Complete Induction

-and-

Graphs

Outline for Today

- **Complete Induction Revisited**
  - Recap from last time.
  - Another example: continued fractions.
- **Graphs**
  - A way of modeling connections between objects.
- **An Overarching Question**
  - How exactly do you *do* math?

Recap from Last Time

Variations on Induction

- ***Starting later:*** Induction still works if the base case is shifted to a later natural number.
- ***Multiple base cases:*** Induction still works if there are many base cases, not just one.
- ***Larger steps:*** Induction still works if you take larger steps (though you need to be sure you cover all the numbers you care about!)

Complete Induction

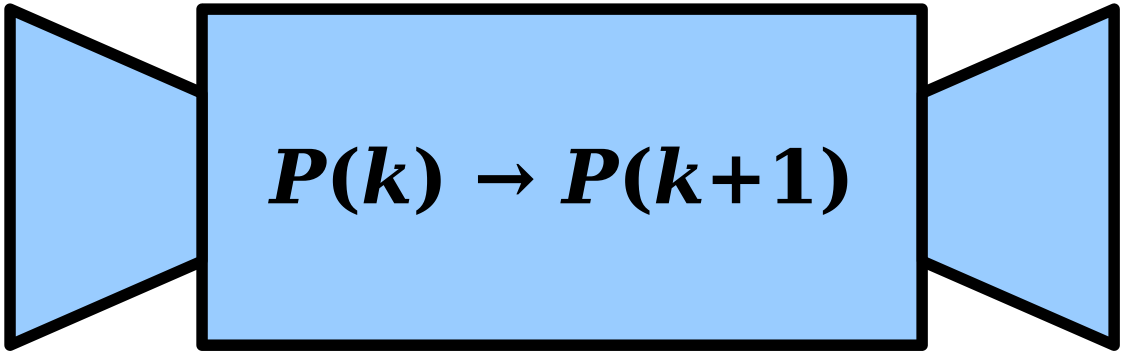
- If the following are true:
  - $P(0)$ is true, and
  - If $P(0), P(1), P(2), \ldots, P(k)$ are true, then $P(k+1)$ is true as well.

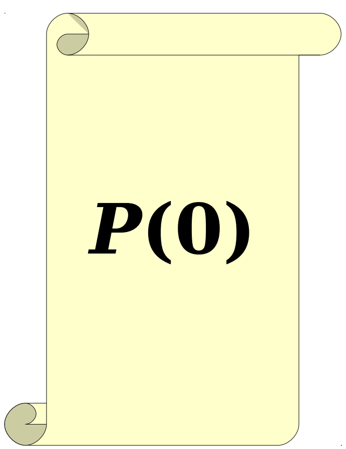
  then $P(n)$ is true for all $n \in \mathbb{N}$.
- This is called the ***principle of complete induction*** or the ***principle of strong induction***.
  - (A note: this also works starting from a number other than 0; just modify what you're assuming appropriately.)

That's a *lot* of assumptions to make!

Why is this legal?

Review: Induction as a Machine





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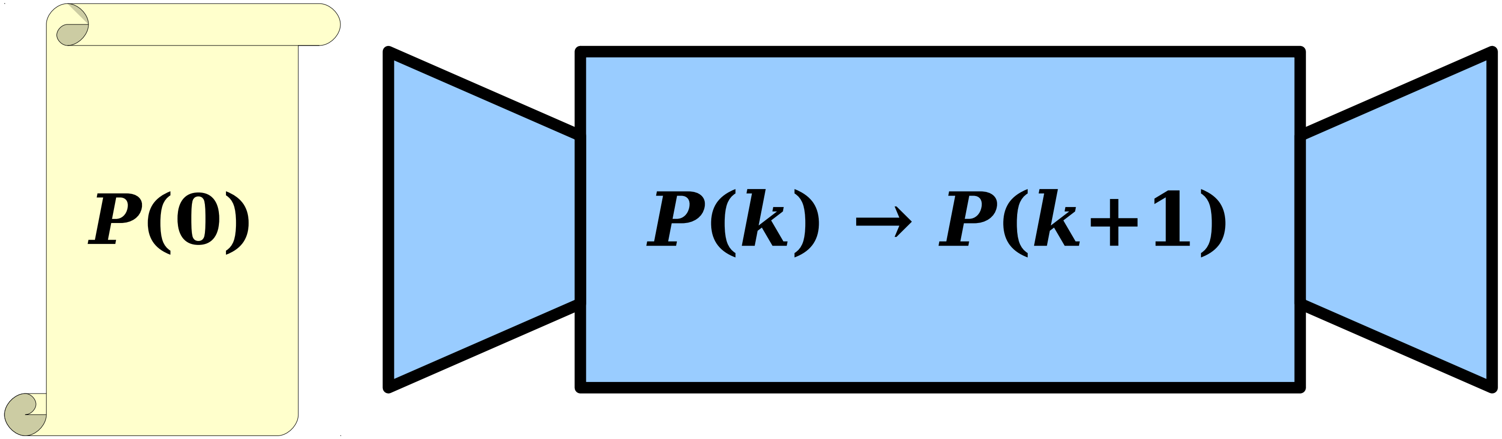

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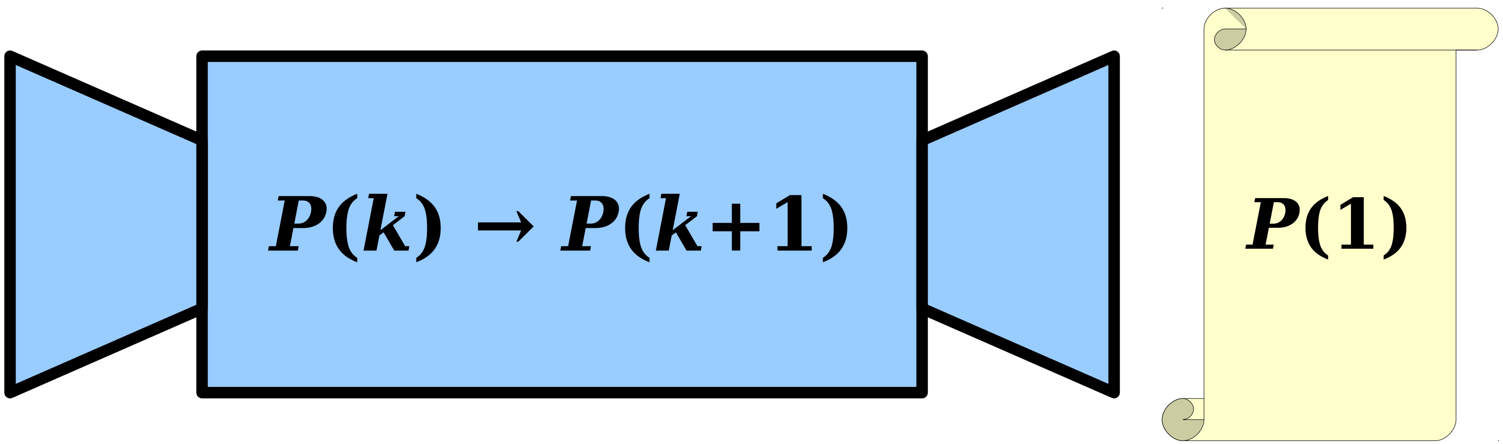

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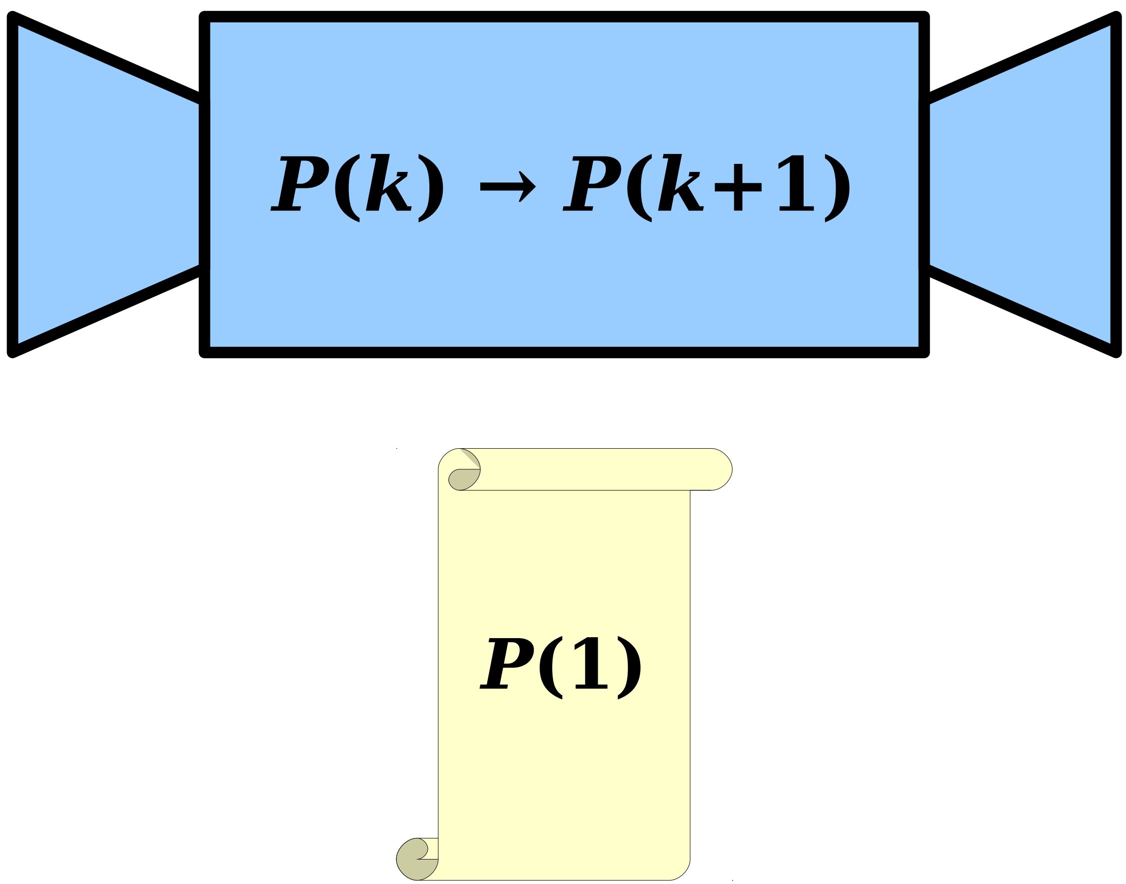

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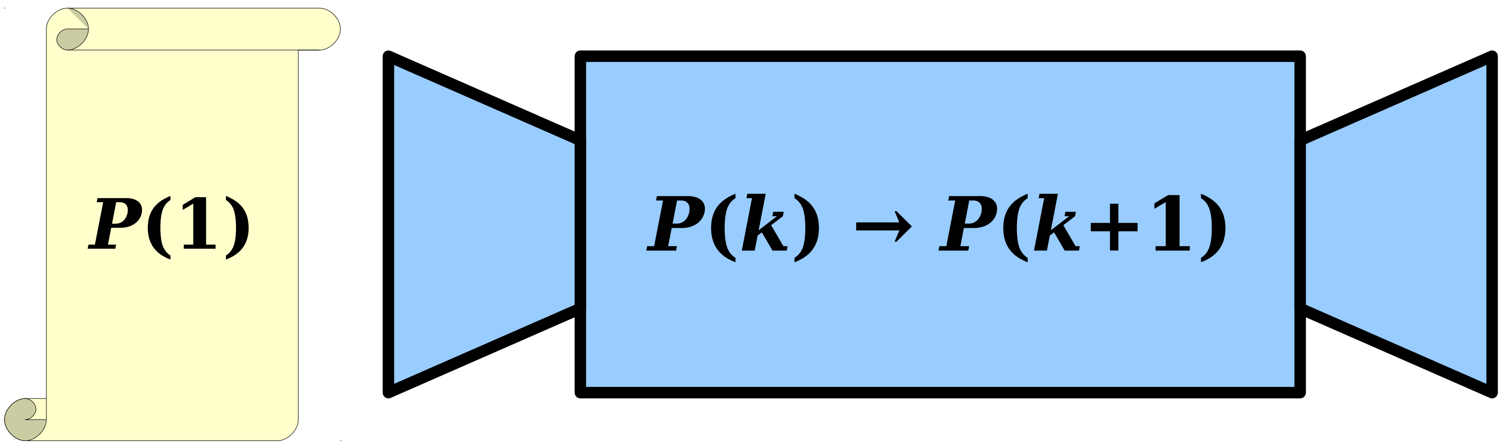

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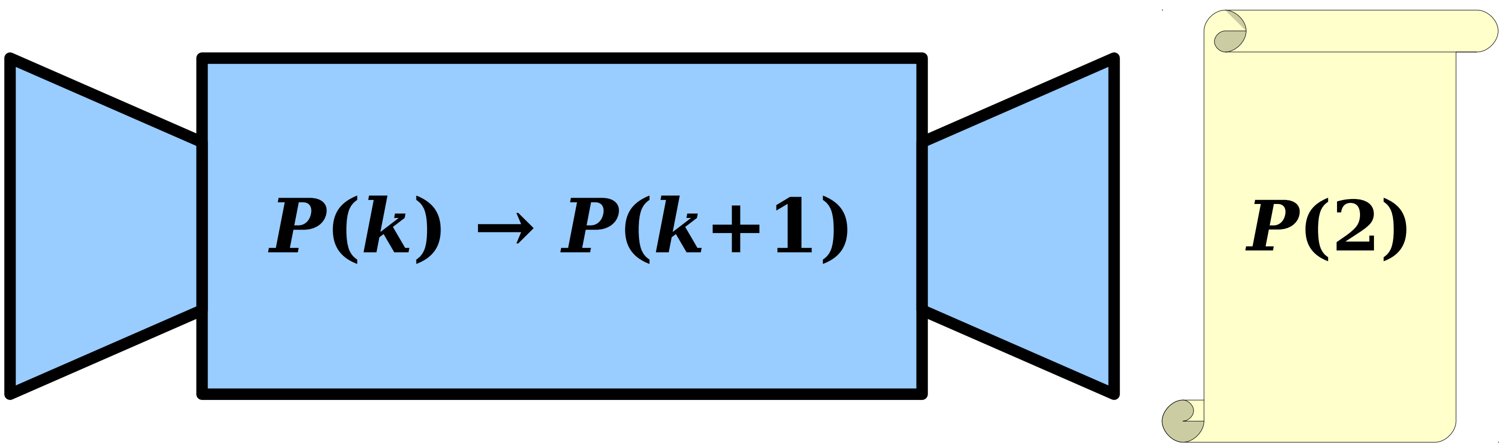

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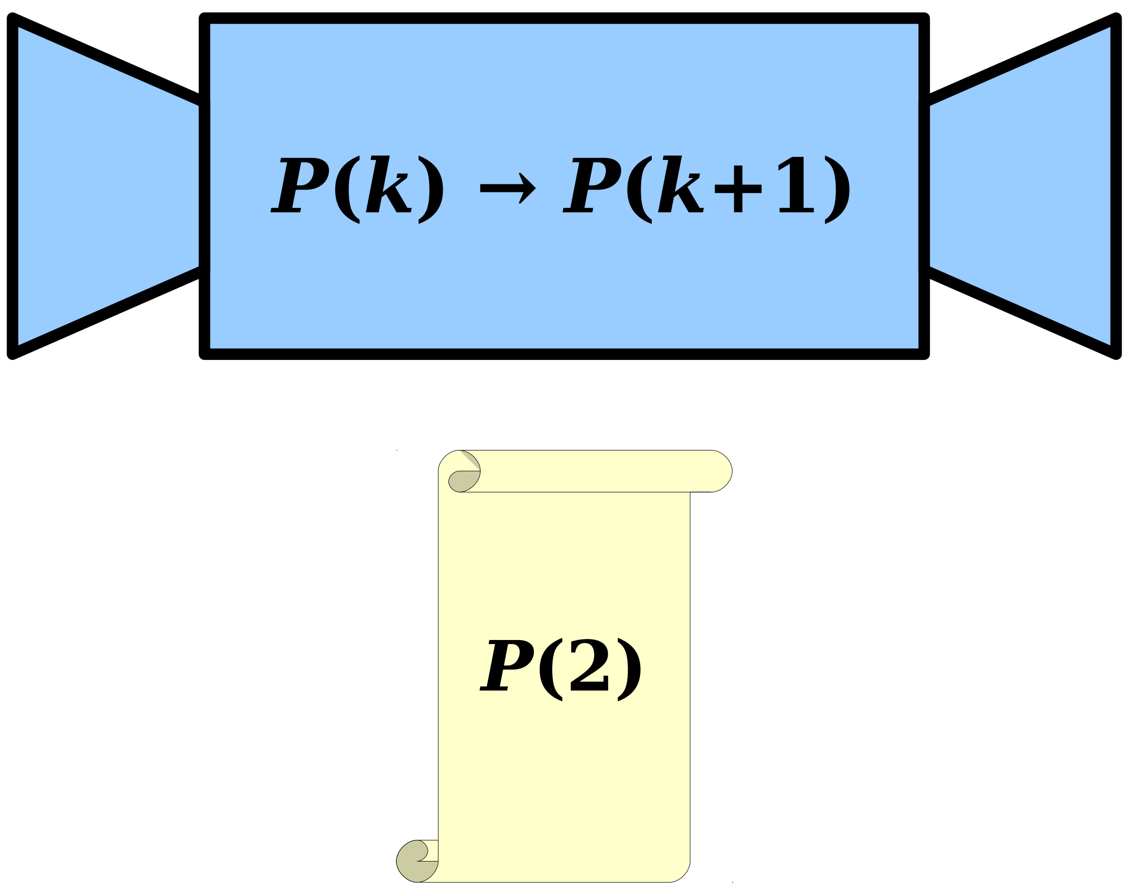

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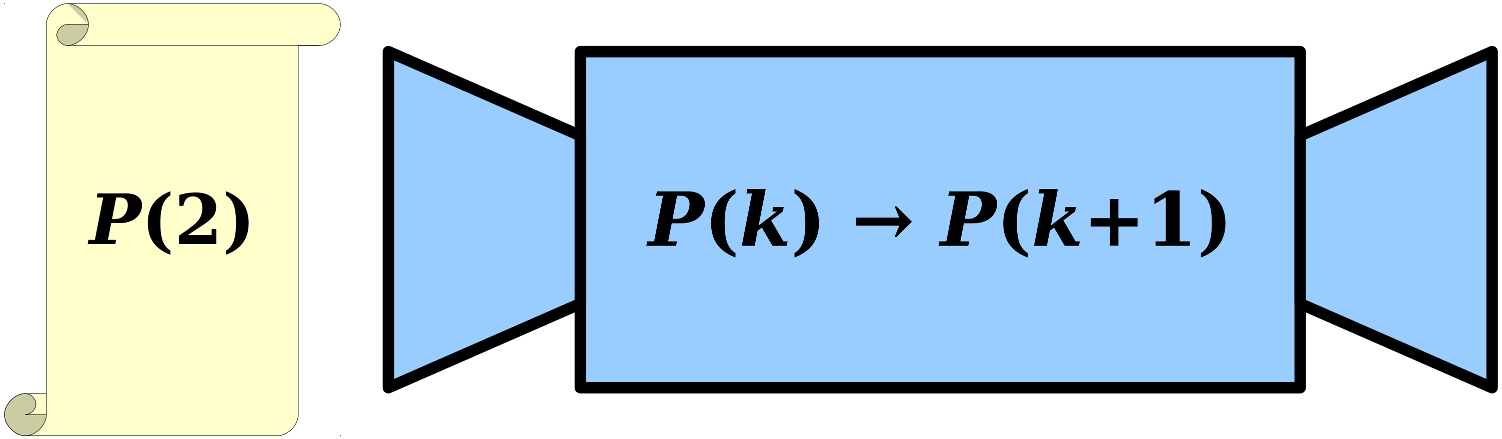

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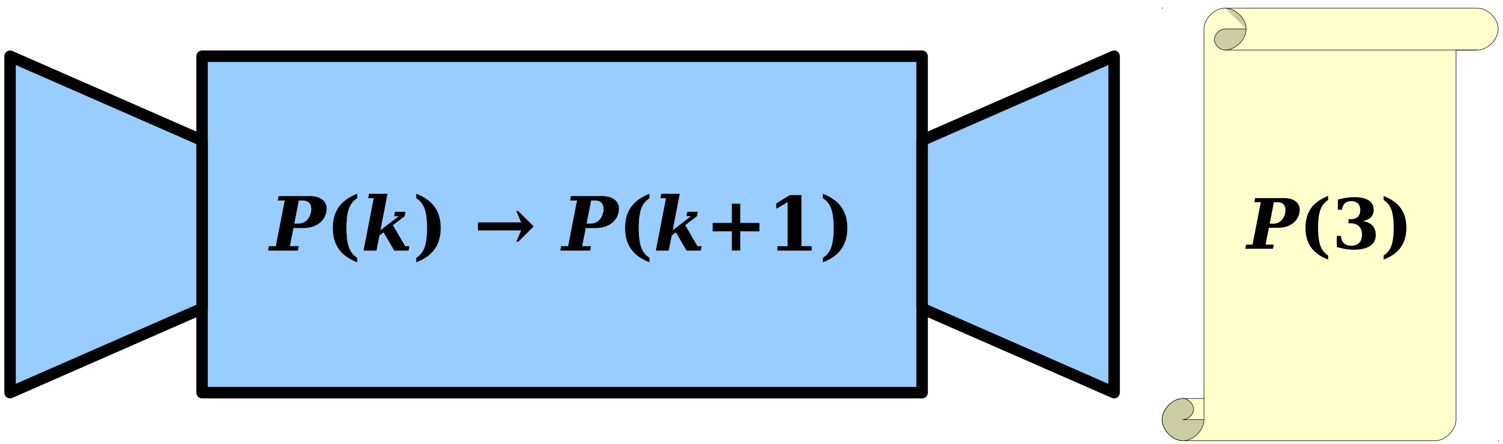
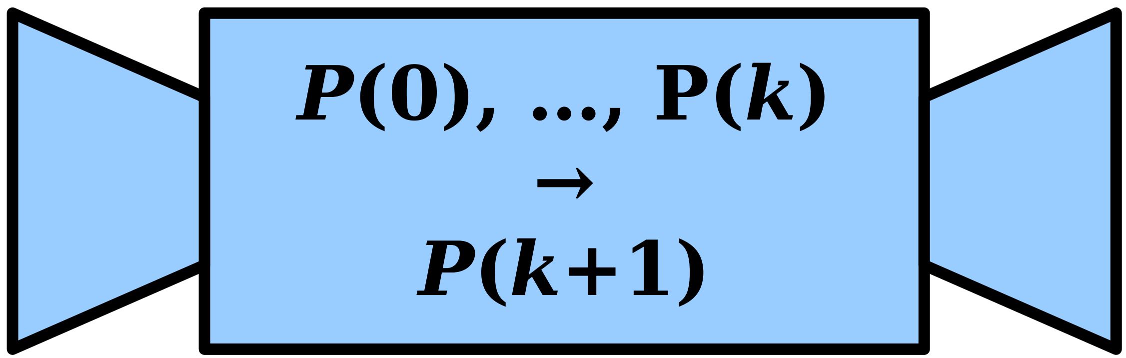
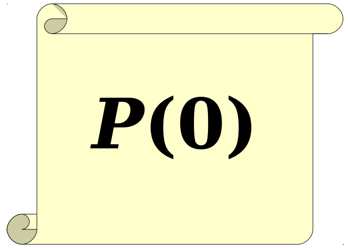


