UNOFFICIAL SOLUTIONS BY TheLongCat

B2. SYMMETRY AND RELATIVITY

TRINITY TERM 2020

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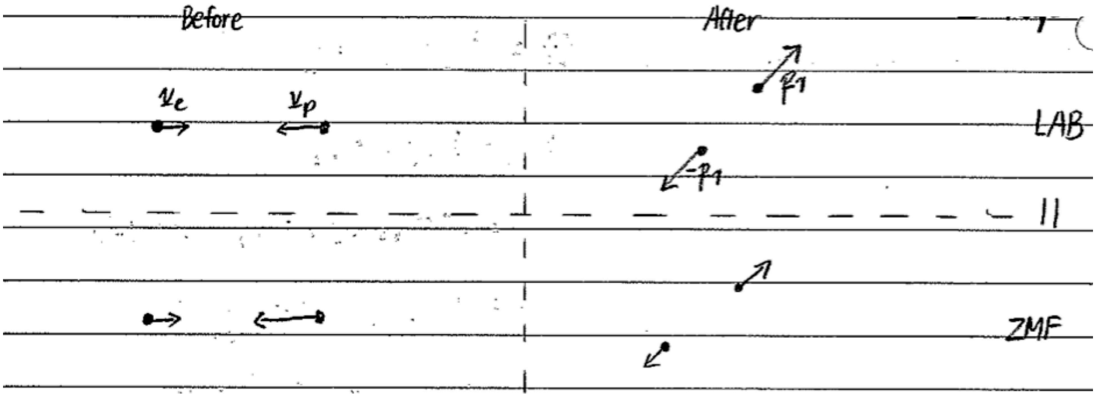
Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. (DRAFT)



(a) Conservation of 4-momentum:

$$\mathsf{P}^{\mu}_{\text{th}} + \mathsf{P}^{\mu}_{M} = \mathsf{P}^{\mu}_{\text{sys}}$$

$$\underbrace{\left(\mathsf{P}^{\mu}_{\text{th}} + \mathsf{P}^{\mu}_{M}\right)^2}_{\substack{\text{evaluate in LAB}\\ \text{before collision}}} = \underbrace{\left(\mathsf{P}^{\mu}_{\text{sys}}\right)^2}_{\substack{\text{evaluate in ZMF}\\ \text{after collision}}}$$

$$-\left(\frac{E_{\text{th}}}{c} + Mc\right)^2 + \underbrace{p^2_{\text{th}}}_{\frac{1}{c^2}\left[E^2_{\text{th}} - m^2c^4\right]} = -\left(\sum_i m_i c\right)^2$$

$$-2E_{\text{th}}M - M^2c^2 - m^2c^2 = -c^2\left(\sum_i m_i\right)^2$$

$$E_{\text{th}} = -\frac{\left[M^2 + m^2 - \left(\sum_i m_i\right)^2\right]c^2}{2M}$$

(b) For the process $p + p \to p + p + \pi^0$,

$$E_{\text{th}} = \frac{\left[(2m_p + m_\pi)^2 - 2m_p^2\right]c^2}{2m_p}$$

$$\Rightarrow \text{Threshold KE } K_{\text{th}} = E_{\text{th}} - m_pc^2$$

$$= \frac{\left[(2m_p + m_\pi)^2 - 4m_p^2\right]c^2}{2m_p}$$

$$= \frac{\left(2m_pm_\pi + m_\pi^2\right)c^2}{2m_p}$$

$$= \left(m_\pi + \frac{m_p i^2}{m_p}\right)c^2$$

$$= 154.4\,\text{MeV} > m_\pi c$$

Momentum:

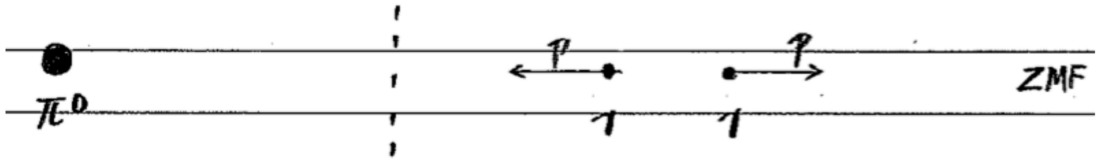
$$p_{\text{th}} = \frac{1}{c}\sqrt{E^2_{\text{th}} - m_p^2c^4}$$

$$p_{\text{th}}c = 776.6\,\text{MeV}$$

ZMF speed = pion v:

$$\beta = \frac{pc}{E} = 0.64$$
$$\Rightarrow v = 0.64c$$

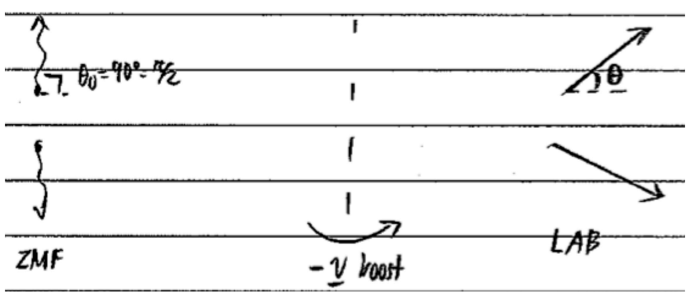
(c) Sketch of the pion decay in ZMF:



Conservation of energy:

$$m_\pi c^2 = 2pc$$
$$p = \frac{m_\pi}{2}c = \frac{h}{\lambda}$$
$$\Rightarrow \lambda = \frac{h}{p} = \frac{2h}{m_\pi c}$$
$$= 1.84 \times 10^{-8}\,\mathrm{m}$$

Consider photons at $\pi/2$ to π^0 motion:



4-wavevector $\mathsf{K}^\mu = (\omega/c, k\cos\theta, k\sin\theta, 0)$

Lorentz transformation gives:

$$\begin{pmatrix} \omega/c \\ k\cos\theta \\ k\sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma & & \\ \beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \begin{pmatrix} \omega_0/c \\ k_0\cos\theta_0 \\ k_0\sin\theta_0 \\ 0 \end{pmatrix}$$

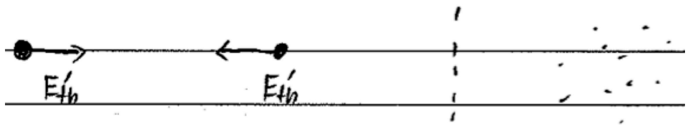
$$\Rightarrow \frac{\omega}{c} = \frac{\gamma\omega_0}{c} + \beta\gamma k_0\cos\theta_0 \tag{1}$$
$$\Rightarrow k\cos\theta = \frac{\beta\gamma\omega_0}{c} + \gamma k_0\cos\theta_0 \tag{2}$$

(2)÷(1) and invoke dispersion $\omega = ck$:

$$\cos\theta = \frac{\beta\gamma + \gamma\cos\theta_0}{\gamma + \beta\gamma\cos\theta_0} = \beta \quad \text{for } \theta_0 = 90^\circ$$
$$\Rightarrow \theta = \arccos\left(\frac{v}{c}\right) \quad \text{for half of the photons}$$

(d) To redice E_{th}, we should make ZMF tha LAB frame, i.e. let the target move opposite to the photon.

In that case, the final particles are all at rest.



Conservation of 4-momentum:

$$\left(\mathsf{P}^{\mu}_{\text{sys}}\right)^2 = -\left(\frac{2E'_{\text{th}}}{c}\right)^2 = -\left(\sum_i m_i c\right)^2$$
$$E'_{\text{th}} = \frac{\sum_i m_i c^2}{2}$$

For $pp \to pp\pi^0$,

$$E'_{\text{th}} = \frac{(2m_p + m_\pi)\,c^2}{2}$$
$$\frac{E'_{\text{th}}}{E_{\text{th}}} = \frac{(2m_p + m_\pi)\cancel{c^2}}{\cancel{2}} \cdot \frac{\cancel{2}m_p}{\left[(2m_p + m_\pi)^2 - 2m_p^2\right]\cancel{c^2}}$$
$$= \frac{(2m_p + m_\pi)\,m_p}{(2m_p + m_\pi)^2 - 2m_p^2}$$
$$= 0.83$$

So 17% energy saved.

2. (DRAFT)

(a) 4-velocity:

$$\begin{aligned}\mathsf{U}^\mu &= \frac{\mathrm{d}\mathsf{X}^\mu}{\mathrm{d}\tau} \\ &= \gamma\left(c, \mathbf{v}\right)\end{aligned}$$

where $\mathsf{X}^\mu = (ct, \mathbf{x})$ is 4-position, τ is proper time.

4-force:

$$\begin{aligned}\mathsf{F}^\mu &= \frac{\mathrm{d}\mathsf{P}^\mu}{\mathrm{d}\tau} \\ &= \gamma\left(\dot{\gamma}mc + \gamma\dot{m}c, \dot{\gamma}m\mathbf{v} + \gamma\dot{m}\mathbf{v} + \gamma m\dot{\mathbf{v}}\right)\end{aligned}$$

where P^μ is 4-momentum.

