UNOFFICIAL SOLUTIONS BY TheLongCat

C2: LASER SCIENCE AND QUANTUM INFORMATION PROCESSING

TRINITY TERM 2018

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Turn over as you please – we are NOT under exam conditions here.

1. Chirp-pulsed amplification and the effects of phase delay in a Michelson interferometer.

(a) Phase acquired by a pulse in a dispersive medium of length z (assuming homogeneity):

$$\begin{aligned}\phi(\omega) &= k(\omega)\cdot z \\ &= \phi(\omega_0) + \phi^{(1)}(\omega_0)\cdot(\omega-\omega_0) + \frac{1}{2}\phi^{(2)}(\omega_0)\cdot(\omega-\omega_0)^2 + \dots\end{aligned}$$

by Taylor expanding around the carrier frequency ω_0.

Here group delay is:

$$\begin{aligned}\phi^{(1)} &= \left.\frac{\mathrm{d}\phi}{\mathrm{d}\omega}\right|_{\omega_0} \\ &= \left.\frac{\mathrm{d}k}{\mathrm{d}\omega}\right|_{\omega_0}\cdot z \\ &= \frac{z}{v_g(\omega_0)} \quad \text{where } v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} \text{ is group velocity}\end{aligned}$$

Group delay dispersion (GDD):

$$\begin{aligned}\phi^{(2)} &= \left.\frac{\mathrm{d}^2\phi}{\mathrm{d}\omega^2}\right|_{\omega_0} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{z}{v_g}\right)\right|_{\omega_0} \\ &= \left.\frac{\mathrm{d}v_g^{-1}}{\mathrm{d}\omega}\right|_{\omega_0}\cdot z\end{aligned}$$

Now consider the time taken for a spectral component of frequency ω to pass through the medium:

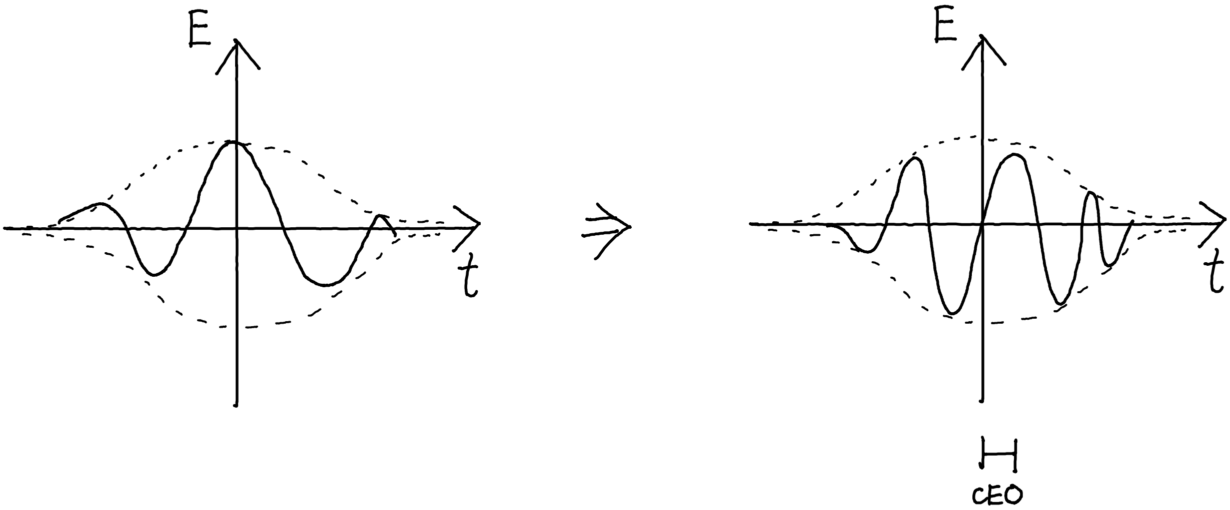
$$\phi(\omega) = \int T(\omega)\,\mathrm{d}\omega$$

$$\Rightarrow T(\omega) = \frac{\partial\phi}{\partial\omega}$$

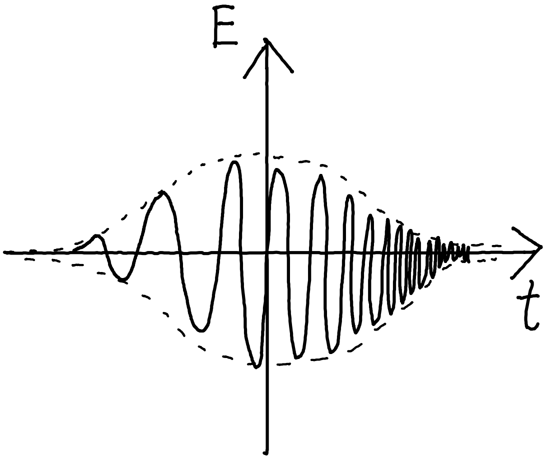
$$= T(\omega_0) + \left.\frac{\partial T}{\partial\omega}\right|_{\omega_0}(\omega-\omega_0) + \dots \quad \text{by Taylor expansion}$$

$$= \phi^{(1)} + \phi^{(2)}\cdot(\omega-\omega_0) + \dots$$

In general phase velocity $v_p = \dfrac{\omega_0}{k(\omega_0)} \neq v_g$, so what $\phi^{(1)}$ is responsible for is introducing the carrier-envelope offset (CEO) where the peak of a pulse drifts away from the envelope peak.



As for $\phi^{(2)}$, note that it has a linear contribution to $T(\omega) \Rightarrow \phi^{(2)}$ separates each frequency component and distorts the pulse, e.g. for $\phi^{(2)} > 0$ we have positive chirp where higher frequencies are pushed towards the back of a pulse.



Ultrashort laser pulses are typically generated by modelocked lasers, which usually possess low energy ($\sim$ nJ). However, owing to its short pulse duration, simple amplification would lead to extremely large peak intensities.