Intuiting Complete Induction





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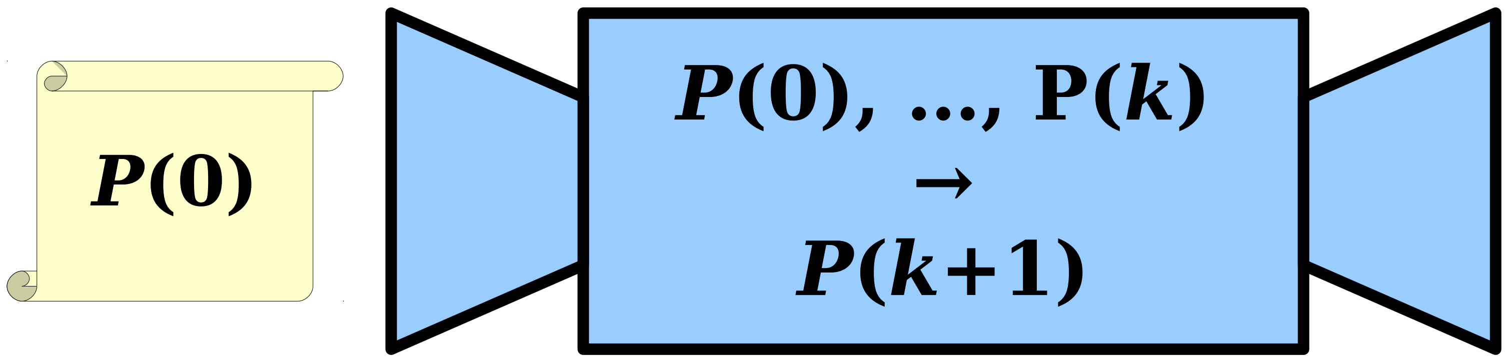

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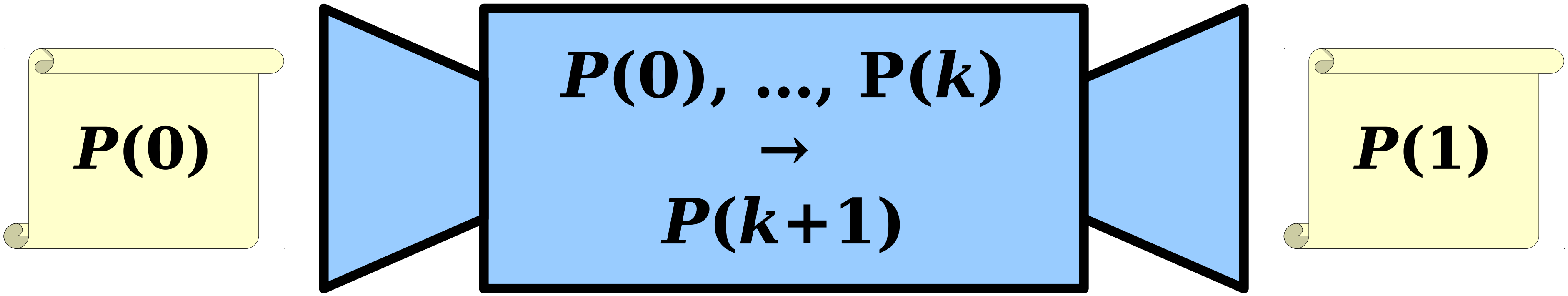

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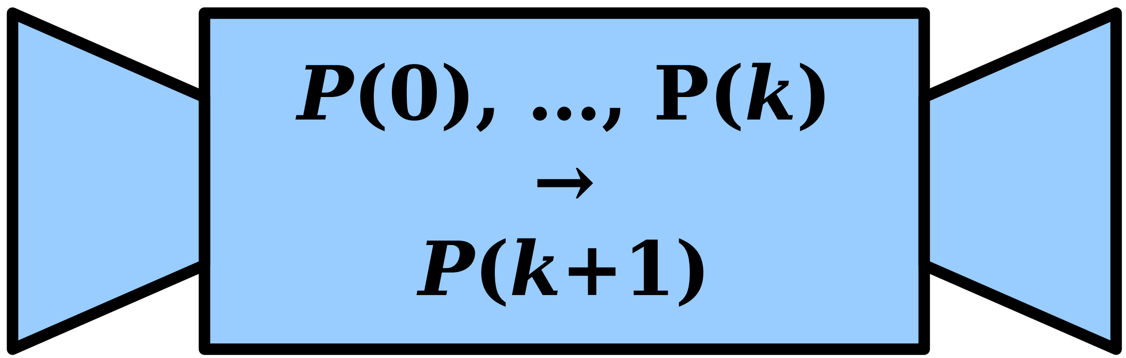

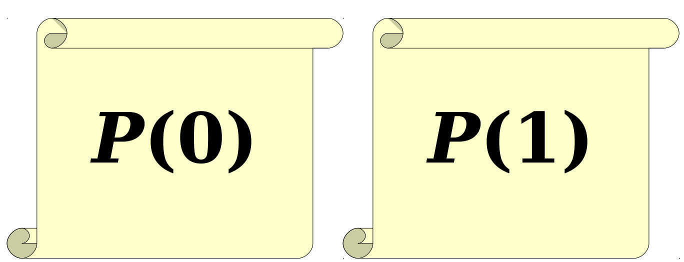

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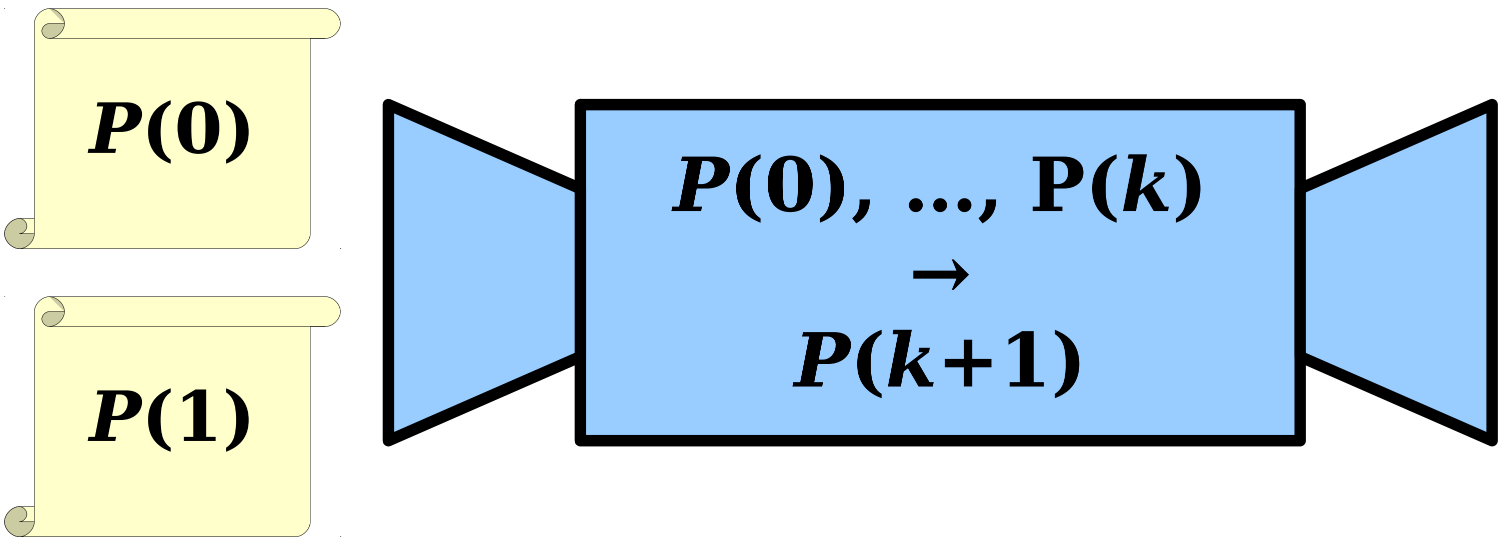

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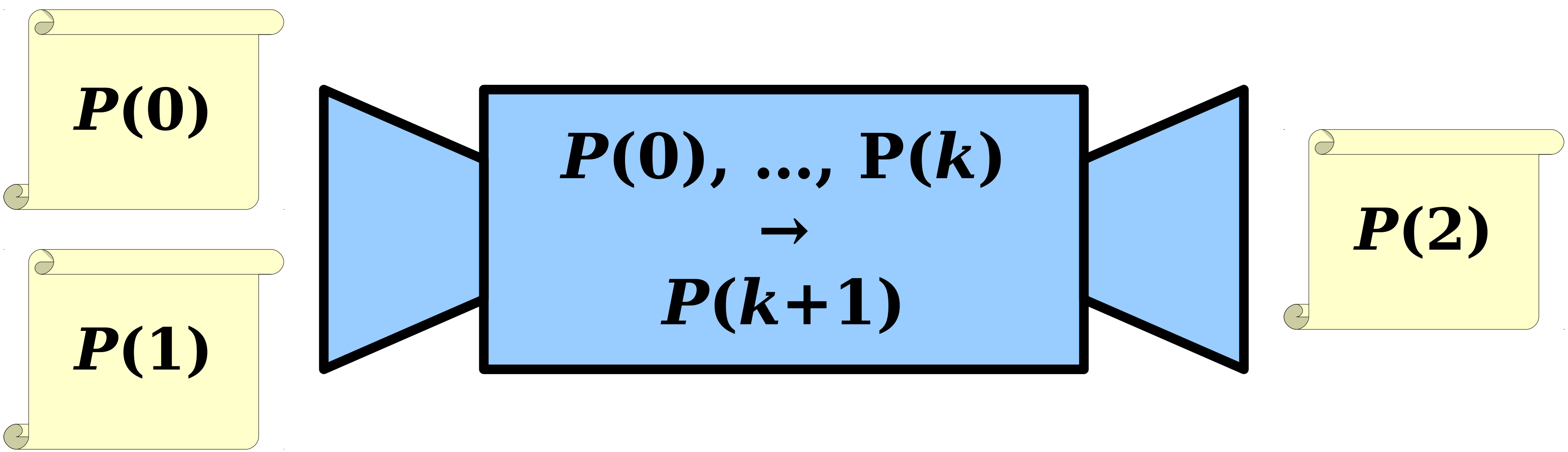

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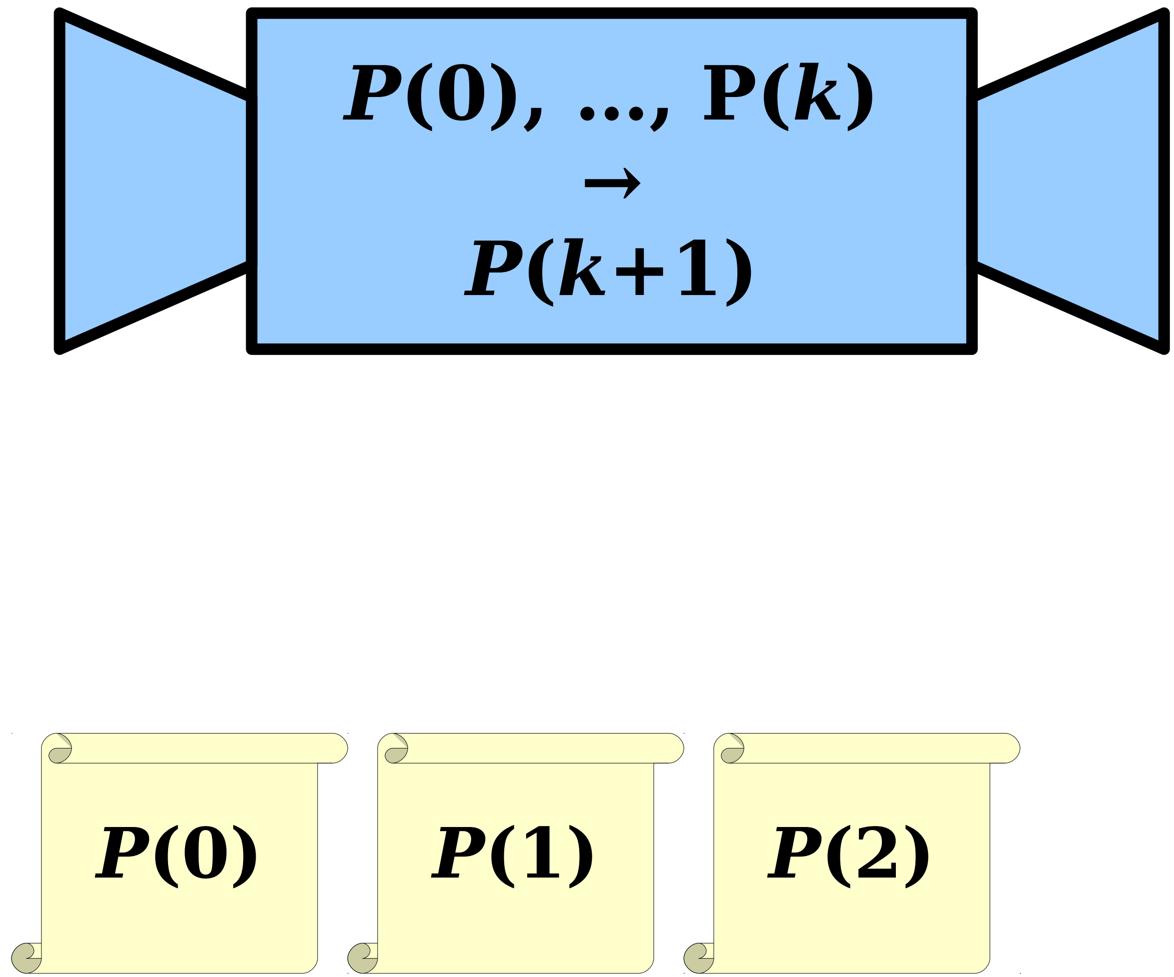

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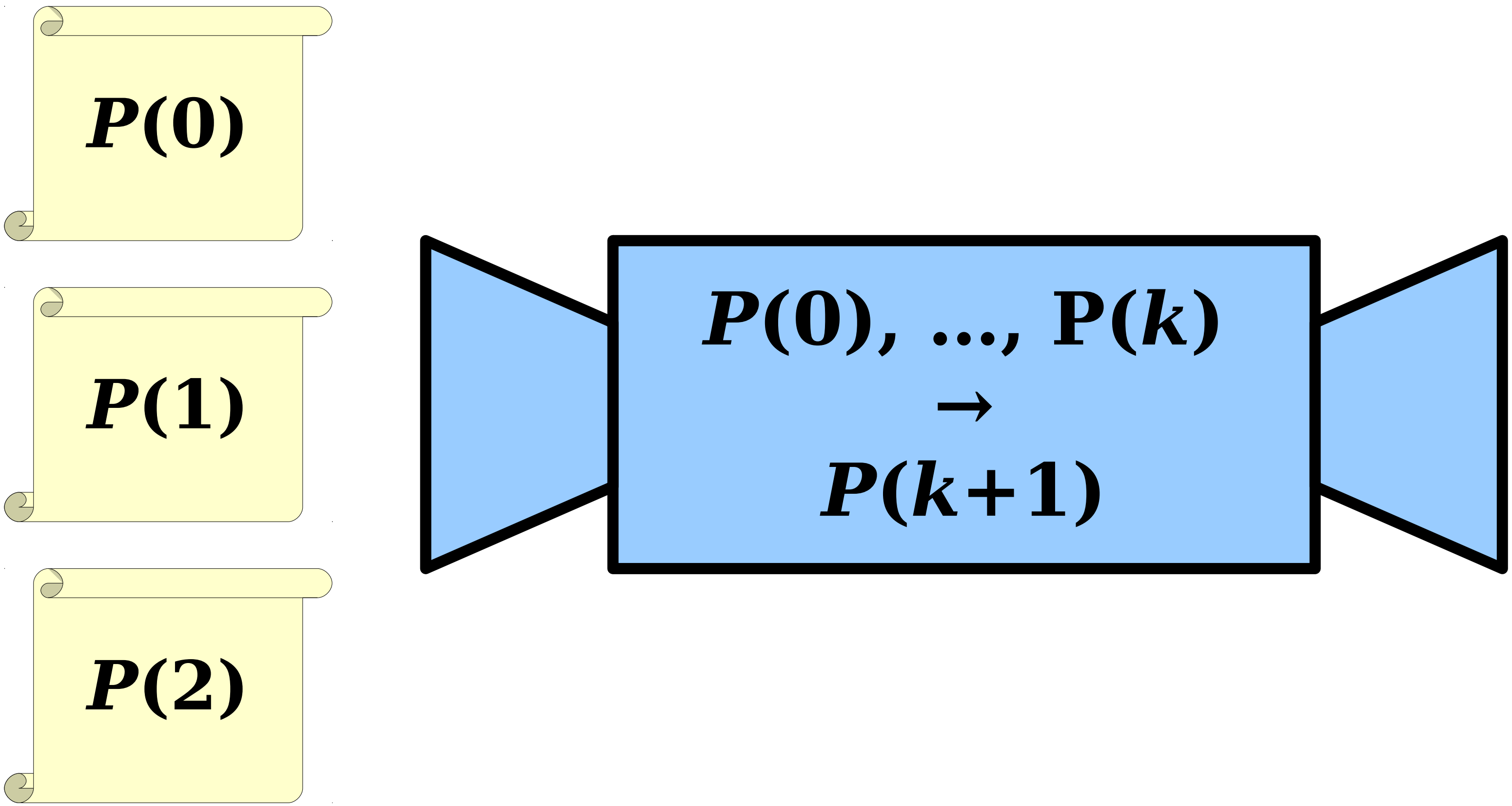

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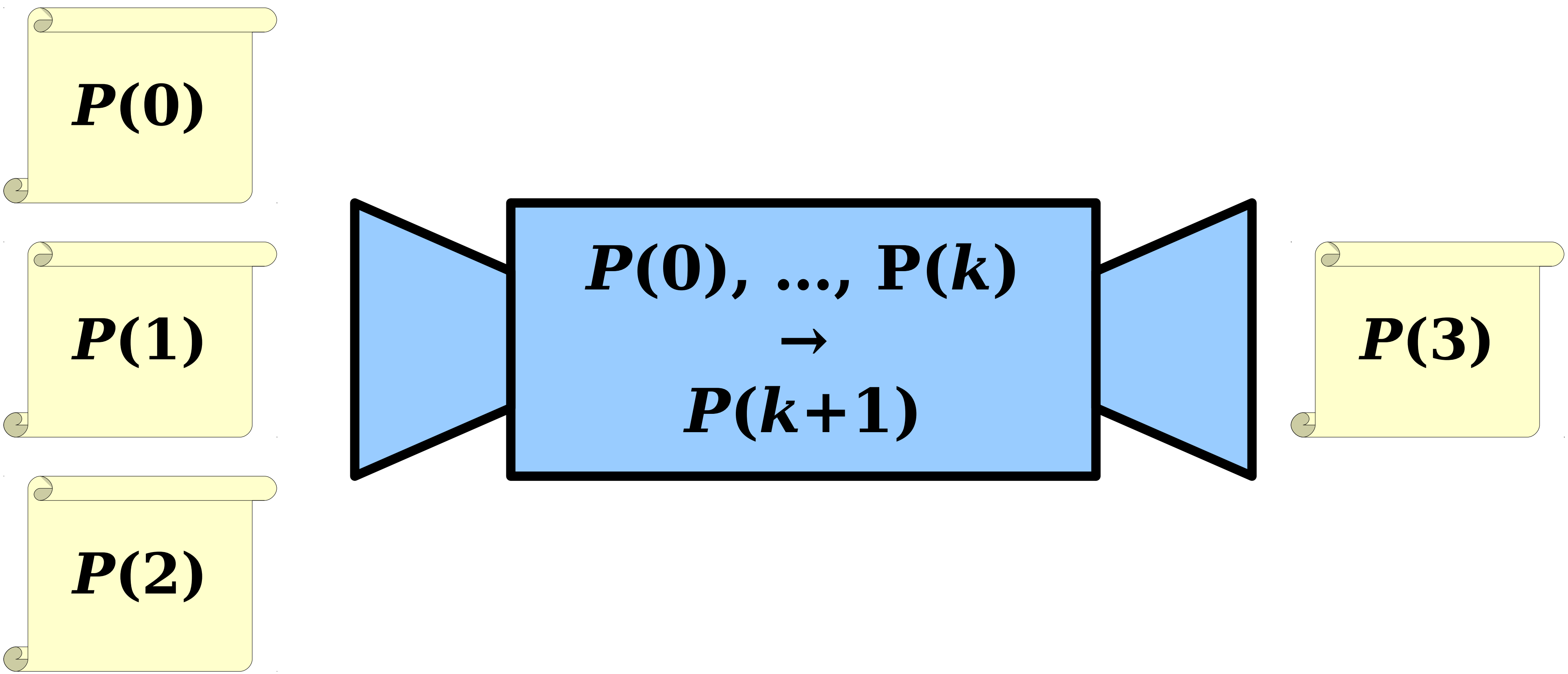

