Finite Automata

Part Three

Recap from Last Time

A language L is called a ***regular language*** if there exists a DFA D such that $\mathscr{L}(D) = L$.

NFAs

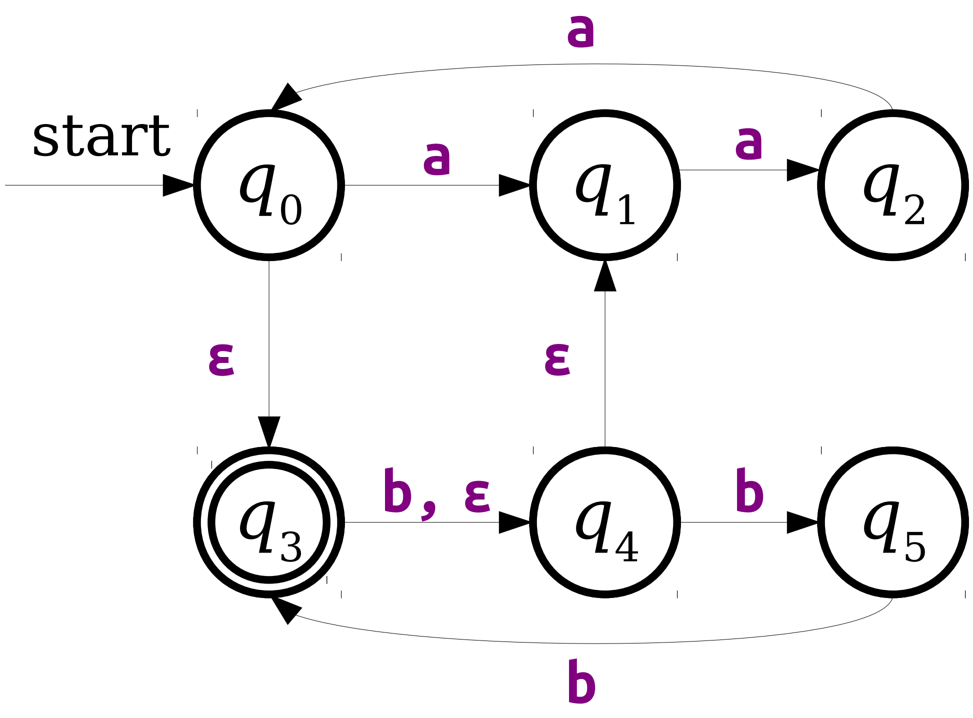
- An **NFA** is a
  - **N**ondeterministic
  - **F**inite
  - **A**utomaton
- Conceptually similar to a DFA, but equipped with the vast power of *nondeterminism*.

(Non)determinism

- A model of computation is ***deterministic*** if at every point in the computation, there is exactly one choice that can make.
- The machine accepts if that series of choices leads to an accepting state.
- A model of computation is ***nondeterministic*** if the computing machine may have multiple decisions that it can make at one point.
- The machine accepts if ***any*** series of choices leads to an accepting state.

ε-Transitions

- NFAs have a special type of transition called the **ε-transition**.
- An NFA may follow any number of ε-transitions at any time without consuming any input.



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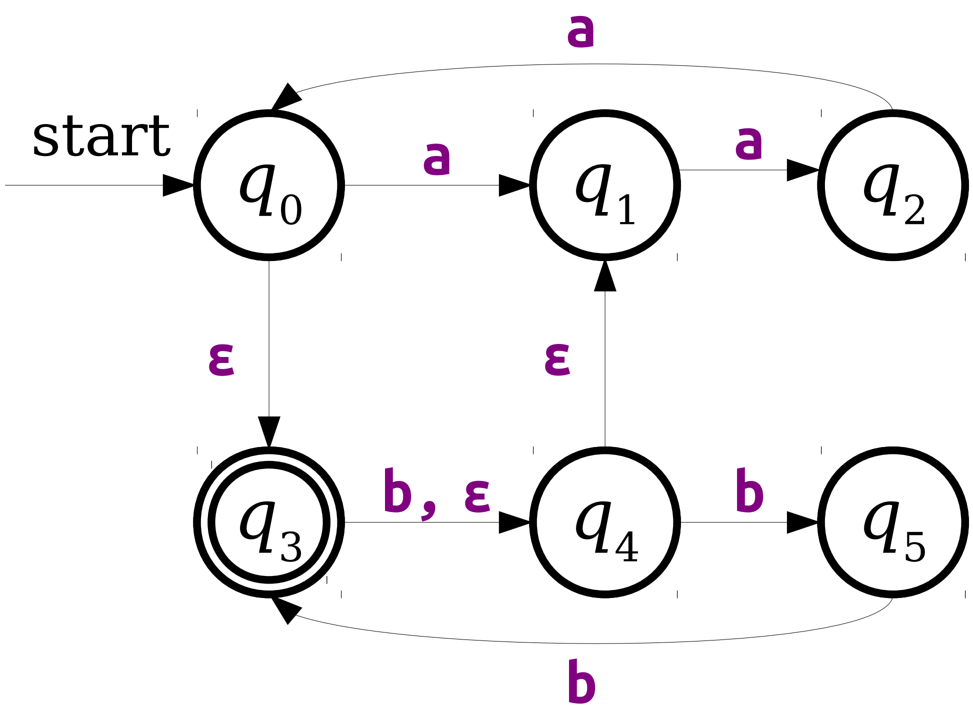

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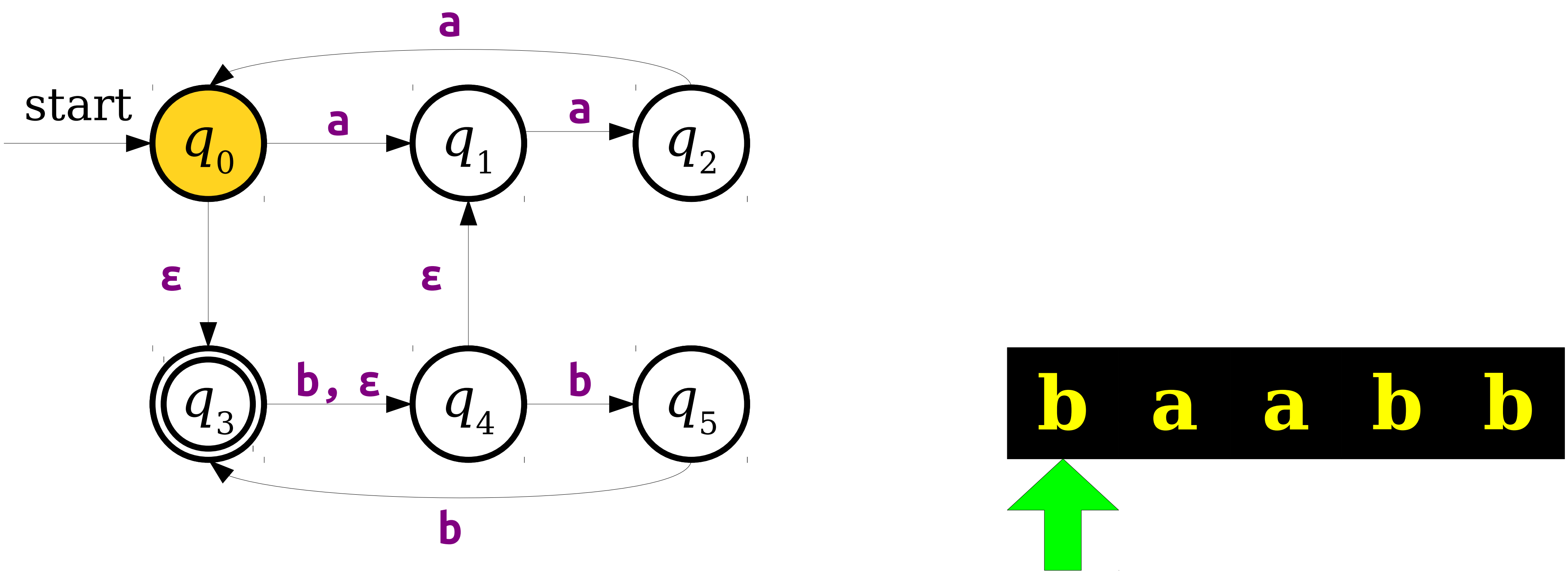

