UNOFFICIAL SOLUTIONS BY TheLongCat

B2: SYMMETRY AND RELATIVITY

TRINITY TERM 2018

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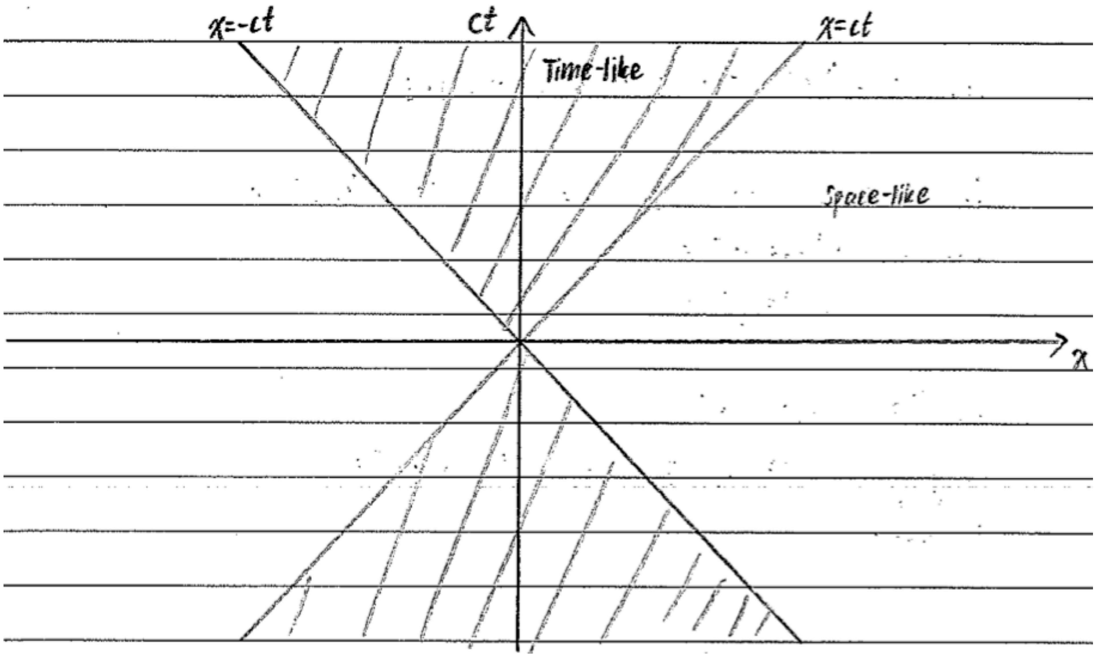
Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. (DRAFT)

(a) Space-like interval: $(\mathsf{D}^\mu - \mathsf{G}^\mu)^2 > 0$ Time-like interval: $(\mathsf{D}^\mu - \mathsf{G}^\mu)^2 < 0$

Nothing stops the 2 events from being connected by a null vector so it is possible, unless they are connected by causality then the interval ≤ 0.



Within the shaded time cone, the spacetime interval between an event and the origin < 0 and must remain so in all frames. Conversely, the events outside of the light cone would be space-like and therefore unreachable by signals.

Now:

$$\begin{aligned}
\mathsf{Y}_\mu \mathsf{X}^\mu &= 0 \\
\Rightarrow -c^2 t t' + \mathbf{x} \cdot \mathbf{x}' &= 0 \\
\mathbf{x} \cdot \mathbf{x}' &= -c^2 t t' \\
\text{For this relation to hold, } \mathbf{x} \cdot (\mathbf{x} + \Delta \mathbf{x}) &= c^2 (t + \Delta t)\, t \\
x^2 + \mathbf{x} \cdot \Delta \mathbf{x} &= c^2 t^2 + c^2 t \Delta t \\
\Rightarrow \mathsf{X}_\mu \mathsf{X}^\mu + \mathsf{X}_\mu (\mathsf{Y} - \mathsf{X})^\mu &= 0 \\
\Rightarrow \mathsf{X}^\mu &= \mathsf{X}^\mu - \mathsf{Y}^\mu \\
\Rightarrow \mathsf{Y}^\mu &= 0
\end{aligned}$$

By symmetry, X^μ could also be 0, or that $\mathbf{x} \cdot \mathbf{x}' = c^2 t t'$.

(b) 4-acceleration: $\mathsf{A}^\mu = {}^{\mathrm{d}\mathsf{U}^\mu}/_{\mathrm{d}\tau} = \gamma\left(\dot{\gamma} c, \dot{\gamma} \mathbf{v} + \gamma \mathbf{a}\right)$ 4-wavevector: $\mathsf{K}^\mu = (\omega / c, \mathbf{k})$ 4-current: $\mathsf{J}^\mu = \rho \mathsf{U}^\mu = (\gamma \rho c, \gamma \rho \mathbf{v})$ So:

$$\mathsf{A}_\mu \mathsf{J}^\mu = -\dot{\gamma} \gamma^2 \rho c^2 + \gamma^2 \dot{\gamma} \rho v^2 + \gamma^3 \rho \mathbf{a} \cdot \mathbf{v}$$

In a rest frame, $\gamma = 1$, $\mathbf{v} = 0 \Rightarrow \mathsf{A}_\mu \mathsf{J}^\mu = -\dot{\gamma} \rho c^2 \neq 0$.

Likewise $\mathsf{A}^\mu \mathsf{A}_\mu = a_0^2 \neq 0$ and $\mathsf{J}_\mu \mathsf{J}^\mu = \rho^2 c^2$.

Also $\mathsf{K}_\mu \mathsf{K}^\mu = -\omega^2/c^2 + k^2$, and we have the following dispersion for light *in vacuo*:

$$\omega = ck \Rightarrow \mathsf{K}_\mu \mathsf{K}^\mu = -k^2 + k^2 = 0$$

(c) 4-momentum:

$$\begin{aligned}\mathsf{P}^\mu &= m\mathsf{U}^\mu = m\frac{\mathrm{d}\mathsf{X}^\mu}{\mathrm{d}\tau} \quad \text{where } \mathsf{U}^\mu = \frac{\mathrm{d}\mathsf{X}^\mu}{\mathrm{d}\tau} \text{ is the 4-velocity} \\ &= \gamma m\left(c, \mathbf{v}\right) \quad \text{where } \mathbf{v} \text{ is 3-velocity}\end{aligned}$$

4-acceleration:

$$\mathsf{A}^\mu = \frac{\mathrm{d}\mathsf{U}^\mu}{\mathrm{d}\tau} = \gamma\left(\dot{\gamma}c, \dot{\gamma}\mathbf{v} + \gamma\mathbf{a}\right) \quad \text{where } \mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \text{ is 3-acceleration}$$

Then,

$$\mathsf{P}_\mu \mathsf{P}^\mu = -\gamma^2 m^2 c^2 + \gamma^2 m^2 v^2$$

Evaluating in rest frame gives $\gamma = 1$, $\mathbf{v} = 0$ so $\mathsf{P}_\mu \mathsf{P}^\mu = -m^2c^2$.

Next we have:

$$\mathsf{A}^\mu \mathsf{A}_\mu = -\gamma^2\dot{\gamma}^2c^2 + \gamma^2\dot{\gamma}^2v^2 + 2\gamma^3\dot{\gamma}\mathbf{v}\cdot\mathbf{a} + \gamma^4a^2$$

Noting that:

$$\begin{aligned}\gamma &= \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \\ \dot{\gamma} &= -\frac{1}{2}\underbrace{\left(1 - \frac{v^2}{c^2}\right)^{-3/2}}_{\gamma^3}\left(-\frac{2v}{c^2}\cdot\frac{\mathrm{d}v}{\mathrm{d}t}\right) \\ &= \frac{v}{c^2}\gamma^3 a\end{aligned}$$