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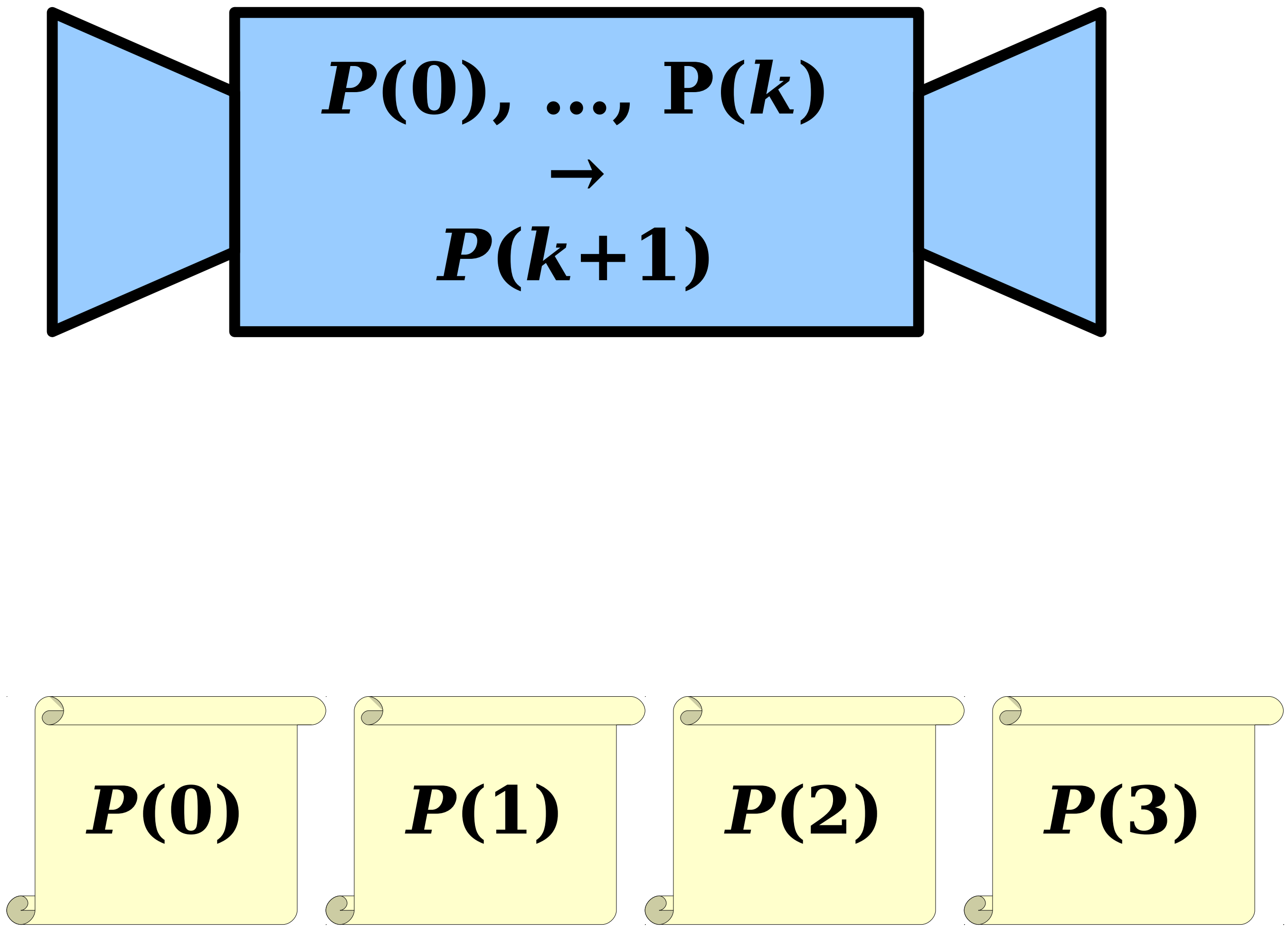


Application: **Continued Fractions**

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{\boxed{1 + \cfrac{1}{2}}}}$$

Continued Fractions

$$\cfrac{1}{4 + \cfrac{1}{\cfrac{3}{2}}}$$

Continued Fractions

$$\frac{1}{4 + \boxed{\dfrac{1}{\dfrac{3}{2}}}}$$

Continued Fractions

$$\frac{1}{4 + \frac{2}{3}}$$

Continued Fractions

$$\frac{1}{\boxed{4 + \frac{2}{3}}}$$

Continued Fractions

$$\frac{1}{\frac{14}{3}}$$

Continued Fractions

$$\cfrac{1}{\cfrac{14}{3}}$$

Continued Fractions

$$\frac{3}{14}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\boxed{4 + \cfrac{1}{2}}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{9}{2}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \boxed{\cfrac{1}{\cfrac{9}{2}}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{2}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\boxed{1 + \cfrac{2}{9}}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\cfrac{11}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{1}{\cfrac{11}{9}}}$$

Continued Fractions

$$3 + \cfrac{1}{3 + \cfrac{9}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{\boxed{3 + \cfrac{9}{11}}}$$

Continued Fractions

$$3 + \cfrac{1}{\cfrac{42}{11}}$$

Continued Fractions

$$3 + \cfrac{1}{\cfrac{42}{11}}$$

Continued Fractions

$$3 + \frac{11}{42}$$

Continued Fractions

$$3 + \frac{11}{42}$$

Continued Fractions

$$\frac{137}{42}$$

Continued Fractions

- A ***continued fraction*** is an expression of the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\dots + \cfrac{1}{a_n}}}}$$

- Formally, a continued fraction is either
  - An integer *x*, or
  - *x* + 1 / *F*, where *x* is an integer and *F* is a continued fraction.
- Continued fractions have numerous applications in number theory and computer science.
- (They're also really fun to write!)

Fun with Continued Fractions

- Every rational number has at least one continued fraction representation.
- Every *irrational* number has an infinite continued fraction representation.
- **Interesting fact:** If we truncate an infinite continued fraction for an irrational number, we can get progressively better approximations of that number.

π as a Continued Fraction

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{\cdots}}}}}}}}}}$$

Approximating π

Approximating π

$\pi = 3$

3 = **3**.0000...

Approximating π

$\pi = 3$

$3 = \mathbf{3}.0000\ldots$

And he made the Sea of cast bronze, ten cubits from one brim to the other; it was completely round. [… A] line of thirty cubits measured its circumference.

1 Kings 7:23, New King James Translation

Approximating π

$$\pi = 3 + \frac{1}{7}$$

3 = **3**.0000...

22/7 = **3.14**2857...

Approximating π

$$\pi = 3 + \frac{1}{7}$$

$3 = \mathbf{3}.0000...$

$22/7 = \mathbf{3.14}2857...$

Greek mathematician **Archimedes** knew of this approximation, circa 250 BCE

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

355/113 = **3.141592**92...

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1}}}$$

3 = **3**.0000...

22/7 = **3.14**2857...

336/106 = **3.1415**094...

355/113 = **3.141592**92...

Chinese mathematician 祖沖之 (Zu Chongzhi) discovered this approximation in the early fifth century; this was the best approximation of π for around one thousand years.

Approximating π

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292}}}}$$

3 = **3**.0000...

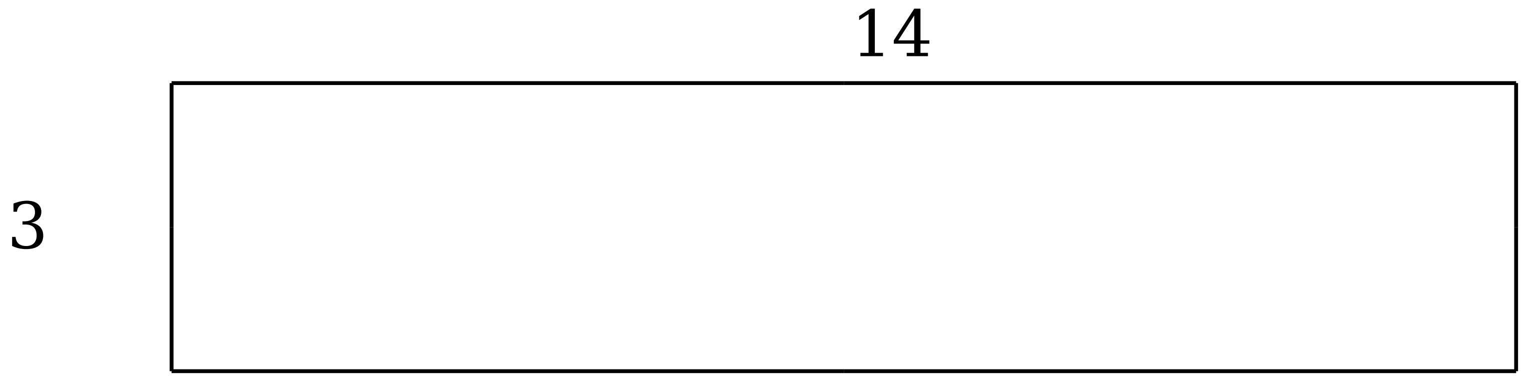
22/7 = **3.14**2857...

336/106 = **3.1415**094...

355/113 = **3.141592**92...

103993/33102 = **3.141592653**0...

More Continued Fractions



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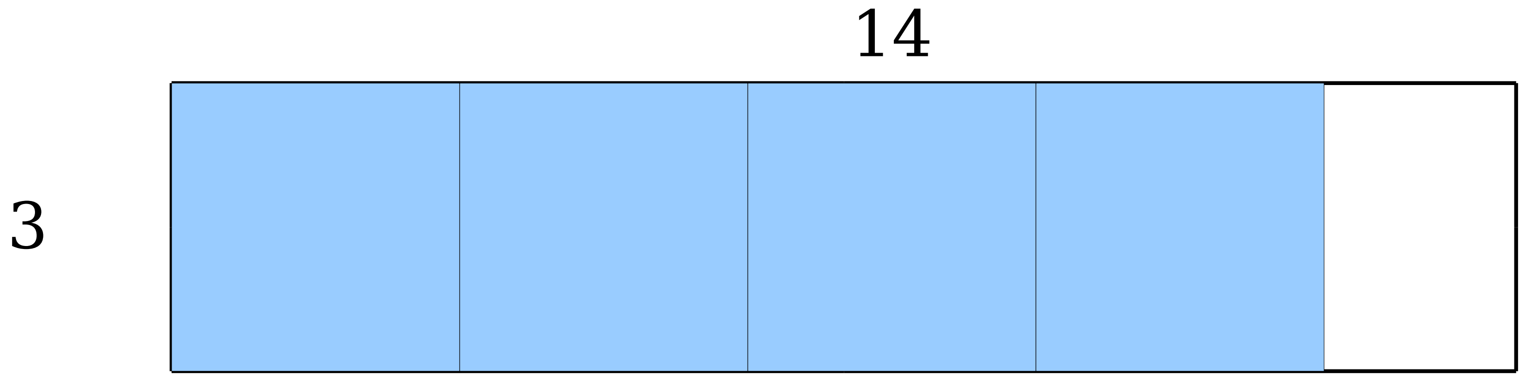

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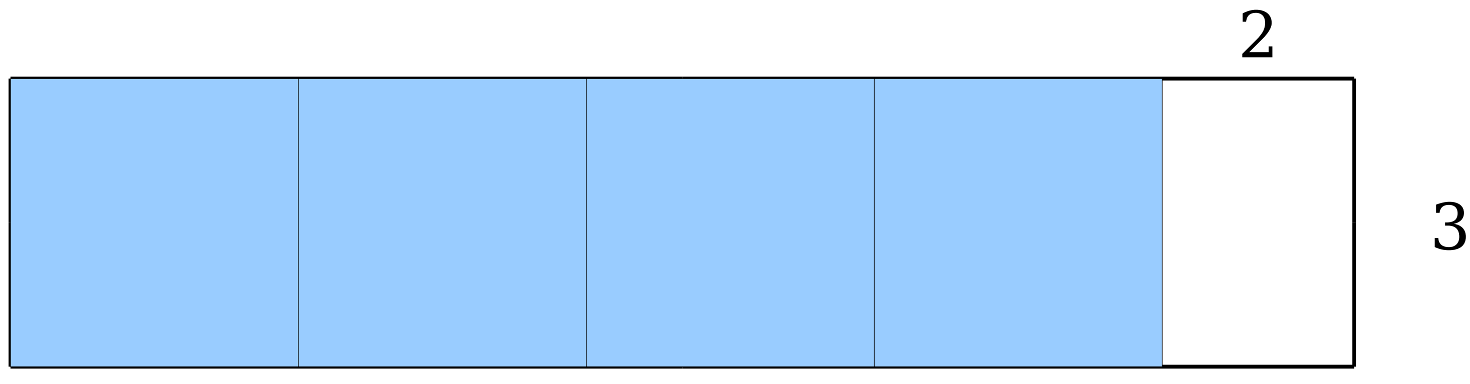

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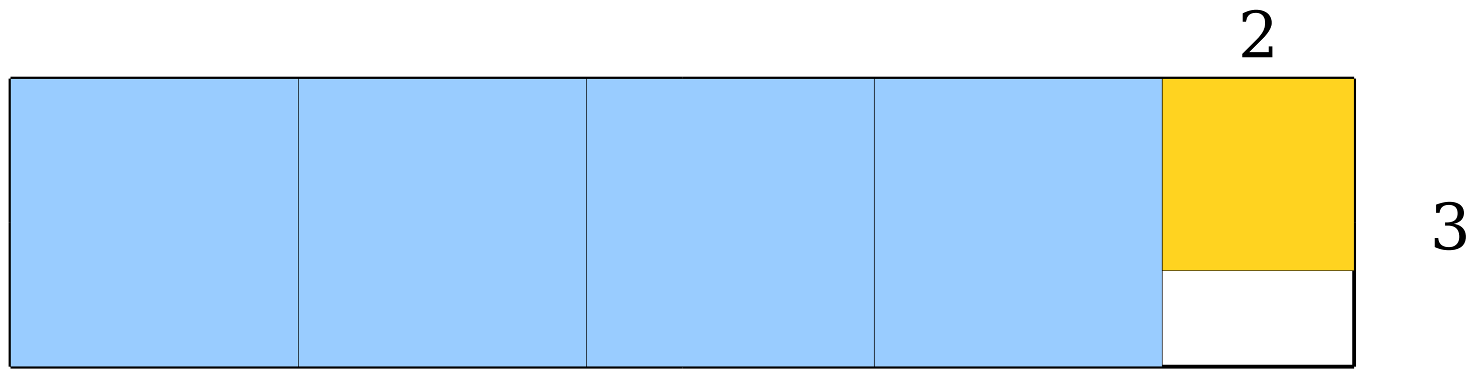

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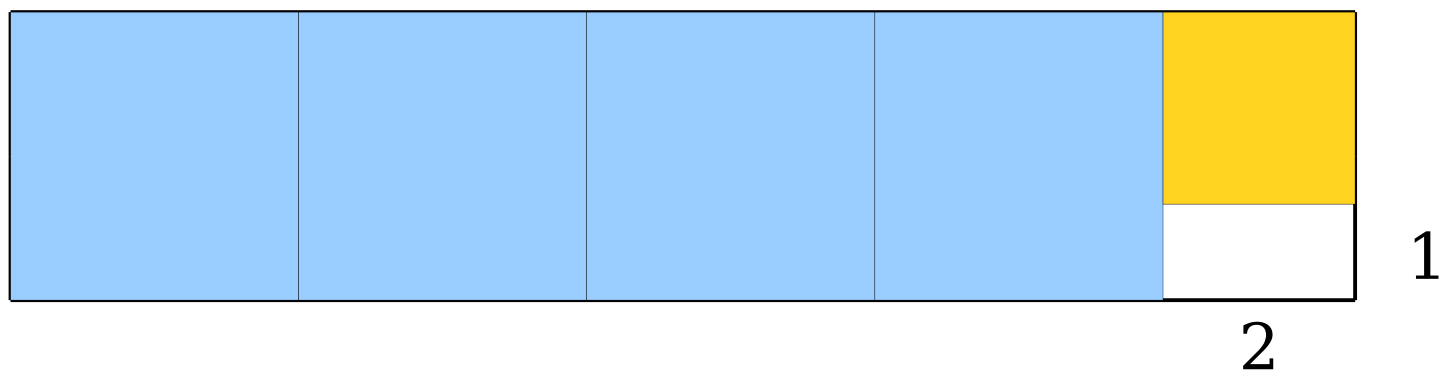

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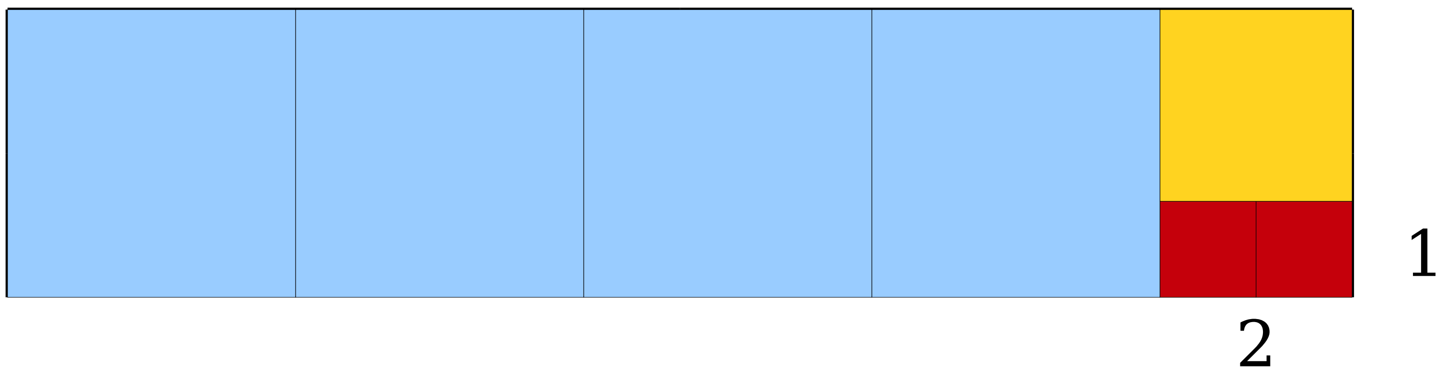

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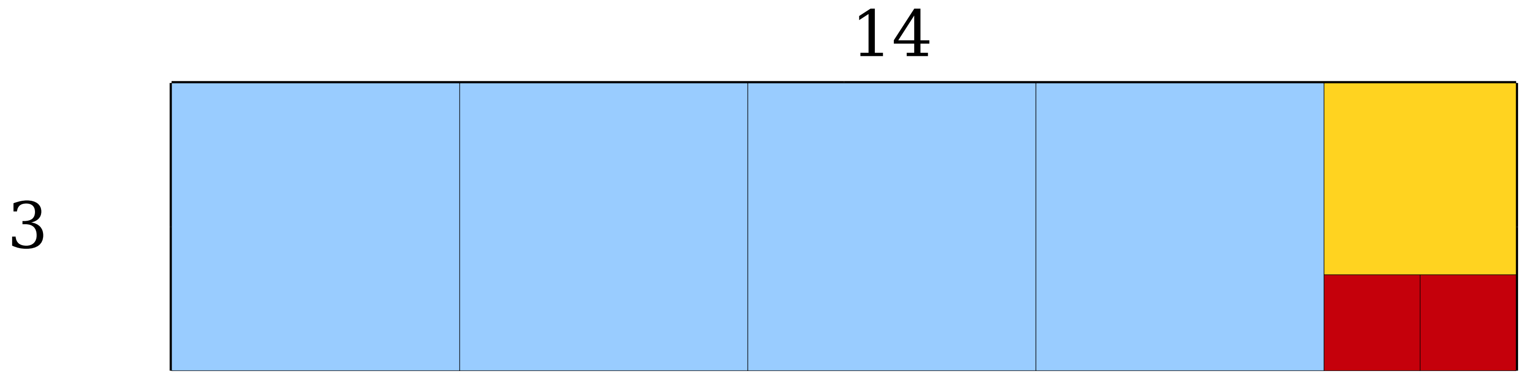

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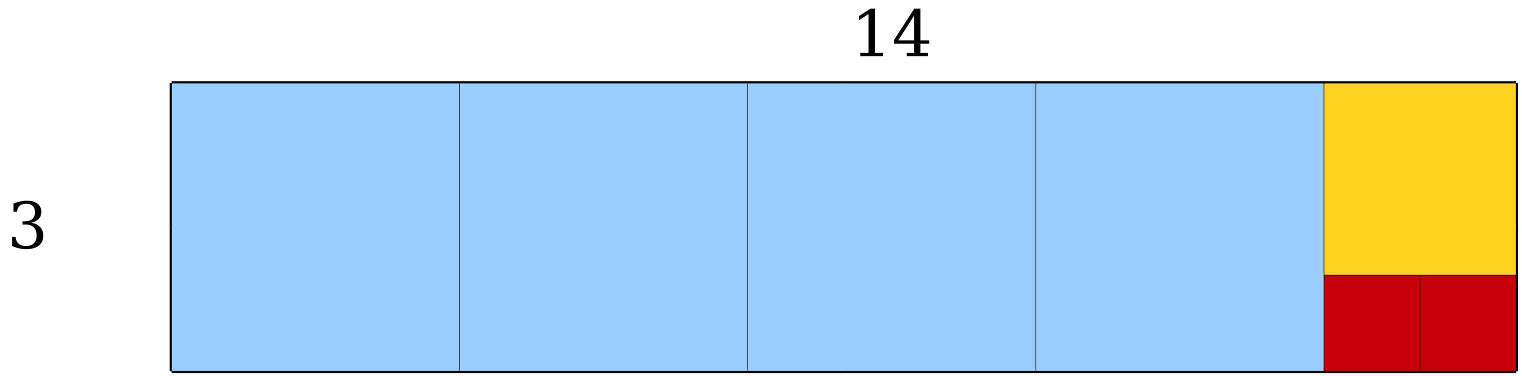
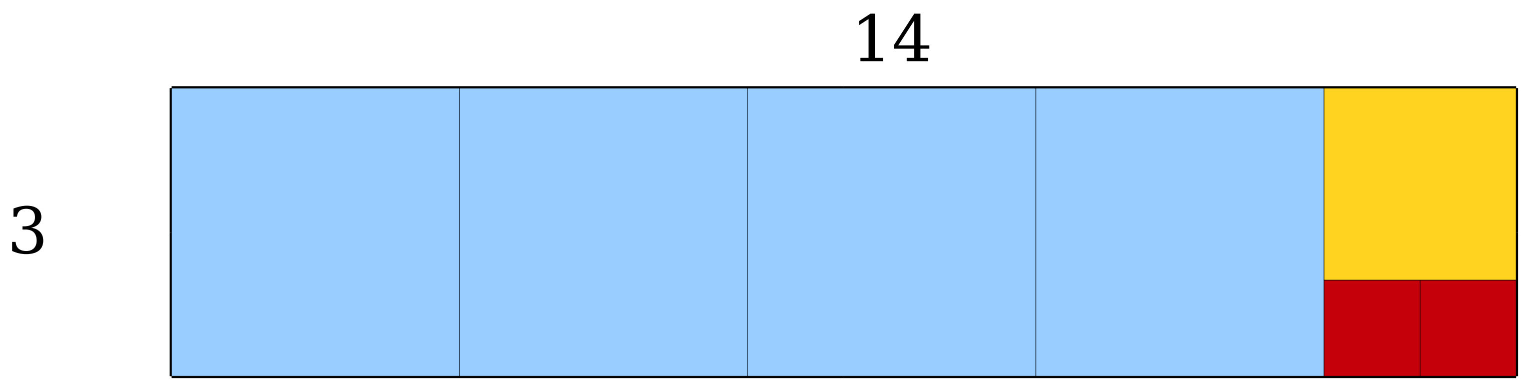


$$\frac{3}{14} = \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}$$

More Continued Fractions



$$\frac{3}{14} = \cfrac{1}{\mathbf{4} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{2}}}}$$