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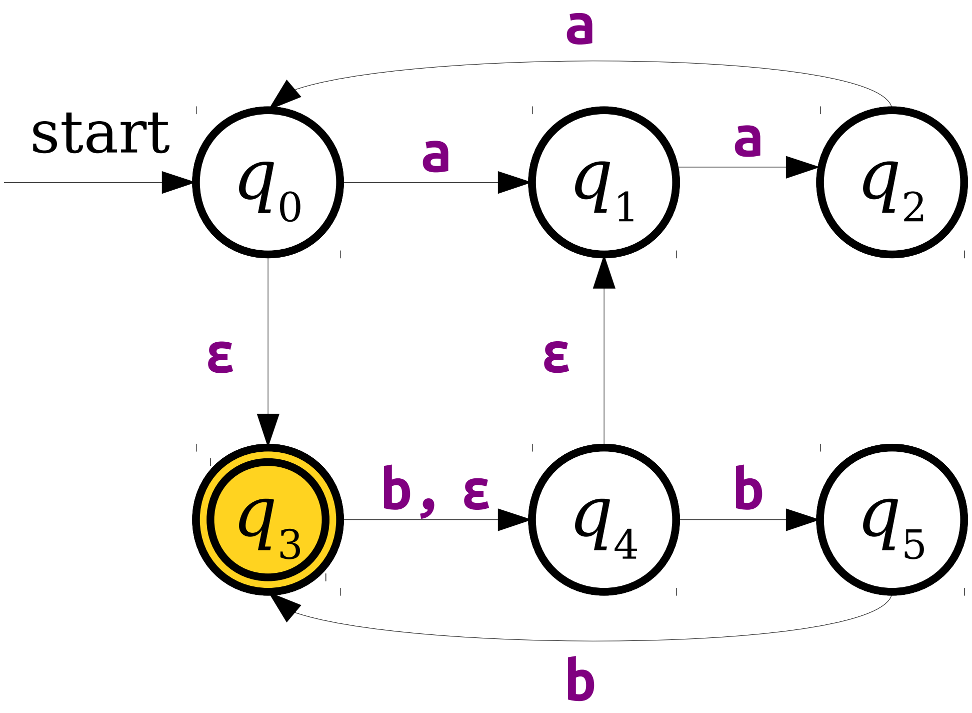

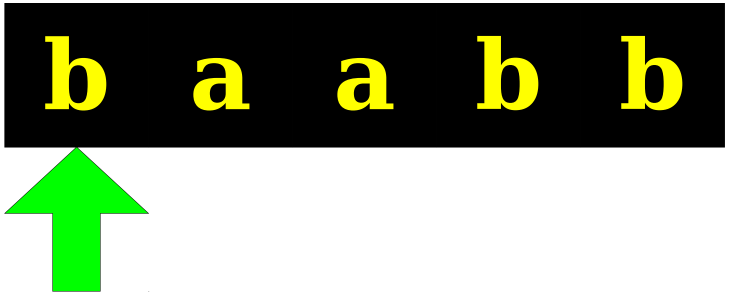

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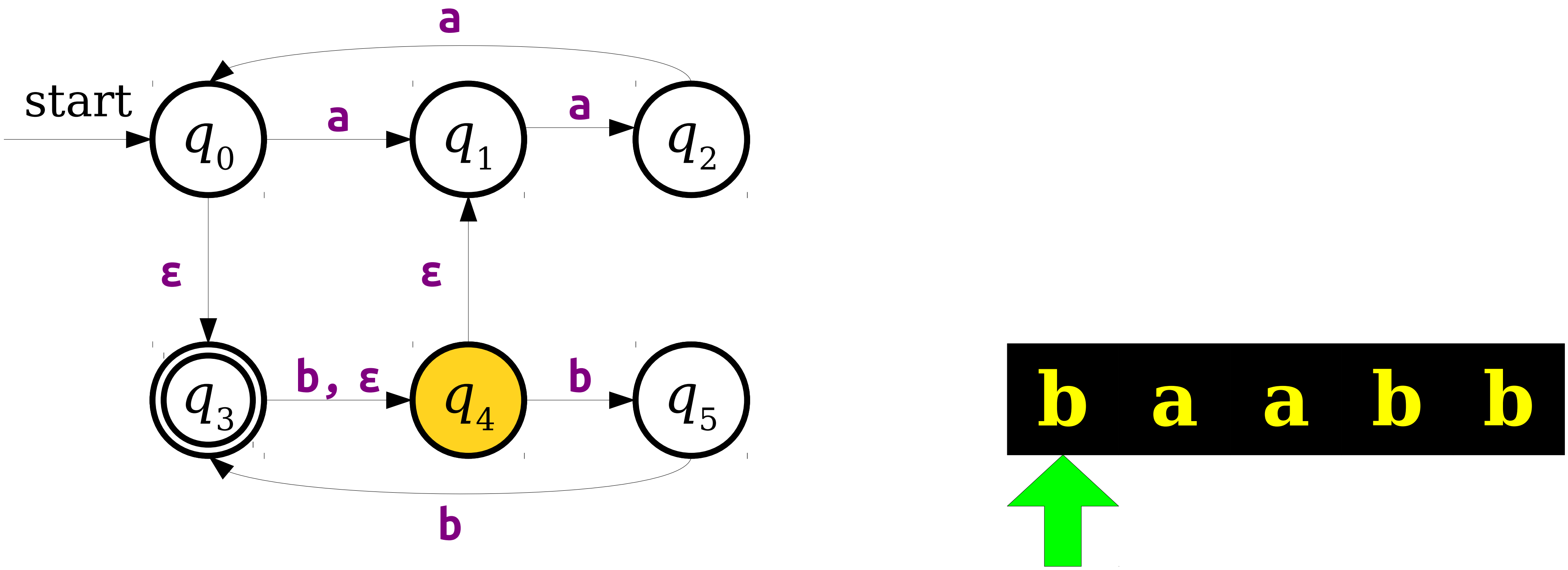

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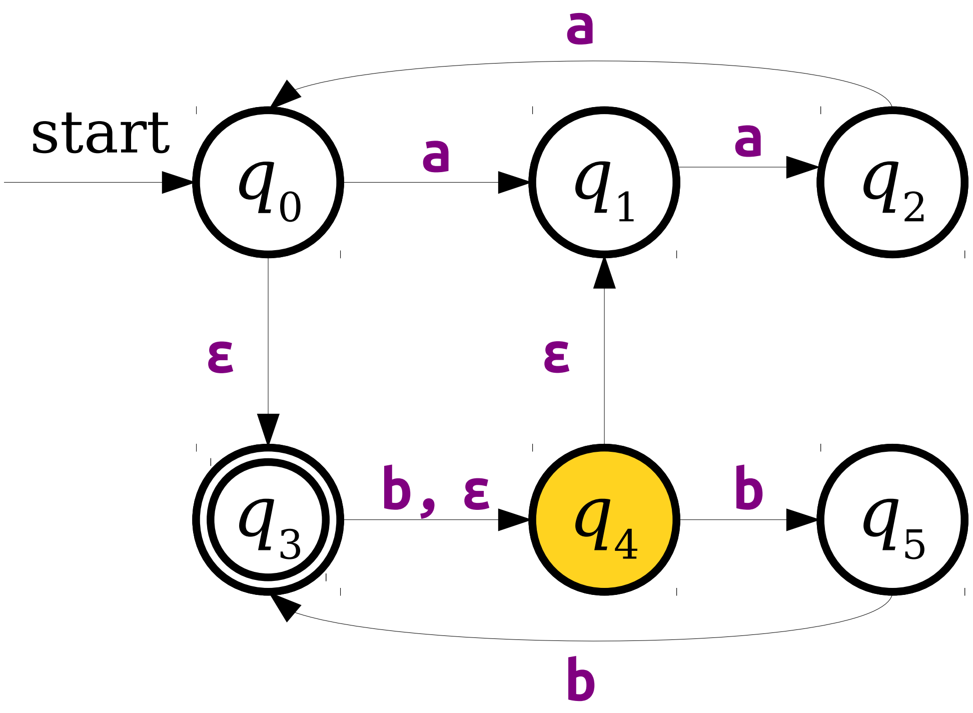

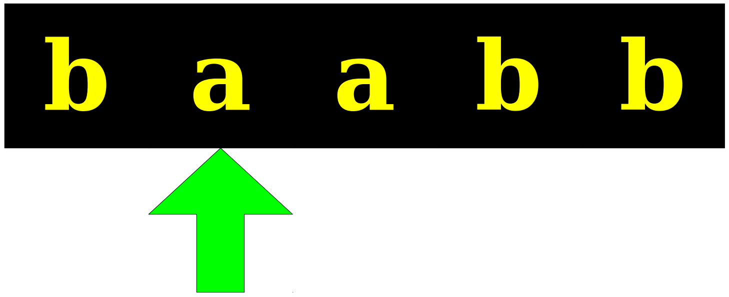

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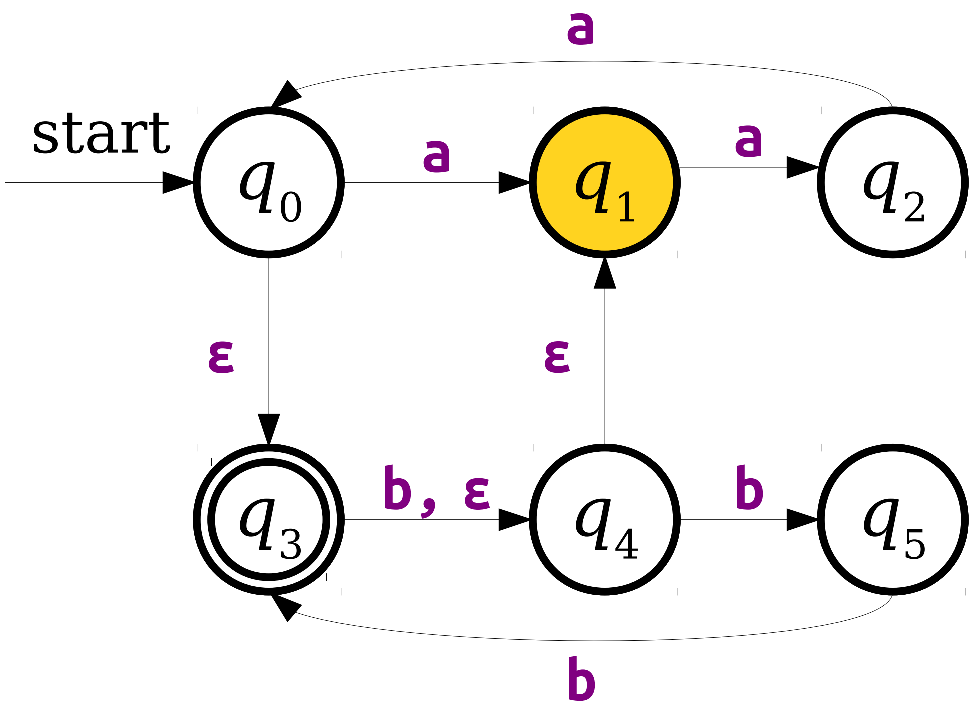

