UNOFFICIAL SOLUTIONS BY TheLongCat

B4: SUB-ATOMIC PHYSICS

TRINITY TERM 2019

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Turn over as you please – we are NOT under exam conditions here.

1. (DRAFT)

(a) i. A nuclear form factor is an additional factor to the scattering amplitude. It enters through the matrix element to encode the spatial extent of a charge distribution (i.e. the Fourier transform of the charge distribution).

ii. Spherically symmetric charge distribution has density:

$$\rho = \frac{q}{\frac{4}{3}\pi r^3} = \frac{3q}{4\pi r^3}$$

$$\begin{aligned} F(k) &= \int_{-\infty}^{\infty} \rho e^{-i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{d}^3\mathbf{x} \\ &= \int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta \int_0^{\infty} \frac{3q}{4\pi r} r^2 \sin\theta e^{ikr\cos\theta}\,\mathrm{d}r \\ &= 2\pi \int_0^{\infty} \mathrm{d}r \int_0^{\pi} \frac{3q}{4\pi r} \sin\theta e^{ikr\cos\theta}\,\mathrm{d}\theta \end{aligned}$$

Substituting $u = \cos\theta \Rightarrow \mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$, the limits map from $[0, \pi] \to [1, -1]$:

$$\begin{aligned} \Rightarrow F(k) &= 2\pi \int_0^{\infty} \frac{3q}{4\pi r}\,\mathrm{d}r \int_{-1}^{1} e^{ikru}\,\mathrm{d}u \\ &= 2\pi \int_0^{\infty} \frac{3q}{4\pi r}\,\mathrm{d}r \left[\frac{e^{ikru}}{ikr}\right]_{u=-1}^{1} \\ &= 2\pi \int_0^{\infty} \frac{3q}{4\pi r} \cdot \frac{1}{kr} \cdot \underbrace{\frac{e^{ikr} - e^{-ikr}}{i}}_{2\sin(kr)}\,\mathrm{d}r \\ &= \frac{4\pi}{k} \int_0^{\infty} \frac{3q}{4\pi r^2} \sin(kr)\,\mathrm{d}r \\ &= -\frac{4\pi}{k} \int_0^{\infty} r\rho(r)\sin(kr)\,\mathrm{d}r \end{aligned}$$

where k is the scattering wavevector, ρ is the charge density. (sign check)

For uniformly charge sphere of radius R,

$$\begin{aligned} F(k) &= \frac{4\pi}{k} \int_0^{R} r \frac{3q}{4\pi R^3} \sin(kr)\,\mathrm{d}r \\ &= \frac{3q}{kR^3} \int_0^{R} r\sin(kr)\,\mathrm{d}r \end{aligned}$$

Integration by parts:

$$\begin{aligned} A &= r \\ \mathrm{d}B &= \sin(kr)\,\mathrm{d}r \end{aligned} \Rightarrow \begin{aligned} \mathrm{d}A &= \mathrm{d}r \\ B &= -\tfrac{1}{k}\cos(kr) \end{aligned}$$

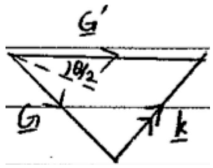
$$\begin{aligned}\Rightarrow F(k) &= \frac{3q}{kR^3}\left\{\left[-\frac{r}{k}\cos(kr)\right]_{r=0}^{R} + \frac{1}{k}\int_0^R \cos(kr)\,\mathrm{d}r\right\} \\ &= \frac{3q}{kR^3}\left\{-\frac{R}{k}\cos(kR) + \frac{1}{k^2}\sin(kR)\right\} \\ &= 3q\left\{-\frac{1}{(kR)^2}\cos(kR) + \frac{1}{(kR)^3}\sin(kR)\right\}\end{aligned}$$

So $C = 3q$, $x = kR$. (Should be e^{-ikx}, perhaps wrong FT)

(b) i. For $F(k)$ to be 0 (scattering amplitude without $F(k)$ drops slowly),

$$\begin{aligned}\frac{1}{(kR)^2}\cos(kR) &= \frac{1}{(kR)^3}\sin(kR) \\ \Rightarrow \tan(kR) &= kR\end{aligned}$$

Now $\mathbf{k} = \mathbf{G'} - \mathbf{G} \Rightarrow k = 2G\sin\frac{\theta}{2}$ with total scattering angle θ.