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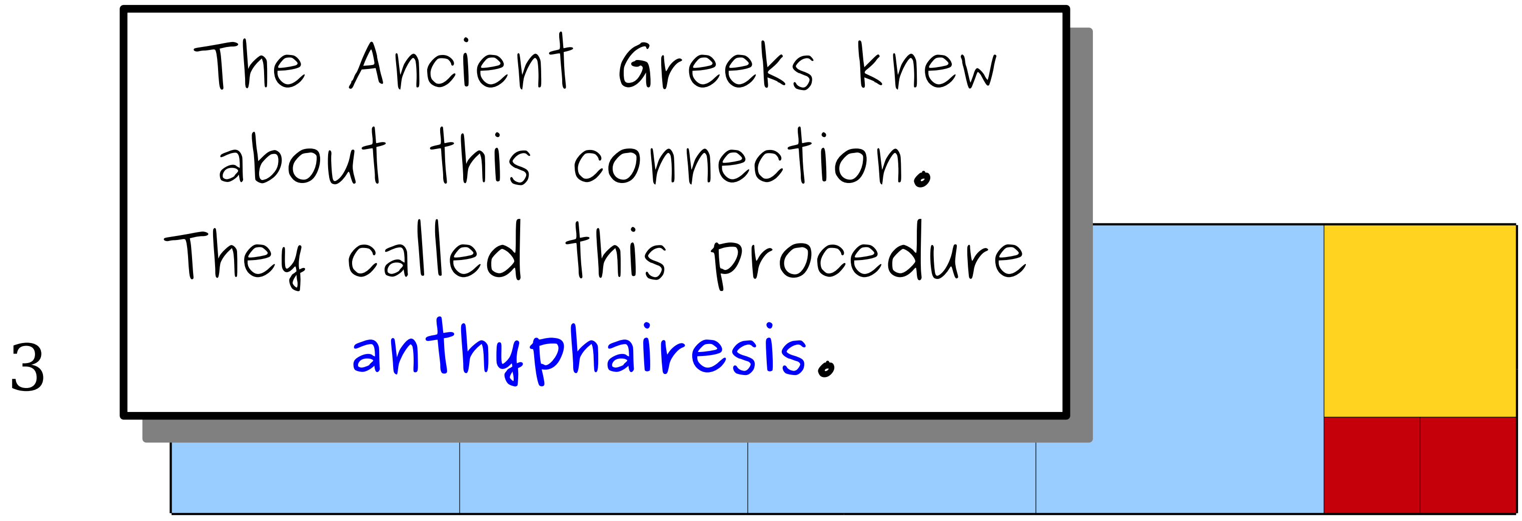


$$\frac{3}{14} = \cfrac{1}{\mathbf{4} + \cfrac{1}{\mathbf{1} + \cfrac{1}{\mathbf{2}}}}$$

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}}}}}}}$$

An Interesting Continued Fraction

$x = 1$

1 / 1

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 \cdot }$$

1 / 1

2 / 1

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1}}$$

1 / 1

2 / 1

3 / 2

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 \cdots}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

13 / 8

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

13 / 8

21 / 13

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}}$$

1 / 1

2 / 1

3 / 2

5 / 3

8 / 5

13 / 8

21 / 13

34 / 21

An Interesting Continued Fraction

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}}}}}}$$

1 / 1
2 / 1
3 / 2
5 / 3
8 / 5
13 / 8
21 / 13
34 / 21

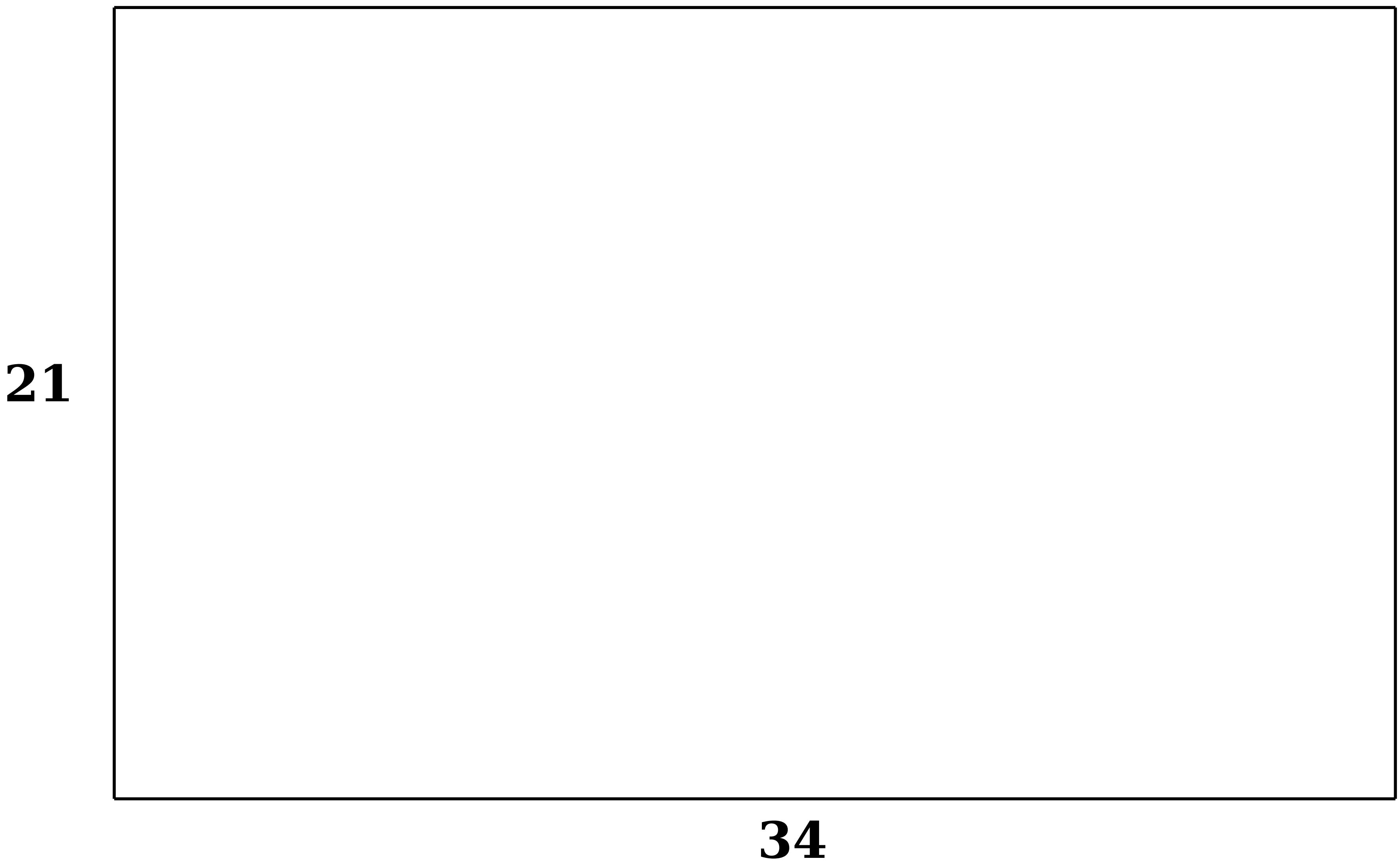
Each fraction is the ratio of consecutive Fibonacci numbers!

The Golden Ratio

$$\varphi = \frac{1+\sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{\cdots}}}}$$

$$\varphi \approx 1.61803399$$

The Golden Ratio



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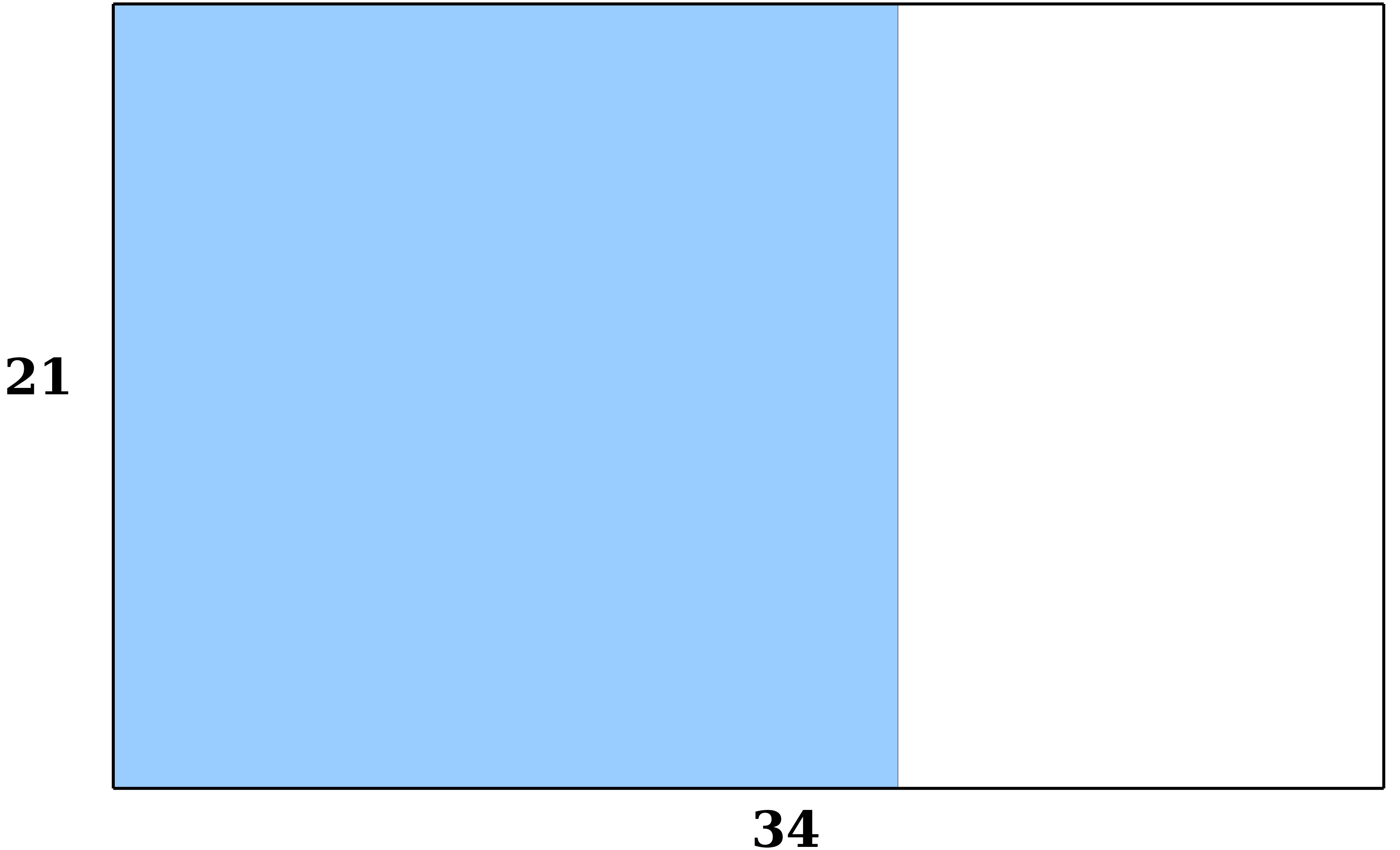

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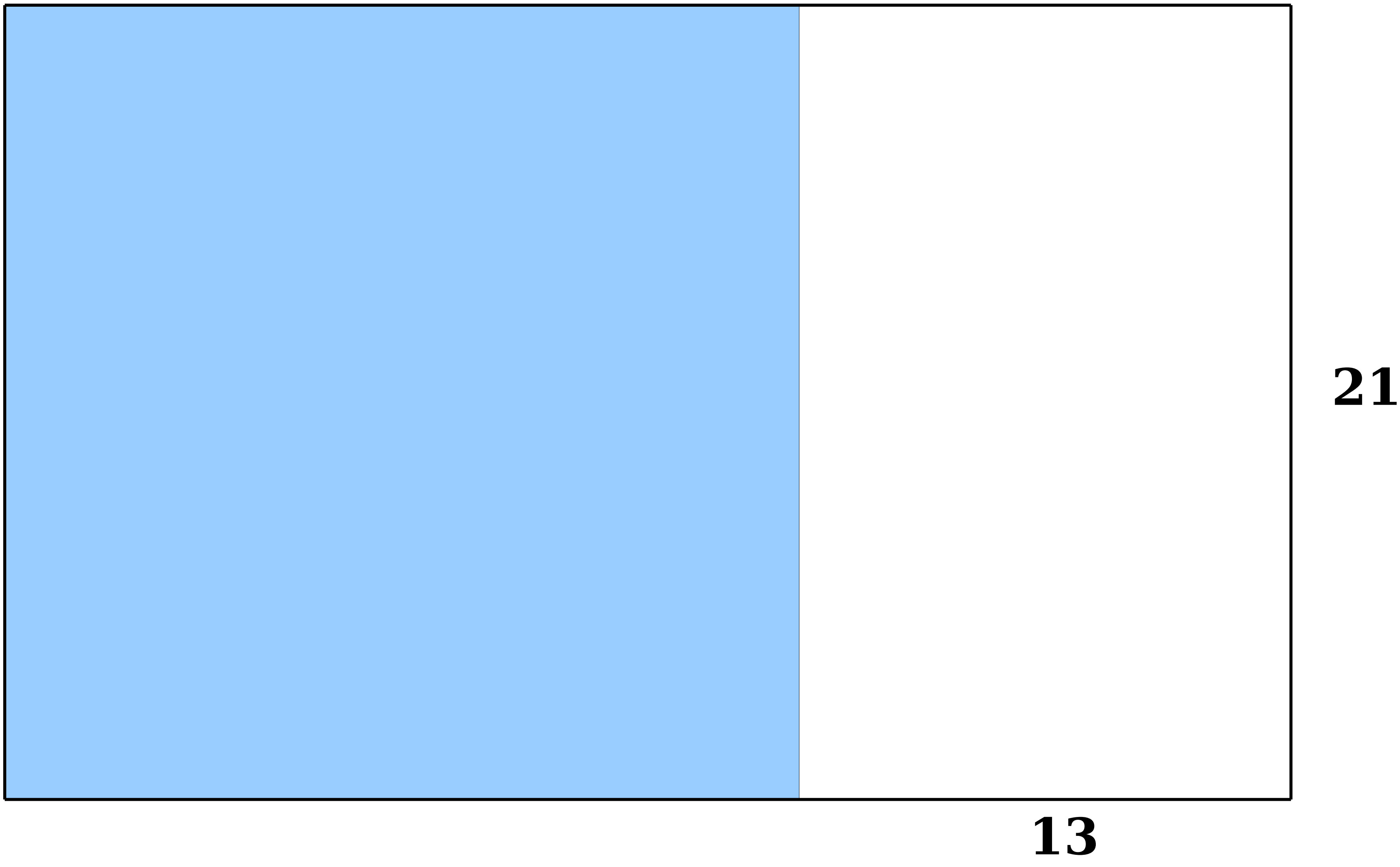

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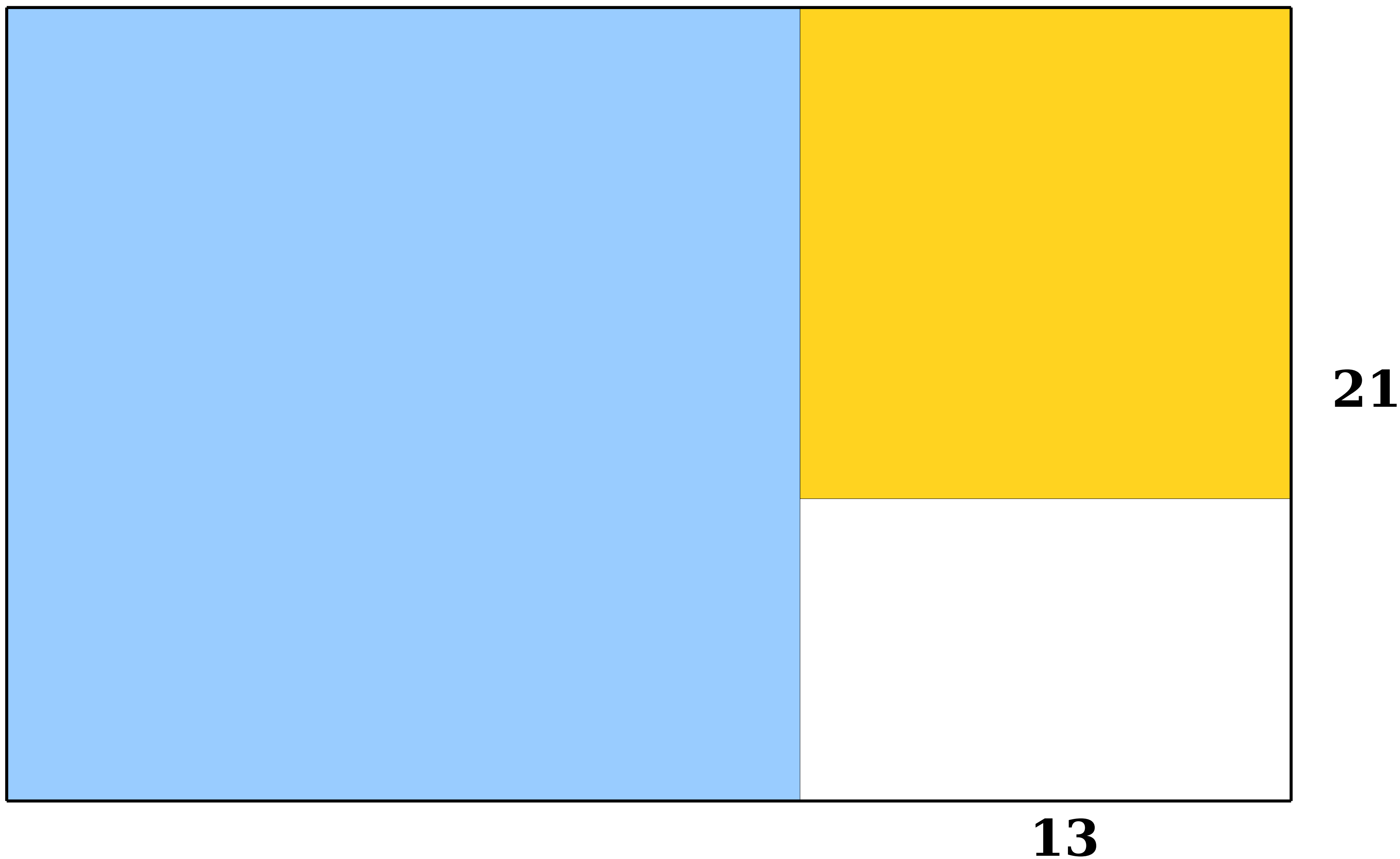

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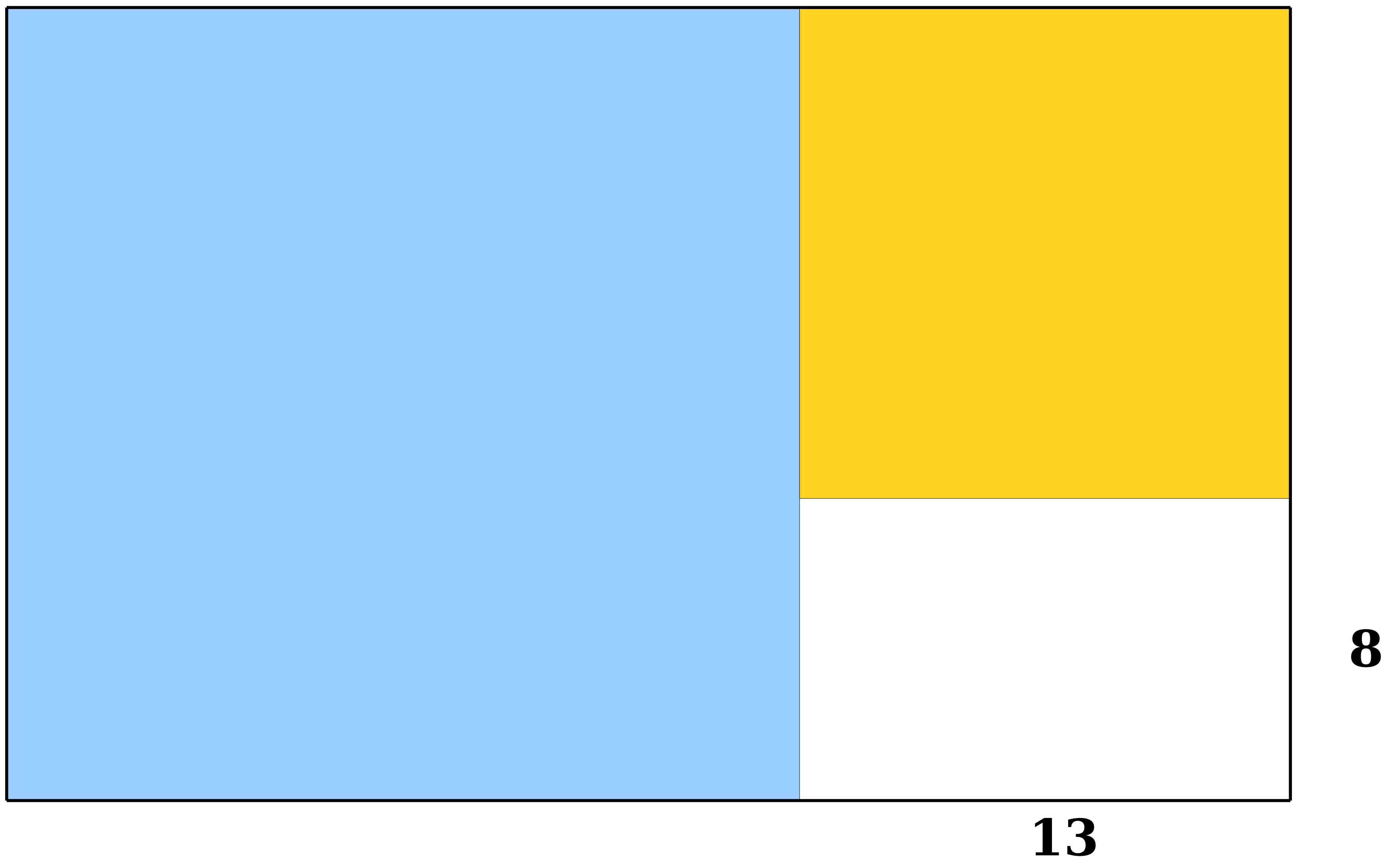

