UNOFFICIAL SOLUTIONS BY TheLongCat

B4. SUB-ATOMIC PHYSICS

TRINITY TERM 2016

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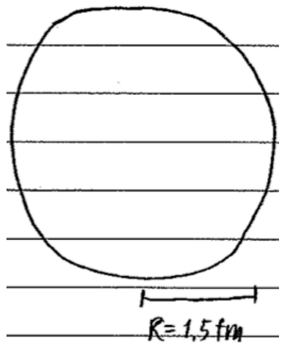
Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. (DRAFT)

(a) Neutron (udd) sits in the $J^P = \frac{1}{2}^+$ octet.

i. Sketch of a quark pudding of radius $R = 1.5\,\mathrm{fm}$:



Uniform distribution $\Rightarrow$ Probability of a quark in radius r:

$$P(r) = \left(\frac{r}{R}\right)^3 \quad \text{for } r \leq R \text{ else } 1 \quad \text{[CDF]}$$
$$\mathbb{P}(r) = \frac{3r^2}{R^3}\,\mathrm{d}r \quad \text{for } r \leq R \text{ else } 0 \quad \text{[PDF]}$$

So:

$$\begin{aligned}\langle r^2 \rangle &= \int_0^\infty \mathbb{P} \cdot r^2 \,\mathrm{d}r \\ &= \int_0^R \frac{3r^4}{R^3}\,\mathrm{d}r \\ &= \frac{3}{5}\left[\frac{r^5}{R^3}\right])_0^R \\ &= \frac{3}{5}R^2\end{aligned}$$

$\Rightarrow$ RMS:

$$\sqrt{\langle r^2 \rangle} = \sqrt{\frac{3}{5}}R = 1.16\,\mathrm{fm}$$

ii. Uncertainty principle gives: $\Delta x \Delta p \geq \frac{\hbar}{2}$

So uncertainty in p:

$$\begin{aligned}\Delta p &\simeq \frac{\hbar}{2\sqrt{\langle r^2 \rangle}} \\ &= \frac{\hbar c}{2\sqrt{\frac{3}{5}}Rc} \\ &= 84.92\,\mathrm{MeV/c}\end{aligned}$$

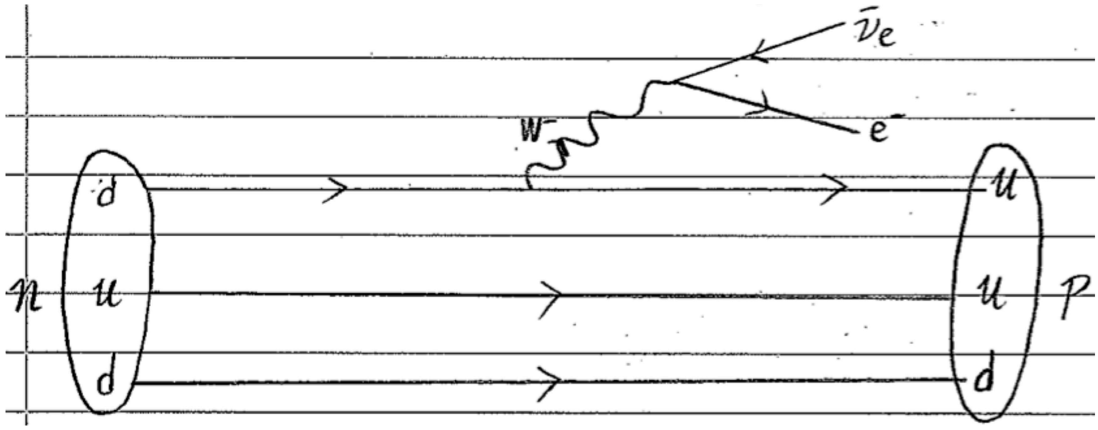
Assuming $\Delta p = \langle p \rangle$, then mean energy:

$$\begin{aligned}\langle E \rangle &= \sqrt{M^2c^4 + \langle p \rangle^2 c^2} \\ &= 85.06\,\text{MeV}\end{aligned}$$