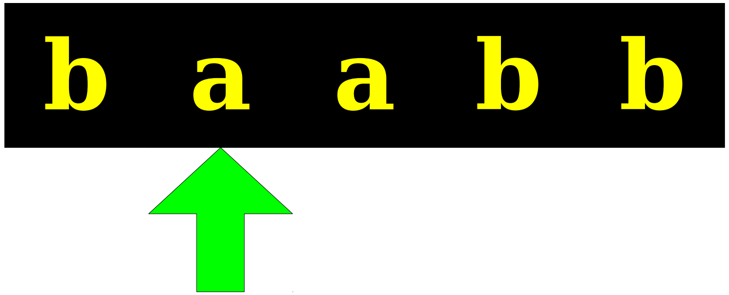

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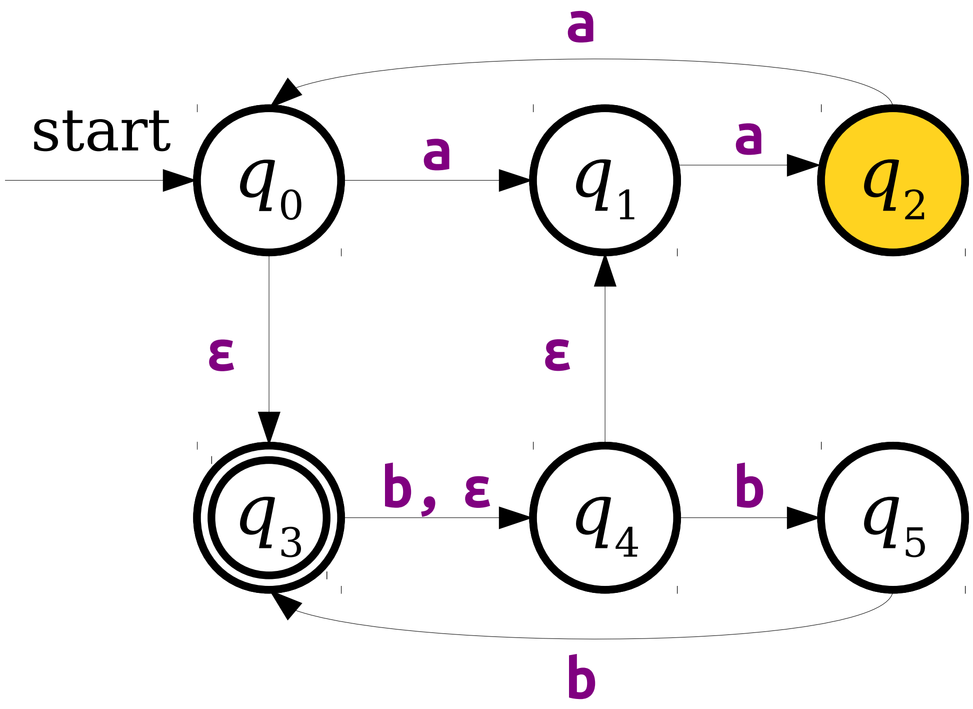

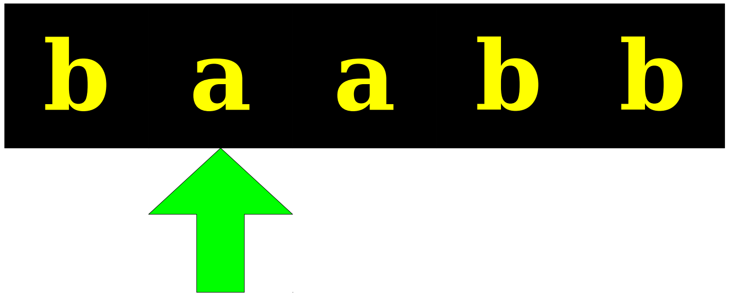

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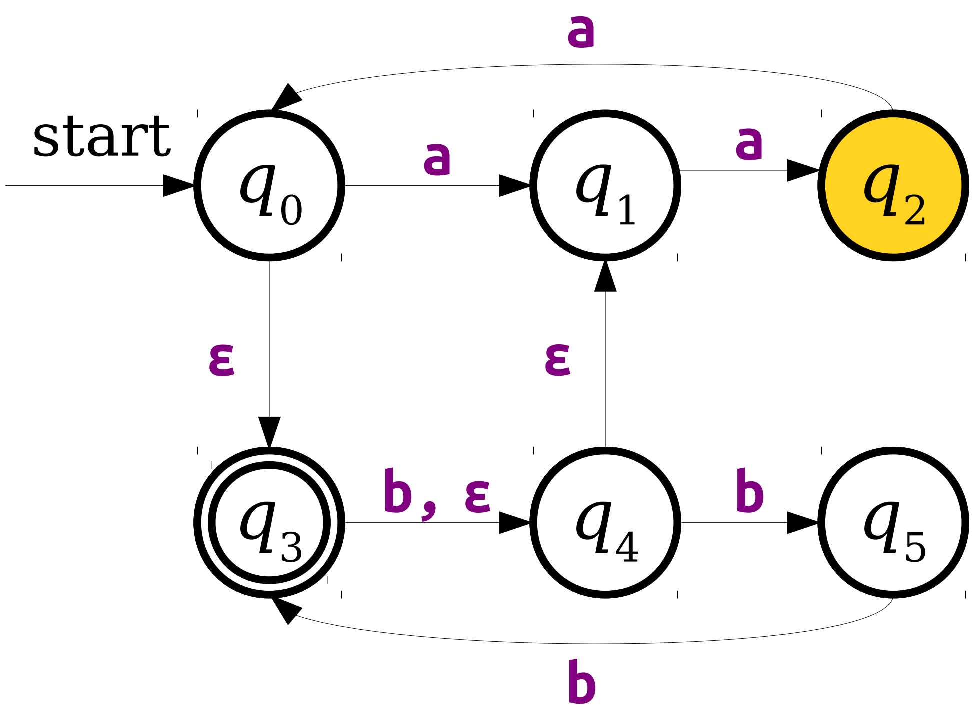

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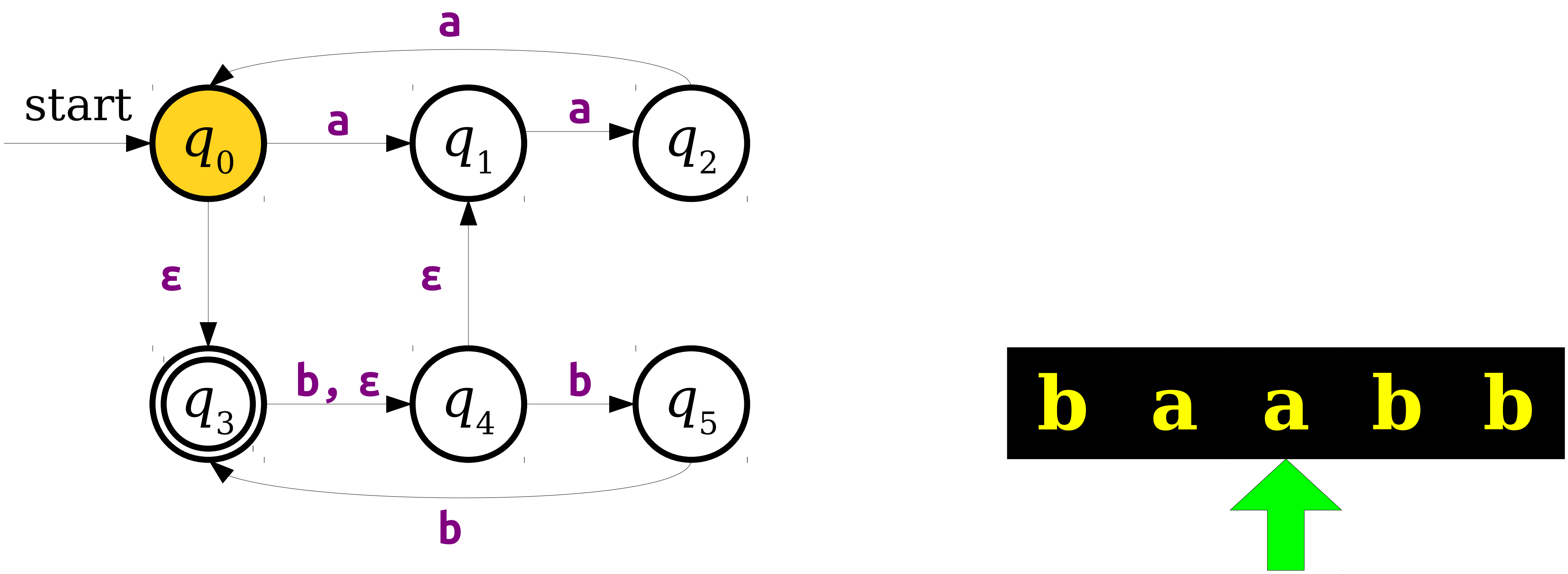

