Trees

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CS 106B
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In our Last Episode . . .

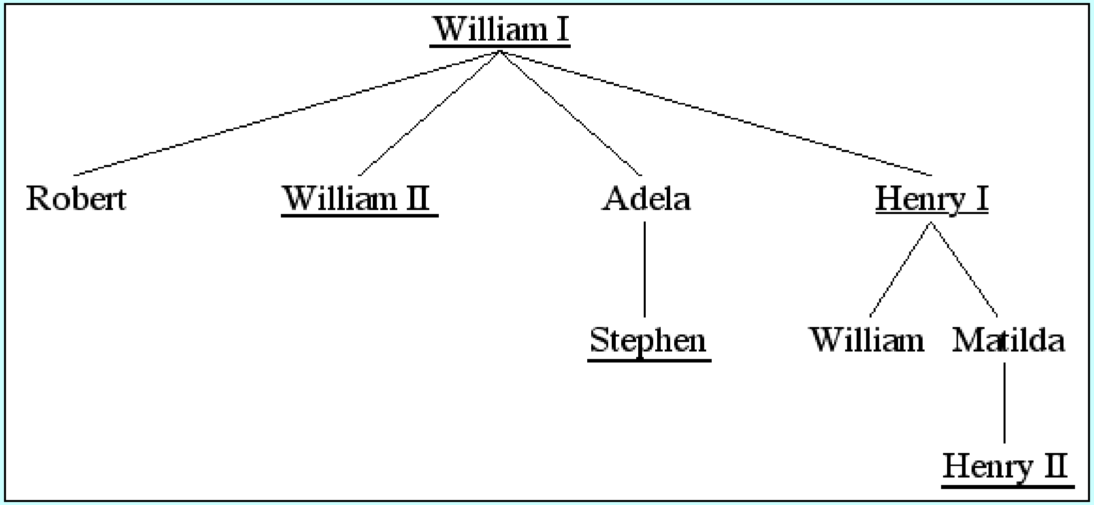
- In Friday's class, I showed how hashing makes it possible to implement the **`get`** and **`put`** operations for a map in $O(1)$ time.
- Despite its extraordinary efficiency, hashing is not always the best strategy for implementing maps, because of the following limitations:
  - Hash tables depend on being able to compute a hash function on some key. Expanding the hash-function idea so that it applies to types other than strings is subtle.
  - Using the range-based **`for`** on hash tables does not deliver the keys in any sensible order. Even when the keys have a natural order (such as the lexicographic order used with strings), the hash table code for iteration cannot take advantage of that fact.
- The goal for today is to explore another representation that supports iterating through the elements in order.

Analyzing the Failure of Sorted Arrays

- One of the strategies I outlined last Friday for implementing a map was to use a sorted array to hold the key-value pairs. Given that representation, binary search made it possible to find a key in $O(\log N)$ time.
- The problem with the sorted array strategy was that inserting a new key required $O(N)$ time to maintain the order.
- In the editor buffer, linked lists solved the insertion problem. Unfortunately, turning a sorted array into a linked list makes it impossible to apply binary search because there is no way to find the middle element.
- But what if you could point to the middle element in a linked list? That question gives rise to a new structure called a ***tree***, which provides the key to implementing a map with $O(\log N)$ performance for both the **`get`** and **`put`** operations.

Trees

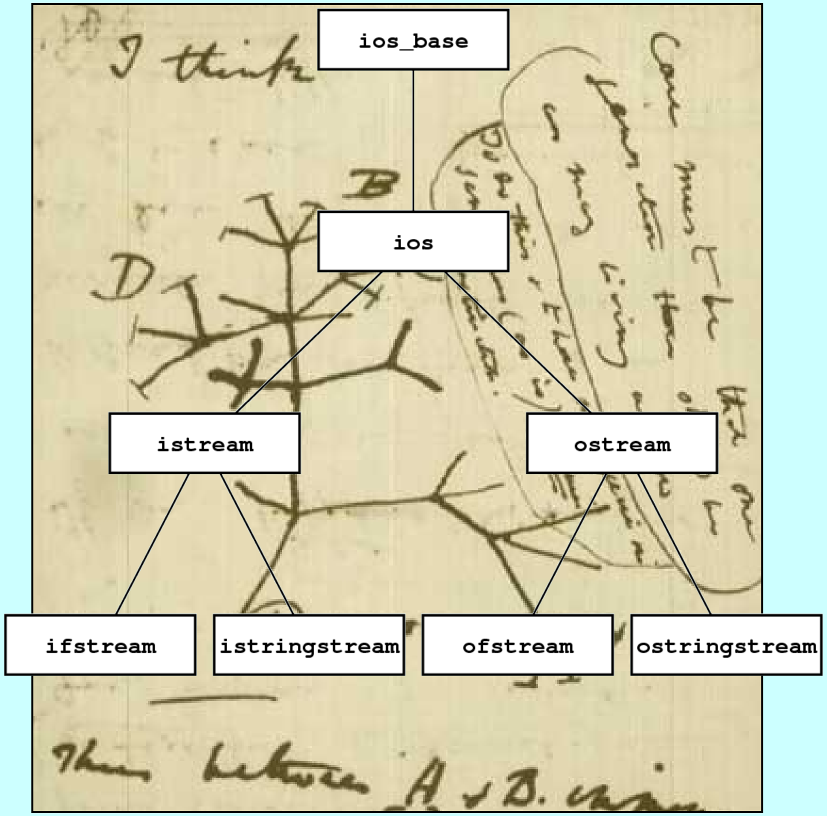
- In the text, the first example I use to illustrate tree structures is the royal family tree of the House of Normandy:



- This example is useful for defining terminology:
  - William I is the ***root*** of the tree.
  - Adela is a ***child*** of William I and the ***parent*** of Stephen.
  - Robert, William II, Adela, and Henry I are ***siblings***.
  - Henry II is a ***descendant*** of William I, Henry I, and Matilda.
  - William I is an ***ancestor*** of everyone else in this tree.

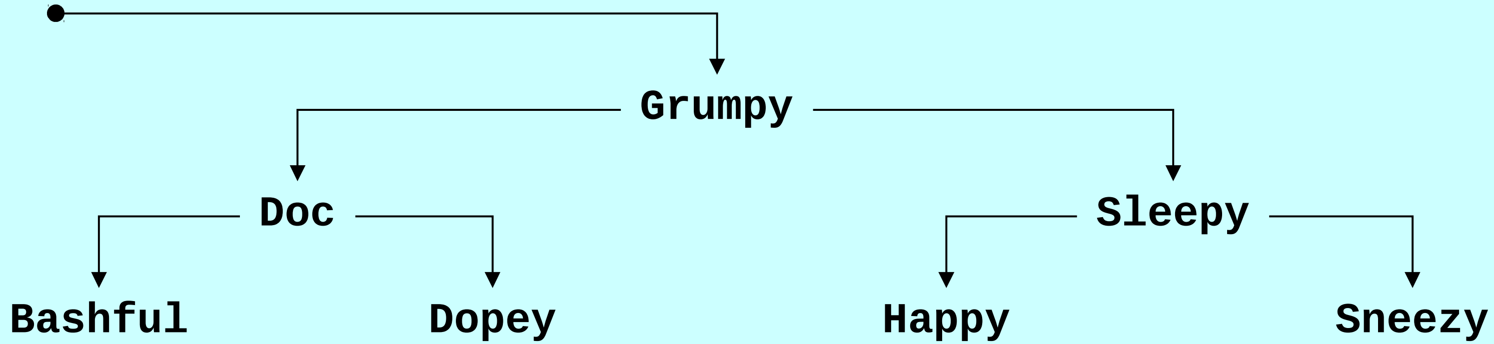
Trees Are Everywhere

- Trees appear in many familiar contexts beyond family trees. The picture at the right comes from Darwin's notebooks and shows his early conception of an evolutionary tree.
- Trees also form the basis for the class hierarchies used in object-oriented programming languages like Java and C++.
- In each of these contexts, trees begin with a single root node and then branch outward repeatedly to encompass any other nodes in the tree.



Binary Search Trees

- The tree that supports the implementation of the `Map` class is called a ***binary search tree*** (or ***BST*** for short). Each node in a BST has exactly two subtrees: a ***left subtree*** that contains all the nodes that come before the current node and a ***right subtree*** that contains all the nodes that come after it. Either or both of these subtrees may be `NULL`.
- The classic example of a binary search tree uses the names from Walt Disney's *Snow White and the Seven Dwarves*:



A Simple BST Implementation

- To get a sense of how binary search trees work, it is useful to start with a simple design in which keys are always strings.
- Each node in the tree is then a structure containing a key and two subtrees, each of which is either **`NULL`** or a pointer to some other node. This design suggests the following definition:

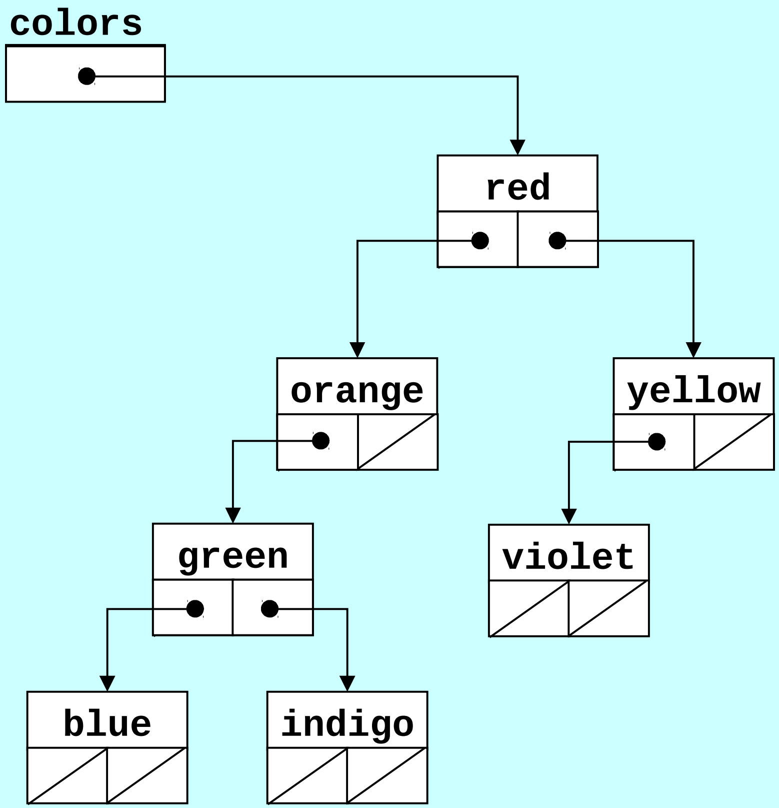
```
struct Node {
   string key;
   Node *left, *right;
};
```

- The code for finding a node in a tree begins by comparing the desired key with the key in the root node. If the strings match, you've found the correct node; if not, you simply call yourself recursively on the left or right subtree depending on whether the key you want comes before or after the current one.

