UNOFFICIAL SOLUTIONS BY TheLongCat

C3: CONDENSED MATTER PHYSICS

TRINITY TERM 2017

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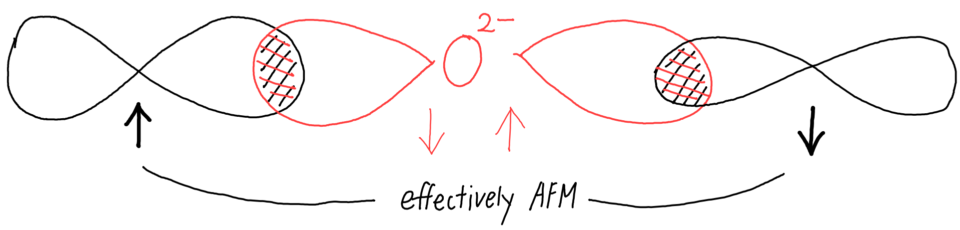
Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.

Turn over as you please – we are NOT under exam conditions here.

1. Classic magnetism question.

(a) Superexchange: in some materials where the magnetic species have no direct orbital overlaps, an intermediate non-magnetic species (e.g. O^{2-}) will facilitate an indirect exchange such that there is a kinematic advantage in an antiferromagnetic order.



Heisenberg exchange: $\mathcal{H} = \sum_{i,j} J/2\,\mathbf{S}_i \cdot \mathbf{S}_j$ where 1/2 is due to overcounting.

In mean field theory we may treat the spins as classical vectors, so:

$$\begin{aligned}\mathcal{H}_{J_2} &= 4 \cdot \frac{J_2}{2} S^2 \cos\theta \\ &= 2J_2 S^2 \cos\theta \qquad (1)\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{J_1} &= 2 \cdot \frac{J_1}{2} S^2 \cos 2\theta \qquad (2) \\ &= J_1 S^2 \underbrace{\left[2\cos^2\theta - 1\right]}_{f(\theta)}\end{aligned}$$

So energy per spin in spiral order:

$$\begin{aligned}E_{\text{spiral}}(\theta) &= 2J_2 S^2 \cos\theta + J_1 S^2 \left[2\cos^2\theta - 1\right] \\ \frac{\partial E_{\text{spiral}}}{\partial\theta} &= -2J_2 S^2 \sin\theta - 4J_1 S^2 \cos\theta\sin\theta = 0 \quad \text{at equilibrium} \\ &\Rightarrow -2S^2 \sin\theta \left[J_2 + 2J_1\cos\theta\right] = 0 \\ &\Rightarrow \sin\theta = 0 \quad \text{(AFM state)} \qquad \text{or} \qquad \cos\theta = -\frac{J_2}{2J_1} \qquad (3)\end{aligned}$$

Note that the spiral solution vanishes when $|\cos\theta| > 1$:

$$\begin{aligned}-\frac{J_2}{2J_1} &< -1 \\ \Rightarrow \frac{J_1}{J_2} &< \frac{1}{2}\end{aligned}$$

Thus the spiral state would be more stable than the AFM state for $J_1/J_2 > 1/2$.

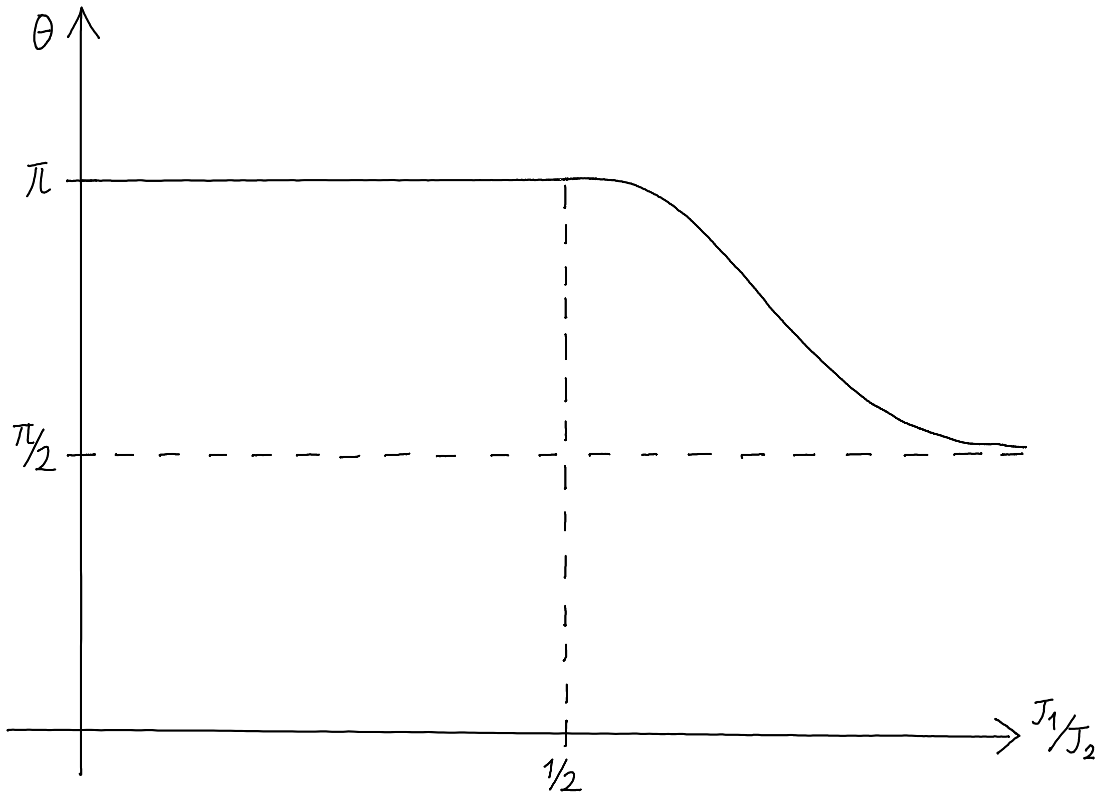
Also for large J_1/J_2:

$$\lim_{J_1\to\infty} \cos\theta = \lim_{J_1\to\infty} -\frac{J_2}{2J_1} = 0$$

Thus we have a limiting value of

$$\lim_{J_1/J_2\to\infty} \theta = \frac{\pi}{2}$$

Sketch of θ against J_1/J_2:



(b) Similarly, by realising that Figure II depicts just the component $S\sin\alpha$, we may rewrite the interaction terms (1) and (2):

$$\mathcal{H}_{J_1} = 2 \cdot \frac{J_1}{2} S^2 \left[\cos 2\theta \sin^2\alpha + \cos^2\alpha\right]$$

$$\mathcal{H}_{J_2} = 4 \cdot \frac{J_2}{2} S^2 \left[\cos \theta \sin^2\alpha + \cos^2\alpha\right]$$

So the energy terms become:

$$E = E_{\text{spiral}} \sin^2\alpha + (J_1 + 2J_2) S^2 \cos^2\alpha + \underbrace{(-g\mu_{\text{B}}\mathbf{S}) \cdot \mathbf{B}}_{-g\mu_{\text{B}}BS\cos\alpha}$$

$$\frac{\partial E}{\partial \alpha} = 2E_{\text{spiral}} \sin\alpha\cos\alpha - (J_1 + 2J_2)\, 2S^2 \sin\alpha\cos\alpha + g\mu_{\text{B}}BS\sin\alpha = 0 \quad \text{at equilibrium}$$

$$\Rightarrow \sin\alpha = 0 \quad \text{(paramagnetic regime)} \qquad \text{or} \qquad \cos\alpha = \frac{g\mu_{\text{B}}BS}{2\,(J_1 + 2J_2)\, S^2 - 2E_{\text{spiral}}}$$

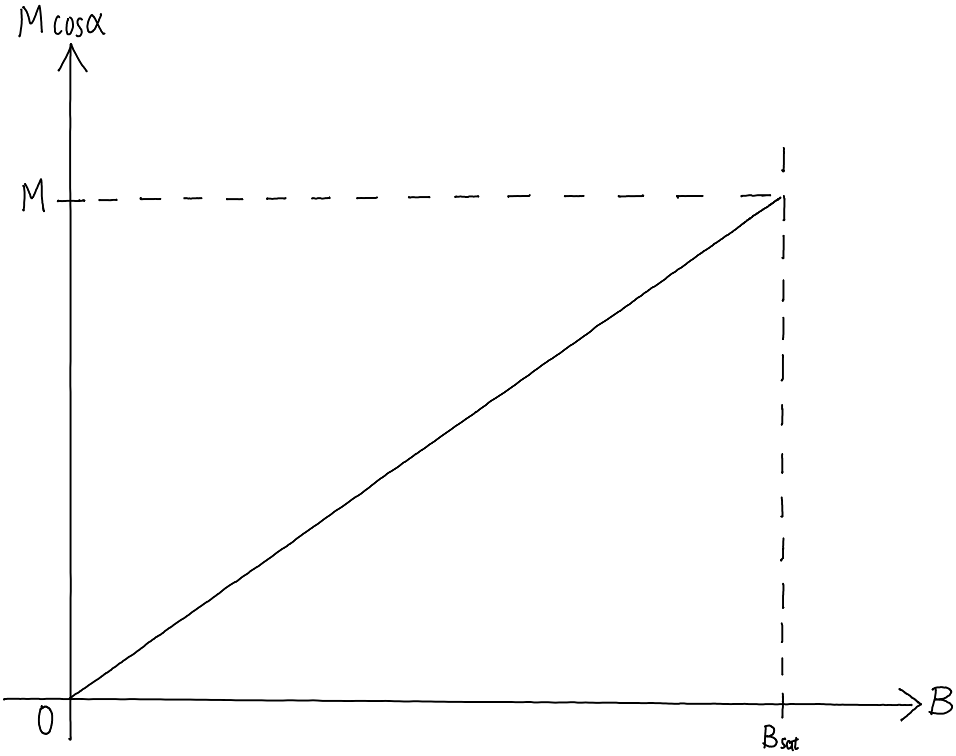
$$\begin{aligned}\cos\alpha &= \frac{g\mu_{\text{B}}B}{2\,(J_1 + 2J_2)\, S - 4J_2 S\cos\theta_{\text{eq}} - 2J_1 S\,(2\cos^2\theta_{\text{eq}} - 1)} \\ &= \frac{g\mu_{\text{B}}B}{\left[2\,(J_1 + 2J_2) + \frac{J_2^2}{J_1} + 2J_1\right] S}\end{aligned}$$

where θ_{eq} is the value of θ at equilibrium from (3).

Saturation occurs when $\cos\alpha = 1 \Rightarrow \alpha = 0$:

$$\Rightarrow B_{\text{sat}} = \frac{2S}{g\mu_{\text{B}}}\left[J_1 + 2J_2 + \frac{J_2^2}{2J_1} + J_1\right]$$
$$= \frac{2S}{g\mu_{\text{B}}}\left[2J_1 + 2J_2 + \frac{J_2^2}{2J_1}\right]$$