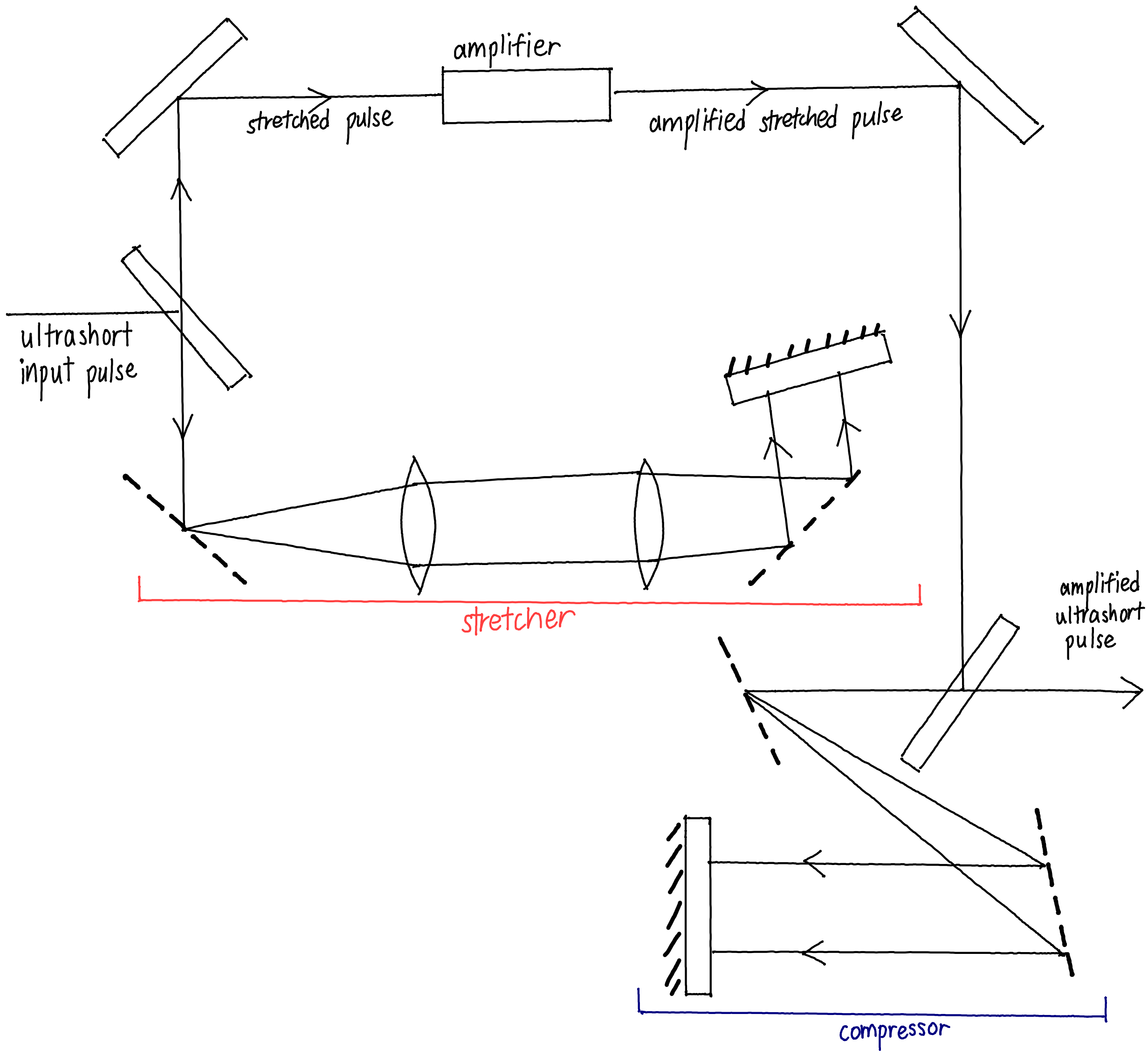
Multiple complications arise as a results:

1. First we have self-modulation of the pulse due to the large B-integral. This leads to complicated phase structure that is difficult to undo.

2. We also have self-focusing that leads to regions of intensities so great that the optical elements may be damaged.

To counter this, chirped-pulse amplification (CPA) is used instead to stretch the input pulse before amplification, thereby avoiding large peak intensities. The amplified pulse is then compressed before output, leading to high power ultrashort laser pulse.

Schematic of a CPA system:



The stretcher applies positive GDD since most optical elements along the chain usually possess positive dispersion, hence avoiding accidental compression of the pulse. The stretched pulse is then amplified and passed to the compressor where negative GDD compresses the pulse back to the original length.

(b) From the given Gaussian pulse:

$$\begin{aligned} E(t) &= E_0 \exp\left(-\Gamma t^2\right) \exp\left(-i\omega_0 t\right) \\ &= E_0 \exp\left(-at^2\right) \exp\left(-i(\omega_0 + bt)t\right) \end{aligned}$$

At HWHM $t = t_{1/2}$, we have:

$$\begin{aligned}
\left|E(t_{1/2})\right|^2 &= \frac{1}{2}\left|E(0)\right|^2 \\
\Rightarrow \exp\left(-2at_{1/2}\right) &= \frac{1}{2} \\
t_{1/2} &= \left[\frac{\ln 2}{2a}\right]^{\frac{1}{2}}
\end{aligned}$$

Thus FWHM τ is:

$$\begin{aligned}
\tau &= 2t_{1/2} \\
&= 2\left[\frac{\ln 2}{2a}\right]^{\frac{1}{2}} \\
&= \left[\frac{2\ln 2}{a}\right]^{\frac{1}{2}}
\end{aligned}$$

We also have phase gained by the pulse in time t: $\phi(t) = (\omega_0 + bt)t$.

So we have instantaneous frequency:

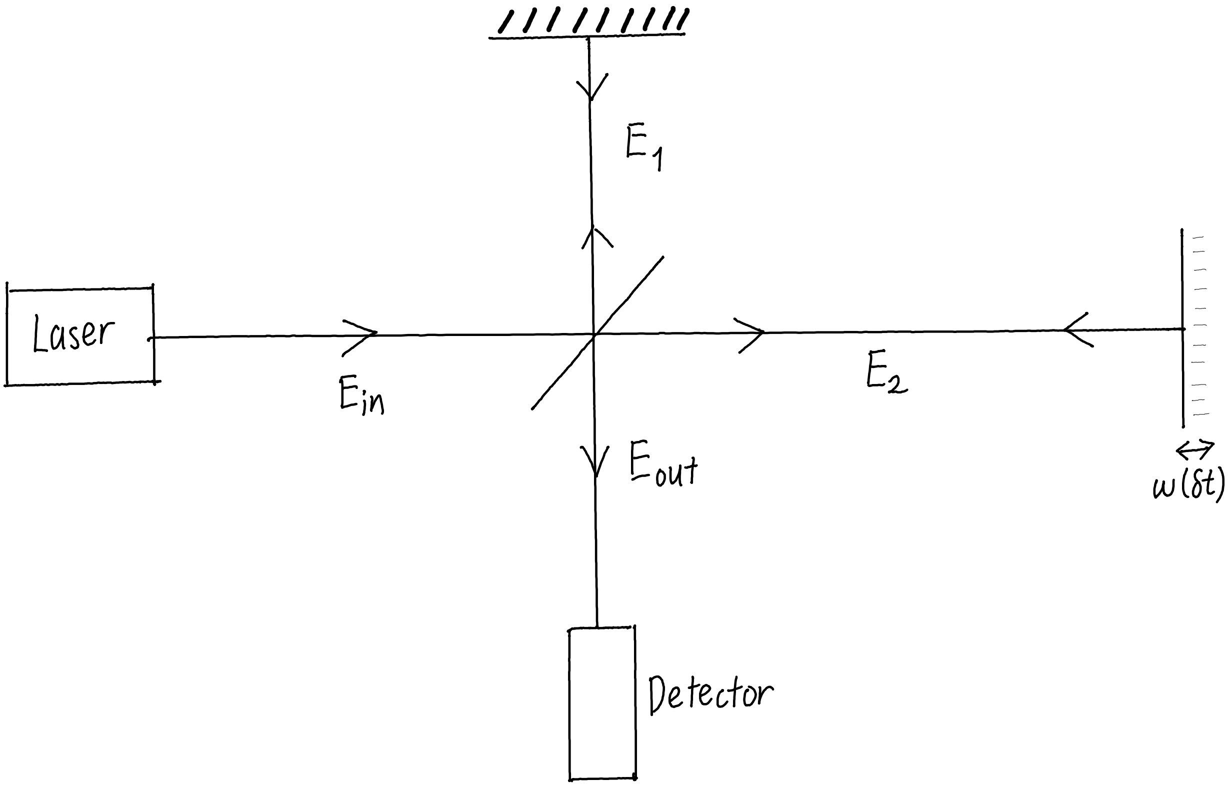
$$\begin{aligned}
\omega(t) &= \frac{\partial \phi}{\partial t} \\
&= \omega_0 + 2bt
\end{aligned}$$