Exercise: Building a Binary Search Tree

Diagram the BST that results from executing the following code:

```
Node *colors = NULL;
insertNode(colors, "red");
insertNode(colors, "orange");
insertNode(colors, "yellow");
insertNode(colors, "green");
insertNode(colors, "blue");
insertNode(colors, "indigo");
insertNode(colors, "violet");
```



A Simple BST Implementation

```
/*
 * Type: Node
 * ----------
 * This type represents a node in the binary search tree.
 */

struct Node {
   string key;
   Node *left, *right;
};

/*
 * Function: findNode
 * Usage: Node *node = findNode(t, key);
 * -------------------------------------
 * Returns a pointer to the node in the binary search tree t than contains
 * a matching key.  If no such node exists, findNode returns NULL.
 */

Node *findNode(Node *t, string key) {
   if (t == NULL) return NULL;
   if (key == t->key) return t;
   if (key < t->key) {
      return findNode(t->left, key);
   } else {
      return findNode(t->right, key);
   }
}
```

A Simple BST Implementation

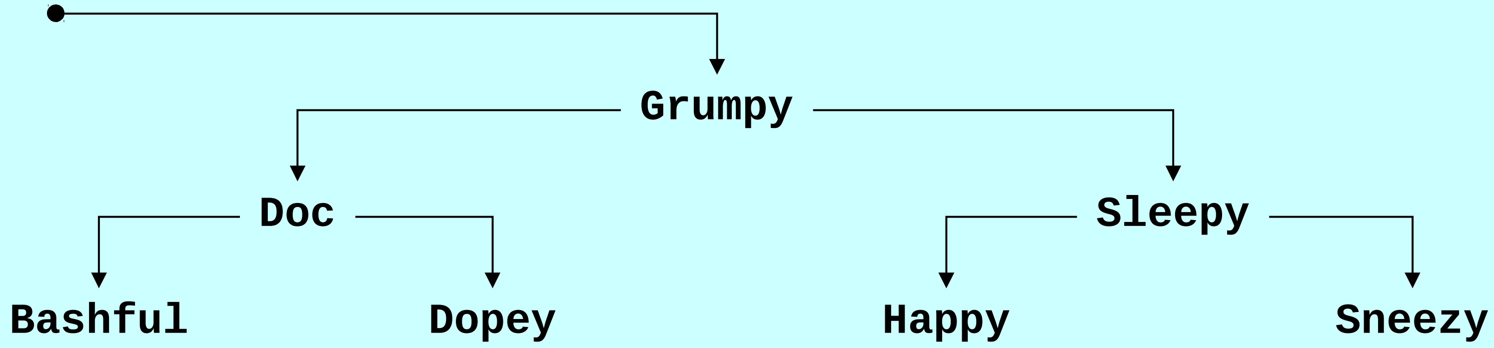
```
/*
 * Function: insertNode
 * Usage: insertNode(t, key);
 * --------------------------
 * Inserts the specified key at the appropriate location in the
 * binary search tree rooted at t.  Note that t must be passed
 * by reference, since it is possible to change the root.
 */

void insertNode(Node * & t, string key) {
   if (t == NULL) {
      t = new Node;
      t->key = key;
      t->left = t->right = NULL;
      return;
   }
   if (key == t->key) return;
   if (key < t->key) {
      insertNode(t->left, key);
   } else {
      insertNode(t->right, key);
   }
}
```

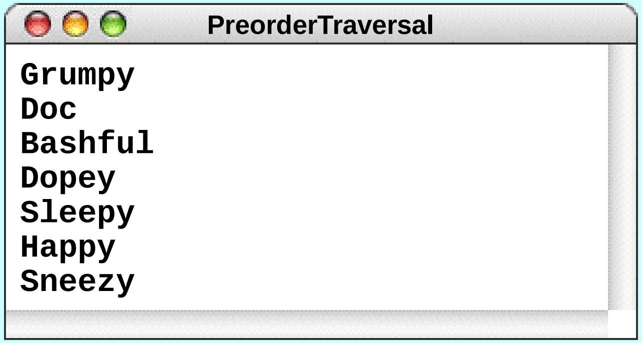
Traversal Strategies

- It is easy to write a function that performs some operation for every key in a binary search tree, because recursion makes it simple to apply that operation to each of the subtrees.
- The order in which keys are processed depends on when you process the current node with respect to the recursive calls:
  - If you process the current node before either recursive call, the result is a ***preorder traversal***.
  - If you process the current node after the recursive call on the left subtree but before the recursive call on the right subtree, the result is an ***inorder traversal***. In the case of the simple BST implementation that uses strings as keys, the keys will appear in lexicographic order.
  - If you process the current node after completing both recursive calls, the result is a ***postorder traversal***. Postorder traversals are particularly useful if you are trying to free all the nodes in a tree.

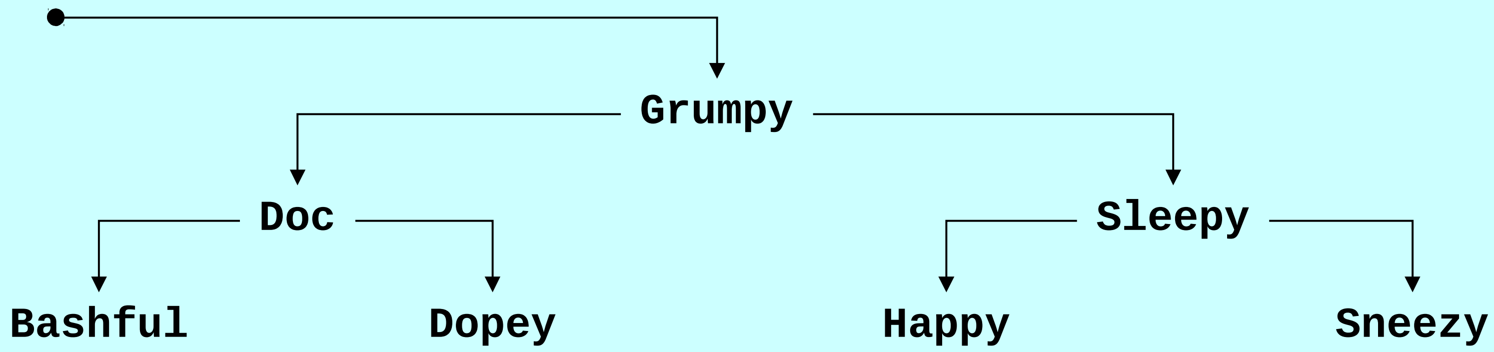
Exercise: Preorder Traversal



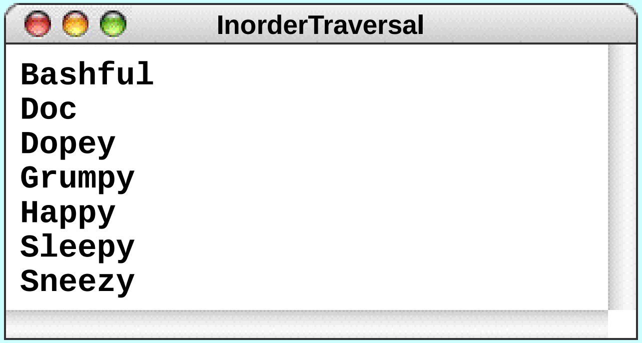

```
void preorderTraversal(Node *t) {
   if (t != null) {
      cout << t->key << endl;
      preorderTraversal(t->left);
      preorderTraversal(t->right);
   }
}
```



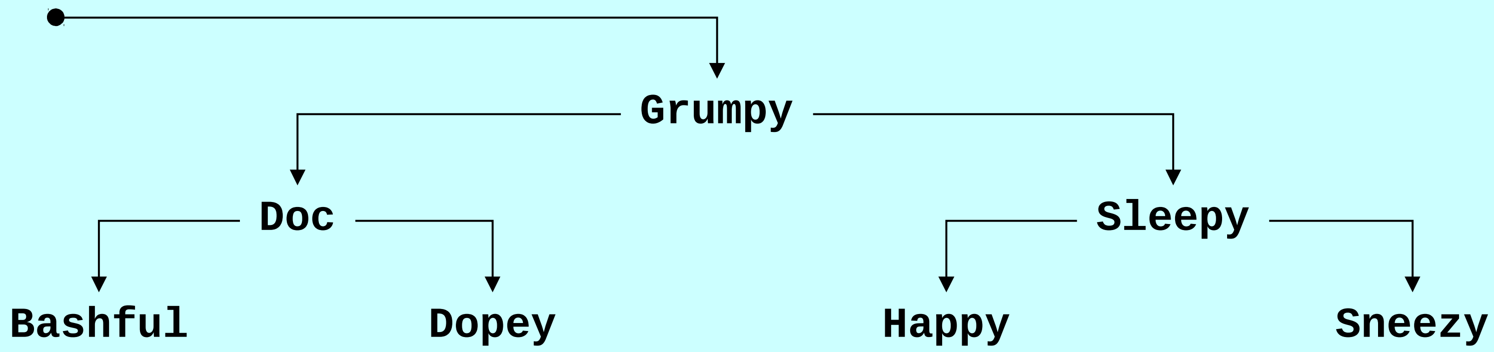
Exercise: Inorder Traversal



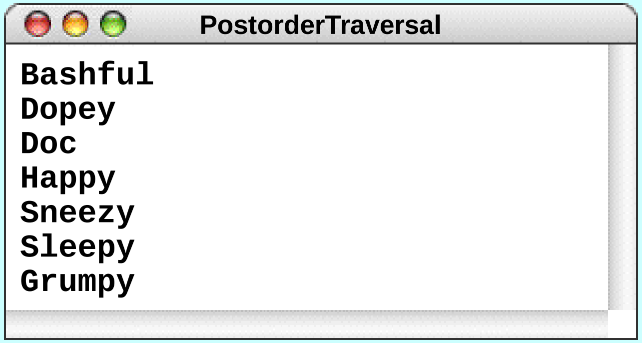

```
void inorderTraversal(Node *t) {
   if (t != null) {
      inorderTraversal(t->left);
      cout << t->key << endl;
      inorderTraversal(t->right);
   }
}
```