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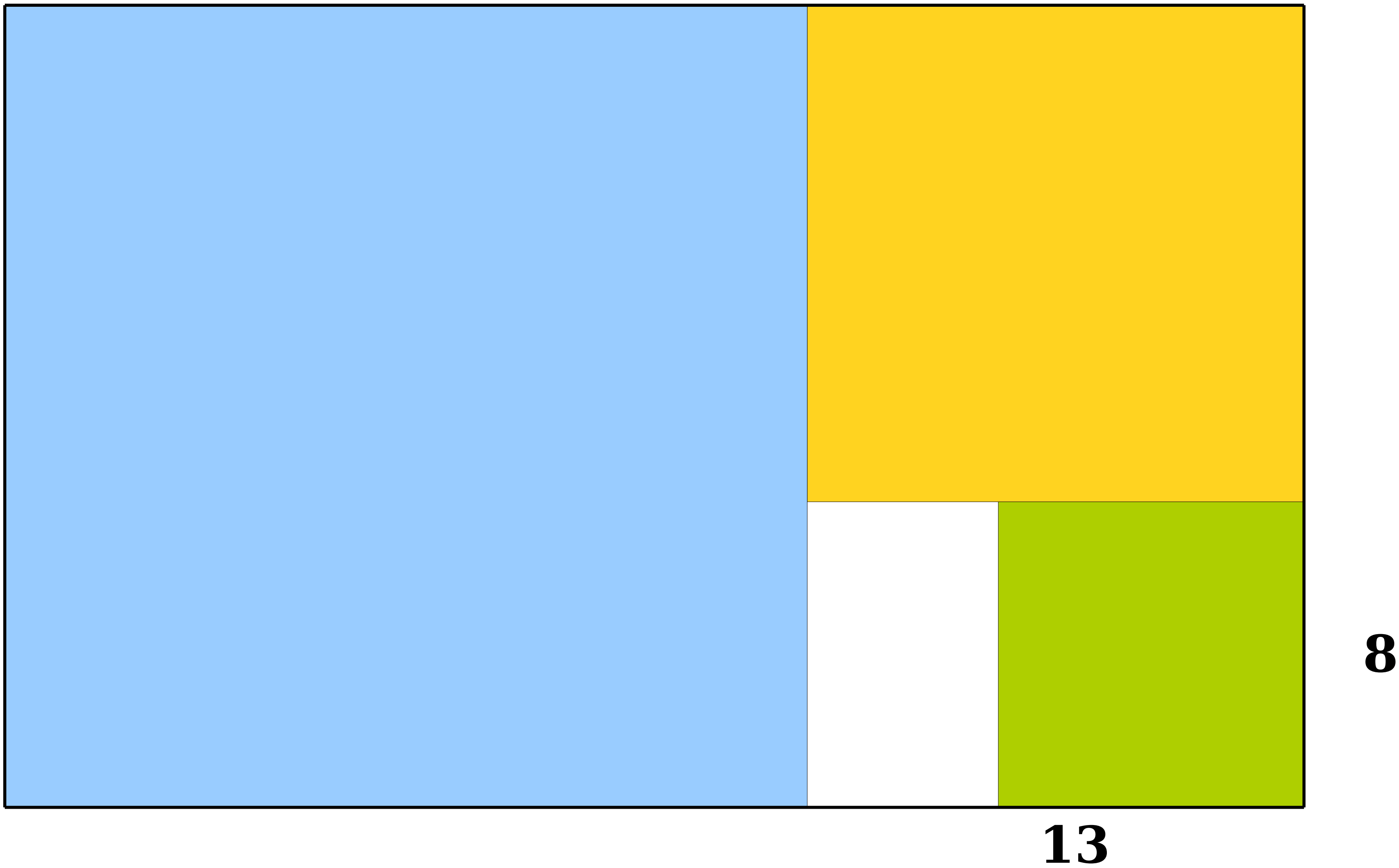

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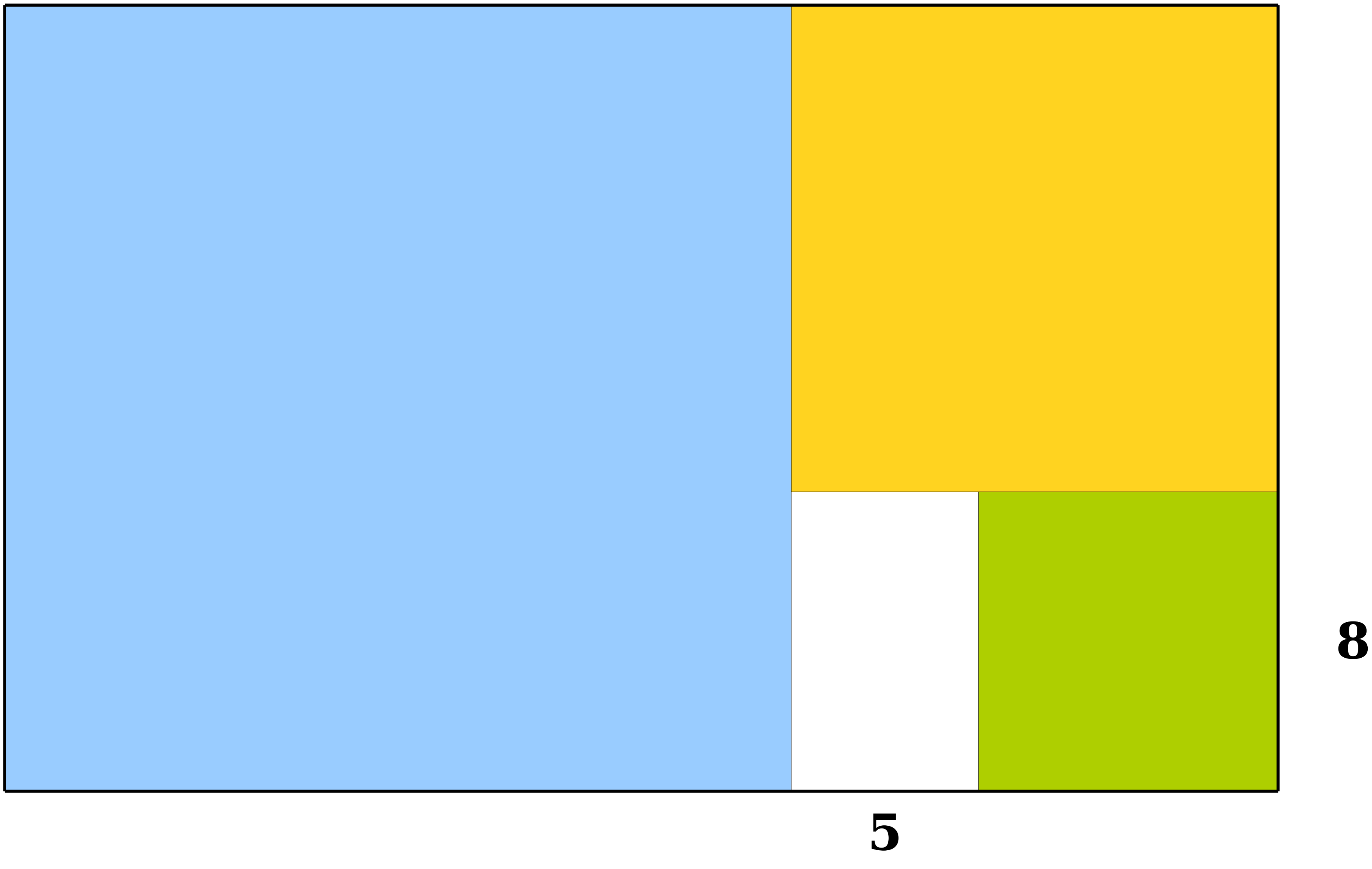

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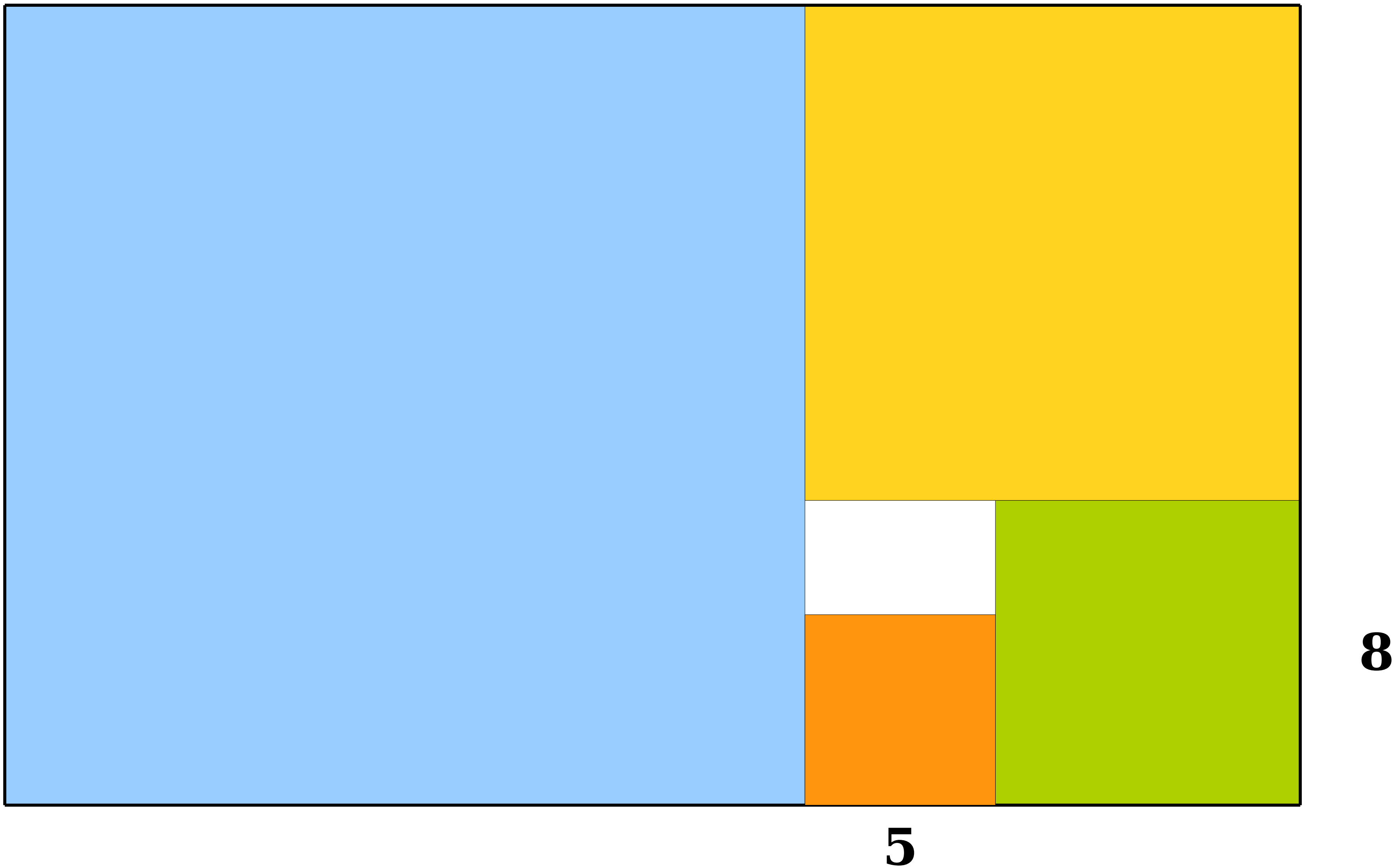

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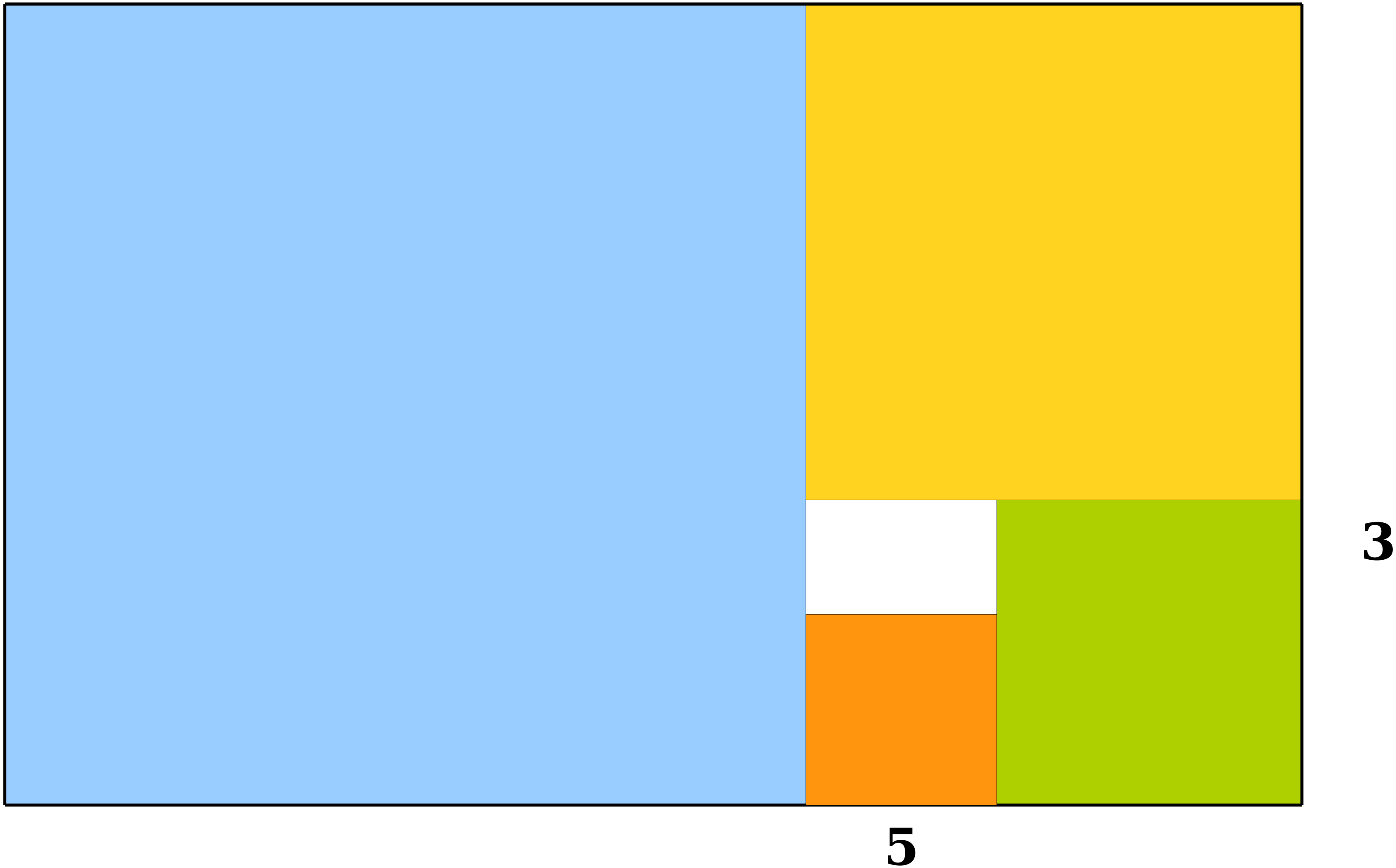

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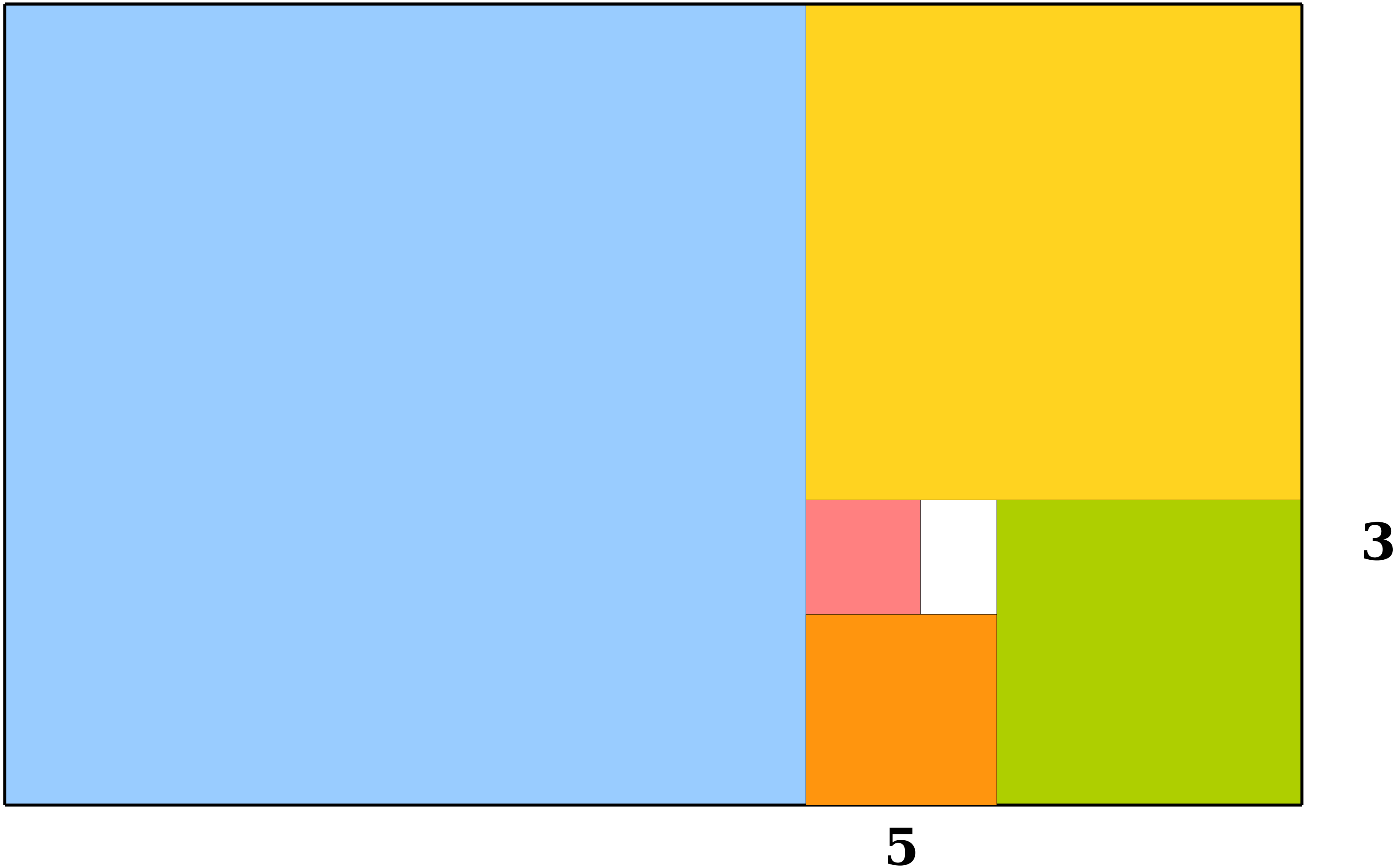

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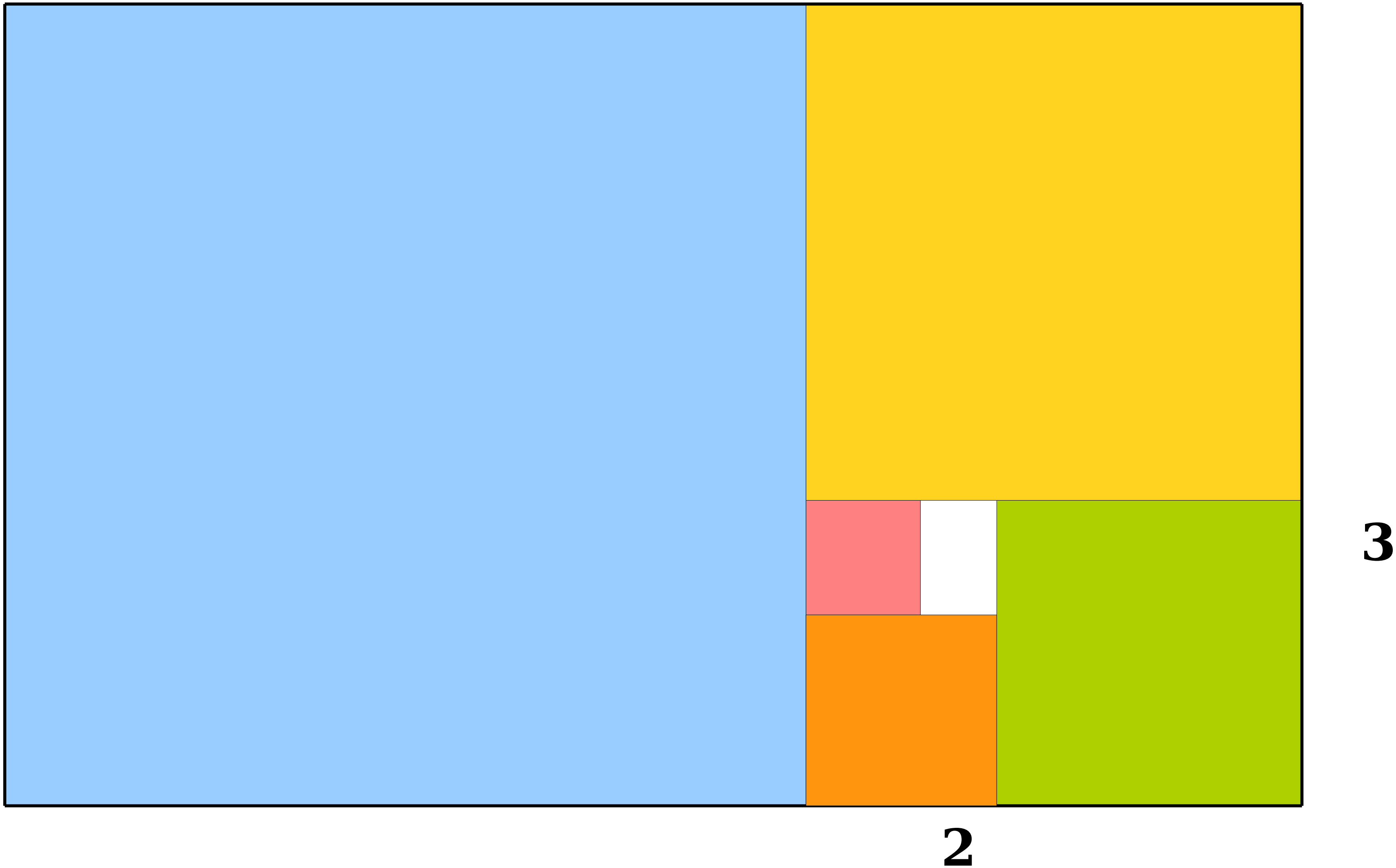