Incoming wavevector $\mathbf{G} = \dfrac{\mathbf{p}}{\hbar}$ where $\mathbf{p}$ is electron momentum. Electron energy:

$$\begin{aligned}E &= \sqrt{(m_e c^2)^2 + (pc)^2} \\ \Rightarrow pc &= \sqrt{E^2 - m_e^2 c^4} \\ \Rightarrow p &= \frac{1}{c}\sqrt{(250\,\mathrm{MeV})^2 - (0.511\,\mathrm{MeV})^2} \\ &= 250\,\mathrm{MeV/c}\end{aligned}$$

Energy $\gg m_e c^2$, can argue this is true in ultra-relativistic limit.

So $G = \dfrac{250\,\mathrm{MeV}}{\hbar c} = 1.267\,\mathrm{fm}^{-1}$.

For $\theta = 53°$,

$$\begin{aligned}k &= 2G\sin\frac{53°}{2} \\ &= 2(1.267\,\mathrm{fm}^{-1})\sin 26.5° \\ &= 1.131\,\mathrm{fm}^{-1} \\ \Rightarrow R &\simeq 3.975\,\mathrm{fm} \quad \text{by trial and error} \quad \leftarrow \text{Ca-48}\end{aligned}$$

For $\theta = 57°$,

$$\begin{aligned}k &= 1.209\,\mathrm{fm}^{-1} \\ \Rightarrow R &= 3.720\,\mathrm{fm} \quad \leftarrow \text{Ca-40}\end{aligned}$$

ii. The liquid drop model states that nuclear radius $r = r_0 A^{1/3}$ where $r_0 = 1.2\,\mathrm{fm}$, A is atomic number.

For Ca-40, $r \simeq 4.10\,\mathrm{fm} \rightarrow$ % error: 9.3%

For Ca-48, $r \simeq 4.36\,\mathrm{fm} \rightarrow$ % error: 8.8%

So there exists significant discrepancy between the experimental results and the liquid drop model, this is likely due to the effects of the shell model.

2. (DRAFT)

(a) i. In the quark mode, the particle states known as quarks form an approximate SU(3) [for light quarks] symmetry under strong interaction.

The quarks are fermions, so they abide Pauli Exclusion Principle and possess spin $\frac{1}{2}$. For mesons, which is made up of 2 quarks, there can be either $S = 0$ on $S = 1$ states. Which corresponds to lightest pseudoscalar octet?

Also in the quark model, a particle and antiparticle have opposing parity, so a meson, which is a composite of a quark-antiquark pair, would have parity -1 inherently.

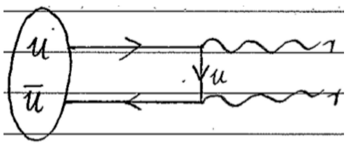
Strangeness is a quantum number assigned to the s quark in the quark model to mark the symmetry breaking as particles with non-zero strangeness would have longer lifetime compared to its counterpart with the d quark.

Which quarts are involed in pseudoscaler ortet?

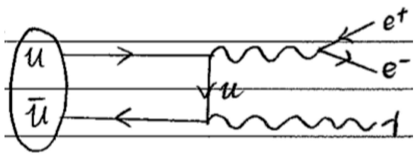
What strangeness numbers are allowed? What is strangeness of s quark?

More details needed for full 6 marks.

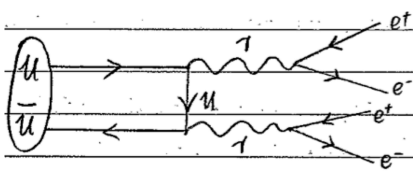
ii. $\pi^0 \to 2\gamma$:



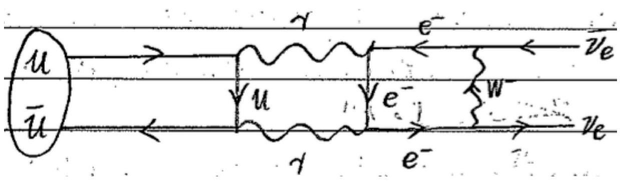
$\pi^0 \to e^+e^-\gamma$:



$\pi^0 \to e^+e^-e^+e^-$:



$\pi^0 \to \nu\bar{\nu}$:



Z^0 weak decay?

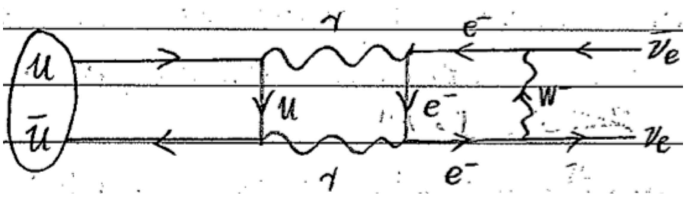
Note that further down the row, more and more EM vertices are added, so the primary decay mode would be the one with the fewest vertices – $\pi^0 \to 2\gamma$.