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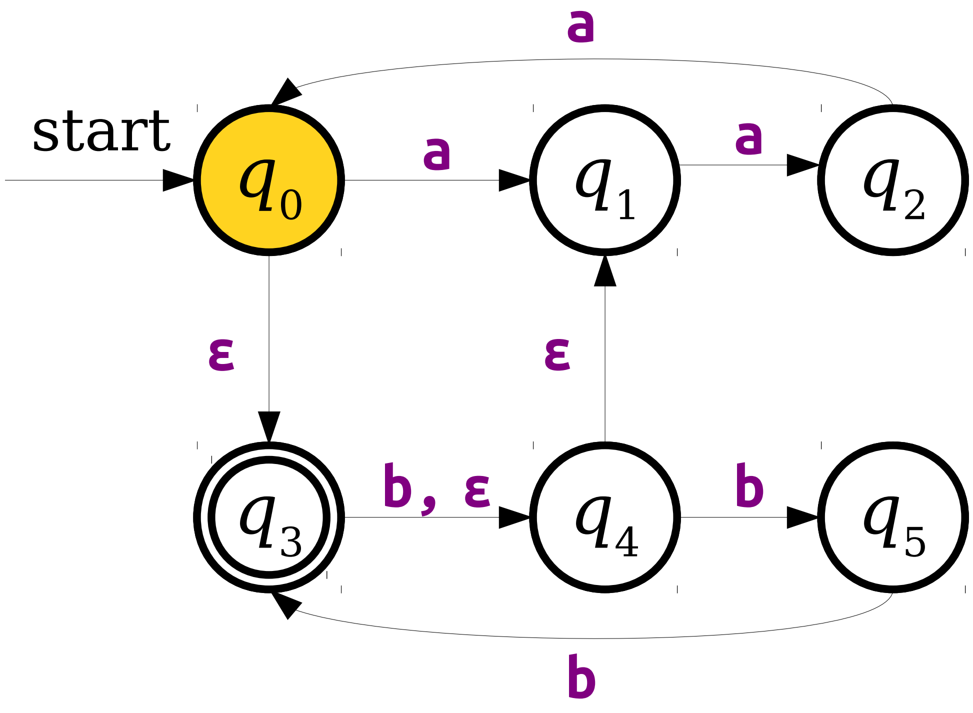

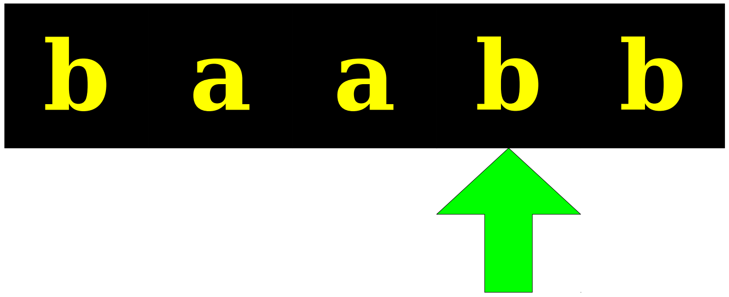

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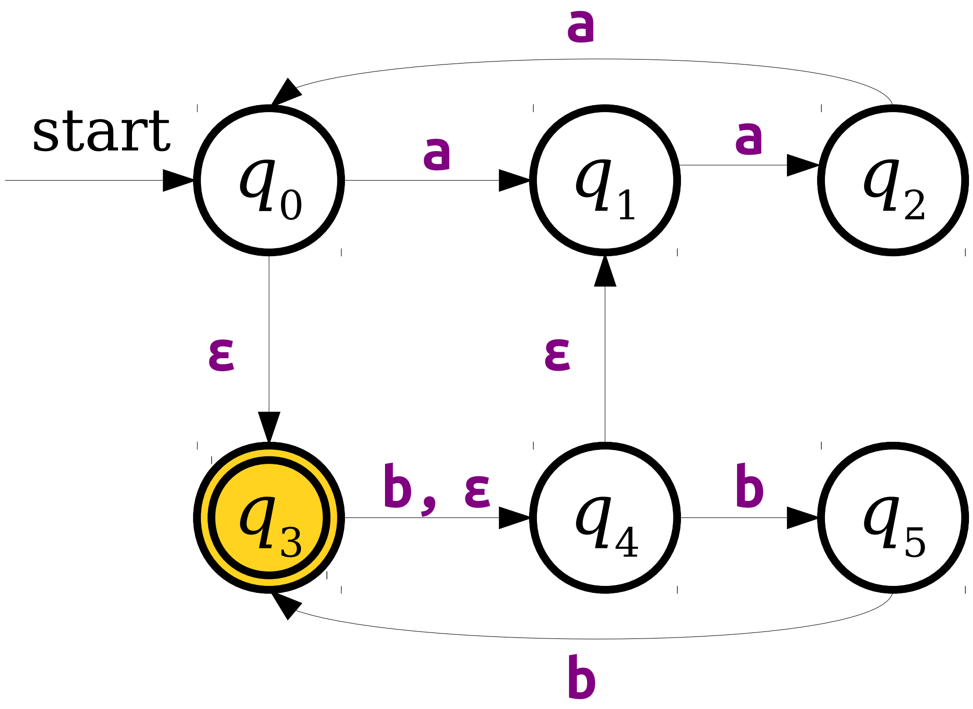

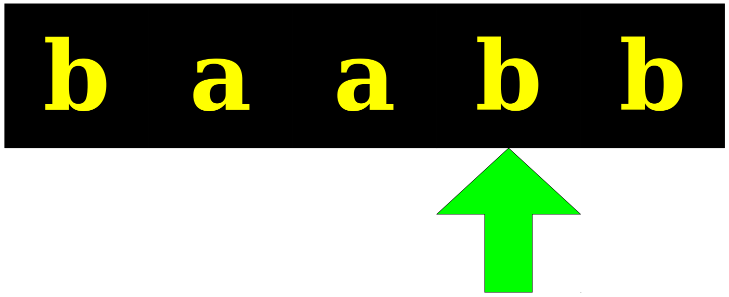

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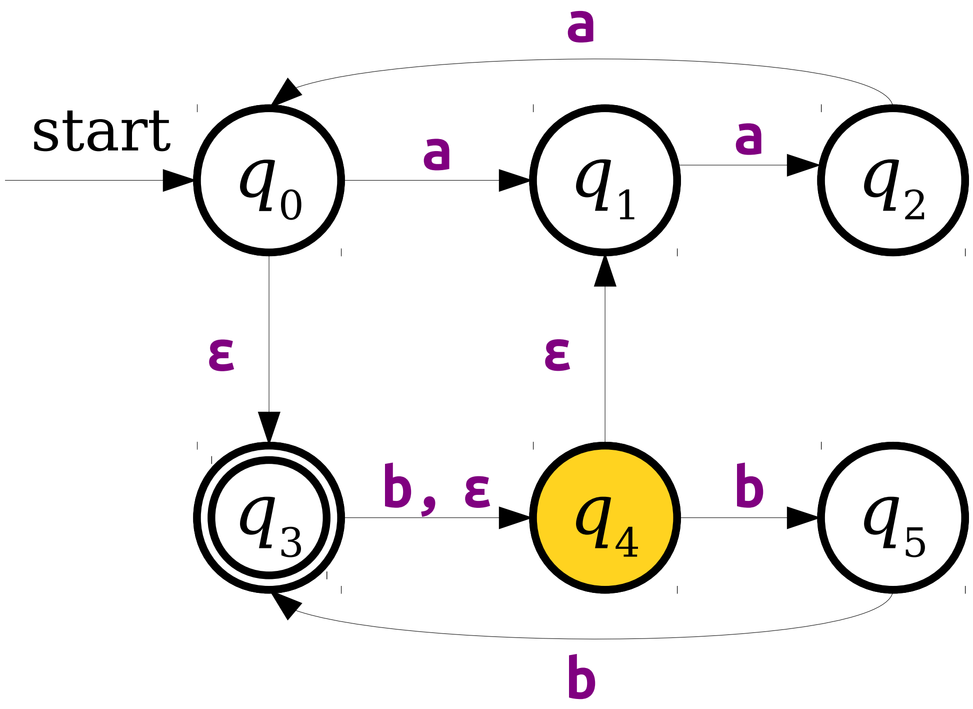

