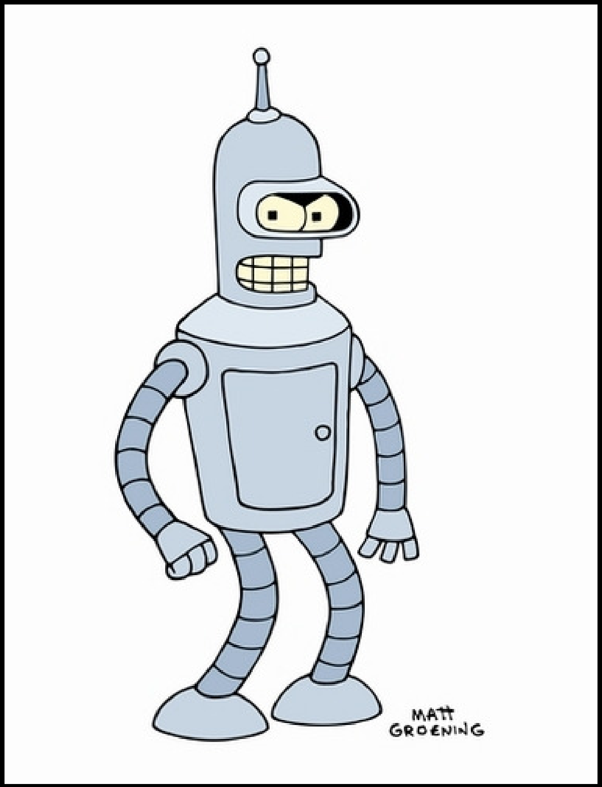
CS 154

Finite Automata vs Regular Expressions, Non-Regular Languages

Deterministic Finite Automata



Computation with finite memory

Non-Deterministic Finite Automata

Computation with finite memory

and "guessing"

Regular Languages are closed under all of the following operations:

- **Union:** $A \cup B = \{ w \mid w \in A \text{ or } w \in B \}$
- **Intersection:** $A \cap B = \{ w \mid w \in A \text{ and } w \in B \}$
- **Complement:** $\neg A = \{ w \in \Sigma^* \mid w \notin A \}$
- **Reverse:** $A^R = \{ w_1 \ldots w_k \mid w_k \ldots w_1 \in A \}$
- **Concatenation:** $A \cdot B = \{ vw \mid v \in A \text{ and } w \in B \}$
- **Star:** $A^* = \{ w_1 \ldots w_k \mid k \geq 0 \text{ and each } w_i \in A \}$

Regular Expressions

Computation as simple, logical description

A totally different way of thinking about computation:

What is the complexity of
describing the strings in the language?

Inductive Definition of Regexp

Let Σ be an alphabet. We define the regular expressions over Σ inductively:

For all $\sigma \in \Sigma$, σ is a regexp

ε is a regexp

$\varnothing$ is a regexp

If R_1 and R_2 are both regexps, then

(R_1R_2), $(R_1 + R_2)$, and $(R_1)^*$ are regexps

Precedence Order: $*$

then $\cdot$

then $+$

Example: $R_1{}^*R_2 + R_3 = ((R_1{}^*)\cdot R_2) + R_3$

Definition: Regexps Represent Languages

The regexp $\sigma \in \Sigma$ *represents* the language $\{\sigma\}$

The regexp ε represents $\{\varepsilon\}$

The regexp $\varnothing$ represents $\varnothing$

If R_1 and R_2 are regular expressions representing L_1 and L_2 then:

(R_1R_2) represents $L_1 \cdot L_2$

$(R_1 + R_2)$ represents $L_1 \cup L_2$

$(R_1)^*$ represents L_1^*

Example: $(10 + 0^*1)$ represents $\{0^k1 \mid k \geq 0\} \cup \{10\}$

Regexps Represent Languages

For every regexp R, define L(R) to be the language that R represents

A string $w \in \Sigma^*$ is *accepted by R* (or, *w matches R*) if $w \in L(R)$

Example: 01010 matches the regexp (01)*0

Assume $\Sigma = \{0,1\}$

{ w | w has exactly a single 1 }

0^*10^*

Assume $\Sigma = \{0,1\}$

What language does
the regexp $\emptyset^*$ represent?

$\{\varepsilon\}$

$\boxed{\text{Assume } \Sigma = \{0,1\}}$

$\{ w \mid w$ has length ≥ 3 and its 3rd symbol is $0 \}$

$$(0+1)(0+1)0(0+1)^*$$

Assume $\Sigma = \{0,1\}$

$\{ w \mid \text{every odd position in } w \text{ is a } 1 \}$

$$(1(0 + 1))^*(1 + \varepsilon)$$

DFAs $\equiv$ NFAs $\equiv$ Regular Expressions!

L can be represented by some regexp

$\Leftrightarrow$ L is regular

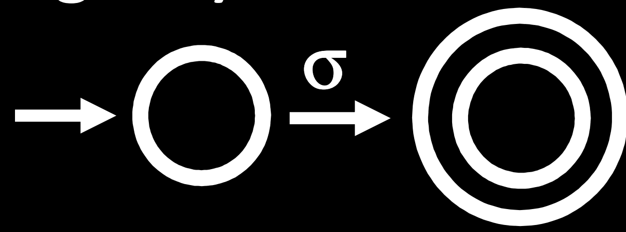
L can be represented by some regexp
$\Rightarrow$ **L is regular**

Given any regexp R, we will construct an NFA N s.t. N accepts *exactly* the strings accepted by R

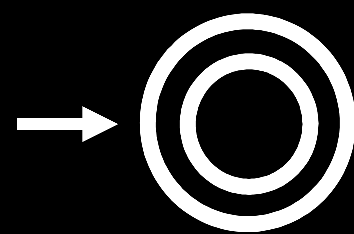
Proof by induction on the *length* of the regexp R:

Base Cases (R has length 1):

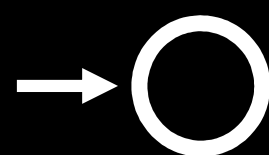
R = σ



R = ε



R = ∅



Induction Step: Suppose every regexp of length $< k$ represents some regular language.

Consider a regexp R of length $k > 1$

Three possibilities for R:

$R = R_1 + R_2$

$R = R_1 R_2$

$R = (R_1)^*$

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By induction, R_1 and R_2 represent some regular languages, L_1 and L_2

But $L(R) = L(R_1 + R_2) = L_1 \cup L_2$

so L(R) is regular, by the union theorem!

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so L(R) is regular by the *concatenation theorem*

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But $L(R) = L(R_1^*) = L_1^*$

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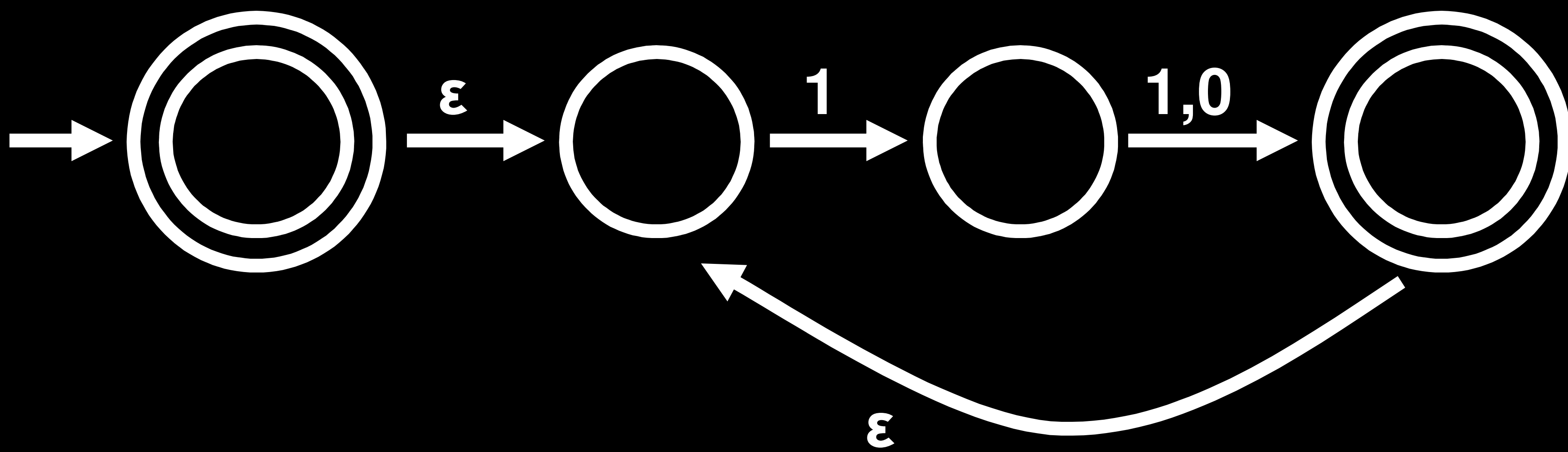
Therefore: If L is represented by a regexp, then L is regular