$\pi^0 \to e^+e^-\gamma$ involves a further EM interaction vertex, hence the matrix element would be $(\sqrt{\alpha})^2 = \alpha = \dfrac{1}{137}$ smaller approximately. The same reasoning goes for $\pi^0 \to e^+e^-e^+e^-$.

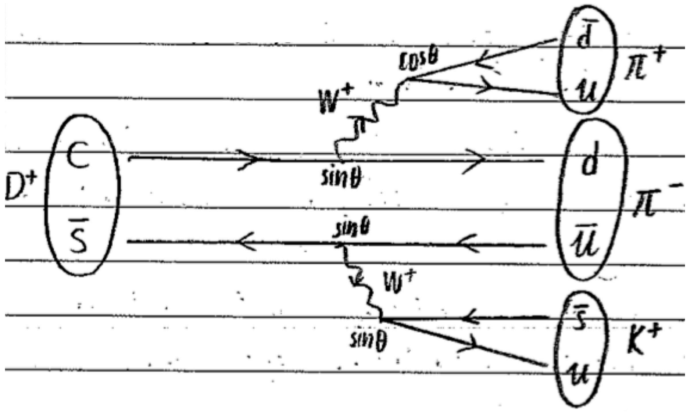
The process $\pi^0 \to \nu\bar{\nu}$ further involves 2 weak interaction vertices, further reducing the decay rate as $g_W \ll \alpha$, where g_W is the weak interaction constant.

Do an actual rough calculation or the decay rate for each, ie. each additional vertex means $\Gamma = \Gamma_0 \alpha n$, where n is the number of additional EM vertices. If final decays is weak-allowed then propagator also relevant.

(b) i. $D^+ \to K^-\pi^+\pi^+$:



$D^+ \to K^+\pi^-\pi^+$:



Under Cabibbo mixing, the light quark states are mixed with:

$$\begin{pmatrix} d' \\ s' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{\text{Cabibbo matrix}} \begin{pmatrix} d \\ s \end{pmatrix}$$

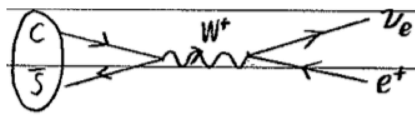
Assuming no extension to heavy quarks, and similar phase space, the ratio of decay rate is simply ratio of matrix element $|M_{if}|^2$.

For $D^+ \to K^-\pi^+\pi^+$, $|M_{if}|^2 \propto |\cos\theta \cdot \cos\theta \cdot \sin\theta \cdot \cos\theta|^2$

For $D^+ \to K^+\pi^-\pi^+$, $|M_{if}|^2 \propto |\cos\theta \cdot \sin\theta \cdot \sin\theta \cdot \sin\theta|^2$

$$\begin{aligned}\Rightarrow \frac{\Gamma_{\mathrm{D}^+\to\mathrm{K}^-\pi^+\pi^+}}{\Gamma_{\mathrm{D}^+\to\mathrm{K}^+\pi^-\pi^+}} &= \frac{\cos^6\theta\sin^2\theta}{\cos^2\theta\sin^6\theta} \\ &= \cot^4\theta \\ &= \frac{7688}{59} = 130.31 \\ \Rightarrow \theta &= 16.5^\circ\end{aligned}$$

ii. The process is not forbidden, however as ν_e is near massless and right-handed the process is helicity suppressed.



3. (DRAFT)

(a) i. The formation of $e^+e^- \to \mu^+\mu^-$ is fixed within the probed energy range (EM channel only). So the R ratio can offer insights into quark formation.

As the probed range is lower than masses of the W and Z gauge bosons, the quarks can only form via EM interaction, hadronisation occurs after the pair formation with gluon emission. W and Z can be virtual, and gluons are strong interaction.

Note that the graph has 2 plateaux, and this is due to the additional formation of c and b quarks respectively. Hence the existence of quark is demonstratable. Can you prove the CoM energy of the plateau is correct for these?

The idea of colour may also be validated by measuring the increase in pluteau height and comparing that against the expected increase, there should he a degeneray of 3 encoded in the hadronisation.

8 marks so more details.

ii. Width of the peak Γ (FWHM): $\Gamma \simeq 0.4\,\mathrm{GeV}$

Lifetime:

$$\begin{aligned}\tau &= \frac{\hbar}{\Gamma} \\ &= \frac{\hbar c}{\Gamma c} \\ &= \frac{197.33\,\mathrm{MeV\,fm}}{400\,\mathrm{MeV} \cdot 3 \times 10^{8}\,\mathrm{m\,s^{-1}}} \\ &= 1.64 \times 10^{-24}\,\mathrm{s}\end{aligned}$$

Seems very low?

