UNOFFICIAL SOLUTIONS BY TheLongCat

C2: LASER SCIENCE AND QUANTUM INFORMATION PROCESSING

TRINITY TERM 2024

Last updated: 30th May 2025

Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

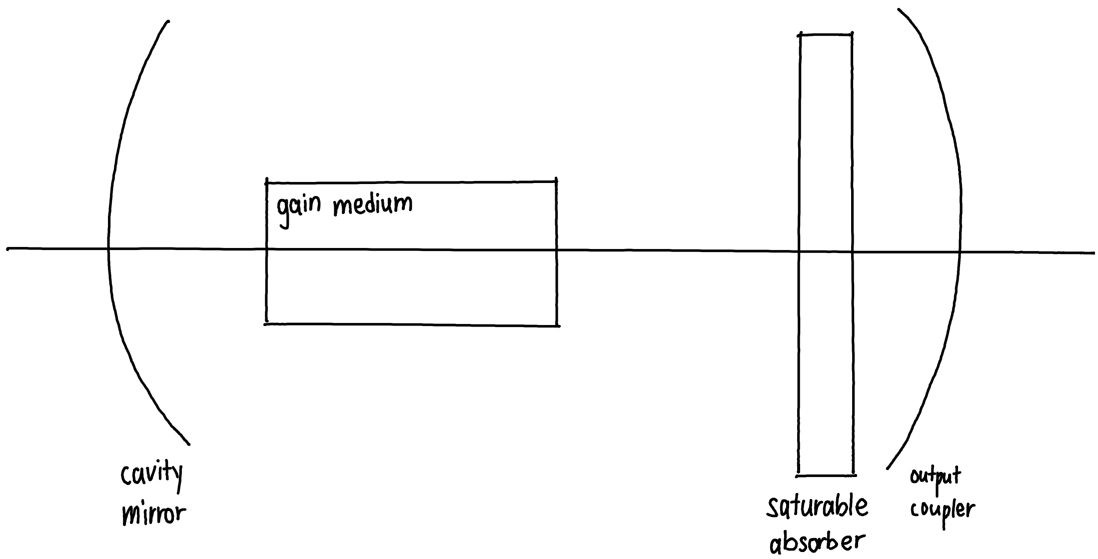
1. Yet another classic Q-switching question. If you keep your head cool then you should be able to bag at least 20 marks from this question.

(a) *Mostly copypasta from Q1 2019.*

Q-switching is a technique by which the quality factor of a laser cavity is periodically varied to intentionally build up population inversion N^* beyond the equilibrium value, thus producing a laser output with *large peak intensity* and *short pulse length.*

Q-switching may be achieved by employing a saturable absorber at the output coupler. As the absorption of the absorber varies with the intensity I, it is able to suppress lasing until saturation, thereby achieving Q-switching without active clock source.

Sketch of the cavity setup:



(b) Usual bookwork, from the rate equations we immediately see that the contribution in cavity photon density due to lasing is given by $N^*\sigma_{21}I/\hbar\omega$.

We also know that intensity $I = n\hbar\omega c$, together with the careful consideration that the lasing medium takes up only a fraction f_c of the cavity:

$$\frac{\mathrm{d}n_{\text{lasing}}}{\mathrm{d}t} = f_c N^* \sigma_{21} n c \tag{1}$$

Taking cavity loss into account then yields the total rate of change in cavity photon density:

$$\begin{aligned}\frac{\mathrm{d}n}{\mathrm{d}t} &= \frac{\mathrm{d}n_{\text{lasing}}}{\mathrm{d}t} - \frac{n}{\tau_c} \\ &= \left(f_c N^* \sigma_{21} c \tau_c - 1\right) \frac{n}{\tau_c} \\ &= \left(\frac{N^*}{N^*_{\text{th}}} - 1\right) \frac{n}{\tau_c}\end{aligned} \tag{2}$$

where τ_c is the cavity lifetime, and $N^*_{\text{th}} = \left(f_c \sigma_{21} c \tau_c\right)^{-1}$ is the threshold population inversion which signifies the minimum population inversion required for the cavity photon density to grow.

(c) Another bookwork. We consider the simplified rate equations by ignoring the pump and spontaneous terms:

$$\frac{\mathrm{d}N_2}{\mathrm{d}t} = -\frac{N^*}{N^*_{\mathrm{th}}}\frac{n}{f_c\tau_c} \tag{3}$$

$$\frac{\mathrm{d}N_1}{\mathrm{d}t} = \frac{N^*}{N^*_{\mathrm{th}}}\frac{n}{f_c\tau_c} \tag{4}$$

We then have population inversion $N^* = N_2 - g_2/g_1 N_1$, so

$$\frac{\mathrm{d}N^*}{\mathrm{d}t} = -\underbrace{\left(1+\frac{g_2}{g_1}\right)}_{\beta}\frac{N^*}{N^*_{\mathrm{th}}}\frac{n}{f_c\tau_c} \tag{5}$$
$$= -\beta\sigma_{21}cN^*n$$

To find the output energy, we first consider the power output P of the cavity:

$$\begin{aligned} P &= (\text{rate of photon ejection}) \times \hbar\omega \\ &= \frac{nV_c\hbar\omega}{\tau_c} \end{aligned} \tag{6}$$

where V_c is the cavity volume.

Then we integrate (6), noting the usual trick of dummy substitution[1]:

$$\begin{aligned} E &= \int_{-\infty}^{\infty} P\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \mathrm{d}t\,\hbar\omega V_c\frac{n}{\tau_c} \\ &= \int_{-\infty}^{\infty} \mathrm{d}t\,\hbar\omega V_c\frac{N^*}{N^*_{\mathrm{th}}}\frac{n}{\tau_c} \\ &= -\int_{N_i^*}^{N_f^*} \mathrm{d}N^*\,\hbar\omega V_c\frac{f_c}{\beta} \\ &= \left(N_i^* - N_f^*\right)\hbar\omega V_c\frac{f_c}{\beta} \\ &= \underbrace{\frac{1-N_f^*/N_i^*}{\beta}}_{\eta}V_g\hbar\omega \end{aligned}$$

where η is the energy utilisation factor – it encodes how efficient a laser system is in extracting energy from population inversion.

Observe that the case with 3-level laser corresponds to severe bottlenecking where $\beta > 1$, making it less efficient than an otherwise similar 4-level system where $\beta = 1$.

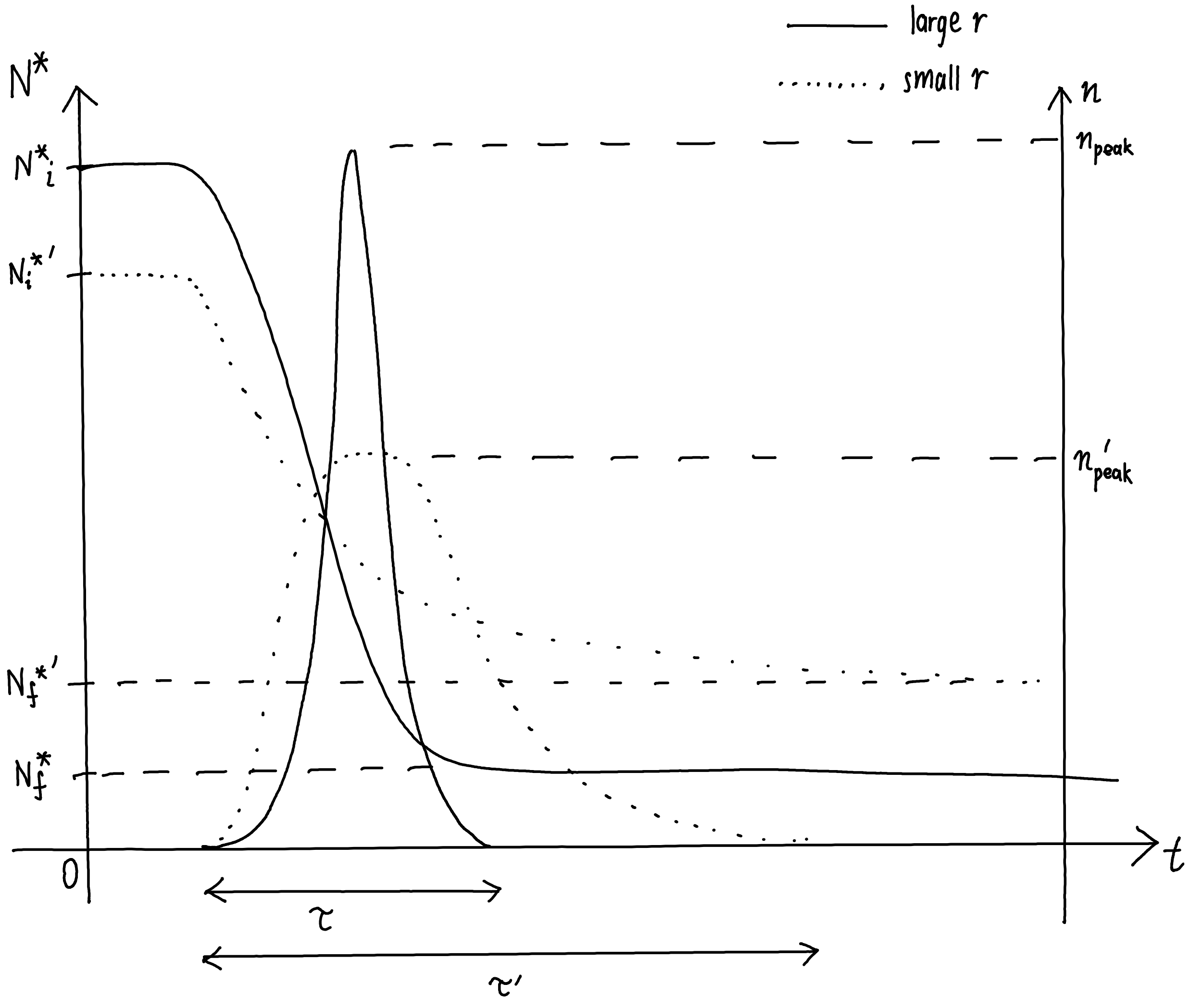
[1] Also a potential footgun here: N_f^* is not the same as N^*_{th}! The author had the misfortune of stepping on this trap and lost a couple of marks...