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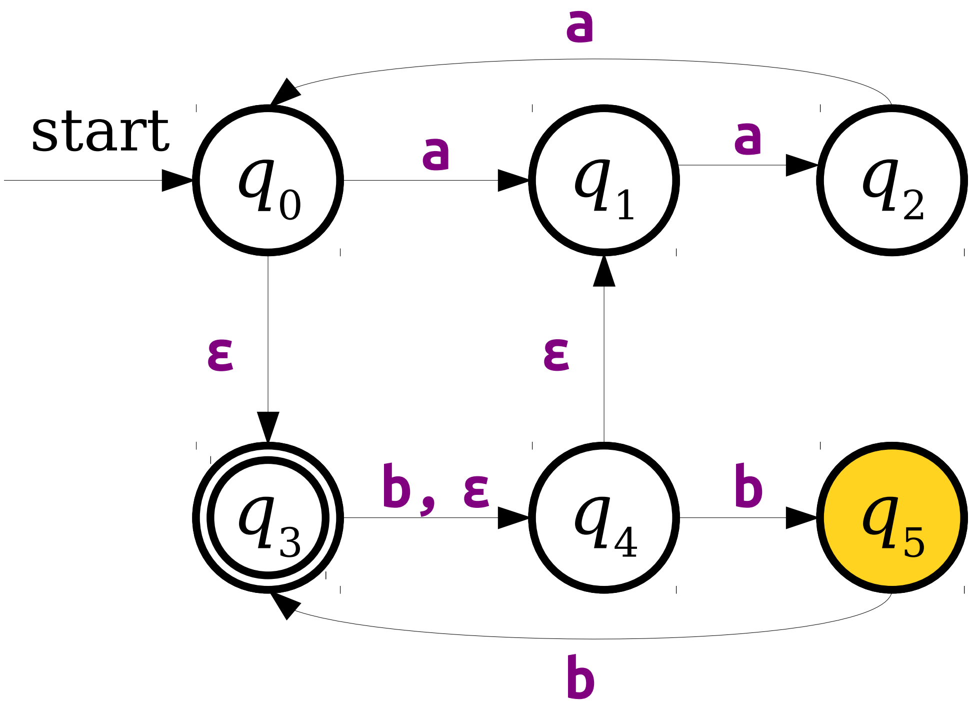

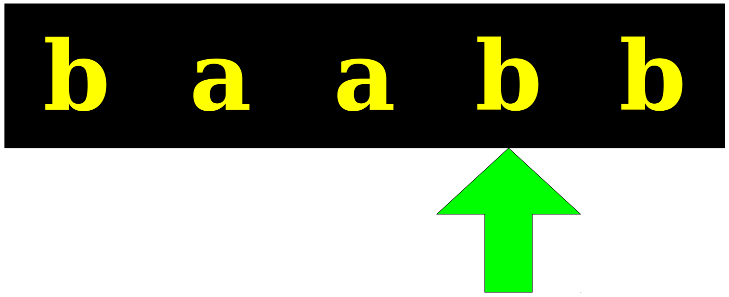

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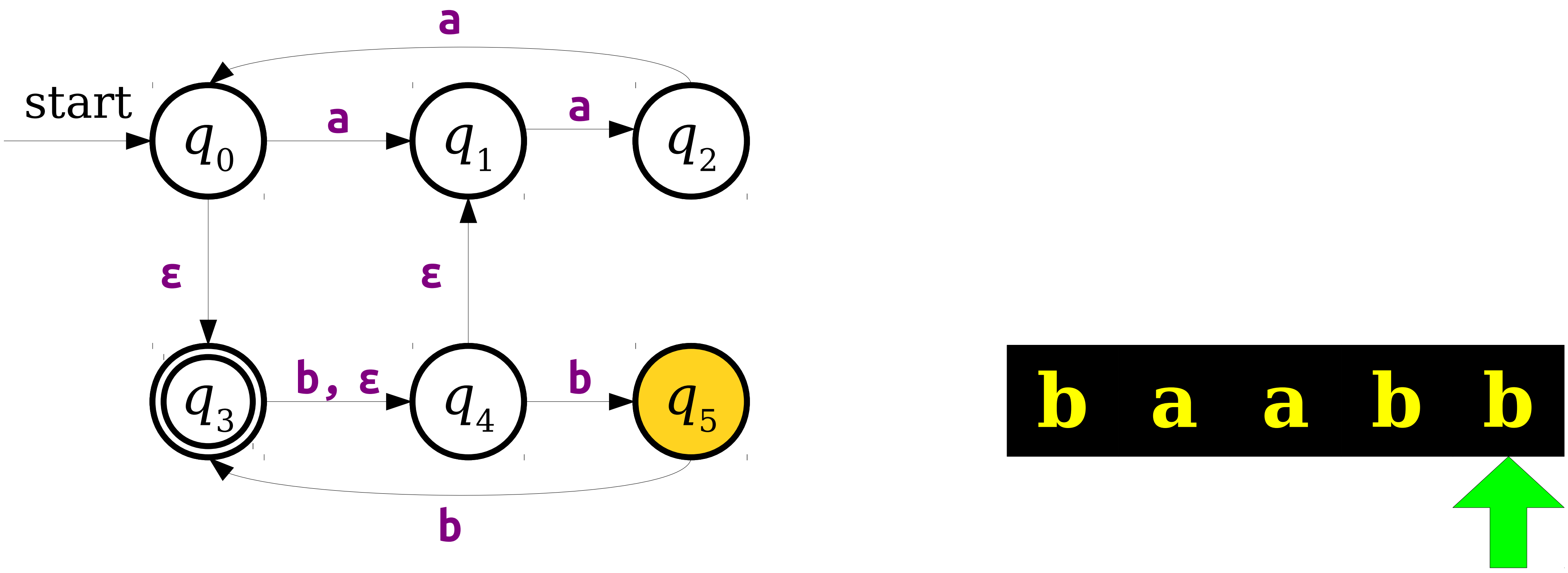

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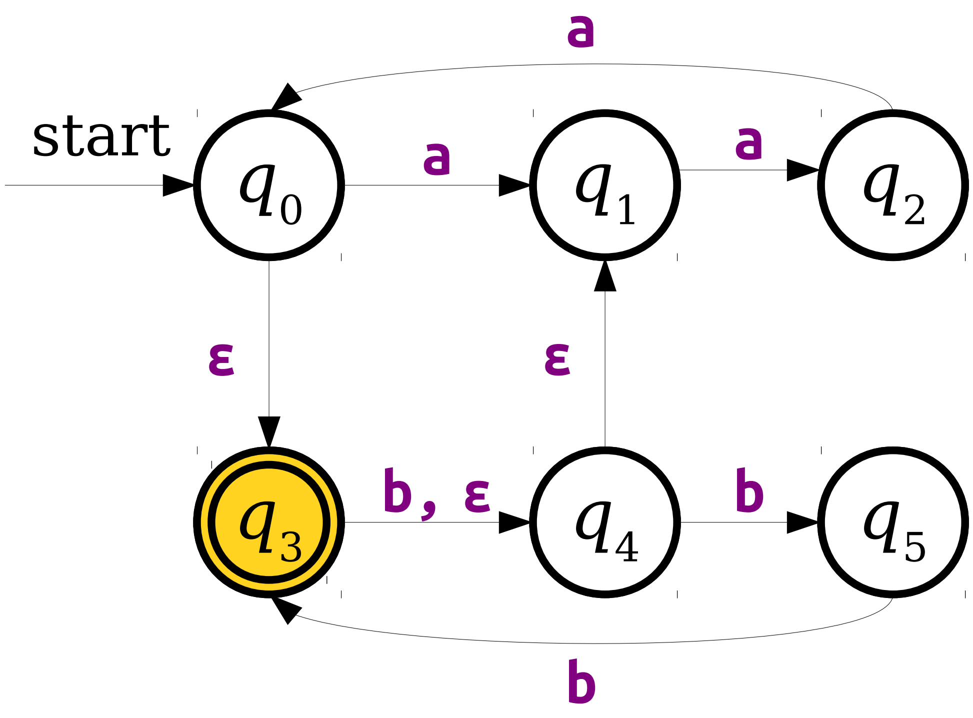

