Cardinality

Outline for Today

- **Recap from Last Time**
  - Revisiting our first lecture with our new techniques!
- **Equal Cardinalities**
  - How do we know when two sets have the same size?
- **Ranking Cardinalities**
  - When can we say one set is no larger than another?
- **Unequal Cardinalities**
  - How do we prove two sets *don't* have the same size?

Injections and Surjections

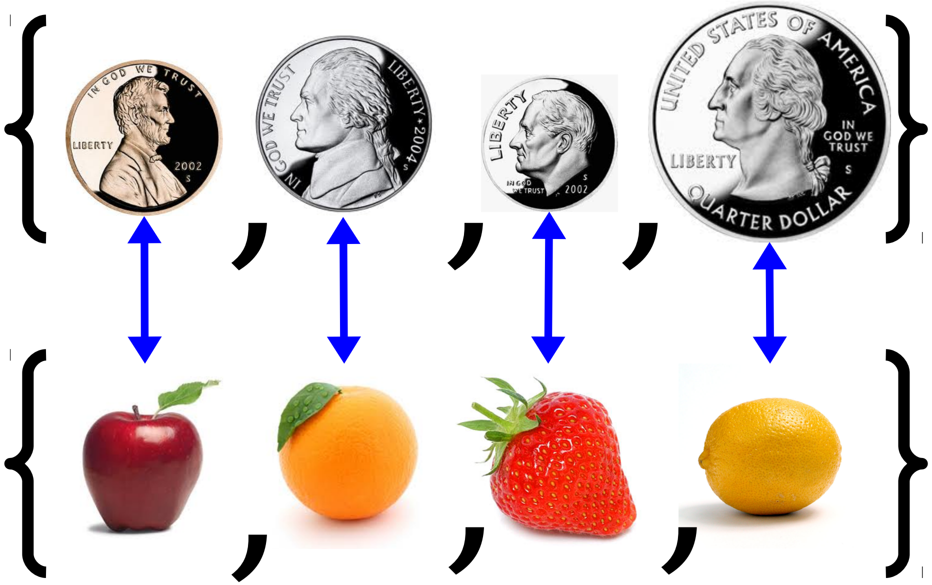
- An ***injective function*** associates ***at most*** one element of the domain with each element of the codomain.
- A ***surjective function*** associates ***at least*** one element of the domain with each element of the codomain.
- A ***bijection*** is a function that is injective and surjective.

Cardinality Revisited

Comparing Cardinalities

- The relationships between set cardinalities are defined in terms of functions between those sets.
- $|S| = |T|$ is defined using bijections.

$|S| = |T|$ if there exists a *bijection* $f : S \to T$



The Cartesian Product

- The ***Cartesian product*** of $A \times B$ of two sets is defined as

$$A \times B = \{ (a, b) \mid a \in A \text{ and } b \in B \}$$

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$$\{0, 1, 2\} \qquad \{\text{a, b, c}\}$$

A B

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$$\underbrace{\left\{ 0, 1, 2 \right\}}_{A} \times \underbrace{\left\{ a, b, c \right\}}_{B} =$$

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$$\underset{A}{\{\, 0,\ 1,\ 2 \,\}} \times \underset{B}{\{\, a,\ b,\ c \,\}} =$$

	a	**b**	**c**
0			
1			
2			

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$$\underbrace{\{\ 0,\ 1,\ 2\ \}}_{A} \times \underbrace{\{\ a,\ b,\ c\ \}}_{B} =$$

	a	**b**	**c**
0	**(0, a)**	**(0, b)**	**(0, c)**
1	**(1, a)**	**(1, b)**	**(1, c)**
2	**(2, a)**	**(2, b)**	**(2, c)**

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$$\underbrace{\{\, 0,\ 1,\ 2 \,\}}_{A} \times \underbrace{\{\, a,\ b,\ c \,\}}_{B} = \left\{ \begin{array}{l} (0,\ a),(0,\ b),(0,\ c), \\ (1,\ a),(1,\ b),(1,\ c), \\ (2,\ a),(2,\ b),(2,\ c) \end{array} \right\}$$

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- We denote $A^2 = A \times A$

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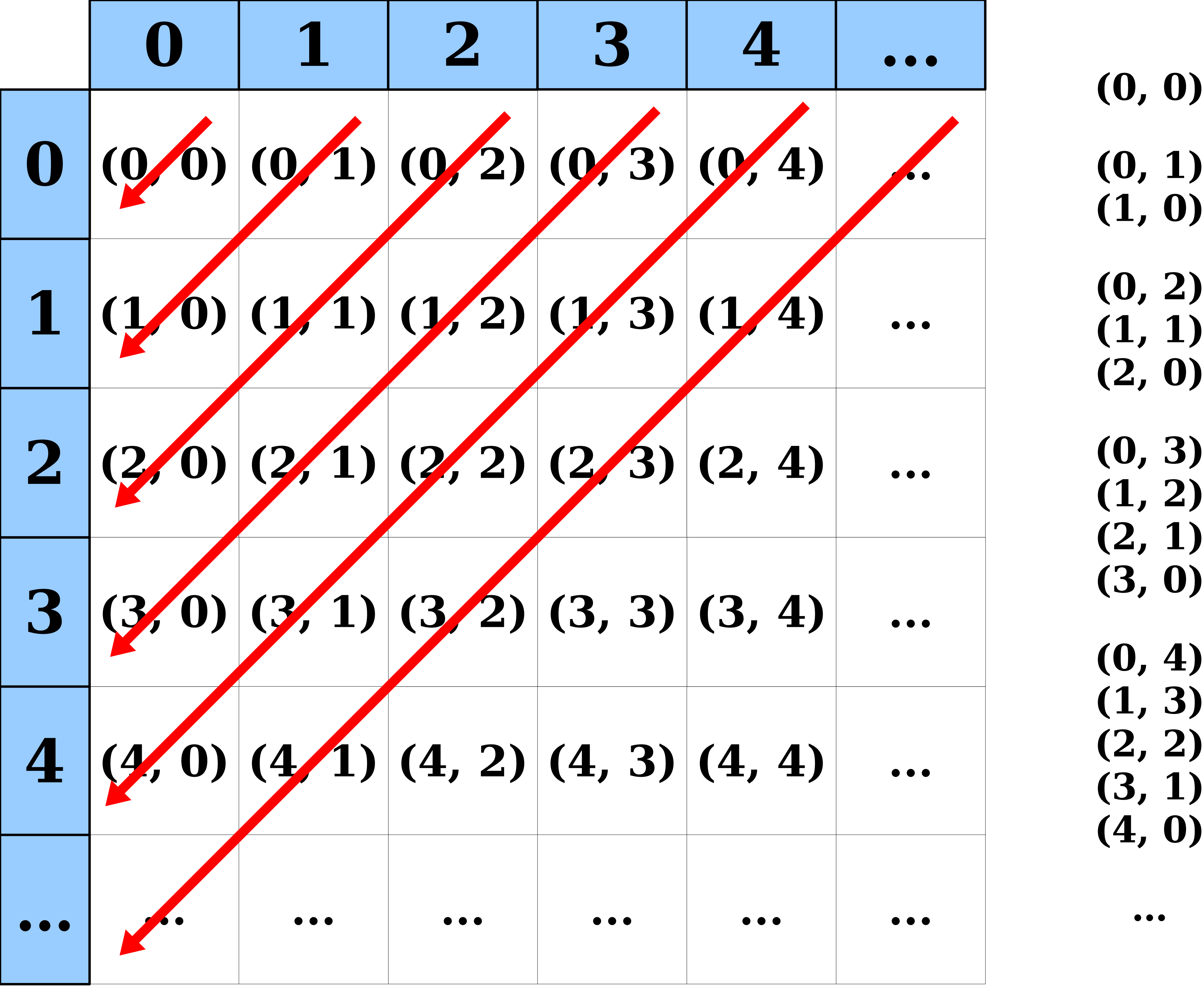
$$\underbrace{\{0, 1, 2\}^2}_{A^2} = \left\{ \begin{array}{l} (0, 0),(0, 1),(0, 2), \\ (1, 0),(1, 1),(1, 2), \\ (2, 0),(2, 1),(2, 2) \end{array} \right\}$$

What is $|\mathbb{N}^2|$?


0 1 2 3 4 ...
0 (0, 0) (0, 1) (0, 2) (0, 3) (0, 4) ...
1 (1, 0) (1, 1) (1, 2) (1, 3) (1, 4) ...
2 (2, 0) (2, 1) (2, 2) (2, 3) (2, 4) ...
3 (3, 0) (3, 1) (3, 2) (3, 3) (3, 4) ...
4 (4, 0) (4, 1) (4, 2) (4, 3) (4, 4) ...
... ...
(0, 0)
(0, 1)
(1, 0)
(0, 2)
(1, 1)
(2, 0)
(0, 3)
(1, 2)
(2, 1)
(3, 0)
(0, 4)
(1, 3)
(2, 2)
(3, 1)
(4, 0)
...

Diagonal 0

$f(0, 0) = 0$

Diagonal 1

$f(0, 1) = 1$
$f(1, 0) = 2$

Diagonal 2

$f(0, 2) = 3$
$f(1, 1) = 4$
$f(2, 0) = 5$

Diagonal 3

$f(0, 3) = 6$
$f(1, 2) = 7$
$f(2, 1) = 8$
$f(3, 0) = 9$

Diagonal 4

$f(0, 4) = 10$
$f(1, 3) = 11$
$f(2, 2) = 12$
$f(3, 1) = 13$
$f(4, 0) = 14$

$$f(a, b) = \begin{array}{c}\text{The number of elements on all previous diagonals} \\ + \\ \text{The index of the current pair on its diagonal}\end{array}$$

Diagonal 0

$f(0, 0) = 0$

Diagonal 1

$f(0, 1) = 1$
$f(1, 0) = 2$

Diagonal 2

$f(0, 2) = 3$
$f(1, 1) = 4$
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$f(0, 4) = 10$
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$f(4, 0) = 14$

$$f(a, b) = \begin{array}{c} (a + b)(a + b + 1) / 2 \\ + \\ \text{The index of the current} \\ \text{pair on its diagonal} \end{array}$$

Diagonal 0

$f(0, 0) = 0$

Diagonal 1

$f(0, 1) = 1$
$f(1, 0) = 2$

Diagonal 2

$f(0, 2) = 3$
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Diagonal 4

$f(0, 4) = 10$
$f(1, 3) = 11$
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$f(3, 1) = 13$
$f(4, 0) = 14$

$$f(a, b) = \begin{matrix} (a + b)(a + b + 1) / 2 \\ + \\ a \end{matrix}$$

Diagonal 0

$f(0, 0) = 0$

Diagonal 1

$f(0, 1) = 1$
$f(1, 0) = 2$

Diagonal 2

$f(0, 2) = 3$
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$$f(a, b) = (a + b)(a + b + 1) / 2 + a$$

This function is called
Cantor's Pairing Function.