For $\mathbf{a} \parallel \mathbf{v}$, $\mathbf{a}\cdot\mathbf{v} = av$, thus:

$$\begin{aligned}\mathsf{A}^\mu \mathsf{A}_\mu &= -\gamma^2\left(\frac{v}{c^2}\gamma^3 a\right)^2 c^2 \\ &\quad + \gamma^2\left(\frac{v}{c^2}\gamma^3 a\right)^2 v^2 \\ &\quad + 2\gamma^3\left(\frac{v}{c^2}\gamma^3 a\right)va + \gamma^4a^2 \\ &= -\gamma^8a^2v^2\cdot\frac{1}{c^2} \\ &\quad + \gamma^8a^2v^4\cdot\frac{1}{c^4} \\ &\quad + 2\gamma^6a^2v^2\cdot\frac{1}{c^2} + \gamma^4a^2 \\ &= \gamma^6a^2\left[-\gamma^2\beta^2 + \gamma^2\beta^4 + 2\beta^2 + \frac{1}{\gamma^2}\right] \\ &= \gamma^6a^2\left[-\beta^2\cancelto{1}{\gamma^2\left(1-\beta^2\right)} + 2\beta^2 + 1 - \beta^2\right] \\ &= \gamma^6a^2\end{aligned}$$

(d) 3-force: $\mathbf{f} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \dot{\gamma} m\mathbf{v} + \gamma m\mathbf{a}$

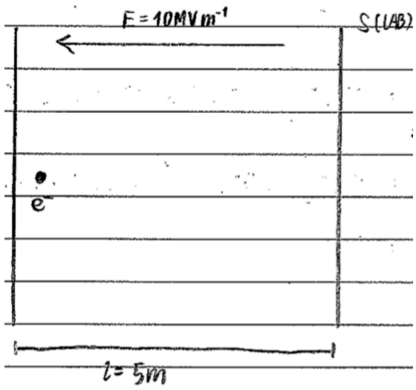
For central force, $\dot{\gamma} = 0$:

$$\Rightarrow \mathbf{f} = \gamma m\mathbf{a} = \frac{\alpha \mathbf{r}}{r^3}$$
$$\Rightarrow \mathsf{F}^\mu = (0, \gamma\mathbf{f}) \quad \text{since central force does no work}$$

Potential of $\mathbf{f}$ is given by $\int_{-\infty}^{r} \alpha/r^2 \,\mathrm{d}r = -\alpha/r$.

Total energy of the system $\gamma mc^2 - \alpha/r$ is thus conserved. One example of central force is Coulomb attraction between an electron and a nucleus.

(e) Sketch of the setup:



In S, the e$^-$ gains energy qEl, thus at the end gaining a total energy of:

$$E' = E_0 + \Delta E$$
$$\gamma mc^2 = mc^2 + qEl$$
$$\gamma = 1 + \frac{qEl}{mc^2} = 97.7$$

In e$^-$ rest frame, we have $E'_\parallel = \gamma E_\parallel$ assuming $B = 0$:

$$\Rightarrow \text{Lorentz force } \mathbf{f} = qE'$$
$$\Rightarrow ma_0 = \gamma qE$$
$$a_0 = \frac{\gamma qE}{m}$$

Also $\gamma^6 a^2 = a_0^2$ from before, so:

$$\gamma^6 a^2 = \frac{\gamma^2 q^2 E^2}{m^2}$$
$$a^2 = \frac{q^2 E^2}{\gamma^4 m^2}$$
$$a = \frac{qE}{\gamma^2 m}$$

Hence:

$$\begin{aligned}
\dot{\gamma} &= \frac{\overbrace{v}^{\beta/c}}{c^2}\gamma^3 \cdot \frac{qE}{\gamma^2 m} \\
&= \frac{1}{c}\left[1-\gamma^{-2}\right]^{-1/2}\frac{qE}{m}\gamma \\
\frac{\mathrm{d}\gamma}{\mathrm{d}t} &= \frac{qE}{mc}\sqrt{\gamma^2-1} \\
\int_1^\gamma \frac{\mathrm{d}\gamma}{\sqrt{\gamma^2-1}} &= \int_0^t \frac{qE}{mc}\,\mathrm{d}t
\end{aligned}$$

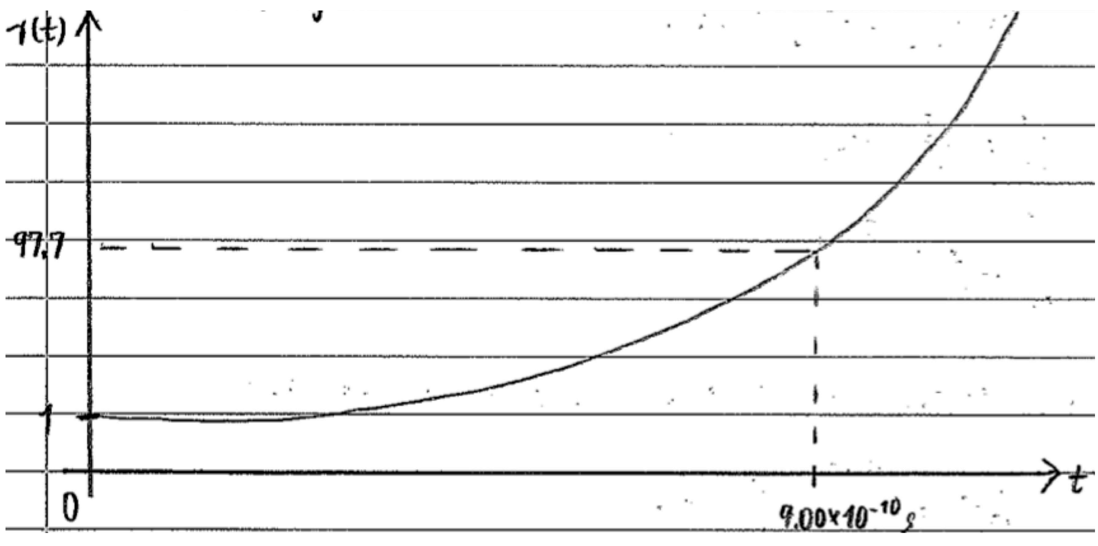
The LHS gives:

$$\begin{aligned}
\frac{\mathrm{d}\gamma}{\sqrt{\gamma^2-1}} &= \operatorname{arcosh}\gamma - \underbrace{\operatorname{arcosh}1}_{0} \\
\Rightarrow \operatorname{arcosh}\gamma &= \frac{qEt}{mc} \\
\gamma &= \cosh\left(\frac{qEt}{mc}\right) \\
\Rightarrow \beta = \left(1-\gamma^{-2}\right)^{1/2} &= \tanh\left(\frac{qEt}{mc}\right) \qquad \text{since rapidity } \rho = qEt/m
\end{aligned}$$

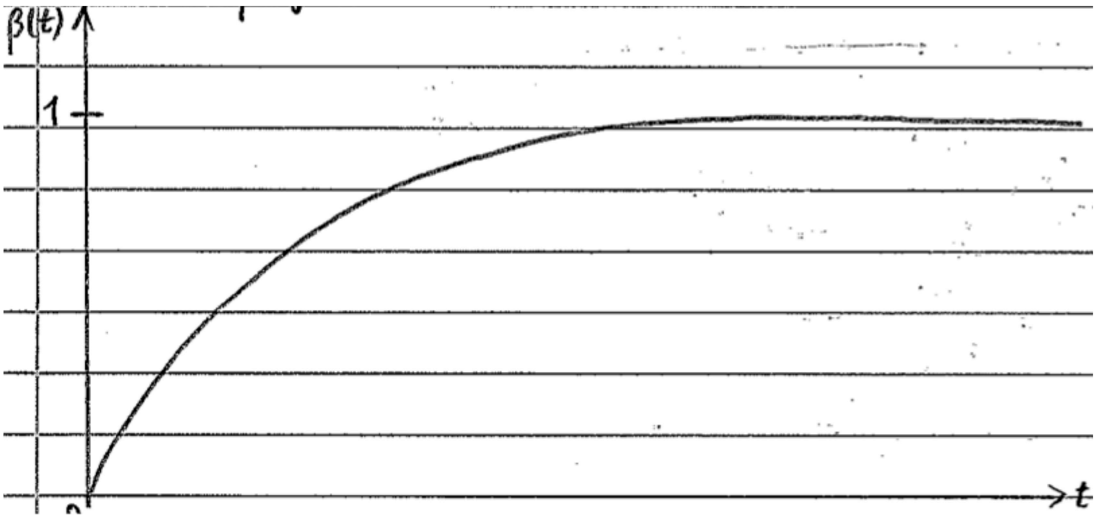
At the end of the gap,

$$\begin{aligned}
t &= \frac{mc}{qE}\operatorname{arcosh}\gamma \\
&= 9.00\times 10^{-10}\,\mathrm{s}
\end{aligned}$$