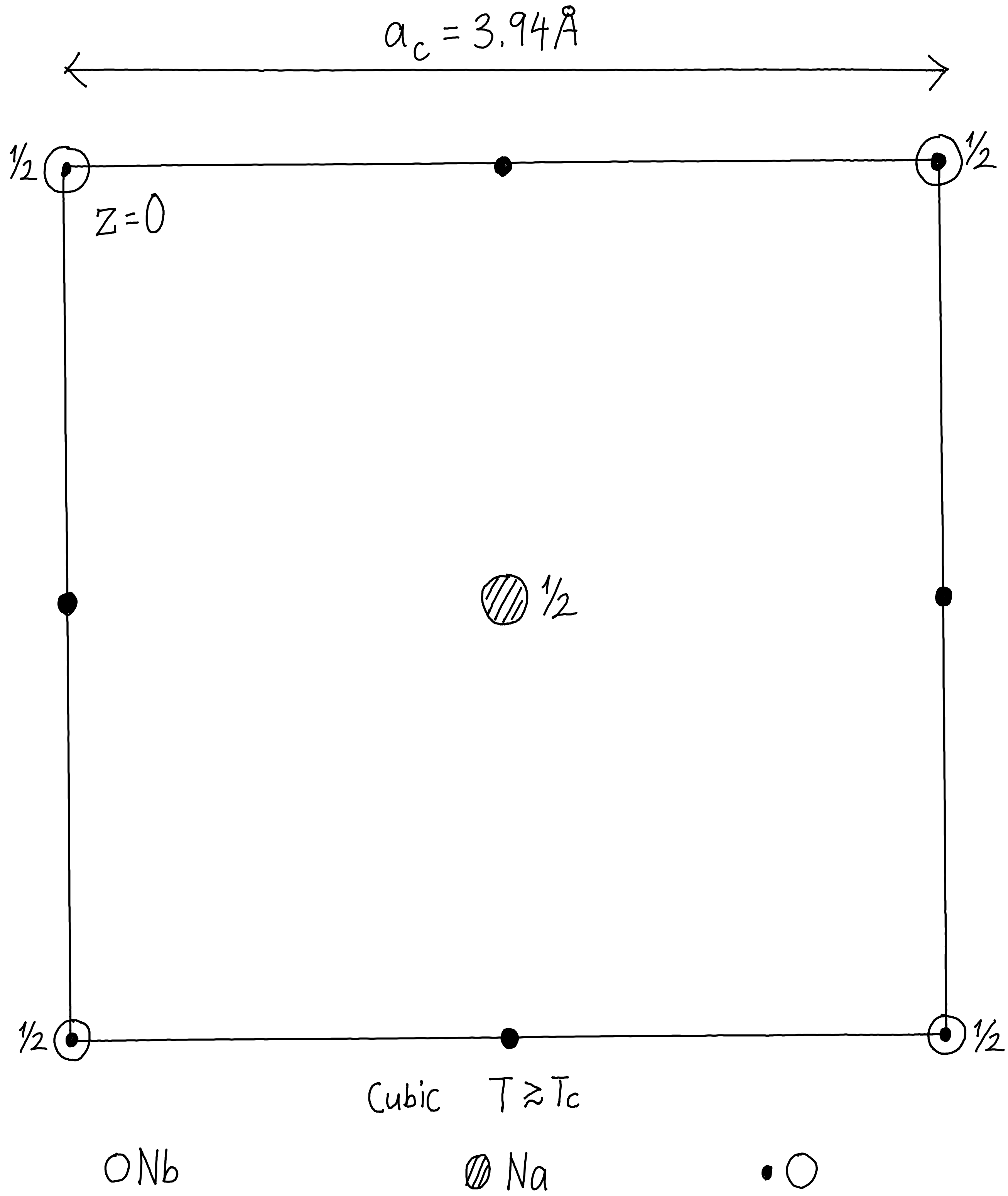
Sketch of $M\cos\alpha$ against B:



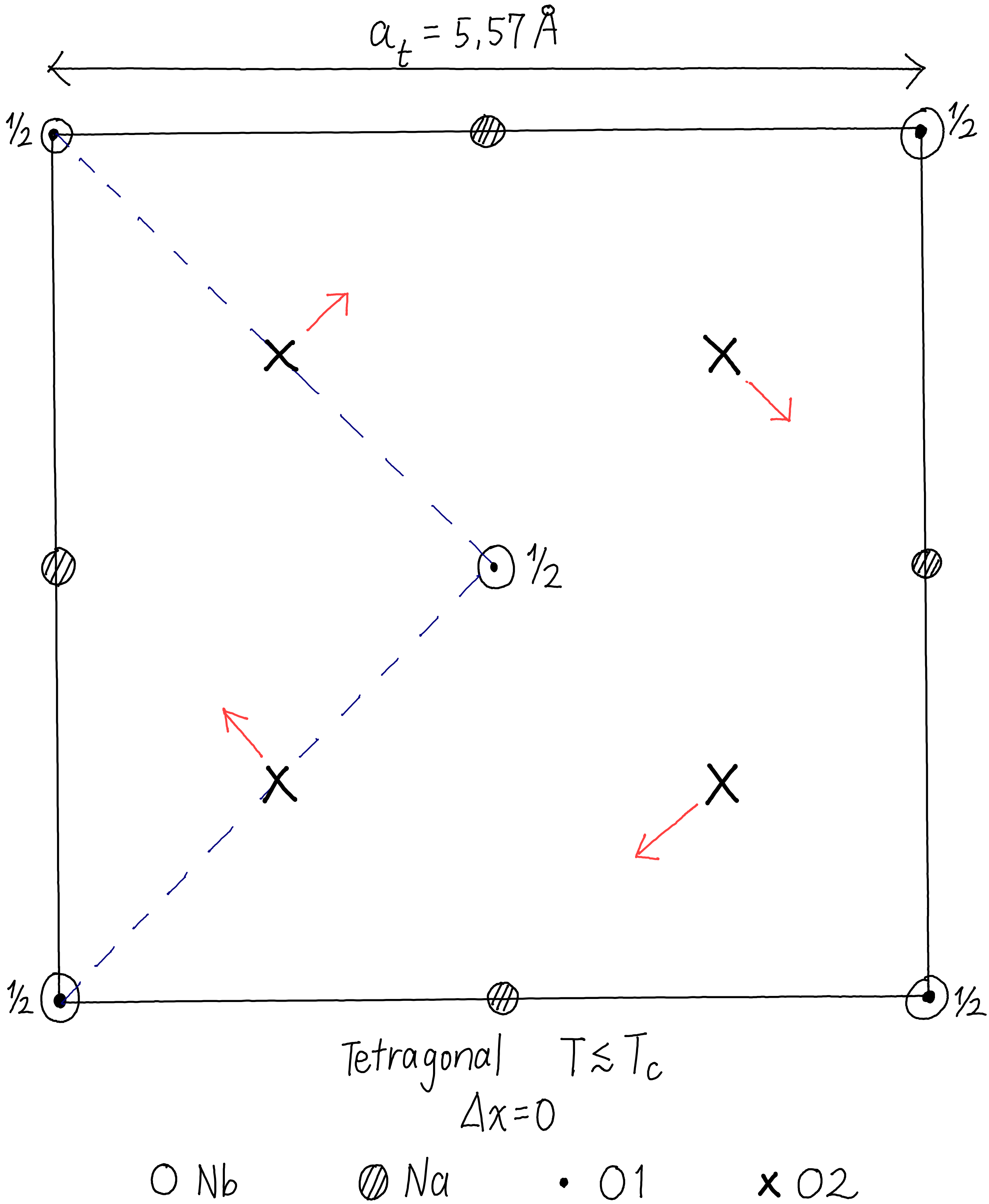
(c) **(TO EXPAND)** To determine the alignment of the spins, one may perform a polarised neutron scattering and measure the relative intensities of scattering.

2. Crystal symmetry and soft phonon modes.

(a) Sketch of lattice in cubic phase:



Sketch of lattice in tetragonal phase:



Note that the cubic UC has rotated 45° to make the new tetragonal UC (see dashed lines). So we have $\mathbf{a}_t = \mathbf{a}_c - \mathbf{b}_c$, $\mathbf{b}_t = \mathbf{a}_c + \mathbf{b}_c$, $\mathbf{c}_t = \mathbf{c}_c$.