Recall that GDD is given by:

$$\begin{aligned}
\phi^{(2)} &= \left.\frac{\partial^2 \phi}{\partial \omega^2}\right|_{\omega_0} \\
&= \left[\frac{\partial}{\partial \omega}\left(\frac{\partial \phi}{\partial t}\cdot\frac{\partial t}{\partial \omega}\right)\right]_{\omega_0} \\
&= \left[\frac{\partial}{\partial \omega}\left[(2b)^{-1}\left(\omega_0 + 2bt\right)\right]\right]_{\omega_0} \\
&= \frac{\partial}{\partial t}\left[(2b)^{-1}\left(\omega_0 + 2bt\right)\right]^{1}_{t=0}\cdot\left.\frac{\partial t}{\partial \omega}\right|_{t=0} \\
&= \frac{1}{2b}
\end{aligned}$$

Hence for positive chirp, $b > 0 \Rightarrow \mathrm{Im}(\Gamma) > 0$.

(c) Schematic of Michelson interferometer:



After the beam splitter, we have $E_{\text{in}} \to \frac{1}{2}E_{\text{in}}$.

Making the substitution $t \to t + \delta t$ then gives:

$$\begin{aligned} E_2(t) &= \frac{E_0}{2}\,\mathrm{e}^{-\Gamma(t+\delta t)^2}\mathrm{e}^{-i\omega_0(t+\delta t)} \\ &= \frac{E_0}{2}\,\mathrm{e}^{-a(t+\delta t)^2}\mathrm{e}^{-ib(t+\delta t)^2}\mathrm{e}^{-i\omega_0(t+\delta t)} \end{aligned}$$

Assuming $\delta t \ll t$, we then have $\mathrm{e}^{-a(t+\delta t)^2} \approx \mathrm{e}^{-at^2}$:

$$\begin{aligned} \Rightarrow E_2(t) &= \frac{E_0}{2}\,\mathrm{e}^{-at^2}\mathrm{e}^{-i\omega_0 t}\mathrm{e}^{-ibt^2-2ibt(\delta t)-ib(\delta t)^2-i\omega_0(\delta t)} \\ &= \frac{E_0}{2}\,\mathrm{e}^{-at^2}\mathrm{e}^{-i[\omega_0+2b(\delta t)+bt]t}\mathrm{e}^{-i\left[b(\delta t)^2+\omega_0(\delta t)\right]} \end{aligned}$$

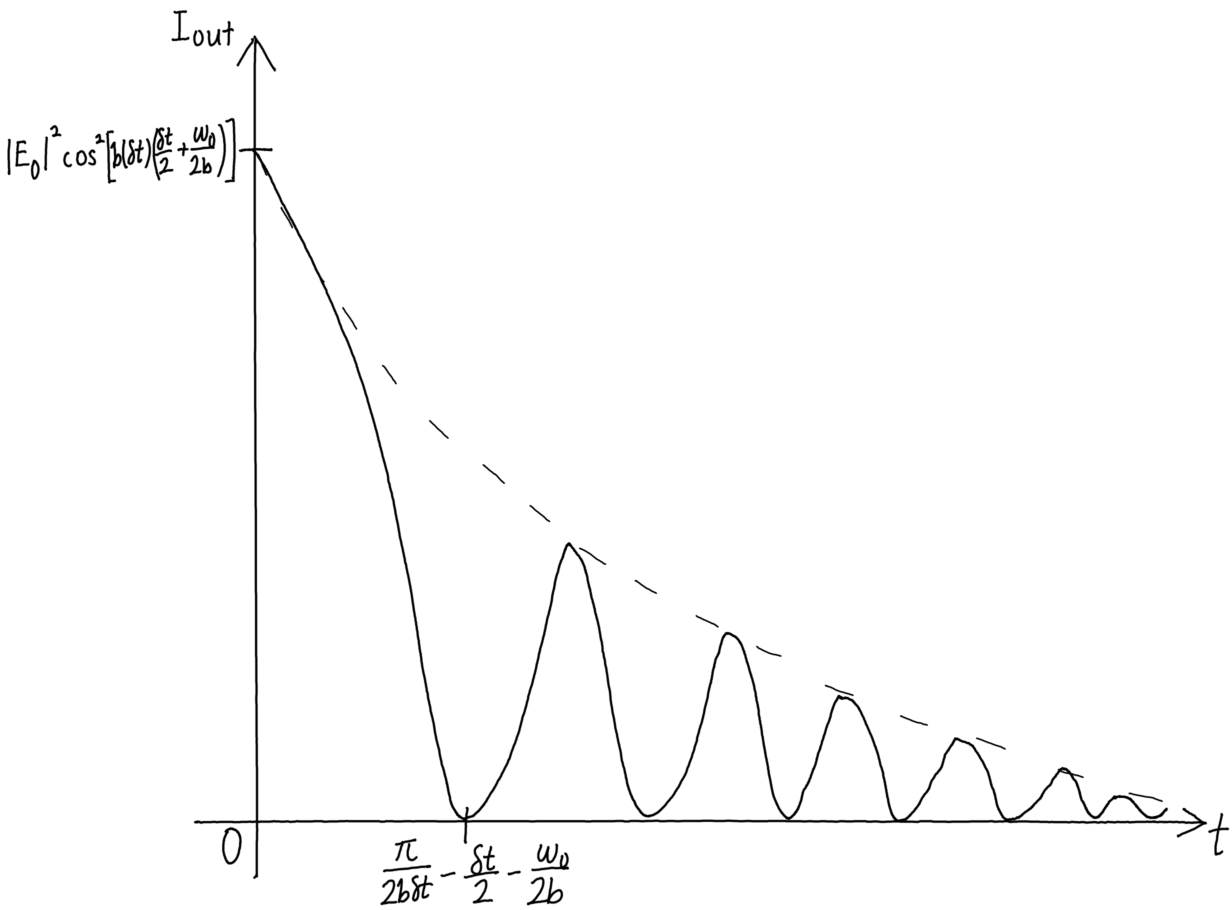
Hence the output E field is:

$$
\begin{aligned}
E_{\mathrm{out}}(t) &= E_1 + E_2 \\
&= \frac{E_0}{2}\,\mathrm{e}^{-at^2}\left[\mathrm{e}^{-i(\omega_0+bt)t} + \mathrm{e}^{-i[\omega_0+2b(\delta t)+bt]t}\mathrm{e}^{-i\left[b(\delta t)^2+\omega_0(\delta t)\right]}\right] \\
&= E_1 + E_2 \\
&= \frac{E_0}{2}\,\mathrm{e}^{-at^2}\mathrm{e}^{-i(\omega_0+bt)t}\left[1 + \mathrm{e}^{-i\left[2b(\delta t)t+b(\delta t)^2+\omega_0(\delta t)\right]}\right] \\
&= \frac{E_0}{2}\,\mathrm{e}^{-at^2}\mathrm{e}^{-i(\omega_0+bt)t}\mathrm{e}^{-i/2\left[2b(\delta t)t+b(\delta t)^2+\omega_0(\delta t)\right]}\cdot 2\cos\left[\frac{[2b(\delta t)t+b(\delta t)^2+\omega_0(\delta t)]}{2}\right] \\
&= E_0\mathrm{e}^{-at^2}\cos\left[b(\delta t)\left(t+\frac{\delta t}{2}+\frac{\omega_0}{2b}\right)\right]\mathrm{e}^{-i\left[\omega_0 t+\underbrace{\left(bt^2+2b(\delta t)t+b(\delta t)^2+\omega_0(\delta t)\right)}_{\psi(t)=b(t+\delta t)^2+\omega_0(\delta t)}\right]}
\end{aligned}
$$

The output intensity is then:

$$
\begin{aligned}
I_{\mathrm{out}} &= |E_{\mathrm{out}}|^2 \\
&= |E_0|^2\,\mathrm{e}^{-2at^2}\cos^2\left[b(\delta t)\left(t+\frac{\delta t}{2}+\frac{\omega_0}{2b}\right)\right]
\end{aligned}
$$

Sketch of I_{out} against t:



The intensity has a frequency of $b(\delta t)$. Hence the interval between adjacent peaks is:

$$(\Delta t) \cdot b(\delta t) = \pi$$
$$\Rightarrow \Delta t = \frac{\pi}{b(\delta t)}$$

(d) We know the frequency spectrum of E_{in} is given by the Fourier transform:

$$\begin{aligned}
a(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E_{\text{in}}(t) e^{i\omega t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} E_0 e^{\left(i\Omega/2\Gamma^{1/2}\right)} \int_{-\infty}^{\infty} e^{-\Gamma t^2 - i\Omega t - \left(i\Omega/2\Gamma^{1/2}\right)}\, dt \quad \text{where } \Omega = \omega_0 - \omega \\
&= \frac{1}{\sqrt{2\pi}} E_0 e^{-\Omega^2/4\Gamma} \underbrace{\int_{-\infty}^{\infty} e^{-\left(\Gamma^{1/2} t + i\Omega/2\Gamma^{1/2}\right)^2}\, dt}_{\int_{-\infty}^{\infty} e^{\Gamma(t + i\Omega/2\Gamma)^2}\, dt = \sqrt{\frac{\pi}{\Gamma}}} \\
&= \frac{1}{\sqrt{2\pi}} E_0 \sqrt{\frac{\pi}{\Gamma}}\, e^{-\Omega^2/4\Gamma}
\end{aligned}$$

Thus the power spectrum is:

$$\begin{aligned}
P(\omega) &= |a(\omega)|^2 \\
&= \frac{1}{2\pi} |E_0|^2 \cdot \frac{\pi}{|\Gamma|} \left| e^{-(\omega_0 - \omega)^2/4\Gamma} \right|^2
\end{aligned}$$

Now for a Michelson interferometer, we have total average intensity[1]:

$$I(\delta t) = \int_0^{\infty} P(\omega) \cdot \frac{1 + \cos(\omega(\delta t))}{2}\, d\omega \tag{1}$$

Now we have:

$$\begin{aligned}
e^{-(\omega_0 - \omega)^2/4\Gamma} &= e^{-(\omega_0 - \omega)^2/4 \cdot a - ib/a^2 + b^2} \\
&= e^{-(\omega_0 - \omega)^2/4(a^2 + b^2) \cdot a} e^{+i(\omega_0 - \omega)^2/4(a^2 + b^2) \cdot b} \\
\Rightarrow \left| e^{(\omega_0 - \omega)^2/4\Gamma} \right|^2 &= e^{-a(\omega_0 - \omega)^2/2(a^2 + b^2)}
\end{aligned}$$

[1]Point of confusion: $I(t)$ or $I(\delta t)$? It turns out that when we are integrating $P(\omega)$ what we are doing is to find the average intensity over a long time (see page 223 of Brooker)

(1) then becomes:

$$\begin{aligned}
I(\delta t) &= \int_0^\infty \frac{|E_0|^2 \pi}{2\pi |\Gamma|} \mathrm{e}^{-a(\omega_0-\omega)^2/2(a^2+b^2)} \cdot \frac{1+\cos(\omega(\delta t))}{2} \, \mathrm{d}\omega \\
&\propto \int_{-\infty}^{\infty} \mathrm{e}^{-a(\omega_0-\omega)^2/2(a^2+b^2)} \left[1+\cos(\omega(\delta t))\right] \mathrm{d}\omega \quad \text{since the integrand is even in } \omega \\
&\propto \mathrm{Re}\left[\int_{-\infty}^{\infty} \mathrm{e}^{-a(\omega_0-\omega)^2/2(a^2+b^2)} \mathrm{e}^{-i\omega(\delta t)} \, \mathrm{d}\omega\right] \quad \text{since the first term is constant and the second term may be written as } \cos(\omega(\delta t)) + i\sin(\omega(\delta t)) \text{ with the imaginary integrand vanishing due to it being odd} \\
&= \mathrm{Re}\left[\int_{-\infty}^{\infty} \mathrm{e}^{-A\omega^2-(i(\delta t)-2A\omega_0)\omega-A\omega_0^2} \, \mathrm{d}\omega\right] \quad \text{where } A = \frac{a}{2(a^2+b^2)} \\
&= \mathrm{Re}\left[\int_{-\infty}^{\infty} \mathrm{e}^{-A\omega^2-(i(\delta t)-2A\omega_0)\omega-\left(\frac{i(\delta t)-2A\omega_0}{2A^{1/2}}\right)^2} \mathrm{e}^{\left(\frac{i(\delta t)-2A\omega_0}{2A^{1/2}}\right)^2-A\omega_0^2} \, \mathrm{d}\omega\right] \\
&= \mathrm{Re}\left[\mathrm{e}^{-(\delta t)^2/4A} \mathrm{e}^{-i\omega_0(\delta t)} \underbrace{\int_{-\infty}^{\infty} \mathrm{e}^{-A(\omega+\dots)^2} \, \mathrm{d}\omega}_{\sqrt{\frac{\pi}{A}}}\right] \\
&= \sqrt{\frac{\pi}{A}} \mathrm{e}^{-(\delta t)^2/4A} \cos(\omega_0(\delta t))
\end{aligned}$$

So $I(\delta t)$ has frequency $\omega_0 \to \omega_0 + 2bt$ to account for the shifting of mean sample frequency due to chirping.