Sketch of γ against t:



Sketch of β against t:



Replace $E \to 0.8E$,

$$\beta = \tanh\left(0.8\frac{qEt}{mc}\right)$$
$$\Rightarrow \rho = 0.8\frac{qEt}{mc} = 4.22$$
$$\Rightarrow \beta = \tanh 4.22$$
$$\Rightarrow \frac{\beta}{\beta_0} = \frac{\tanh 4.22}{\tanh\left(\operatorname{arcosh} 97.7\right)} = 0.9996$$

Replace $m \to m_p$,

$$\rho = \frac{qEt}{m_p c} = 2.87 \times 10^{-3}$$
$$\Rightarrow \gamma = \cosh 2.87 \times 10^{-3} = 1.000\,004$$
$$\beta = \tanh 2.87 \times 10^{-3} = 2.87 \times 10^{-3}$$

2. (DRAFT)

(a) 4-wavevector $\mathsf{K}^\mu = (\omega/c, \mathbf{k})$

Dispersion *in vacuo* means that $\omega = ck \Rightarrow \mathsf{K}_\mu \mathsf{K}^\mu = 0$.

Phase velocity v_p and group velocity v_g are given by:

$$\begin{aligned} v_p &= \frac{\omega}{k} \\ &= \frac{c\mathsf{K}^0}{\sqrt{\mathsf{K}_i\mathsf{K}^i}} = \frac{c\mathsf{K}^0}{\mathsf{K}^1} \quad \text{in standard config} \end{aligned}$$

$$\begin{aligned} v_g &= \frac{\partial \omega}{\partial k} \\ &= \partial_i \partial^i c\mathsf{K}^0 \quad \text{for } \partial^i = \frac{\partial}{\partial k_i} \\ &= \partial_1 \partial^1 c\mathsf{K}^0 \quad \text{in standard config} \end{aligned}$$

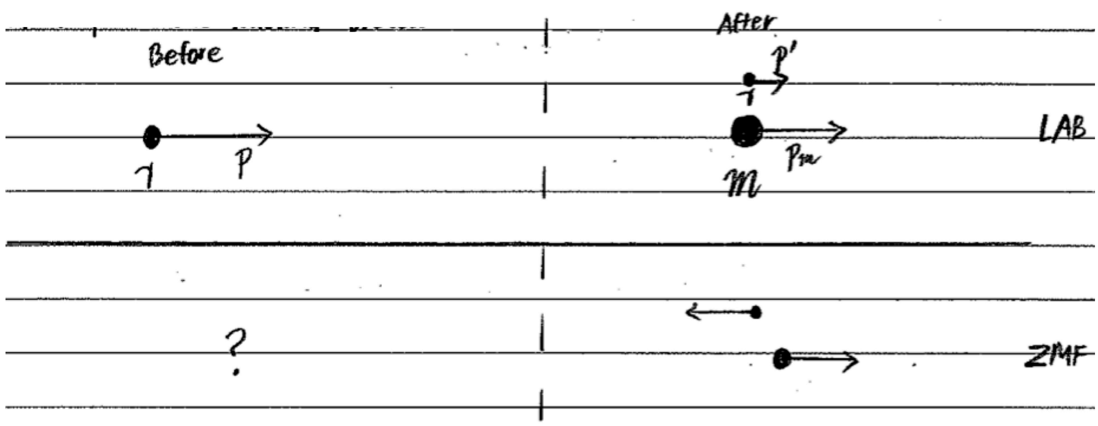
Also for a 4-position $\mathsf{X}^\mu = (ct, \mathbf{x})$, the phase ϕ is given as:

$$\begin{aligned} \phi &= \mathsf{K}_\mu \mathsf{X}^\mu \\ &= -\omega t + \mathbf{k}\cdot\mathbf{x} \end{aligned}$$

However as the relation of ϕ with ωt and $\mathbf{k}\cdot\mathbf{x}$ may be written in terms of the contraction of two 4-vectors, ϕ is Lorentz invariant.

For $v_p = c\mathsf{K}^0/\sqrt{\mathsf{K}_i\mathsf{K}^i}$, since there is no full contraction between two 4-vectors, it is not Lorentz invariant, but it follows Lorentz transformation as $v_p' = c\Lambda^\kappa_0\mathsf{K}^0/\sqrt{\Lambda^i_j\Lambda^j_i\mathsf{K}_i\mathsf{K}^i}$.

(b) Propose the following process:



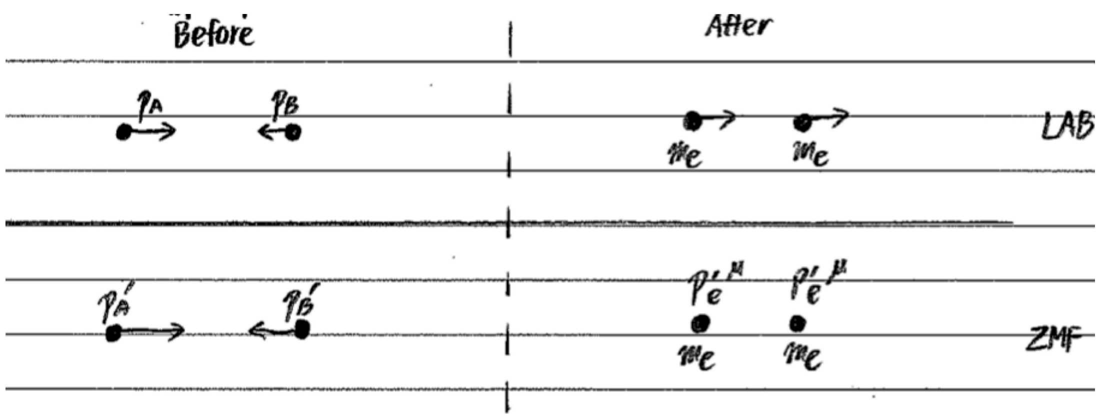
Since there exsits no rest frame for photons (for $\mathsf{P}_\mu \mathsf{P}^\mu = 0$), the above process is kinematically forbidden as per the first postulate of SR.

In ZMF,

$$
\begin{aligned}
\mathsf{P}^{\mu} &= \mathsf{P}'^{\mu}_{m} + \mathsf{P}'^{\mu}_{\gamma} \\
\underbrace{\mathsf{P}_{\mu}\mathsf{P}^{\mu}}_{0} &= \mathsf{P}_{\mu}\mathsf{P}'^{\mu}_{m} + \mathsf{P}_{\mu}\mathsf{P}'^{\mu}_{\gamma} \\
\Rightarrow -\frac{E'}{c}\cdot\frac{E'_m}{c} + \frac{E'}{c}p'_m &= -\frac{E'}{c}\cdot\frac{E'_\gamma}{c} + \frac{E'}{c}\cdot\frac{E'_\gamma}{c} \\
\Rightarrow \frac{E'_m}{c} &= p'_m \\
\Rightarrow m &= 0 \quad \text{like photons!}
\end{aligned}
$$

The issue with one photon is the fact that the system 4-momentum has null norm $\Rightarrow$ minimum of 2 photons required for mass generation.