(d) Yet another bookwork. Overpumping ratio r is defined as

$$r = \frac{N_i^*}{N_{\text{th}}^*} \tag{7}$$

which encodes how far away the initial population inversion before pulse is from the threshold value.

Following the sketch below, we see that higher r will lead to a more intense and shorter pulse.



Now to relate between r and E, we need to first eliminate N_f^* – this can be done by dividing (2) with (5) to get:

$$\frac{\mathrm{d}n}{\mathrm{d}N^*} = -\frac{f_c}{\beta}\frac{N^*/N_{\text{th}}^* - 1}{N^*/N_{\text{th}}^*} \tag{8}$$

Solving (8) then gives:

$$\begin{aligned}
\int_{n(t=0)}^{n(t)} \mathrm{d}n &= -\int_{N_i^*}^{N^*(t)} \mathrm{d}N^* \, \frac{f_c}{\beta}\left(1 - \frac{N_{\text{th}}^*}{N^*}\right) \\
n(t) - 0 &= -\frac{f_c}{\beta}\left[\left(N^*(t) - N_i^*\right) - N_{\text{th}}^*\left(\ln N^*(t) - \ln N_i^*\right)\right] \\
\Rightarrow n(t) &= \frac{f_c}{\beta}\left[N_{\text{th}}^* \ln\left(\frac{N^*(t)}{N_i^*}\right) - \left(N^*(t) - N_i^*\right)\right] \qquad (9)
\end{aligned}$$

Substituting $n(t \to \infty) = 0$ and $N^*(t \to \infty) = N_f^*$ into (9) then gives:

$$\begin{aligned}
N_f^* - N_i^* &= N_{\text{th}}^* \ln\left(\frac{N_f^*}{N_i^*}\right) \\
\Rightarrow \underbrace{1 - \frac{N_f^*}{N_i^*}}_{\beta\eta} &= -\frac{1}{r} \ln \underbrace{\left(\frac{N_f^*}{N_i^*}\right)}_{1-\beta\eta}
\end{aligned}$$

Hence for the energy output to reach 95 % of its maximum, we need:

$$\begin{aligned}
\beta\eta &= 95\,\% \\
\Rightarrow 0.95 &= -\frac{1}{r}\ln\left(1 - 0.95\right) \\
\Rightarrow r &= 3.15
\end{aligned}$$

2. After nearly a decade of absence, non-linear optics finally appears again this year, albeit with a new perspective on wave generation!

(a) Explanation of how a non-linear response in a medium gives rise to second harmonic generation.

We begin by inspecting the definition of second-order susceptibility $\chi^{(2)}$ in a non-linear crystal:

$$P_i^{(2)} = \epsilon_0 \chi_{ijk}^{(2)} E_j E_k \tag{10}$$

The interaction between E_j and E_k to produce P_i is called wave-mixing. In the case where both terms possess the same frequency, (10) tells us that components with frequencies $\omega \pm \omega$ are produced – thereby producing a second harmonics and an optically-rectified field.

Phase difference Δk is given as:

$$\Delta k = k_3 - 2k_1$$

(b) Bookwork – this should be very familiar for those who had done parametric wave mixing for their projects.

Starting from the given $\partial A_3/\partial x$, we invoke the non-depleting-pump approximation where A_1 is constant, thus:

$$\begin{aligned}
\frac{\partial A_3}{\partial x} &= \beta A_1^2 \exp(-i\Delta k x) \\
\Rightarrow A_3(x) - A_3(0) &= \beta A_1^2 \frac{\exp(-i\Delta k x) - \exp(0)}{-i\Delta k} \\
A_3(x) &= \beta A_1^2 \frac{\exp(-i\Delta k x) - 1}{-i\Delta k} \\
&= \beta A_1^2 \exp\left(\frac{-i\Delta k x}{2}\right) \frac{\sin\left(\frac{\Delta k x}{2}\right)}{i\Delta k} \\
&= 2\beta A_1^2 \exp\left(\frac{-i\Delta k x}{2}\right) x \,\mathrm{sinc}\left(\frac{\Delta k x}{2}\right)
\end{aligned} \tag{11}$$

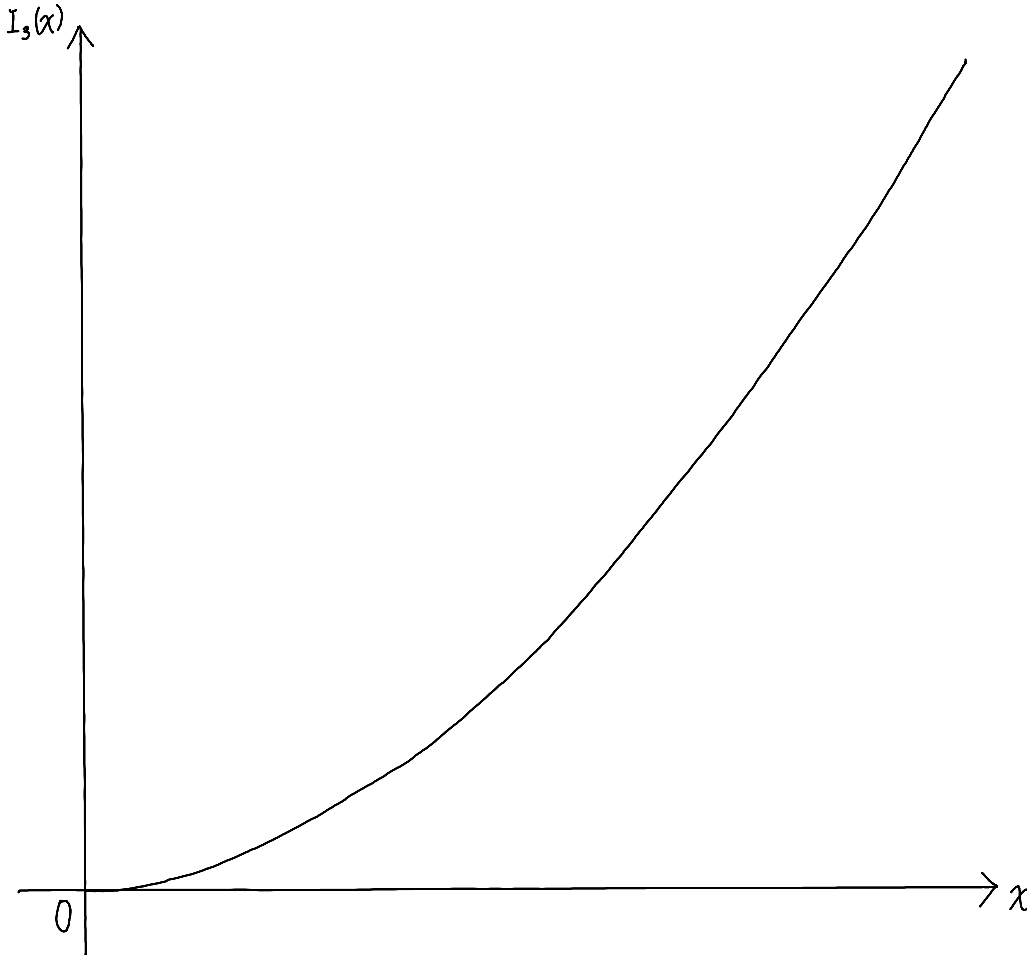
where $A_3(0)$ is assumed to be null.

