

UNOFFICIAL SOLUTIONS BY TheLongCat

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**B3. QUANTUM, ATOMIC AND MOLECULAR PHYSICS**

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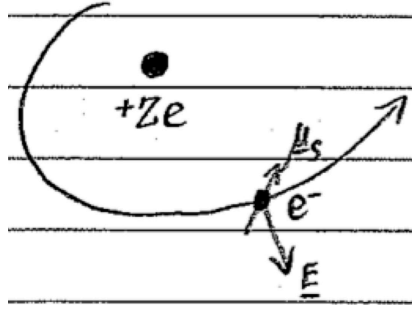
**TRINITY TERM 2017**

**Last updated: 30th May 2025**

*Disclaimer: due to its unofficial nature, the author does not warrant the accuracy of the presented solutions in any form. However, the author is happy to discuss the typos and errors should one arises.*

**Turn over as you please – we are NOT under exam conditions here.**

## 1. (DRAFT)



(a) In  $e^-$  rest frame, the magnetic field due to nucleus is:

$$\begin{aligned}
 \mathbf{B} &= -\frac{\mathbf{v} \times \mathbf{E}}{c^2} \quad \text{by Lorentz transformation} \\
 &= -\frac{1}{m_e c^2} \mathbf{p} \times \frac{1}{e} \frac{\partial U}{\partial r} \frac{\mathbf{r}}{r} \quad \text{for } \mathbf{p} = m\mathbf{v} \text{ momentum and the central field } -eE = -\frac{\partial U}{\partial r} \hat{\mathbf{r}} \\
 &= \frac{1}{m_e c^2} \left( \frac{1}{er} \frac{\partial U}{\partial r} \right) \hbar \mathbf{l} \quad \text{for } \mathbf{r} \times \mathbf{p} = \hbar \mathbf{l}
 \end{aligned}$$

Intrinsic magnetic moment of  $e^-$ :

$$\boldsymbol{\mu}_s = -g_s \mu_B \mathbf{S}$$

with  $g_s \simeq 2$ ,  $\mu_B = \frac{e\hbar}{2m_e}$ .

Dipole interaction:

$$\begin{aligned}
 \hat{H}_{\text{SO}} &= -\boldsymbol{\mu}_s \cdot \mathbf{B} \\
 &= g_s \mu_B \cdot \frac{1}{m_e c^2} \left( \frac{1}{er} \frac{\partial U}{\partial r} \right) \hbar \mathbf{s} \cdot \mathbf{l} \\
 &\rightarrow (g_s - 1) \underbrace{\frac{e\hbar}{2m_e}}_{\frac{1}{2}\alpha c e a_0} \cdot \frac{1}{m_e c^2} \left( \frac{1}{er} \frac{\partial U}{\partial r} \right) \hbar \mathbf{s} \cdot \mathbf{l} \quad \text{by Thomas precession} \\
 &= \underbrace{(g_s - 1)}_{\simeq 1} \frac{\hbar^2}{2m_e^2 c^2} \left( \frac{1}{r} \frac{\partial U}{\partial r} \right) \mathbf{s} \cdot \mathbf{l} \\
 &= \frac{\hbar \alpha a_0}{2m_e c} \left( \frac{1}{r} \frac{\partial U}{\partial r} \right) \mathbf{s} \cdot \mathbf{l}
 \end{aligned}$$

So:

$$\begin{aligned}
 \Delta E_{\text{SO}} &= \frac{\hbar^2}{2m_e^2 c^2} \left( \left\langle \frac{1}{r} \frac{Ze^2}{4\pi\epsilon_0 r^2} \right\rangle \right) \langle \mathbf{s} \cdot \mathbf{l} \rangle \\
 &= \frac{\hbar^2}{2m_e^2 c^2} \cdot Z\alpha\hbar c \left\langle \frac{1}{r^3} \right\rangle \langle \mathbf{s} \cdot \mathbf{l} \rangle \\
 &= \frac{\hbar^2}{2m_e^2 c^2} \cdot Z\alpha\hbar c \cdot \frac{1}{l(l + \frac{1}{2})(l + 1)} \left( \frac{Z}{na_0} \right)^3 \cdot \langle \mathbf{s} \cdot \mathbf{l} \rangle \\
 &= \frac{\hbar\alpha a_0}{2m_e c} \cdot \frac{Z^4\alpha\hbar c}{l(l + \frac{1}{2})(l + 1)} \cdot \frac{1}{(na_0)^3} \langle \mathbf{s} \cdot \mathbf{l} \rangle \\
 &= \frac{Z^4\alpha^2\hbar^2 \langle \mathbf{s} \cdot \mathbf{l} \rangle}{2m_e n^3 a_0^2 l(l + \frac{1}{2})(l + 1)} \\
 &= \frac{\overbrace{Z^4\alpha^2}^{\beta} \overbrace{\hbar^2}^{\beta}}{n^3 l(l + 1) \underbrace{m_e a_0^2 (2l + 1)}_{\frac{1}{2}[j(j+1)-l(l+1)-s(s+1)] \text{ for } \mathbf{j}=\mathbf{l}+\mathbf{s}}} \langle \mathbf{s} \cdot \mathbf{l} \rangle
 \end{aligned}$$