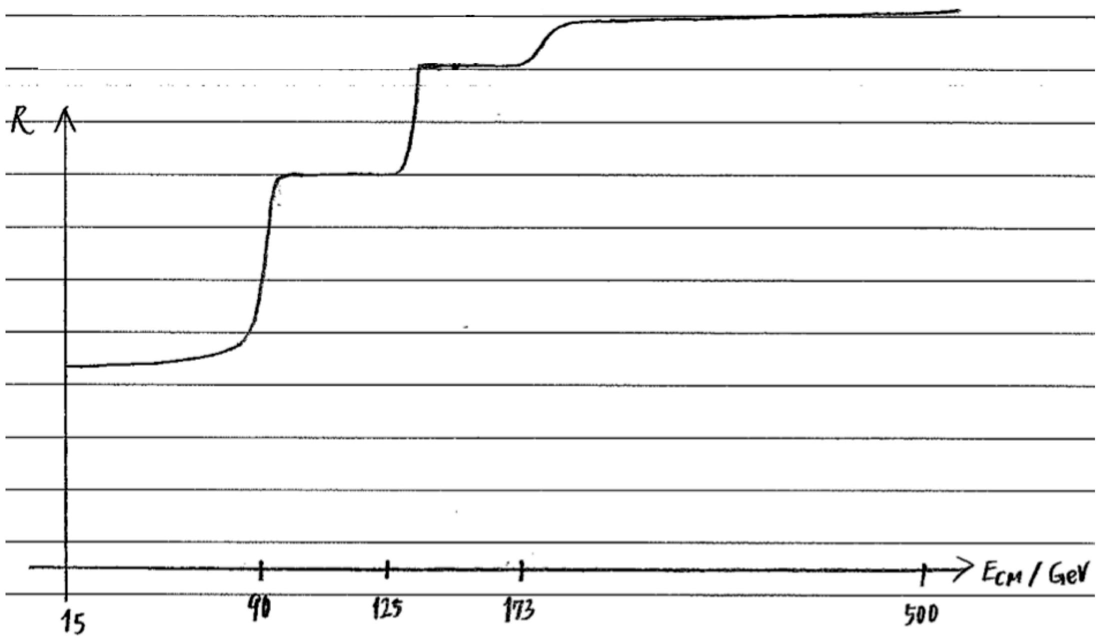
(b) As the available energy increases, more formation would be possible as the masses of the gauge bosons are overcome:

- At around 90 GeV, Z^0 formation becomes possible and R should double as Z^0 couples all fermions (in pairs).
- At around 125 GeV, H^0 formation becomes possible and R should further double for the same reason.

Some calculations to demo R of formed particles.

Apart from having new formation channels, we should also observe a jump in the plateau in several spots where new quarks are formed:

- Around 173 GeV due to the t quark.



(c) Practically at high energies, e^+ and e^- would emit bremsstrahlung due to the acceleration in synchrotron (required since collision between particles is difficult so need multiple rounds) and shed away its energy, which in turn requires higher power to the boosters in the storage ring.

Does cross section for collision.

4. (DRAFT)

(a) i. Partial width refers to the contribution to the total width due to a particular process i, e.g. Γ_i is the width due to the initial state and Γ_f is that due to the final state in Breit-Wigner equation.

Total width is simply the sum of all partial widths, and is the FWHM one measures from data: $\Gamma = \sum_j \Gamma_j$ Circular definition, what do the partial/total widths correspond to physically?

For Breit-Wigner resonance to occur, the incoming wave must be plane wave, and that the time spent in the interaction potential is long enough that the probability of transitions is random.

ii. At resonance,

$$\begin{aligned}\sigma_{\text{peak}} &= \pi g \left(\frac{\lambda}{2\pi}\right)^2 \frac{\Gamma_i \Gamma_f}{\frac{1}{4}\Gamma^2} \\ &= \frac{g\lambda^2}{\pi}\frac{\Gamma_i\Gamma_f}{\Gamma^2} \end{aligned} \tag{1}$$

For FWHM $\Gamma = 2\delta$,

$$\begin{aligned}\frac{1}{2}\sigma_{\text{peak}} &= \pi g \left(\frac{\lambda}{2\pi}\right)^2 \frac{\Gamma_i \Gamma_f}{\delta^2 + \frac{1}{4}\Gamma^2} \\ \Rightarrow \sigma_{\text{peak}} &= \frac{g\lambda^2}{2\pi}\frac{\Gamma_i\Gamma_f}{\delta^2 + \frac{1}{4}\Gamma^2} \end{aligned} \tag{2}$$