Therefore the intensity of the second harmonics is:

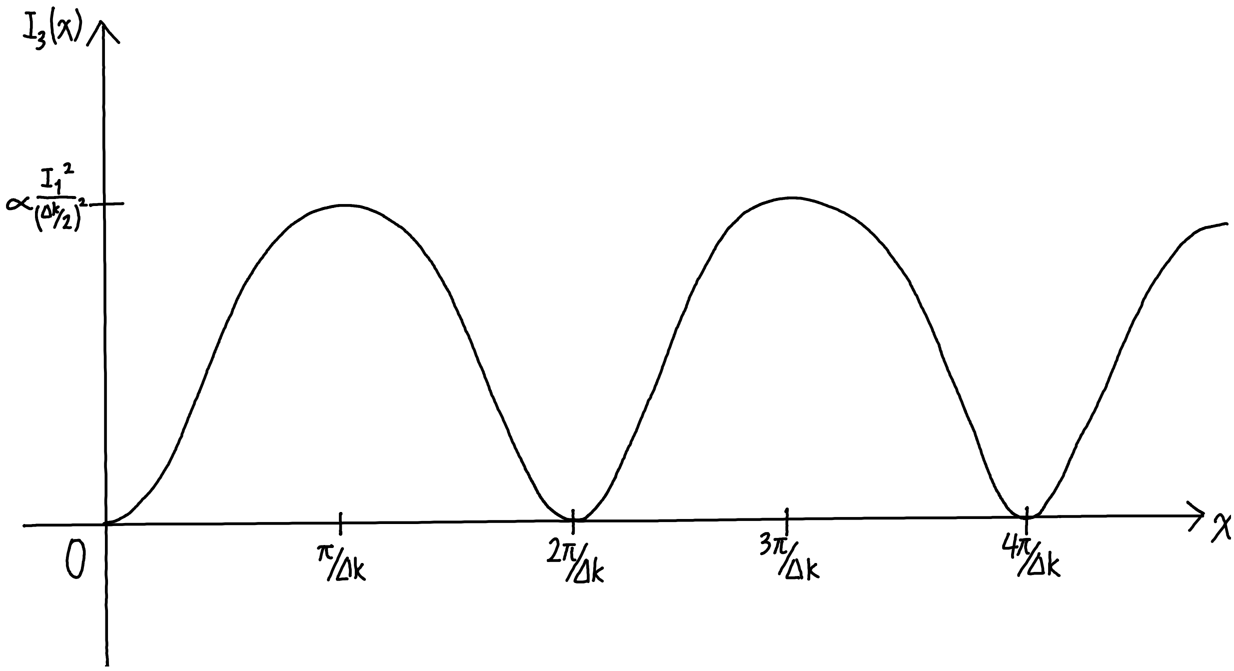
$$\begin{aligned}
I_3(x) &\propto |A_3|^2 \\
&\propto 4\beta^2 \left|A_1^2\right|^2 x^2 \mathrm{sinc}^2\left(\frac{\Delta k x}{2}\right) \\
&\propto I_1^2 \, x^2 \,\mathrm{sinc}^2\left(\frac{\Delta k x}{2}\right)
\end{aligned}$$

Now onto the sketches:

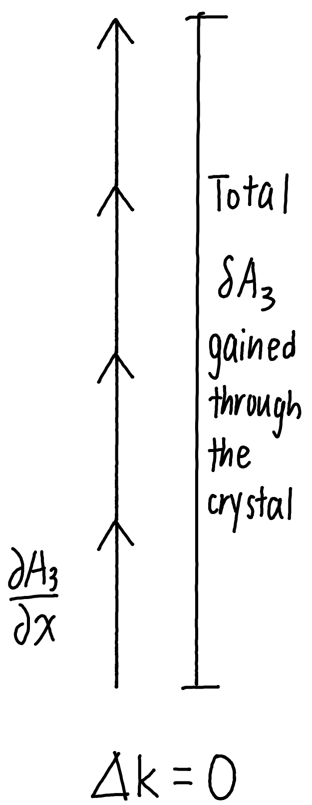
i. For $\Delta k = 0$, the sinc term vanishes, hence we have $I_3(x) \propto x^2$:

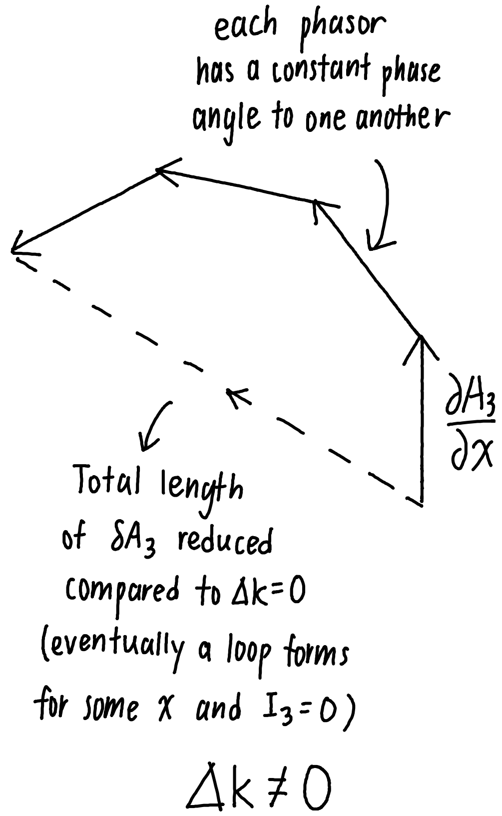


ii. For $\Delta k \neq 0$, the sinc term does not vanish and thus $I_3(x) \propto \sin^2(\Delta k x/2)$:



The difference between these two cases may be illustrated by the phasor diagram below:





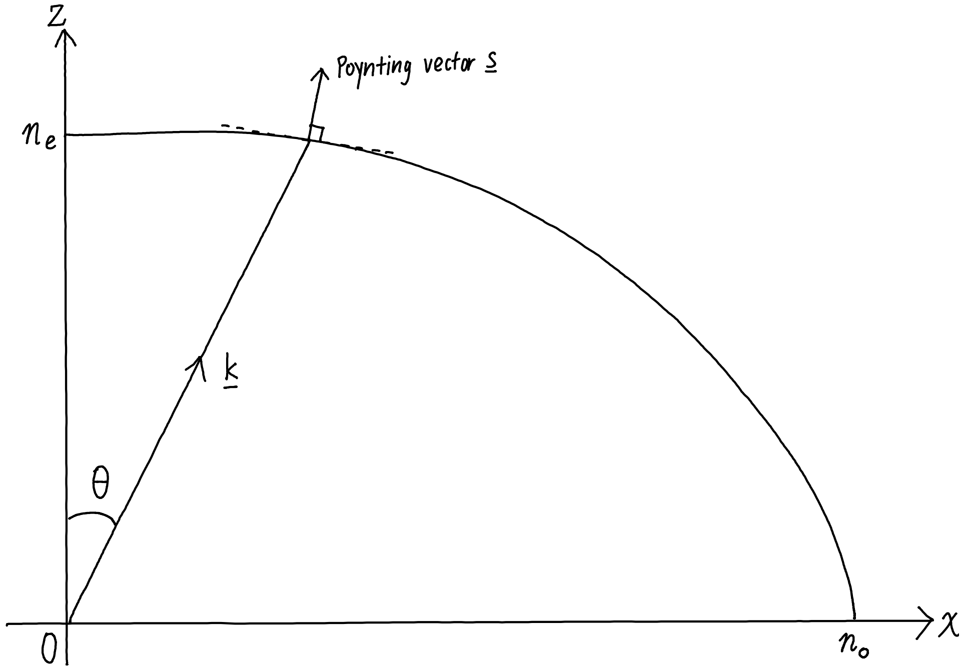
In the case of $\Delta k = 0$, the phasors in each $\mathrm{d}x$ step are perfectly aligned to one another, thereby maximising the growth in A_3 as we integrate along x.

However for $\Delta k \neq 0$, the phasors in each $\mathrm{d}x$ step would be at an angle to one another, thereby causing an oscillatory behaviour where the beam growth is stunted.

Phase-matching is thus a technique by which the condition $\Delta k = 0$ is achieved. In the optical regime, we may align the optic axis of an uniaxial crystal at an angle θ to the incoming wavevector to vary the refractive index $n(\theta)$, thereby adjusting the resulting wavevector $k = nk_0$ where k_0 is the wavevector in vacuum.

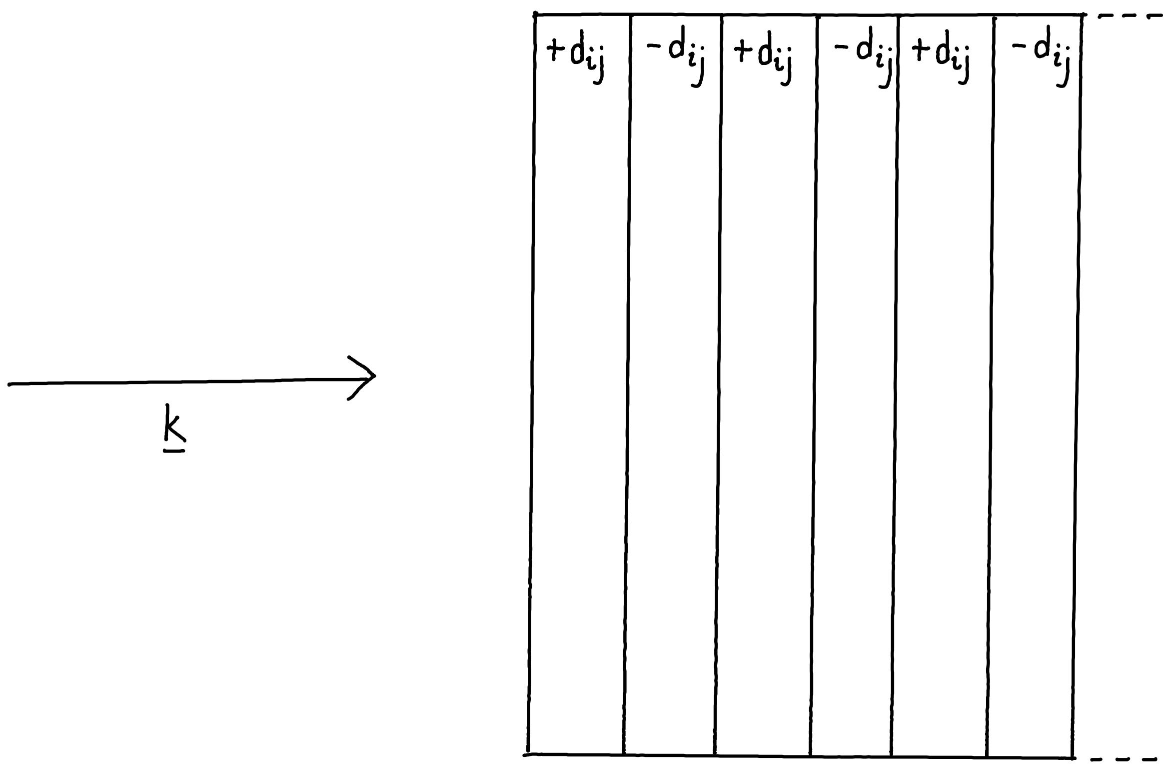
The relationship between $n(\theta)$ and θ may be illustrated by the indicatrix below (only a quadrant is drawn for simplicity):

$$\frac{1}{n^2(\theta)} = \frac{\sin^2\theta}{n_o^2} + \frac{\cos^2\theta}{n_e^2}$$



(c) The wording here can be rather confusing – the reversal of crystal axes means the inversion of the sign in d_{ij}!