Energy separation:

$$\begin{aligned}
 \Delta E &= \frac{\beta}{2} \left[ \left( l + \frac{1}{2} \right) \left( l + \frac{3}{2} \right) - l(l + 1) - s(s + 1) \right. \\
 &\quad \left. - \left( l - \frac{1}{2} \right) \left( l + \frac{1}{2} \right) + l(l + 1) + s(s + 1) \right] \\
 &= \frac{\beta}{2} \left[ l^2 + 2l + \frac{3}{4} - l^2 + \frac{1}{4} \right] \\
 &= \frac{\beta}{2} (2l + 1) \\
 &= \frac{Z^4\alpha^2}{n^3 l(l + 1)} \cdot \frac{\hbar^2}{\underbrace{2m_e a_0^2}_{hcR_\infty?}}
 \end{aligned}$$

Configuration 2p:  $l = 1$ ,  $s = 1/2$

$$\begin{aligned}
 \Rightarrow \Delta E &= \frac{\alpha^2 hc R_\infty}{2^3 1(2)} \\
 &= \frac{13.606 \text{ eV}}{8 \cdot 2 \cdot (137.04)^2} = 4.528 \times 10^{-5} \text{ eV}
 \end{aligned}$$

Magnetic field:

$$\begin{aligned}
 -\Delta \boldsymbol{\mu}_s \cdot \mathbf{B} &= \Delta E \\
 B &= \frac{\Delta E}{g_s \mu_B \left( \frac{1}{2} + \frac{1}{2} \right)} \\
 &= \frac{\alpha^2 hc R_\infty}{32 \mu_B} = 0.024 \text{ T}
 \end{aligned}$$

For  $n = 2$ ,  $2s$  has no splitting since  $l = 0$ .  $2p$  was already done.

For  $n = 4$ ,  $4s$  has no splitting.  $4p$ ,  $4d$ ,  $4f$  do.

$4p$ :  $l = 1$ ,  $s = 1/2$

$$\begin{aligned}\Delta E &= \frac{\alpha^2 hc R_\infty}{4^3 \cdot 1(2)} \\ &= \frac{13.606 \text{ eV}}{128 (137.04)^2} = 5.660 \times 10^{-6} \text{ eV}\end{aligned}$$

$4d$ :  $l = 2$ ,  $s = 1/2$

$$\begin{aligned}\Delta E &= \frac{\alpha^2 hc R_\infty}{4^3 \cdot 2(3)} \\ &= \frac{13.606 \text{ eV}}{384 (137.04)^2} = 1.887 \times 10^{-6} \text{ eV}\end{aligned}$$

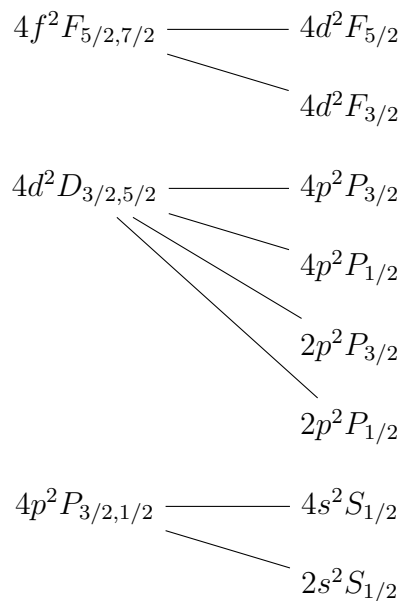
$4f$ :  $l = 3$ ,  $s = 1/2$

$$\begin{aligned}\Delta E &= \frac{\alpha^2 hc R_\infty}{4^3 \cdot 3(4)} \\ &= \frac{13.606 \text{ eV}}{768 (137.04)^2} = 9.434 \times 10^{-7} \text{ eV}\end{aligned}$$