Give an NFA that accepts the language represented by (1(0 + 1))*



Regular expression: (1(0+1))*

Generalized NFAs (GNFA)

L can be represented by a regexp

$\Leftarrow$

L is a regular language

Idea: Transform an NFA for L into a regular expression by removing states and re-labeling the arcs with *regular expressions*

Rather than reading in just 0 or 1 letters from the string on a step, we can read in *entire substrings*

A GNFA is a 5-tuple $G = (Q, \Sigma, R, q_{start}, q_{accept})$

Q, Σ are states and alphabet

$R : (Q-\{q_{accept}\}) \times (Q-\{q_{start}\}) \rightarrow \mathcal{R}$
is the transition function

$q_{start} \in Q$ is the start state

$q_{accept} \in Q$ is the (unique) accept state

$\boxed{\mathcal{R} \quad = \text{set of all regular expressions over } \Sigma}$

A GNFA is a 5-tuple $G = (Q, \Sigma, R, q_{start}, q_{accept})$

Let $w \in \Sigma^*$ and let G be a GNFA.

G accepts w if w can be written as $w = w_1 \cdots w_k$

where $w_i \in \Sigma^*$ and there is a sequence

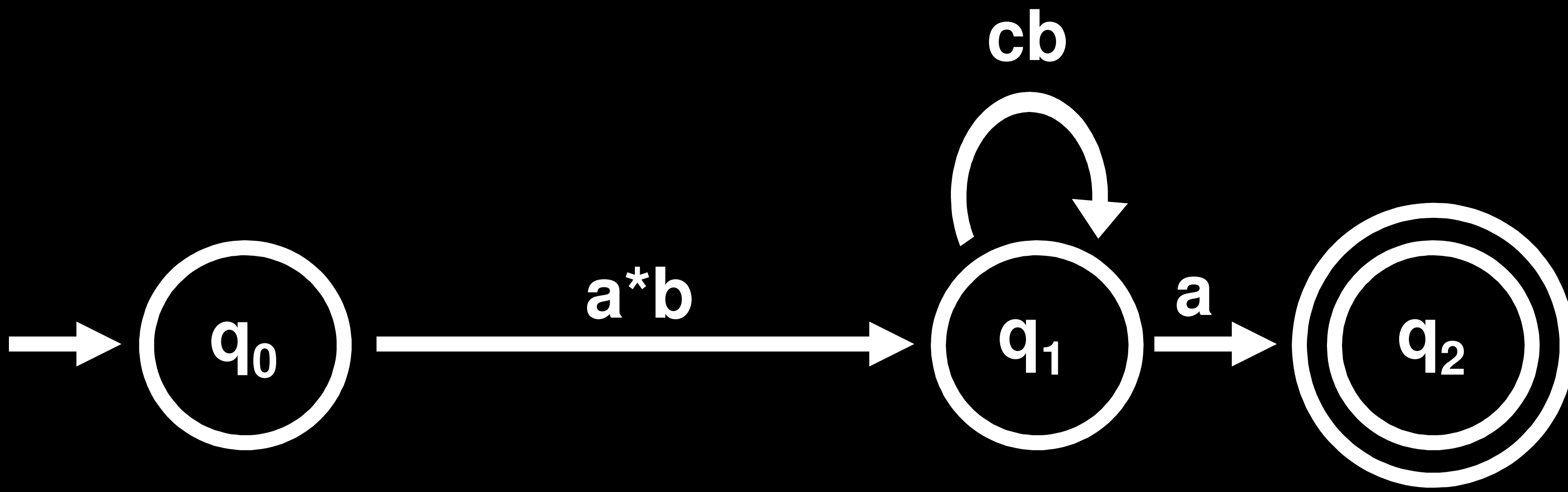
$r_0, r_1, \ldots, r_k \in Q$ such that

- $r_0 = q_{start}$
- w_i matches $R(r_{i-1}, r_i)$ for all $i = 1, \ldots, k$, and
- $r_k = q_{accept}$

$L(G)$ = set of all strings that G accepts

= "the language recognized by G"

Generalized NFA (GNFA)

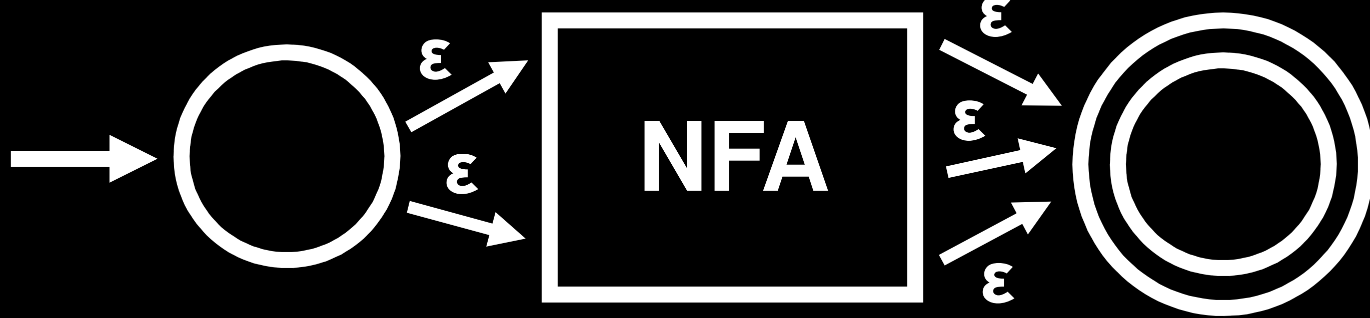


This GNFA recognizes L(a*b(cb)*a)

Is aaabcbcba accepted or rejected?

Is bba accepted or rejected?

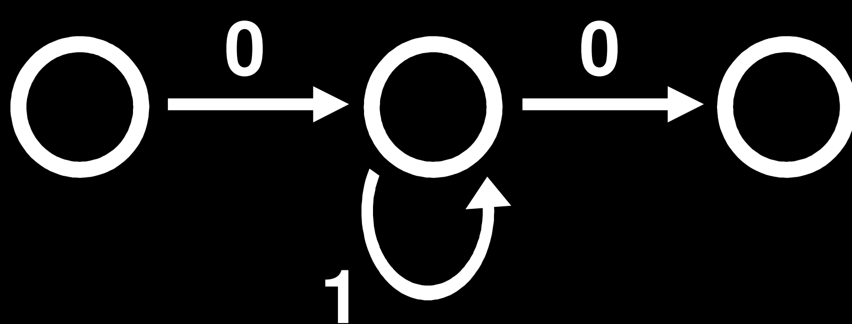
Is bcba accepted or rejected?