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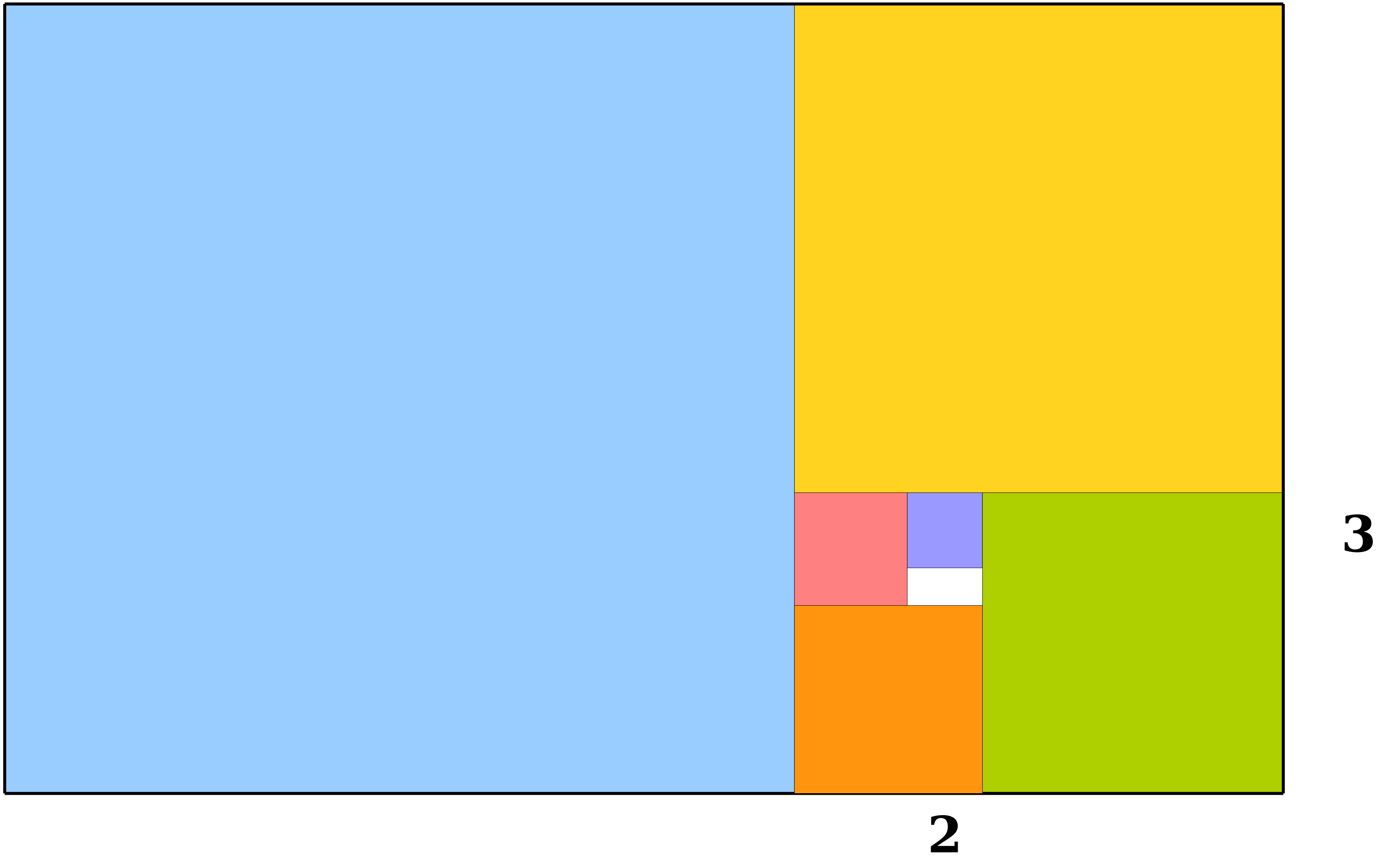

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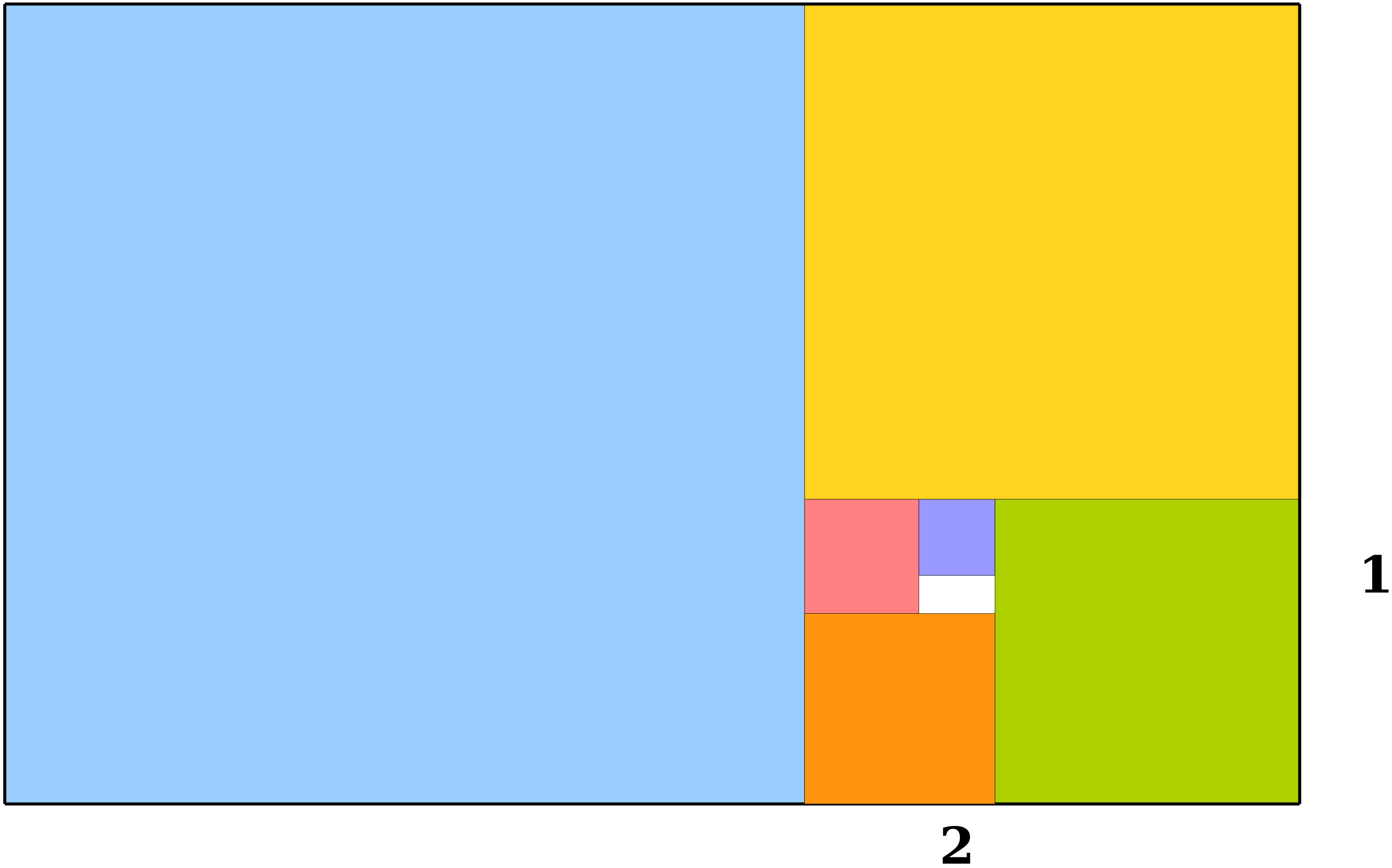

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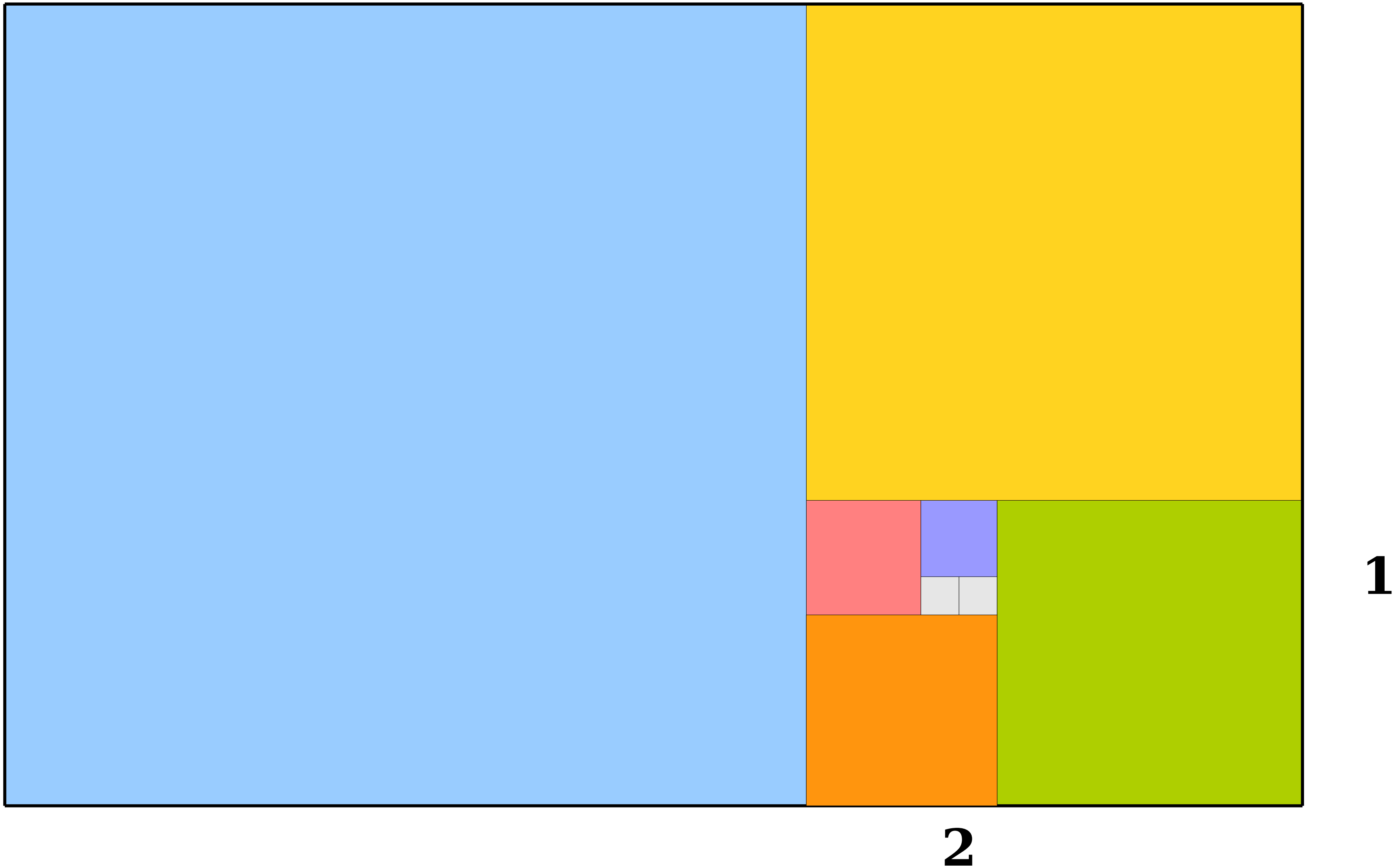
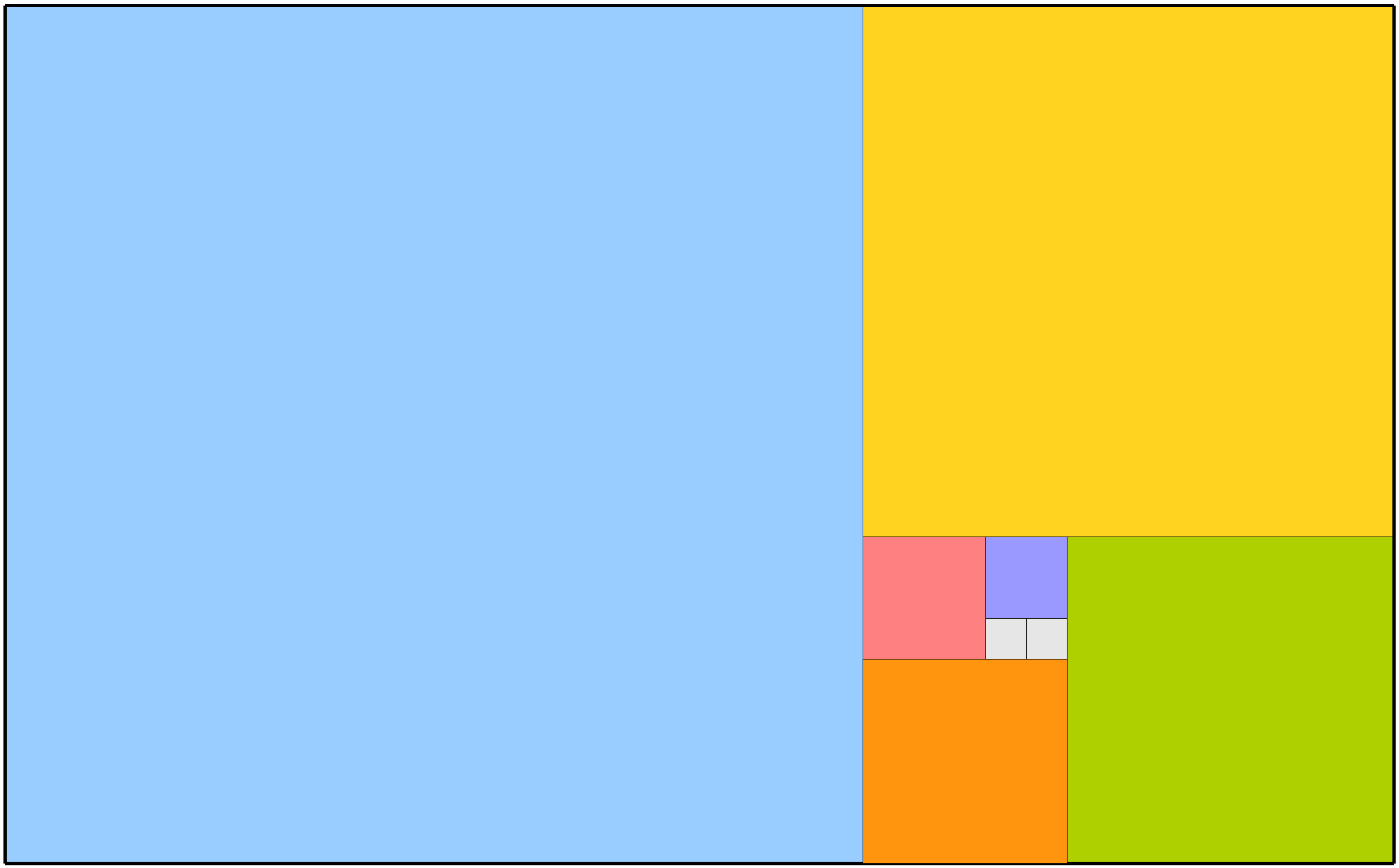
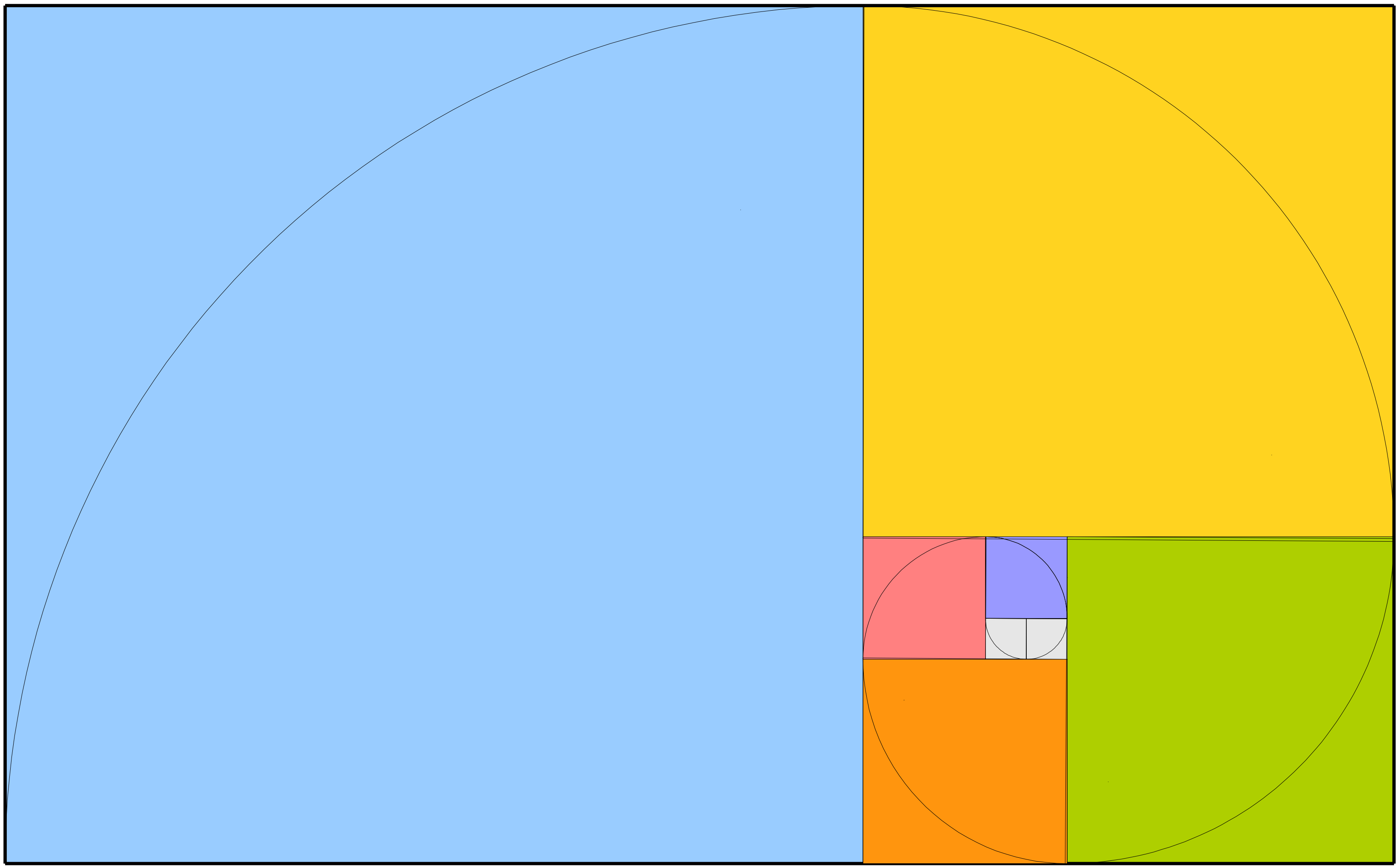


The Golden Ratio



The Golden Spiral



Question: How do we prove all rational numbers have continued fractions?

Where We're Going

- First, we're going to devise an *algorithm* for constructing a continued fraction from a rational number.
- Next, we're going to look at that algorithm to try to see why it works.
- Finally, we're going to prove that all rational numbers have continued fractions.
  - The proof will essentially describe the algorithm and use the justification we found in the second step.
- This approach is *very useful* for proving results inductively. We highly recommend it on the problem set!

Constructing a Continued Fraction

$$\frac{107}{103}$$

Constructing a Continued Fraction

$$\frac{107}{103} = 1 + \frac{4}{103}$$

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$$\frac{4}{3}$$

$$\frac{3}{1}$$

Constructing a Continued Fraction

- Suppose we have rational number a / b.
- If a / b is an integer, it's its own continued fraction.
- Otherwise, compute the quotient q and remainder r of a / b and write

$$a / b = q + r / b$$

- Equivalently:

$$a / b = q + 1 / (b / r)$$

- Construct a continued fraction F for b / r.
- The overall continued fraction is then

$$\boldsymbol{a / b = q + 1 / F}$$

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Equivalently:

How do we know that
this is possible?

- Construct a continued fraction F for b / r.

The overall continued fraction is then

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$$\frac{\mathbf{3}}{\mathbf{1}}$$

107 > 103 > 4 > 3

103 > 4 > 3 > 1

Observation: In this case, each rational number has a smaller numerator and denominator than the previous one.

This helps explain why we eventually “bottom out” - these numbers can't decrease forever.

Question: Is this just a coincidence?

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$$\frac{7}{9} = 0 + \cfrac{1}{\cfrac{9}{7}}$$

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$$\mathbf{9 > 7 > 2 > 1}$$

Observation: In this case, each rational number has a smaller ~~numerator and~~ denominator than the previous one.

This helps explain why we eventually "bottom out" - these numbers can't decrease forever.

Question: Is this just a coincidence?

$$\frac{a}{b}$$

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r is the remainder of a divided by b.
It's guaranteed to be smaller than b.

Fact: Using our algorithm, each continued fraction has a smaller denominator than the previous one.

This helps explain why we eventually “bottom out” - these numbers can't decrease forever.

Question: How do we turn this into a proof?

A Helpful Intuition

- If you see something of the form

  “keep repeating *X* until…”

  try proving it by induction.
- Use the inductive hypothesis to “assume away” future steps.
- Example: Counterfeit coins.
  - Process: “Keep splitting the coins into thirds and throwing away coins until only one's left.”
  - Proof: “Assume that it works for 3^k coins and prove that it works for 3^{k+1} coins.”

From Intuition to Proof

- In our case, the intuition is

"Keep constructing continued fractions until the denominator becomes 1."

- We'll prove this by using the following inductive hypothesis:

"We can construct a continued fraction for any rational number with denominator *k* or less."

- In our inductive step, we'll show that we can build continued fractions for rational numbers with denominator $k+1$ by using a continued fraction for a rational number with a smaller denominator.

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Take any rational number with denominator $k+1$; let it be a / ($k+1$). Compute the quotient q and remainder r when a is divided by $k+1$.

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Since $1 \leq r \leq k$, by our inductive hypothesis we know that (k+1) / r has a continued fraction representation; call it F.

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In both cases, we see a / (k+1) has a continued fraction representation. Therefore, P(k+1) holds, completing the induction.

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In both cases, we see $a / (k+1)$ has a continued fraction representation. Therefore, $P(k+1)$ holds, completing the induction. ■

When Use Strong Induction?

- Normal (*weak*) induction is good for when you are shrinking the problem size by exactly one.
  - Peeling one final term off a sum.
  - Making one weighing on a scale.
  - Considering one more action on a string.
- Strong induction is good when you are shrinking the problem, but you can't be sure by how much.
  - Splitting a tournament into two smaller tournaments.
  - Taking the remainder of one number divided by another.

For more on continued fractions:

http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/cfINTRO.html

Time-Out for Announcements!

Problem Set Two

- Problem Set One was due at the start of class.
  - Due on Monday if you use your one late period.
- Problem Set Two goes out now.
  - Checkpoint problem due at the start of Monday's lecture.
  - Remaining problems due at the start of next Friday's lecture.
  - Play around with induction in all its forms!

Group Submissions

- The Scoryst folks have been working hard and in upcoming assignments, you should be able to explicitly submit as a group.
- Have one group member submit the assignment. They can then list teammates when submitting.
- Then, everyone in the group can look at the feedback.

Casual CS Dinner

- WiCS will be hosting their first biquarterly Casual CS Dinner on **Wednesday, October 15** from 6PM – 8PM on the fifth floor of the Gates building.
- This is a really fantastic event and everyone is welcome.
- Interested in attending? RSVP through the link sent out earlier today.