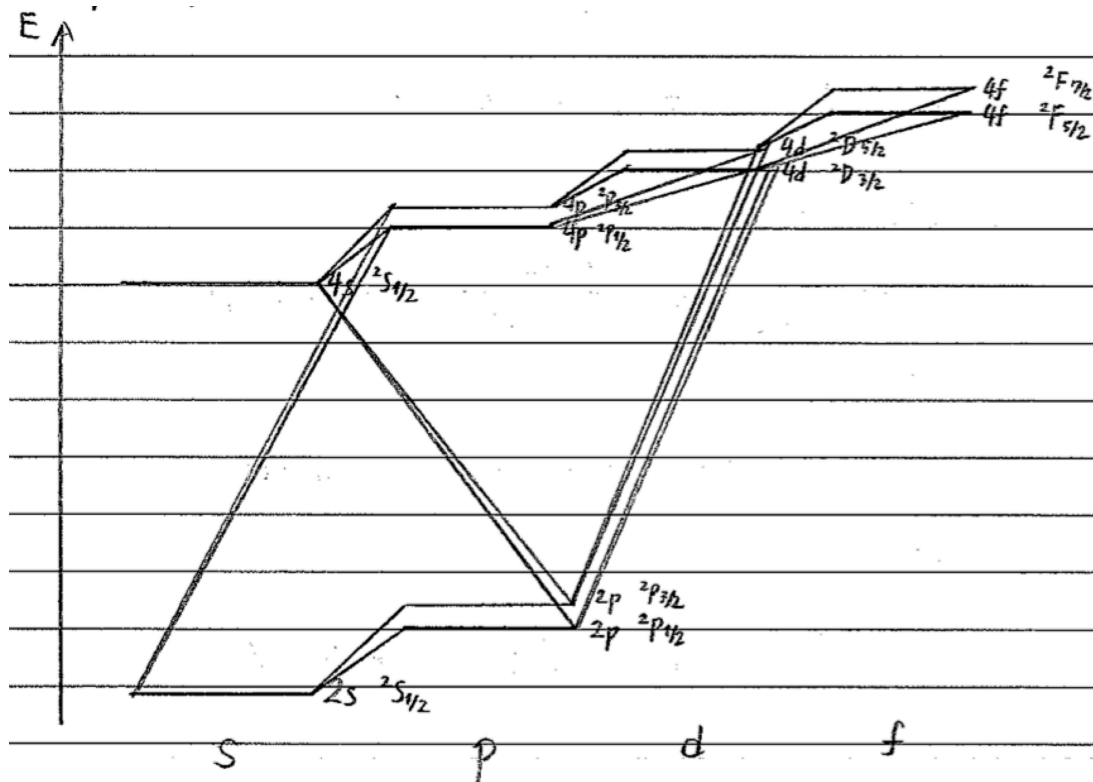
Electric dipole selection rules:

- 1  $e^-$  transition
- $\Delta l = \pm 1$
- $\Delta n = \text{any}$
- $\Delta J = \Delta j = 0, \pm 1$  ( $0 \nrightarrow 0$ )

So we have the following transitions:



Energy diagram:



From gross structure energy,  $E_n = -\frac{R_\infty hc}{n^2}$

Balmer- $\beta$  line should possess energy:

$$R_\infty hc \left[ \frac{1}{2^2} - \frac{1}{4^2} \right] = \frac{3}{16} R_\infty hc = \frac{hc}{\lambda}$$

$$\Rightarrow \lambda = \frac{16}{3R_\infty} = 4.860 \times 10^{-7} \text{ m}$$

$$= 486 \text{ nm}$$

Doppler broadening has  $\frac{\Delta\omega}{\omega} = \frac{\Delta E}{E} = \frac{v}{c}$  with  $v$  the velocity of atoms.

Equipartition theorem gives:

$$\frac{1}{2}mv^2 = \frac{3}{2}k_B T$$

$$v = \sqrt{\frac{3k_B T}{m}}$$

So:

$$\frac{\Delta E}{E} = \sqrt{\frac{3k_B T}{m_H c^2}} = 1.173 \times 10^{-5} @ 500 \text{ K}$$

Fine structure has  $\frac{\Delta E}{E} = \frac{4.528 \times 10^{-5} \text{ eV}}{2.551 \text{ eV}} = 1.775 \times 10^{-5}$  so they are just barely resolved, we could consider Doppler-free spectroscopy to eliminate this.