(b)

$$\begin{aligned} S_{\mathrm{Na}} &= f_{\mathrm{Na}}\left[\mathrm{e}^{i2\pi(0+k/2+l/2)}+\mathrm{e}^{i2\pi(h/2+0+l/2)}\right] \\ &= f_{\mathrm{Na}}\mathrm{e}^{i\pi l}\left[\mathrm{e}^{i\pi k}+\mathrm{e}^{i\pi h}\right] \end{aligned}$$

So Na needs k and h to be both odd/even to have non-zero S.

$$\begin{aligned} S_{\mathrm{Nb}} &= f_{\mathrm{Nb}}\left[\mathrm{e}^{i0}+\mathrm{e}^{i2\pi(h/2+k/2+0)}\right] \\ &= f_{\mathrm{Nb}}\left[1+\mathrm{e}^{i\pi(h+k)}\right] \end{aligned}$$

Selection rule for Nb: $h+k=$ even.

$$\begin{aligned} S_{\mathrm{O1}} &= f_{\mathrm{O}}\left[\mathrm{e}^{i2\pi(0+0+l/2)}+\mathrm{e}^{i2\pi(h/2+k/2+l/2)}\right] \\ &= f_{\mathrm{O}}\mathrm{e}^{i\pi l}\left[1+\mathrm{e}^{i\pi(h+k)}\right] \end{aligned}$$

So O1 has the same selection rule as Nb.

$$\begin{aligned} S_{\mathrm{O2}} = f_{\mathrm{O}}\Bigg[& \mathrm{e}^{i2\pi\left(h/4+\underbrace{3k/4}_{-k/4}+0+h\Delta x+k\Delta x\right)}+\mathrm{e}^{i2\pi(-h/4+k/4+0-h\Delta x-k\Delta x)}+\mathrm{e}^{i2\pi(h/4+k/4+0-h\Delta x+k\Delta x)} \\ & +\mathrm{e}^{i2\pi\left(\underbrace{3h/4}_{-h/4}-k/4+0+h\Delta x-k\Delta x\right)}\Bigg] \\ = f_{\mathrm{O}}\Bigg[& 2\cos\left(\frac{h-k}{2}+2(h+k)\Delta x\right)\pi+2\cos\left(\frac{h+k}{2}+2(-h+k)\Delta x\right)\pi\Bigg] \end{aligned}$$

i. Note that for h odd, k even (or vice versa), $S_{\mathrm{Na}}=S_{\mathrm{Nb}}=S_{\mathrm{O1}}=0$, however $S_{\mathrm{O2}}\neq 0$ if $\Delta x\neq 0$. So this family of reflections shall be exclusive to O2 for $\Delta x\neq 0$.

ii. Also since the lattice is tetragonal, for the parity between h and k to differ, it is also subject to further constraints of $h\neq 0$, $k\neq 0$.

(c) For $(2,1,0)$, we have $S_{\mathrm{Na}}=S_{\mathrm{Nb}}=S_{\mathrm{O1}}=0$ from b.

So we have:

$$\begin{aligned} I(2,1,0) &\propto \left|S_{\mathrm{O2}}\right|^2\times M_{(2,1,0)} \\ &= 4f_{\mathrm{O}}^2\left[\cos\left(\frac{1}{2}+6\Delta x\right)\pi+\cos\left(\frac{3}{2}-2\Delta x\right)\right]^2\times 8 \end{aligned}$$

For $(0,0,1)$:

$$
\begin{aligned}
S_{\mathrm{Na}} &= f_{\mathrm{Na}}\,(-1)\,(1+1) = -2f_{\mathrm{Na}} \\
S_{\mathrm{Nb}} &= f_{\mathrm{Nb}}\,(1+1) = 2f_{\mathrm{Nb}} \\
S_{\mathrm{O1}} &= f_{\mathrm{O}}\,(-1)\,(1+1) = -2f_{\mathrm{O}} \\
S_{\mathrm{O2}} &= 2f_{\mathrm{O}}\,[\cos 0 + \cos 0] = 4f_{\mathrm{O}} \\
\Rightarrow I(0,0,1) &\propto \left|\sum S\right|^2 \times M_{(0,0,1)} \\
&= 2^2\left(-f_{\mathrm{Na}} + f_{\mathrm{Nb}} + f_{\mathrm{O}}\right)^2 \times 2
\end{aligned}
$$

Hence:

$$
\begin{aligned}
\frac{I(2,1,0)}{I(0,0,1)} &= \frac{4f_{\mathrm{O}}^2}{\left(-f_{\mathrm{Na}} + f_{\mathrm{Nb}} + f_{\mathrm{O}}\right)^2}\left[\cos\left(\frac{1}{2} + 6\Delta x\right)\pi + \cos\left(\frac{3}{2} - 2\Delta x\right)\right]^2 \\
&= A\left[\cos\frac{\pi}{2}\cos 6\pi\Delta x - \sin\frac{\pi}{2}\sin 6\pi\Delta x + \cos\frac{3\pi}{2}\cos 2\pi\Delta x + \sin\frac{3\pi}{2}\sin 2\pi\Delta x\right]^2 \\
&= A\left[\sin 2\pi\Delta x + \sin\underbrace{6\pi\Delta x}_{m=3}\right]^2
\end{aligned}
$$

where

$$
\begin{aligned}
A &= \frac{4f_{\mathrm{O}}^2}{\left(-f_{\mathrm{Na}} + f_{\mathrm{Nb}} + f_{\mathrm{O}}\right)^2} \\
&= 0.178 \quad \text{assuming } f_X \propto Z_X \text{ the atomic number of } X
\end{aligned}
$$

If neutrons are used instead, the factor A would be in terms of the scattering length b_X, which varies rather unpredictably between elements.