Also take note that the setup here is different from the birefringent case[2]: here both the second harmonics and the pump have the same polarisation, thus the β factor differs – I shall write $\beta_{\text{pole}} \propto d_{33}$ in the poled crystal case and $\beta_{\text{uni}} \propto d_{31}$ in the uniform crystal case.



[2]In the phasor picture, we are essentially changing the direction of the rotation for every L_p so that the phasors wiggle around the net gain – this article from RP Photonics has a pretty neat explanation: `https://www.rp-photonics.com/quasi_phase_matching.html`

As suggested by the sketch of I_3 against x above, the optimal zone length L_p should be the value of x at which A_3 is maximised:

$$\begin{aligned} \sin\left(\frac{\Delta k x}{2}\right) &= 1 \\ \Rightarrow \frac{\Delta k L_p}{2} &= \frac{\pi}{2} \quad \text{picking the minimal root} \\ \Rightarrow L_p &= \frac{\pi}{\Delta k} \end{aligned} \tag{12}$$

Now suppose $\Delta k = 0$.[3]

After each zone, (11) tells us the increment in A_3 will be:

$$\delta A_3 = 2\beta_{\text{pole}} A_1^2 L_p \tag{13}$$

For N such zones, (13) tells us that the total A_3 shall be:

$$A_{3_{\text{pole}}}(NL_p) = 2\beta_{\text{pole}} A_1^2 N L_p$$

The corresponding A_3 for optimal phase matching in a uniform crystal is then:

$$A_{3_{\text{uni}}}(NL_p) = 2\beta_{\text{uni}} A_1^2 N L_p$$

The intensity ratio is then:

$$\begin{aligned} \frac{I_{3_{\text{pole}}}}{I_{3_{\text{uni}}}} &= \frac{|A_{3_{\text{pole}}}|^2}{|A_{3_{\text{uni}}}|^2} \\ &= \frac{\beta_{\text{pole}}^2}{\beta_{\text{uni}}^2} \\ &= \left(\frac{d_{33}}{d_{31}}\right)^2 \\ &= \left(\frac{d_{33}/\epsilon_0}{d_{31}/\epsilon_0}\right)^2 = 32.3 \end{aligned}$$

(d) Now recall from part b we argued that for birefringent phase-matching to work, the wavevectors between the pump and the second harmonics have to be at an angle to one another.

However the d_{33} component implies that we need both wavevectors to be polarised along the z-axis, which violates the assumption above. Therefore birefringent phase-matching is not possible for this setup.

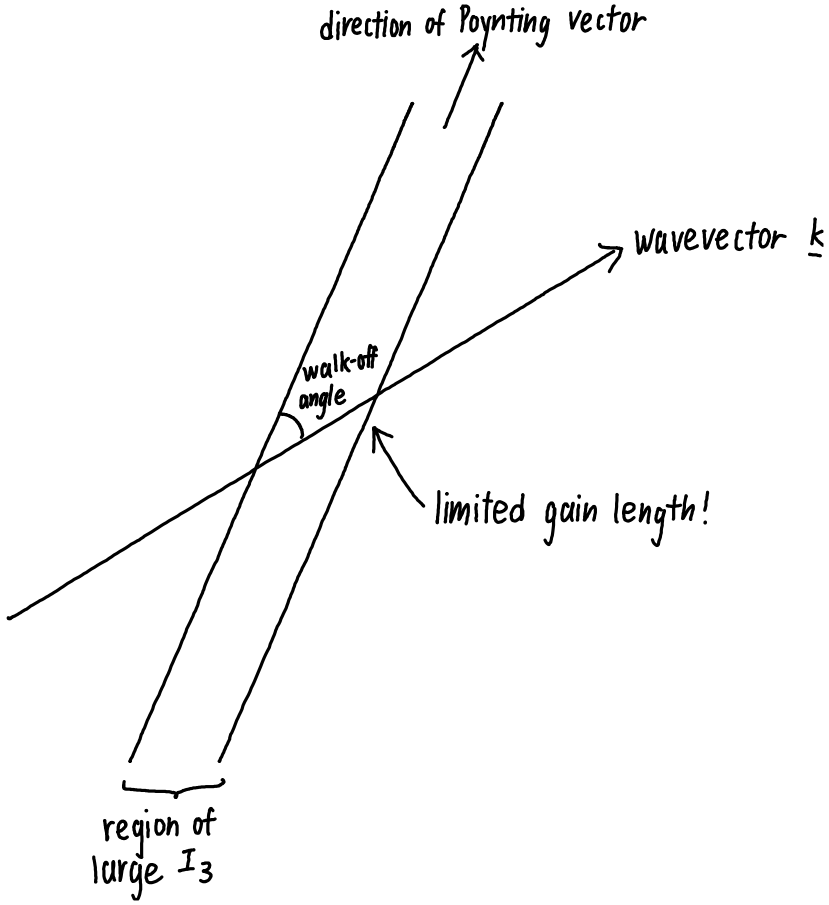
To find the value of L_p in this scenario, we first calculate Δk:

$$\begin{aligned} \Delta k &= k_e^{2\omega} - 2k_e^{\omega} \\ &= n_e^{2\omega} k_{e_0}^{2\omega} - 2n_e^{\omega} k_{e_0}^{\omega} \\ &= 2.2232 \times \frac{2\pi}{532\,\text{nm}} - 2.1470 \times \frac{2\pi}{1064\,\text{nm}} \\ &= 1.36 \times 10^7\ \text{m}^{-1} \end{aligned}$$

[3] Yeah I know the wording for this section is rather confusing especially under exam conditions – I didn't do it the right way round too!

Plugging this into (12) then gives $L_p = 2.31 \times 10^{-7}\,\text{m} = 0.231\,\mu\text{m}$, which seems reasonable given the current fabrication technique!

(e) One problem with birefringent phase-matching is the beam walk-off – note that the shape of the indicatrix implies that the direction of beam propagation (i.e. wavevector) and the intensity distribution (i.e. Poynting vector/the normal of indicatrix) differ, as indicated by the sketch below:



However by periodically reversing the axes in a poled crystal, we may reverse the walk-off periodically and keep the deviation minimal – thereby increasing the useful gain length of the crystal.

3. (DRAFT) Rabi flopping in the context of atomic/ionic quantum computing.

(a) Mathematical proof as to why the evolution of a closed quantum system is unitary.

From the spectral theorem, we may immediately quote that for an operator $\mathcal{H}$ that commutes with its conjugate, there exists a complete set[4] of orthonormal eigenbasis:

$$\mathcal{H} = \sum_i \lambda_i \left|i\right\rangle \left\langle i\right| \tag{14}$$

We then have the TISE which relates (14) to the following:

$$\mathcal{H}\left|\psi\right\rangle = E\left|\psi\right\rangle$$

where we rewrite $\left|i\right\rangle \to \left|\psi\right\rangle$, and E takes the role of energy of state $\left|\psi\right\rangle$.

We require energy to be real to make physical sense, hence the eigenvalues of $\mathcal{H}$ must be real.

This means that $\mathcal{H}$ is Hermitian, i.e. $\mathcal{H} = \mathcal{H}^\dagger$. Hence the propagator $U = \exp(-i\mathcal{H}t/\hbar)$ satisfies the following relation:

$$\begin{aligned} UU^\dagger &= \exp\left(-\frac{i\mathcal{H}t}{\hbar}\right)\exp\left(\frac{i\mathcal{H}^\dagger t}{\hbar}\right) \\ &= \exp\left(-\frac{i\mathcal{H}t}{\hbar} + \frac{i\mathcal{H}t}{\hbar}\right) \quad \text{since } [\mathcal{H},\, \mathcal{H}^\dagger] = 0 \\ &= \exp(0) \\ &= \mathbb{1} \end{aligned}$$

Thus U is unitary.