**2. (DRAFT)**

(a) For a multi-e<sup>-</sup> atom, we have the Hamiltonian:

$$\hat{H} = \sum_i \left[ \frac{\hat{\mathbf{p}}_i^2}{2m_e} - \frac{Ze^2}{4\pi\epsilon_0\hat{r}_i} + \sum_{j>i} \frac{e^2}{4\pi\epsilon_0\hat{r}_{ij}} \right]$$

that describes the kinetic energy, the potential energy with nucleus and the inter-e<sup>-</sup> mutual repulsion.

We may introduce a central field  $S(r)$  such that:

$$\hat{H} = \hat{H}_{\text{CF}} + \Delta\hat{H}_{\text{RE}}$$

where

$$\begin{aligned} \hat{H}_{\text{CF}} &= \sum_i \left[ \frac{\hat{\mathbf{p}}_i^2}{2m_e} - \frac{Ze^2}{4\pi\epsilon_0\hat{r}_i} + S(r_i) \right] \quad \text{is the central field,} \\ \Delta\hat{H}_{\text{RE}} &= \sum_i \left[ -S(r_i) + \sum_{j>i} \frac{e^2}{4\pi\epsilon_0\hat{r}_{ij}} \right] \quad \text{is residual electrostatic.} \end{aligned}$$

Under this approximation, we may then treat the electronic system as acting a perturbation  $\Delta\hat{H}_{\text{RE}}$  on top of  $\hat{H}_{\text{CF}}$ , which separates the Schrödinger equation to form configuration.

This requires that  $\Delta\hat{H}_{\text{RE}} \ll \hat{H}_{\text{CF}}$ , and under this perturbation, as orbital angular momenta couple, it is not a good quantum number. However since  $\Delta\hat{H}_{\text{RE}}$  is an internal interaction, the total orbital angular momentum  $\mathbf{L}$  is conserved, making it a good quantum number.

Also  $S$  is a good quantum number since  $\Delta\hat{H}_{\text{RE}}$  does not act on spins. Hence we may label the eigenstates as  $|LM_L SM_S\rangle$  or  $|LSJM_J\rangle$ . This is  $LS$  coupling.

For configuration  $1s^2 2s^2 2p^6 3s^2 3p 4p$ ,  $L = 0, 1, 2$ ,  $S = 0, 1$ .

So we have terms  $^1S, ^3S, ^1P, ^3P, ^1D, ^3D$ .

(b) For a multi-e<sup>-</sup> atom, we have spin-orbit interaction:

$$\Delta\hat{H}_{\text{SO}} = \sum_i \beta_i \hat{\mathbf{l}}_i \cdot \hat{\mathbf{s}}_i$$

Wigner-Eckart theorem then allows us to equate  $\hat{\mathbf{l}}_i$  and  $\hat{\mathbf{s}}_i$  with  $\hat{\mathbf{L}}$  and  $\hat{\mathbf{S}}$  with constant of proportionality akin to vector projection:

$$\hat{\mathbf{l}}_i \rightarrow \frac{\mathbf{l}_i \cdot \mathbf{L}}{L^2} \hat{\mathbf{L}} \quad \hat{\mathbf{s}}_i \rightarrow \frac{\mathbf{s}_i \cdot \mathbf{S}}{S^2} \hat{\mathbf{S}}$$

So:

$$\begin{aligned} \Delta\hat{H}_{\text{SO}} &= \sum_i \beta_i \frac{\mathbf{l}_i \cdot \mathbf{L}}{L^2} \hat{\mathbf{L}} \cdot \hat{\mathbf{S}} \frac{\mathbf{s}_i \cdot \mathbf{S}}{S^2} \\ &= \beta_{\text{SO}} \hat{\mathbf{L}} \cdot \hat{\mathbf{S}} \end{aligned}$$

where  $\beta_{\text{SO}} = \sum_i \beta_i \frac{\mathbf{l}_i \cdot \mathbf{L}}{L^2} \cdot \frac{\mathbf{s}_i \cdot \mathbf{S}}{S^2}$

(c) External magnetic Hamiltonian:

$$\Delta \hat{H}_z = -\boldsymbol{\mu}_{\text{tot}} \cdot \mathbf{B}_{\text{ext}}$$

where  $\boldsymbol{\mu}_{\text{tot}} = \sum_i -\mu_B \mathbf{l}_i - g_s \mu_B \mathbf{S}_i = -\mu_B \mathbf{L} - \mu_B g_s \mathbf{S}$ .