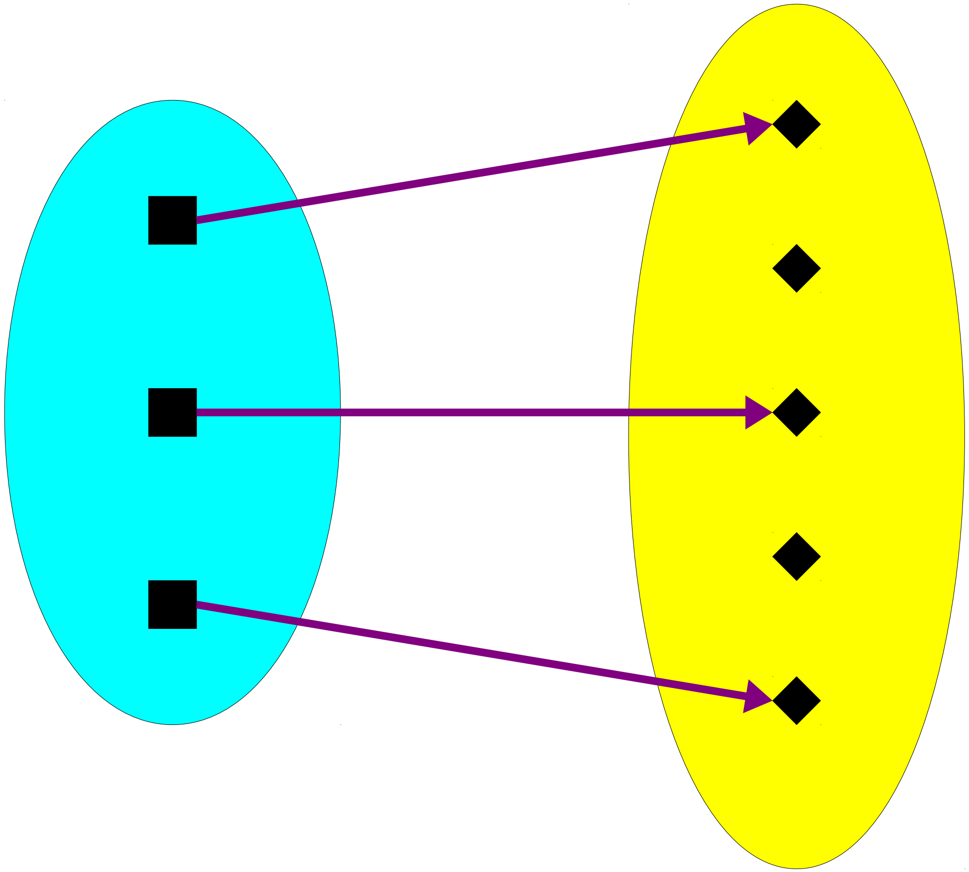
$\mathbb{N}$ and $\mathbb{N}^2$

- ***Theorem:*** $|\mathbb{N}| = |\mathbb{N}^2|$.
- The proof involves showing that the Cantor pairing function actually is a bijection.
- Lots of icky tricky math to prove this; it's left as an Exercise to the Very Bored Reader.

Ranking Cardinalities

- We define $|S| \leq |T|$ as follows:

  $|S| \leq |T|$ if there is an injection $f : S \rightarrow T$



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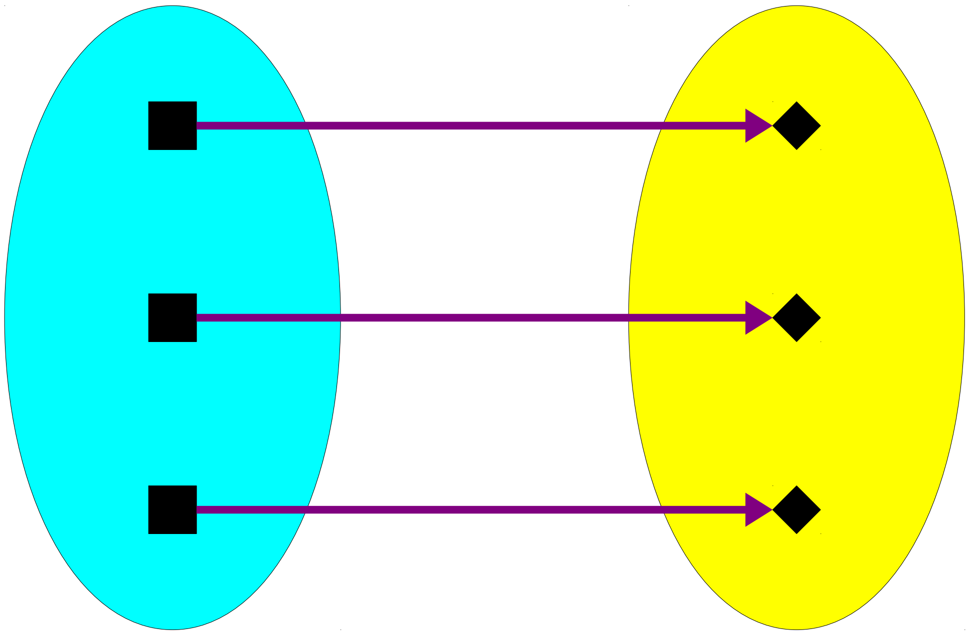


Ranking Cardinalities

- We define $|S| \leq |T|$ as follows:

  $|S| \leq |T|$ if there is an injection $f : S \to T$

- For any sets R, S, and T:
  - $|S| \leq |S|$.
  - If $|R| \leq |S|$ and $|S| \leq |T|$, then $|R| \leq |T|$.
  - Either $|S| \leq |T|$ or $|T| \leq |S|$.

Theorem (Cantor-Bernstein-Schroeder): If S and T are sets where $|S| \leq |T|$ and $|T| \leq |S|$, then $|S| = |T|$

(This was first proven by Richard Dedekind.)

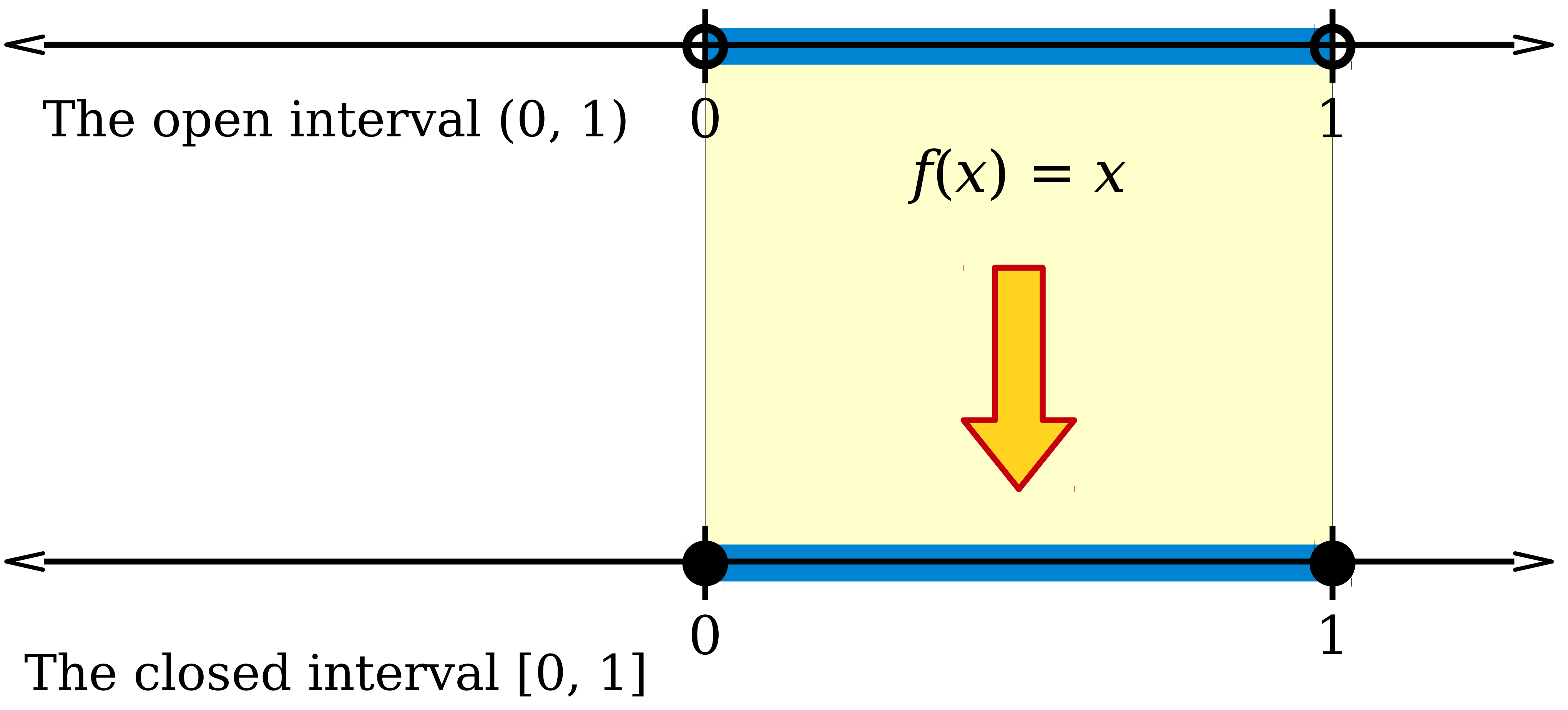
The CBS Theorem

- ***Theorem:*** If $|S| \leq |T|$ and $|T| \leq |S|$, then $|S| = |T|$.
- Isn't this, kinda, you know, obvious?
- Look at the definitions. What does the above theorem actually say?