Now consider e^+e^- pair production (at threshold):



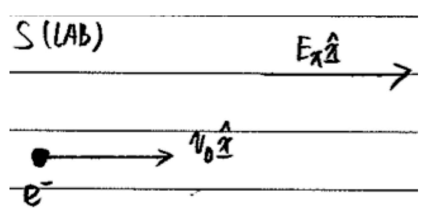
At threshold, e^+e^- pair is at rest in ZMF:

$$
\begin{aligned}
\text{Conservation of 4-momentum: } \mathsf{P}^{\mu}_{\text{ZMF}} &= 2\mathsf{P}'^{\mu}_{\text{e}} \\
-\left(p'_A + p'_B\right)^2 &= -2\left(p'_A + p'_B\right) m_e c \\
\Rightarrow p'_A + p'_B &= 2 m_e c
\end{aligned}
$$

For photons with the same frequency in LAB, which is also now ZMF:

$$
\begin{aligned}
2p' &= 2m_e c \\
p' &= m_e c \\
\Rightarrow E_{\text{CM, min}} &= m_e c^2
\end{aligned}
$$

(c) i. Lab frame sketch:



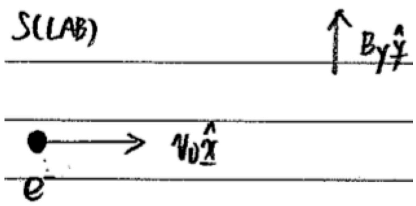
$$\begin{aligned}\text{4-force } \mathsf{F}^\mu &= \frac{\mathrm{d}}{\mathrm{d}\tau}\mathsf{P}^\mu \quad \text{where } \mathsf{P}^\mu = (\gamma mc, \gamma m\mathbf{v}) \text{ is the 4-momentum}\\ &= \gamma\frac{\partial \mathsf{P}^\mu}{\partial t}\\ &= \gamma\left(\dot{\gamma}mc + \gamma \dot{m}c, \dot{\gamma}m\mathbf{v} + \gamma m\mathbf{a}\right) \quad \text{where } \mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \text{ is 3-acceleration}\end{aligned}$$

By Lorentz force,

$$\begin{aligned}\mathbf{f} &= q\left(\mathbf{E} + \cancelto{0}{\mathbf{v}\times\mathbf{B}}\right)\\ &= qE\hat{\mathbf{x}} = -eE\hat{\mathbf{x}} \Rightarrow \mathbf{a} \parallel \mathbf{v}\ \forall t\end{aligned}$$

So $\dot{\gamma} = 0$, $\dot{m} = 0$, $\mathbf{a} \neq 0$ and so none of the components is conserved.

ii. Lab frame sketch:

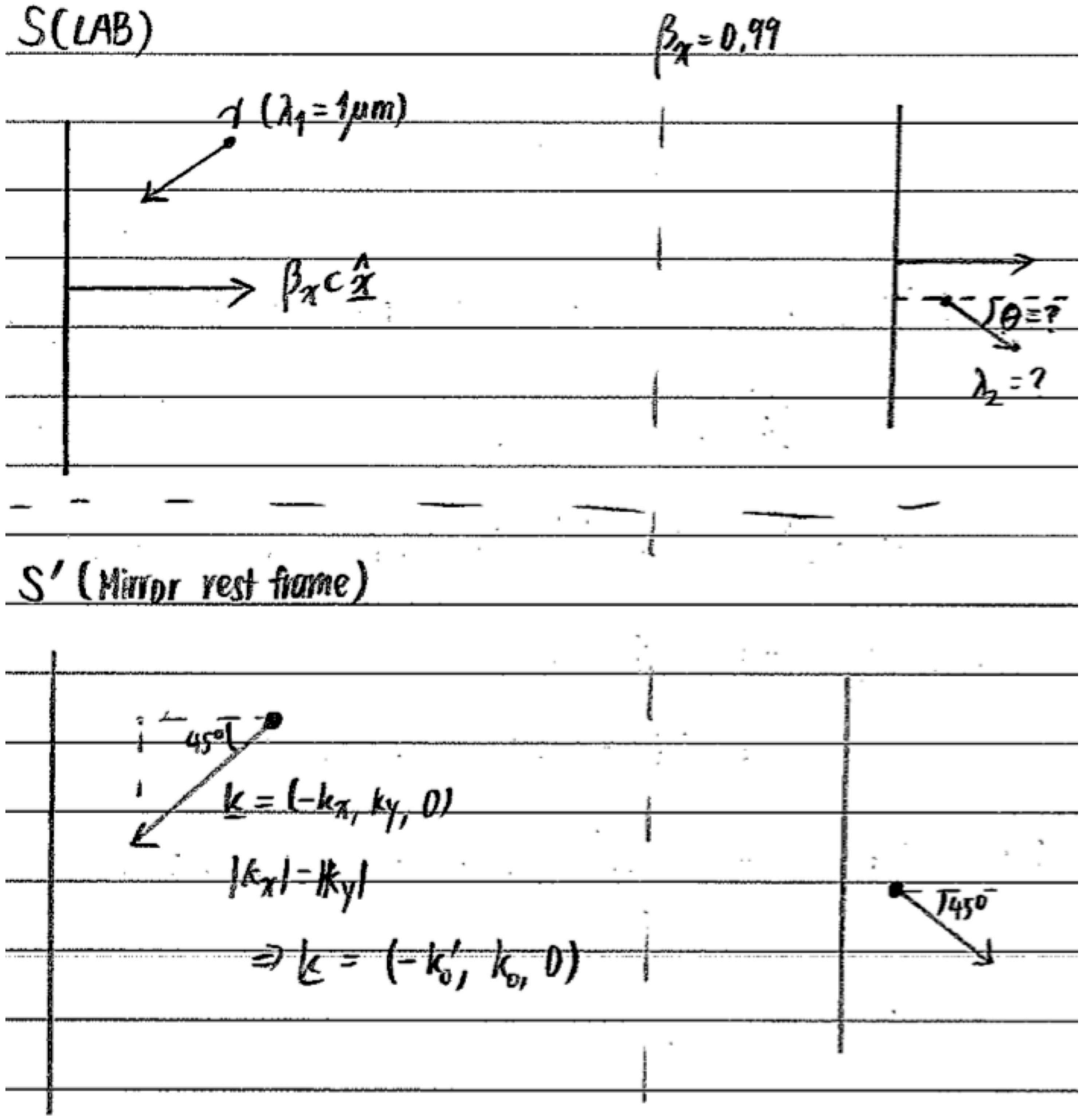


Swapping the situation with $\mathbf{E} = 0$, $\mathbf{B} = B_y\hat{\mathbf{y}}$ gives:

$$\begin{aligned}\mathbf{f} &= -e\left[v_0\hat{\mathbf{x}} \times B_y\hat{\mathbf{y}}\right]\\ &= -ev_0B_y\hat{\mathbf{z}}\\ \Rightarrow \mathbf{f} &\perp \mathbf{v}\ \forall t \quad \text{due to the cross product}\end{aligned}$$

So $\dot{m} = 0$, $\dot{\gamma} = 0$ and so $\mathsf{F}^\mu = (0, \gamma m\mathbf{a}) \Rightarrow \mathsf{P}^0$ the energy component is conserved.

(d) Sketch of both lab and mirror rest frames:



In S', the law of reflection gives the angle of reflection $\theta_0 = 45°$.

Hence the final 4-wavevector in S' is given by:

$$(\mathsf{K}_0)^\mu = \left(\frac{\omega_0}{c}, k_0 \cos\theta_0, k_0 \sin\theta_0, 0\right)$$

where $\omega_0 = ck_0$ is the frequency in S'.

Before reflection, initial 4-wavevectors are realted by $\mathsf{K}_0^\mu = \Lambda^\mu_\nu \mathsf{K}^\nu$:

$$\Rightarrow \frac{\omega_0}{c} = \gamma\frac{\omega}{c} - \beta\gamma k \cos\theta_{\text{bef}} \tag{1}$$

$$k_0 \cos\theta_0 = -\beta\gamma\frac{\omega}{c} + \gamma k \cos\theta_{\text{bef}}$$

$$\Rightarrow \gamma k \cos\theta_{\text{bef}} = k_0 \cos\theta_0 + \beta\gamma\frac{\omega}{c} \tag{2}$$

Combine (1) and (2):

$$\begin{aligned}\frac{\omega_0}{c} &= \frac{\gamma\omega}{c} - k_0\cos\theta_0 - \frac{\beta\gamma\omega}{c}\\ \Rightarrow k_0\left(1+\cos\theta_0\right) &= k\gamma\left(1-\beta\right)\\ k_0 &= k\gamma\frac{1-\beta}{1+\cos\theta_0} \qquad (3)\end{aligned}$$

Boosting the final 4-wavevector back to S:

$$\begin{aligned}\left(\mathsf{K}'\right)^{\mu} &= \Lambda^{\mu}_{\nu}\left(\mathsf{K}'_0\right)^{\nu}\\ \Rightarrow \frac{\omega'}{c} &= \frac{\gamma\omega_0}{c} + \beta\gamma k_0\cos\theta_0 \qquad (4)\\ k'\cos\theta &= \frac{\beta\gamma\omega_0}{c} + \gamma k_0\cos\theta_0 \qquad (5)\end{aligned}$$