Under weak perturbation,  $J$  remains a good quantum number so by Wigner-Eckart theorem:

$$\hat{\mathbf{L}} \rightarrow \frac{\mathbf{L} \cdot \mathbf{J}}{J^2} \hat{\mathbf{J}} \quad \hat{\mathbf{S}} \rightarrow \frac{\mathbf{S} \cdot \mathbf{J}}{J^2} \hat{\mathbf{J}}$$

So:

$$\Delta \hat{H}_z = \left( +\mu_B \frac{L^2 + \mathbf{L} \cdot \mathbf{S}}{J^2} + g_s \mu_B \frac{S^2 + \mathbf{L} \cdot \mathbf{S}}{J^2} \right) \mathbf{J} \cdot \mathbf{B}$$

$$\langle \Delta E_z \rangle = g_J \mu_B M_J B$$

where

$$g_J = \frac{\cancel{L(L+1)} + \frac{1}{2} (J(J+1) - L(L+1) - S(S+1)) + 2S(S+1)}{J(J+1)}$$

$$= \frac{3}{2} + \frac{S(S+1) - L(L+1)}{2J(J+1)}$$

Electric dipole selection rules:

$$\begin{array}{ll} \Delta L = 0, \pm 1 (0 \nrightarrow 0) & \Delta J = 0, \pm 1 (0 \nrightarrow 0) \\ \Delta S = 0 & \Delta M_J = 0, \pm 1 (0 \nrightarrow 0 \text{ iff } \Delta J = 0) \end{array}$$

For transitions with multiple energies but same term, we need  $S = 1$  to have “degeneracy”.

$\Delta S = 0 \Rightarrow$  transition between  $S = 1$  states.

Try  $\underbrace{6s6p^3 P}_{\text{levels: } ^3P_0, ^3P_1, ^3P_2} \rightarrow \underbrace{6s7s^3 S}_{^3S_1}$ .

Try  $^3S_1 \rightarrow ^3P_0$ , we have:

$$\begin{array}{l} M_J = +1 \rightarrow 0 \\ \quad \quad \quad 0 \rightarrow 0 \\ \quad \quad \quad -1 \rightarrow 0 \end{array}$$

So 3 splittings.

$^3S_1 \rightarrow ^3P_1$ ,  $M_J$  (note that there is no  $0 \rightarrow 0$  here since  $\Delta J = 0$ ):

$$\begin{array}{lll} +1 \rightarrow +1 & 0 \rightarrow +1 & -1 \rightarrow 0 \\ +1 \rightarrow 0 & 0 \rightarrow -1 & -1 \rightarrow -1 \end{array}$$

6 splittings here.

$^3S_1 \rightarrow ^3P_2$ ,  $M_J$ :

$$\begin{array}{lll}
+1 \rightarrow +2 & 0 \rightarrow +1 & -1 \rightarrow 0 \\
+1 \rightarrow +1 & 0 \rightarrow 0 & -1 \rightarrow -1 \\
+1 \rightarrow 0 & 0 \rightarrow -1 & -1 \rightarrow -2
\end{array}$$

9 splittings here.

So line A corresponds to  $\underbrace{6s7s^3S_1}_{L=0, S=1, J=1} \rightarrow \underbrace{6s6p^3P_0}_{L=1, S=1, J=0}$ .

$$g_J = \frac{3}{2} + \frac{1(2) - 0}{2 \cdot 1 \cdot 2} = 2 \quad \text{since lower level has no splitting}$$

Line B corresponds to  $\underbrace{6s7s^3S_1}_{L=0, S=1, J=1} \rightarrow \underbrace{6s6p^3P_1}_{L=1, S=1, J=1}$ .

Lower level has:

$$g_J = \frac{3}{2} + \frac{1(2) - 1(2)}{2 \cdot 1 \cdot 2} = \frac{3}{2}$$

Line C corresponds to  $\underbrace{6s7s^3S_1}_{L=0, S=1, J=1} \rightarrow \underbrace{6s6p^3P_2}_{L=1, S=1, J=2}$ .

One method of determining if  $LS$  coupling is good is to verify the Interval Rule, that is  $\Delta E_{\text{SO}}(J) - \Delta E_{\text{SO}}(J-1) \propto J$ .

This implies that:

$$\frac{\Delta E_{\text{SO}}(J) - \Delta E_{\text{SO}}(J-1)}{\Delta E_{\text{SO}}(J-1) - \Delta E_{\text{SO}}(J-2)} = \frac{J}{J-1}$$

assuming  $\beta_{\text{SO}}$  is constant.