Add unique start and accept states

While the machine has more than 2 states:

Pick an internal state, rip it out and
re-label the arrows with regexps,
to account for paths through the missing state



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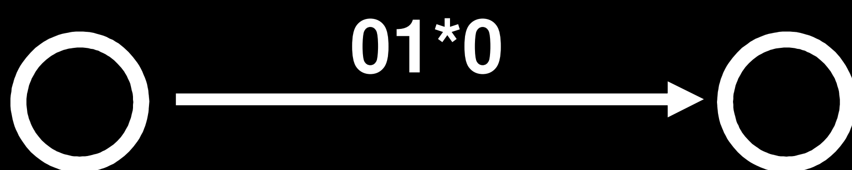
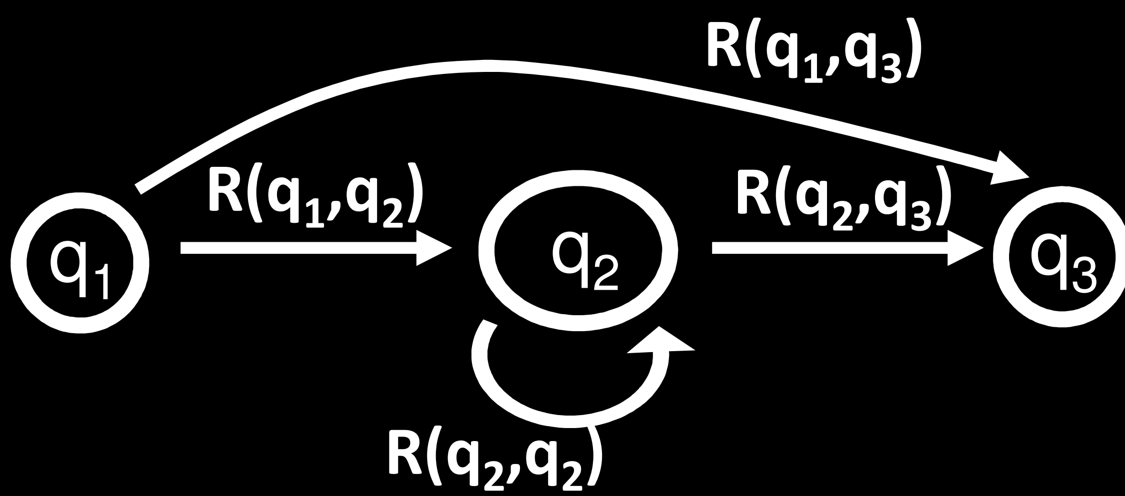


While the machine has more than 2 states:

In general:



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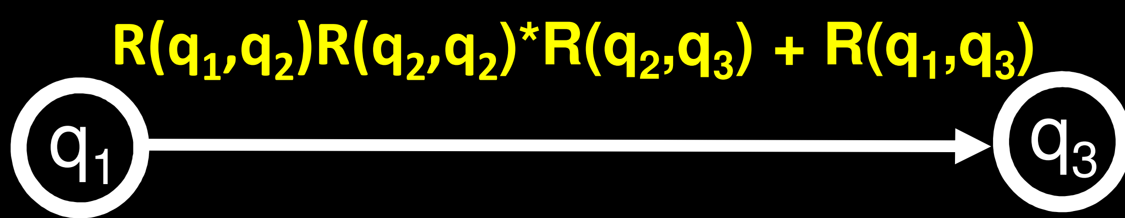
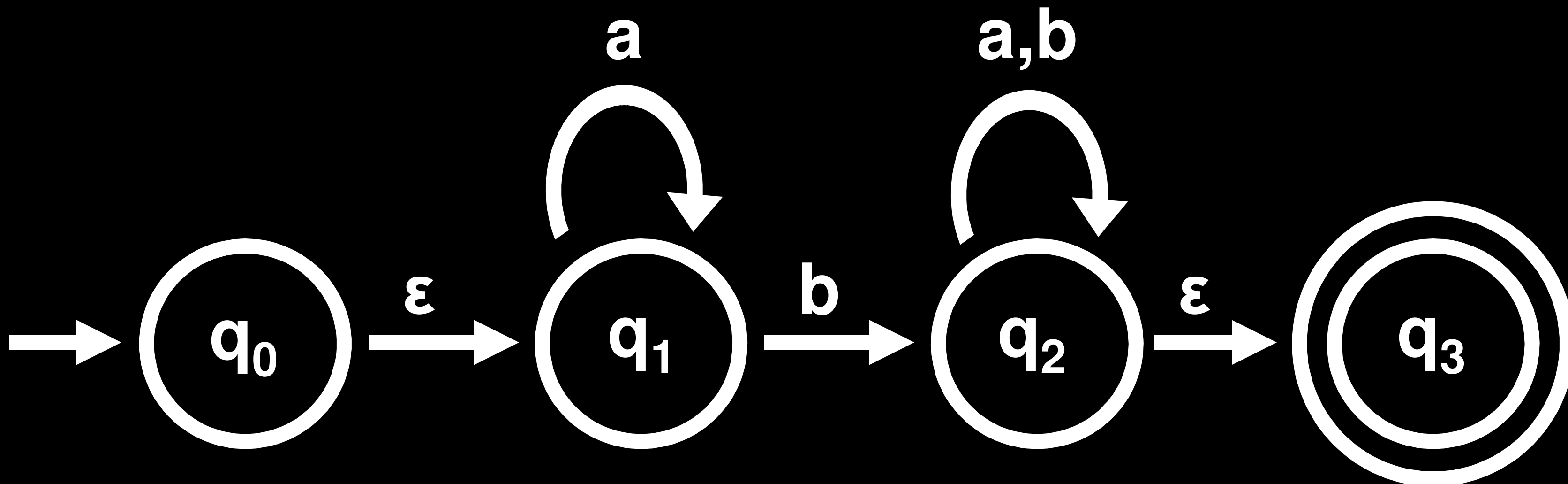
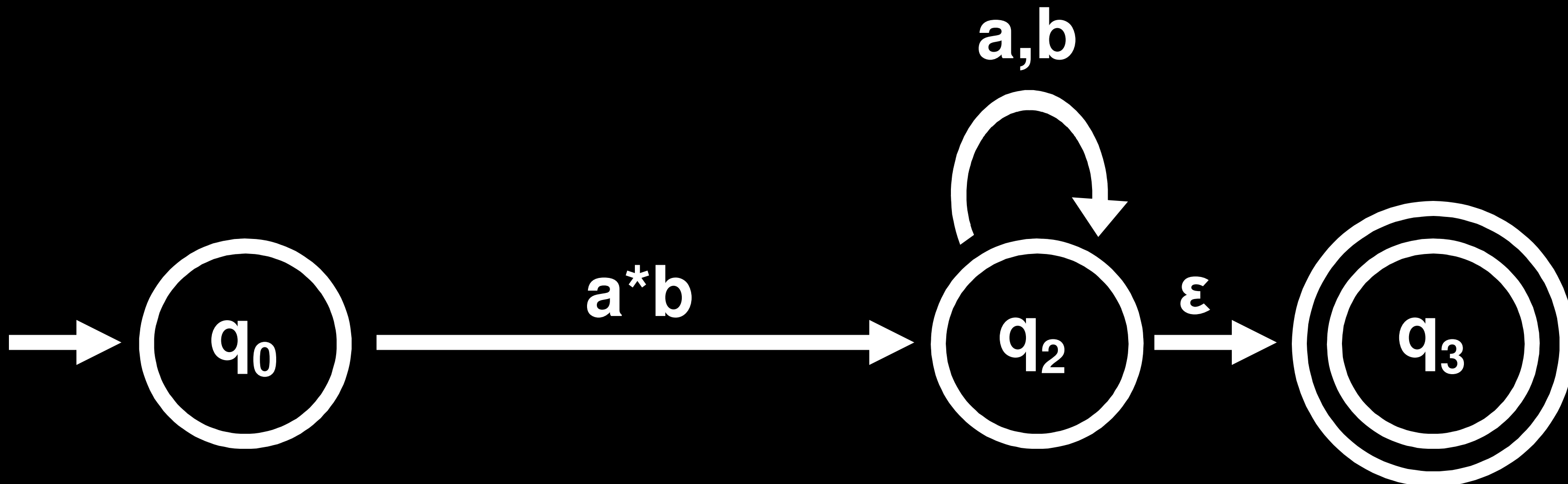