Thus adjacent peaks would have an interval of:

$$\begin{aligned}
2b(\Delta t)(\delta t) &= 2\pi \\
\Delta t &= \frac{\pi}{b(\delta t)}
\end{aligned}$$

which matches the result from part c.

As δt increases, the average intensity $I(\delta t)$ rapidly drops and approaches 0 due to the exponential prefactor, suggesting that there is a limitation in how far a measurement in δt can go.

4. Quantum gates and their fidelities.

(a) Universal set of unitary quantum gates: the set of all possible qubit gates that are unitary (i.e. reversible), which may be approximated up to an arbitrary accuracy.

Clifford set: need 2-qubit gate, e.g. `CX`, `CNOT` to introduce entanglement.

(b)

$$\begin{aligned}\theta_\phi &= \mathrm{e}^{-i\theta\sigma_\phi/2} \\ &= \mathrm{e}^{-i\theta\sigma_x\cos\phi/2}\mathrm{e}^{-i\theta\sigma_y\sin\phi/2} \\ &= \mathbb{1} - i\theta\underbrace{\begin{pmatrix} 0 & \frac{\cos\phi}{2} - i\frac{\sin\phi}{2} \\ \frac{\cos\phi}{2} + i\frac{\sin\phi}{2} & 0 \end{pmatrix}}_{M} + \frac{(-i\theta)^2}{2}M^2 + \dots\end{aligned}$$

Note that $M = \frac{1}{2}\begin{pmatrix} 0 & \mathrm{e}^{-i\phi} \\ \mathrm{e}^{i\phi} & 0 \end{pmatrix} \Rightarrow M^2 = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

$$\begin{aligned}\theta_\phi &= \mathbb{1}\left[1 + \frac{1}{2!}\left(-\frac{i\theta}{2}\right)^2 + \frac{1}{4!}\left(-\frac{i\theta}{2}\right)^4 + \dots\right] + \begin{pmatrix} 0 & \mathrm{e}^{-i\phi} \\ \mathrm{e}^{i\phi} & 0 \end{pmatrix}\left[\left(-\frac{i\theta}{2}\right) + \frac{1}{3!}\left(-\frac{i\theta}{2}\right)^3 + \dots\right] \\ &= \mathbb{1}\cdot\cos\frac{\theta}{2} + \begin{pmatrix} 0 & \mathrm{e}^{-i\phi} \\ \mathrm{e}^{i\phi} & 0 \end{pmatrix}(-i)\sin\frac{\theta}{2} \\ &= \begin{pmatrix} \cos\frac{\theta}{2} & -i\mathrm{e}^{-i\phi}\sin\frac{\theta}{2} \\ -i\mathrm{e}^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}\end{aligned}$$

We then have:

$$\begin{aligned}\pi_{\phi_2}\cdot\pi_{\phi_1} &= \begin{pmatrix} \cos\frac{\pi}{2} & -i\mathrm{e}^{-i\phi_2}\sin\frac{\pi}{2} \\ -i\mathrm{e}^{i\phi_2}\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\pi}{2} & -i\mathrm{e}^{-i\phi_1}\sin\frac{\pi}{2} \\ -i\mathrm{e}^{i\phi_1}\sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{pmatrix} \\ &= \begin{pmatrix} -\mathrm{e}^{i(\phi_1-\phi_2)} & 0 \\ 0 & -\mathrm{e}^{-i(\phi_1-\phi_2)} \end{pmatrix} \\ &= -\mathrm{e}^{i(\phi_1-\phi_2)}\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{-2i(\phi_1-\phi_2)} \end{pmatrix}\end{aligned}$$

Note that for $-2i(\phi_1 - \phi_2) = \pi/4 \Rightarrow \phi_2 = \phi_1 + \pi/8$, we recover a T gate up to a global phase.

Next we examine HT^4:

$$\begin{aligned}\mathrm{HT}^4 &= \mathrm{HZ} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \\ &= \mathrm{H}\cdot 90^\circ_y\end{aligned}$$

This is similar to a Hadamard gate, which can in fact be recovered by the observing that $(\mathrm{HT}^4)\mathrm{T}^4 = \mathrm{H}$ up to a global phase.

From part a, we also know that H and T are universal single-qubit gates. Therefore we only need one distinct rotation (ϕ_1 as ϕ_2 is fixed by the relation above) to perform a general single-qubit manipulation up to an arbitrary precision.

(c) Propose $\mathcal{F} = \dfrac{\left|\mathrm{tr}\left(U^\dagger V\right)\right|}{\mathrm{tr}\left(U^\dagger U\right)}$, this is a good measure since if $V = U$, we have $\mathcal{F} = 1$, and if $V \neq U$, we have $U^\dagger V \neq \mathbb{1}$, hence its trace would be less than $\mathrm{tr}\left(U^\dagger U\right) = \mathrm{tr}\left(\mathbb{1}\right) = \mathrm{rk}\left(\mathbb{1}\right) = 2$, which indicates lower accuracy.

$$
\begin{aligned}
V = \theta'_\phi &= \begin{pmatrix} \cos\frac{\theta'}{2} & -i\mathrm{e}^{-i\phi}\sin\frac{\theta'}{2} \\ -i\mathrm{e}^{i\phi}\sin\frac{\theta'}{2} & \cos\frac{\theta'}{2} \end{pmatrix} \\
\Rightarrow U^\dagger V &= \begin{pmatrix} \cos\frac{\theta}{2} & +i\mathrm{e}^{-i\phi}\sin\frac{\theta}{2} \\ +i\mathrm{e}^{i\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} (\dots) \\
&= \begin{pmatrix} \cos\frac{\theta}{2}\cos\frac{\theta'}{2} + \sin\frac{\theta}{2}\sin\frac{\theta'}{2} & \dots \\ \dots & \sin\frac{\theta}{2}\sin\frac{\theta'}{2} + \cos\frac{\theta}{2}\cos\frac{\theta'}{2} \end{pmatrix}
\end{aligned}
$$

So the trace is:

$$
\begin{aligned}
\mathrm{tr}\left(U^\dagger V\right) &= 2\left[\cos\frac{\theta}{2}\cos\frac{\theta+\epsilon\theta}{2} + \sin\frac{\theta}{2}\sin\frac{\theta+\epsilon\theta}{2}\right] \\
&= 2\left[\cos^2\frac{\theta}{2}\cos\frac{\epsilon\theta}{2} - \cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\frac{\epsilon\theta}{2} + \sin^2\frac{\theta}{2}\cos\frac{\epsilon\theta}{2} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\sin\frac{\epsilon\theta}{2}\right] \\
&= 2\cos\frac{\epsilon\theta}{2}
\end{aligned}
$$

Thus $\mathcal{F}_1$ is:

$$
\begin{aligned}
\mathcal{F}_1 &= \cos\frac{\epsilon\theta}{2} \\
&= 1 - \frac{1}{2}\left(\frac{\epsilon\theta}{2}\right)^2 + \dots
\end{aligned}
$$

For $\theta = \pi$, we then have $\alpha = \pi^2/8$.