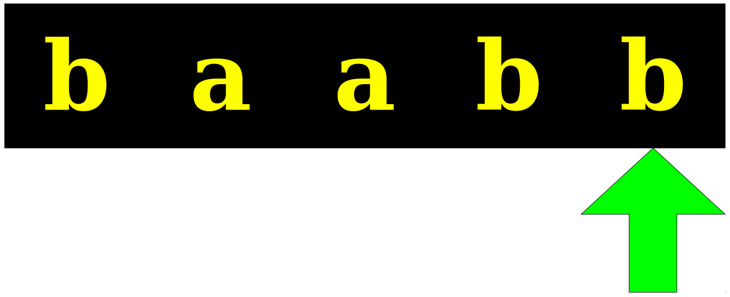

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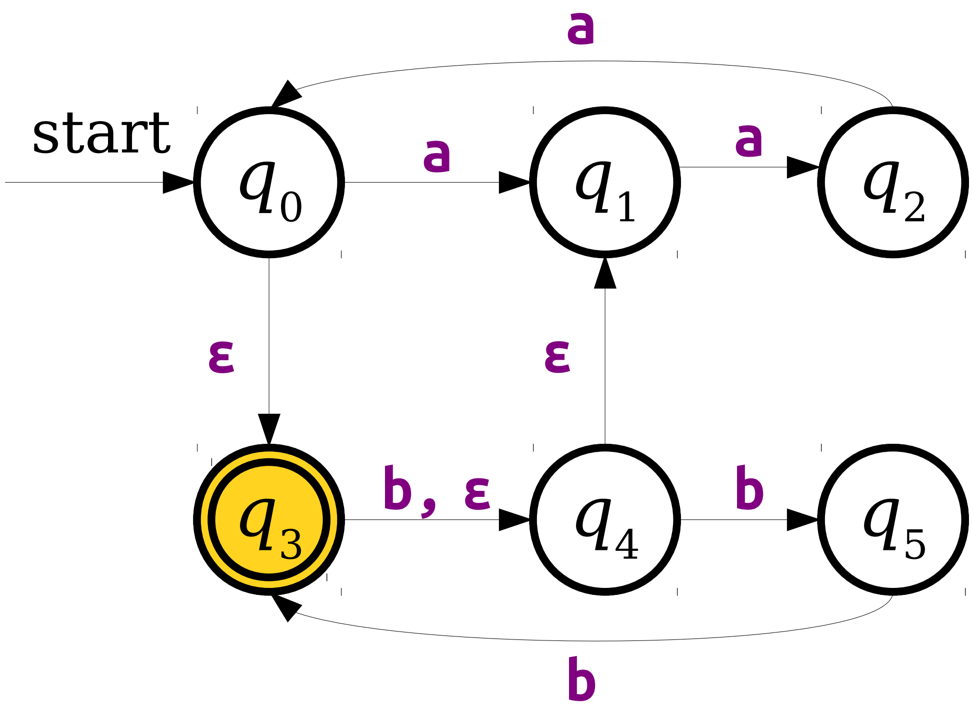


b a a b b

New Stuff!

DFAs and NFAs

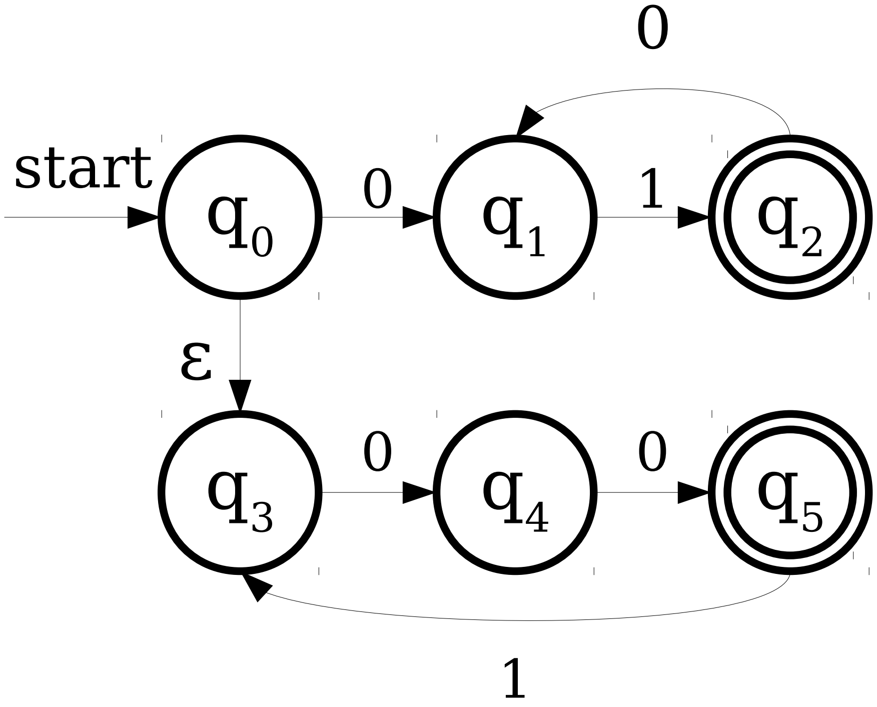
NFAs and DFAs

- Any language that can be accepted by a DFA can be accepted by an NFA.
- Why?
  - Just use the same set of transitions as before.
- ***Question**:* Can any language accepted by an NFA also be accepted by a DFA?
- Surprisingly, the answer is ***yes!***

Finite Automata

- NFAs and DFAs are *finite* automata; there can only be finitely many states in an NFA or DFA.
- An NFA can be in any combination of its states, but there are only finitely many possible combations.
- ***Idea:*** Build a DFA where each state of the DFA corresponds to a *set* of states in the NFA.

Simulating an NFA with a DFA



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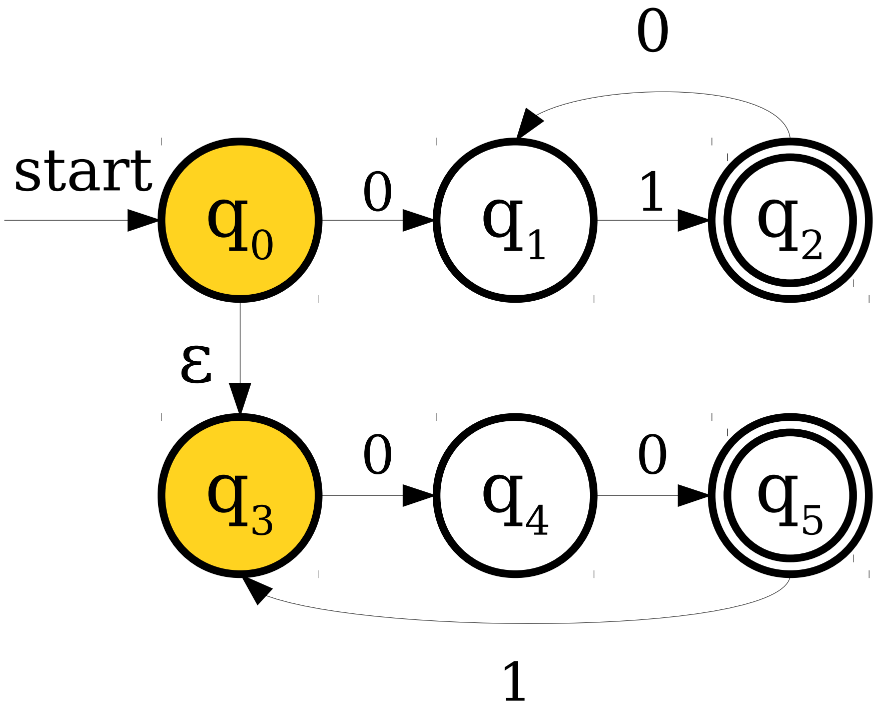

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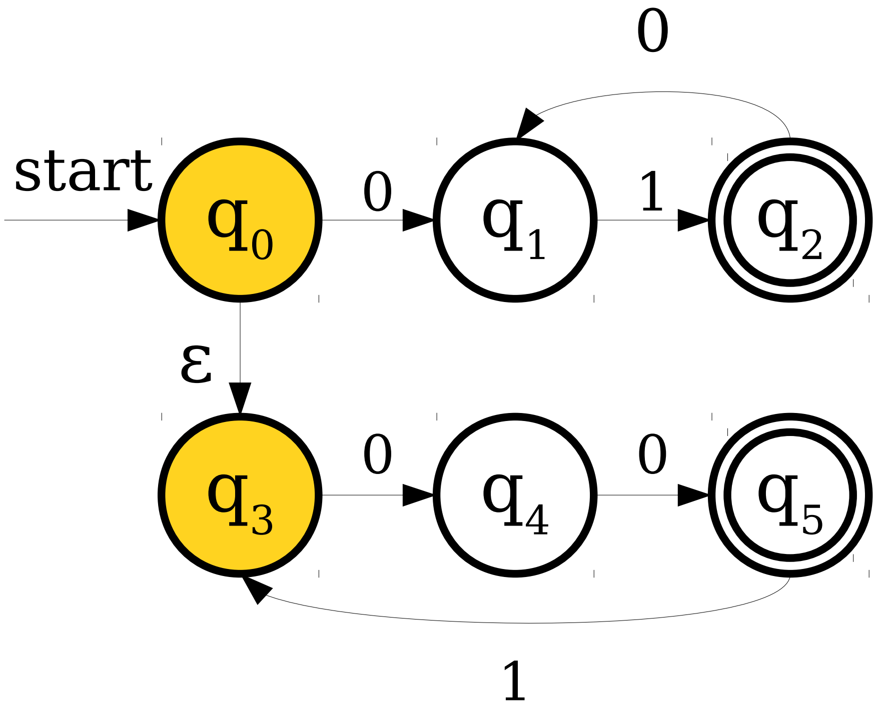