Exercise: Postorder Traversal

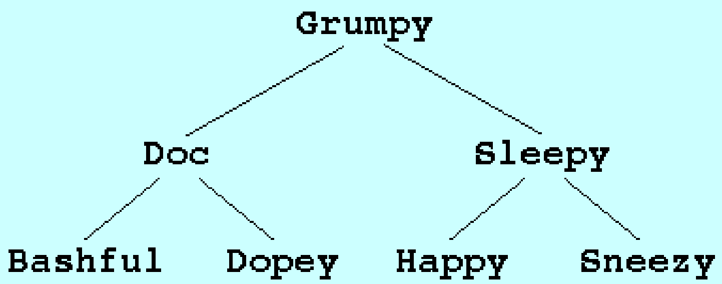



```
void postorderTraversal(Node *t) {
   if (t != null) {
      postorderTraversal(t->left);
      postorderTraversal(t->right);
      cout << t->key << endl;
   }
}
```



A Question of Balance

- Ideally, a binary search tree containing the names of Disney’s seven dwarves would look like this:



- If, however, you happened to enter the names in alphabetical order, this tree would end up being a simple linked list in which all the left subtrees were **`NULL`** and the right links formed a simple chain. Algorithms on that tree would run in $O(N)$ time instead of $O(\log N)$ time.

- A binary search tree is ***balanced*** if the height of its left and right subtrees differ by at most one and if both of those subtrees are themselves balanced.

Illustrating the AVL Algorithm

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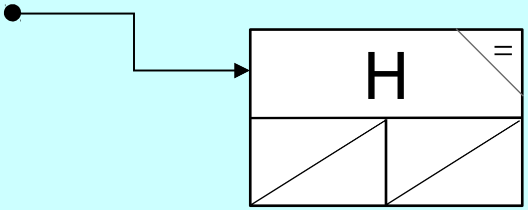

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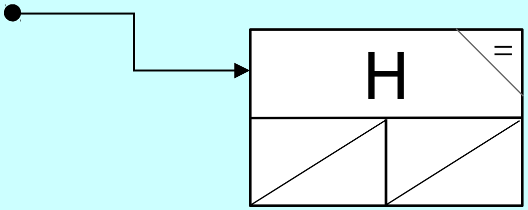

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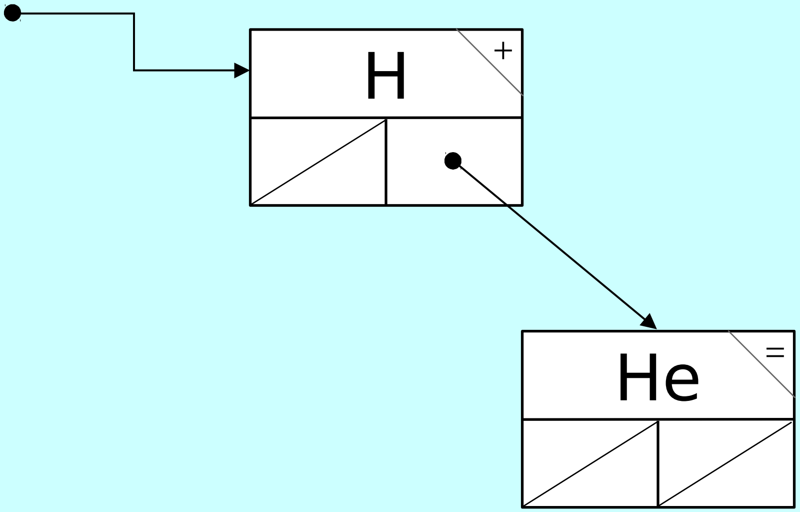

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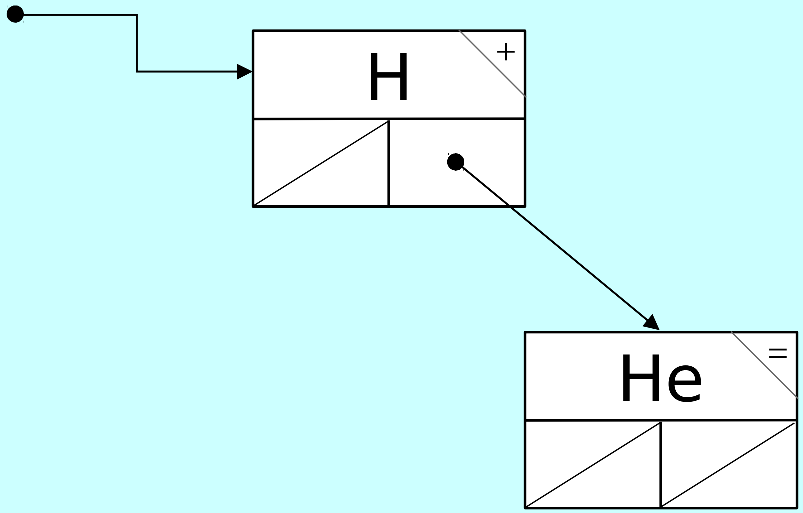

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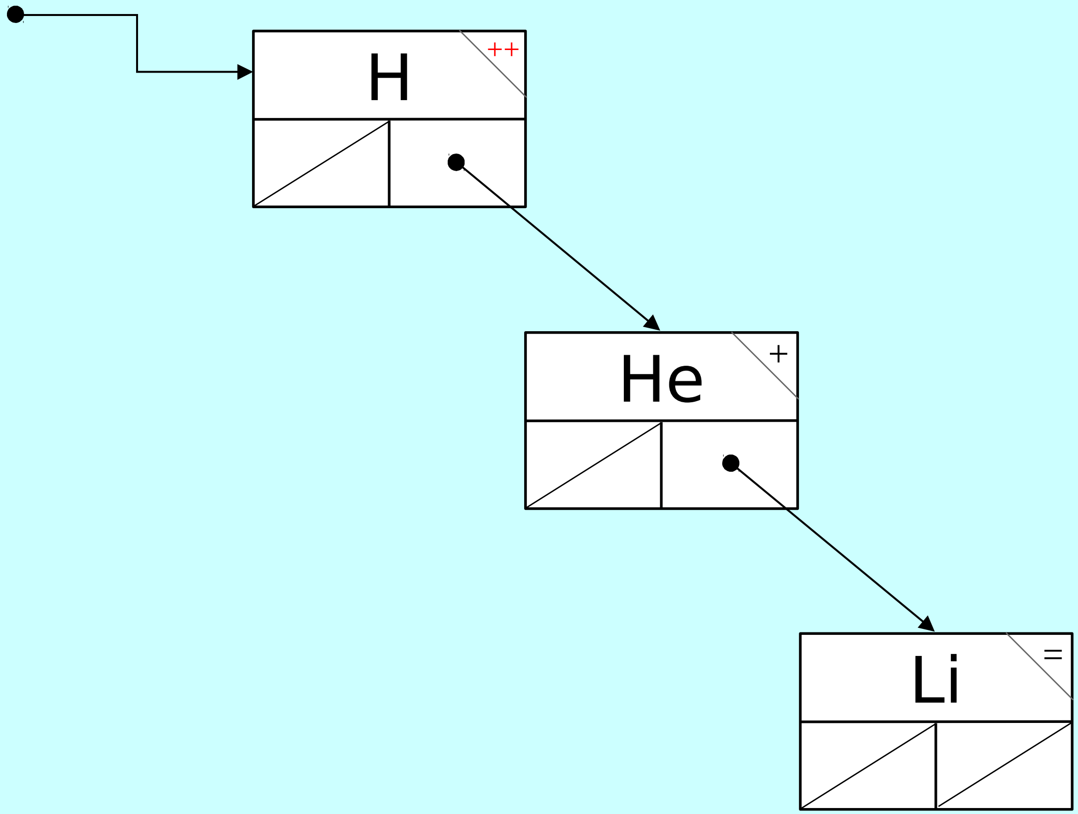

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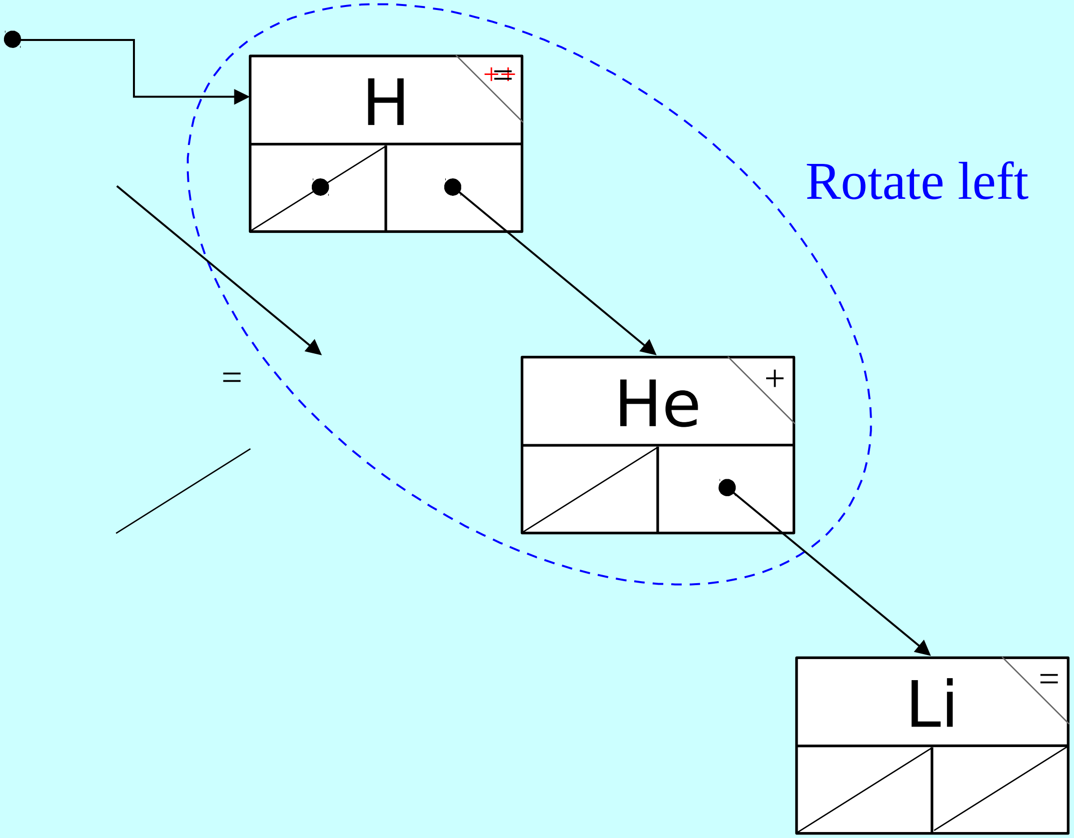

