -\frac{i}{2}\Omega_R e^{i(\omega_0-\omega)t}c_1$ from before:

$$\Rightarrow c_1 = -\frac{2A\omega_A \cos\omega_A t}{\Omega_R}e^{i(\omega-\omega_0)t}$$
$$c_1(0) = 1 \Rightarrow -\frac{2\omega_A}{\Omega_R} = \frac{1}{A}$$
$$\Rightarrow A = -\frac{\Omega_R}{2\omega_A}$$

So:

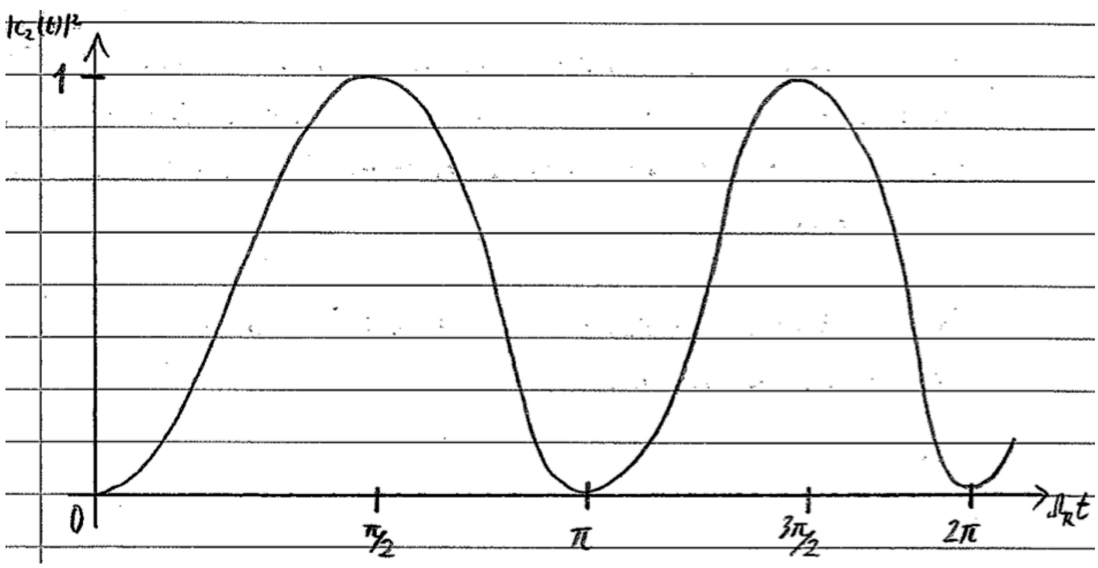
$$c_2(t) = 2i\left(-\frac{\Omega_R}{2\omega_A}\right)\sin\omega_A t$$
$$\Rightarrow |c_2(t)|^2 = 4\left(\frac{\Omega_R}{2\omega_A}\right)^2 \sin^2\omega_A t$$
$$= \left(\frac{\Omega_R}{\sqrt{\Omega_R^2+(\delta\omega)^2}-\delta\omega}\right)^2 \sin^2\left(\frac{\sqrt{\Omega_R^2+(\delta\omega)^2}-\delta\omega}{2}t\right)$$

So $\Omega_{\text{eff}} = \sqrt{\Omega_R^2+(\delta\omega)^2}-\delta\omega$.

(c) For $\delta\omega = 0$,

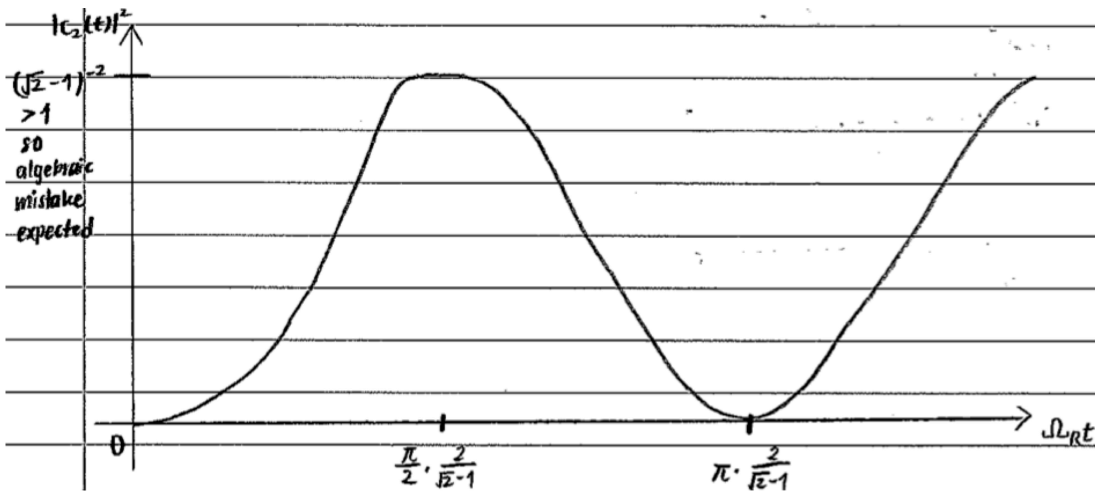
$$|c_2(t)|^2 = \left(\frac{\Omega_R}{\Omega_R}\right)^2 \sin^2\left(\frac{\Omega_R}{2}t\right)$$
$$= \sin^2\left(\frac{\Omega_R}{2}t\right)$$

So Rabi oscillations with periodicity $t = \frac{\pi}{\Omega_R}$.