(4) and (3) gives:

$$\begin{aligned}\frac{\omega'}{c} &= \gamma k_0 + \beta\gamma k_0\cos\theta_0\\ \Rightarrow k' &= \gamma^2 k\frac{1-\beta}{1+\cos\theta_0}\left(1+\beta\cos\theta_0\right) \qquad (6)\end{aligned}$$

(5) and (3) gives:

$$\begin{aligned}k'\cos\theta &= \beta\gamma k_0 + \gamma k_0\cos\theta_0\\ \xrightarrow{\div(6)} \cos\theta &= \frac{\gamma k_0\left(\beta+\cos\theta_0\right)}{\gamma k_0\left(1+\beta\cos\theta_0\right)} = \frac{\beta+\cos\theta_0}{1+\beta\cos\theta_0} \qquad (7)\end{aligned}$$

Wavelength:

$$\begin{aligned}\lambda &= \frac{2\pi}{k} \Rightarrow k = \frac{2\pi}{\lambda_1}\\ \xrightarrow{(6)} \lambda_2 &= \lambda_1\frac{1+\cos\theta_0}{\left(1-\beta\right)\left(1+\beta\cos\theta_0\right)\underbrace{\gamma^2}_{\left(1-\beta^2\right)^{-1}}}\end{aligned}$$

$$\lambda_2 = \overbrace{\lambda_1}^{1\,\mu\text{m}}\frac{\left(1+\cos 45^\circ\right)\left(1+0.99\right)}{1+0.99\cos 45^\circ} = 1.9983\,\mu\text{m}$$

And finally:

$$\begin{aligned}\theta &= \arccos\frac{\beta+\cos\theta_0}{1+\beta\cos\theta_0}\\ &= \arccos\frac{0.99+\cos 45^\circ}{1+0.99\cos 45^\circ}\\ &= 3.36^\circ\end{aligned}$$

3. **(DRAFT)**

(a)

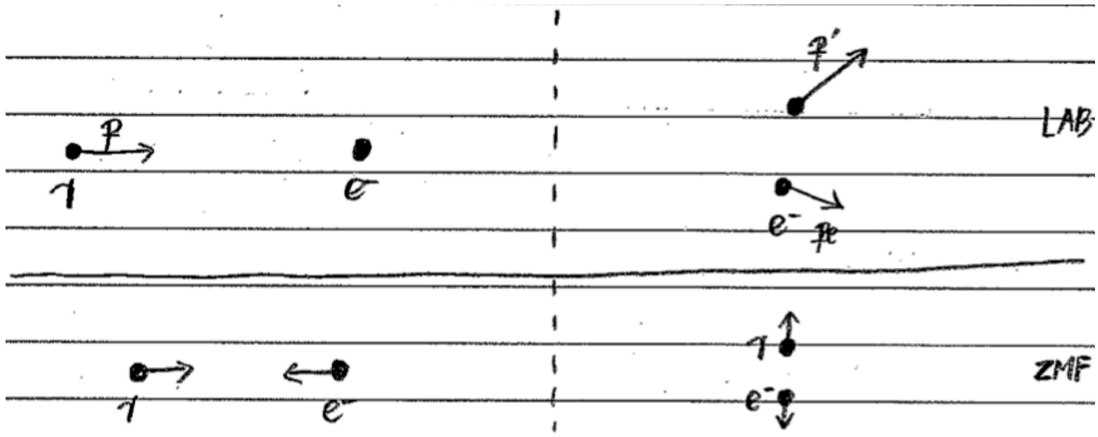
$$\text{4-acceleration } \mathsf{A}^\mu = \frac{\mathrm{d}\mathsf{X}^\mu}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}\mathsf{X}^\mu}{\mathrm{d}t} \quad \text{with } \mathsf{X}^\mu \text{ the 4-velocity}$$
$$= \gamma\left(\dot{\gamma}c, \dot{\gamma}\mathbf{v} + \gamma\mathbf{a}\right) \quad \text{with } \mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} \text{ the 3-acceleration}$$

$$\text{4-force } \mathsf{F}^\mu = \frac{\mathrm{d}\mathsf{P}^\mu}{\mathrm{d}\tau} = \gamma\frac{\mathrm{d}\mathsf{P}^\mu}{\mathrm{d}t} \quad \text{with } \mathsf{P}^\mu \text{ the 4-momentum}$$
$$= \gamma\left(\dot{\gamma}mc + \gamma\dot{m}c, \dot{\gamma}m\mathbf{v} + \gamma\dot{m}\mathbf{v} + \gamma m\mathbf{a}\right)$$

Note that $\gamma \to \infty$ as $v \to c$, so SR places no bound on the possible sizes of the force or acceleration.

$$\text{Phase velocity } v_p = \frac{\omega}{k}$$
$$= \frac{c\mathsf{K}^0}{\sqrt{\mathsf{K}_i\mathsf{K}^i}} \quad \text{with } \mathsf{K}^\mu = \left(\frac{\omega}{c}, \mathbf{k}\right) \text{ the 4-wavevector}$$

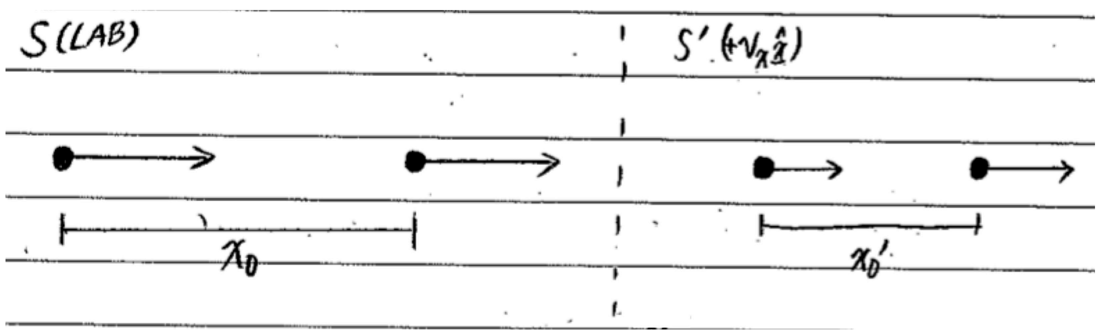
Note that there is no constraint on what form v_p should take, so SR does not restrict v_p, the confusion comes from the fact that $v_p \leq c$ only happens *in vacuo*.



Conservation of 4-momentum tells us:

$$\mathsf{P}_\gamma^\mu + \mathsf{P}_\mathrm{e}^\mu = \mathsf{P}_\gamma'^\mu + \mathsf{P}_\mathrm{e}'^\mu \quad \text{in LAB}$$
$$\Rightarrow \left(\mathsf{P}_\gamma^\mu + \mathsf{P}_\mathrm{e}^\mu - \mathsf{P}_\gamma'^\mu\right)^2 = \left(\mathsf{P}_\mathrm{e}'^\mu\right)^2$$
$$\cancelto{0}{\left(\mathsf{P}_\gamma\right)_\mu \left(\mathsf{P}_\gamma\right)^\mu} + \left(\mathsf{P}_\mathrm{e}\right)_\mu \left(\mathsf{P}_\mathrm{e}\right)^\mu + \cancelto{0}{\left(\mathsf{P}_\gamma'\right)_\mu \left(\mathsf{P}_\gamma'\right)^\mu}$$
$$+2\left[\left(\mathsf{P}_\gamma\right)_\mu \left(\mathsf{P}_\mathrm{e}\right)^\mu - \left(\mathsf{P}_\gamma\right)_\mu \left(\mathsf{P}_\gamma'\right)^\mu - \left(\mathsf{P}_\mathrm{e}\right)_\mu \left(\mathsf{P}_\gamma'\right)^\mu\right] = -m_\mathrm{e}^2c^2$$
$$-m_\mathrm{e}^2c^2 + 2\left[-pm_\mathrm{e}c + pp' - pp'\cos\theta + p'm_\mathrm{e}c\right] = -m_\mathrm{e}^2c^2$$
$$p'\left[p - p\cos\theta + m_\mathrm{e}c\right] = pm_\mathrm{e}c$$
$$p' = \frac{pm_\mathrm{e}c}{p\left(1-\cos\theta\right) + m_\mathrm{e}c}$$
$$\Rightarrow E' = \frac{E}{E/m_\mathrm{e}c^2\left(1-\cos\theta\right) + 1}$$
$$\Rightarrow f' = \frac{f}{\hbar f/m_\mathrm{e}c^2\left(1-\cos\theta\right) + 1}$$