(b) Diagonalising $\mathcal{H}_2$ gives:

$$\begin{aligned} \begin{vmatrix} -\lambda & \hbar V/2 \\ \hbar V/2 & \hbar\delta - \lambda \end{vmatrix} &= 0 \\ \lambda^2 - \hbar\delta\lambda - \frac{\hbar^2 V^2}{4} &= 0 \\ \lambda &= \frac{\hbar\delta \pm \sqrt{\hbar^2\delta^2 + \hbar^2 V^2}}{2} \end{aligned} \tag{15}$$

For the case where $V = 0$, we have:

$$\lambda = \hbar\delta \Rightarrow \text{Eigenvector: } \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \lambda = 0 \Rightarrow \text{Eigenvector: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Hence a system with initial state $\left|0\right\rangle$ or $\left|1\right\rangle$ remains stationary over time in this case.

Similarly for the case where $\delta = 0$, we have:

$$\begin{aligned} \lambda = \frac{\hbar V}{2} &\Rightarrow \text{Eigenvector: } \left|+\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda = -\frac{\hbar V}{2} &\Rightarrow \text{Eigenvector: } \left|-\right\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{aligned}$$

[4]Mathematically referred to as the *spectrum* of the operator. Surprise!

So a system undergoes Rabi flopping over time:

$$|\psi(t)\rangle = A\,|+\rangle \exp\left(-\frac{iVt}{2}\right) + B\,|-\rangle \exp\left(\frac{iVt}{2}\right)$$

Note that $|0\rangle = \big(|+\rangle\ +\ |-\rangle\big)/\sqrt{2}$ and $|1\rangle = \big(|+\rangle\ -\ |-\rangle\big)/\sqrt{2}$.

An initial state of $|0\rangle$ then gives $A = B = 1/\sqrt{2}$, hence:

$$|\psi(t)\rangle = \cos\left(\frac{Vt}{2}\right)|0\rangle - i\sin\left(\frac{Vt}{2}\right)|1\rangle$$

Whereas an initial state of $|1\rangle$ gives $A = -B = 1/\sqrt{2}$:

$$|\psi(t)\rangle = -i\sin\left(\frac{Vt}{2}\right)|0\rangle + \cos\left(\frac{Vt}{2}\right)|1\rangle$$

In the case where $V \ll \delta$, we may apply perturbation theory onto the case with $V = 0$. So we rewrite $\mathcal{H} = \mathcal{H}_0 + \Delta\mathcal{H}$ where $\mathcal{H}_0$ is the Hamiltonian of the unperturbed two-level system, $\Delta\mathcal{H}$ is the control Hamiltonian with the off-diagonal terms.

The first order change in the state is given as:

$$\Delta\,|0\rangle = \sum_{i\neq 0} \frac{\langle i|\,\Delta\mathcal{H}\,|0\rangle}{E_0 - E_i} = -\frac{V}{2\delta}\,|1\rangle \sim 0$$
$$\Delta\,|1\rangle = \sum_{i\neq 1} \frac{\langle i|\,\Delta\mathcal{H}\,|1\rangle}{E_1 - E_i} = \frac{V}{2\delta}\,|0\rangle \sim 0$$

Therefore taking the eigenvectors from the case $V = 0$ is a valid approach.

We proceed to solve the TDSE with the *exact* eigenvalue[5] from (15) to get detuned Rabi flopping:

$$\begin{aligned}
|\psi(t)\rangle &= C\,|0\rangle \exp\left(-\frac{i\lambda_+ t}{\hbar}\right) + D\,|1\rangle \exp\left(-\frac{i\lambda_- t}{\hbar}\right) \\
&= C\,|0\rangle \exp\left(-it\,\frac{\delta + \sqrt{\delta^2 + V^2}}{2}\right) + D\,|1\rangle \exp\left(-it\,\frac{\delta - \sqrt{\delta^2 + V^2}}{2}\right) \\
&= \exp(-i\delta t/2)\left[C\,|0\rangle \exp\left(-it\sqrt{\delta^2+V^2}/2\right) + D\,|1\rangle \exp\left(it\sqrt{\delta^2+V^2}/2\right)\right] \qquad (16)
\end{aligned}$$

(c) The very first assumption we made to analyse an atomic/ionic quantum system is to assume that it is a two-level system.

We also changed from the lab frame into the rotating frame to get rid of the oscillating exponentials.

In addition, we have also invoked the rotating wave approximation where we ignored the counter-rotating component of the incoming wave: $\cos\omega t = {}^1\!/_2\big(\mathrm{e}^{i\omega t} + \underbrace{\mathrm{e}^{-i\omega t}}_{\text{counter-rotating component}}\big)$.

[5] Not a huge fan of the wording but this is the most logical thing I can think of!

The difference between optical and microwave transitions is the lifetime of the levels involved – as Rabi oscillation requires a fairly strongly-driven interaction to be observed, we need high-power laser to offset the inherent short lifetime of the upper optical level.

On the other hand, the microwave transitions are usually electric dipole forbidden, which implies that they are of higher order and thus less likely to occur. In addition, focusing a microwave beam to address an atom is difficult as the Abbe limit states that the minimum spot diameter $d \gtrsim \lambda/2$.

(d) In Dirac notation, we may write the Hamiltonian of the three-level system as:

$$\mathcal{H}_3 = \frac{\hbar V}{2}\Big(|2\rangle\langle 0| \ + \ |2\rangle\langle 1|\Big) + \hbar\delta\,|2\rangle\langle 2| + \text{complex conjugate (c.c.)}$$

Rewriting $|\pm\rangle = \big(|0\rangle \pm |1\rangle\big)/\sqrt{2}$ then gives:

$$\mathcal{H}_3 = \frac{\hbar V}{2}\,|2\rangle \times \Big(\sqrt{2}\,\langle +|\Big) + \hbar\delta\,|2\rangle\langle 2| + \text{c.c.}$$

$$= \hbar \begin{array}{c} \begin{array}{ccc} \langle -| & \langle +| & \langle 2| \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & V/\sqrt{2} \\ 0 & V/\sqrt{2} & \delta \end{pmatrix} \end{array}$$

Note that the off-diagonal interaction only occurs between the state $|+\rangle$ and $|2\rangle$, hence we may quote the previous part (16) for the time evolution of the state:

$$\begin{aligned} |\psi(t)\rangle =& X\,|-\rangle \exp(0) \\ &+ \exp(-i\delta t/2)\Big[Y\,|+\rangle \exp\Big(-it\sqrt{\delta^2+V^2}/2\Big) + Z\,|2\rangle \exp\Big(it\sqrt{\delta^2+V^2}/2\Big)\Big] \end{aligned}$$

For an initial state of $|0\rangle$, we have $X = Y = 1/\sqrt{2}$ and $Z = 0$. Hence the evolution will be given as:

$$\begin{aligned} |\psi(t)\rangle &= \frac{1}{\sqrt{2}}\left[|-\rangle \ + \ |+\rangle \exp\left(-it\,\frac{\delta + \sqrt{\delta^2+V^2}}{2}\right)\right] \\ &= \frac{1}{2}\Big[\big(\exp(-i\Omega t)+1\big)\,|0\rangle \ + \ \big(\exp(-i\Omega t)-1\big)\,|1\rangle\Big] \\ &= \exp\left(-\frac{i\Omega t}{2}\right)\left[\cos\left(\frac{\Omega t}{2}\right)|0\rangle - i\sin\left(\frac{\Omega t}{2}\right)|1\rangle\right] \end{aligned}$$

And this is simply a detuned Rabi flopping with effective frequency $\Omega = {}^1\!/_2\,\big(\delta + \sqrt{\delta^2+V^2}\big)$.

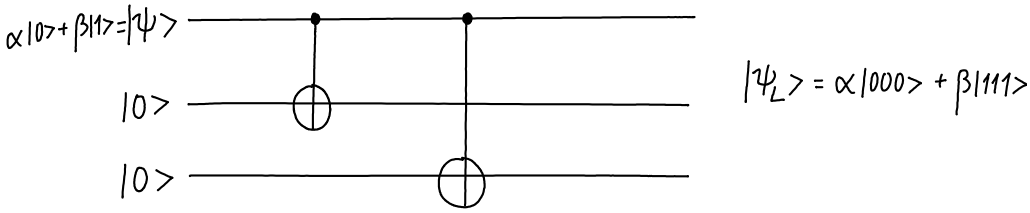
(e) The setup above corresponds to Raman transitions where an additional third level is used to facilitate transitions between two low-lying levels where direct transition is forbidden via the two-photon process.

The result shows that even though direct transition is forbidden, we may still induce Rabi flopping between the states $|0\rangle$ and $|1\rangle$.

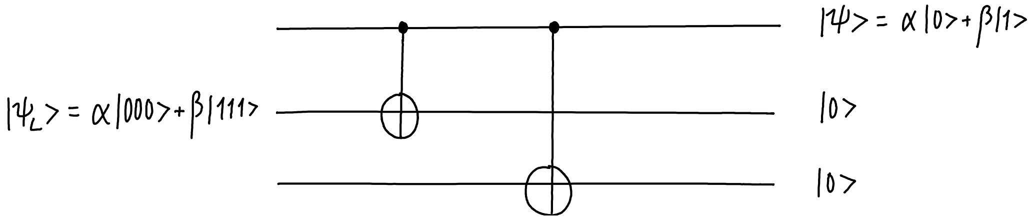
4. **(DRAFT)** Quantum error correction – I recommend Chapter 7 of the notes from A. Ekert[6] et al: `https://qubit.guide/7-stabilisers`

You may also find the notes from the MMathPhys course useful: `https://zhenyucai.com/post/intro_to_qi/AdditionalMaterials/QECNotes.pdf`

(a) Bookwork – to encode $|\psi\rangle$ into $|\psi_L\rangle$, we pass it through the network below:



To decode, we simply reverse the network:



(b) For a logical qubit $|\psi_L\rangle = \alpha|0_L\rangle + \beta|1_L\rangle = \alpha|000\rangle + \beta|111\rangle$, the following bit-flips are possible:

- 0 bit flips

  In this case we simply have $|\psi_L\rangle = \alpha|000\rangle + \beta|111\rangle$ with a probability of $(1-p)^3$.