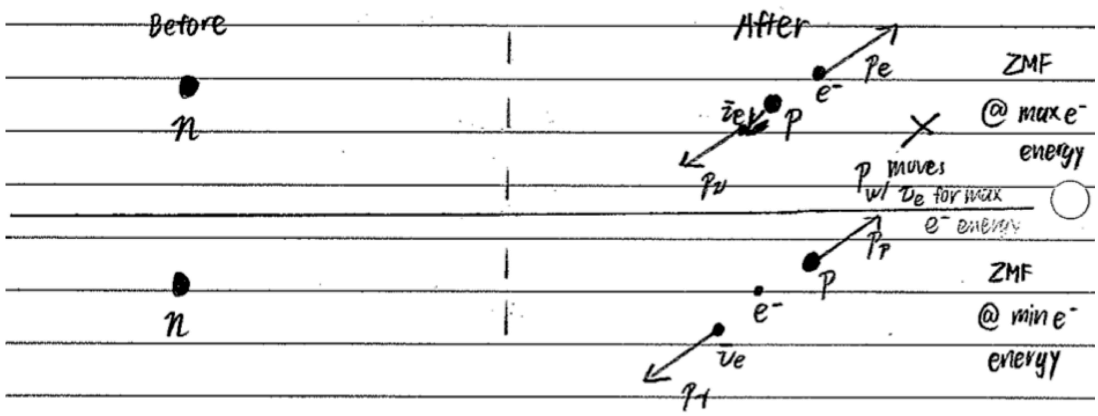
(b) Assuming total symmetry between u and d quarks, total constituent mass is then $3\langle E \rangle = 255.2\,\text{MeV}$.

Ratio of mass: $\dfrac{3\langle E \rangle}{m_n c^2} = 0.27$ so not in agreement! However modelling the quarks as gases is an unphysical model as the quarks interact heavily under strong interaction. It would be greater to treat the system quantum-mechanically with the use of $\langle E \rangle = \langle \psi | \hat{H} | \psi \rangle$ where $|\psi\rangle$ is a normalised system wavefunction, $\hat{H}$ is the Hamiltonian of the system.

Free neutron decay: $n \longrightarrow p + \text{e}^- + \bar{\nu}_e$.



There is no similar process for protons as it is the lightest baryon – there are no lighter baryons to decay into.



For max e$^-$ energy, proton must move with $\bar{\nu}_e$ ($\bar{\nu}_e$ assumed to be massless so can't be at rest).

For min e$^-$ energy, e$^-$ must be at rest instead to allow for proton movement.

For max e$^-$ energy, conservation of 4-momentum gives:

$$
\begin{aligned}
(\mathsf{P}_n)^\mu &= (\mathsf{P}_p)^\mu + (\mathsf{P}_\nu)^\mu + (\mathsf{P}_e)^\mu \\
\Rightarrow \begin{pmatrix} m_n c \\ \mathbf{0} \end{pmatrix} &= \begin{pmatrix} m_p c + p_\nu + \frac{E_e}{c} \\ \mathbf{p}_\nu + \mathbf{p}_e \end{pmatrix} \\
\Rightarrow |\mathbf{p}_\nu| &= |\mathbf{p}_e| = p' \\
\Rightarrow m_n c &= m_p c + \underbrace{p'}_{\frac{1}{c}\sqrt{E_e^2 - m_e^2 c^4}} + \frac{E_e}{c} \\
\Rightarrow \cancel{E_e^2} - m_e^2 c^4 &= \left((m_n - m_p)c^2 - E_e^2\right) \\
&= \cancel{E_e^2} - 2E_e(m_n - m_p)c^2 + (m_n - m_p)^2 c^4 \\
\Rightarrow E_{e,\max} &= \frac{\left[(m_n - m_p)^2 + m_e^2\right] c^4}{2(m_n - m_p)c^2} \\
&= \frac{\left[(m_n - m_p)^2 + m_e^2\right]}{2(m_n - m_p)} c^2
\end{aligned}
$$