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Fuse!box

- Fuse!box is a mobile hackerspace for teaching engineering to local, low-income high-school students.
- Run through Stanford ESW (Engineers for a Sustainable World).
- Interested in joining the team? Contact Courtney Noh at cnoh@stanford.edu and check out the application at http://bit.ly/FuseboxApp.

Your Questions

“If we get stuck on a problem, should we go to office hours or ask about a specific approach on Piazza or email a TA?”

“If, hypothetically speaking, you were in a garage band, which instrument would you play?”

“What is your favorite programming language and why is it Python?”

“How do you make sure you don't overlook anything or forget any of the principles you've been taught?”

“What is the meaning of life,
the universe, and everything?”

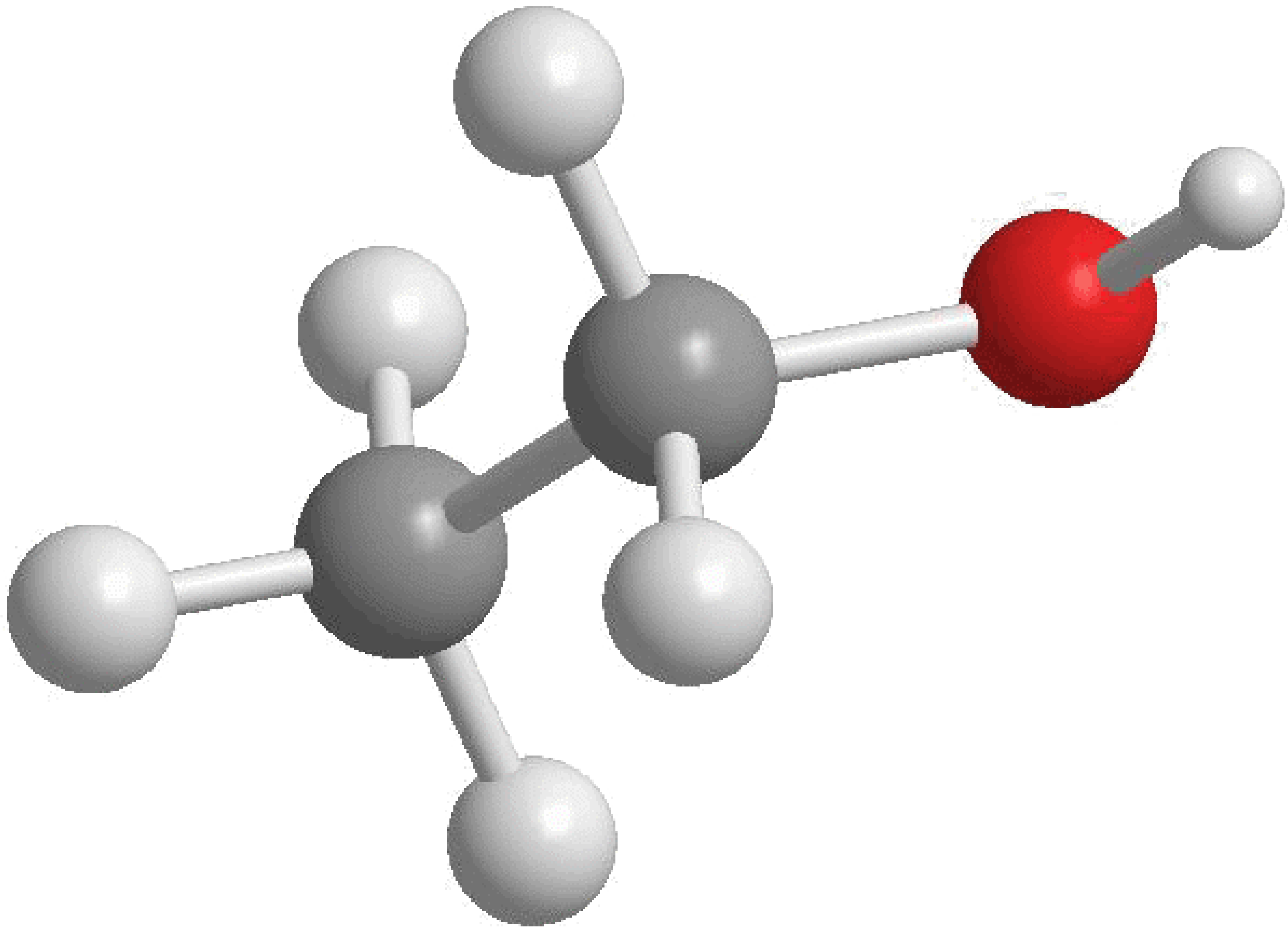
Back to CS103!

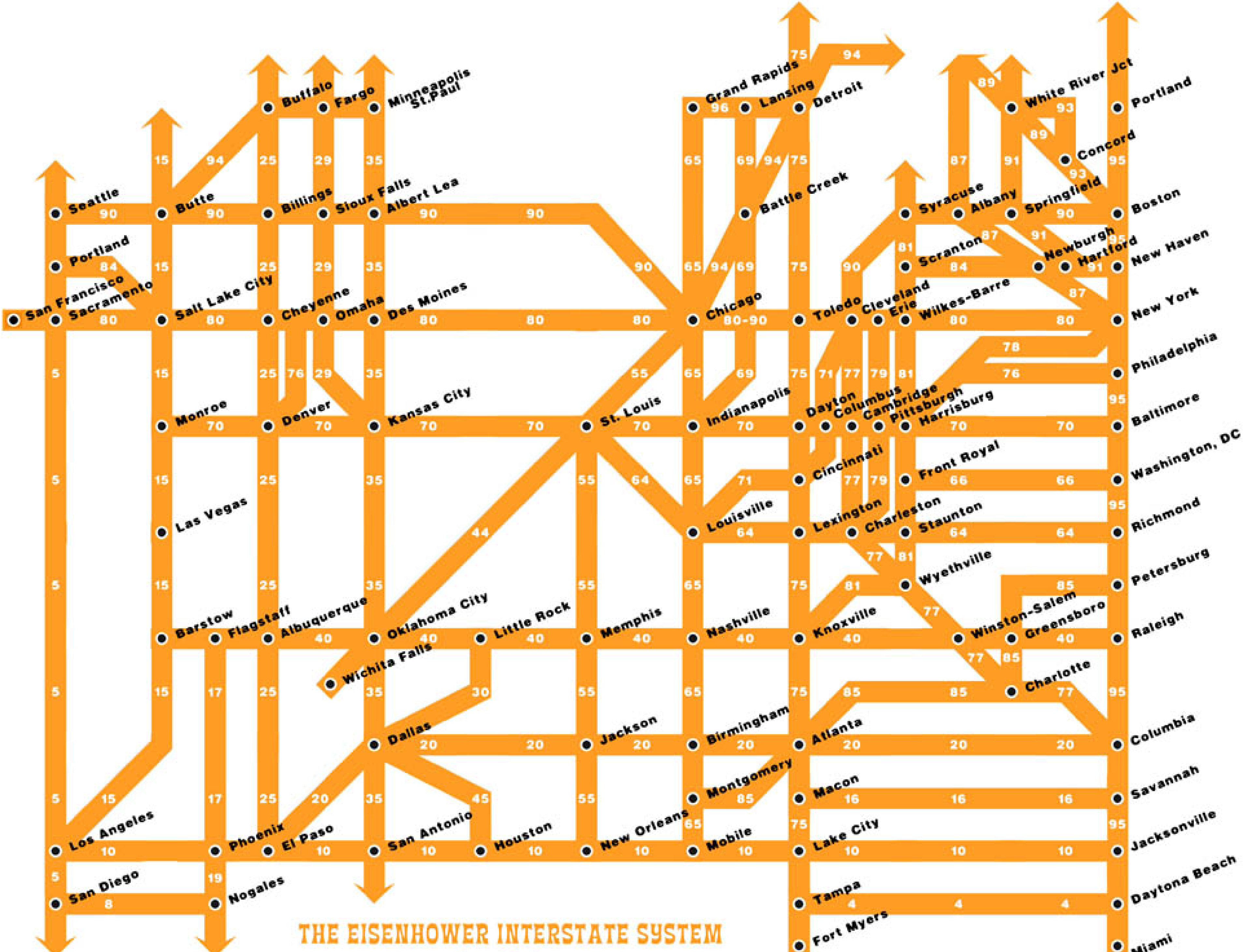
Graphs

Mathematical Structures

- Just as there are common data structures in programming, there are common mathematical structures in discrete math.
- So far, we've seen simple structures like sets and natural numbers, but there are many other important structures out there.
- Over the next few weeks, we'll explore several of them.

Chemical Bonds





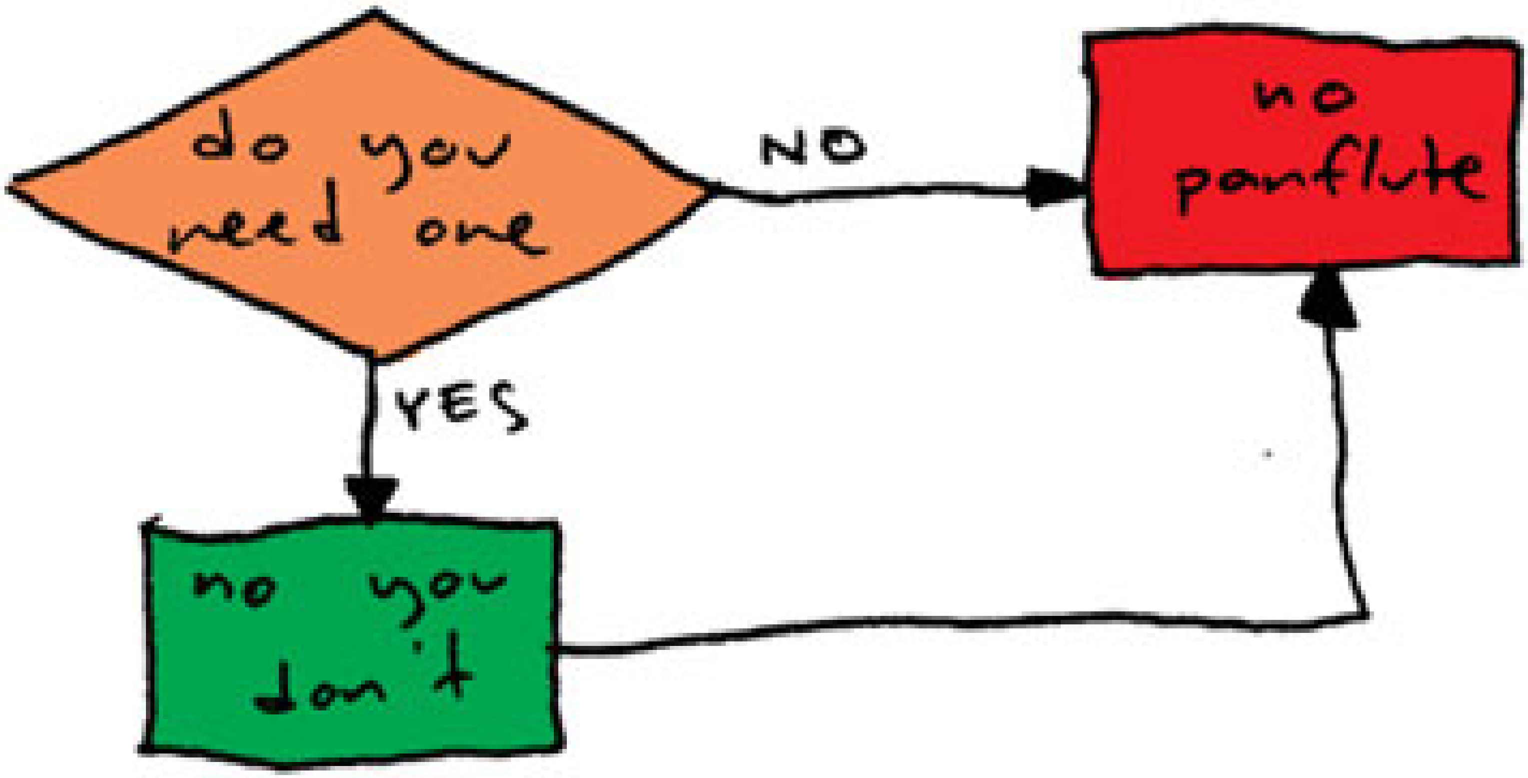
THE EISENHOWER INTERSTATE SYSTEM

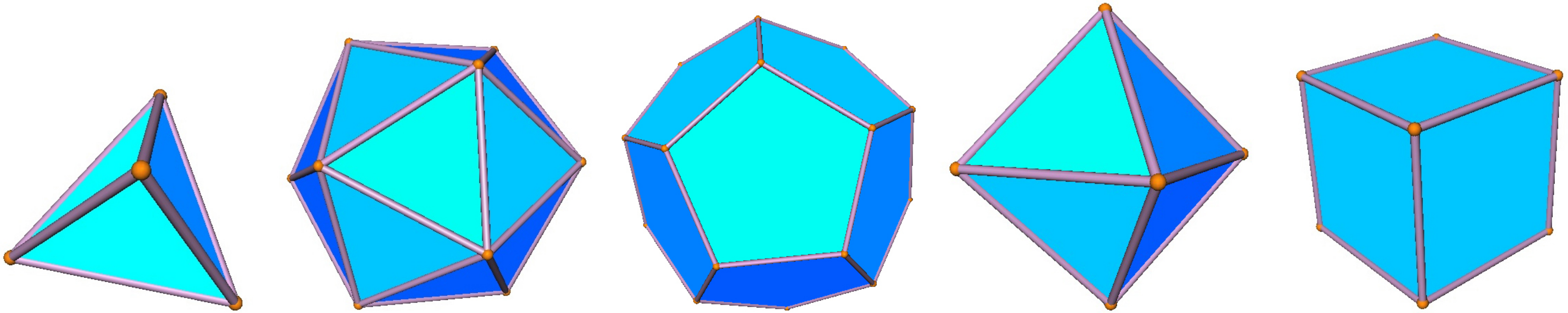
(simplified)

CHRIS YATES 2007

http://strangemaps.files.wordpress.com/2007/02/fullinterstatemap-web.jpg

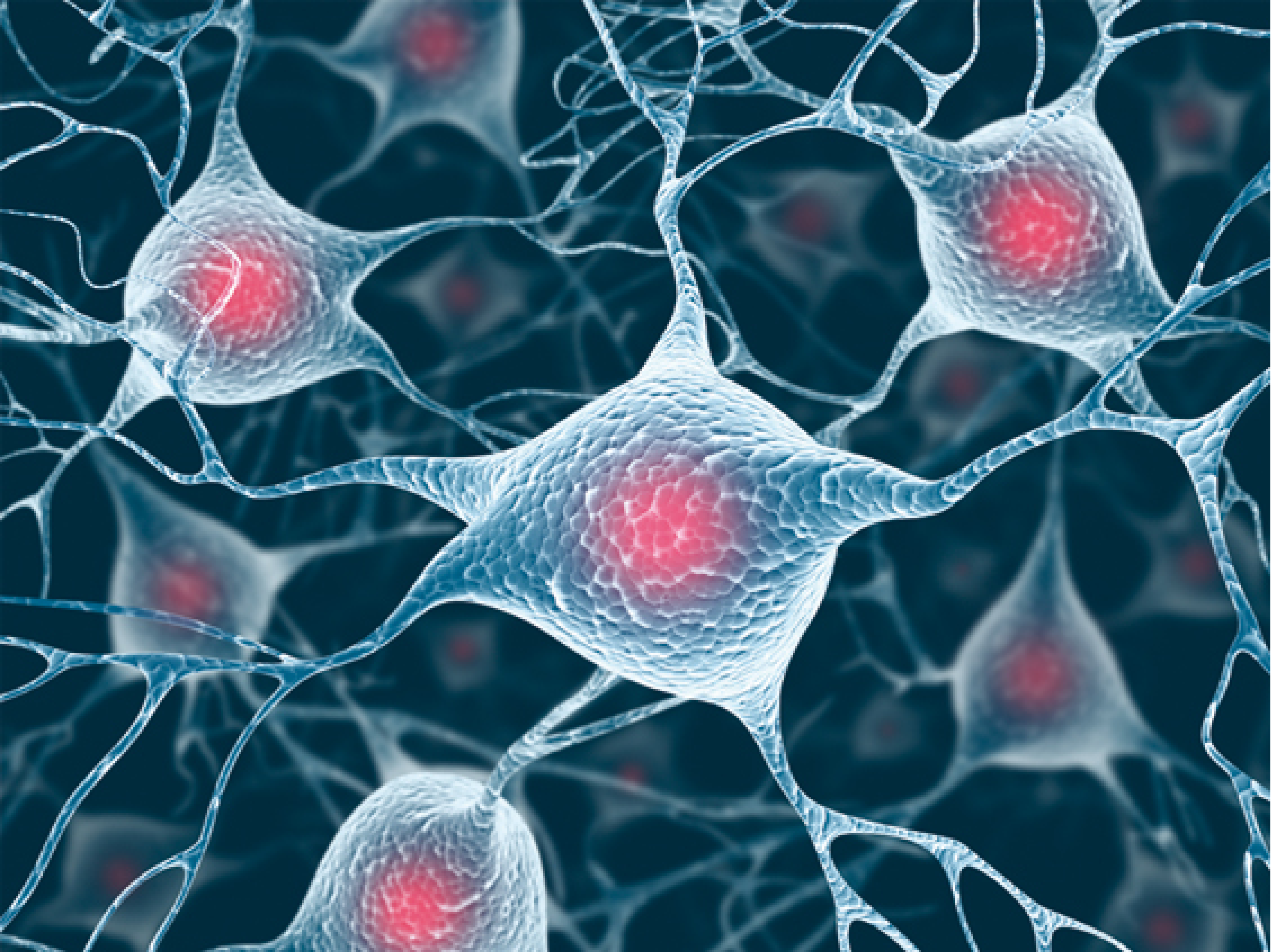
PANFLUTE FLOWCHART





Tetrahedron Icosahedron Dodecahedron Octahedron Cube

http://www.princeton.edu/pr/pictures/l-r/packingproblem/pu-platonic-solids.jpg



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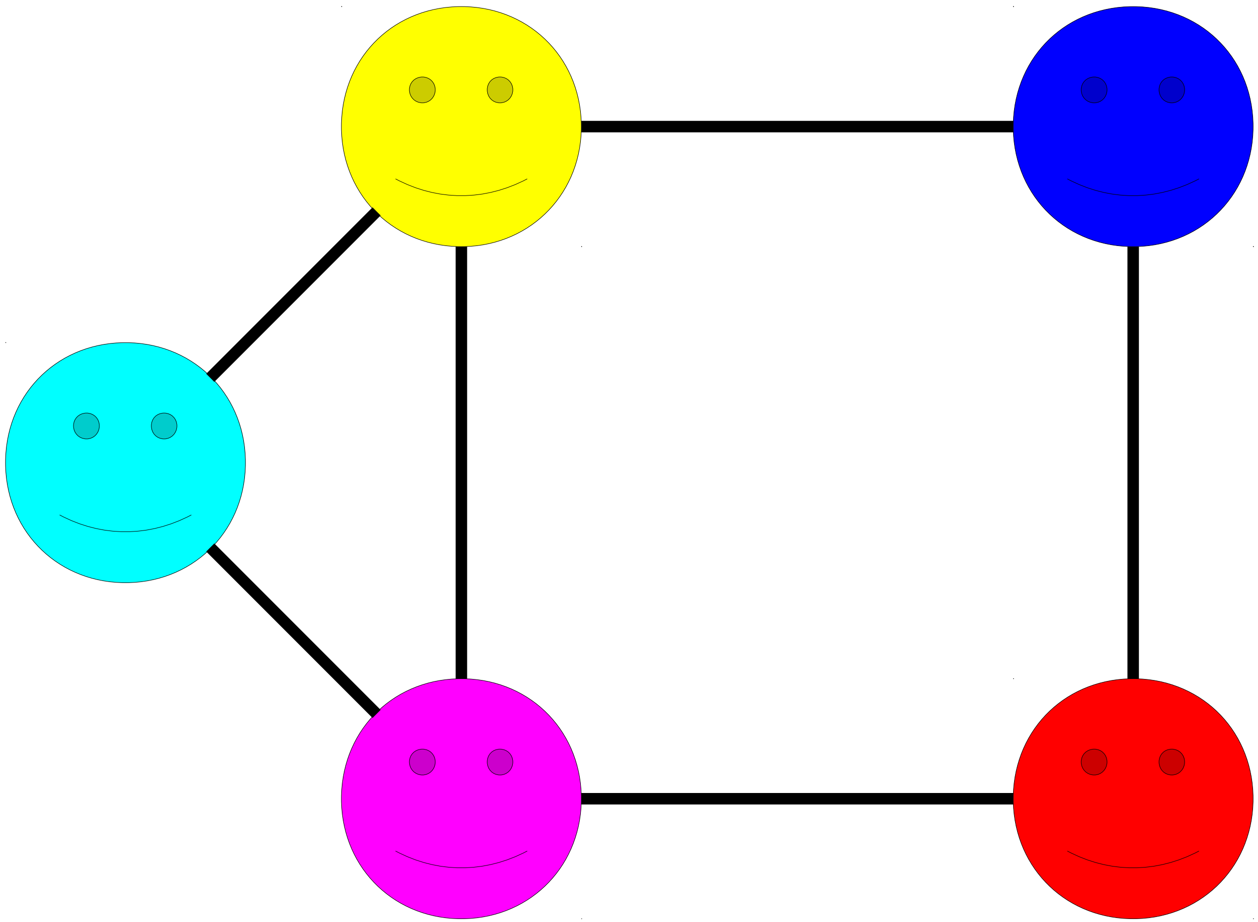
Me too!

What's in Common

- Each of these structures consists of
  - Individual objects and
  - Links between those objects.
- Goal: find a general framework for describing these objects and their properties.

A **graph** is a mathematical structure for representing relationships.