- 1 bit flip

  Three cases arise.

$$|\psi_L\rangle = \begin{cases} \alpha|001\rangle + \beta|110\rangle & \text{3rd qubit flips} \\ \alpha|010\rangle + \beta|101\rangle & \text{2nd qubit flips} \\ \alpha|100\rangle + \beta|011\rangle & \text{1st qubit flips} \end{cases}$$

  Each of which has a probability of $p(1-p)^2$ occurring.

- 2 bit flips: Similar to the 1 bit flip case.

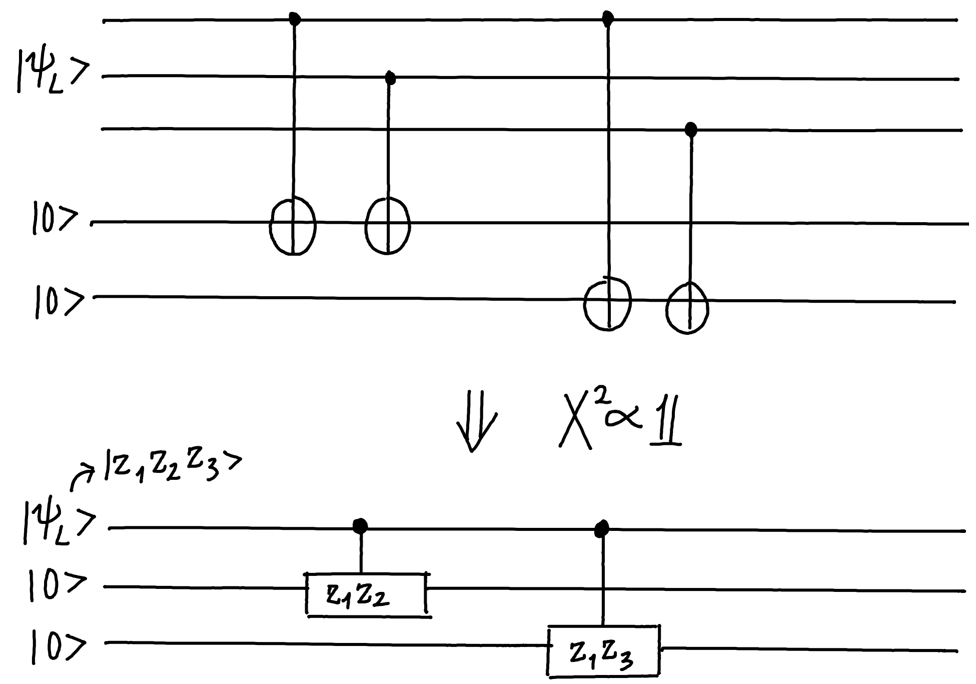
$$|\psi_L\rangle = \begin{cases} \alpha|011\rangle + \beta|100\rangle & \text{2nd and 3rd qubits flip} \\ \alpha|101\rangle + \beta|010\rangle & \text{1st and 3rd qubits flip} \\ \alpha|110\rangle + \beta|001\rangle & \text{1st and 2nd qubits flip} \end{cases}$$

  Each of which has a probability of $p^2(1-p)$ occurring.

- 3 bit flips: Similar to the 0 bit flip case, we have $|\psi_L\rangle = \alpha|111\rangle + \beta|000\rangle$ with a probability of p^3.

[6]The one who came up with the `Ekert91` protocol indeed!

(c) From the point of view of the ancillas, we are effectively measuring the parity between each physical qubit.



$$Z_i Z_j = \begin{cases} +1 & \text{Physical qubits } i \text{ and } j \text{ have identical parity} \\ -1 & \text{Physical qubits } i \text{ and } j \text{ have opposing parity} \end{cases}$$

As suggested by the diagram above, by measuring the ancillas, we may obtain the value of $z_1 z_2$ and $z_1 z_3$, which may be related to the single qubit error as follows:

z_1z_2	z_1z_3	Physical Qubits	Correction Required
$+1$	$+1$	$\alpha\|000\rangle + \beta\|111\rangle$	$\mathbb{1}$
$+1$	-1	$\alpha\|001\rangle + \beta\|110\rangle$	X_3
-1	$+1$	$\alpha\|010\rangle + \beta\|101\rangle$	X_2
-1	-1	$\alpha\|100\rangle + \beta\|011\rangle$	X_1

The encoded qubit is unaffected by the measurement as the operations above are of Pauli group $\mathcal{P}$ which have the properties of $\mathcal{P}^2 = \pm\mathbb{1}$.

One thing of note is that the table above applies to the vector space[7] spanned by the corresponding basis. Hence if there were 2 or 3 errors, we would have applied the wrong correction operation to the qubits.

In the 2 bit flips case, we would have the table above with the coefficients α and β swapped. After the erroneous correction, we will end up with the 3 bit flips case where $|\psi_L\rangle = \beta|000\rangle + \alpha|111\rangle$.

This is not a significant concern as the probability of individual bit flip p is usually very small, therefore a simultaneous n bit flips would have a probability proportional to p^n which approaches 0 quickly.

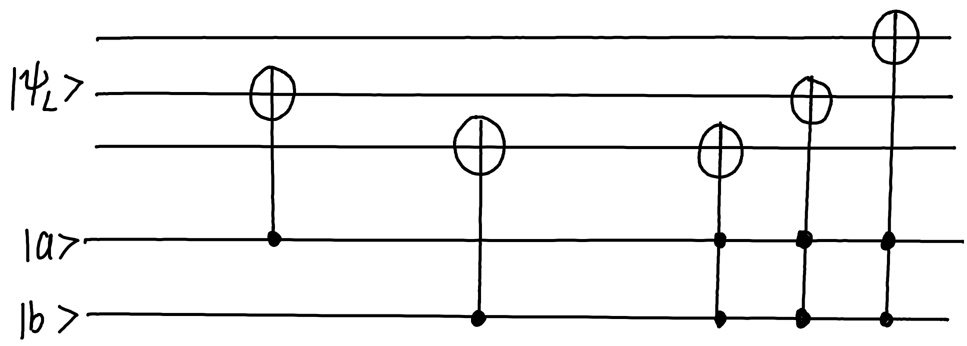
[7] A detour: the vector space described by a stabiliser is known as the code space. The operations we are doing here are simply to check if the qubit resides in the code space since the Pauli group guarantees the eigenvalue to be ± 1.

(d) If a Z error were to occur, the network will be unable to detect it as it only measures the parity of the z component between each qubit.

Conversely, if a Y error were to occur, the network will be able to detect it as we know Y = iZX, and ±Z stabilises the Z basis states of $|0\rangle$ and $|1\rangle$. Hence an error in Y would contain a distinguishable error in Z, which can then be reliably detected by the network above.

(e) We also know that HXH = Z where H is the Hadamard gate. Therefore to detect an Z error we simply add a pair of Hadamard gates in front and back of the `CNOT` gates.

(f) **(TO BE VERIFIED)** The table above suggests that we may construct a feedback circuit of `CNOT` gates to automatically correct the X error as follows:



This approach has the advantage of not having to perform measurement on the ancillas, thereby providing greater isolation to the QEC circuit which helps in guarding against decoherence. However, the lack of measurement also makes the assessment of fidelity difficult as we no longer have the explicit states of the ancillas.

(g) Nothing to do with error correction here! The best description we can have for a qubit whose state is unknown due to a random process is naturally a density matrix.

So in the first case we may construct the density matrix as follows:

$$\begin{aligned}
\rho_{\pm\phi} &= \frac{1}{2}\left|\psi^{+\phi}\right\rangle\left\langle\psi^{+\phi}\right| + \frac{1}{2}\left|\psi^{-\phi}\right\rangle\left\langle\psi^{-\phi}\right| \\
&= \frac{1}{2}\left[\begin{pmatrix}\alpha \\ \beta\mathrm{e}^{+i\phi}\end{pmatrix}\begin{pmatrix}\alpha^* & \beta^*\mathrm{e}^{-i\phi}\end{pmatrix} + \begin{pmatrix}\alpha \\ \beta\mathrm{e}^{-i\phi}\end{pmatrix}\begin{pmatrix}\alpha^* & \beta^*\mathrm{e}^{+i\phi}\end{pmatrix}\right] \\
&= \frac{1}{2}\left[\begin{pmatrix}|\alpha|^2 & \alpha\beta^*\mathrm{e}^{-i\phi} \\ \alpha^*\beta\mathrm{e}^{+i\phi} & |\beta|^2\end{pmatrix} + \begin{pmatrix}|\alpha|^2 & \alpha\beta^*\mathrm{e}^{+i\phi} \\ \alpha^*\beta\mathrm{e}^{-i\phi} & |\beta|^2\end{pmatrix}\right] \\
&= \begin{pmatrix}|\alpha|^2 & \alpha\beta^*\cos\phi \\ \alpha^*\beta\cos\phi & |\beta|^2\end{pmatrix}
\end{aligned}$$