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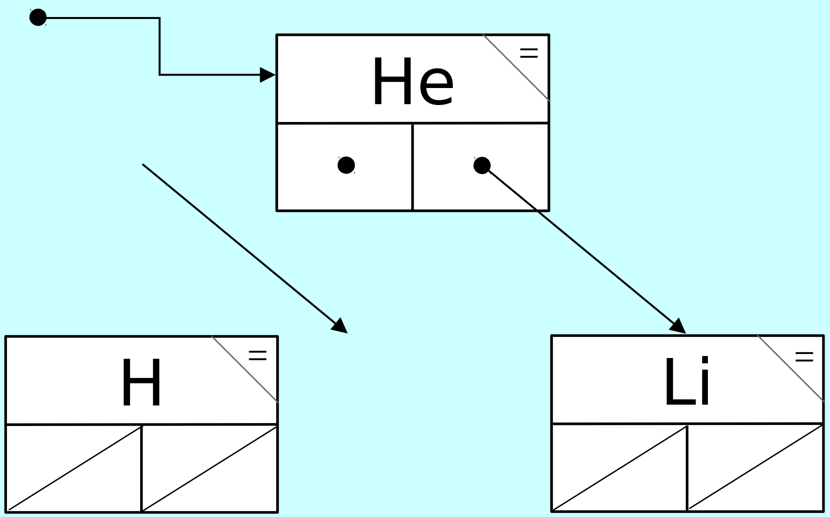

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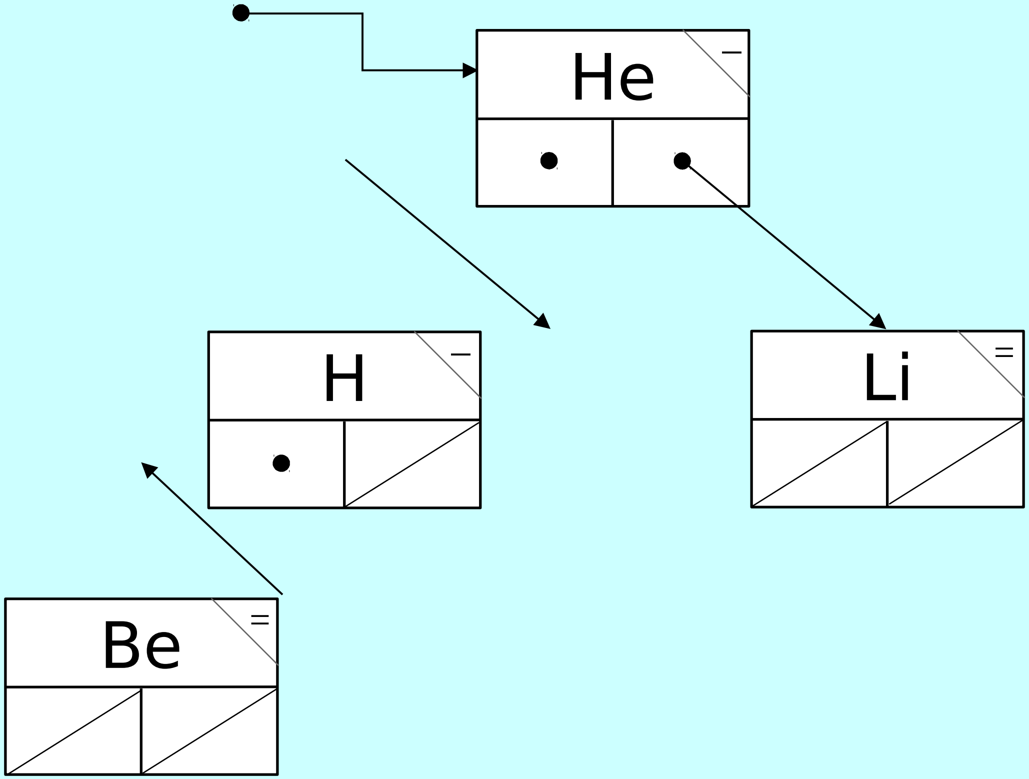

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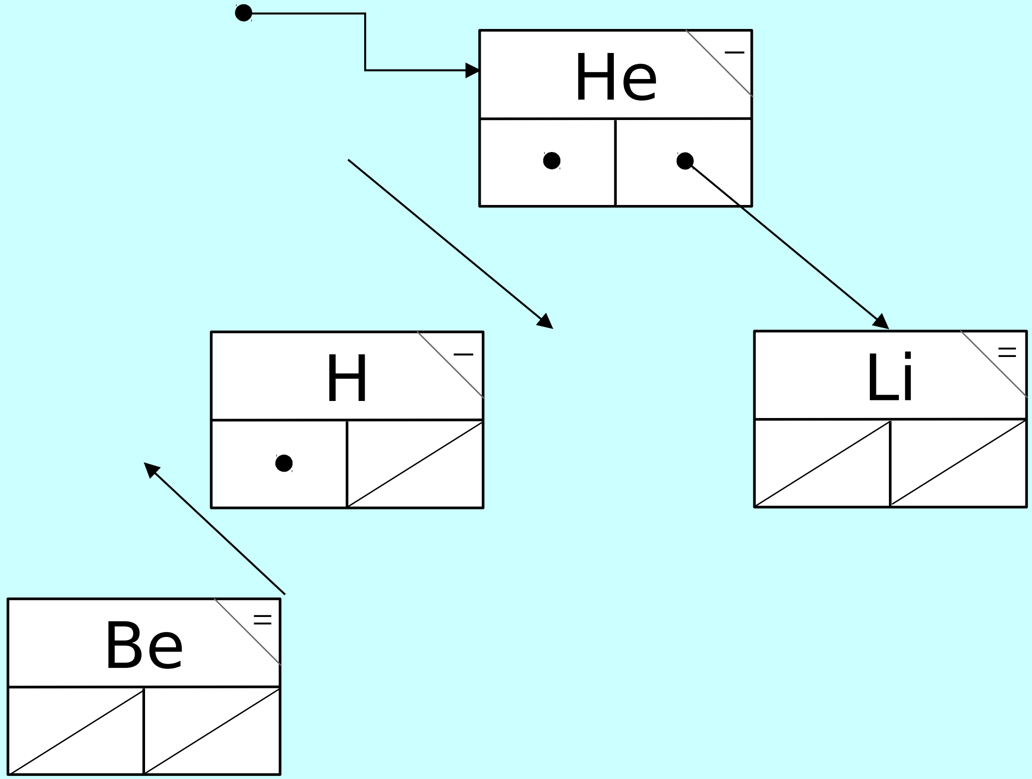

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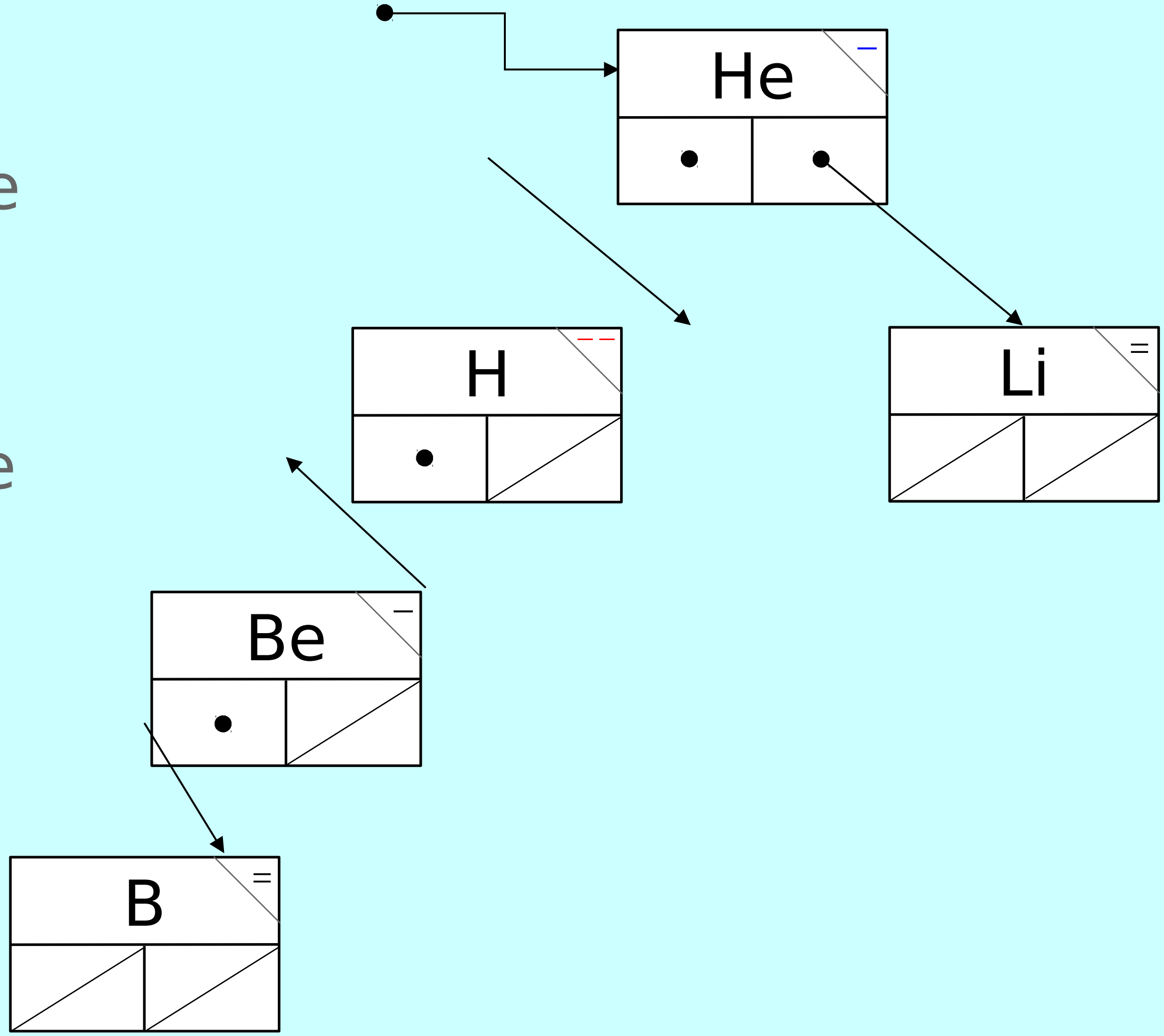