(d) From part b, we have:

$$
\begin{aligned}
\pi_\phi\pi_{2\phi}\pi_\phi &= \begin{pmatrix} 0 & -i\mathrm{e}^{-i\phi} \\ -i\mathrm{e}^{i\phi} & 0 \end{pmatrix} \begin{pmatrix} -\mathrm{e}^{-i\phi} & 0 \\ 0 & -\mathrm{e}^{i\phi} \end{pmatrix} \\
&= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\
&= \mathrm{e}^{i\pi/2} \cdot \mathrm{X}
\end{aligned}
$$

Also from the given fidelity, we shall seek the value of ϕ such that the lowest order error term is 0:

$$
\begin{aligned}
1 + 2\cos\phi &= 0 \\
\Rightarrow \cos\phi &= -\frac{1}{2} \\
\phi &= \pm\frac{2\pi}{3}
\end{aligned}
$$

Hence the maximum achievable fidelity shall be:

$$\begin{aligned}\max(\mathcal{F}_3) &= 1 + \frac{\pi^4}{384}\underbrace{\left(21 + 40\cos\frac{2\pi}{3} + 20\cos\frac{4\pi}{3}\right)}_{-9}\epsilon^4 \dots \\ &= 1 - \frac{3\pi^4}{128}\epsilon^4 + \dots\end{aligned}$$

5. One of the rare cases where the question on NMR quantum computing is relatively easy to answer.

(a) Zeeman energy for spin:

$$E = \hbar\gamma B = \hbar\omega$$

NMR Hamiltonian:

$$\mathcal{H} = \hbar\omega_1 \frac{\sigma_{1z}}{2} + \hbar\omega_2 \frac{\sigma_{2z}}{2} + \hbar\omega_{12} \frac{\sigma_{1z}\sigma_{2z}}{4}$$

At thermal equilibrium, the kinetic population shall be described by the Boltzmann distribution:

$$\frac{n_\uparrow}{n_\downarrow} = \mathrm{e}^{-\hbar\omega/k_\mathrm{B}T}$$

$$\Rightarrow \begin{cases} p_{\uparrow\uparrow} = \frac{1}{4} + \mathrm{e}^{-2\hbar\omega/k_\mathrm{B}T} \approx \frac{1}{4} - \frac{2\hbar\omega}{k_\mathrm{B}T} \\ p_{\downarrow\downarrow} = \frac{1}{4} + \frac{2\hbar\omega}{k_\mathrm{B}T} \end{cases} \qquad \text{to 1st order}$$

Thus we have density matrix:

$$\begin{aligned} \rho_\mathrm{th} &= \frac{1}{4} \left|\uparrow\downarrow\right\rangle \left\langle\uparrow\downarrow\right| + \frac{1}{4} \left|\downarrow\uparrow\right\rangle \left\langle\downarrow\uparrow\right| + p_{\uparrow\uparrow} \left|\uparrow\uparrow\right\rangle \left\langle\uparrow\uparrow\right| + p_{\downarrow\downarrow} \left|\downarrow\downarrow\right\rangle \left\langle\downarrow\downarrow\right| \\ &= \frac{1}{4} \begin{pmatrix} 1 + \frac{\hbar\omega}{k_\mathrm{B}T} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 - \frac{\hbar\omega}{k_\mathrm{B}T} \end{pmatrix} \end{aligned}$$

(b) From the question, we have $\texttt{CTG} \equiv \texttt{crush} \cdot |1\rangle\langle 1| \otimes \mathrm{X}_\theta$.

We then have:

$$\begin{aligned} \mathrm{X}_\theta &= \mathrm{e}^{-i\theta\sigma_x/2} \\ &= \mathbb{1} - i\theta \cdot \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2} \cdot \left(\frac{i\theta}{2}\right)^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \dots \\ &= \mathbb{1} \cos\frac{\theta}{2} - i\sigma_x \sin\frac{\theta}{2} \\ &= \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \end{aligned}$$

Hence controlled-X_θ may be written as:

$$\mathrm{CX}_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{X_θ}} \\ 0 & 0 & & \end{pmatrix}$$

We thus have the following CP map:

$$\begin{aligned}
\rho &\to (\mathrm{CX}_\theta)\rho(\mathrm{CX}_\theta)^\dagger \\
&= \begin{pmatrix} 1 & & \\ & 1 & \\ & & \mathrm{X}_\theta \end{pmatrix} \begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1 & \\ & & \mathrm{X}_\theta^\dagger \end{pmatrix} \\
&= \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ & & -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} a & & & \\ & b & & \\ & & c & \\ & & & d \end{pmatrix} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \cos\frac{\theta}{2} & i\sin\frac{\theta}{2} \\ & & i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \\
&= \dots \begin{pmatrix} a & & & \\ & b & & \\ & & c\cos\frac{\theta}{2} & ic\sin\frac{\theta}{2} \\ & & id\sin\frac{\theta}{2} & d\cos\frac{\theta}{2} \end{pmatrix} \\
&= \begin{pmatrix} a & & & \\ & b & & \\ & & c\cos^2\frac{\theta}{2} + d\sin^2\frac{\theta}{2} & \dots \\ & & \dots & c\sin^2\frac{\theta}{2} + d\cos^2\frac{\theta}{2} \end{pmatrix} \xRightarrow{\texttt{crush}} \begin{pmatrix} a & & & \\ & b & & \\ & & c' & \\ & & & d' \end{pmatrix}
\end{aligned}$$

Thus we have:

$$\begin{aligned}
c' &= c\cos^2\frac{\theta}{2} + d\sin^2\frac{\theta}{2} \\
&= c\left(\frac{1+\cos\theta}{2}\right) + d\left(\frac{1-\cos\theta}{2}\right) \\
&= \frac{(c+d)+\cos\theta(c-d)}{2}
\end{aligned}$$

Similarly we have $d' = \dfrac{(c+d)-\cos\theta(c-d)}{2}$.

If the upper qubit is acted upon instead, we simply swap b and c'.

a is unaffected since `CTG` is a conditional gate and a corresponds to the case where both qubits are 0.

(c) For ρ_{th}, we have $a \equiv {}^1\!/_4(1+\epsilon)$, $b \equiv {}^1\!/_4 \equiv c$, $d \equiv {}^1\!/_4(1-\epsilon)$.

After `CTG`$(\theta_1 = \arccos({}^1\!/_3), \text{lower})$, we have density matrix:

$$\begin{pmatrix} a & & & \\ & b & & \\ & & c' & \\ & & & d' \end{pmatrix}$$

where $c' = \dfrac{(c+d)+{}^1\!/_3(c-d)}{2}$, $d' = \dfrac{(c+d)-{}^1\!/_3(c-d)}{2}$.