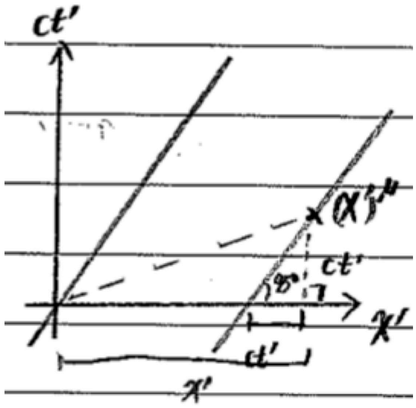
(b) Sketch of the lab frame and another travelling at $+v_x\hat{\mathbf{x}}$ relative to it:



Events separation $\mathsf{X}^\mu = [0, x_0]$

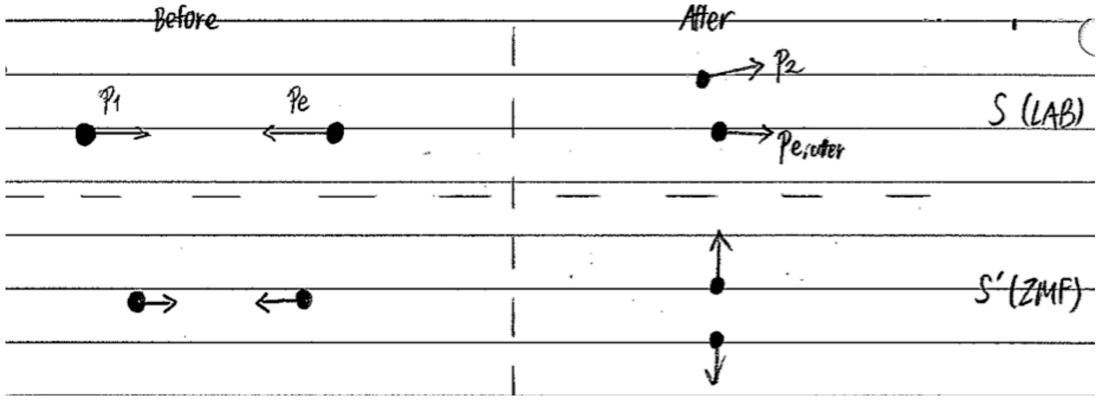
Boosting by $+v_x\hat{\mathbf{x}}$ gives:

$$(\mathsf{X}')^\mu = \Lambda^\mu_\nu \mathsf{X}^\nu$$
$$\Rightarrow ct' = -\beta\gamma x_0$$
$$x' = \gamma x_0$$



From the space-time diagram,

$$\begin{aligned} x_0' &= -ct' + x' \\ &= (\gamma + \beta\gamma)\, x_0 \\ &= \frac{1 + v_x/c}{\sqrt{1 - v_x^2/c^2}} x_0 \end{aligned}$$



(c) We know that $p_1 = hc/\lambda_1$. Conservation of 4-momentum gives:

$$\mathsf{P}_1^\mu + \mathsf{P}_\mathrm{e}^\mu = \mathsf{P}_2^\mu + \mathsf{P}_{\mathrm{e,after}}^\mu$$

$$\Rightarrow \left(\mathsf{P}_1^\mu + \mathsf{P}_\mathrm{e}^\mu - \mathsf{P}_2^\mu\right)^2 = \left(\mathsf{P}_{\mathrm{e,after}}^\mu\right)^2$$

$$\cancelto{0}{(\mathsf{P}_1)_\mu (\mathsf{P}_1)^\mu} + (\mathsf{P}_\mathrm{e})_\mu (\mathsf{P}_\mathrm{e})^\mu - \cancelto{0}{(\mathsf{P}_2)_\mu (\mathsf{P}_2)^\mu}$$

$$+2\left[(\mathsf{P}_1)_\mu (\mathsf{P}_\mathrm{e})^\mu - (\mathsf{P}_1)_\mu (\mathsf{P}_2)^\mu - (\mathsf{P}_\mathrm{e})_\mu (\mathsf{P}_1)^\mu\right] = (\mathsf{P}_{\mathrm{e,after}})_\mu (\mathsf{P}_{\mathrm{e,after}})^\mu$$

$$-m_\mathrm{e}^2c^2 + 2\left[-p_1\gamma m_\mathrm{e}c - p_1\gamma \underbrace{\gamma m_\mathrm{e} v}_{1/c\sqrt{\gamma^2 m_\mathrm{e}^2 c^4 - m_\mathrm{e}^2 c^4} = m_\mathrm{e}c\sqrt{\gamma^2-1}} + p_1p_2 - p_1p_2\cos\theta + p_2\gamma m_\mathrm{e}c + p_2\gamma m_\mathrm{e}v\cos\theta\right] = -m_\mathrm{e}^2c^2$$

$$\Rightarrow p_1\left(-\gamma m_\mathrm{e}c - m_\mathrm{e}c\sqrt{\gamma^2-1}\right) + p_1p_2 - p_1p_2\cos\theta + p_2\left(\gamma m_\mathrm{e}c + m_\mathrm{e}c\sqrt{\gamma^2-1}\cos\theta\right) = 0$$

$$p_2 = \frac{p_1\left(\gamma + \sqrt{\gamma^2-1}\right)m_\mathrm{e}c}{p_1 - p_1\cos\theta + m_\mathrm{e}c\left[\gamma + \sqrt{\gamma^2-1}\right]\cos\theta}$$

$$= \frac{p_1\left(\gamma + \sqrt{\gamma^2-1}\right)}{p_1/m_\mathrm{e}c\,(1-\cos\theta) + \left(\gamma + \sqrt{\gamma^2-1}\right)\cos\theta}$$

$$= p_1 \cdot \frac{\gamma + \sqrt{\gamma^2-1}}{\gamma + \sqrt{\gamma^2-1}\cos\theta} \quad \text{since } p_1c \ll m_\mathrm{e}c^2$$

$$\Rightarrow \lambda_2 = \lambda_1\frac{\gamma + \sqrt{\gamma^2-1}\cos\theta}{\gamma + \sqrt{\gamma^2-1}}$$

$$= \frac{\gamma - \sqrt{\gamma^2-1}}{\gamma + \sqrt{\gamma^2-1}}\lambda_1$$

$$= \frac{1}{\gamma^2 + \gamma^2 - 1 + 2\gamma\sqrt{\gamma^2-1}}\lambda_1$$

$$= \frac{1}{2\gamma^2 + 2\gamma^2}\lambda_1 \quad \text{for } \gamma \gg 1$$

$$= \frac{\lambda_1}{4\gamma^2}$$

For $\beta = 0.999$,

$$
\begin{aligned}
\gamma &= \left[1 - 0.999^2\right]^{-1/2} \\
&= 22.4 \\
\Rightarrow \lambda_2 &= \frac{\lambda_1}{4 \cdot 22.4^2} \\
&= 4.00 \times 10^{-3}\,\mu\mathrm{m} = 4.00\,\mathrm{nm}
\end{aligned}
$$

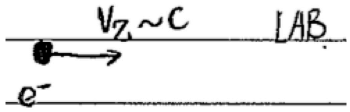
(d) As the wave propagates in x direction (or by symmetry in the yz plane), we have:

$$
\begin{aligned}
\mathsf{F}^{\mu\nu} &= \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix} \\
&= \partial^\mu \mathsf{A}^\nu - \partial_\nu \mathsf{A}^\mu \\
&= 0 \quad \text{for } \mu\nu = (0,1), (1,0), (2,3), (3,2) \\
\Rightarrow \mathsf{A}^\mu &= \left(\frac{\phi}{c}, A, 0, 0\right)
\end{aligned}
$$