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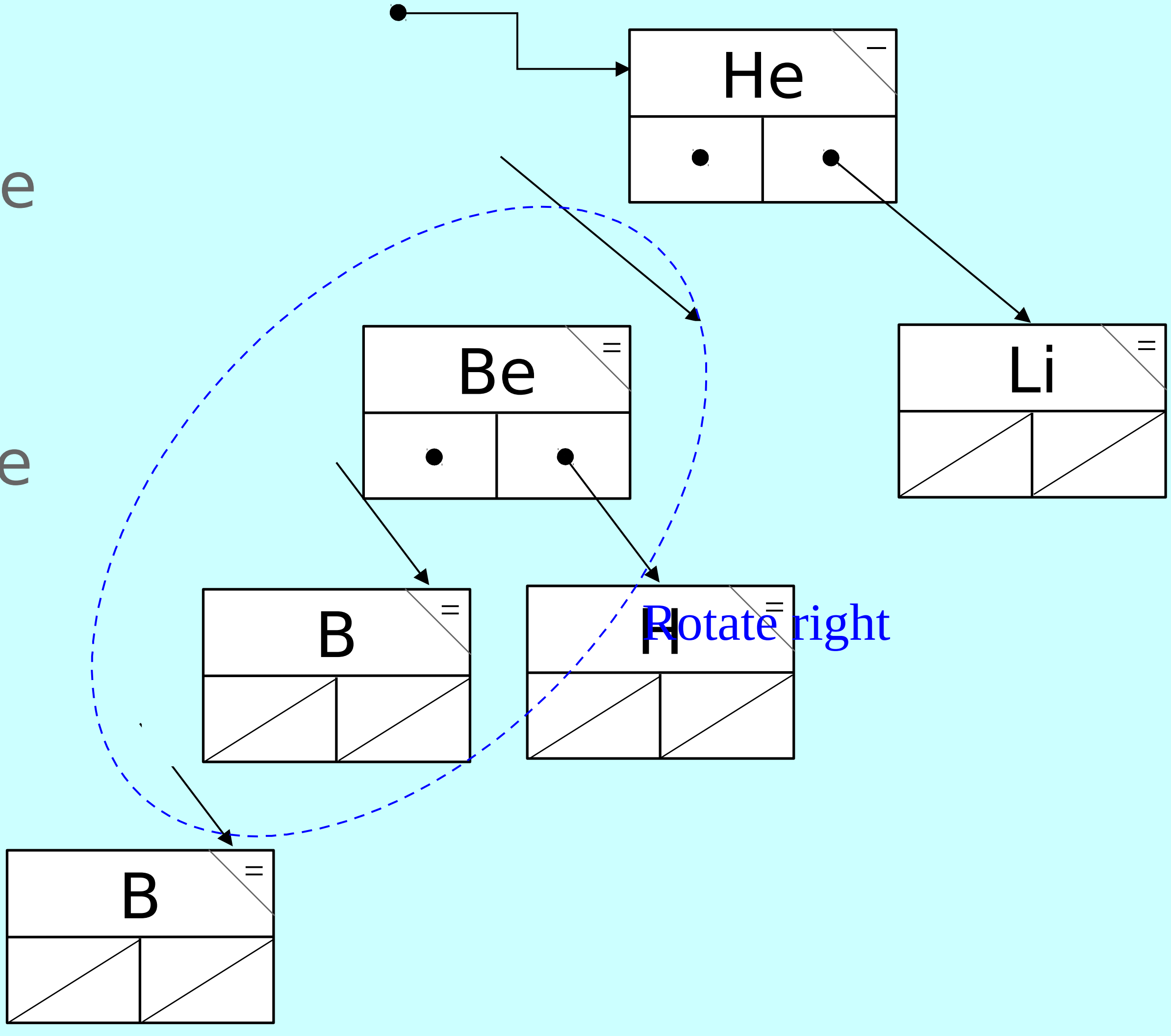

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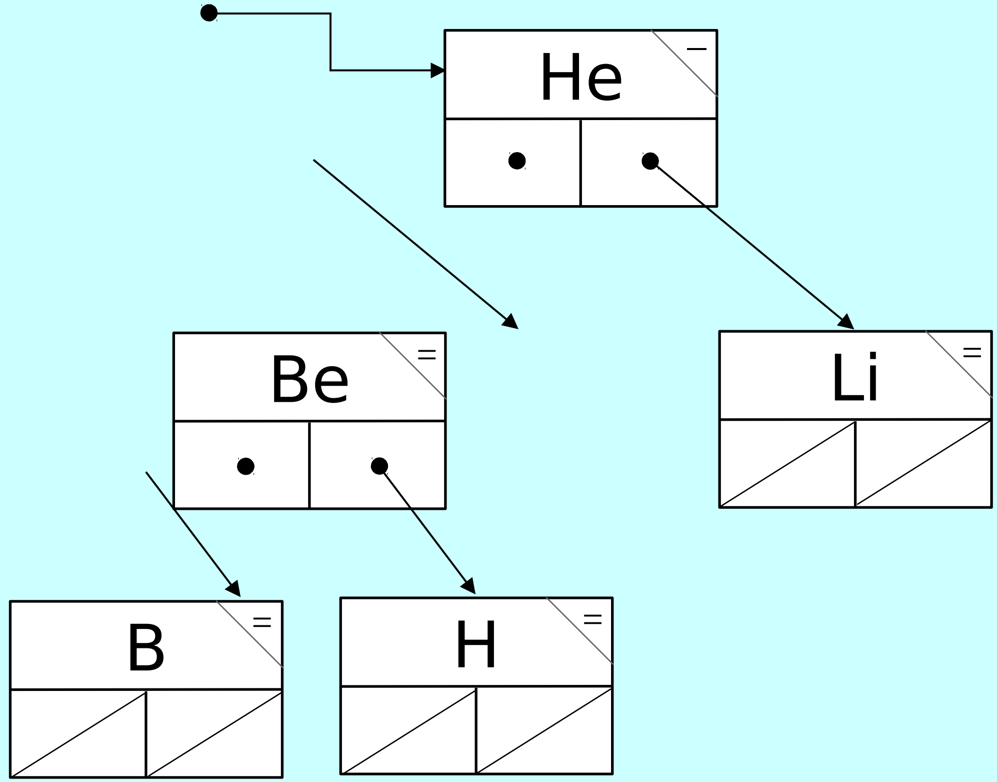

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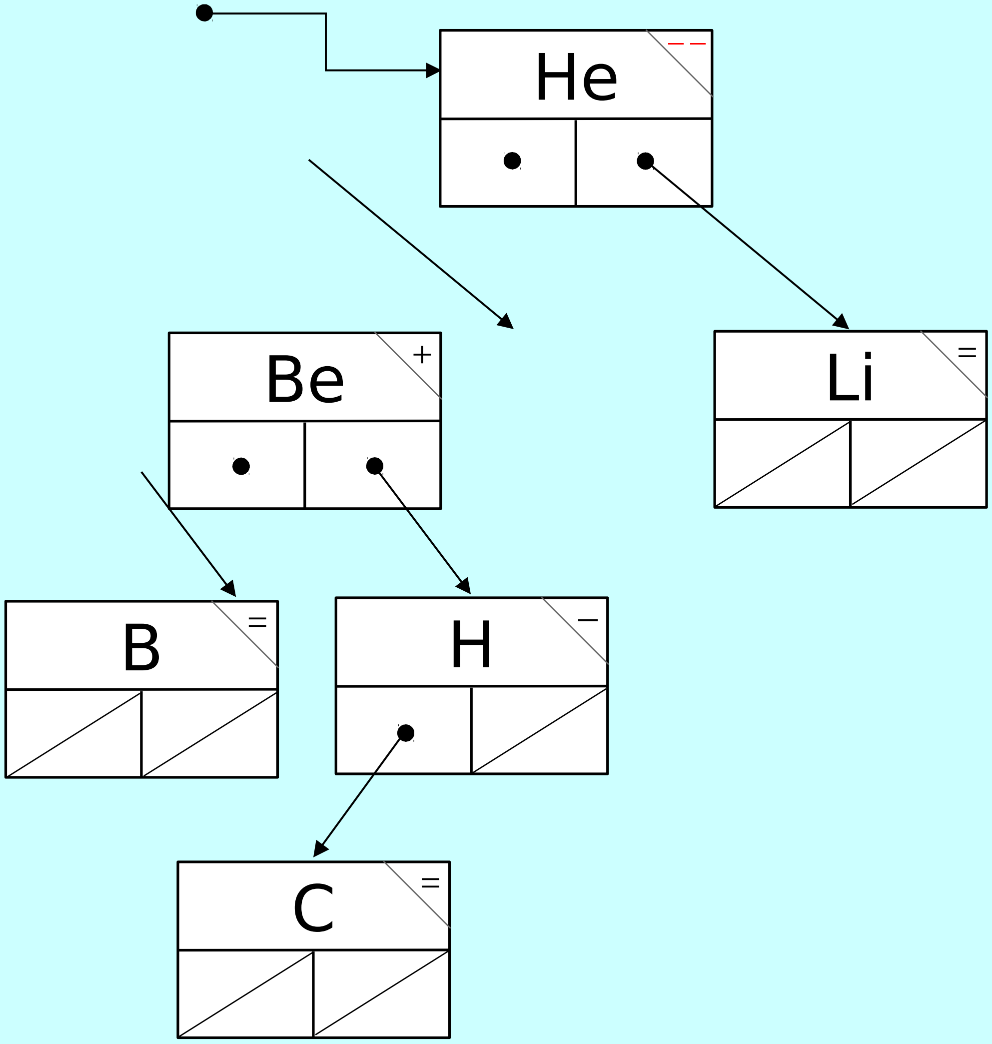