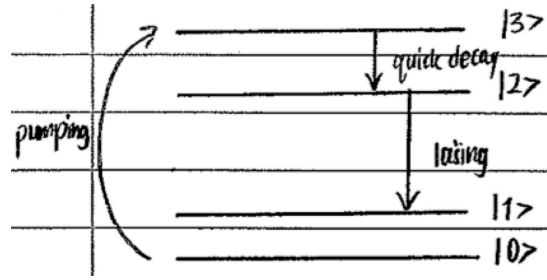
$$\begin{aligned}
\Delta E_{\text{SO}}(J=2) - \Delta E_{\text{SO}}(J=1) &= -\frac{hc}{\lambda_C} + \frac{hc}{\lambda_B} \\
\Delta E_{\text{SO}}(J=1) - \Delta E_{\text{SO}}(J=0) &= -\frac{hc}{\lambda_B} + \frac{hc}{\lambda_A} \\
\Rightarrow \frac{\Delta E_{\text{SO}}(J=2) - \Delta E_{\text{SO}}(J=1)}{\Delta E_{\text{SO}}(J=1) - \Delta E_{\text{SO}}(J=0)} &= \frac{1/\lambda_B - 1/\lambda_C}{1/\lambda_A - 1/\lambda_B} \\
&= 2.63 \neq 2
\end{aligned}$$

with  $\sim 30\%$  deviation!

So  $LS$  coupling does not describe mercury well.



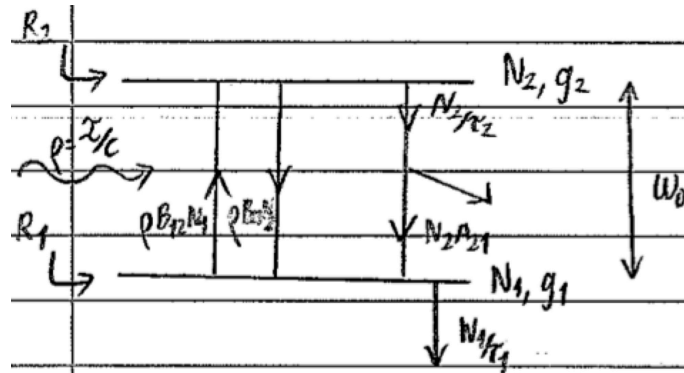
## 4. (DRAFT)



- (a) Both three-level and four-level lasers operate by pumping a ground state population  $|0\rangle$  to an unstable state  $|3\rangle$ , which decays immediately to the upper lasing state  $|2\rangle$ . The lasing causes the population to drop to lower state  $|1\rangle$ .

Three-level lasers have the energy gap between  $|0\rangle$  and  $|1\rangle$  within thermal energy  $k_B T$ , making them essentially a single level. Otherwise it is a four-level laser.

The reason why three-level laser has a substantially higher power requirement is that population at  $|1\rangle \neq 0$  initially, requiring higher power to induce population inversion. Four-level laser has the initial population = 0, so it requires lower power.



Rate equations:

$$\begin{aligned}\frac{dN_2}{dt} &= R_2 + \rho B_{12}N_1 - \rho B_{21}N_2 - \frac{N_2}{\tau_2} \\ \frac{dN_1}{dt} &= R_1 - \rho B_{12}N_1 + \rho B_{21}N_2 + A_{21}N_2 - \frac{N_1}{\tau_1}\end{aligned}$$

For a radiation of width  $\delta\omega$ , energy change after passing through a volume element  $A dz$ :

$$\begin{aligned}d\mathcal{I}\delta\omega A &= (\rho B_{21}N_2 - \rho B_{12}N_1) g(\omega - \omega_0) \delta\omega \cdot \hbar\omega \cdot A dz \\ \frac{d\mathcal{I}}{dz} &= \left( N_2 - \frac{g_2}{g_1} N_1 \right) \underbrace{\rho}_{\frac{\mathcal{I}}{c}} B_{21} g(\omega - \omega_0) \hbar\omega \\ &= N^* \sigma_{21}(\omega - \omega_0) \mathcal{I}\end{aligned}$$

where  $\mathcal{I}$  is spectral density,  $N^* = N_2 - \frac{g_2}{g_1} N_1$ ,  $\sigma_{21}(\omega - \omega_0) = B_{21} g(\omega - \omega_0) \frac{\hbar\omega}{c}$ .