In vacuum, $\mathbf{B} \perp \mathbf{E}\mathbf{k}$ as per Maxwell's equations. So choose $\mathbf{B} \parallel \hat{\mathbf{y}}$, $\mathbf{E} \parallel \hat{\mathbf{z}}$:

$$
\begin{aligned}
B_y &= \mathsf{F}^{31} \\
&= \partial^3 \mathsf{A}^1 - \partial^1 \cancelto{0}{\mathsf{A}^3} \\
&= \frac{\partial A}{\partial z}
\end{aligned}
$$

$$
\begin{aligned}
E_z &= \mathsf{F}^{03} \\
&= \partial^0 \cancelto{0}{\mathsf{A}^3} - \partial^3 \mathsf{A}^0 \\
&= -\frac{\partial \phi / c}{\partial c t} \\
&= -\frac{1}{c^2} \frac{\partial \phi}{\partial t}
\end{aligned}
$$



(e) In LAB, $\mathsf{A}^\mu = (\phi/c, \mathbf{0})$:

$$
\begin{aligned}
\mathsf{F}^{\mu\nu} &= \partial^\mu \mathsf{A}^\nu - \partial_\nu \mathsf{A}^\mu \\
&= \begin{bmatrix} 0 & {}^{1}\!/_{c}\, {}^{\partial}\!/_{\partial t} \mathsf{A}^1 - {}^{\partial \mathsf{A}^0}\!/_{\partial x} & \cdots & \cdots \\ \vdots & 0 & \cdots & \cdots \\ \vdots & \vdots & 0 & \cdots \\ \vdots & \vdots & \vdots & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & 0 & -{}^{\partial}\!/_{\partial z}(\phi/c) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ {}^{\partial}\!/_{\partial z}(\phi/c) & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial z}\left(\frac{\phi}{c}\right) &= \frac{1}{c}\frac{\partial}{\partial z}\left(\phi_0 \cos\left(k_u z\right)\right) \\
&= -\frac{\phi_0}{c} k_u \sin k_u z
\end{aligned}
$$

$$
\Rightarrow \mathbf{E} = \frac{\phi_0}{c} k_u \sin k_u z \hat{\mathbf{z}} \qquad \mathbf{B} = 0
$$

Boosting into the e^- rest frame gives $\mathbf{B}_\parallel = 0$, $\mathbf{E}_\perp = 0$.

$$
\begin{aligned}
\mathbf{B}_\perp &= \gamma \left(\mathbf{B}_\perp^{\text{LAB}} + \frac{\mathbf{v} \times \mathbf{E}}{c}\right) = 0 \\
\mathbf{E}_\parallel &= \gamma \left(\mathbf{E}_\parallel - \mathbf{v} \times \mathbf{B}\right) \\
&= \gamma \mathbf{E}_\parallel \\
&= \frac{\gamma \phi_0}{c} k_u \sin k_u \ \hat{\mathbf{z}}
\end{aligned}
$$

where $\gamma = [1 - (v_z/c)^2]^{-1/2}$.

For an EM wave, $\mathsf{F}_{\mu\nu}\mathsf{F}^{\mu\nu} \propto B^2 - E^2/c^2 = 0$, however in this case $B^2 - E^2/c^2 \neq 0$ in the e^- rest frame, so the field is not an EM wave.

4. (DRAFT)

(a) Define parity operator $\hat{\rho}$ such that it maps a 3-vector $(x, y, z) \to (-x, -y, -z)$. So we expect for a vector $\mathbf{v}$, $\hat{\rho}\hat{\rho}\mathbf{v} = \mathbf{v}$, but then 2 possibilities arise:

1. A polar vector obeys the transformation above such that $\hat{\rho}\mathbf{v} = -\mathbf{v}$;
2. An axial vector, however, does not flip its sign upon $\hat{\rho}$: $\hat{\rho}\mathbf{v} = \mathbf{v}$

Lorentz force $\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

Since $\mathbf{f}$ should revese upon $\hat{\rho}$, $\mathbf{E}$ should be a polar vector, however $\hat{\rho}\mathbf{v} = -\mathbf{v}$ so $\mathbf{B}$ has to be axial for $\mathbf{v} \times \mathbf{B}$ to be polar vector.

(b) With $\mathsf{U}^\mu = \gamma(c, \mathbf{v})$ the 4-velocity, we have 4-current:

$$\begin{aligned}\mathsf{J}^\mu &= \rho \mathsf{U}^\mu \\ &= (\rho c, \mathbf{j})\end{aligned}$$

Continuity equation then tells us:

$$\begin{aligned} +\frac{\partial \rho}{\partial t} + \div \cdot \mathbf{j} &= 0 \\ \Rightarrow +\frac{1}{c}\frac{\partial}{\partial t}(\rho c) + \div \cdot \mathbf{j} &= 0 \\ \Rightarrow \partial_\mu \mathsf{J}^\mu &= 0 \quad \text{where } \partial_\mu = \left(+\frac{1}{c}\frac{\partial}{\partial t}, \mathbf{v}\right)\end{aligned}$$

3-force component of f_μ:

$$\mathbf{f} = \rho\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right) = \rho\mathbf{E} + \mathbf{j} \times \mathbf{B} \quad \text{since } \rho\mathbf{v} = \mathbf{j}$$

EM tensor:

$$\mathsf{F}^{\mu\nu} = \begin{bmatrix} 0 & & \mathbf{E}/c & \\ & 0 & B_z & -B_y \\ -\mathbf{E}/c & -B_z & 0 & B_x \\ & B_y & -B_x & 0 \end{bmatrix}$$

$$\Rightarrow \mathsf{f}_i = \mathsf{J}^j \mathsf{F}_{ji} \quad \text{satisfies } \mathbf{j} \times \mathbf{B}$$

Extending to 4-vector gives $\mathsf{f}_\mu = \mathsf{J}^\nu \mathsf{F}_{\nu\mu}$.

f_0 refers to the power density due to the current: $\mathsf{J}^\nu \mathsf{F}_{\nu 0} = -\mathbf{j} \cdot \mathbf{E}/c$

(c) Know:

$$\begin{aligned}\mathsf{A}^\mu &= \left(0, A_0 \cos\left[\mathsf{K}^\mu \mathsf{X}_\mu\right], A_0 \sin\left[\mathsf{K}^\mu \mathsf{X}_\mu\right], 0\right) \\ \mathsf{X}^\mu &= (ct, x, y, z) \\ \mathsf{K}^\mu &= \left(\frac{\omega}{c}, 0, 0, k_z\right)\end{aligned}$$

$$\begin{aligned}\Rightarrow \mathsf{K}^\mu \mathsf{X}_\mu &= -\omega t + z k_z \\ \Rightarrow \mathsf{A}^\mu &= \left(0, A_0 \cos\left(k_z z - \omega t\right), A_0 \sin\left(k_z z - \omega t\right), 0\right)\end{aligned}$$

We then have EM tensor:

$$
\begin{aligned}
\mathsf{F}^{\mu\nu} &= \partial^\mu \mathsf{A}^\nu - \partial^\nu \mathsf{A}^\mu \\
&= \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{bmatrix}
\end{aligned}
$$

So:

$$
\begin{aligned}
\frac{E_x}{c} &= \partial^0 \mathsf{A}^1 - \cancelto{0}{\partial^1 \mathsf{A}^0} \\
&= -\frac{1}{c}\frac{\partial}{\partial t}\left[A_0 \cos\left(k_z z - \omega t\right)\right] \\
&= -\frac{\omega}{c} A_0 \sin\left(k_z z - \omega t\right) \\
\frac{E_y}{c} &= \partial^0 \mathsf{A}^2 - \cancelto{0}{\partial^2 \mathsf{A}^0} \\
&= -\frac{1}{c}\frac{\partial}{\partial t}\left[A_0 \sin\left(k_z z - \omega t\right)\right] \\
&= \frac{\omega}{c} A_0 \cos\left(k_z z - \omega t\right) \\
-B_y &= \cancelto{0}{\partial^1 \mathsf{A}^3} - \partial^3 \mathsf{A}^1 \\
&= -\frac{\partial}{\partial z} A_0 \cos\left(k_z z - \omega t\right) \\
&= A_0 k_z \sin\left(k_z - \omega t\right) \\
B_x &= \cancelto{0}{\partial^2 \mathsf{A}^3} - \partial^3 \mathsf{A}^2 \\
&= -\frac{\partial}{\partial z} A_0 \sin\left(k_z z - \omega t\right) \\
&= -A_0 k_z \cos\left(k_z - \omega t\right)
\end{aligned}
$$