For min e$^-$ energy, we get similarly:

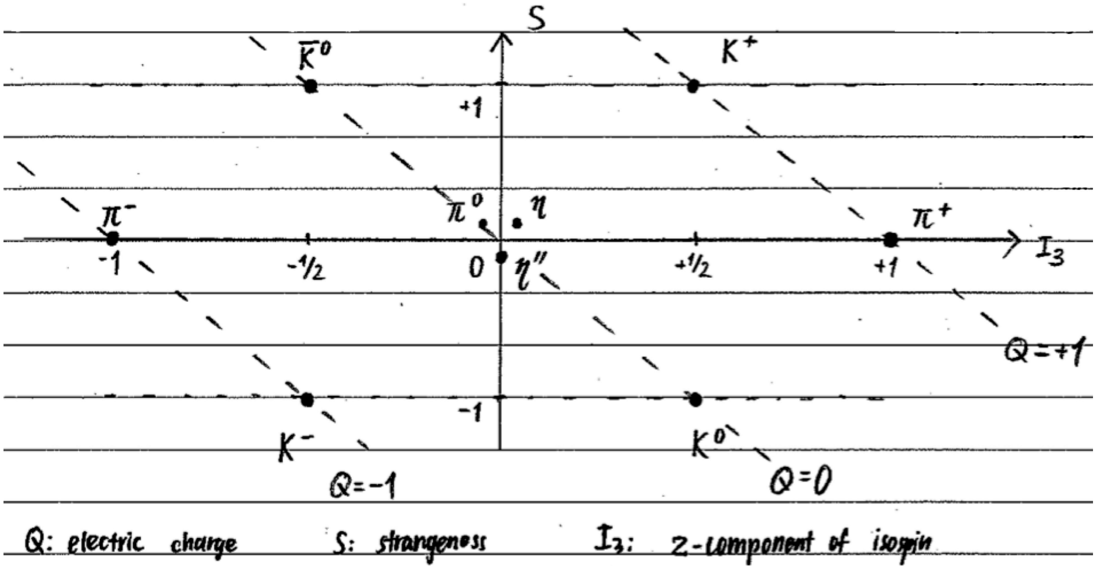
$$
\begin{aligned}
\begin{pmatrix} m_n c \\ \mathbf{0} \end{pmatrix} &= \begin{pmatrix} \frac{E_p}{c} + \mathbf{p}_\nu + m_e c \\ \mathbf{p}_\nu + \mathbf{p}_p \end{pmatrix} \\
\Rightarrow E_{e,\min} &= m_e c^2
\end{aligned}
$$

Decay width:

$$
\begin{aligned}
\Gamma &= \frac{\hbar}{\tau} \\
&= \frac{\hbar c}{c(15 \cdot 60\,\mathrm{s})} \\
&= \frac{197.33\,\mathrm{MeV\,fm}}{2.7 \times 10^{26}\,\mathrm{fm}} \\
&= 7.31 \times 10^{-25}\,\mathrm{MeV}
\end{aligned}
$$

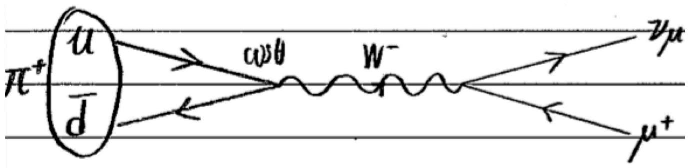
2. (DRAFT)

Meson $J^P = 0^-$ nonet:

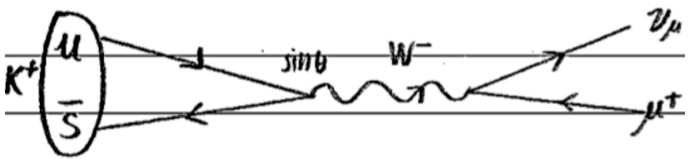


where Q is electric charge, S is strangeness, I_3 is the z-component of isospin.