From (1), neglecting other channels so $\Gamma = \Gamma_i + \Gamma_f$:

$$\sigma_{\text{peak}} = \frac{g\lambda^2}{\pi}\frac{\Gamma_i(\Gamma - \Gamma_i)}{\Gamma^2}$$

$$\begin{aligned} g &= \frac{2\cdot 1 + 1}{(2\cdot\frac{3}{2}+1)(2\cdot\frac{1}{2}+1)} \\ &= \frac{1}{8} \end{aligned}$$

$$\begin{aligned}
p &= \frac{h}{\lambda} \\
\Rightarrow \lambda &= \frac{h}{p} \quad \text{where } p \text{ is proton momentum} \\
T &= \frac{p^2}{2m_\mathrm{p}} \\
\Rightarrow p &= \sqrt{2m_\mathrm{p}T} \quad \text{at non-relativistic regime} \\
\Rightarrow \lambda &= \frac{h}{\sqrt{2m_\mathrm{p}T}} \\
&= \frac{2\pi\hbar c}{\sqrt{2m_\mathrm{p}c^2T}} \\
&= \frac{2\pi(197.33\,\mathrm{MeV\,fm})}{\sqrt{2(938.3\,\mathrm{MeV})(1.4\,\mathrm{MeV})}} \\
&= 24.2\,\mathrm{fm}
\end{aligned}$$

So:

$$\begin{aligned}
\sigma_\mathrm{peak}\cdot\pi\cdot\Gamma^2 &= g\lambda^2\Gamma_i(\Gamma-\Gamma_i) \\
\Rightarrow g\lambda^2\Gamma_i^2 - g\lambda^2\Gamma\Gamma_i + \Gamma^2\pi\sigma_\mathrm{peak} &= 0 \\
\Gamma_i &= \frac{g\lambda^2\Gamma \pm \sqrt{g^2\lambda^4\Gamma^2 - 4g\lambda^2\Gamma^2\pi\sigma_\mathrm{peak}}}{2g\lambda^2} \\
&= 1.059\Gamma \quad \text{or} \quad -0.059\Gamma \quad \text{(unphysical)} \\
&= 1.218\,\mathrm{MeV} \quad \text{or} \quad -0.06785\,\mathrm{MeV}
\end{aligned}$$

Doesn't make sense as $\Gamma_i + \Gamma_f = \Gamma > \Gamma_i$

iii. Instead of α decay, one might also observe β decay and γ emission.

(b) Gamow factor G encodes the probability of an alpha particle tunneling through the Coulombic barrier of the nucleus, e^{2G}.

Now consider the Coulomb potential of the nucleus sans α-particle: $\dfrac{(Z-2)e}{4\pi\epsilon_0 r}$

So the energy barrier required would be:

$$V \sim \frac{2(Z-2)e^2}{4\pi\epsilon_0 r}$$

with $r \sim r_0 A^{1/3}$ where $r_0 \simeq 1.2\,\mathrm{fm}$ and A is the atomic #.

So:

$$V = \frac{2(Z-2)e^2}{4\pi\epsilon_0 r_0 A^{1/3}}$$

For ^{8}Be, $Z = 4,\ A = 8$:

$$\begin{aligned}
\Rightarrow V &= \frac{4e^2}{4\pi\epsilon_0 r_0 \underbrace{8^{1/3}}_{2}} \\
&= \frac{2e^2}{4\pi\epsilon_0 r_0} \\
&= \frac{2\alpha\hbar c}{r_0} \\
&= \frac{2(\frac{1}{137})(197.33\,\mathrm{MeV\,fm})}{1.2\,\mathrm{fm}} \\
&= 2.4\,\mathrm{MeV} \gg Q = 91\,\mathrm{keV}
\end{aligned}$$

$$\begin{aligned}
G &= \sqrt{\frac{2m_\alpha}{\hbar^2}} \int \sqrt{V(r) - Q}\,\mathrm{d}r \\
&\simeq \sqrt{\frac{2m_\alpha}{\hbar^2}} \int_0^\infty \sqrt{\frac{4e^2}{4\pi\epsilon_0 r}}\,\mathrm{d}r \\
&= \sqrt{\frac{2m_\alpha}{\hbar^2}} \int_0^\infty \sqrt{\frac{4\alpha\hbar c}{r}}\,\mathrm{d}r \\
&= \sqrt{\frac{2m_\alpha}{\hbar^2}} \sqrt{4\alpha\hbar c}\,\left[r^{1/2}\right]_{r=0}^{\infty}
\end{aligned}$$