And for the case where the qubit experiences a phase flip with probability q, we have:

$$\begin{aligned}
\phi_{\mathrm{Z}} &= (1-q)\left|\psi\right\rangle\left\langle\psi\right| + q\left|\psi^{\mathrm{Z}}\right\rangle\left\langle\psi^{\mathrm{Z}}\right| \\
&= (1-q)\begin{pmatrix}\alpha \\ \beta\end{pmatrix}\begin{pmatrix}\alpha^* & \beta^*\end{pmatrix} + q\begin{pmatrix}\alpha \\ -\beta\end{pmatrix}\begin{pmatrix}\alpha^* & -\beta^*\end{pmatrix} \\
&= (1-q)\begin{pmatrix}|\alpha|^2 & \alpha\beta^* \\ \alpha^*\beta & |\beta|^2\end{pmatrix} + q\begin{pmatrix}|\alpha|^2 & -\alpha\beta^* \\ -\alpha^*\beta & |\beta|^2\end{pmatrix} \\
&= \begin{pmatrix}|\alpha|^2 & (1-2q)\,\alpha\beta^* \\ (1-2q)\,\alpha^*\beta & |\beta|^2\end{pmatrix}
\end{aligned}$$

Comparing the terms simply yields:

$$\begin{aligned}
1-2q &= \cos\phi \\
\Rightarrow q &= \frac{1-\cos\phi}{2}
\end{aligned}$$

5. Classic Jaynes-Cummings question. The recent topic of separable/non-separable vector spaces from 2023 makes this question a rather familiar one[8].

(a) We begin by considering a cavity with (single mode) EM field energy density:

$$U = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2$$

By second quantisation, we may make the (up to some constant that makes up $\hbar\omega$) substitutions $E \to {}^1\!/\!_{\sqrt{2}}(a + a^\dagger)$ and $B \to {}^i\!/\!_{\sqrt{2}}(a - a^\dagger)$. The resulting Hamiltonian is then:

$$\begin{aligned}
\mathcal{H} &= \frac{1}{4}\hbar\omega\left[\left(a + a^\dagger\right)^2 - \left(a - a^\dagger\right)^2\right] \\
&= \frac{\hbar\omega}{4}\left[2\underbrace{aa^\dagger}_{1+a^\dagger a} + 2a^\dagger a\right] \\
&= \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)
\end{aligned}$$

This corresponds nicely to a quantum harmonic oscillator – in fact the offset of ${}^1\!/\!_2$ suggests that energy is non-zero even when there is no photon!

(b) Usual Jaynes-Cummings Hamiltonian *on resonance*, be careful about the Rabi frequency though[9]:

$$\mathcal{H} = \underbrace{\hbar\omega\frac{\sigma_z}{2}}_{\text{Atomic energy}} + \underbrace{\hbar\omega a^\dagger a}_{\text{Photonic energy}} + \underbrace{\frac{\hbar\Omega_a}{2}(a^\dagger\sigma_- + a\sigma_+)}_{\text{Mode } a \text{ coupling}} + \underbrace{\frac{\hbar\Omega_b}{2}(a^\dagger\sigma_- + a\sigma_+)}_{\text{Mode } b \text{ coupling}} \tag{17}$$

(c) For this part we set $\Omega_b = 0$, then solve the TDSE with Jaynes-Cummings Hamiltonian.

My personal favourite way of solving this is by diagonalising the Hamiltonian (here I ignored mode b for clarity):

$$\begin{array}{cc} & \begin{array}{cc} \langle \mathrm{e},\, 0_a| & \langle \mathrm{g},\, 1_a| \end{array} \\ H = & \begin{pmatrix} \frac{\hbar\omega}{2} & \frac{\hbar\Omega_a}{2} \\ \frac{\hbar\Omega_a}{2} & \hbar\omega - \frac{\hbar\omega}{2} \end{pmatrix} \end{array}$$

$$\Rightarrow \begin{vmatrix} \frac{\hbar\omega}{2} - E & \frac{\hbar\Omega_a}{2} \\ \frac{\hbar\Omega_a}{2} & \frac{\hbar\omega}{2} - E \end{vmatrix} = 0$$

$$\Rightarrow E_\pm = \frac{\hbar\omega}{2} \pm \frac{\hbar\Omega_a}{2}$$

[8] That being said, the author still had the ingenuity of quoting double the value of Rabi frequency and muddling up the last part!

[9] Tip: since Rabi flopping is an observable quantity, the frequency that goes into the amplitude should be halved so that the correct frequency is recovered upon squaring (you can see that $\cos^2$ is frequency doubling by writing it in exponentials).

And the corresponding dressed states are $|\pm\rangle = {}^1\!/\!_{\sqrt{2}}\big(|\mathrm{e},\,0_a\rangle \pm |\mathrm{g},\,1_a\rangle\big)$.

TDSE then tells us the general solution would be:

$$|\psi(t)\rangle = A\exp\left(-\frac{iE_+t}{\hbar}\right)|+\rangle + B\exp\left(-\frac{iE_-t}{\hbar}\right)|-\rangle$$

Substituting the initial condition of $|\psi(0)\rangle = |\mathrm{e},\,0_a\rangle = {}^1\!/\!_{\sqrt{2}}\big(|+\rangle + |-\rangle\big)$ then gives:

$$A = B = \frac{1}{\sqrt{2}}$$

Further simplification then gives the familiar Rabi flopping:

$$\begin{aligned}|\psi(t)\rangle &= \frac{1}{\sqrt{2}}\exp\left(-\frac{i\omega t}{2}\right)\left[\frac{1}{\sqrt{2}}\left(\exp\left(-\frac{\Omega_a t}{2}\right) + \exp\left(\frac{\Omega_a t}{2}\right)\right)|\mathrm{e},\,0_a\rangle\right.\\ &\qquad\left. + \frac{1}{\sqrt{2}}\left(\exp\left(-\frac{\Omega_a t}{2}\right) - \exp\left(\frac{\Omega_a t}{2}\right)\right)|\mathrm{g},\,1_a\rangle\right]\\ &= \exp\left(-\frac{i\omega t}{2}\right)\left[\cos\left(\frac{\Omega_a t}{2}\right)|\mathrm{e},\,0_a\rangle - i\sin\left(\frac{\Omega_a t}{2}\right)|\mathrm{g},\,1_a\rangle\right]\end{aligned} \tag{18}$$

(d) At $t = \tau = \pi/2\Omega_a$, (18) becomes:

$$|\psi(\tau)\rangle = \exp\left(-\frac{i\omega\pi}{4\Omega_a}\right)\left[\cos\left(\frac{\pi}{4}\right)|\mathrm{e},\,0_a\rangle - i\sin\left(\frac{\pi}{4}\right)|\mathrm{g},\,1_a\rangle\right]$$

Now we turn mode a off and mode b on, now we have coupling between the states $|\mathrm{e},\,0_b\rangle$ and $|\mathrm{g},\,1_b\rangle$.

Since a and b do not have any direct coupling, the state $|\mathrm{g},\,1_a\rangle$ remains stationary.

Repeating the same procedure as the previous part then yields:

$$|\psi'(t)\rangle = C\exp\left(-\frac{iE'_+t}{\hbar}\right)|+'\rangle + D\exp\left(-\frac{iE'_-t}{\hbar}\right)|-'\rangle$$

where the primed variables are of mode b instead of a.

However this time we have a different initial state of $|\psi'(0)\rangle = {}^1\!/\!_{\sqrt{2}}\,|\mathrm{e},\,0_b\rangle = {}^1\!/\!_2\big(|+'\rangle + |-'\rangle\big)$:

$$C = D = \frac{1}{2}$$

So the new state at time t shall be:

$$\begin{aligned}|\Psi(t)\rangle = \frac{1}{\sqrt{2}}\exp\left(-\frac{i\omega t}{2}\right)&\left[\cos\left(\frac{\Omega_b t}{2}\right)|\mathrm{e},\,0_a,\,0_b\rangle - i\sin\left(\frac{\Omega_b t}{2}\right)|\mathrm{g},\,0_a,\,1_b\rangle\right.\\ &\left. - i\exp\left(-\frac{i\omega\pi}{4\Omega_a}\right)|\mathrm{g},\,1_a,\,0_b\rangle\right]\end{aligned}$$

Given the atom is measured to be in the ground state, the state of the field is then the following mixed state:

$$\rho = \frac{{}^1\!/\!_2}{{}^1\!/\!_2[\sin^2(\Omega_b t/2) + 1]}\left[\sin^2\left(\frac{\Omega_b t}{2}\right)|0_a,\,1_b\rangle\langle 0_a,\,1_b| + |1_a,\,0_b\rangle\langle 1_a,\,0_b|\right]$$

where the denominator simply normalises the probabilities of the density matrix.