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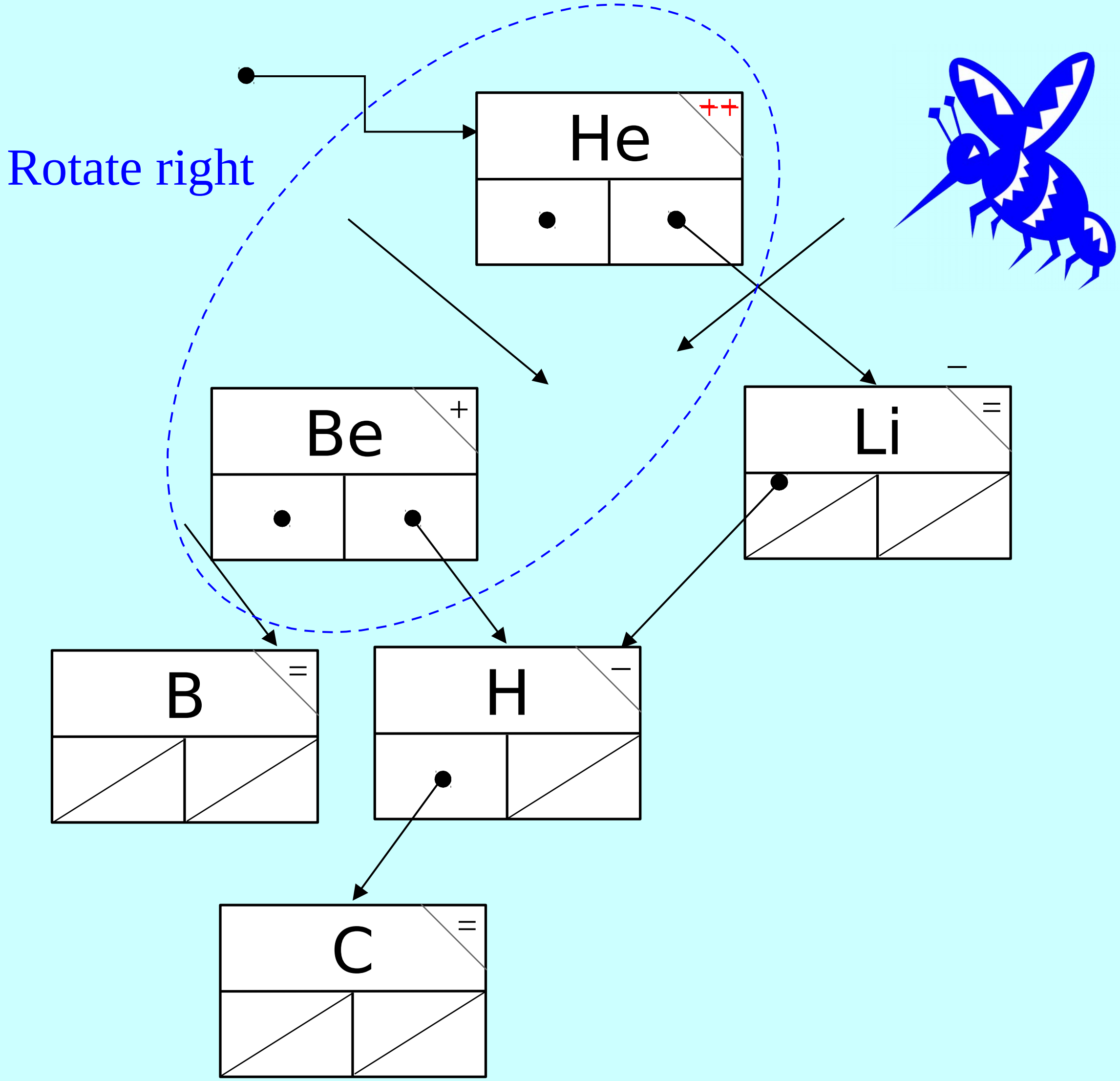

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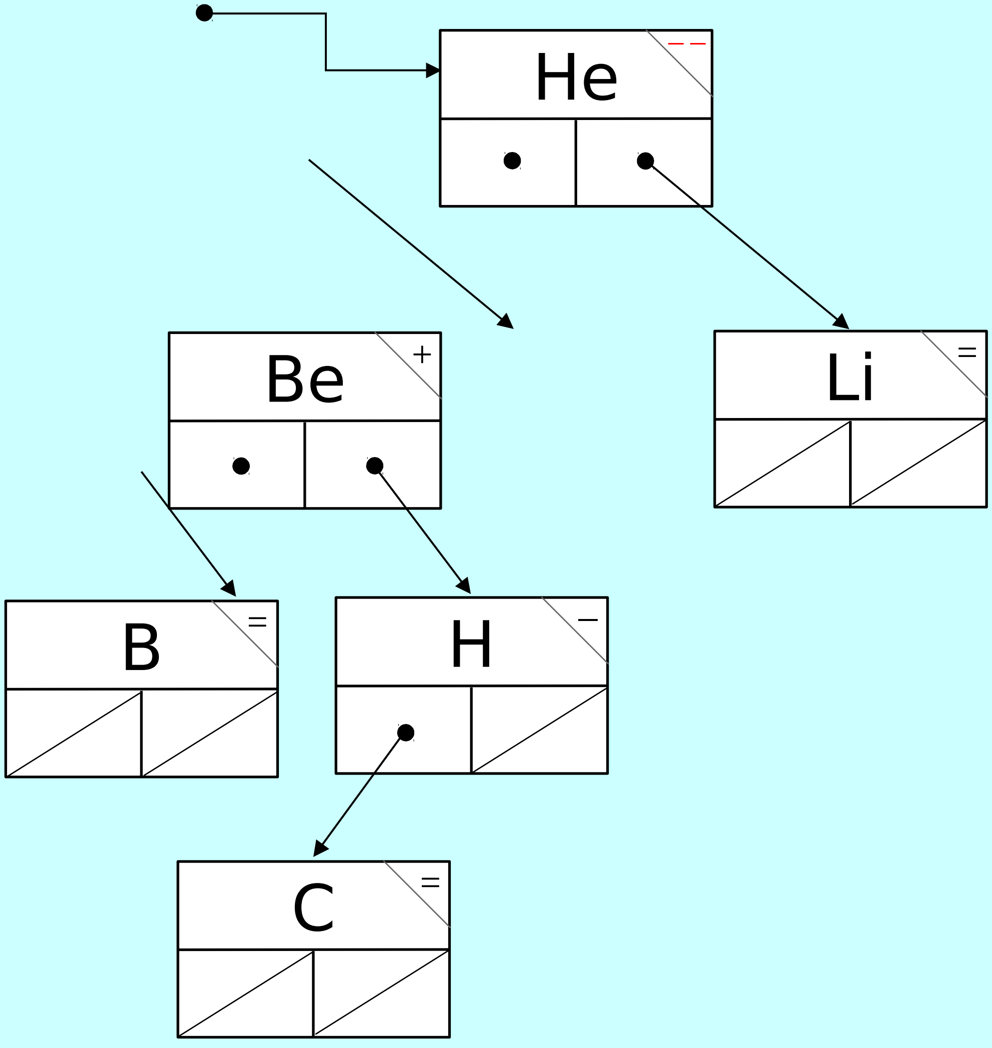

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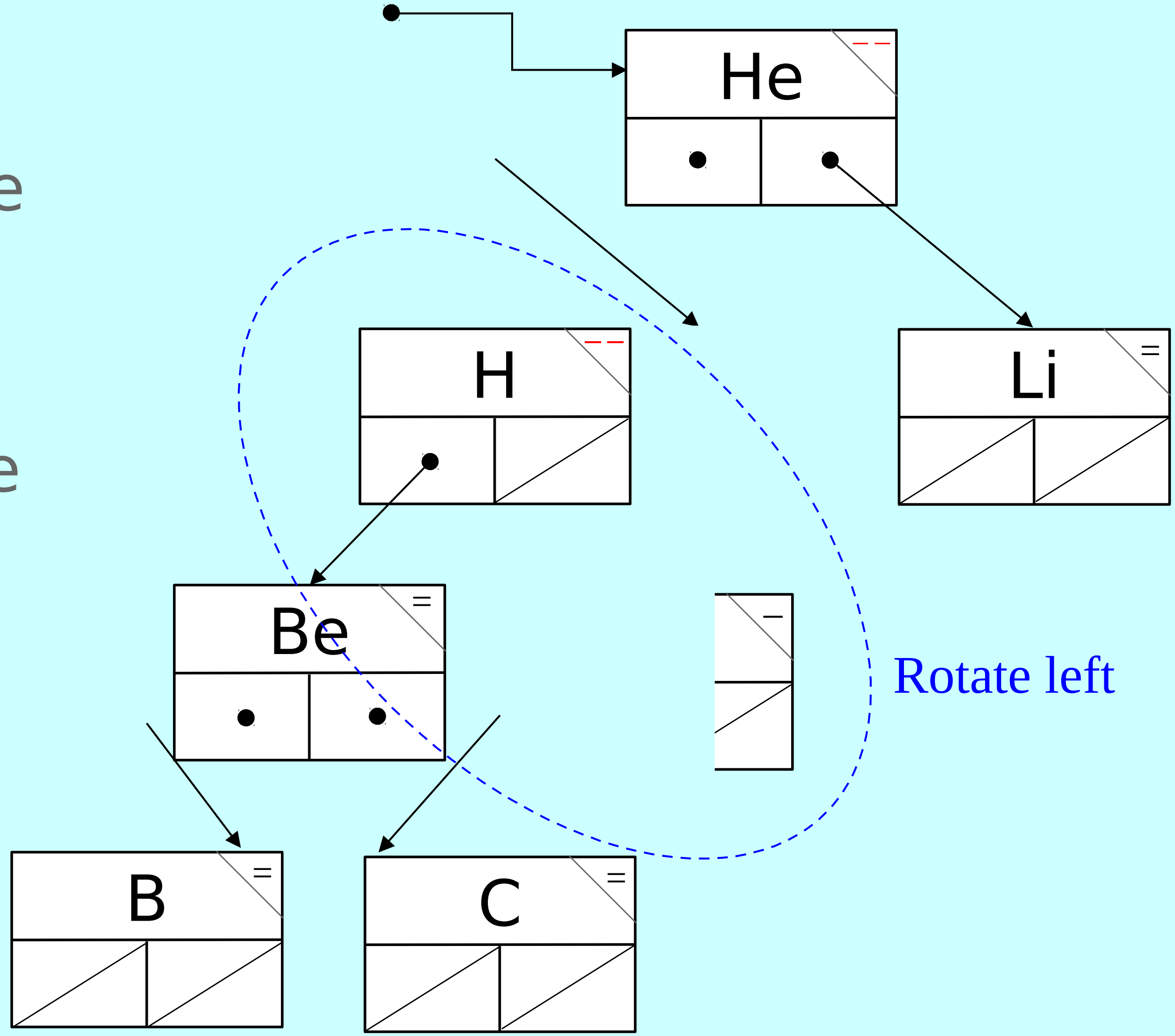