After $\texttt{CTG}(\theta_2 = \pi/2, \text{upper})$, we have:

$$\begin{pmatrix} a & & & \\ & b' & & \\ & & c' & \\ & & & d'' \end{pmatrix}$$

where $b' = \dfrac{(b+d') + 0(b-d')}{2}$, $d'' = \dfrac{(b+d') - 0(b-d')}{2}$.

Simplifying the expressions then gives:

$$\begin{aligned} c' &= \frac{2}{3}c + \frac{1}{3}d = \frac{1}{6} + \frac{1}{12}(1-\epsilon) \\ d' &= \frac{1}{3}c + \frac{2}{3}d = \frac{1}{12} + \frac{1}{6}(1-\epsilon) \\ b' &= d'' = \frac{b+d'}{2} = \frac{1}{6} + \frac{1}{12}(1-\epsilon) \end{aligned}$$

So we have the resulting density matrix:

$$\begin{aligned} \rho_{\text{th}} &= \begin{pmatrix} \frac{1}{4}(1+\epsilon) & \\ & \operatorname{diag}\left(\frac{1}{6} + \frac{1}{12}(1-\epsilon)\right) \end{pmatrix} \\ &= \operatorname{diag}\left(\frac{1}{6} + \frac{1}{12}(1-\epsilon)\right) + \begin{pmatrix} \frac{1}{12} - \frac{1}{12} + \frac{\epsilon}{4} + \frac{\epsilon}{12} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \\ &= \frac{1}{4}\operatorname{diag}\left(1 - \frac{\epsilon}{3}\right) + \begin{pmatrix} \frac{\epsilon}{3} & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix} \end{aligned}$$

Thus $p = \epsilon/3 = \hbar\omega/3k_{\text{B}}T$.

Now we note that p scales as the inverse of the system size (ρ has dimension 2^n): $p \propto \frac{1}{2^n}$. Therefore scaling up the system decreases p quickly, rendering large scale NMR computing via pseudo-pure state unfeasible.

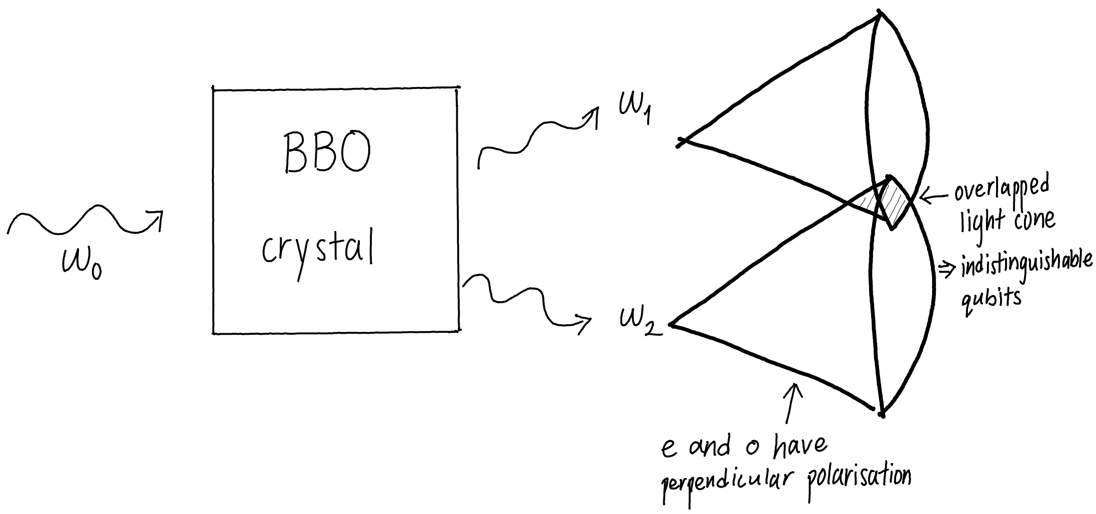
(d) To interact with the NMR system, one may employ an RF pulse at the appropriate ω and applying it for some time τ that corresponds to the desired phase. In addition, by rotating the polarisation of the RF pulse, one may construct a general gate.

The catch with homonuclear system is that the transition energies are exactly the same, so spatial separation is required for qubit addressing. Whereas for heteronuclear system, there are only a finite number of elements, which then imposes an upper limit to the number of qubits available.

NMR computing is not scalable due to the ever shrinking effective purity, as explored in the previous part.

6. Bell function for a Bell basis with extra phase factor.

(a) To produce an entangled polarisation pair, we exploit the non-linear optics in BBO crystal to generate a pair of down-converted photons travelling along θ from the optic axis such that the conservation of energy and momentum are satisfied.



(b) Starting from the given state:

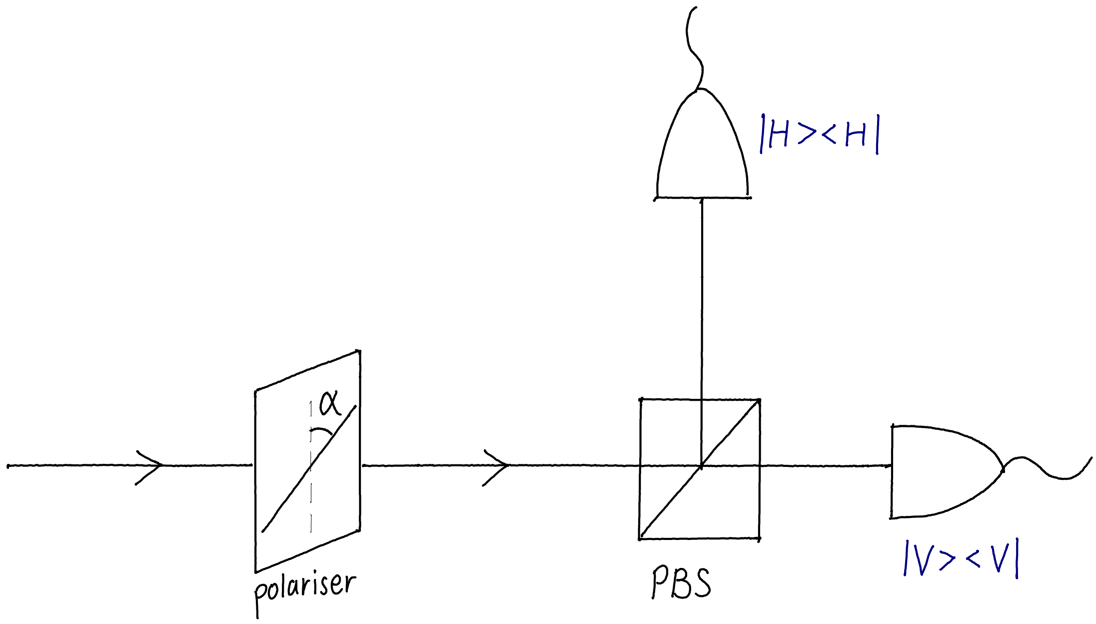
$$|\psi^{\varphi}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + \mathrm{e}^{i\varphi}|10\rangle\right)$$

$$\Rightarrow \begin{array}{c} \begin{array}{cccc} \langle 0_\mathrm{A}1_\mathrm{B}| & \langle 0_\mathrm{A}1_\mathrm{B}| & \langle 1_\mathrm{A}0_\mathrm{B}| & \langle 1_\mathrm{A}1_\mathrm{B}| \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2}\mathrm{e}^{i\varphi} & 0 \\ 0 & \frac{1}{2}\mathrm{e}^{-i\varphi} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = \rho_{\mathrm{sys}} \quad \text{the system density matrix}$$