For pure force, $\dot{m} = 0$:

$$\Rightarrow \mathsf{F}^\mu = \gamma\left(\dot{\gamma}mc, \dot{\gamma}m\mathbf{v} + \gamma m\dot{\mathbf{v}}\right)$$

But note that:

$$\begin{aligned}\frac{\mathrm{d}\gamma}{\mathrm{d}t} &= -\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-1/2} \cdot \left(-\frac{2v}{c^2}\right) \cdot \mathbf{a} \\ &= \gamma^3 \frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\end{aligned}$$

where $\mathbf{a} = \dot{\mathbf{v}}$.

Hence:

$$\mathsf{F}^\mu = \left(\underbrace{\gamma^4 mc\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}}_{\gamma\frac{\mathrm{d}E}{\mathrm{d}t}}, \underbrace{\gamma^4 m\mathbf{v}\frac{\mathbf{v}\cdot\mathbf{a}}{c^2} + \gamma m\mathbf{a}}_{\gamma\mathbf{f}}\right)$$

Try:

$$\begin{aligned}\mathbf{f}\cdot\mathbf{v} &= \gamma^3 m v^2 \frac{\mathbf{v}\cdot\mathbf{a}}{c^2} + m\mathbf{v}\cdot\mathbf{a} \\ &= m\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\left[\gamma^3 v^2 + c^2\right] \\ &= m\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\left[\frac{v^2}{\left(1 - \frac{v^2}{c^2}\right)^{3/2}} + c^2\right] \\ \mathsf{U}_\mu\mathsf{F}^\mu &= -\gamma^2\frac{\mathrm{d}E}{\mathrm{d}t} + \gamma^2\mathbf{f}\cdot\mathbf{u}\end{aligned}$$

In rest frame,

$$\begin{aligned}E &= mc^2 \\ \Rightarrow \frac{\mathrm{d}E}{\mathrm{d}t} &= \dot{m}c^2 \\ \Rightarrow \mathsf{U}_\mu\mathsf{F}^\mu &= -\dot{m}c^2\end{aligned}$$

For pure force,

$$\begin{aligned} \dot{m} &= 0 \\ \Rightarrow \mathsf{U}_\mu \mathsf{F}^\mu &= 0 \\ \Rightarrow \frac{\mathrm{d}E}{\mathrm{d}t} &= \mathbf{f}\cdot\mathbf{u} \end{aligned}$$

(b) In rest frame,

$$\begin{aligned} \mathrm{d}v &= a_0\,\mathrm{d}\tau \\ \Rightarrow \frac{\mathrm{d}\beta}{\mathrm{d}\tau} &= \frac{a_0}{c} \end{aligned}$$

Since rapidity is defined by:

$$\begin{aligned} \tanh(\rho) &= \beta \\ \Rightarrow \operatorname{sech}^2(\rho)\,\mathrm{d}\rho &= \mathrm{d}\beta \\ \frac{\mathrm{d}\rho}{\mathrm{d}\beta} &= \cosh^2(\rho) \\ \frac{\mathrm{d}\rho}{\mathrm{d}\tau} &= \frac{a_0\cosh^2(\rho)}{c} \\ \rho \simeq \frac{a_0\tau}{c} \quad \rho &= 0 \text{ in rest frame} \\ \beta &= \tanh\left(\frac{a_0\tau}{c}\right) \end{aligned}$$

Also we know $\mathrm{d}t = \gamma\,\mathrm{d}\tau$:

$$\begin{aligned} \Rightarrow t &= \int \cosh\left(\frac{a_0\tau}{c}\right)\mathrm{d}\tau \\ &= \frac{c}{a_0}\sinh\left(\frac{a_0\tau}{c}\right) \end{aligned}$$

Furthermore,

$$\begin{aligned} \sinh\rho &= \beta\gamma \\ &= \frac{v}{c}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \end{aligned}$$

Hence:

$$\begin{aligned} t &= \frac{v}{a_0}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \\ a_0^2t^2\left(1-\frac{v^2}{c^2}\right) &= v^2 \\ \left(1+\frac{a_0^2t^2}{c^2}\right)v^2 &= a_0^2t^2 \\ v &= \frac{a_0t}{\sqrt{1+\frac{a_0^2t^2}{c^2}}} \end{aligned}$$

We also know:

$$
\begin{aligned}
\frac{a_0 t}{c} &= \sinh\theta \\
\Rightarrow a_0 \,\mathrm{d}t &= \cosh\theta \,\mathrm{d}\theta \\
\Rightarrow x &= \int \frac{c\sinh\theta}{\cancel{\sqrt{1+\sinh^2\theta}}} \frac{\cancel{\cosh\theta}}{a_0} \,\mathrm{d}\theta \\
&= \frac{c}{a_0}\cosh\theta \\
\Rightarrow x^2 &= \frac{c^2}{a_0^2}\left(1 + \frac{a_0^2 t^2}{c^2}\right)
\end{aligned}
$$