(d)

$$
\begin{aligned}
\mathbf{a}_t^* &= \frac{2\pi}{V_t}\mathbf{b}_t \times \mathbf{c}_t \\
&= \frac{2\pi}{V_t}\left(\mathbf{a}_c + \mathbf{b}_c\right) \times \mathbf{c}_c \\
&= \frac{2\pi}{V_t}\left(-\frac{\mathbf{b}_c^*}{2\pi} + \frac{\mathbf{a}_c^*}{2\pi}\right)\cdot V_c
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{b}_t^* &= \frac{2\pi}{V_t}\mathbf{c}_t \times \mathbf{a}_t \\
&= \frac{2\pi}{V_t}\mathbf{c}_c \times \left(\mathbf{a}_c - \mathbf{b}_c\right) \\
&= \frac{2\pi}{V_t}\left(\frac{\mathbf{b}_c^*}{2\pi} + \frac{\mathbf{a}_c^*}{2\pi}\right)\cdot V_c
\end{aligned}
$$

$$\begin{aligned}
\mathbf{c}_t^* &= \frac{2\pi}{V_t}\mathbf{a}_t \times \mathbf{b}_t \\
&= \frac{2\pi}{V_t}\left(\mathbf{a}_c - \mathbf{b}_c\right) \times \left(\mathbf{a}_c + \mathbf{b}_c\right) \\
&= \frac{2\pi}{V_t}\left(2\mathbf{a}_c \times \mathbf{b}_c\right) \\
&= \frac{2\pi}{V_t} \cdot 2\mathbf{c}_c^* \cdot \frac{V_c}{2\pi}
\end{aligned}$$

where V_c is the volume of the tetragonal UC $= 2V_c$ that of cubic UC.

Therefore:

$$\begin{aligned}
\mathbf{a}_t^* &= (\mathbf{a}_c^* - \mathbf{b}_c^*) \times \frac{1}{2} \\
\mathbf{b}_t^* &= (\mathbf{a}_c^* + \mathbf{b}_c^*) \times \frac{1}{2} \\
\mathbf{c}_t^* &= \mathbf{c}_c^*
\end{aligned}$$

Note that the only high symmetry point intersected by $\mathbf{a}_t^*$ and $\mathbf{b}_t^*$ is M, so $\mathbf{k}_s$ is M.

3. **(DRAFT)** Cyclotron motion and Landau levels.

(a) Under a magnetic field, an electron undergoes cyclotron motion and this results in a series of quantised energy states called Landau levels. $E = (l + 1/2)\hbar\omega_c$ where $\omega_c = eB/m_{\text{CR}}$ is the cyclotron frequency with $m_{\text{CR}} = \hbar^2/2\pi\ \partial A/\partial E$ the cyclotron mass.

Low temperature is required for the electron to complete an orbit before scattering: $\omega_c\tau \gg 1$.

Note that Landau levels have units of meV, whereas a Fermi surface is of order eV, we may invoke the correspondence principle and approximate the energy difference of adjacent levels near Fermi surface as that of classical orbit:

$$E_{l+1} - E_l = \hbar\omega_c = \hbar eB \cdot \frac{2\pi}{\hbar^2}\frac{\partial E}{\partial A}$$

Approximating $E_{l+1} - E_l$ as δE then gives:

$$\delta A = \frac{2\pi}{\hbar}eB$$

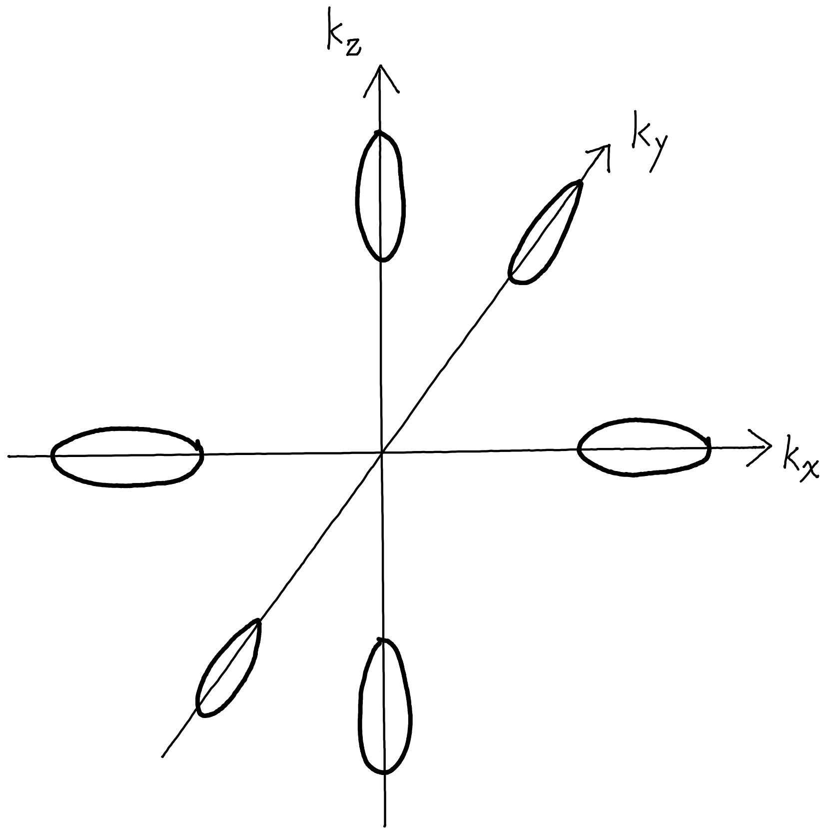
$$\Rightarrow A_\lambda = (l + \lambda) \cdot \frac{2\pi eB}{\hbar} \quad \text{is the quantised k-space area of orbit } \lambda$$

When A_λ crosses (tangents) the Fermi surface, there would be a spike in d.o.s., thus causing most properties of the material to oscillate:

$$A_{\text{ext}} = (l + \lambda)\,\frac{2\pi}{\hbar}eB$$

$$\Rightarrow \Delta\left(\frac{1}{B}\right) = \frac{2\pi}{\hbar}e\frac{1}{A_{\text{ext}}} \quad \text{where } A_{\text{ext}} \text{ is the extremal area of Fermi surface}$$

(b) Constant energy surface is an ellipsoid:



Volume of ellipsoid: $V = {}^4\!/_3\pi k_x k_\perp^2$ where $k_\perp$ is the semi major axis length in the y, z axes.