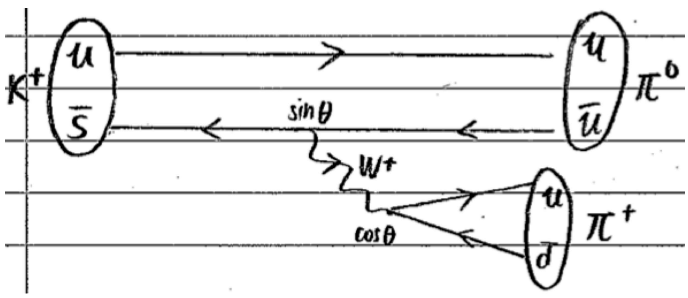
$\pi^+ \to \mu^+\nu_\mu$ decay:



$K^+ \to \mu^+\nu_\mu$ decay:



$K^+ \to \pi^+\pi^0$ decay:



Note that weak charged current has a helicity preference, and since neutrinos are largely left-handed, but since π^+ has spin 0, the daughters must have anti-aligned spins, together with conservation of momentum to give right-handedness requirement. So the pion decay is heavily suppressed compared to kaon: $\dfrac{m_e^2}{(m_p - m_e)^2}$ is the suppression factor where m_p is the mass of parent.

For $K^+ \to \pi^+\pi^0$, since it involves a coupling across generations, a penalty of $\sin\theta$ is added compared to $\pi^+ \to \mu^+\nu_\mu$. Therefore the decay is Cabibbo suppressed.

(a) Assuming $B_{\text{in}} = 100\%$ at each resonance:

$$\sigma_\rho \propto (2+1)\frac{\Gamma_{\text{tot}}^2}{(E^2 - m_p^2)^2 + E^2\Gamma_{\text{tot}}^2}$$
$$\sigma_K \propto (0+1)\frac{0.69\Gamma_{\text{tot}}^2}{(E^2 - m_K^2)^2 + E^2\Gamma_{\text{tot}}^2}$$

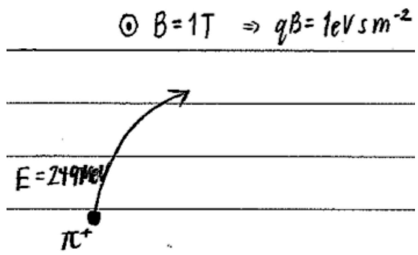
where $\Gamma_{\text{tot}} = \Gamma_K + \Gamma_\rho$ is assumed.

$$\Rightarrow \frac{\sigma_\rho}{\sigma_K} = \frac{2}{0.69}\frac{(E^2 - m_K^2)^2 + E^2\Gamma_{\text{tot}}^2}{(E^2 - m_p^2)^2 + E^2\Gamma_{\text{tot}}^2}$$
$$= 0$$

for $E = 249\,\text{MeV} \times 2 = m_K$.

(b) For $E = 250\,\text{MeV} \times 2$, $\dfrac{\sigma_\rho}{\sigma_K} = 0.129$.

It is impractical to study K^0 resonance with $\pi^+\pi^-$ collision as the partial width is so small that it is overwhelmed by other resonances within an error window.