So substitute  $N^*$ ,  $\sigma_{21}$ ,  $\mathcal{I}$  into the rate equations and integrating over all  $\omega$  to get total intensity  $I$ :

$$\begin{aligned}\frac{dN_2}{dt} &= R_2 - N^* \sigma_{21} \frac{I}{\hbar\omega} - \frac{N_2}{\tau_2} \\ \frac{dN_1}{dt} &= R_1 + N^* \sigma_{21} \frac{I}{\hbar\omega} + A_{21} N_2 - \frac{N_1}{\tau_1}\end{aligned}$$

At steady state,

$$\begin{aligned}N_2 &= R_2 \tau_2 - N^* \sigma_{21} \frac{I}{\hbar\omega} \tau_2 \\ N_1 &= R_1 \tau_1 + N^* \sigma_{21} \frac{I}{\hbar\omega} \tau_1 + A_{21} N_2 \tau_1\end{aligned}$$

Hence:

$$\begin{aligned}N^* &= N_2 - \frac{g_2}{g_1} N_1 \\ &= \left(1 - \frac{g_2}{g_1} A_{21} \tau_1\right) N_2 - \frac{g_2}{g_1} R_1 \tau_1 - \frac{g_2}{g_1} N^* \sigma_{21} \frac{I}{\hbar\omega} \tau_1 \\ &= \left(1 - \frac{g_2}{g_1} A_{21} \tau_1\right) R_2 \tau_2 + \left(-\tau_2 + \frac{g_2}{g_1} A_{21} \tau_1 \tau_2 - \frac{g_2}{g_1} \tau_1\right) N^* \sigma_{21} \frac{I}{\hbar\omega} - \frac{g_2}{g_1} R_1 \tau_1 \\ \Rightarrow N^*(I) &= \frac{N^*(0)}{1 + \frac{I}{I_s}}\end{aligned}$$

where

$$\begin{aligned}N^*(0) &= \left(1 - \frac{g_2}{g_1} A_{21} \tau_1\right) R_2 \tau_2 - \frac{g_2}{g_1} R_1 \tau_1 \\ I_s &= \left[ \underbrace{\left(\tau_2 + \frac{g_2}{g_1} \tau_1 - \frac{g_2}{g_1} A_{21} \tau_1\right)}_{\tau_R} \frac{\sigma_{21}}{\hbar\omega} \right]^{-1} \\ &= \frac{\hbar\omega}{\sigma_{21} \tau_R}\end{aligned}$$

is the saturation intensity.

- (b) Natural broadening has a Lorentzian lineshape. Its width is the sum of the width from upper and lower decay widths ( $1/\tau_{\text{tot}} = 1/\tau_{\text{upper}} + 1/\tau_{\text{lower}}$ ).

Einstein relation:

$$\begin{aligned}\frac{A_{21}}{B_{21}} &= \frac{\hbar\omega^3}{\pi^2 c^3} \\ \Rightarrow \sigma_{21}(\omega - \omega_0) &= \frac{\pi^2 c^3}{\hbar\omega^3} A_{21} g(\omega - \omega_0) \cdot \frac{\hbar\omega}{c}\end{aligned}$$

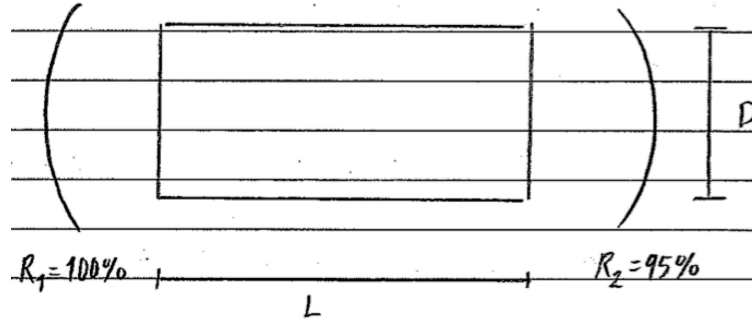
At  $\omega = \omega_0$ ,  $g(0) = \frac{2}{\pi}$  by normalisation (Lorentzian peaks at  $\omega = \omega_0$ ):

$$\begin{aligned}\Rightarrow \sigma_{21}(0) &= \frac{\pi^2 c^3}{\hbar \omega^3} \cdot \frac{2}{\pi} \cdot \frac{\hbar \omega_0}{c} \\ &= \frac{2\pi c^2}{\omega_0^2}\end{aligned}$$

For  $\lambda_0 = 700 \text{ nm}$ ,

$$\begin{aligned}\sigma_{21}(0) &= \frac{2\pi c^2}{\omega_0^2} \\ &= \frac{2\pi}{k_0^2} \\ &= \frac{\lambda_0^2}{2\pi} = 7.799 \times 10^4 \text{ nm}^2\end{aligned}$$

(c) Sketch of the setup:



i.  $L = 4 \text{ cm}$ ,  $D = 0.4 \text{ cm}$ ,  $N_{\text{tot}} = 4 \times 10^{19} \text{ cm}^{-3}$ ,  $\lambda_0 = 694 \text{ nm}$ .