For $\delta\omega = \Omega_R$,

$$|c_2(t)|^2 = \left(\frac{\Omega_R}{(\sqrt{2}-1)\Omega_R}\right)^2 \sin^2\left[\frac{(\sqrt{2}-1)\Omega_R}{2}t\right]$$
$$= (\sqrt{2}-1)^{-2}\sin^2\left[\frac{\sqrt{2}-1}{2}\Omega_R t\right]$$

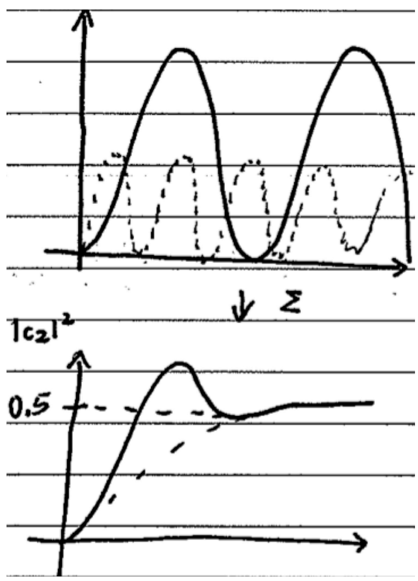


Expected smaller amplitude but faster oscillation, so mistakes in Ω_{eff} expected.

(d) As the detuning increases, the atomic response gets weaker. Thus increasing the saturation intensity for lasing applications. Oscillation frequency $\propto \delta$, sum of incoherent $\Rightarrow$ Einstein's equation applicable!

For a broadband weak radiation, we may replace E_0 with $\frac{1}{2}\epsilon_0 E_0^2 = \int \rho(\omega)\,\mathrm{d}\omega$, where $\rho(\omega)$ is the spectral energy density of the radiation. Furthermore, approximating that $\int \rho(\omega)\,\mathrm{d}\omega = \rho(\omega_0)\Delta\omega$ where $\Delta\omega = |\omega - \omega_0|$ is acceptable due to the narrow response of the atom.

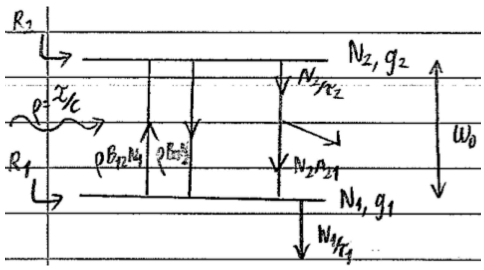
Then by Einstein's coefficient, $B_{12}\rho = |c_2(t)|^2$, we may then calculate the values of B_{12}, B_{21} and A_{21}.



4. (DRAFT)

(a) Gain saturation refers to the termination in laser gain growth as population inversion begins to be burnt out, resulting in a linear beam growth instead.

In a laser amp, as $\rho(\omega)$ overwhelms the system, it can no longer sustain a population inversion, resulting in a dynamic equilibrium where beam can only grow linearly via pumping.



$$\frac{\mathrm{d}I}{\mathrm{d}z} = \alpha I \qquad \alpha = \frac{\alpha_0}{1+\frac{I}{I_s}} \qquad \Longrightarrow \qquad \lim_{I\to\infty}\frac{\mathrm{d}I}{\mathrm{d}z} = \alpha_0$$

(b) Rate equations:

$$\begin{aligned}
\frac{\mathrm{d}N_2}{\mathrm{d}t} &= \mathcal{R}_2 + \left(B_{12}\rho - B_{21}\rho - \frac{1}{\tau_2}\right) N_2 \\
\frac{\mathrm{d}N_1}{\mathrm{d}t} &= \mathcal{R}_1 + \left(B_{21}\rho - B_{12}\rho - \frac{1}{\tau_1}\right) N_1 + A_{21}N_2
\end{aligned}$$

At steady state, ${}^{\mathrm{d}N_2}\!/_{\mathrm{d}t} = {}^{\mathrm{d}N_1}\!/_{\mathrm{d}t} = 0$:

$$\Rightarrow N_2 = -\frac{\mathcal{R}_2}{B_{12}\rho - B_{21}\rho - \dfrac{1}{\tau_2}}$$