$R(q_0,q_3) = (a^*b)(a+b)^*$

represents L(N)



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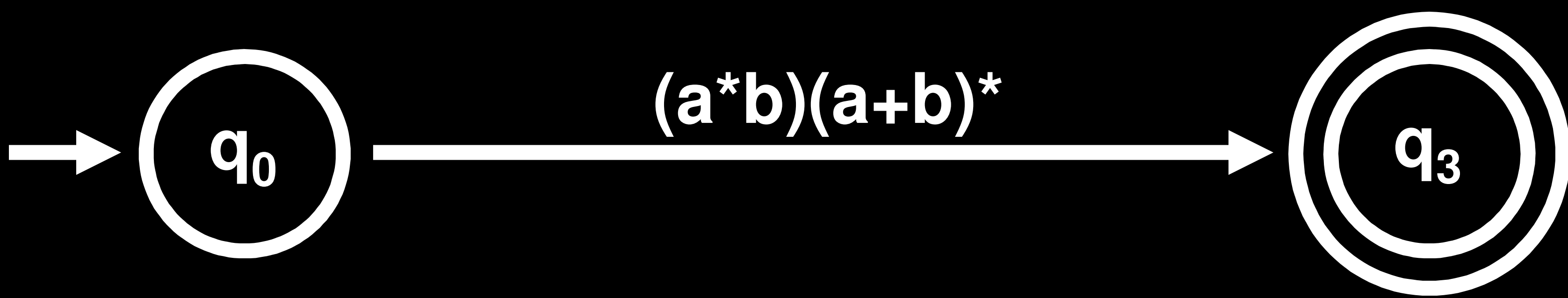


$R(q_0,q_3) = (a^*b)(a+b)^*$

represents $L(N)$

Formally: Given an DFA, add q_{start} and q_{acc} to create G

For all q,q', define R(q,q') to be σ if δ(q,σ) = q', else ∅

CONVERT(G): *(Takes a GNFA, outputs a regexp)*

If #states = 2 **return** $R(q_{start}, q_{acc})$

If #states > 2

select $q_{rip} \in Q$ different from q_{start} and q_{acc}

define $Q' = Q - \{q_{rip}\}$

define R' on $Q'-\{q_{acc}\} \times Q'-\{q_{start}\}$ as:

$$R'(q_i,q_j) = R(q_i,q_{rip})R(q_{rip},q_{rip})^*R(q_{rip},q_j) + R(q_i,q_j)$$

return CONVERT(G′)

defines a new GNFA G′

Claim:

L(G') = L(G)

Theorem: Let R = CONVERT(G). Then L(R) = L(G).

Proof by induction on k, the number of states in G

Base Case: k = 2 CONVERT outputs $R(q_{start}, q_{acc})$ ✓

Inductive Step:

Assume theorem is true for k-1 state GNFAs

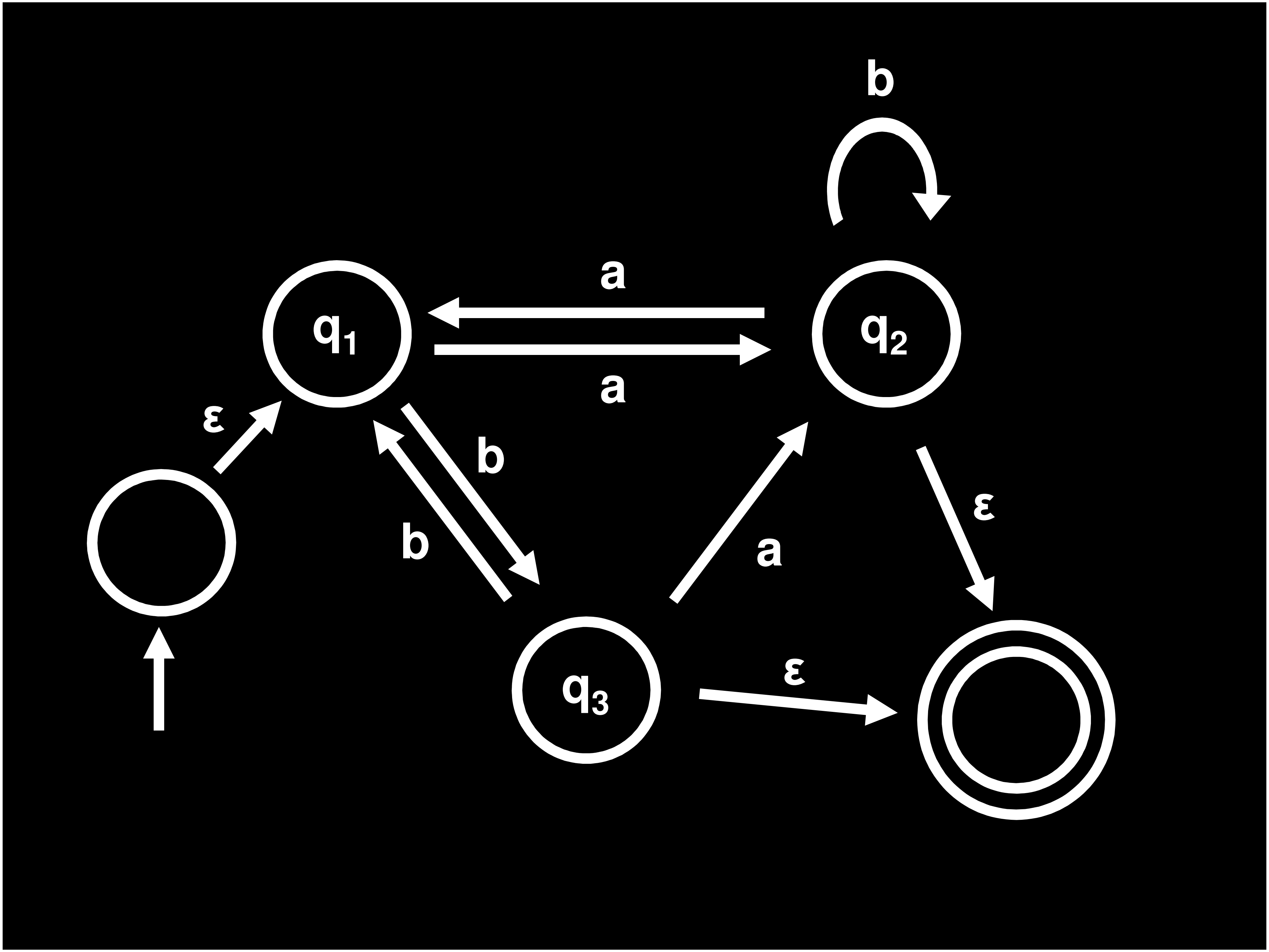
Let G have k states. Let G' be the k-1 state GNFA obtained by ripping out a state.