Mass tensor suggests that (1/2 to account for spin degeneracy):

$$\frac{k_\perp^2}{0.2} = k_x^2$$
$$\Rightarrow V = \frac{4}{3}\cdot 0.2\pi k_x^2 = \frac{1}{2}n \quad \text{where } n \text{ is the density of charge carrier}$$
$$\Rightarrow k_x = \left[\frac{1}{2}n\cdot\frac{3}{4}\cdot 5\cdot\frac{1}{\pi}\right]^{1/3}$$

Normal to (1 0 0), we have a circular cross section, and so A_{ext} is:

$$\begin{aligned}\pi k_\perp^2 &= 0.2\pi k_x^2\\ &= 0.2\pi\left[\frac{15}{8\pi}n\right]^{2/3}\end{aligned}$$

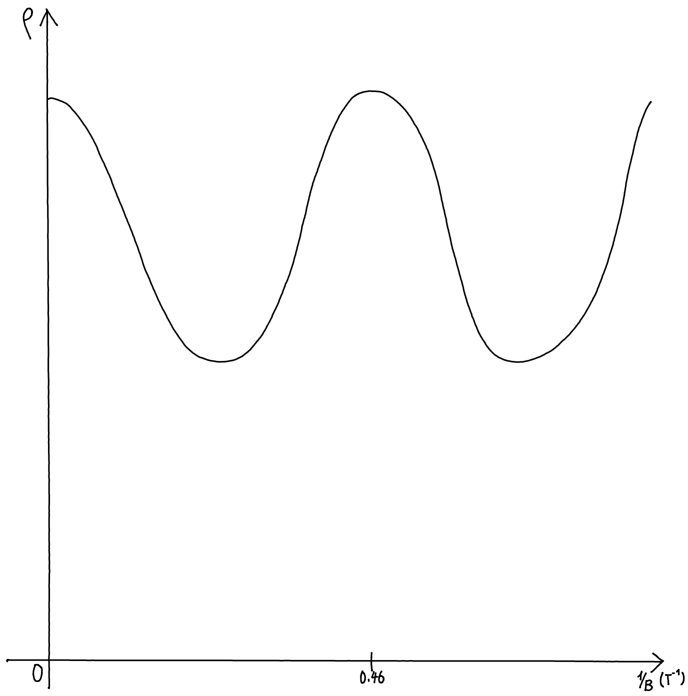
Thus we have the associated period $\Delta\,(1/B)$:

$$\begin{aligned}\Delta\left(\frac{1}{B}\right) &= \frac{2\pi e}{\hbar}\cdot\frac{1}{A_{\text{ext}}}\\ &= \frac{2\pi e}{\hbar}\cdot\frac{5}{\pi}\left[\frac{8\pi}{15n}\right]^{2/3}\\ &= 0.46\,\mathrm{T}^{-1}\end{aligned}$$

And the corresponding frequency is $F = 1/\Delta(1/B) = 2.17\,\mathrm{T}$.

Range: $0 - 10\,\mathrm{T}$ since that is about the maximum B field generated by most laboratory equipments.

Oscillation in resistivity should follow the above oscillation as $\rho \propto 1/J \propto 1/n$ so it is sensitive to the perturbation of Fermi surface:



(c) For Landau levels to be observable, $\omega\tau \gg 1$:

$$\Rightarrow \frac{eB}{m_y} \gg \frac{1}{\tau}$$
$$\Rightarrow T \ll 840\,\mathrm{K}$$

Mobility $\mu = e\tau/m$:

$$\mu B \gg 1$$
$$\Rightarrow \mu \gg 0.1\,\mathrm{T}^{-1}$$

To verify the mass values, we may perform an angle-resolved photoelectron spectroscopy (ARPES) to obtain the band structure near Fermi level. Then we would have the mass tensor given by $M_{ij} = 1/\hbar\, \partial E/\partial k_i\, \partial k_j$.

5. **(DRAFT)** Classic optical properties question.

(a) From Drude, we know:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = f - \frac{p}{\tau}$$
$$m^*\ddot{x} + \gamma m^*\dot{x} = qE\mathrm{e}^{i\omega t}$$

Substitute trial solution $x(t) = x_0\mathrm{e}^{i\omega t}$:

$$-\omega^2 m^* x_0 + i\omega\gamma m^* x_0 = qE$$
$$x_0 = \frac{qE}{(\omega^2 - i\omega\gamma)\, m^*}$$

With the polarisation $P = Nqx$, we then have the displacement field D as:

$$\Rightarrow D = \epsilon_0 E + P + P_{\text{bg}} = \epsilon_0\epsilon_r E \quad \text{with } P_{\text{bg}} \text{ the background polarisation}$$
$$\begin{aligned}\Rightarrow \epsilon_r &= 1 + \frac{P}{\epsilon_0 E} + \chi_{\text{bg}} \\ &= 1 + \frac{Nq}{\epsilon_0 E}\left(\frac{qE}{(\omega^2 - i\omega\gamma)\, m^*}\right) + \chi_{\text{bg}} \\ &= 1 + \frac{Nq^2}{m^*\epsilon_0}\left(\frac{1}{\omega^2 - i\omega\gamma}\right) + \chi_{\text{bg}} \\ &= 1 + \frac{Nq^2}{\epsilon_0 m^*}\left(\frac{\omega^2 + i\omega\gamma}{\omega^4 + \omega^2\gamma^2}\right) + \chi_{\text{bg}}\end{aligned}$$

For $\omega \gg \gamma$,

$$\epsilon_r \simeq \underbrace{1 + \chi_{\text{bg}}}_{\epsilon_\infty} + \underbrace{\frac{Nq^2}{\epsilon_0 m^*}}_{\omega_p^2\epsilon_\infty}\left(\frac{1}{\omega^2} + i\frac{\gamma}{\omega^3}\right)$$

So $\epsilon_r = \epsilon' + i\epsilon''$ where:

$$\epsilon' \simeq \epsilon_\infty - \frac{\omega_p^2}{\omega^2}\epsilon_\infty = \epsilon_\infty\left(1 - \frac{\omega_p^2}{\omega^2}\right)$$
$$\epsilon'' \simeq \epsilon_\infty\omega_p^2\left(\frac{\gamma}{\omega^3}\right) = \epsilon_\infty\left(\frac{\gamma\omega_p^2}{\omega^3}\right) \quad \text{with } \omega_p^2 = \frac{Nq^2}{\epsilon_\infty\epsilon_0 m^*}$$

(b) Reflectivity $R = \left|\frac{\tilde{n}-1}{\tilde{n}+1}\right|^2$ where $\tilde{n} = \sqrt{\epsilon_r}$:

$$\begin{aligned}\Rightarrow R &= 0 \quad \text{when } \tilde{n} = 1 \\ \Rightarrow \epsilon_r &= 1 \\ \epsilon' &\simeq 1 \quad \text{since we assume } \epsilon' \gg \epsilon'' \\ \Rightarrow \epsilon_\infty\left(1 - \frac{\omega_p^2}{\omega^2}\right) &= 1 \\ \frac{\omega_p^2}{\omega^2} &= 1 - \frac{1}{\epsilon_\infty}\end{aligned} \tag{4}$$