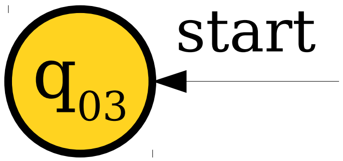

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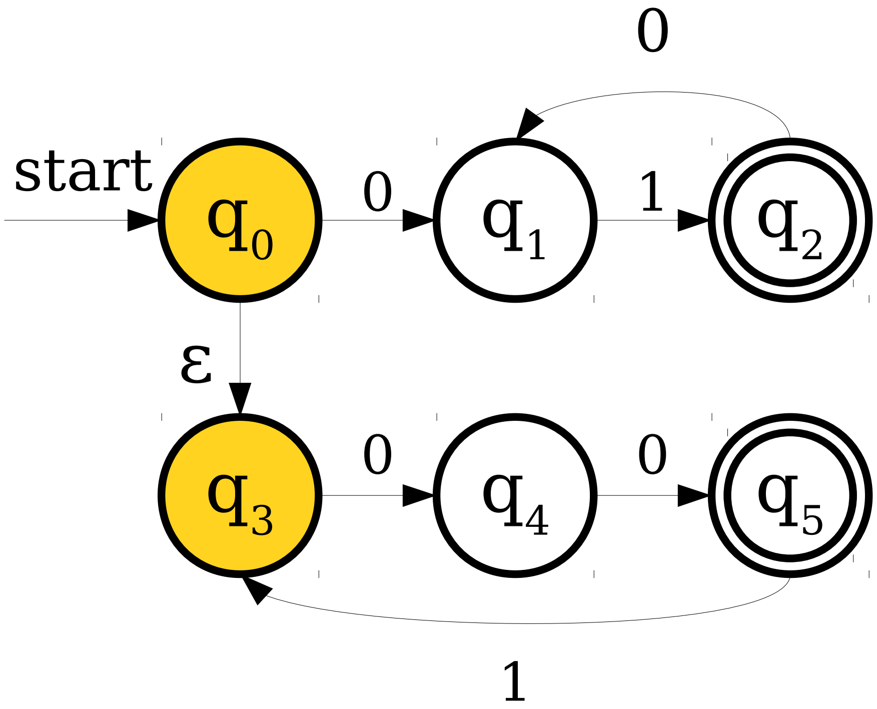

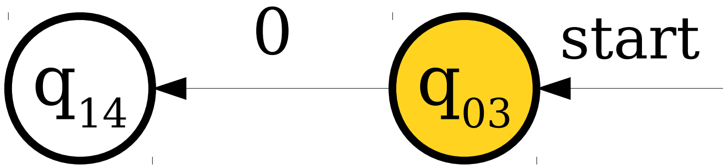

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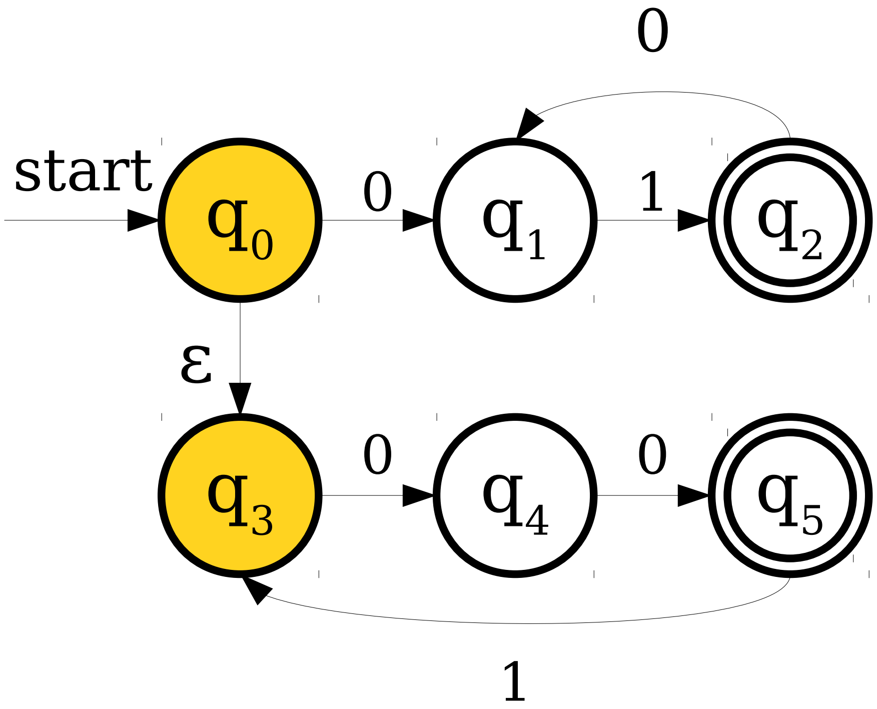

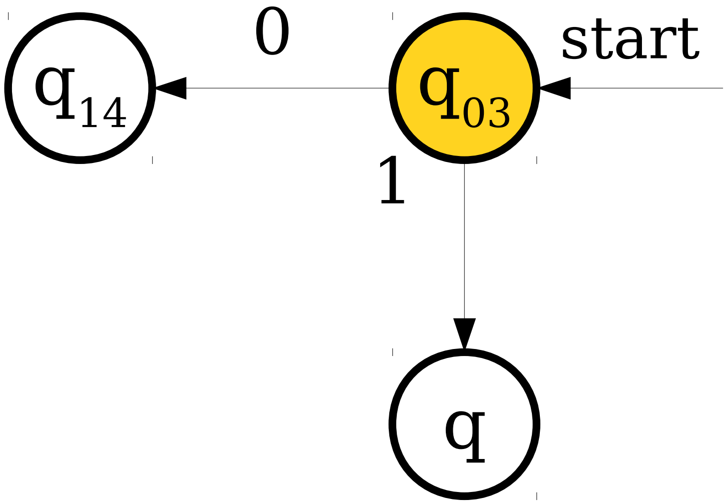

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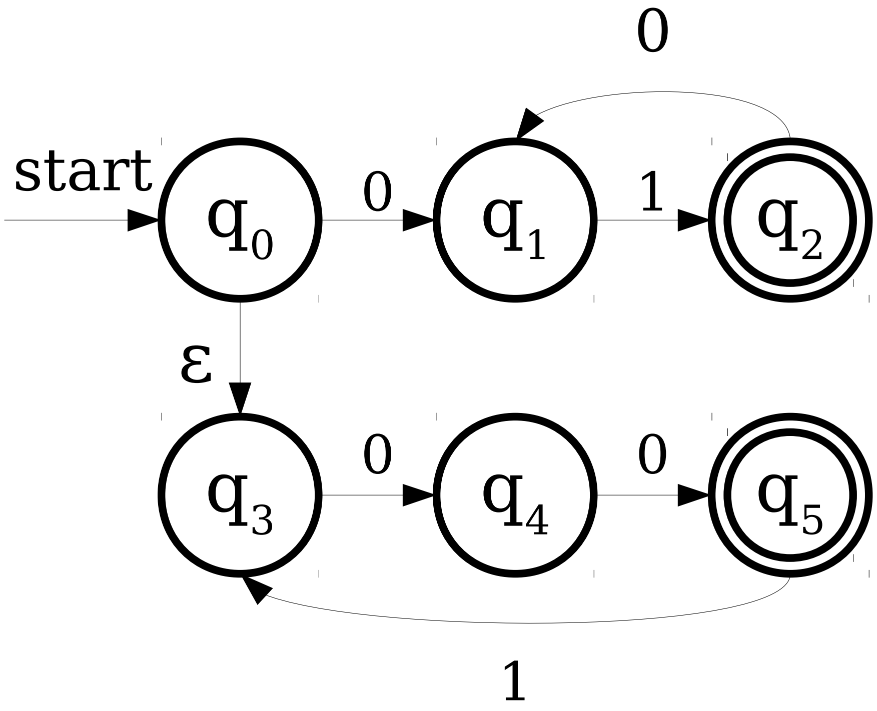

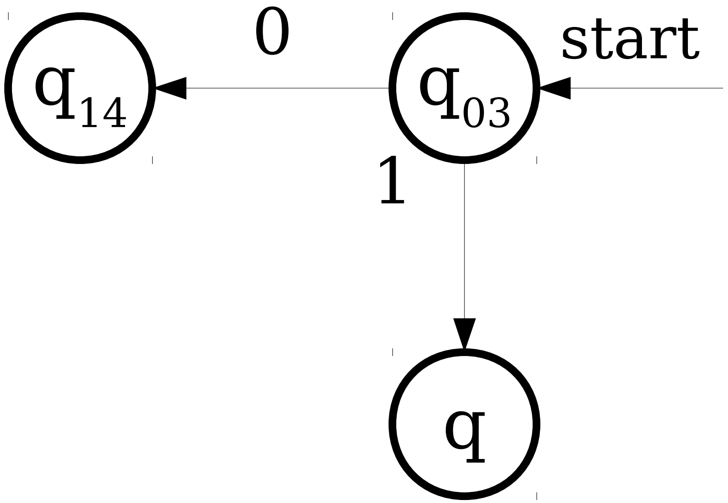

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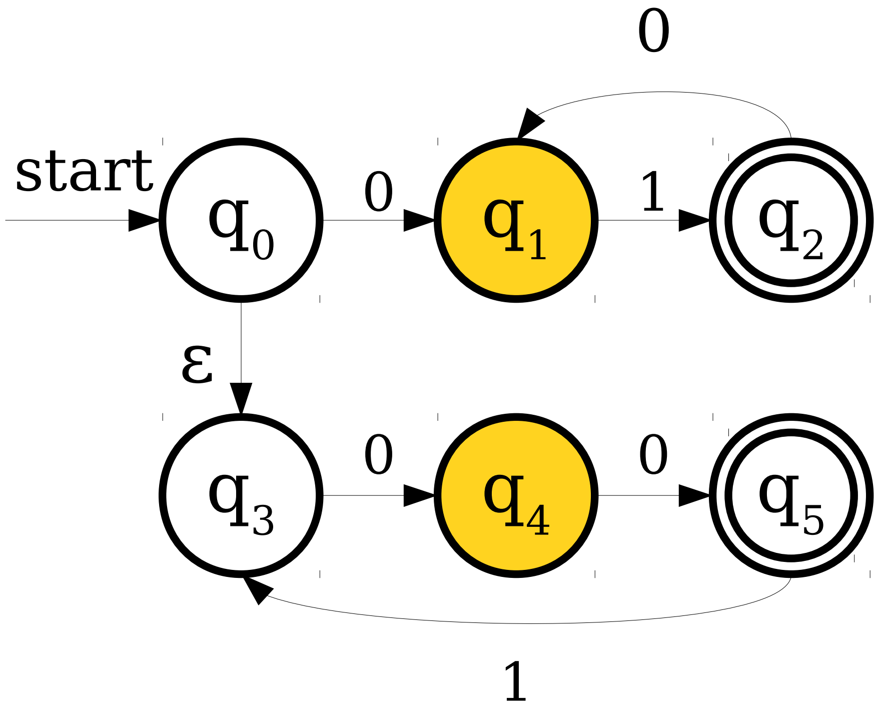