If there is an injection $f : S \to T$ and an injection $g : T \to S$, then there must be some bijection $h : S \to T$.

- This is much less obvious than it looks.

Why CBS is Tricky



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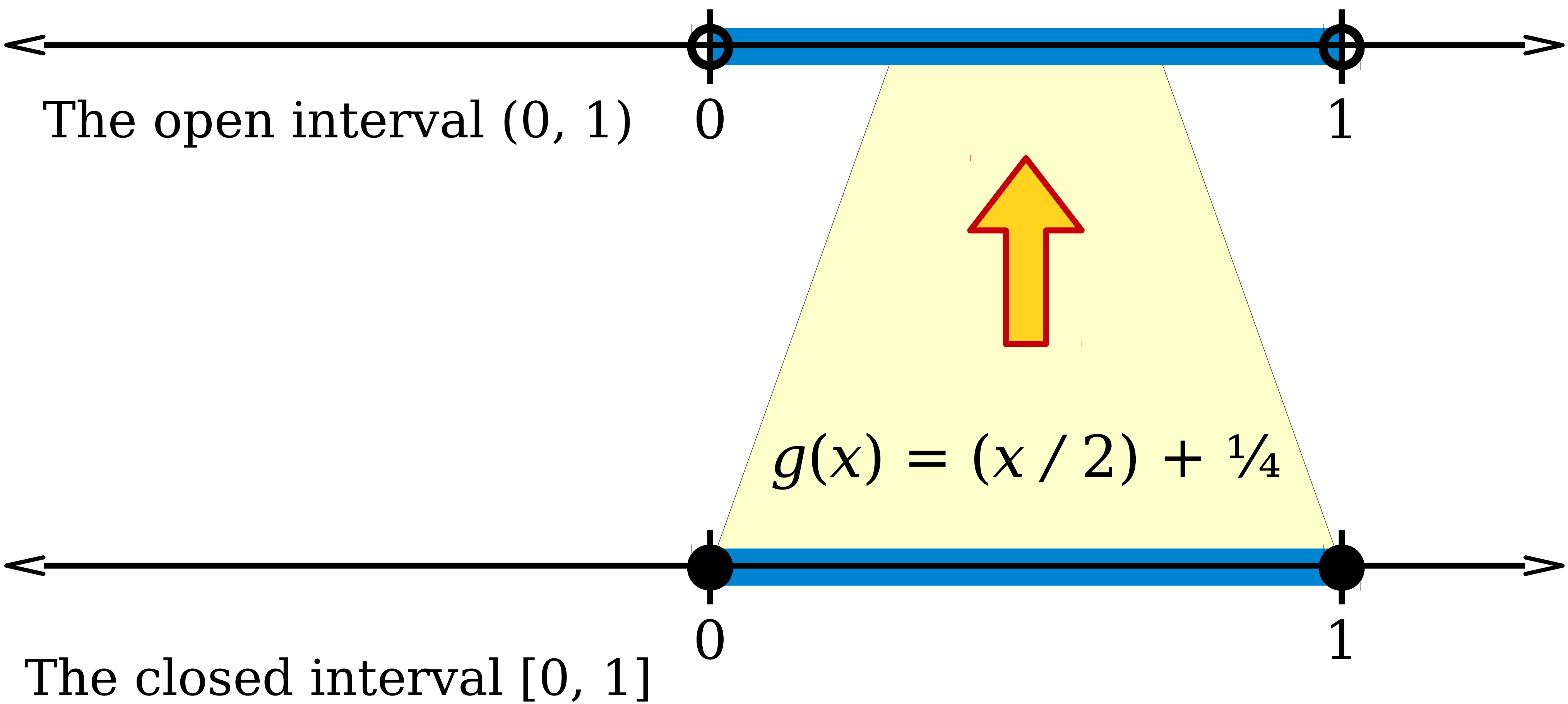

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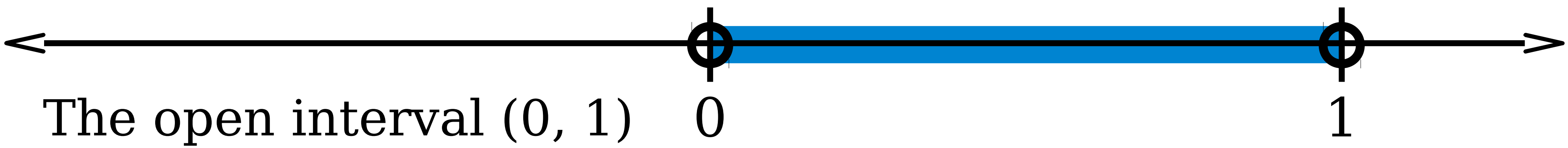
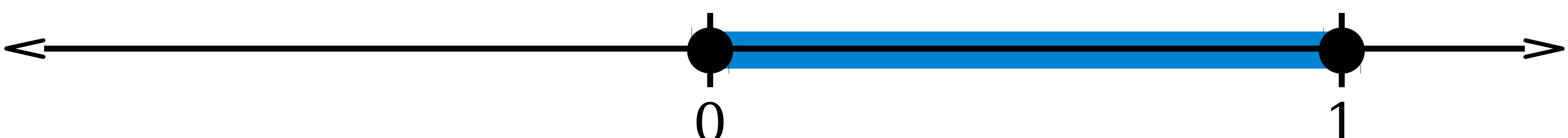


The open interval (0, 1)

There has to be a bijection between these two sets… so what is it?



The closed interval [0, 1]

Proving CBS, Intuitively

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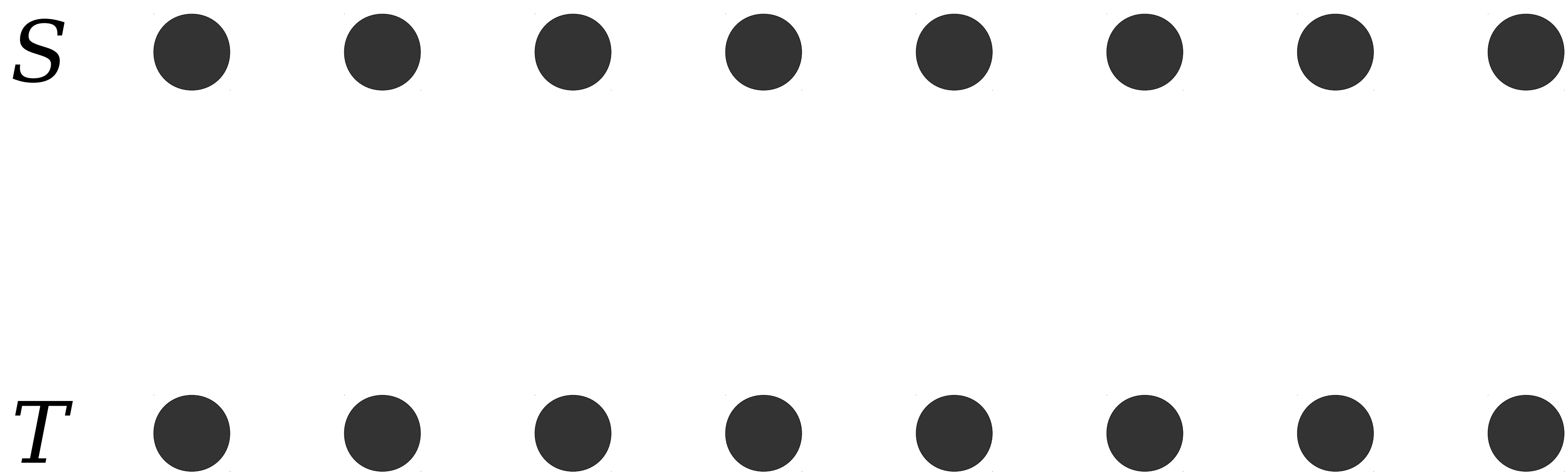

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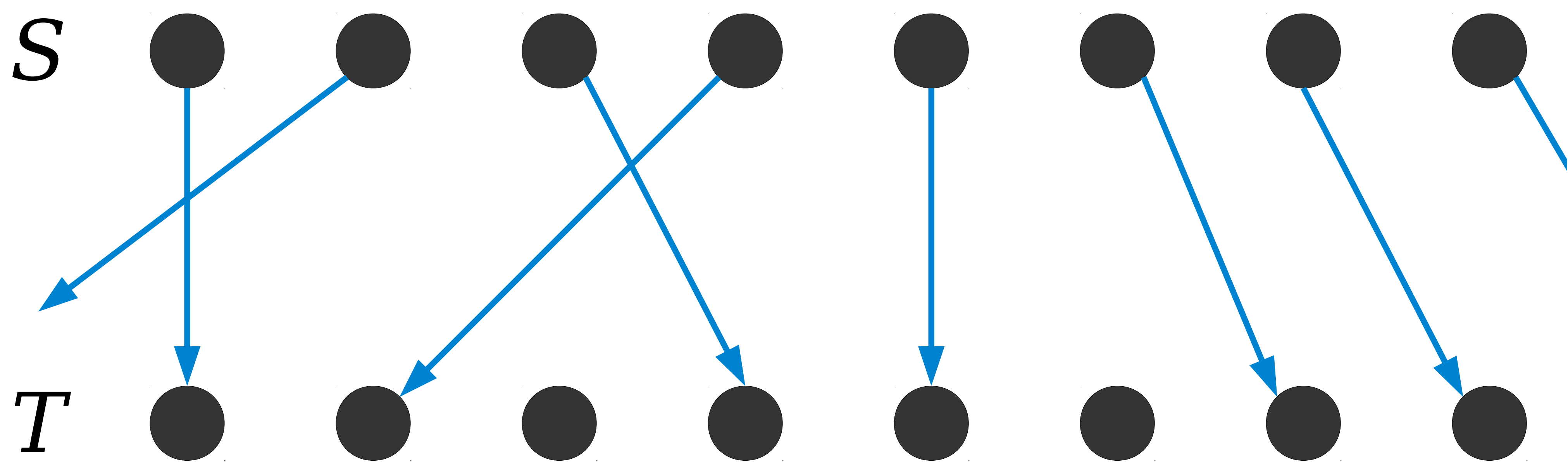