Assuming no saturation, beam growth  $\propto R_1 R_2 e^{2\alpha L}$  with  $\alpha = N^* \sigma_{21}$  the gain coefficient.

At threshold,

$$\begin{aligned}R_1 R_2 e^{2\alpha L} &= 1 \\ 2\alpha L &= -\ln(R_1 R_2) \\ \alpha &= -\frac{1}{2L} \ln(R_1 R_2) \\ N_{\text{th}}^* &= -\frac{\ln(R_1 R_2)}{2L\sigma_{21}} \\ N_2^{\text{th}} - \frac{g_2}{g_1} N_1^{\text{th}} &= N_2^{\text{th}} - (N_{\text{tot}} - N_2^{\text{th}}) \\ \Rightarrow N_2^{\text{th}} &= \frac{1}{2} N_{\text{tot}} - \frac{1}{2} \frac{\ln(R_1 R_2)}{2L\sigma_{21}}\end{aligned}$$

Hence energy density required to achieve population inversion:

$$\begin{aligned}\rho &= N_2^{\text{th}} \cdot \hbar \omega_{\text{pump}} \\ \sigma_{21} &= \frac{\lambda_0^2}{2\pi} \Rightarrow \rho = 2 \times 10^{19} \text{ cm}^{-3} \cdot 3.98 \times 10^{-19} \text{ J}\end{aligned}$$

Total energy required (assuming 100% efficiency):

$$E = \rho \cdot \pi \left( \frac{D}{2} \right)^2 \cdot L$$

$$= 12.56 \text{ J}$$

- ii.  $L = 1 \text{ cm}$ ,  $N_{\text{tot}} = 5 \times 10^{19} \text{ cm}^{-3}$ ,  $\sigma_{21} = 3 \times 10^{-19} \text{ cm}^2$ ,  $\lambda_{\text{pump}} = 532 \text{ nm}$ ,  $D = 100 \mu\text{m}$ ,  $\tau_2 = 3 \times 10^{-6} \text{ s}$ ,  $\tau_1 = 1 \times 10^{-10} \text{ s}$ .

For pulsed laser, we just need to achieve population inversion (not maintain it):

$$\text{Since } N_1 = 0 \text{ for four-level laser, } N_{\text{th}}^* = N_2^{\text{th}} = -\frac{\ln(R_1 R_2)}{2L\sigma_{21}}.$$

So energy density required:

$$\rho = N_2^{\text{th}} \cdot \frac{hc}{\lambda_{\text{pump}}}$$

$$= 8.55 \times 10^{16} \text{ cm}^{-3} \cdot 3.74 \times 10^{-19} \text{ J}$$

Hence pumping energy:

$$E = \rho \pi \left( \frac{D}{2} \right)^2 \cdot L$$

$$= 2.51 \times 10^{-6} \text{ J}$$

For continuous wave, we need:

$$N_2^{\text{th}} = \left( 1 - \frac{g_2}{g_1} \frac{A_{21}}{1} \tau_1 \right) \mathcal{R} \tau_2$$

$$= \left( 1 - \frac{\tau_1}{\tau_2} \right) \mathcal{R} \tau_2 = -\frac{\ln(R_1 R_2)}{2L\sigma_{21}}$$

$$\Rightarrow \mathcal{R} = \frac{-\ln(R_1 R_2)}{\left( 1 - \frac{\tau_1}{\tau_2} \right) \tau_2 \cdot 2L\sigma_{21}}$$

$$= 2.85 \times 10^{22} \text{ cm}^2 \text{ s}^{-1}$$

is the pumping rate.

So pumping power:

$$P = \mathcal{R} \frac{hc}{\lambda_{\text{pump}}} \cdot \pi \left( \frac{D}{2} \right)^2 \cdot L$$

$$= 0.836 \text{ W}$$

Natural width should be  $\Delta\omega = 1/\tau_1 + 1/\tau_2 = 1 \times 10^{10} \text{ rad s}^{-1}$ . Other mechanisms such as phonon broadening can cause extra broadening. But Doppler does not apply here as sapphire is a solid.