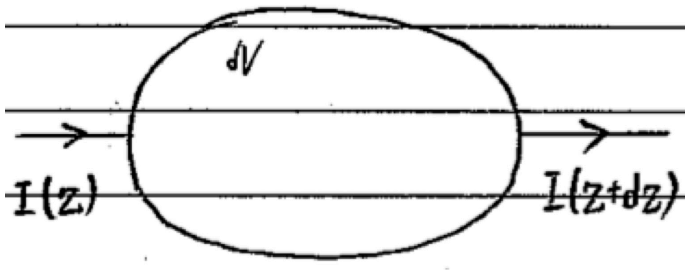
$$\mathcal{R}_1 + \left(B_{21}\rho - B_{12}\rho - \frac{1}{\tau_1}\right) N_1 + \frac{A_{21}\mathcal{R}_2}{B_{12}\rho - B_{21}\rho - \dfrac{1}{\tau_2}} = 0$$

$$\begin{aligned}
\Rightarrow N_1 &= \frac{1}{B_{21}\rho - B_{12}\rho - \frac{1}{\tau_1}} \cdot \left[-\frac{A_{21}\mathcal{R}_2}{B_{12}\rho - B_{21}\rho - \dfrac{1}{\tau_2}} - \mathcal{R}_1\right] \\
&= \tau_1 \cdot (\mathcal{R}_1 - A_{21}\tau_2\mathcal{R}_2)
\end{aligned}$$

since $g_1 = g_2 \Rightarrow B_{12} = B_{21}$.

$$\begin{aligned}
\Rightarrow N^*(0) &= N_2 - N_1 \\
&= +\tau_2\mathcal{R}_2 - \tau_1\left(\mathcal{R}_1 - A_{21}\tau_2\mathcal{R}_2\right) \\
&= \mathcal{R}_2\tau_2\left(1 - A_{21}\tau_1\right) - \mathcal{R}_1\tau_1
\end{aligned}$$

(c) Sketch of the change in intensity:



Energy deposited in $\mathrm{d}z$ of a laser amp:

$$E = I(z) \cdot \sigma_{21}(\omega - \omega_0)$$

Also from rate equations:

$$\begin{aligned} E &= (B_{21} - B_{12})\rho\hbar\omega \\ \Rightarrow (B_{21} - B_{12})\rho &= \frac{E}{\hbar\omega} = \frac{I\sigma_{21}}{\hbar\omega} \end{aligned}$$

So:

$$\begin{aligned} \frac{\mathrm{d}N_2}{\mathrm{d}t} &= \mathcal{R}_2 - \frac{I\sigma_{21}}{\hbar\omega}N^* - \frac{N_2}{\tau_2} \\ \frac{\mathrm{d}N_1}{\mathrm{d}t} &= \mathcal{R}_1 + \frac{I\sigma_{21}}{\hbar\omega}N^* - \frac{N_1}{\tau_1} + A_{21}N_2 \end{aligned}$$

At steady state,

$$\begin{aligned} N_2 &= \mathcal{R}_2\tau_2 - N^*\frac{I\sigma_{21}}{\hbar\omega}\tau_2 \\ N_1 &= \mathcal{R}_1 + N^*\frac{I\sigma_{21}}{\hbar\omega}\tau_1 + A_{21}\left(\mathcal{R}_2\tau_2 - N^*\frac{I\sigma_{21}}{\hbar\omega}\tau_2\right)\tau_1 \end{aligned}$$

$$\begin{aligned} \Rightarrow N^* &= N_2 - N_1 \\ &= \mathcal{R}_2\tau_2 - N^*\frac{I\sigma_{21}}{\hbar\omega}\tau_2 - \mathcal{R}_1\tau_1 - N^*\frac{I\sigma_{21}}{\hbar\omega}\tau_1 \\ &\quad - A_{21}\left(\mathcal{R}_2\tau_2 - N^*\frac{I\sigma_{21}}{\hbar\omega}\tau_2\right)\tau_1 \end{aligned}$$

$$\left[1 + \frac{I\sigma_{21}}{\hbar\omega}(\tau_2 + \tau_1 - A_{21}\tau_2\tau_1)\right]N^* = \mathcal{R}_2\tau_2(1 - A_{21}\tau_1) - \mathcal{R}_1\tau_1$$

So:

$$I_s = \frac{\hbar\omega}{\sigma_{21}(\tau_2 + \tau_1 - A_{21}\tau_2\tau_1)}$$

is the saturation intensity, this is a measure of the intensity required to burn out population inversion. As atomic response to external field is frequency dependent, $I_s(\omega)$ naturally follows since the further away one is from the resonance frequency, the weaker perturbution it gets $\rightarrow$ larger I_s.