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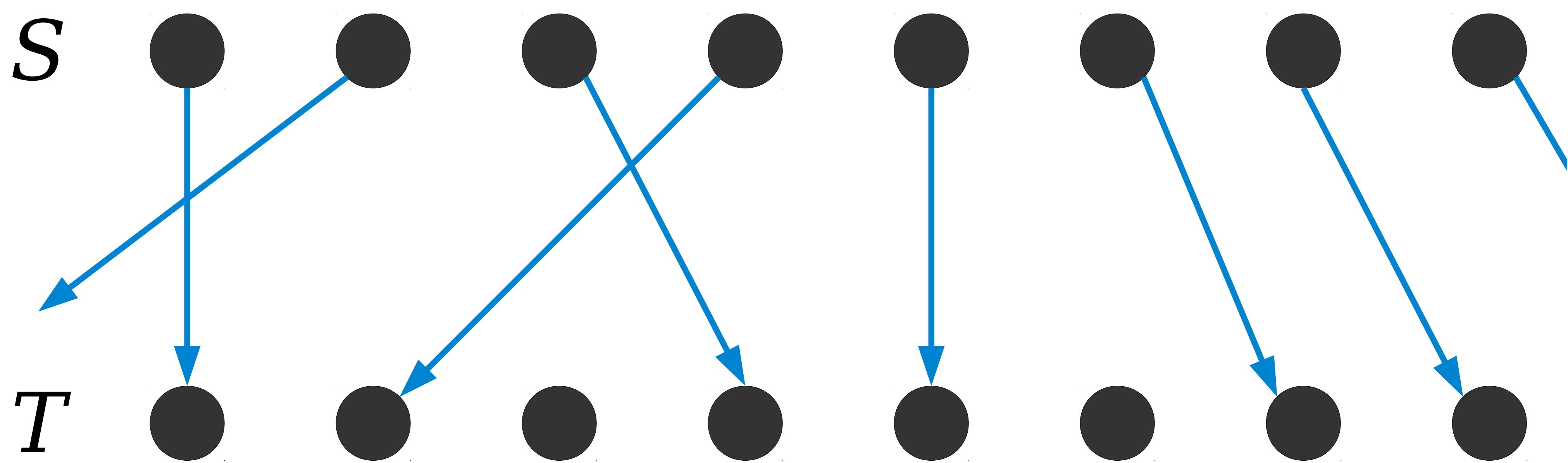
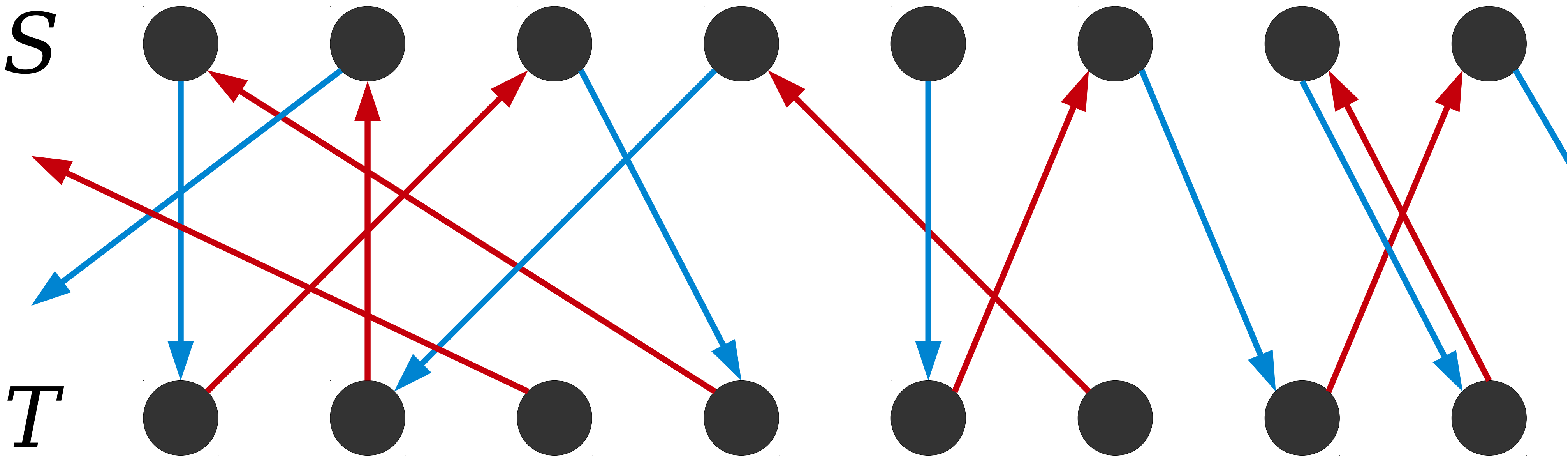


Blue lines represent the injection $f : S \to T$

Proving CBS, Intuitively



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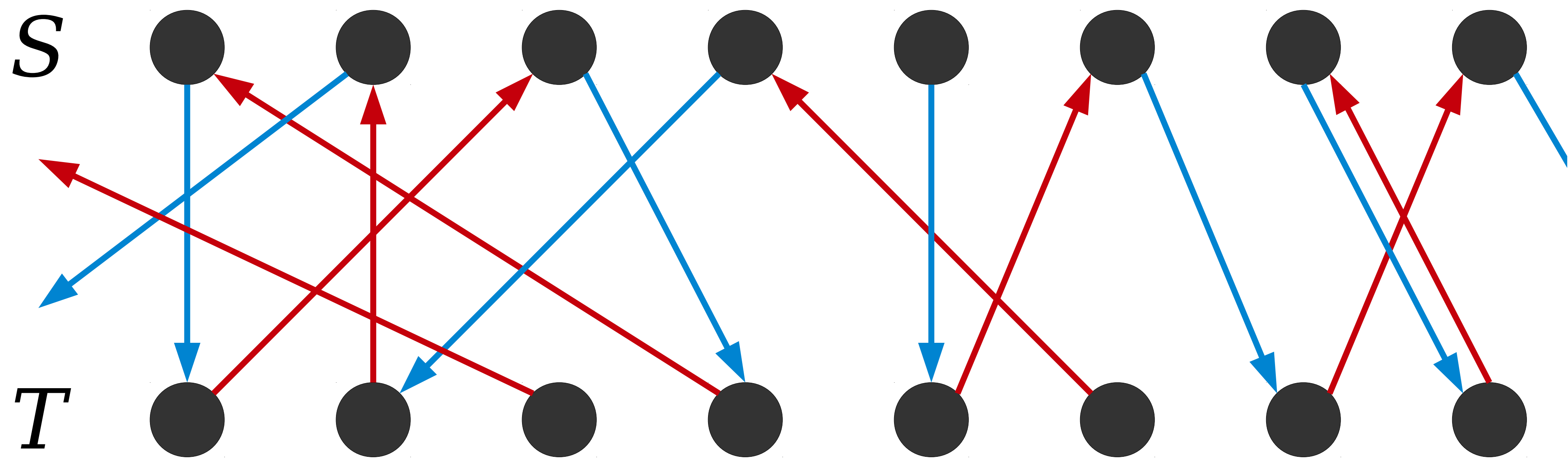
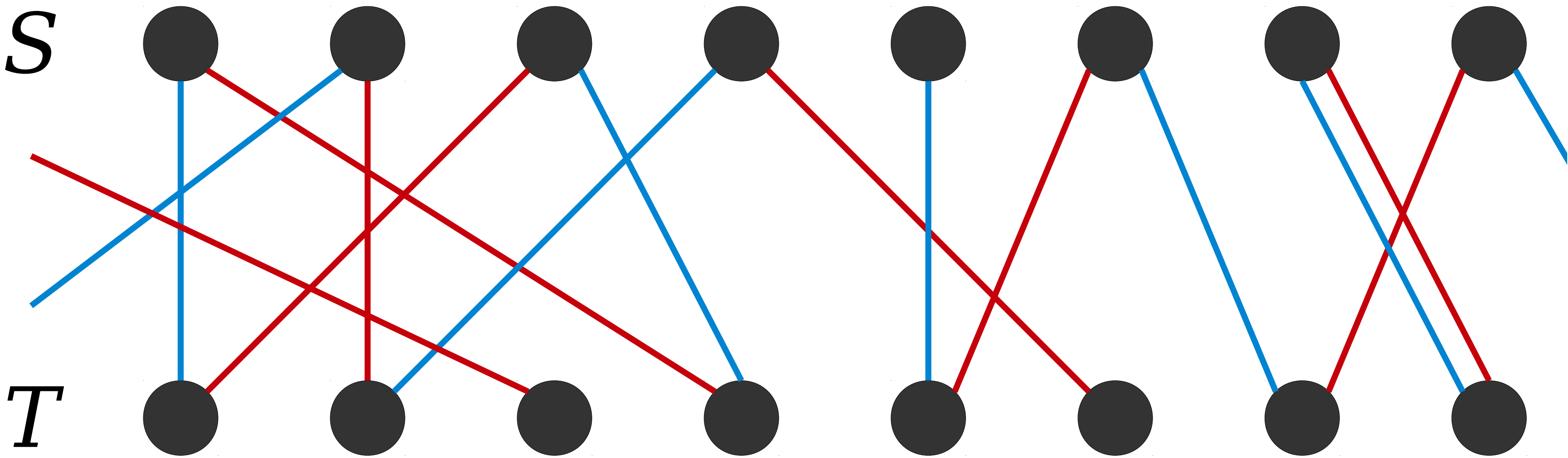