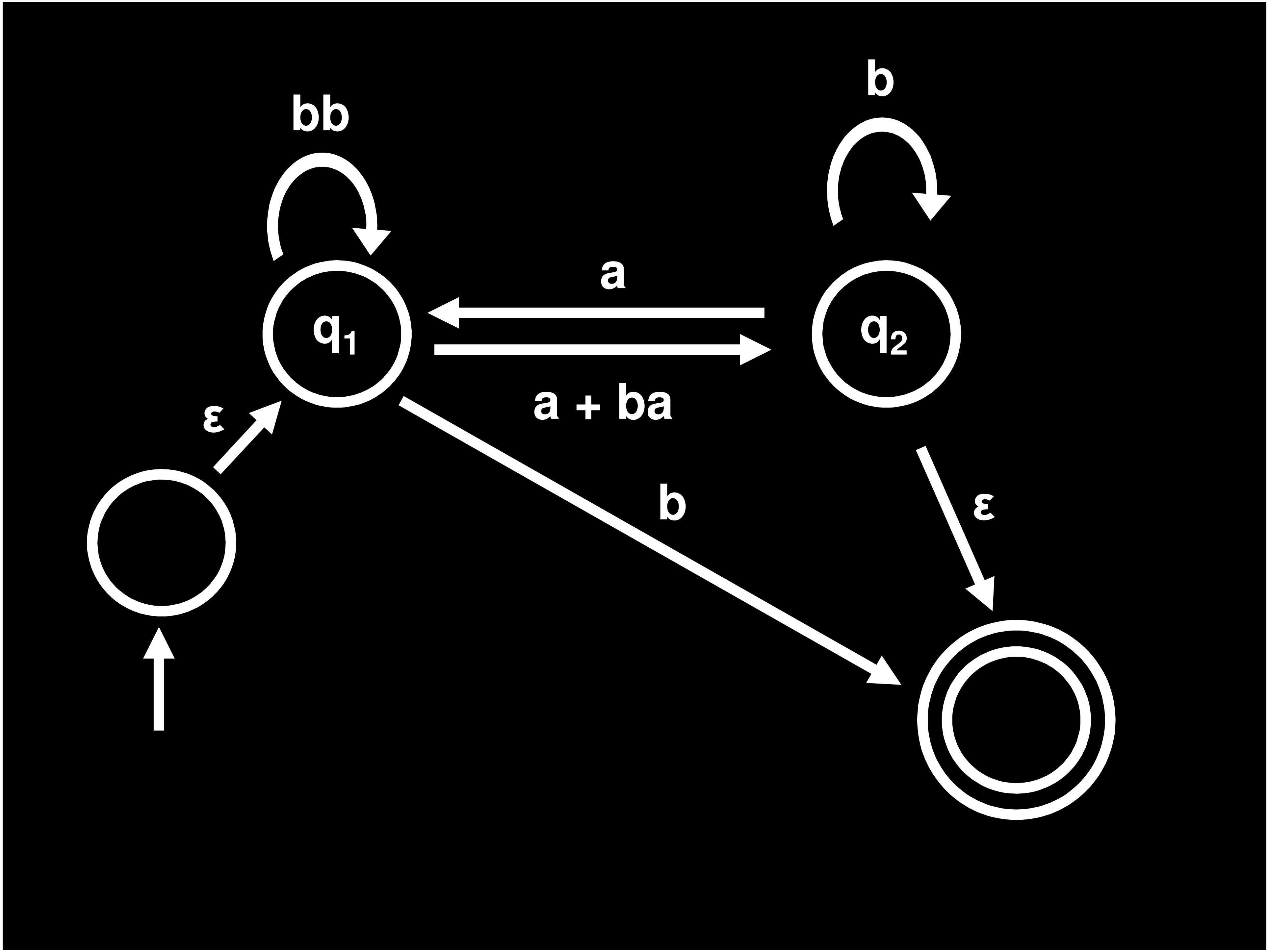
We already claimed that L(G) = L(G′)

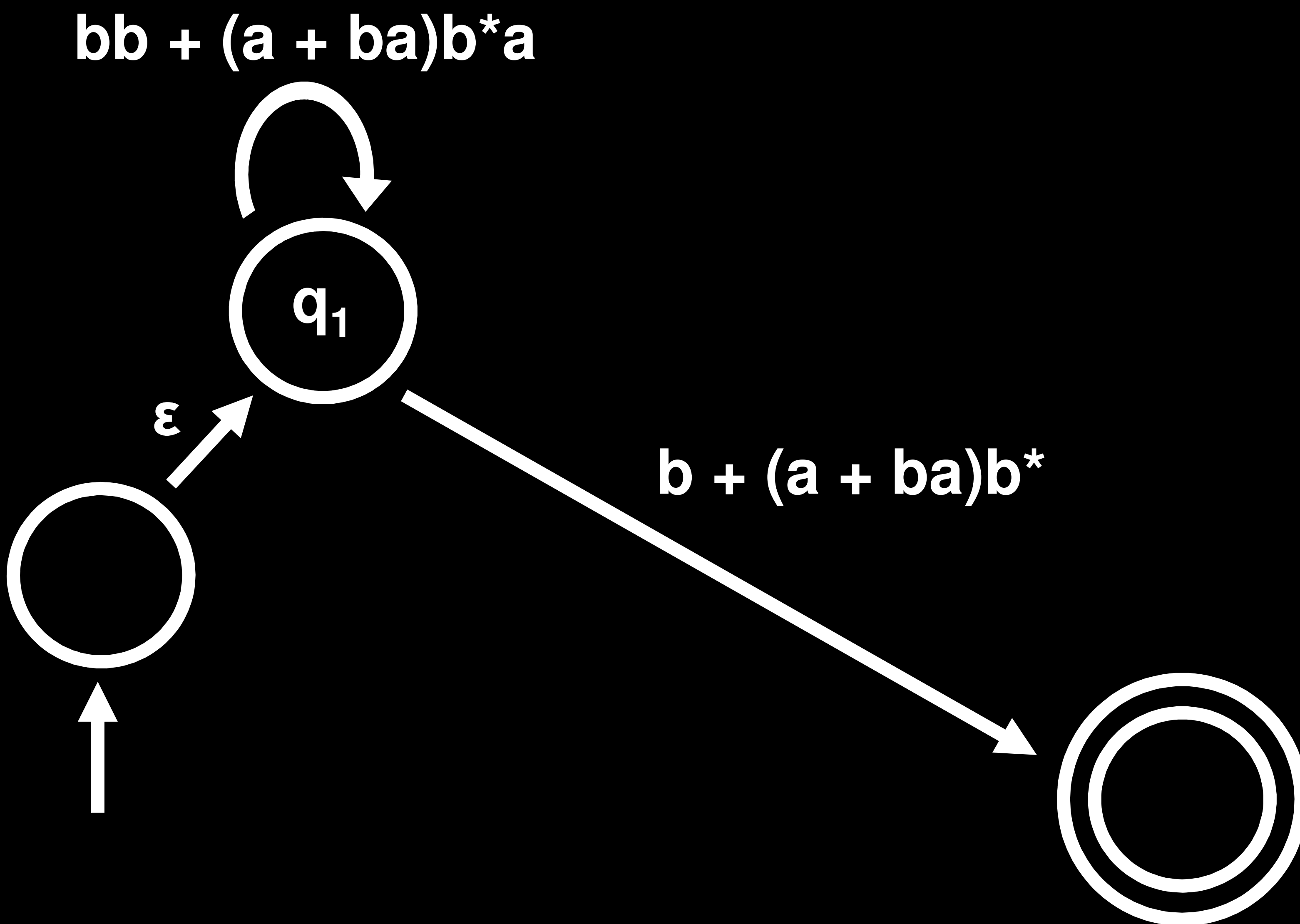
G' has k-1 states, so by induction,

L(G′) = L(CONVERT(G')) = L(R)