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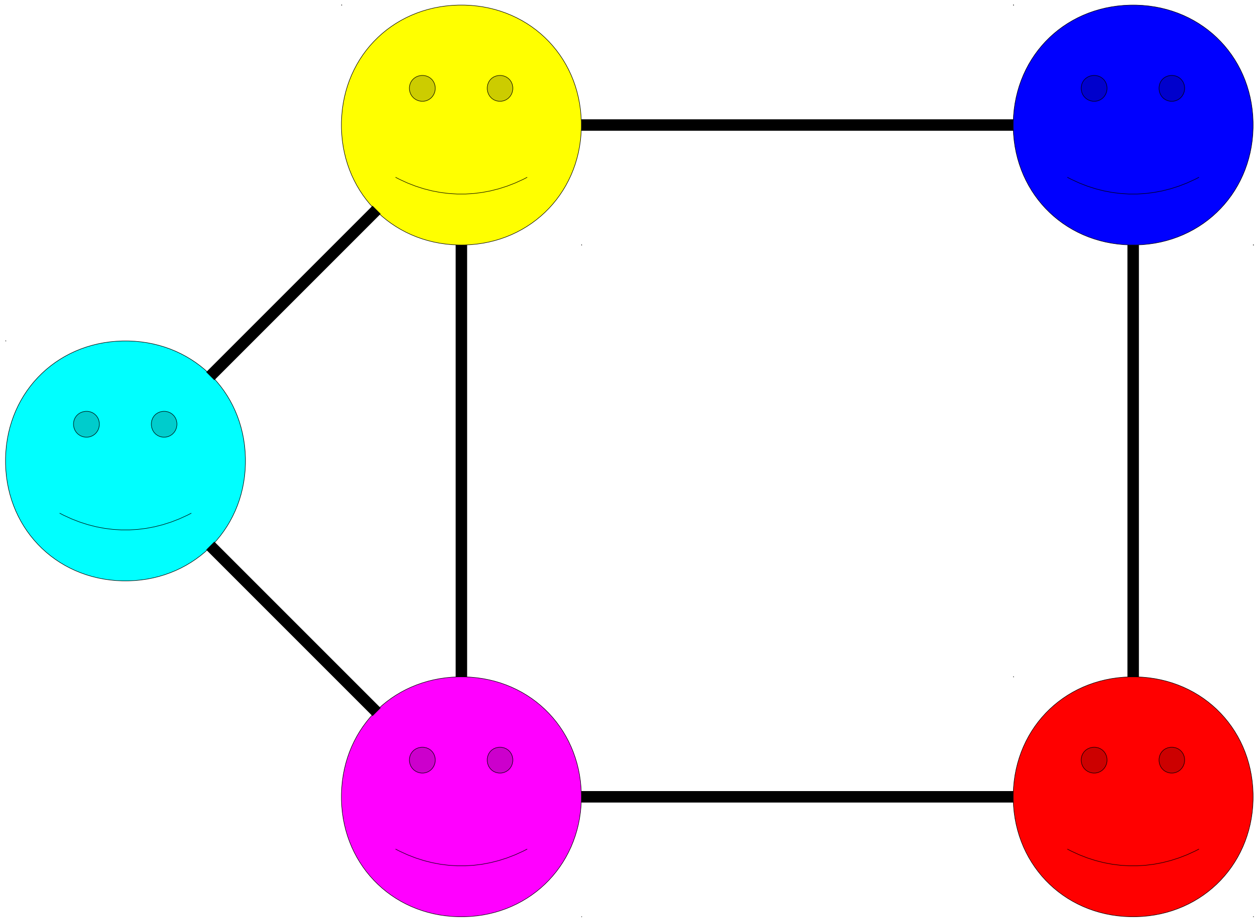

A graph consists of a set of **nodes** (or ***vertices***) connected by **edges** (or ***arcs***)

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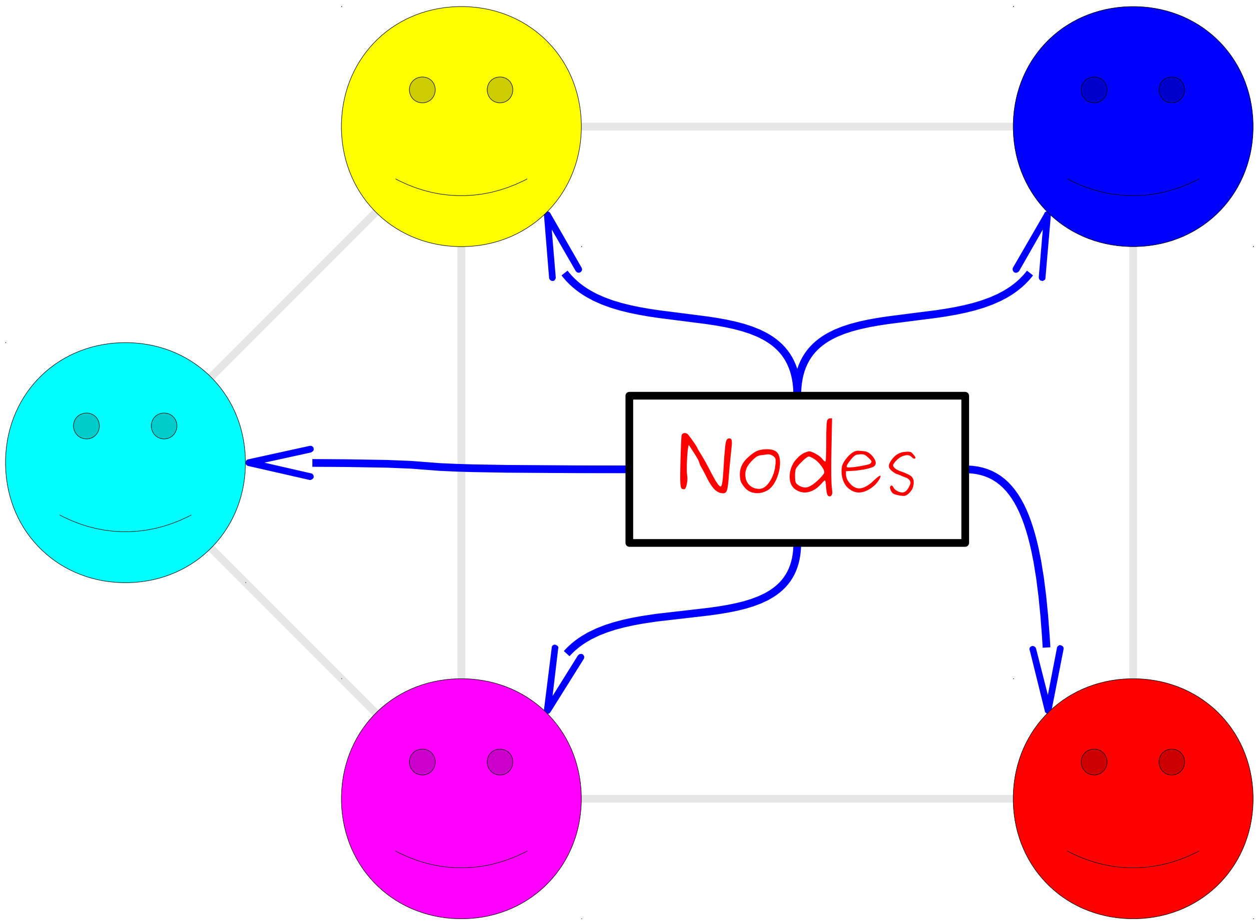

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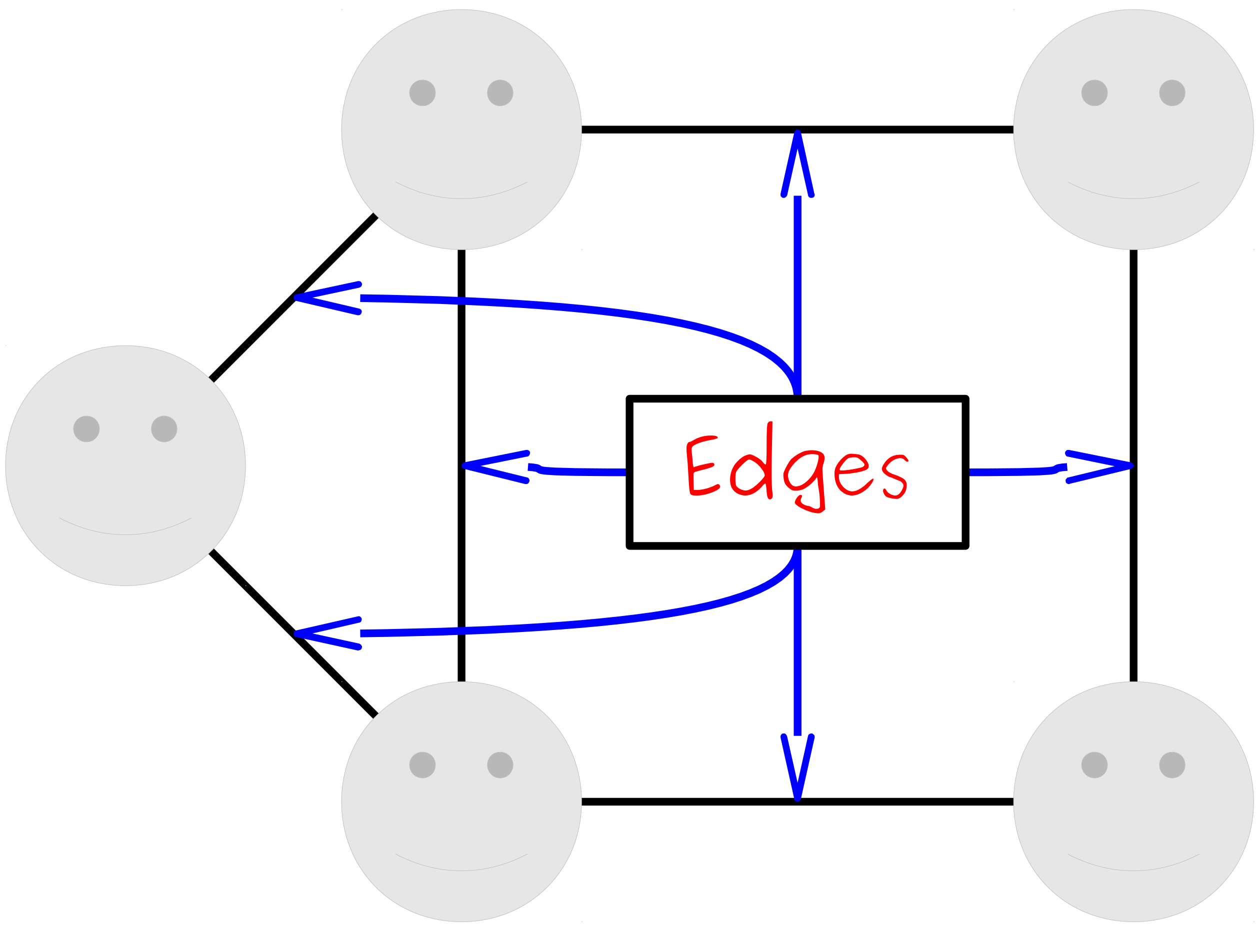
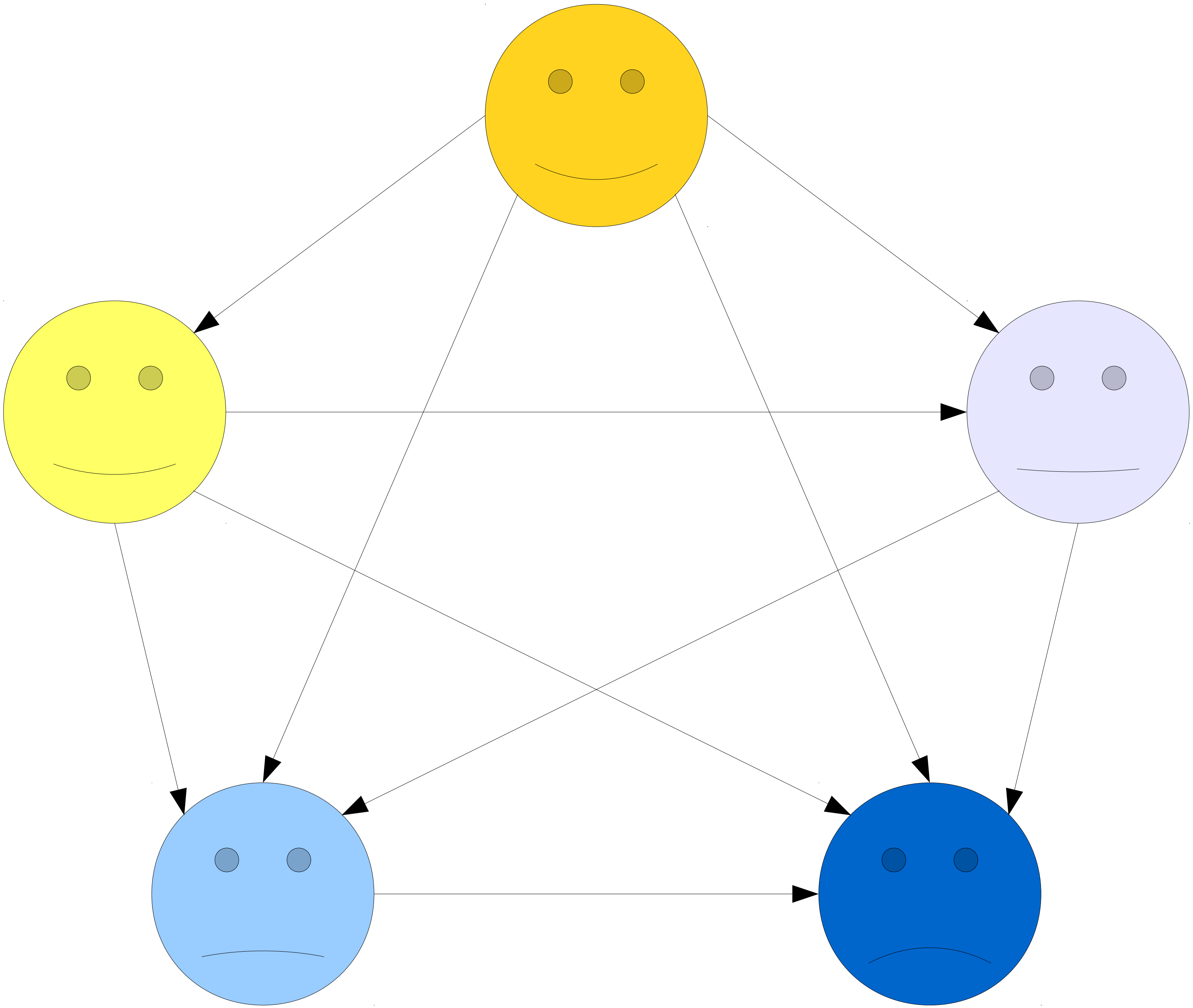
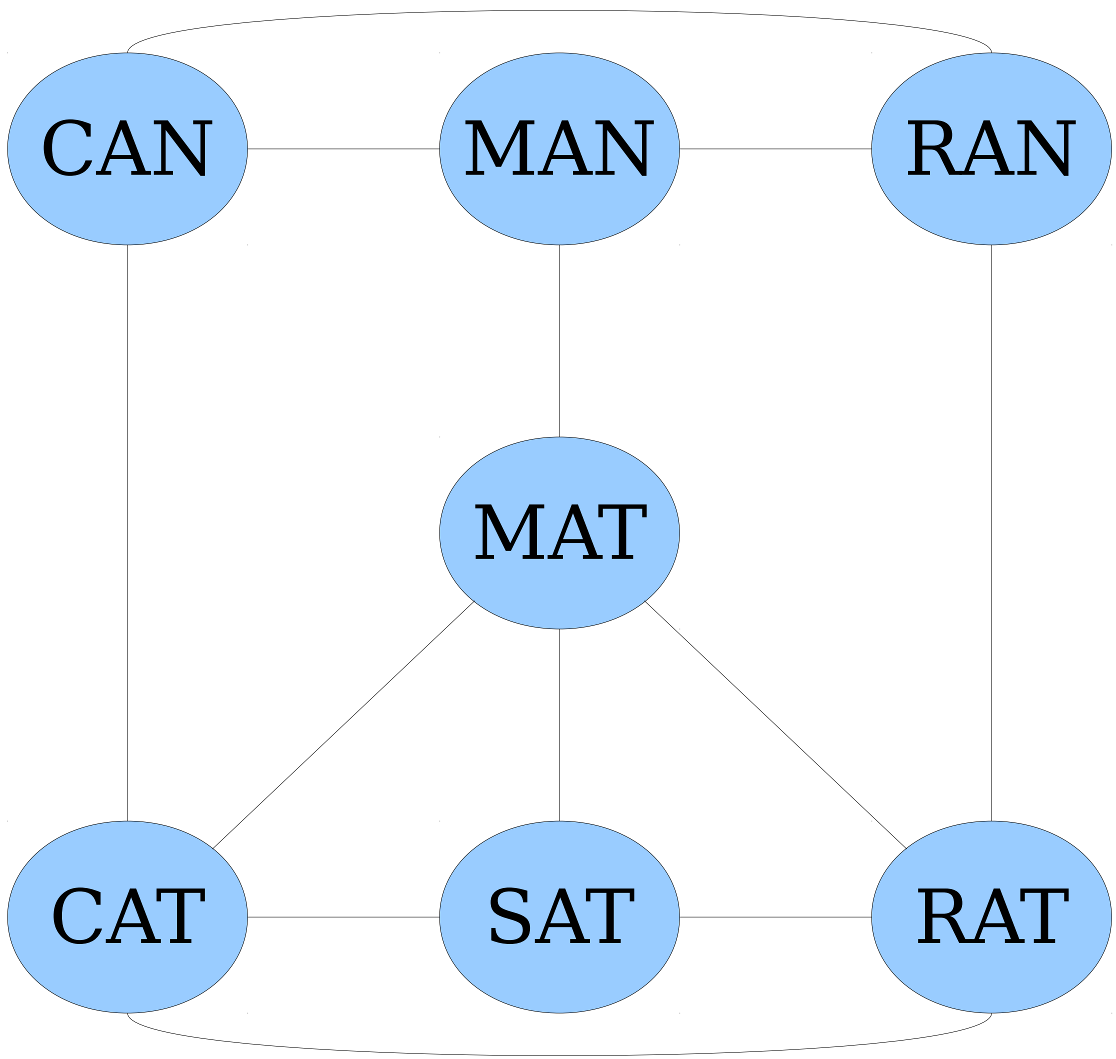


A graph consists of a set of **nodes** (or ***vertices***) connected by **edges** (or ***arcs***)

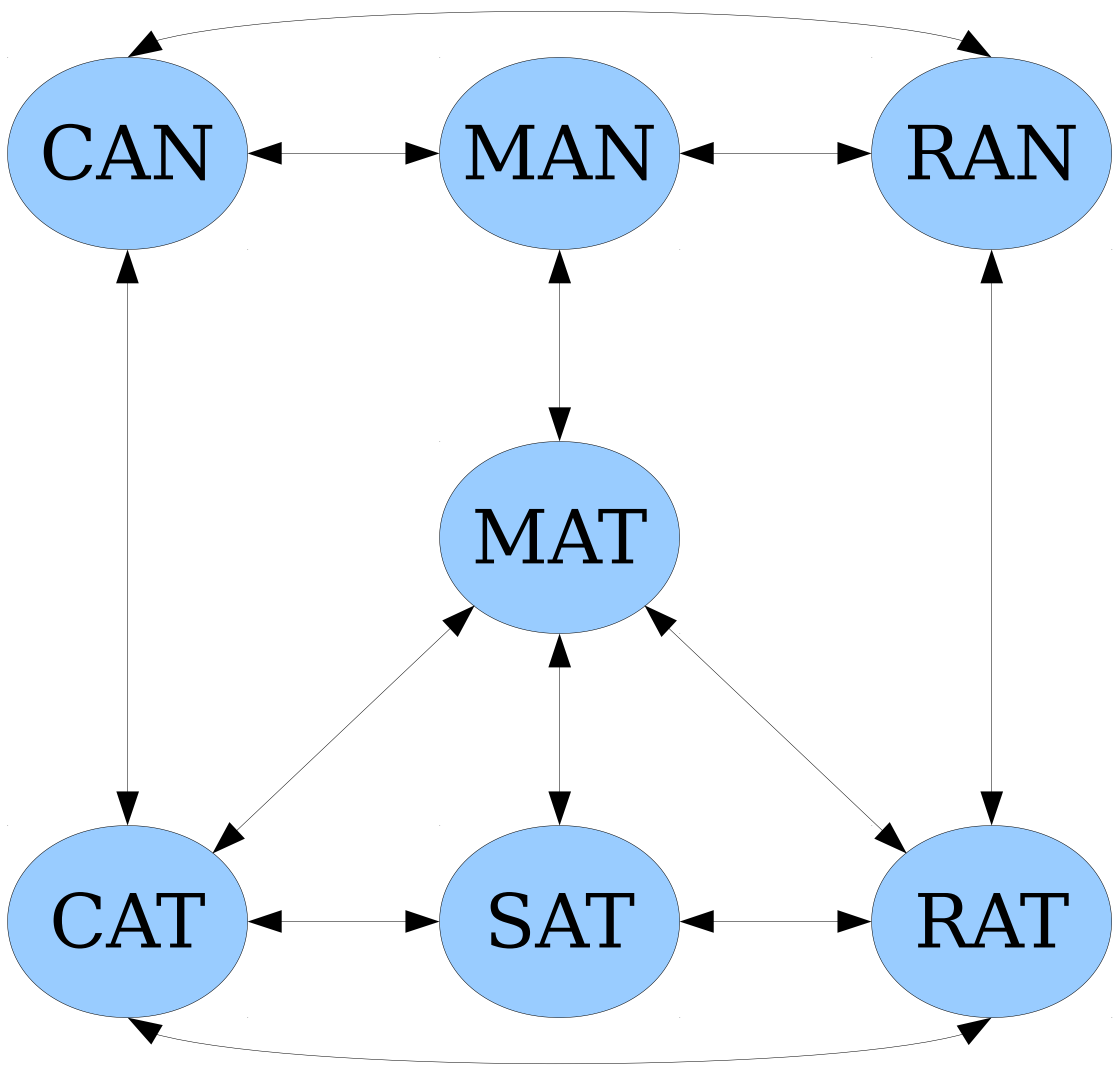
Some graphs are **directed**.



Some graphs are **undirected**.



Some graphs are **undirected**.



You can think of them as directed graphs with edges both ways.

Formalizing Graphs

- How might we define a graph mathematically?
- Need to specify
  - What the nodes in the graph are, and
  - What the edges are in the graph.
- The nodes can be pretty much anything.
- What about the edges?

Ordered and Unordered Pairs

- An ***unordered pair*** is a set $\{a, b\}$ of two elements (remember that sets are unordered).
  - $\{0, 1\} = \{1, 0\}$
- An ***ordered pair*** (a, b) is a pair of elements in a specific order.
  - $(0, 1) \neq (1, 0)$.
  - Two ordered pairs are equal iff each of their components are equal.

Formalizing Graphs

- Formally, a ***graph*** is an ordered pair $G = (V, E)$, where
  - V is a set of nodes.
  - E is a set of edges.
- G is defined as an *ordered* pair so it's clear which set is the nodes and which is the edges.
- V can be any set whatsoever.
- E is one of two types of sets:
  - A set of *unordered* pairs of elements from V.
  - A set of *ordered* pairs of elements from V.

Next Time

- **More About Graphs**
  - Undirected connectivity.
  - Connected components.
  - Planar graphs.
  - Graph coloring.