(c) Lorentz force $\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$:

$$\Rightarrow qBv = \gamma\frac{mv^2}{r} \quad \text{for circular motion}$$
$$\frac{v}{r} = \frac{qB}{m}$$
$$\Rightarrow r = \frac{mv}{qB}$$
$$= \frac{p}{qB} \quad \text{where } p \text{ is momentum}$$
$$= \frac{\sqrt{E^2 - m_\pi^2 c^4}}{qBc}$$
$$= 0.687\,\text{m}$$

3. **(DRAFT)**

(a) Deuterium production: $\mathrm{p} + \mathrm{p} \to \mathrm{np} + \mathrm{e}^+ + \nu_\mathrm{e}$

Q-value:

$$\begin{aligned} Q &= -(m_{np} + m_e)c^2 + (2m_pc^2) \quad \text{assuming } m_\nu \text{ negligible} \\ &\simeq (2m_p - m_n - m_p - m_e)c^2 \quad \text{assuming negligible binding energy} \\ &= -1.811\,\text{MeV?} \end{aligned}$$

(b) Total energy released $E = WT$ where $W = 3.86 \times 10^{26}\,\text{W}$, $T = 4.6 \times 10^9\,\text{yr}$.

Each fusion has energy nett output of:

$$\mathcal{E} = Q + 2\epsilon_e - 2\epsilon_\nu$$

where $Q = 24.68\,\text{MeV}$, $\epsilon_e = 1.02\,\text{MeV}$, $\epsilon_\nu = 0.26\,\text{MeV}$.

Total energy produced if all H was used:

$$\mathbb{E} = N_0\mathcal{E}$$

where $N_0 = \frac{1}{4} \times 9 \times 10^{56}$.

Lifetime of the helium production $T_\text{total} = \frac{\mathbb{E}}{W}$.

$\Rightarrow$ Remaining time:

$$\begin{aligned} T_\text{left} &= T_\text{total} - T \\ &= \frac{\mathbb{E}}{W} - T \\ &= \frac{N_0\left[Q + 2\epsilon_e - 2\epsilon_\nu\right]}{W} - T \\ &= \left(7.8 \times 10^{10} - 4.6 \times 10^9\right)\text{yr} \\ &= 7.3 \times 10^{10}\,\text{yr} \end{aligned}$$

(c) SEMF assumes the validity of the liquid drop model, so:

- a_v is the volume term that states the strong field is short-ranged, hence each nucleon may only interact with its neighbours and renders the potential $\propto$ the volume of nucleus.
- a_s is the surface term that provides correction for the nucleons at the surface of the nucleus since they have fewer neighbours.
- a_c is the Coulomb term that accounts for the electrostatic repulsion between protons.
- a_a is the asymmetry term, it accounts for the fact that nucleons are fermions and must thus occupy different energy levels, for large nucleus it is favourable to have neutrons and protons equal in numbers to avoid energy penalty in the levels.
- a_p is the pairing term that accounts for the overlap of the nucleon wavefunction due to spin alignment, even-even nucteus is favoured since it forms $S = 0$ singlet and allows the nucleons to overlap each other.

$Z(m_p + m_e)$ accounts for the proton and electron mass, $(A-Z)m_n$ accounts for the neutron masses.

Since strong field is attractive, a_v is $-$ve, a_s is +ve for it being a correction term.

a_c is +ve since the EM interaction is repulsive.

a_a is +ve since energy penalty is added for asymmetry.

a_p is +ve or $-$ve depending on the # of nucleons, odd-odd is +ve since $S = 1$ has higher energy state.

(d)

$$^{235}_{92}U + n \rightarrow {}^{236}_{92}U \rightarrow {}^{92}_{37}Rb + {}^{140}_{55}Cs + 4n$$

Q-value of the fission:

$$\begin{aligned} Q &= [M(92,236) - M(37,92) - M(55,140) - 4m_n]\, c^2 \\ &= 152.38\,\text{MeV} - 12(236)^{1/2}\text{MeV} - 0\,\text{MeV} - 0\,\text{MeV} \\ &= -31.97\,\text{MeV}? \end{aligned}$$

There appears to be a sign error in the a_p terms.

Nevertheless, total energy production by the reactor in a year: $E = WT$ with $W = 100\,\text{MW}$, $T = 1\,\text{yr}$.