Therefore L(R)=L(G). QED

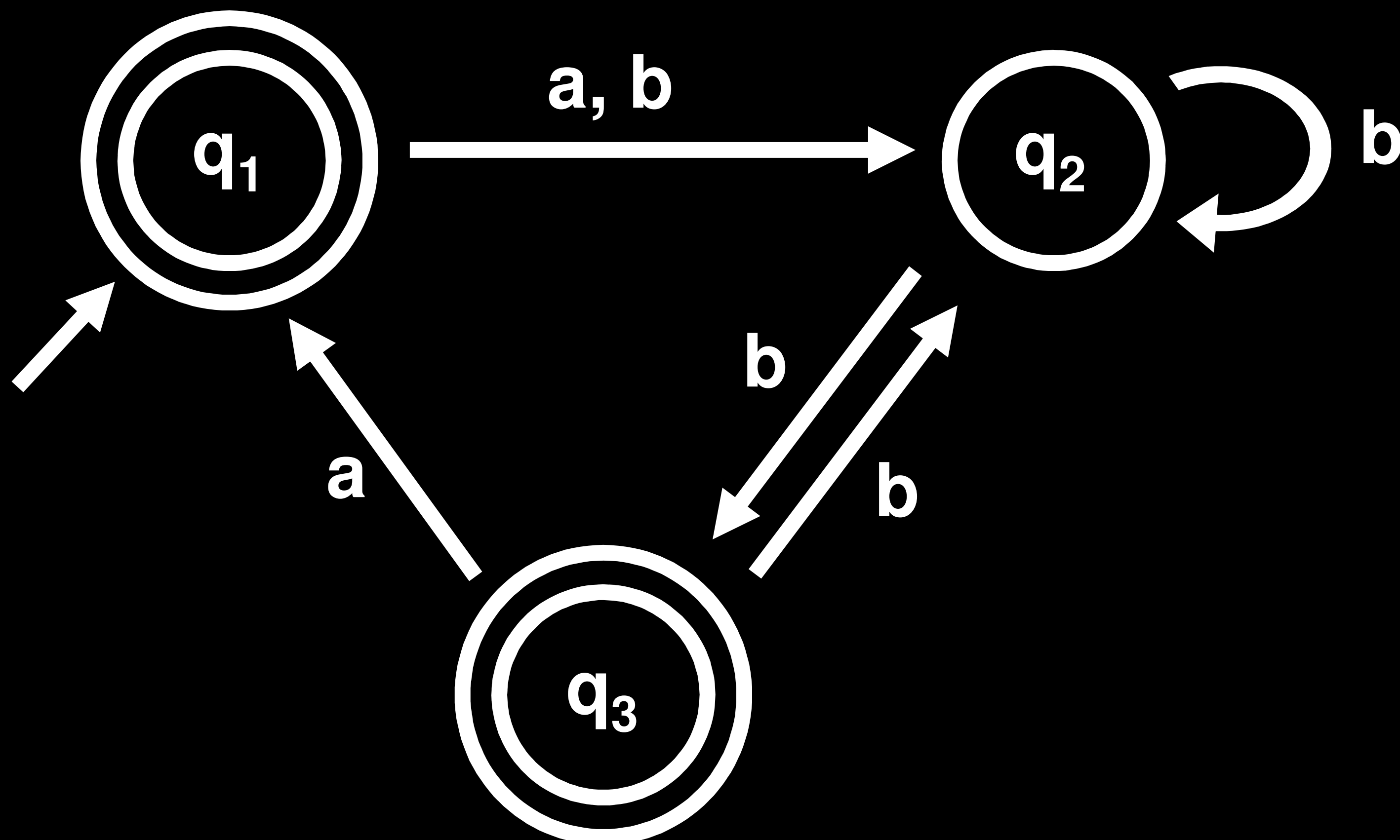

b
a
q1
q2
a
ε
b
b
ε
a
q3
ε


bb
b
a
q1
q2
a + ba
ε
b
ε



(bb + (a + ba)b*a)* (b + (a + ba)b*)

Convert the NFA to a regular expression



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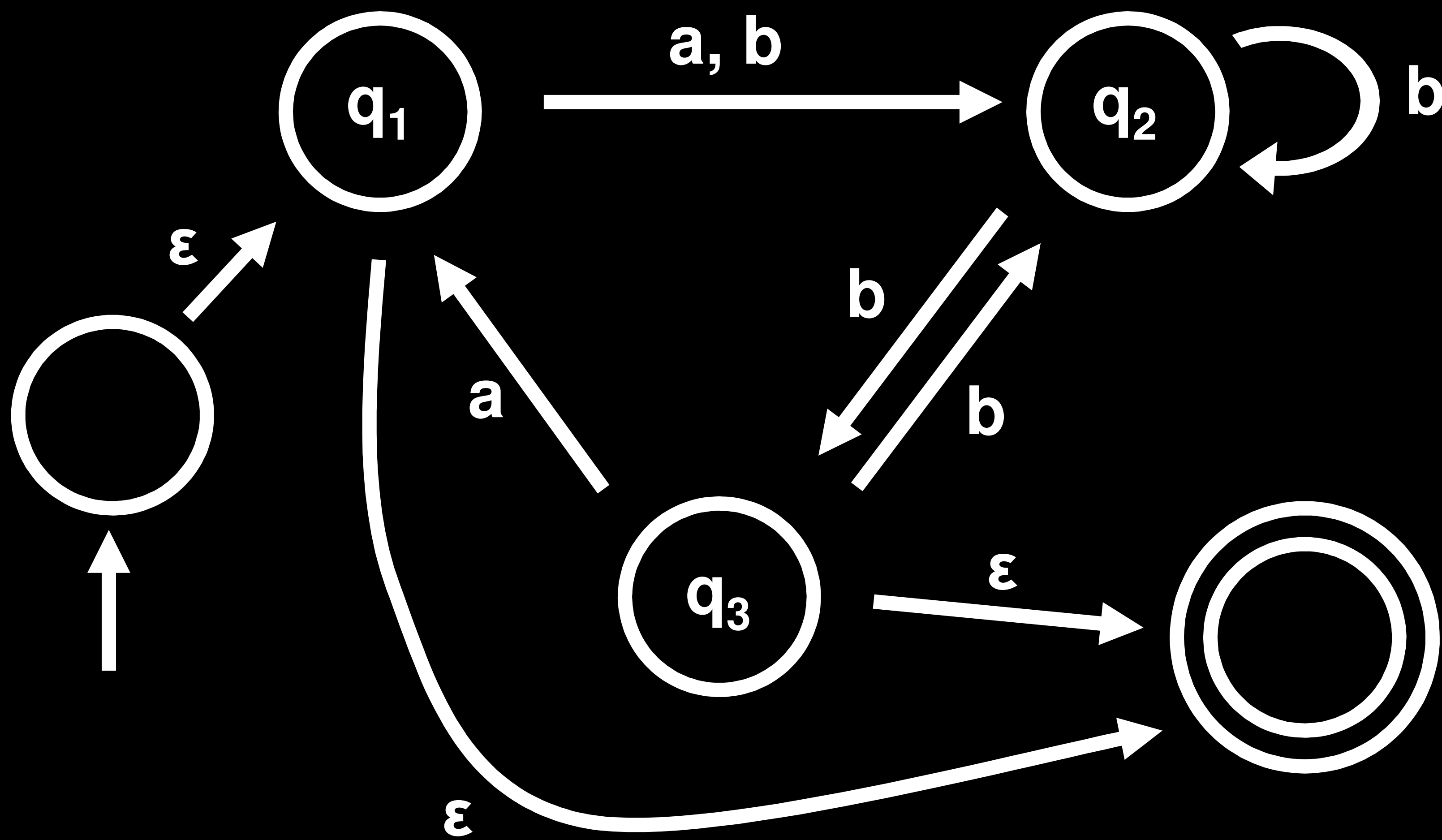