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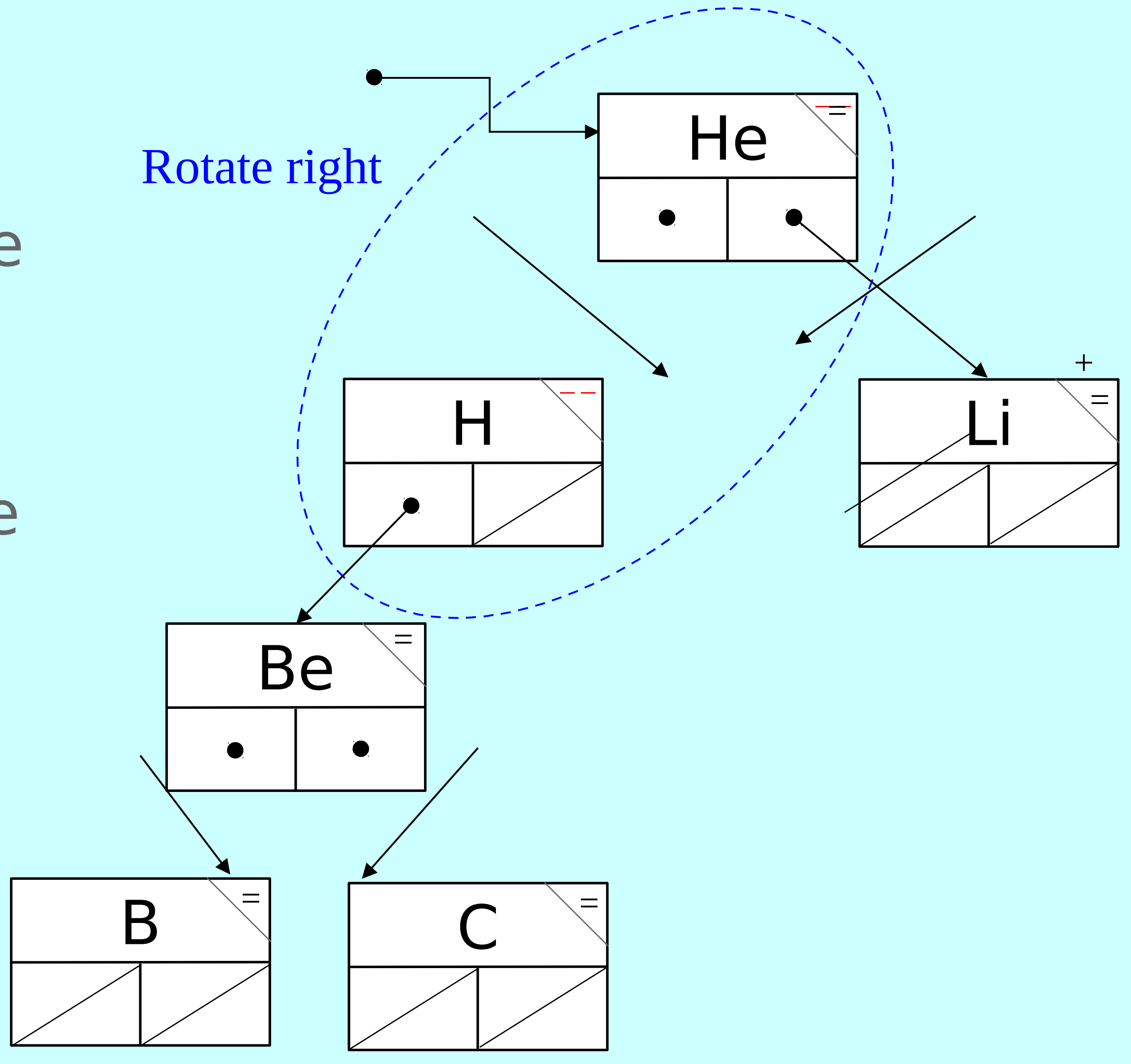


Tree-Balancing Algorithms

- The AVL algorithm was the first tree-balancing strategy and has been superseded by newer algorithms that are more effective in practice. These algorithms include:
  - Red-black trees
  - 2-3 trees
  - AA trees
  - Fibonacci trees
  - Splay trees

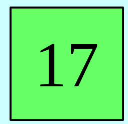
- In CS 106B, the important thing to know is that it is *possible* to keep a binary tree balanced as you insert nodes, thereby ensuring that lookup operations run in $O(\log N)$ time. If you get really excited about this kind of algorithm, you'll have the opportunity to study them in more detail in CS 161.

The Heap Algorithm

- Assignment #5 asks you to implement a data structure called a ***priority queue*** in several different ways.
- The standard algorithm for implementing priority queues uses a data structure called a ***heap***, which makes it possible to implement priority queue operations in $O(\log N)$ time.
- A heap is an array-based representation of a ***partially ordered tree***, which is a binary tree with three additional properties:
  - The tree is ***complete***, which means that it is not only completely balanced but that each level of the tree is filled as far to the left as possible.
  - The root node of the tree has higher priority than the root of either of its subtrees.
  - Every subtree is also a partially ordered tree.

Exercise: Tracing the Heap Algorithm

Insert in order: 17 93 20 42 68 11



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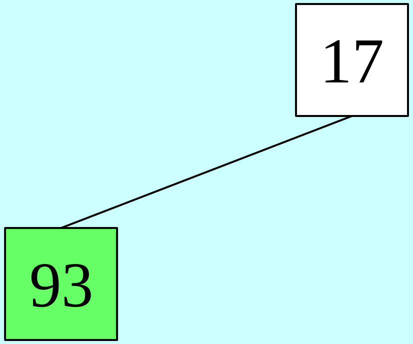

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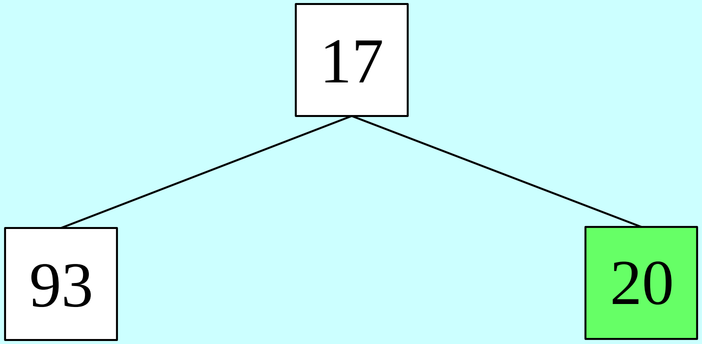

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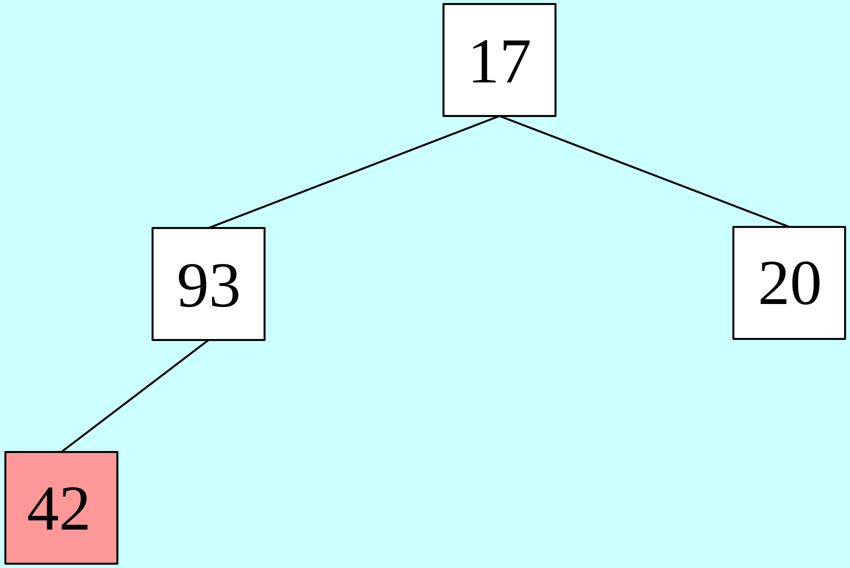

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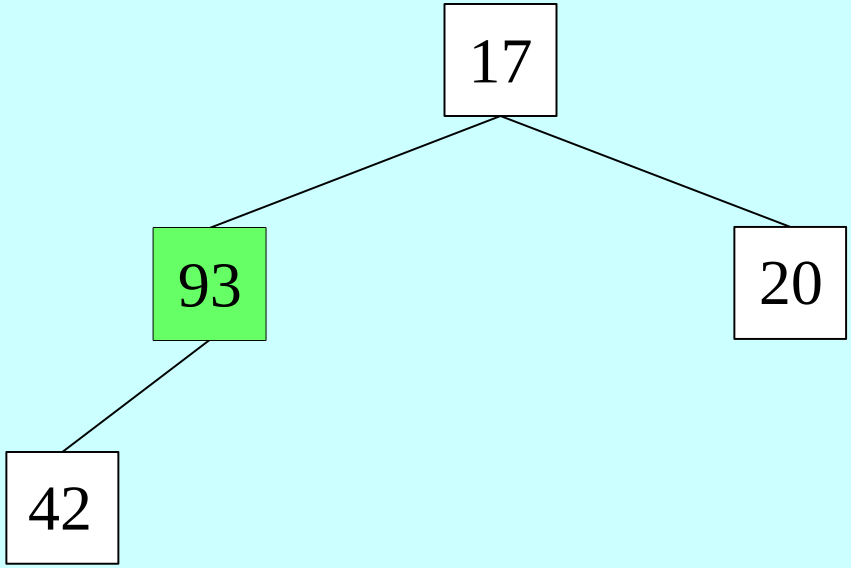