Blue lines represent the injection $f : S \to T$
Red lines represent the injection $g : T \to S$

Proving CBS, Intuitively



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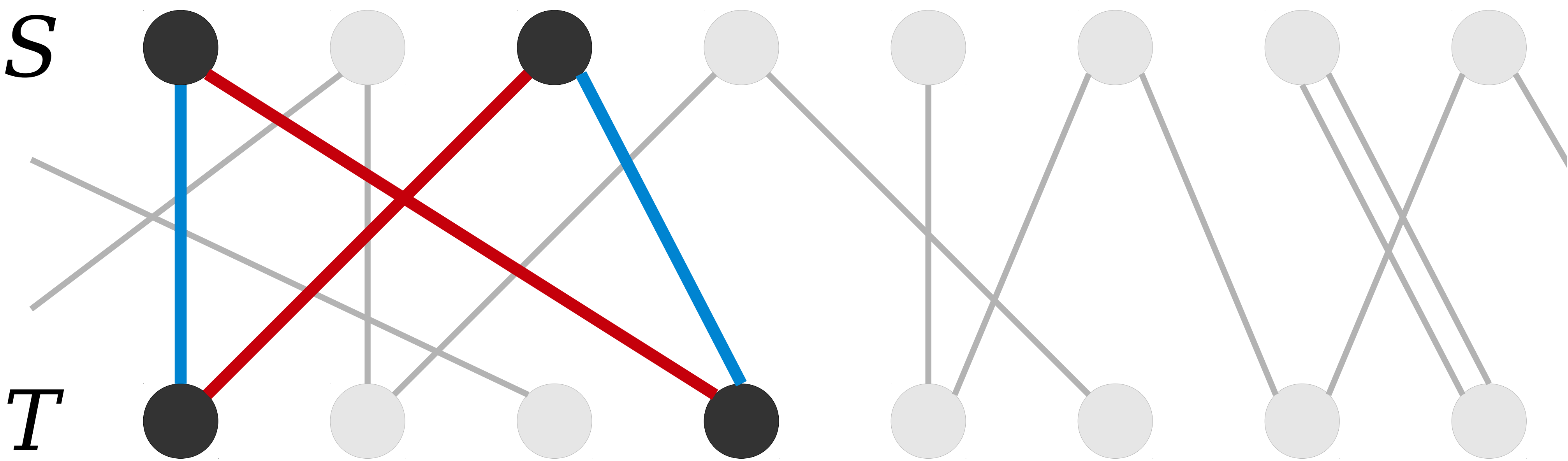

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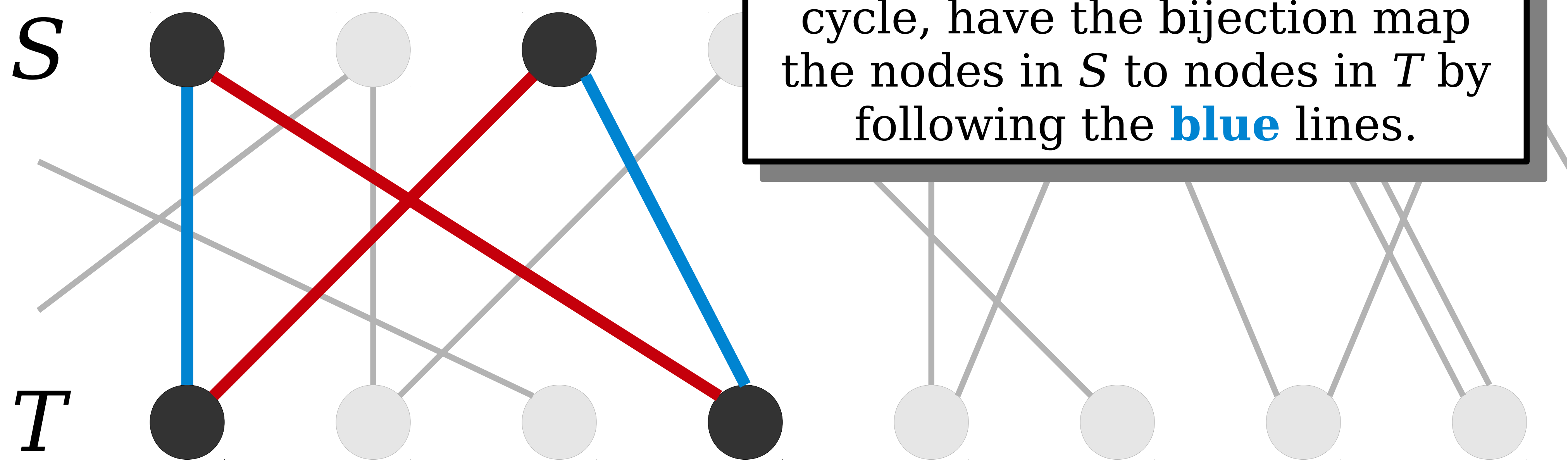
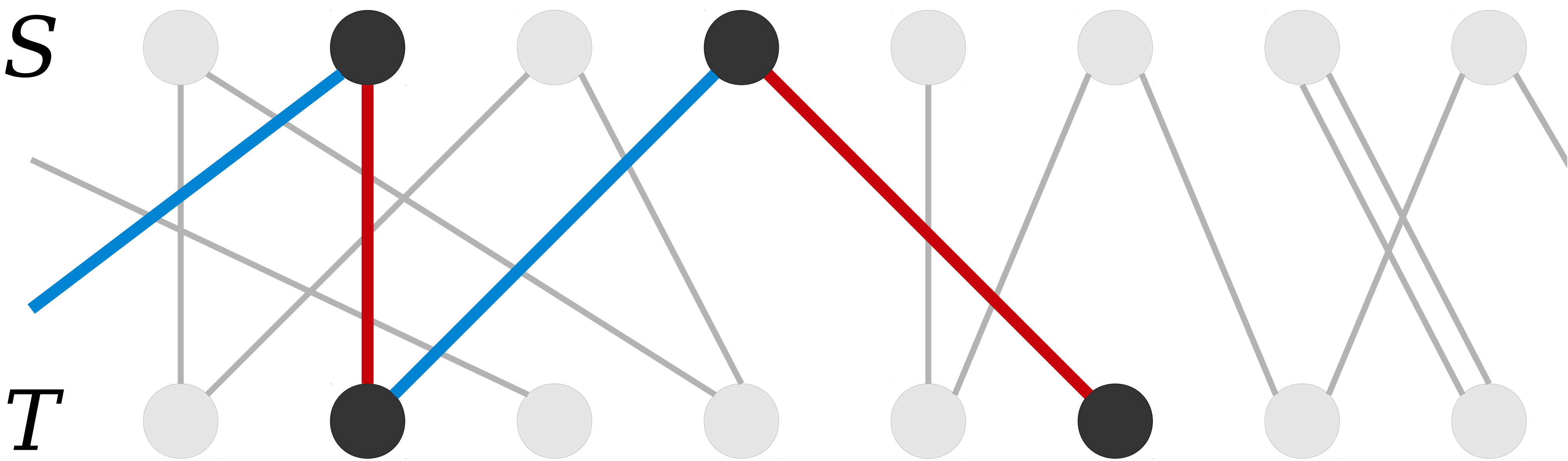


If the connected component is a cycle, have the bijection map the nodes in S to nodes in T by following the **blue** lines.

Blue lines represent the injection $f : S \to T$
Red lines represent the injection $g : T \to S$

Proving CBS, Intuitively



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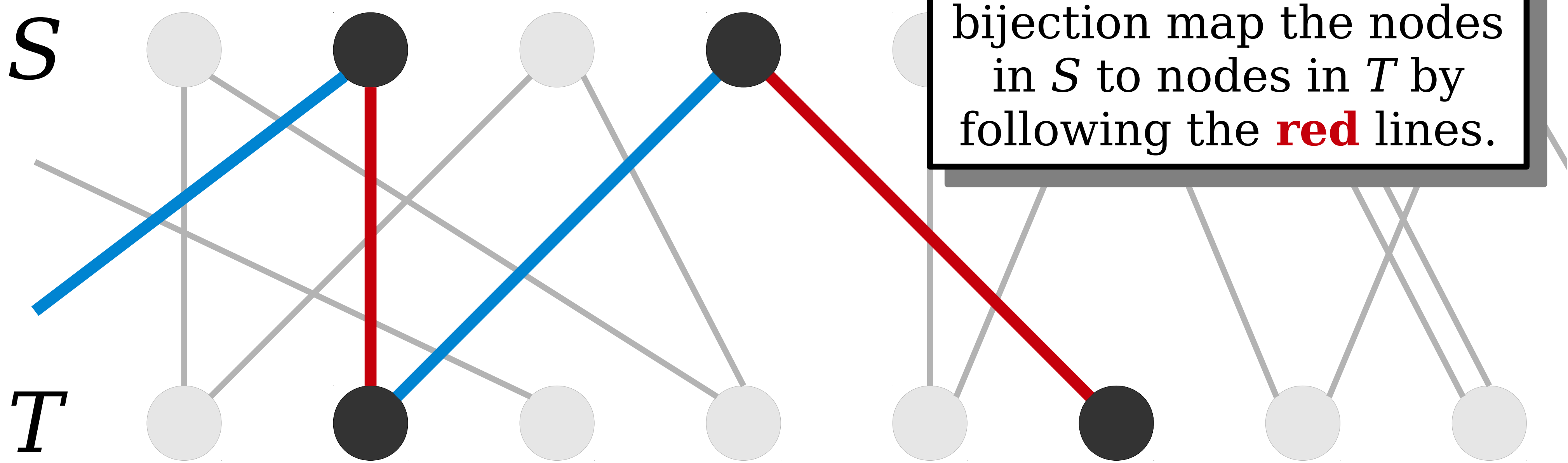
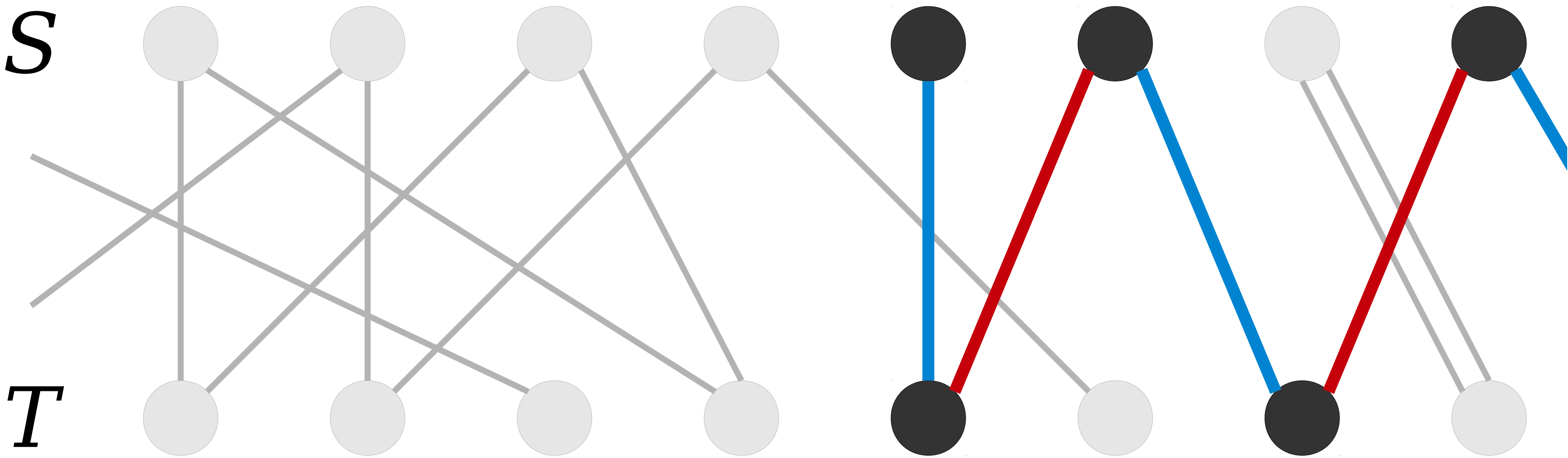
Proving CBS, Intuitively