The rest is 0 so $\mathsf{F}^{\mu\nu}$ is now:

$$
\begin{bmatrix}
0 & -{}^{\omega}\!/_{c} A_0 \sin\left(k_z z - \omega t\right) & {}^{\omega}\!/_{c} A_0 \cos\left(k_z z - \omega t\right) & 0 \\
{}^{\omega}\!/_{c} A_0 \sin\left(k_z z - \omega t\right) & 0 & 0 & A_0 k_z \sin\left(k_z - \omega t\right) \\
-{}^{\omega}\!/_{c} A_0 \cos\left(k_z z - \omega t\right) & 0 & 0 & -A_0 k_z \cos\left(k_z - \omega t\right) \\
0 & -A_0 k_z \sin\left(k_z - \omega t\right) & A_0 k_z \cos\left(k_z - \omega t\right) & 0
\end{bmatrix}
$$

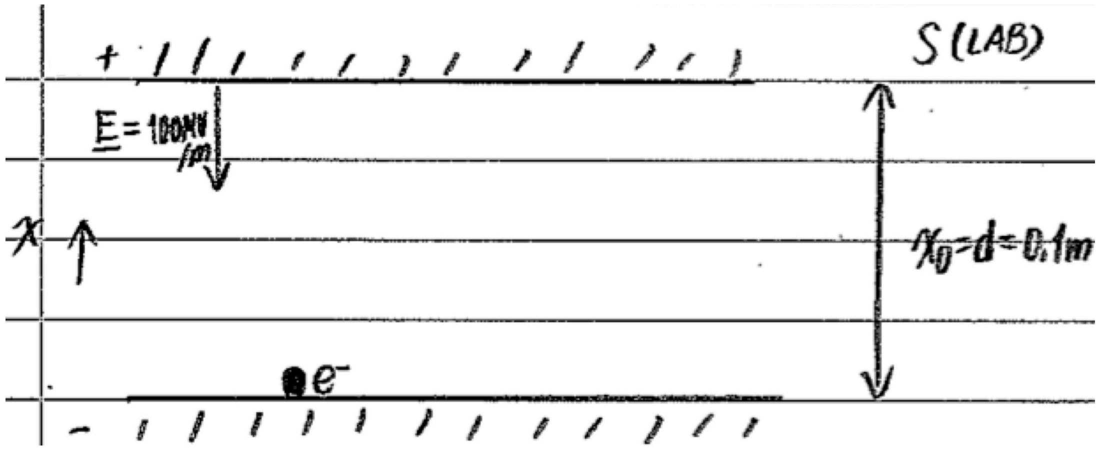
(d) E-field transformation:

$$
\begin{aligned}
\mathbf{E}'_\parallel &= \gamma\left(\mathbf{E}_\parallel - \mathbf{v}\times\mathbf{B}\right) \\
\mathbf{E}'_\perp &= \mathbf{E}_\perp
\end{aligned}
$$

B-field transformation:

$$
\begin{aligned}
\mathbf{B}'_\parallel &= \mathbf{B}_\parallel \\
\mathbf{B}'_\perp &= \gamma\left(\mathbf{B}_\perp - \frac{\mathbf{v}\times\mathbf{E}}{c^2}\right)
\end{aligned}
$$

where $\mathbf{X}_\parallel$ is the X-field component parallel to $\mathbf{v}$, $\mathbf{X}_\perp$ is that perpendicular to $\mathbf{v}$.



(e) i. At the other end of the capacitor, the e$^-$ would have gained kinetic energy $T = e\epsilon d$:

$$\begin{aligned}\text{Total energy } E &= m_\mathrm{e}c^2 + T \\ &= m_\mathrm{e}c^2 + e\epsilon d \\ &= 0.5110\,\mathrm{MeV} + 100\,\mathrm{MeV\,m^{-1}} \cdot 0.1\,\mathrm{m} \\ &= 100.511\,\mathrm{MeV}\end{aligned}$$

The velocity would be:

$$\begin{aligned}\gamma &= \frac{E}{m_\mathrm{e}c^2} \\ \left(1-\beta^2\right)^{-1/2} &= \frac{E}{m_\mathrm{e}c^2} \\ \beta &= \sqrt{1-\left(\frac{m_\mathrm{e}c^2}{E}\right)^2} \\ v &= c\sqrt{1-\left(\frac{0.511\,\mathrm{MeV}}{100.511\,\mathrm{MeV}}\right)^2} = 0.999\,987c\end{aligned}$$

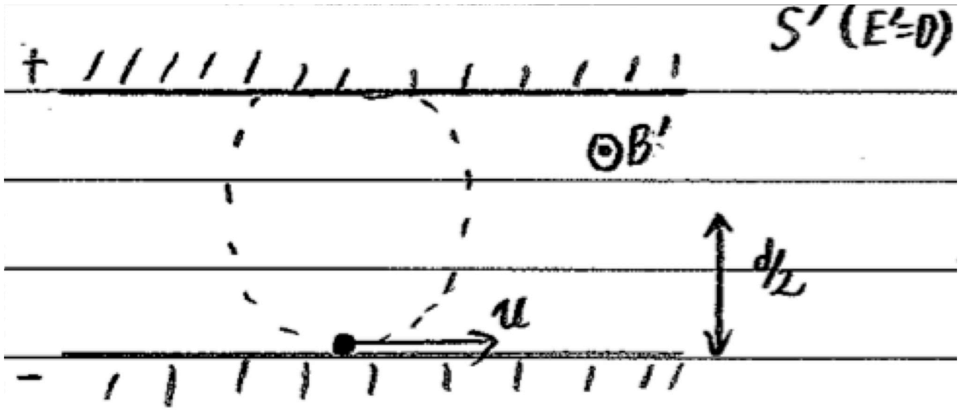
ii. By applying sufficient magnetic field in the yz-plane, we should be able to trap the e$^-$ in between the plates such that the energy in *bremsstrahlung* is compensated by the E-field.

iii. As explained above, yz-plane to induce a curvature in e$^-$ motion.

iv. Boosting along, say $\hat{\mathbf{y}}$, and $\mathbf{B} \parallel \hat{\mathbf{z}}$, to a frame where $E' = 0$:

$$E'_\perp = \gamma_u \left(E_\perp - \mathbf{u}\times\mathbf{B}\right) \quad \text{with boost } -\mathbf{u} \text{ along } \hat{\mathbf{y}}$$
$$\Rightarrow uB = E \Rightarrow u = \frac{E}{B}$$

In this frame,



In this pure B' field, e^- would undergo circular motion with radius $d/2$.

Since perpendicular dimensions are not contracted, $d/2$ remains invariant across different frames.

$$\begin{aligned}
\text{Lorentz force } \mathbf{f} &= q\mathbf{v} \times \mathbf{B} \\
\frac{m_\mathrm{e} u^2}{r} &= (-e)\, \mathbf{u} \times \mathbf{B}' \\
\frac{2m_\mathrm{e} u^2}{d} &= euB' \\
\Rightarrow B' &= \frac{2m_\mathrm{e} u}{ed}
\end{aligned}$$

Boosting back to S for consistent solution:

$$\begin{aligned}
\mathbf{B}_\perp &= \gamma_u \left(\mathbf{B}'_\perp + \frac{\mathbf{u} \times \mathbf{E}'}{c^2} \right) \\
\Rightarrow B &= \gamma_u \left(\frac{2m_\mathrm{e} u}{ed} + 0 \right) \\
&= \frac{2m_\mathrm{e} u}{ed\sqrt{1-(u/c)^2}} \\
&= \frac{2m_\mathrm{e}}{ed} \frac{E/B}{\sqrt{1-(E/Ec)^2}} \\
\Rightarrow B^2 \sqrt{1 - \left(\frac{E}{Bc}\right)^2} &= \frac{2m_\mathrm{e}}{ed} \\
\Rightarrow B^4 - \frac{E^2 B^2}{c^2} &= \frac{2m_\mathrm{e}}{ed} \\
\Rightarrow B^2 &= \frac{E^2/c^2 \pm \sqrt{E^4/c^4 + 8m_\mathrm{e} E/ed}}{2} = \begin{cases} 0.351 \\ -0.129 \quad \text{(unphysical)} \end{cases} \\
\Rightarrow B &= 0.593\,\mathrm{T}
\end{aligned}$$