On Alice's side, she would observe:

$$\begin{aligned} \rho_\mathrm{A} &= \mathrm{tr}_\mathrm{B}\rho_{\mathrm{sys}} \\ &= \begin{pmatrix} \mathrm{tr}\begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} & \mathrm{tr}\begin{pmatrix} 0 & 0 \\ \frac{1}{2}\mathrm{e}^{i\varphi} & 0 \end{pmatrix} \\ \mathrm{tr}\begin{pmatrix} 0 & \frac{1}{2}\mathrm{e}^{-i\varphi} \\ 0 & 0 \end{pmatrix} & \mathrm{tr}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \rho_\mathrm{B} \quad \text{by symmetry} \end{aligned}$$

Note that $\mathrm{tr}(\rho_\mathrm{A}^2) = \mathrm{tr}(\rho_\mathrm{B}^2) = 1/2 \neq 1$ and this indicates that we have an entangled state since each subsystem is not pure $\Rightarrow$ system is not separable.



By setting up the chain above, one may perform a measurement of $\sigma_\alpha = \cos\alpha\,\sigma_z + \sin\alpha\,\sigma_x$. Alternatively, putting the PBS at an angle α to the vertical will achieve the same.

Calculation of the expectation value:

$$
\begin{aligned}
\langle \sigma_\alpha \sigma_\beta \rangle &= \langle \psi^\varphi | \, \sigma_\alpha \sigma_\beta \, | \psi^\varphi \rangle \\
&= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \mathrm{e}^{-i\varphi} & 0 \end{pmatrix} \cdot \\
&\quad \begin{pmatrix} \cos\alpha \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix} & \sin\alpha \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix} \\ \sin\alpha \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix} & -\cos\alpha \begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix} \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \mathrm{e}^{i\varphi} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \mathrm{e}^{-i\varphi} & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\alpha\sin\beta + \mathrm{e}^{i\varphi}\sin\alpha\cos\beta \\ -\cos\alpha\cos\beta + \mathrm{e}^{i\varphi}\sin\alpha\sin\beta \\ \sin\alpha\sin\beta - \mathrm{e}^{i\varphi}\cos\alpha\cos\beta \\ \sin\alpha\cos\beta + \mathrm{e}^{i\varphi}\cos\alpha\sin\beta \end{pmatrix} \\
&= \frac{1}{2}\left[-\cos\alpha\cos\beta + \mathrm{e}^{i\varphi}\sin\alpha\sin\beta + \mathrm{e}^{-i\varphi}\sin\alpha\sin\beta - \cos\alpha\cos\beta \right] \\
&= -\cos\alpha\cos\beta + \cos\varphi\sin\alpha\sin\beta
\end{aligned}
$$

We have Bell function given as: $B = \langle QS \rangle - \langle RS \rangle - \langle RT \rangle - \langle QT \rangle$.

Next we make identifications $Q \equiv \sigma_0$, $R \equiv \sigma_{\pi/2}$, $S \equiv \sigma_{-3\pi/4}$, $T \equiv \sigma_{-\pi/4}$.

Suppose Q, R, S, T are related by local realism, then each random variable should be independent of one another $\Rightarrow \langle AB \rangle = \langle A \rangle \langle B \rangle$.

Now we know each individual measurement has a possible value of ± 1, so upon perfect correlation we should have $B = 1 - 1 - 1 - 1 = -2$. And similarly $B = -1 + 1 + 1 + 1 = +2$ for anti-correlation. So $|B| \leq 2$ is mandated by local realism.

Now let's calculate them via QM:

$$\begin{aligned}
\langle QS\rangle &= \left\langle \sigma_0\, \sigma_{-3\pi/4}\right\rangle \\
&= -\cos 0 \cos\left(-\frac{3\pi}{4}\right) + \cos\varphi \sin 0 \sin\left(-\frac{3\pi}{4}\right) \\
&= -\cos\left(-\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\langle RS\rangle &= \left\langle \sigma_{\pi/2}\, \sigma_{-3\pi/4}\right\rangle \\
&= \cos\varphi \sin\frac{\pi}{2} \sin\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}\cos\varphi
\end{aligned}$$

$$\begin{aligned}
\langle RT\rangle &= \left\langle \sigma_{\pi/2}\, \sigma_{-\pi/4}\right\rangle \\
&= \cos\varphi \sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}\cos\varphi
\end{aligned}$$

$$\begin{aligned}
\langle QT\rangle &= \left\langle \sigma_0\, \sigma_{-\pi/4}\right\rangle \\
&= -\cos\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow B &= \frac{1}{\sqrt{2}}\left(1 + \cos\varphi + \cos\varphi + 1\right) \\
&= \sqrt{2}\left(1 + \cos\varphi\right)
\end{aligned}$$

Note that for $\sin^2(\varphi/2) \geq 1/\sqrt{2}$, $|B| \geq 2$ and thus local realism is violated.

Maximal violation when $\varphi = \pi \Rightarrow B = \sqrt{2}(1+1) = 2\sqrt{2}$.

(c) Average of B over $[-\varphi_0, \varphi_0]$:

$$\begin{aligned}
\langle B\rangle &= \frac{1}{2\varphi_0}\int_{-\varphi_0}^{\varphi_0} \sqrt{2}\left(\cos\varphi + 1\right)\, \mathrm{d}\varphi \\
&= \frac{1}{\sqrt{2}\varphi_0}\left[\sin\varphi + \varphi\right]_{-\varphi_0}^{\varphi_0} \\
&= \frac{\sqrt{2}}{\varphi_0}\left(\sin\varphi_0 + \varphi_0\right) \\
&= \sqrt{2}\left(\operatorname{sinc}\varphi_0 + 1\right)
\end{aligned}$$

This method works since we are finding B for an uniform ensemble of entangled pairs with no correlation between each pair.

For local realism to be violated, we need:

$$\begin{gathered}
|\operatorname{sinc}\varphi_0 + 1| > \sqrt{2} \\
\Rightarrow \operatorname{sinc}\varphi_0 > \sqrt{2} - 1 = 0.414
\end{gathered}$$

From the graph, we have such $\varphi_0 \approx 2.1\,\mathrm{rad}$ as the max value beyond which $B_{\varphi_0} < 2 \Rightarrow$ local realism holds.