If the CC is an infinite path starting with a node in T, have the bijection map the nodes in S to nodes in T by following the **red** lines.

Blue lines represent the injection $f : S \to T$
Red lines represent the injection $g : T \to S$

Proving CBS, Intuitively



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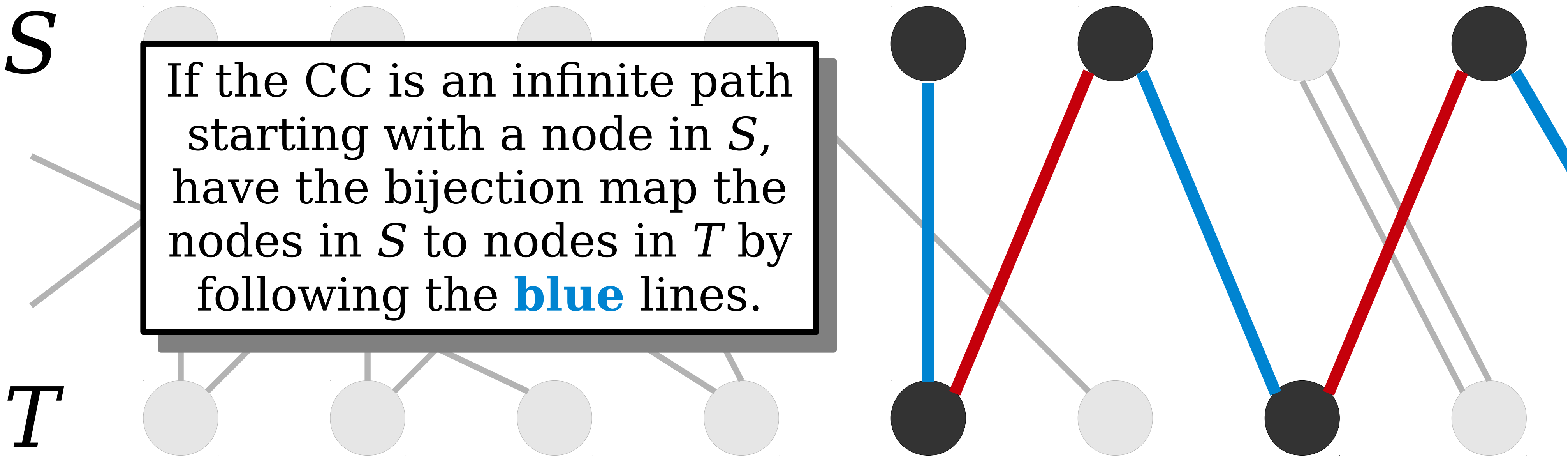

If the CC is an infinite path starting with a node in S, have the bijection map the nodes in S to nodes in T by following the **blue** lines.

Blue lines represent the injection $f : S \to T$
Red lines represent the injection $g : T \to S$

Why This Matters

- I chose to sketch out the proof of the CBS theorem because it combines so many different pieces:
  - Bipartite graphs, connected components, paths, cycles, etc.
- Don't worry too much about the specifics of this proof. Think of it as more of a “math symphony.” ☺

Time-Out for Announcements!

Problem Set Four

- Problem Set Four goes out right now.
  - Checkpoint problem due on Monday at the start of class.
  - Full problem set due one week from Monday at the start of class.
- Play around with functions, cardinality, the pigeonhole principle, and more!

Midterm Logistics

- Midterm exam is next Thursday, 6PM – 9PM.
- Room locations divvied up by last name:
  - Abr – Hsu: Go to **420-040**
  - Hua – Med: Go to **420-041**
  - Mei – Raj: Go to **Art 2**
  - Ram – Saw: Go to **Art 4**
  - Sch – Zie: Go to **Bishop Auditorium**
- Alternate exam locations will be handled over email; please let us know immediately if you need to take the exam at an alternate time and haven't done so already.

Midterm Logistics

- Exam is closed-book, closed-computer.
- You may have one double-sided sheet of notes on 8.5” × 11” paper.
- Covers material up through and including first-order logic.

Extra Review

- We released a set of practice problems on Wednesday; solutions available outside lecture today.
  - SCPD: They should arrive soon.
- New set of practice problems available online. Solutions go out on Monday.

Practice Midterm

- We will be holding a practice midterm exam from 7PM – 10PM on Monday night in Annenberg Auditorium.
- Purely optional, but would be a great way to practice for the exam.
- Course staff will be available outside the practice exam to answer your questions afterwards.
- We'll release the practice exam questions on Monday night / Tuesday morning.

Discussion Sections

- We're converting all discussion sections next week into midterm review sessions.
- Feel free to stop by to work on the extra review problems, ask for general help or advice, etc.

Theorem and Definition Reference

- On the CS103 website, we've posted a *Theorem and Definition Reference* defining important terms and theorems from the course.
- Each entry links to the corresponding lecture slides.
- We hope this helps you review for the upcoming exam (and saves you lots of time searching through slides!)

Updated Regrade Policy

- Want to ask for a regrade? Send an email to this email list:

  cs103-aut1415-regrades@lists.stanford.edu

- Please let us know
  - what you'd like us to look at,
  - why you think the deduction is incorrect, and
  - what you'd like us to do to correct it.
- Please submit regrade requests no more than a week after the problem sets are returned. We'll try to get back to you as soon as we can.

#include Fellowship

- she++ is organizing a mentorship program for high-school students learning CS.
- Interested in being a mentor? Apply online at http://tinyurl.com/advisorsignup, or go to http://sheplusplus.com/include for more information.

Your Questions

“How can CS do more than just change the world? How can it help it in ways it really needs? When I go to EPA, I see more costs than benefits from Silicon Valley.”

“What are your favorite books, be they novels or general reading, touching on topics similar to the course material (or in general)?”

Back to CS103!

Differing Infinities

Unequal Cardinalities

- Recall: $|A| = |B|$ if the following statement is true:

  There exists a bijection $f : A \to B$

- What does it mean for $|A| \neq |B|$?

  No function $f : A \to B$ is a bijection.

- Need to show that, out of all the (potentially infinitely many) functions from A to B, *not one of them* is a bijection.

What is the relation between $|\mathbb{N}|$ and $|\mathbb{R}|$?

Theorem: $|\mathbb{N}| \neq |\mathbb{R}|$

Our Goal

- We need to show the following:

  No function $f : \mathbb{N} \to \mathbb{R}$ is a bijection

- To prove it, we will do the following:
  - Choose an arbitrary function $f : \mathbb{N} \to \mathbb{R}$.
  - Show that f cannot be a surjection by finding some $r \in \mathbb{R}$ that is not mapped to by f.
  - Conclude that this arbitrary function f is not a bijection, so no bijections from $\mathbb{N}$ to $\mathbb{R}$ exist.

The Intuition

- Suppose we have a function $f : \mathbb{N} \to \mathbb{R}$.
- We can then list off an infinite sequence of real numbers

$$r_0, r_1, r_2, r_3, r_4, \ldots$$

by setting $r_n = f(n)$.

- We will show that we can always find a real number d such that

$$\textbf{If } \boldsymbol{n \in \mathbb{N}}\textbf{, then } \boldsymbol{r_n \neq d}\textbf{.}$$

Rewriting Our Constraints

- Our goal is to find some $d \in \mathbb{R}$ such that

$$\boldsymbol{\text{If } n \in \mathbb{N}, \text{ then } r_n \neq d.}$$

- In other words, we want to pick d such that

$$r_0 \neq d$$

$$r_1 \neq d$$

$$r_2 \neq d$$

$$r_3 \neq d$$

$$\ldots$$

The Critical Insight

- **Key Proof Idea:** Build the real number d out of infinitely many "pieces," with one piece for each number r_n.
  - Choose the 0th piece such that $r_0 \neq d$.
  - Choose the 1st piece such that $r_1 \neq d$.
  - Choose the 2nd piece such that $r_2 \neq d$.
  - Choose the 3rd piece such that $r_3 \neq d$.
  - …
- Building a "frankenreal" out of infinitely many pieces of other real numbers.