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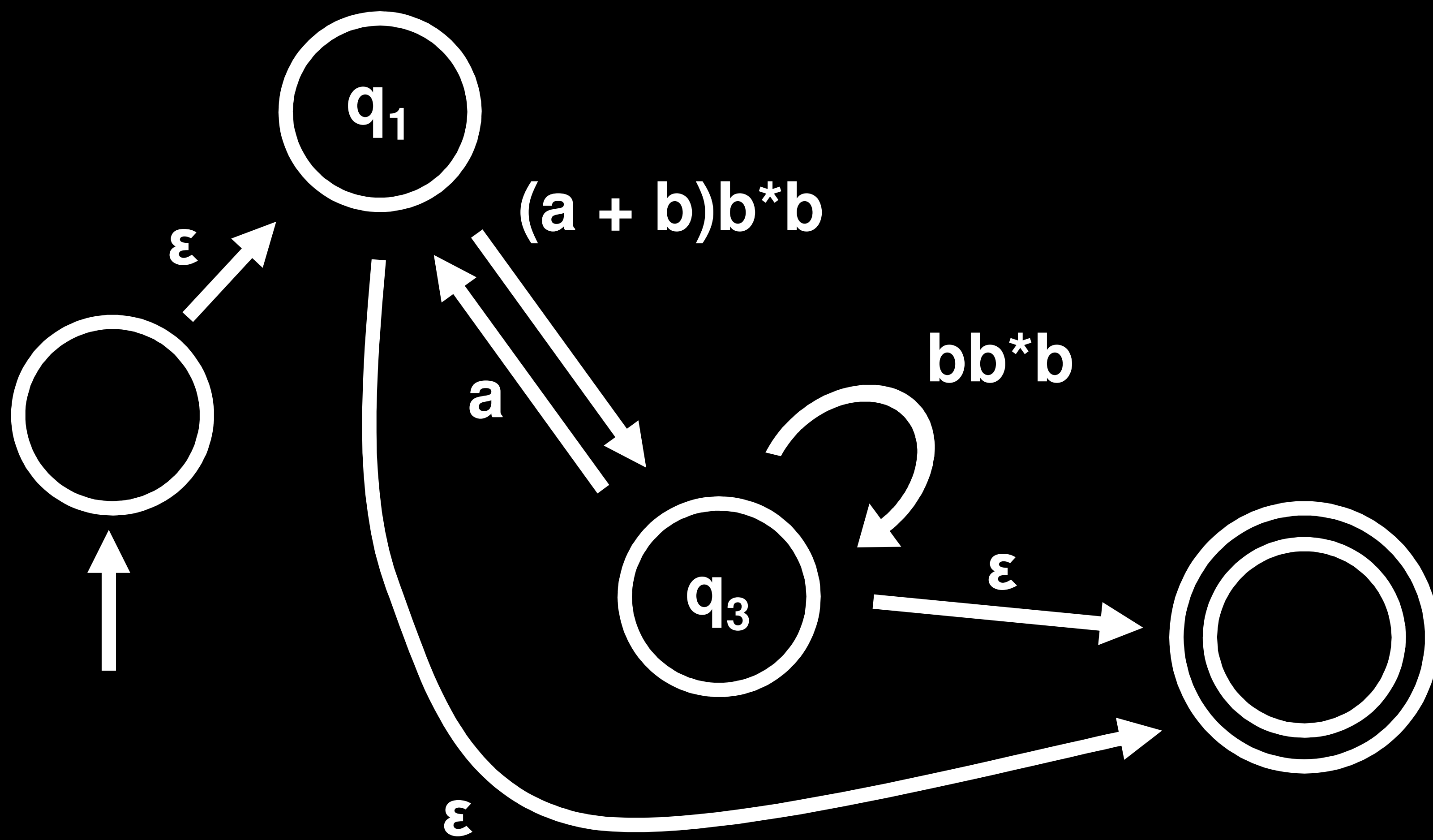

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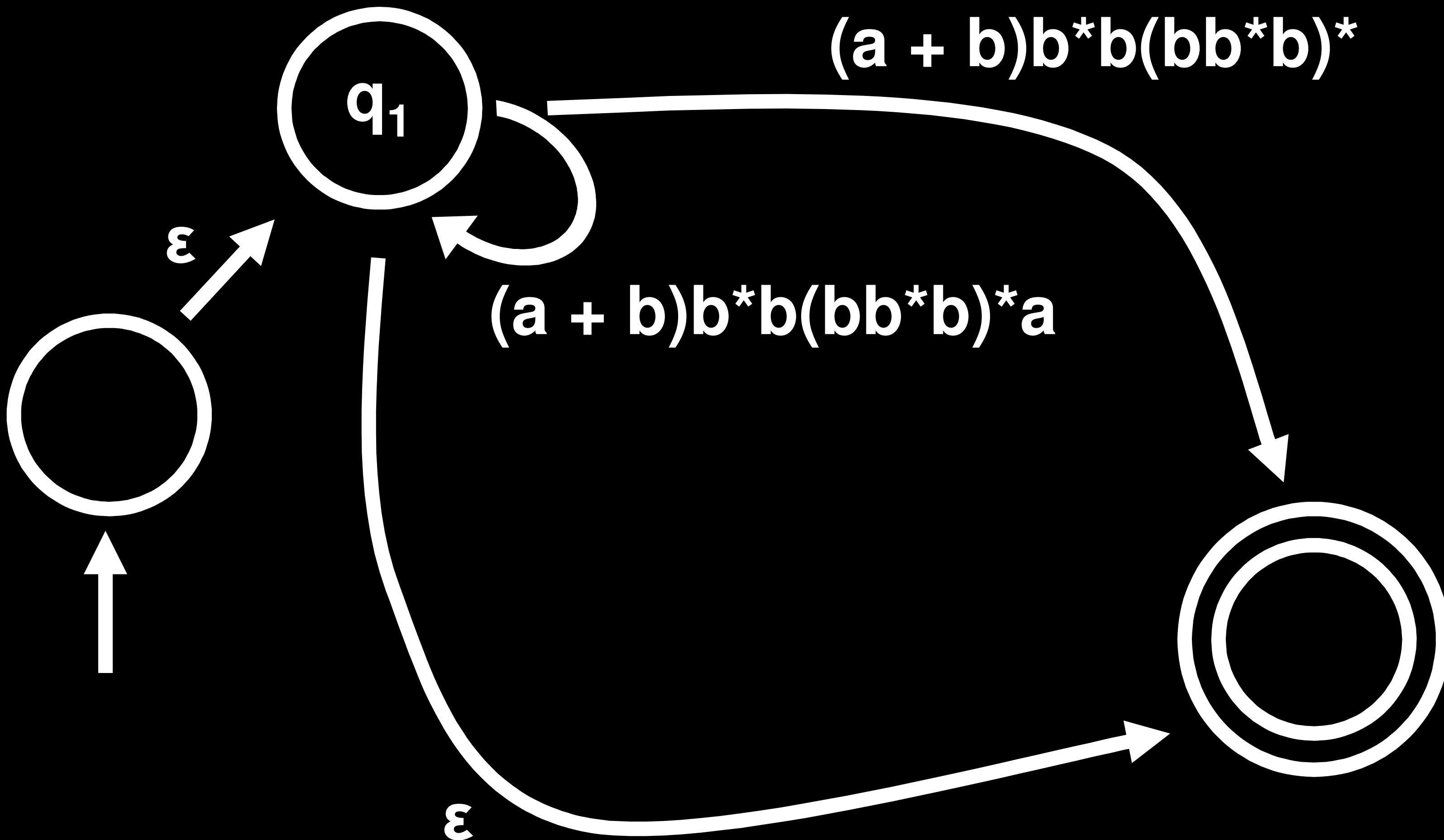
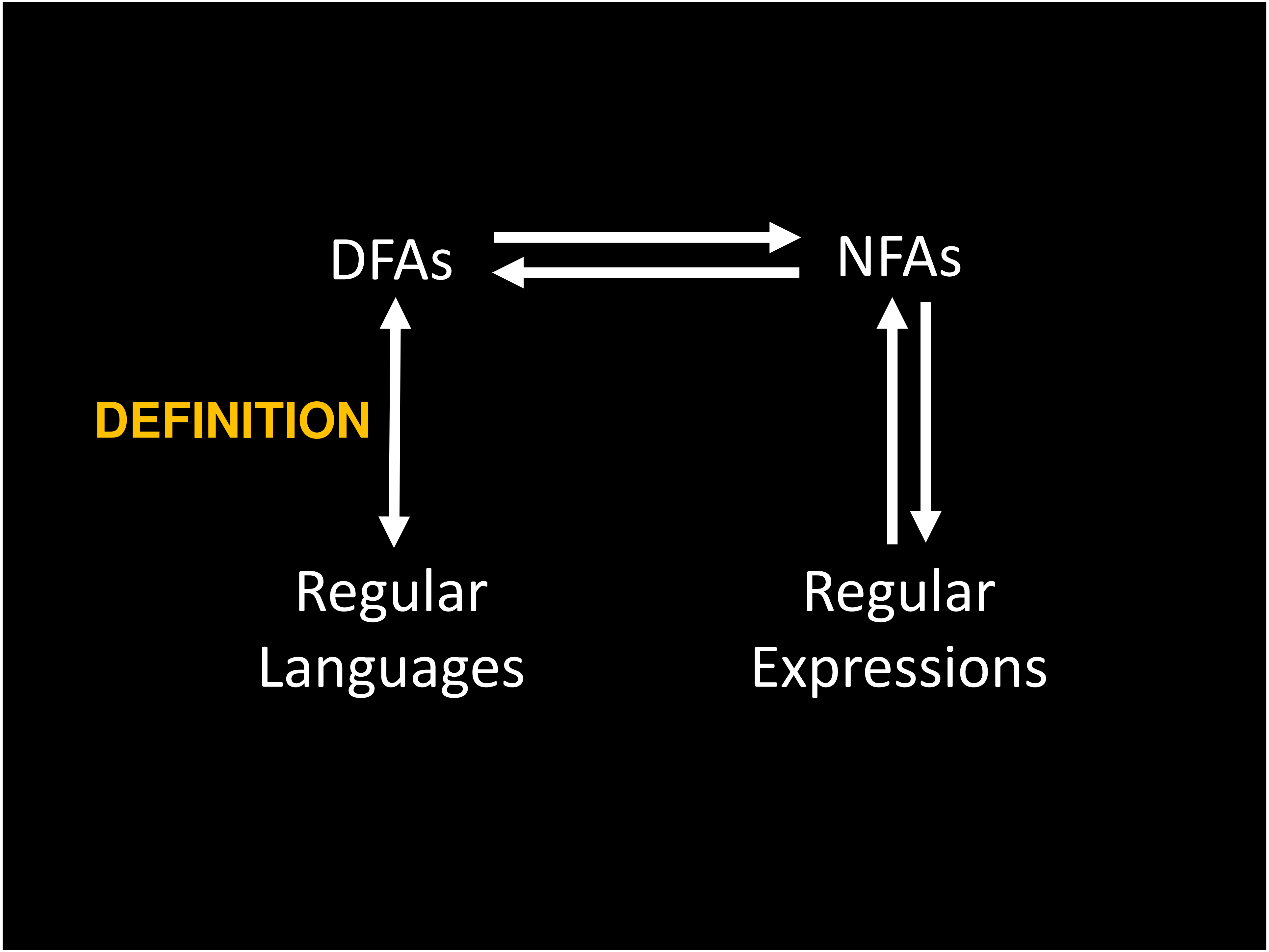