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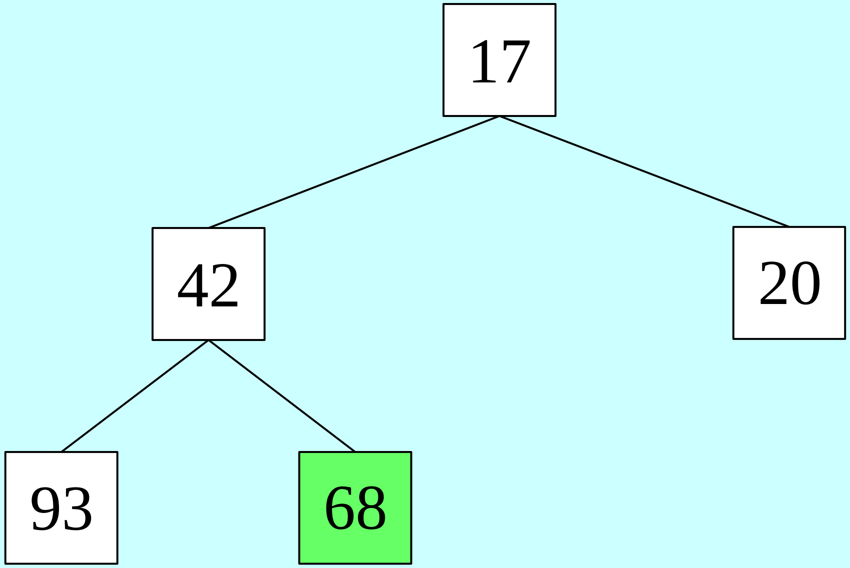

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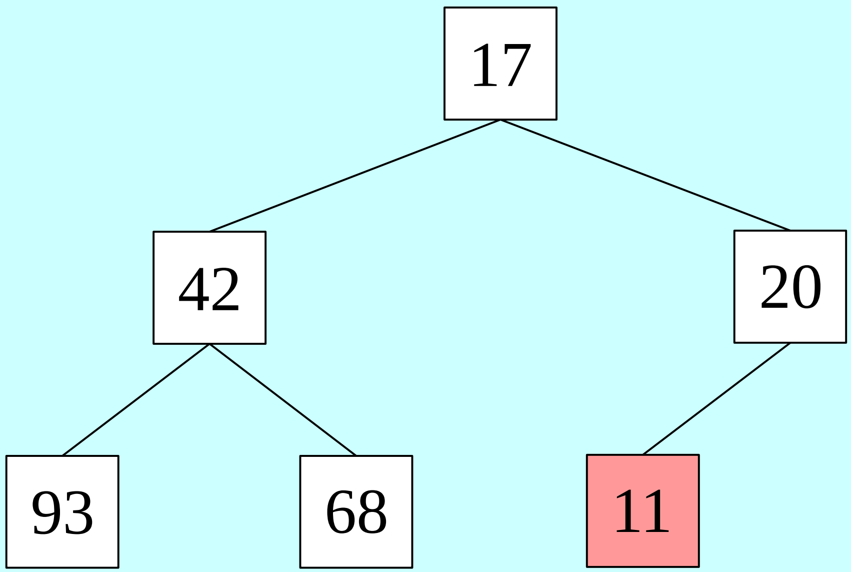

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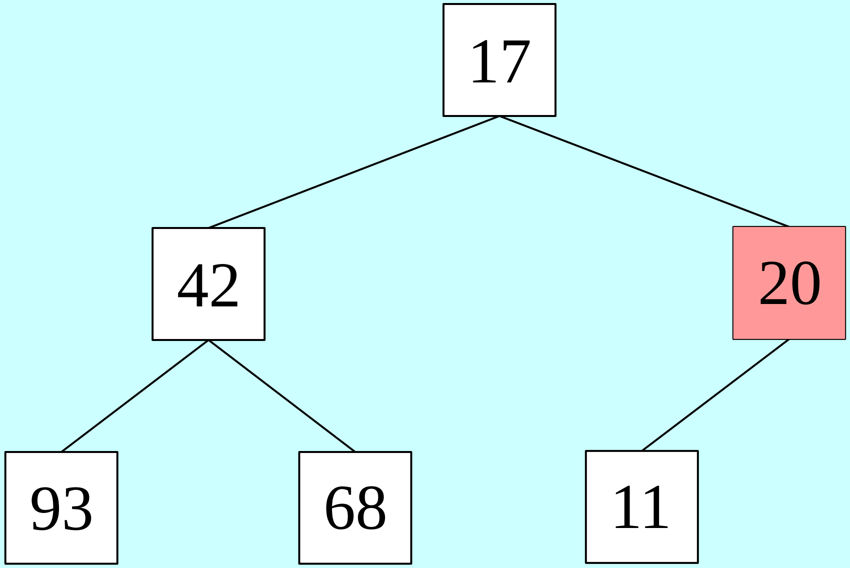
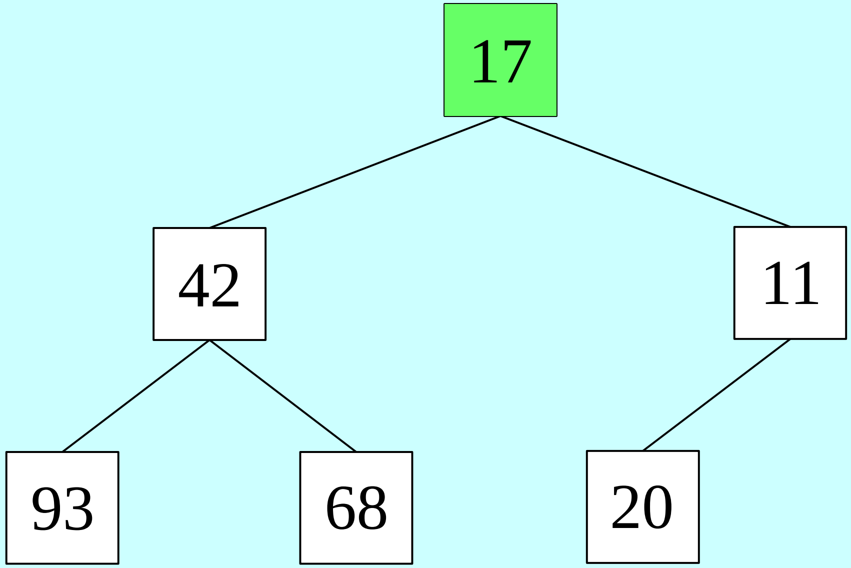


Exercise: Tracing the Heap Algorithm

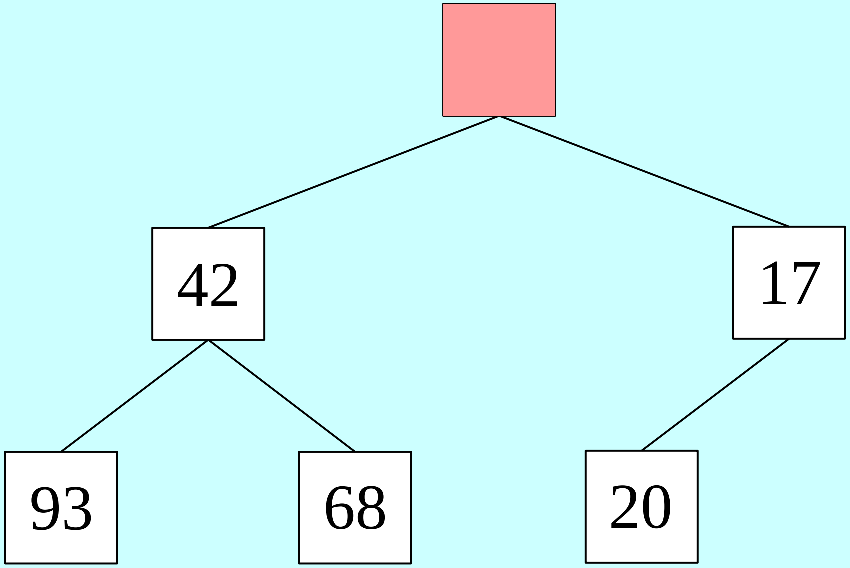
Insert in order: 17 93 20 42 68 11



Exercise: Tracing the Heap Algorithm

Insert in order: 17 93 20 42 68 11

Dequeue the top priority element → 11



Exercise: Tracing the Heap Algorithm

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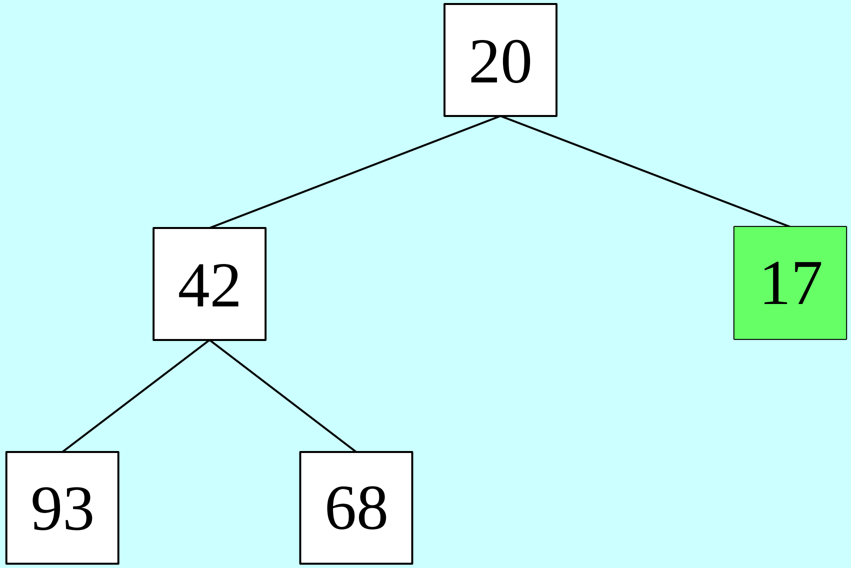
Dequeue the top priority element $\rightarrow$ 11



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