Building our "Frankenreal"

- Goal: build "frankenreal" d out of infinitely many pieces, one for each r_k.
- One idea: Define d via its decimal representation.
- Choose the digits of d as follows:
  - The 0^{th} digit of d is not the same as the 0^{th} digit of r_0.
  - The 1^{st} digit of d is not the same as the 1^{st} digit of r_1.
  - The 2^{nd} digit of d is not the same as the 2^{nd} digit of r_2.
  - …
- So $d \neq r_n$ for any $n \in \mathbb{N}$.

Building our “Frankenreal”

- If r is a real number, define $r[n]$ as follows:
  - $r[0]$ is the integer part of r.
  - $r[n]$ is the nth decimal digit of r, if $n > 0$.
- Examples:
  - $\pi[0] = 3$ $\quad (-e)[0] = -2$ $\quad 5[0] = 5$
  - $\pi[1] = 1$ $\quad (-e)[1] = 7$ $\quad 5[1] = 0$
  - $\pi[2] = 4$ $\quad (-e)[2] = 1$ $\quad 5[2] = 0$
  - $\pi[3] = 1$ $\quad (-e)[3] = 8$ $\quad 5[3] = 0$

Building our "Frankenreal"

- We can now build our frankenreal d.
- Define $d[n]$ as follows:

$$d[n]=\begin{cases}1 & \text{if } r_n[n]=0\\ 0 & \text{otherwise}\end{cases}$$

- Now, $d \neq r_n$ for any $n \in \mathbb{N}$:
  - If $r_n[n] = 0$, then $d[n] = 1$, so $r_n \neq d$.
  - If $r_n[n] \neq 0$, then $d[n] = 0$, so $r_n \neq d$.

0	$\longleftrightarrow$	8.	6	7	5	3	0	...
1	$\longleftrightarrow$	3.	1	4	1	5	9	...
2	$\longleftrightarrow$	0.	1	2	3	5	8	...
3	$\longleftrightarrow$	-1.	0	0	0	0	0	...
4	$\longleftrightarrow$	2.	7	1	8	2	8	...
5	$\longleftrightarrow$	1.	6	1	8	0	3	...
...	$\longleftrightarrow$	...	...	...	...	...	...	...

		d_0	d_1	d_2	d_3	d_4	d_5	...
0	$\longleftrightarrow$	8.	6	7	5	3	0	...
1	$\longleftrightarrow$	3.	1	4	1	5	9	...
2	$\longleftrightarrow$	0.	1	2	3	5	8	...
3	$\longleftrightarrow$	-1.	0	0	0	0	0	...
4	$\longleftrightarrow$	2.	7	1	8	2	8	...
5	$\longleftrightarrow$	1.	6	1	8	0	3	...
...	$\longleftrightarrow$	...	...	...	...	...	...	...

	d_0	d_1	d_2	d_3	d_4	d_5	...
0	8.	6	7	5	3	0	...
1	3.	1	4	1	5	9	...
2	0.	1	2	3	5	8	...
3	-1.	0	0	0	0	0	...
4	2.	7	1	8	2	8	...
5	1.	6	1	8	0	3	...
...	...	...	...	...	...	...	...

	d_0	d_1	d_2	d_3	d_4	d_5	...
0	8.	6	7	5	3	0	...
1	3.	1	4	1	5	9	...
2	0.	1	2	3	5	8	...
3	-1.	0	0	0	0	0	...
4	2.	7	1	8	2	8	...
5	1.	6	1	8	0	3	...
...	...	...	...	...	...	...	...

	d_0	d_1	d_2	d_3	d_4	d_5	...
0	8.	6	7	5	3	0	...
1	3.	1	4	1	5	9	...
2	0.	1	2	3	5	8	...
3	-1.	0	0	0	0	0	...
4	2.	7	1	8	2	8	...
5	1.	6	1	8	0	3	...
...	...	...	...	...	...	...	...

	d_0	d_1	d_2	d_3	d_4	d_5	...
0	8.	6	7	5	3	0	...
1	3.	1	4	1	5	9	...
2	0.	1	2	3	5	8	...
3	-1.	0	0	0	0	0	...
4	2.	7	1	8	2	8	...
5	1.	6	1	8	0	3	...
...	...	...	...	...	...	...	...

8.	1	2	0	2	3	...

	d_0	d_1	d_2	d_3	d_4	d_5	...
0	8.	6	7	5	3	0	...
1	3.	1	4	1	5	9	...
2	0.	1	2	3	5	8	...
3	-1.	0	0	0	0	0	...
4	2.	7	1	8	2	8	...
5	1.	6	1	8	0	3	...
...	...	...	...	...	...	...	...

Set all nonzero values to 0 and all 0s to 1.

0.	0	0	1	0	0	...

	d_0	d_1	d_2	d_3	d_4	d_5	...
0	8.	6	7	5	3	0	...
1	3.	1	4	1	5	9	...
2	0.	1	2	3	5	8	...
3	-1.	0	0	0	0	0	...
4	2.	7	1	8	2	8	...
5	1.	6	1	8	0	3	...
...	...	...	...	...	...	...	...

0.	0	0	1	0	0	...

	d_0	d_1	d_2	d_3	d_4	d_5	...
0	8.	6	7	5	3	0	...
1	3.	1	4	1	5	9	...
2	0.	1	2	3	5	8	...
3	-1.	0	0	0	0	0	...
4	2.	7	1	8	2	8	...
5	1.	6	1	8	0	3	...
...	...	...	...	...	...	...	...

0.	0	0	1	0	0	...

Which natural number is paired with this real number?

	d_0	d_1	d_2	d_3	d_4	d_5	...
0	8.	6	7	5	3	0	...
1	3.	1	4	1	5	9	...
2	0.	1	2	3	5	8	...
3	-1.	0	0	0	0	0	...
4	2.	7	1	8	2	8	...
5	1.	6	1	8	0	3	...
...	...	...	...	...	...	...	...

0.	0	0	1	0	0	...

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5	1.	6	1	8	0	3	...
...	...	...	...	...	...	...	...

Which natural number is paired with this real number?

0.	0	0	1	0	0	...

	d_0	d_1	d_2	d_3	d_4	d_5	...
0	8.	6	7	5	3	0	...
1	3.	1	4	1	5	9	...
2	0.	1	2	3	5	8	...
3	-1.	0	0	0	0	0	...
4	2.	7	1	8	2	8	...
5	1.	6	1	8	0	3	...
...	...	...	...	...	...	...	...

0.	0	0	1	0	0	...

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...	...	...	...	...	...	...	...

Which natural number is paired with this real number?

0.	0	0	1	0	0	...

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...	...	...	...	...	...	...	...

Which natural number is paired with this real number?

0.	0	0	1	0	0	...

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...	...	...	...	...	...	...	...

0.	0	0	1	0	0	...

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...	...	...	...	...	...	...	...

0.	0	0	1	0	0	...

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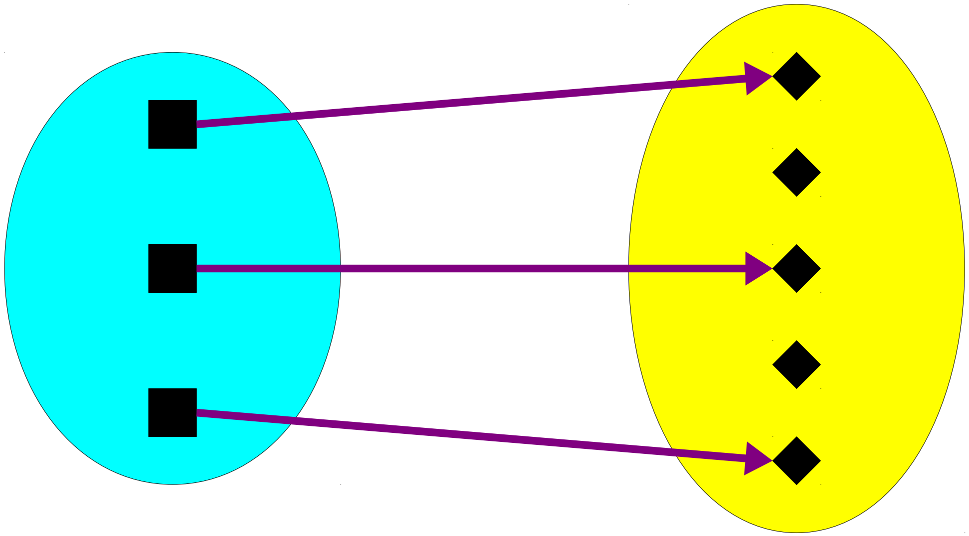
A Silly Observation…

Comparing Cardinalities

- Formally, we define < on cardinalities as

$$|S| < |T| \text{ iff } |S| \leq |T| \text{ and } |S| \neq |T|$$

- In other words:
  - There is an injection from S to T.
  - There is no bijection between S and T.



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- In other words:
  - There is an injection from S to T.
  - There is no bijection between S and T.
- ***Theorem:*** For any sets S and T, exactly one of the following is true:

$$\mathbf{|S| < |T| \qquad |S| = |T| \qquad |S| > |T|}$$

Theorem: $|\mathbb{N}| \leq |\mathbb{R}|$.

Proof: We exhibit an injection from $\mathbb{N}$ to $\mathbb{R}$. Let $f(n) = n$. Then $f : \mathbb{N} \to \mathbb{R}$, since every natural number is also a real number.

We further claim that f is an injection. To see this, suppose that for some $n_0, n_1 \in \mathbb{N}$ that $f(n_0) = f(n_1)$. We will prove that $n_0 = n_1$. To see this, note that

$$n_0 = f(n_0) = f(n_1) = n_1$$

Thus $n_0 = n_1$, as required, so f is an injection from $\mathbb{N}$ to $\mathbb{R}$. Therefore, $|\mathbb{N}| \leq |\mathbb{R}|$. ■

Cantor's Theorem Revisited

Cantor's Theorem

- ***Cantor's Theorem*** is the following:

  If S is a set, then $|S| < |\wp(S)|$

- This is how we concluded that there are more problems to solve than programs to solve them.
- We informally sketched a proof of this in the first lecture.
- Let's now formally prove Cantor's Theorem.