$\Rightarrow$ # of uranium burnt:

$$\begin{aligned} N &= \frac{E}{Q} \\ &= 6.16 \times 10^{26} \end{aligned}$$

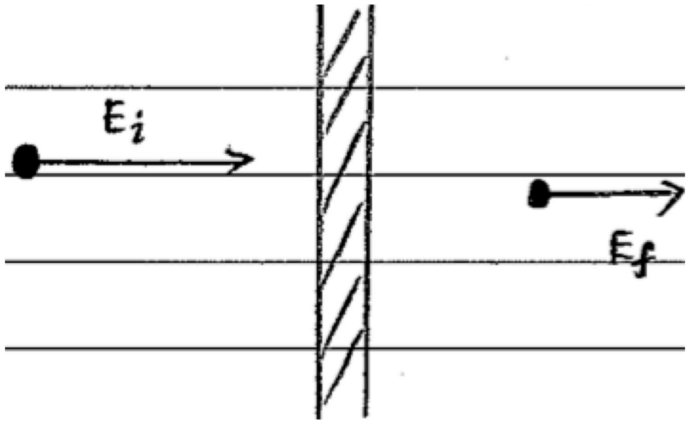
Mass of ^{235}U: $235u$

$\Rightarrow$ Mass of ^{235}U burnt:

$$\begin{aligned} M &= 235Nu \\ &= 240\,\text{kg} = 240\,000\,\text{g} \end{aligned}$$

OR if a_p is wrong then $Q = 336.73\,\text{MeV}$:

$$\Rightarrow N = 5.85 \times 10^{25}$$
$$\begin{aligned} M &= 22.81\,\text{kg} \\ &= 22\,810\,\text{g} \end{aligned}$$



(e) Reduction ratio:

$$R = \frac{m_c^2 + m_n^2}{(m_c + m_n)^2}$$
$$= 0.857$$

For n collisions, final energy $E_f = R^n E_i$.

For $E_i = 2\,\mathrm{MeV},\ E_f = 0.025\,\mathrm{eV}$:

$$\Rightarrow n = \frac{\ln(E_f) - \ln(E_i)}{\ln(R)}$$
$$= 117.8 \simeq 118$$

So 118 collisions required to thermalise the emitted neutrons.

4. (DRAFT)

(a) 1st gen in SM:

Quarks: u, d

Leptons: e^-, μ_e

So Z decays:

$\mathrm{Z}^0 \to \mathrm{u\bar{u}}$, $\mathrm{Z}^0 \to \mathrm{d\bar{d}}$, $\mathrm{Z}^0 \to \mathrm{e}^-\bar{\nu}_\mathrm{e}$, $\mathrm{Z}^0 \to \mathrm{e}^+\nu_\mathrm{e}$

W decays:

$\mathrm{W}^- \to \mathrm{\bar{u}d}$, $W^- \to \mathrm{u\bar{d}}$, $\mathrm{W}^- \to \mathrm{e}^-\bar{\nu}_\mathrm{e}$, $\mathrm{W}^- \to \mathrm{e}^+\nu_\mathrm{e}$

For 1st gen, all particles have masses $\ll m_W c^2$, so 4-point interaction is a good approximation $\Rightarrow$ Sargent's rule applies and d.o.s. $\propto Q^5$.

So:

$$\mathrm{BR}(W \to l\nu_l) = \frac{m_e^5}{m_e^5 + 3m_n^5} = 8.29 \times 10^{-12}$$
$$\mathrm{BR}(W \to qq') = \frac{3m_u^5}{m_e^5 + 3m_u^5} = 1$$

(b) For each generation, we have degeneracy of 3 from colours + 1 from leptons so:

$$4n = \frac{2085}{232} \quad \text{assuming similar partial widths}$$
$$n \simeq 2.25$$

So there must be at least 2 generations of particles.

(c) For 4th gen to occur, their masses must be so massive ($\sim m_W c^2$) so that BR is negligible.