Wavelength $\lambda = \dfrac{2\pi}{k}$ and $\dfrac{\omega}{k} = cn = c$ since $\epsilon_r = 1 \Rightarrow n = 1$:

$$\Rightarrow \lambda = \frac{2\pi c}{\omega} \tag{5}$$

Combining (4) and (5) gives:

$$\begin{aligned}\lambda_{\text{min}} &= 2\pi c \cdot \left[\frac{1}{\omega_p^2}\left(1 - \frac{1}{\epsilon_\infty}\right)\right]^{1/2} \\ &= 2\pi c \omega_p^{-1}\left(1 - \epsilon_\infty^{-1}\right)^{1/2}\end{aligned}$$

We estimate λ_{min} from the given graph:

$$\lambda_{\text{min}} = \begin{cases} 23.5\,\mu\text{m} & \text{(A)} \\ 20\,\mu\text{m} & \text{(B)} \\ 15\,\mu\text{m} & \text{(C)} \\ 9.5\,\mu\text{m} & \text{(D)} \end{cases}$$

Thus:

$$\begin{aligned}\lambda_{\text{min}}^2 &= (2\pi c)^2\left(1 - \epsilon_\infty^{-1}\right) \cdot \frac{\epsilon_\infty \epsilon_0 m^*}{Nq^2} \\ \Rightarrow m^* &= \left(\frac{(2\pi c)^2\left(\epsilon_\infty - 1\right)\epsilon_0}{Nq^2\lambda_{\text{min}}^2}\right)^{-1}\end{aligned}$$

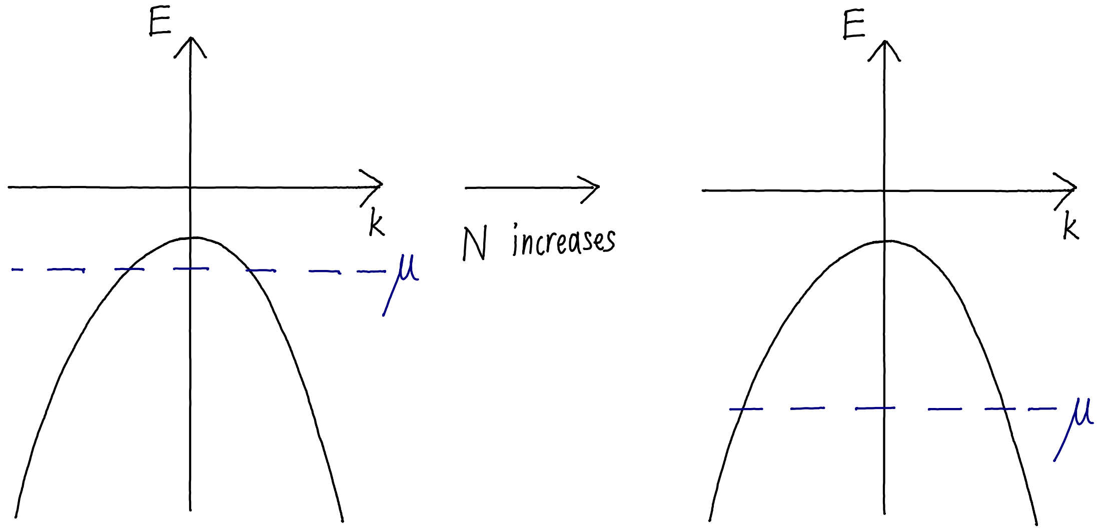
From the graph we estimate $R(\omega \to \infty) = \dfrac{1}{2}$:

$$\begin{aligned}\Rightarrow \left|\frac{\sqrt{\epsilon_\infty} - 1}{\sqrt{\epsilon_\infty} + 1}\right|^2 &= \frac{1}{2} \\ \sqrt{\epsilon_\infty} - 1 &= \sqrt{\frac{1}{2}}\sqrt{\epsilon_\infty} + \sqrt{\frac{1}{2}} \\ \sqrt{\epsilon_\infty} &= \frac{1 + \sqrt{1/2}}{1 - \sqrt{1/2}} \\ \epsilon_\infty &= \left(\frac{1 + \sqrt{1/2}}{1 - \sqrt{1/2}}\right)^2 = 34.0\end{aligned}$$

So we have:

$$\begin{aligned}m_{\text{A}}^* &= 0.05m_e \\ m_{\text{B}}^* &= 0.06m_e \\ m_{\text{C}}^* &= 0.09m_e \\ m_{\text{D}}^* &= 0.12m_e\end{aligned}$$

Note that the effective mass increases as N increases, this is due to the fact that the valence band is populated with more holes, thereby lowering the chemical potential and render m^* greater:



(c) Mobility $\mu = \dfrac{e\tau}{m^*}$.

We observe that λ_{min} increases as T increases, thereby suggesting m^* decreases from before.

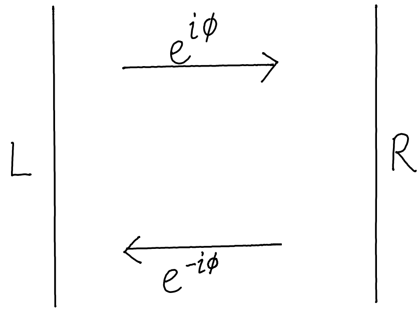
Lower m^* thus increases μ for a constant τ.

However we know τ decreases due to larger thermal agitation so we expect μ to decrease at large T.

6. **(DRAFT)** Question on Josephson Junction and its application on B field measurement.

(a) ϕ is the phase difference of the 2 SC wavefunctions across the junction.

The origin of the JJ equations:



- JJ1 – consider the tunnelling from L→R + R→L:

$$\begin{aligned} I &\propto \mathrm{e}^{i(\phi_2-\phi_1)} - \mathrm{e}^{i(\phi_1-\phi_2)} \\ &\propto \sin\phi \quad \text{where} \quad \phi = \phi_2 - \phi_1 \end{aligned}$$

- JJ2 – consider TDSE on each side of the junction:

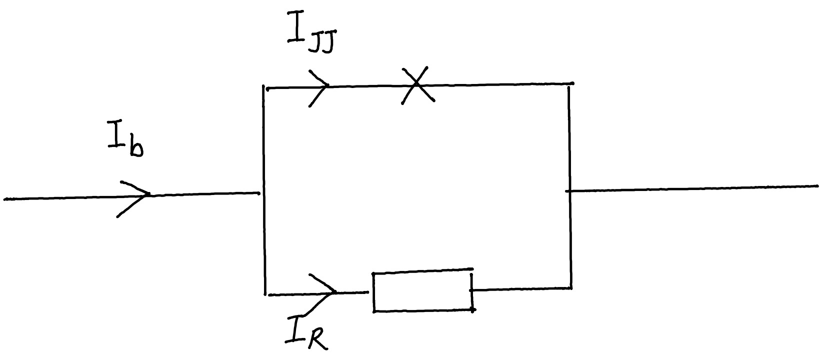
$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\left|\psi\right\rangle &= E\left|\psi\right\rangle \quad \text{where } \left|\psi\right\rangle = \sqrt{n_s}\mathrm{e}^{i\theta} \text{ is the SC wavefunction} \\ -\hbar\frac{\partial\theta}{\partial t}\left|\psi\right\rangle &= E\left|\psi\right\rangle \end{aligned}$$

Subtracting the energies across the junction then yields:

$$\begin{aligned} \Delta E &= -\hbar\frac{\partial\theta}{\partial t} = -2eV \\ V &= \frac{\hbar}{2e}\frac{\partial\theta}{\partial t} \end{aligned}$$

where V is the voltage developed due to the energy difference carried by the Cooper pair.

(b) Sketch of the circuit:



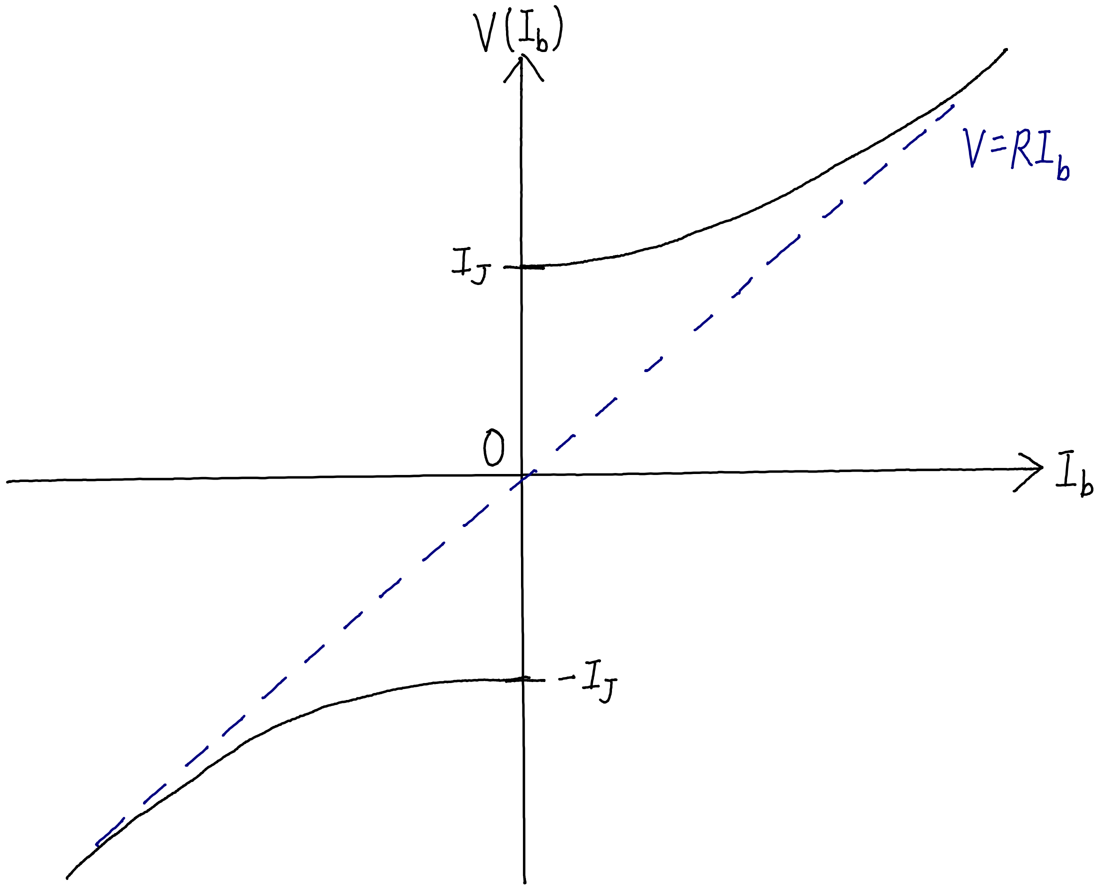
KCL gives:

$$\begin{aligned} I_b &= I_{JJ} + I_R \\ &= I_J \sin\phi + \frac{V}{R} \\ &= I_J \sin\phi + \frac{\hbar}{2eR}\frac{\partial \phi}{\partial t} \end{aligned}$$

For $I_b < I_J$, note we have a steady state solution: $\sin\phi = I_b/I_J$ so no voltage develops.

For $I_b \ll I_J$, we have $\partial\phi/\partial t \simeq 2eR/\hbar I_b \Rightarrow V \simeq I_b R \Rightarrow$ Ohmic behaviour.

Sketch of voltage against I_b:



(c) The flux dependence arises from the quantisation of flux through a loop of superconductors. This is due to the inherent rigidity of wavefunction where the phase must be single-valued upon a rotation of 2π.

2nd Ginzburg-Landau equation gives Φ_0 as the flux quantum.

(d) **(TO EXPAND)**

$$V = I_b R \left[1 - \left(\frac{I_J}{I_b}\right)^2\right]^{1/2}$$

So:

$$\frac{\partial V}{\partial I_b} = 0 = R\left[1 - \left(\frac{I_J}{I_b}\right)^2\right]^{1/2} + \frac{1}{2} I_b R \left[1 - \left(\frac{I_J}{I_b}\right)^2\right]^{-1/2} \left(-2\left(\frac{I_J}{I_b}\right)\left(-\frac{I_J}{I_b^2}\right)\right)$$

$$\Rightarrow R\left[1 - \left(\frac{I_J}{I_b}\right)^2\right] + I_b R \left(\frac{I_J^2}{I_b^3}\right) = 0$$

???

$$\frac{\partial V}{\partial \Phi} = I_b R \cdot \frac{1}{2}\left[1 - \left(\frac{I_J}{I_b}\right)^2\right]^{-1/2} \cdot \left(-2\frac{I_J}{I_b^2}\right)\left(\frac{\partial I_J}{\partial \Phi}\right)$$

$$\frac{\partial I_J}{\partial \Phi} = I_{J0}\left|\sin\left(\frac{2\pi\Phi}{\Phi_0}\right)\right| \cdot \frac{2\pi}{\Phi_0}$$

(e) **(TO EXPAND)** So bias current???