The Key Step

- We need to show the following:

If S is a set, then $|S| \neq |\wp(S)|$.

- To do this, we need to prove this statement:

For any set S, every function $f : S \to \wp(S)$ is not a bijection.

- We'll do this using a proof by diagonalization.

x_0

x_1

x_2

x_3

x_4

x_5

$\ldots$

$$
\begin{aligned}
x_0 &\longleftrightarrow \{\, x_0, x_2, x_4, \ldots \,\} \\
x_1 &\longleftrightarrow \{\, x_0, x_3, x_4, \ldots \,\} \\
x_2 &\longleftrightarrow \{\, x_4, \ldots \,\} \\
x_3 &\longleftrightarrow \{\, x_1, x_4, \ldots \,\} \\
x_4 &\longleftrightarrow \{\, x_0, x_5, \ldots \,\} \\
x_5 &\longleftrightarrow \{\, x_0, x_1, x_2, x_3, x_4, x_5, \ldots \,\} \\
&\ldots
\end{aligned}
$$

x_0	x_1	x_2	x_3	x_4	x_5	…

$x_0 \longleftrightarrow \{ x_0, x_2, x_4, \dots \}$

$x_1 \longleftrightarrow \{ x_0, x_3, x_4, \dots \}$

$x_2 \longleftrightarrow \{ x_4, \dots \}$

$x_3 \longleftrightarrow \{ x_1, x_4, \dots \}$

$x_4 \longleftrightarrow \{ x_0, x_5, \dots \}$

$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**

$x_1 \leftrightarrow \{ x_0, x_3, x_4, \ldots \}$

$x_2 \leftrightarrow \{ x_4, \ldots \}$

$x_3 \leftrightarrow \{ x_1, x_4, \ldots \}$

$x_4 \leftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

		x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	$\leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**...**
x_1	$\leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**...**

$x_2 \leftrightarrow \{ x_4, \ldots \}$

$x_3 \leftrightarrow \{ x_1, x_4, \ldots \}$

$x_4 \leftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

...

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**

$x_2 \leftrightarrow \{ \; x_4, \; \ldots \}$

$x_3 \leftrightarrow \{ x_1, x_4, \ldots \}$

$x_4 \leftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \longleftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \longleftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \longleftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**

$x_3 \longleftrightarrow \{ x_1, x_4, \ldots \}$

$x_4 \longleftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \longleftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \leftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**

$x_3 \leftrightarrow \{ \quad x_1, \quad x_4, \quad \dots \}$

$x_4 \leftrightarrow \{ x_0, x_5, \dots \}$

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \dots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \leftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
$x_3 \leftrightarrow$	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**

$x_4 \leftrightarrow \{ x_0, x_5, \ldots \}$

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
$x_0 \leftrightarrow$	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
$x_1 \leftrightarrow$	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
$x_2 \leftrightarrow$	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
$x_3 \leftrightarrow$	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
$x_4 \leftrightarrow$	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**

$x_5 \leftrightarrow \{ x_0, x_1, x_2, x_3, x_4, x_5, \ldots \}$

…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…							

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**...**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**...**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**...**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**...**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**...**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**...**
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$$\left\{ \boldsymbol{x_0}, \qquad \boldsymbol{x_5}, \ \ldots \right\}$$

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

Y	**N**	**N**	**N**	**N**	**Y**	**…**

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

Flip all Y's to N's and vice-versa to get a new set

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

$$\{\ \boldsymbol{x_1},\ \boldsymbol{x_2},\ \boldsymbol{x_3},\ \boldsymbol{x_4},\ \ldots\ \}$$

Flip all Y's to N's and vice-versa to get a new set

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**...**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**...**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**...**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**...**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**...**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**...**
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**

N	**Y**	**Y**	**Y**	**Y**	**N**	**...**

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	...
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**...**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**...**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**...**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**...**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**...**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**...**
...	**...**	**...**	**...**	**...**	**...**	**...**	**...**

N	**Y**	**Y**	**Y**	**Y**	**Y**	**N**	**...**

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

Which row in the table is paired with this set?

N	**Y**	**Y**	**Y**	**Y**	**N**	…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

Which row in the table is paired with this set?

N	**Y**	**Y**	**Y**	**Y**	**N**	…

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	…
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	…
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	…
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	…
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	…
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	…
…	…	…	…	…	…	…	…

N	**Y**	**Y**	**Y**	**Y**	**N**	…

Which row in the table is paired with this set?

	x_0	x_1	x_2	x_3	x_4	x_5	…
x_0	**Y**	**N**	**Y**	**N**	**Y**	**N**	**…**
x_1	**Y**	**N**	**N**	**Y**	**Y**	**N**	**…**
x_2	**N**	**N**	**N**	**N**	**Y**	**N**	**…**
x_3	**N**	**Y**	**N**	**N**	**Y**	**N**	**…**
x_4	**Y**	**N**	**N**	**N**	**N**	**Y**	**…**
x_5	**Y**	**Y**	**Y**	**Y**	**Y**	**Y**	**…**
…	**…**	**…**	**…**	**…**	**…**	**…**	**…**

N	**Y**	**Y**	**Y**	**Y**	**N**	**…**

What set is this?

The Diagonal Set

- Let $f : S \to \wp(S)$ be an arbitrary function from S to $\wp(S)$.
- Define the set D as follows:

$$\boldsymbol{D = \{\ x \in S \mid x \notin f(x)\ \}}$$

- (A note on the notation: $x \notin f(x)$ means “the element x is not in the set $f(x)$,” not “the element x is not mapped to by $f(x)$.”)
- This is a formalization of the set we found in the previous picture.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

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Proof: Let S be an arbitrary set.

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Consider any arbitrary function $f : S \to \wp(S)$.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$.

Consider any arbitrary function $f : S \to \wp(S)$. Now, define $D = \{ x \in S \mid x \notin f(x) \}$.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$.

Consider any arbitrary function $f : S \to \wp(S)$. Now, define $D = \{ x \in S \mid x \notin f(x) \}$. We've specifically constructed D so that $f(x) \neq D$ for any $x \in S$.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

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$$x \in D \text{ iff } x \notin f(x). \qquad (1)$$

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

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$$x \in D \text{ iff } x \notin f(x). \qquad (1)$$

By assumption, $f(x) = D$. Combined with (1), this tells us

$$x \in D \text{ iff } x \notin D. \qquad (2)$$

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$$x \in D \text{ iff } x \notin f(x). \tag{1}$$

By assumption, $f(x) = D$. Combined with (1), this tells us

$$x \in D \text{ iff } x \notin D. \tag{2}$$

This is impossible, so there must not be any $x \in S$ where $f(x) = D$. Therefore, we see that $f(x) \neq D$ for any $x \in S$.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$.

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$$x \in D \text{ iff } x \notin f(x). \tag{1}$$

By assumption, $f(x) = D$. Combined with (1), this tells us

$$x \in D \text{ iff } x \notin D. \tag{2}$$

This is impossible, so there must not be any $x \in S$ where $f(x) = D$. Therefore, we see that $f(x) \neq D$ for any $x \in S$.

Since $f(x) \neq D$ for any $x \in S$, we know that f is not a surjection, and therefore f cannot be a bijection.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$.

Consider any arbitrary function $f : S \to \wp(S)$. Now, define $D = \{ x \in S \mid x \notin f(x) \}$. We've specifically constructed D so that $f(x) \neq D$ for any $x \in S$. To see this, suppose that this isn't the case and that there is some $x \in S$ where $f(x) = D$. By definition of D, we know that

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$$x \in D \text{ iff } x \notin D. \qquad (2)$$

This is impossible, so there must not be any $x \in S$ where $f(x) = D$. Therefore, we see that $f(x) \neq D$ for any $x \in S$.

Since $f(x) \neq D$ for any $x \in S$, we know that f is not a surjection, and therefore f cannot be a bijection. Since f was chosen arbitrarily, this means that there cannot be *any* bijections $f : S \to \wp(S)$, so $|S| \neq |\wp(S)|$, as required.

Theorem: If S is a set, then $|S| \neq |\wp(S)|$.

Proof: Let S be an arbitrary set. We will prove that $|S| \neq |\wp(S)|$ by showing that there are no bijections from S to $\wp(S)$.

Consider any arbitrary function $f : S \to \wp(S)$. Now, define $D = \{ x \in S \mid x \notin f(x) \}$. We've specifically constructed D so that $f(x) \neq D$ for any $x \in S$. To see this, suppose that this isn't the case and that there is some $x \in S$ where $f(x) = D$. By definition of D, we know that

$$x \in D \text{ iff } x \notin f(x). \tag{1}$$

By assumption, $f(x) = D$. Combined with (1), this tells us

$$x \in D \text{ iff } x \notin D. \tag{2}$$

This is impossible, so there must not be any $x \in S$ where $f(x) = D$. Therefore, we see that $f(x) \neq D$ for any $x \in S$.

Since $f(x) \neq D$ for any $x \in S$, we know that f is not a surjection, and therefore f cannot be a bijection. Since f was chosen arbitrarily, this means that there cannot be *any* bijections $f : S \to \wp(S)$, so $|S| \neq |\wp(S)|$, as required. ■

Concluding the Proof

- We've just shown that $|S| \neq |\wp(S)|$ for any set S.
- To prove $|S| < |\wp(S)|$, we need to show that $|S| \leq |\wp(S)|$ by finding an injection from S to $\wp(S)$.
- Take $f : S \to \wp(S)$ defined as

$$\boldsymbol{f(x) = \{x\}}$$

- Good exercise: prove this function is injective.

Why All This Matters

- Proof by diagonalization is a powerful technique for showing two sets cannot have the same size.
- Can also be adapted for other purposes:
  - Finding specific problems that cannot be solved by computers.
  - Proving Gödel's Incompleteness Theorem.
  - Finding problems requiring some amount of computational resource to solve.
- We will return to this later in the quarter.