To examine if they are separable, we shall inspect the density matrices of the subsystems:

$$\begin{aligned}\rho_{a/b} &= \mathrm{tr}_{a/b}\,\rho \\ &= \frac{1}{\sin^2(\Omega_b t/2)+1}\left[\sin^2\left(\frac{\Omega_b t}{2}\right)|0_a/1_b\rangle\langle 0_a/1_b| + |1_a/0_b\rangle\langle 1_a/0_b|\right]\end{aligned}$$

Note that by symmetry, ρ_a and ρ_b share the same purity:

$$\mathrm{tr}(\rho_{a/b}^2) = \frac{\sin^4(\Omega_b t/2)+1}{[\sin^2(\Omega_b t/2)+1]^2}$$

The states are separable when the purity is 1:

$$\begin{aligned}\sin^4\left(\frac{\Omega_b t}{2}\right) + 2\sin^2\left(\frac{\Omega_b t}{2}\right) + 1 &= \sin^4\left(\frac{\Omega_b t}{2}\right) + 1 \\ \Rightarrow \sin\left(\frac{\Omega_b t}{2}\right) &= 0 \\ \Rightarrow t &= \frac{2n\pi}{\Omega_b}\end{aligned}$$

where $n \in \mathbb{Z}$. At other times the states are entangled.

6. Introductory information theory with qubits.

(a) To examine entanglement, we first inspect the density matrices of the subsystems (Alice and Bob correspond to the left and right side of the notation):

$$\begin{aligned}\rho = &|a|^2 |0+\rangle\langle 0+| \\ &+ ab^* |0+\rangle\langle 1-| \\ &+ ba^* |1-\rangle\langle 0+| \\ &+ |b|^2 |1-\rangle\langle 1-|\end{aligned}$$

In Dirac notation, the trace operation is defined as: $\operatorname{tr} |a\rangle\langle b| = \langle b|a\rangle$, thus we have:

$$\begin{aligned}\rho_{\mathrm{A}} &= \operatorname{tr}_{\mathrm{B}} \rho \\ &= |a|^2 \langle +|+\rangle |0\rangle\langle 0| \ + \ ab^* \langle +|-\rangle |0\rangle\langle 1| \\ &\qquad + ba^* \langle -|+\rangle |1\rangle\langle 0| \ + \ |b|^2 \langle -|-\rangle |1\rangle\langle 1| \\ &= |a|^2 |0\rangle\langle 0| + |b|^2 |1\rangle\langle 1| \\ \\ \rho_{\mathrm{B}} &= |a|^2 |+\rangle\langle +| + |b|^2 |-\rangle\langle -| \quad \text{similarly}\end{aligned}$$

Now we see that both density matrices are simply mixed states for non-zero a and b, so the system state has to be entangled.

Alternatively, calculating the purity for ρ_{A} and ρ_{B} would also work.

(b) The density matrix for the system:

$$\rho = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|\Phi^+\rangle\langle\Phi^+|$$

where $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is a Bell basis state.

In matrix form it is simply:

$$\rho = \begin{pmatrix} 3/4 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/4 & 0 & 0 & 1/4 \end{pmatrix}$$

Following the instruction of the question, we diagonalise the density matrix:

$$\begin{vmatrix} 3/4 - \lambda & 0 & 0 & 1/4 \\ 0 & -\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 0 \\ 1/4 & 0 & 0 & 1/4 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 \left[\left(\frac{3}{4} - \lambda\right)\left(\frac{1}{4} - \lambda\right) - \left(\frac{1}{4}\right)^2 \right] = 0$$

$$\Rightarrow \lambda = 0 \quad \text{(Trivial root)} \qquad\qquad \text{or} \qquad\qquad \begin{aligned} \lambda^2 - \lambda + \frac{1}{8} &= 0 \\ \lambda &= \frac{1 \pm \sqrt{1 - 4 \times 1/8}}{2} \\ &= \frac{1 \pm \sqrt{1/2}}{2} \end{aligned}$$

Now recall the definition of von Neumann entropy:

$$S = -\mathrm{tr}\left(\rho \log_2 \rho\right)$$

And further note that trace is basis invariant, therefore we have:

$$\begin{aligned}\rho \log_2 \rho &= \mathrm{diag}\left(0,\, 0,\, \frac{1+\sqrt{1/2}}{2},\, \frac{1-\sqrt{1/2}}{2}\right) \\ &\quad\cdot \mathrm{diag}\left(\log_2 0,\, \log_2 0,\, \log_2\left(\frac{1+\sqrt{1/2}}{2}\right),\, \log_2\left(\frac{1-\sqrt{1/2}}{2}\right)\right) \\ &= \mathrm{diag}\left(0,\, 0,\, -0.195,\, -0.406\right)\end{aligned}$$

Hence $S(\mathrm{AB}) = 0.601$.

(c) Perform partial trace on the system to get the density matrix for each side. By symmetry we simply consider one side:

$$\begin{aligned}\rho_{\mathrm{A}} &= \begin{pmatrix} \mathrm{tr}\begin{pmatrix} 3/4 & 0 \\ 0 & 0 \end{pmatrix} & \mathrm{tr}\begin{pmatrix} 0 & 1/4 \\ 0 & 0 \end{pmatrix} \\ \mathrm{tr}\begin{pmatrix} 0 & 0 \\ 1/4 & 0 \end{pmatrix} & \mathrm{tr}\begin{pmatrix} 0 & 0 \\ 0 & 1/4 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} 3/4 & 0 \\ 0 & 1/4 \end{pmatrix}\end{aligned}$$

We then have the entropy for each of the subsystem as:

$$\begin{aligned}S(\mathrm{A}) = S(\mathrm{B}) &= -\mathrm{tr}\left[\mathrm{diag}\left(3/4,\, 1/4\right) \cdot \mathrm{diag}\left(\log_2\left(3/4\right),\, \log_2\left(1/4\right)\right)\right] \\ &= -\mathrm{tr}\left[\mathrm{diag}\left(-0.311,\, -0.5\right)\right] \\ &= 0.811\end{aligned}$$

(d) As per the definition of mutual information, we have:

$$\begin{aligned}I(\mathrm{A}:\mathrm{B}) &= S(\mathrm{A}) + S(\mathrm{B}) - S(\mathrm{AB}) \\ &= 1.022\end{aligned}$$

(e) From part c, we have

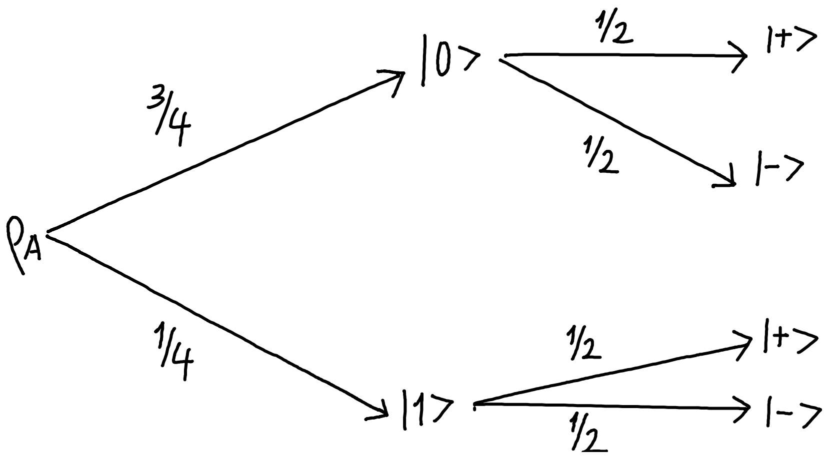
$$\rho_{\mathrm{A}} = \frac{3}{4}\left|0\right\rangle\left\langle 0\right| + \frac{1}{4}\left|1\right\rangle\left\langle 1\right| \tag{19}$$

Rewriting (19) in the basis $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$ gives:

$$\begin{aligned}\rho_{\mathrm{A}} &= \frac{3}{8}\Big(|+\rangle\langle+| \;+\; |+\rangle\langle-| \;+\; |-\rangle\langle+| \;+\; |-\rangle\langle-|\Big) \\ &\quad+ \frac{1}{8}\Big(|+\rangle\langle+| \;-\; |+\rangle\langle-| \;-\; |-\rangle\langle+| \;+\; |-\rangle\langle-|\Big) \\ &= \frac{1}{2}|+\rangle\langle+| \;+\; \frac{1}{4}|+\rangle\langle-| \;+\; \frac{1}{4}|-\rangle\langle+| \;+\; \frac{1}{2}|-\rangle\langle-|\end{aligned}$$

From this we can immediately read that the probability of either outcome is $^1\!/_2$.

Alternatively we may think of the mixed state above as a lucky draw:



From the tree diagram, it is straightforward to see that the probability of each outcome is $^1\!/_2$.

However the very act of measurement has de-cohered Bob's qubit, hence the best description he has for his qubit would be:

$$\begin{aligned}\rho_{\mathrm{B}} &= \mathrm{diag}\left(^1\!/_2,\ ^1\!/_2\right)\\ &= \frac{1}{2}\left|0\right\rangle\left\langle 0\right| + \frac{1}{2}\left|1\right\rangle\left\langle 1\right|\end{aligned}$$

(f) The corresponding entropy for Bob's qubit would then be:

$$\begin{aligned}S(\mathrm{B}) &= -\mathrm{tr}\left[\mathrm{diag}\left(^1\!/_2,\ ^1\!/_2\right)\cdot\mathrm{diag}\left(\log_2\left(^1\!/_2\right),\ \log_2\left(^1\!/_2\right)\right)\right]\\ &= -\mathrm{tr}\left[\mathrm{diag}\left(-0.5,\ -0.5\right)\right]\\ &= 1\end{aligned}$$

Bob's entropy has increased as a result of the decoherence – he no longer has any information on the state of his qubit!