(d) Gain coefficient $\alpha = N^*\sigma_{21}$:

$$\Rightarrow \alpha_1 = \frac{N^*(0)\sigma_{21}(\omega_1)}{1 + \dfrac{I_0}{I_s(\omega_1)}} = 3.2$$

$$\alpha_2 = \frac{N^*(0)\sigma_{21}(\omega_2)}{1 + \dfrac{2I_0}{I_s(\omega_1)}} = 2.1$$

$$\frac{\alpha_1}{\alpha_2} = \frac{\sigma_{21}(\omega_1)\left[1 + \dfrac{2I_0}{I_s(\omega_1)}\right]}{\sigma_{21}(\omega_2)\left[1 + \dfrac{I_0}{I_s(\omega_1)}\right]}$$

Also:

$$\begin{aligned}
I_s(\omega_2) &= \frac{\hbar\left(\omega_1 + \dfrac{1}{2}\Delta\omega\right)}{\sigma_{21}(\omega_2)\,[\ldots]} \\
&= \frac{1}{2}I_s(\omega_1) = \frac{1}{2}\frac{\hbar\omega_1}{\sigma_{21}(\omega_1)\,[\ldots]}
\end{aligned}$$

$$\Rightarrow \frac{\sigma_{21}(\omega_1)}{\sigma_{21}(\omega_2)} = \frac{1}{2}\frac{\omega_1}{\omega_1 + \dfrac{1}{2}\Delta\omega}$$

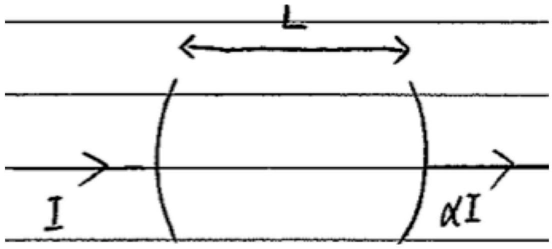
$$\Rightarrow \frac{\alpha_1}{\alpha_2} = \frac{1}{2}\frac{\omega_1}{\omega_1 + \dfrac{1}{2}\Delta\omega} \cdot \frac{1 + \dfrac{2I_0}{I_s(\omega_1)}}{1 + \dfrac{I_0}{I_s(\omega_1)}}$$

$$\frac{\alpha_1}{\alpha_2} + \frac{\alpha_1}{\alpha_2}\frac{I_0}{I_s(\omega_1)} = \frac{\omega_1}{2\omega_1 + \Delta\omega} + \frac{\omega_1}{2\omega_1 + \Delta\omega} \cdot \frac{2I_0}{I_s(\omega_1)}$$

$$\left(\frac{\alpha_1}{\alpha_2} - \frac{\omega_1}{2\omega_1 + \Delta\omega}\right) I_s(\omega_1) = \left(\frac{2\omega_1}{2\omega_1 + \Delta\omega} - \frac{\alpha_1}{\alpha_2}\right) I_0$$

$$\Rightarrow I_s(\omega_1) = \frac{\dfrac{\omega_1}{\omega_1 + \frac{1}{2}\Delta\omega} - \dfrac{\alpha_1}{\alpha_2}}{\dfrac{\alpha_1}{\alpha_2} - \dfrac{\omega_1}{2\omega_1 - \Delta\omega}}$$

Sketch of the gain medium:



Gain coefficient:

$$\begin{aligned}\alpha &= N^*\sigma_{21} \\ &= \frac{N^*(0)\sigma_{21}(\omega)}{1+\dfrac{I}{I_s(\omega)}}\end{aligned}$$

Also:

$$\begin{aligned}\alpha &= \frac{\partial I}{\partial z} \\ \Rightarrow 1+\frac{I}{I_s}\,\mathrm{d}I &= N^*(0)\sigma_{21}(\omega)\,\mathrm{d}z\end{aligned}$$

For ω_1:

$$\ln\left(\frac{I_1}{I_0}\right)+\frac{I_1-I_0}{I_{s_1}} = \alpha_{01}L$$

Same for ω_2:

$$\ln\left(\frac{I_2}{I_0}\right)+\frac{I_2-I_0}{I_{s_2}} = \underbrace{\alpha_{02}}_{\frac{1}{2}\alpha_{01}}L$$

Also:

$$\begin{aligned}\sigma_2 &= \frac{1}{2}\sigma_1 \\ I_s &\propto \frac{1}{\sigma} \quad \text{for optical } (\omega_2 \approx \omega_1 \approx \omega_0)\end{aligned}$$

$$\begin{aligned}\Rightarrow I_{s_1} &= 3.43 I_0 \qquad \alpha_{01} = 36\,\mathrm{m}^{-1} \\ I_{s_2} &= 6.86 I_0 \qquad \alpha_{01} = 18\,\mathrm{m}^{-1}\end{aligned}$$