$((a + b)b^*b(bb^*b)^*a)^*(\varepsilon + (a + b)b^*b(bb^*b)^*)$


DFAs
NFAs
DEFINITION
Regular
Languages
Regular
Expressions

Some Languages Are Not Regular:

Limitations on DFAs

Regular or Not?

C = { w | w has equal number of 1s and 0s}

NOT REGULAR!

D = { w | w has equal number of
occurrences of 01 and 10 }

REGULAR!

{ w | w has equal number of occurrences of 01 and 10}

= { w | w = 1, w = 0, or w = ε, or w starts with a 0 and ends with a 0, or w starts with a 1 and ends with a 1 }

$$1 + 0 + \varepsilon + 0(0+1)^*0 + 1(0+1)^*1$$

Claim:

A string w has equal occurrences of 01 and 10

⇔ w starts and ends with the same bit.

The Pumping Lemma:
Structure in Regular Languages

Let L be a regular language

Then **there is a positive integer P** s.t.

for all strings $w \in L$ with $|w| \geq P$

there is a way to write $w = xyz$, where:

1. $|y| > 0$ (that is, $y \neq \varepsilon$)
2. $|xy| \leq P$
3. For ***all*** $i \geq 0$, $xy^iz \in L$

Why is it called the pumping lemma? The word w gets *pumped* into longer and longer strings…

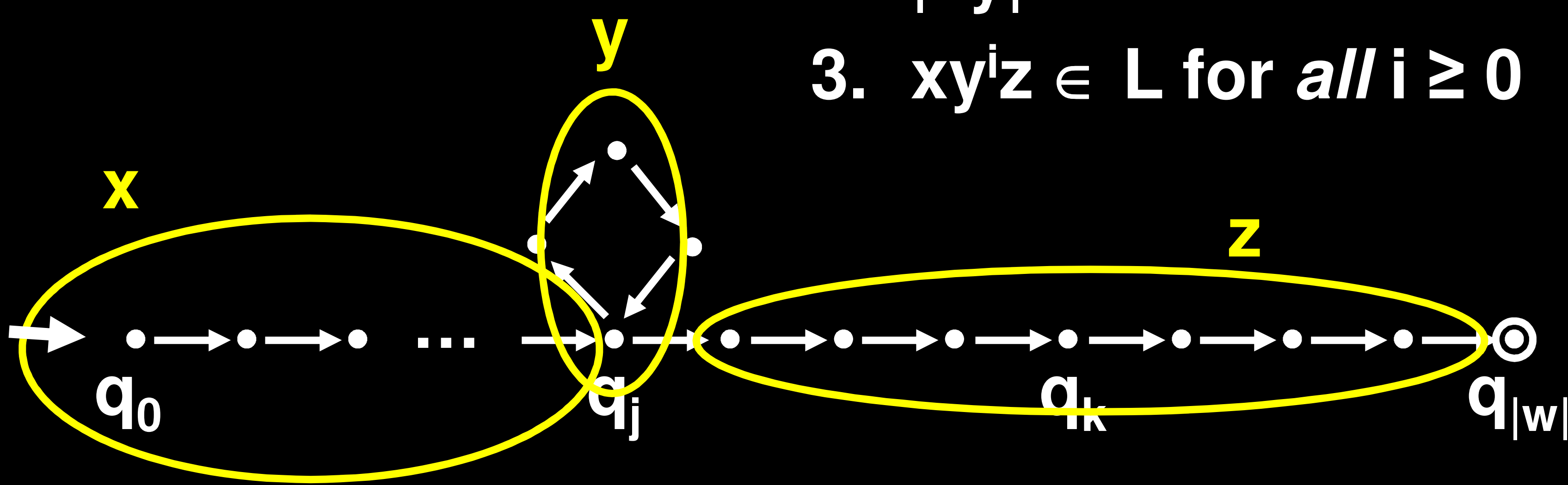
Proof: Let M be a DFA that recognizes L

Let P be the **number of states in M**

Let w be a string where $w \in L$ and $|w| \geq P$

We show: $w = xyz$

1. $|y| > 0$
2. $|xy| \leq P$
3. $xy^iz \in L$ for *all* $i \geq 0$



There must exist j and k such that

$0 \leq j < k \leq P$, and $q_j = q_k$

Applying the Pumping Lemma

Let's prove that

B = $\{0^n1^n \mid n \geq 0\}$ is not regular

By contradiction. Assume B is regular.

Let P be the number of states in a DFA for B.

Let $w = 0^P1^P$

If B is regular, then there is a way to write w as $w = xyz$, $|y| > 0$, $|xy| \leq P$, and for all $i \geq 0$, xy^iz is *also* in B

Claim: The string y must be all zeroes.

Why? Because $|xy| \leq P$ and $w = xyz = 0^P1^P$

But then xyyz has more 0s than 1s *Contradiction!*

end