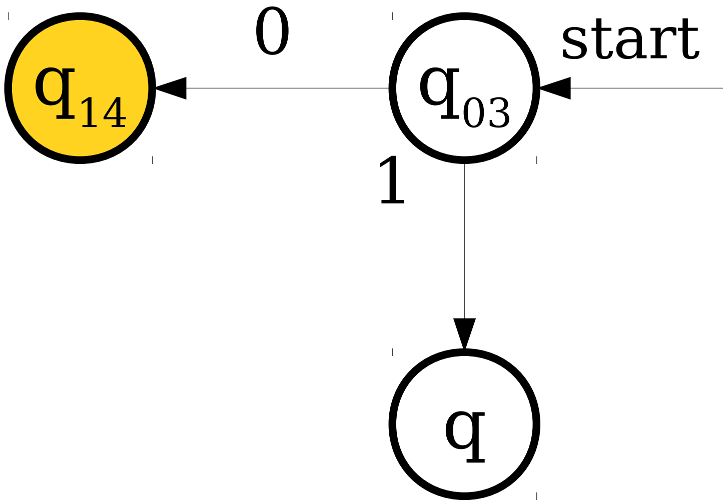

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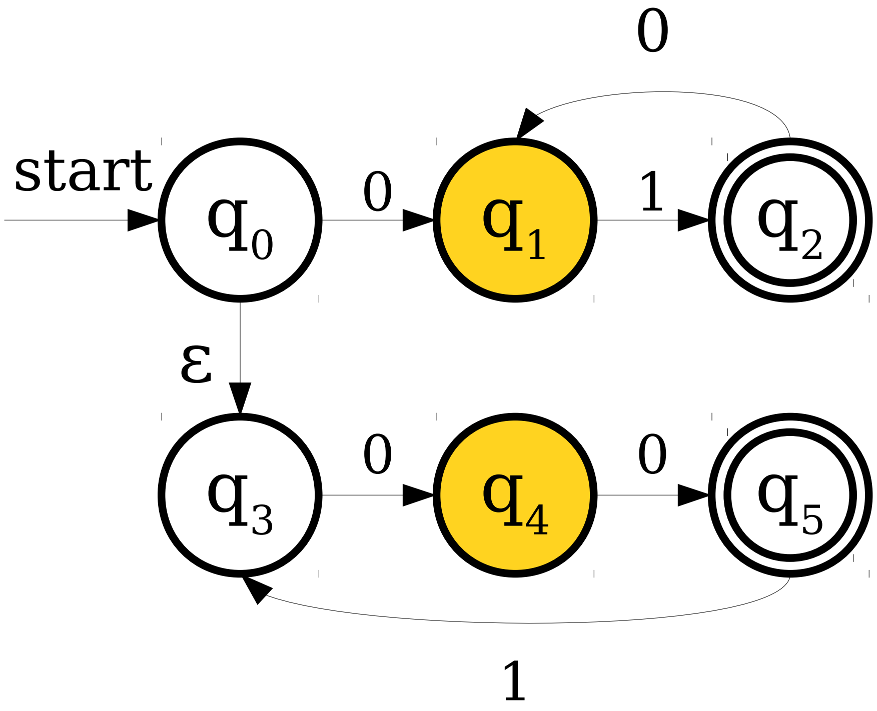

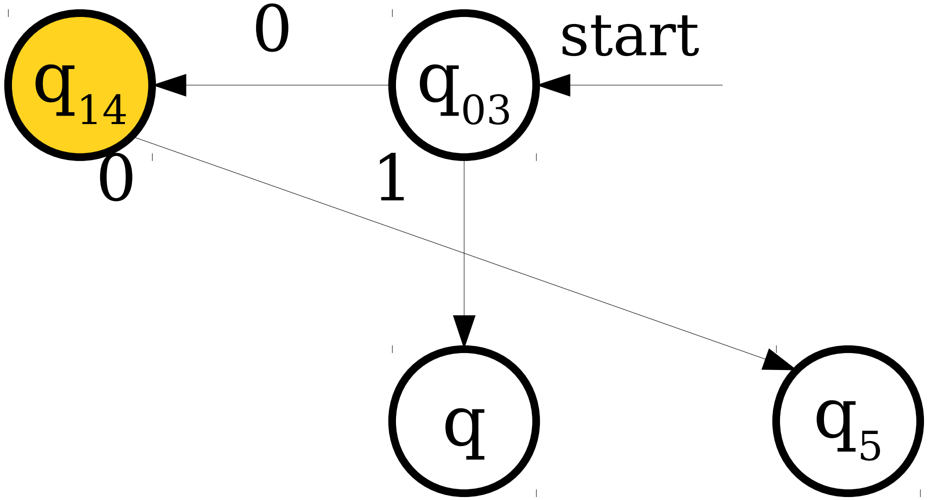

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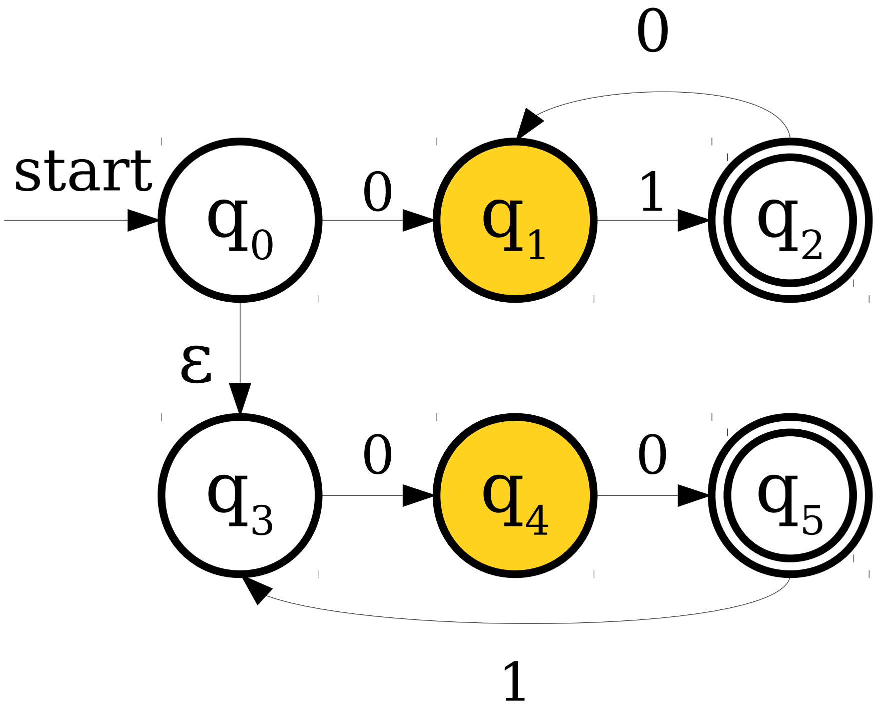

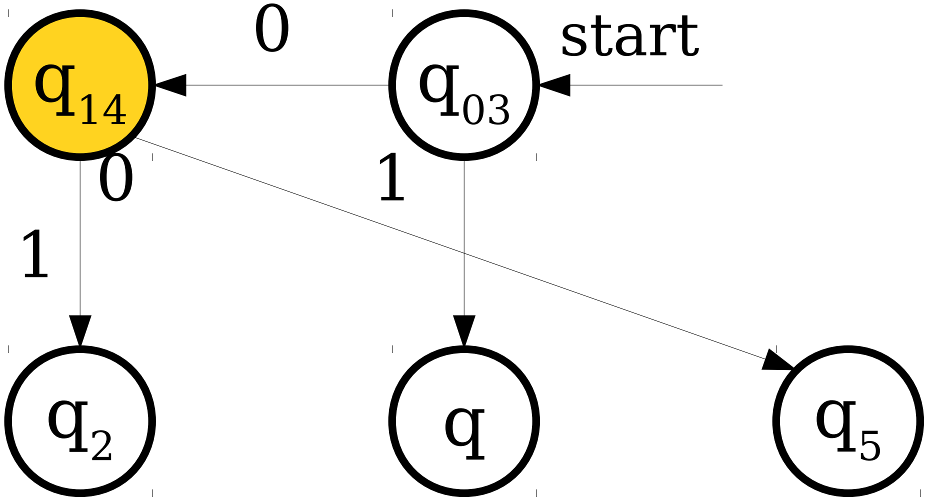

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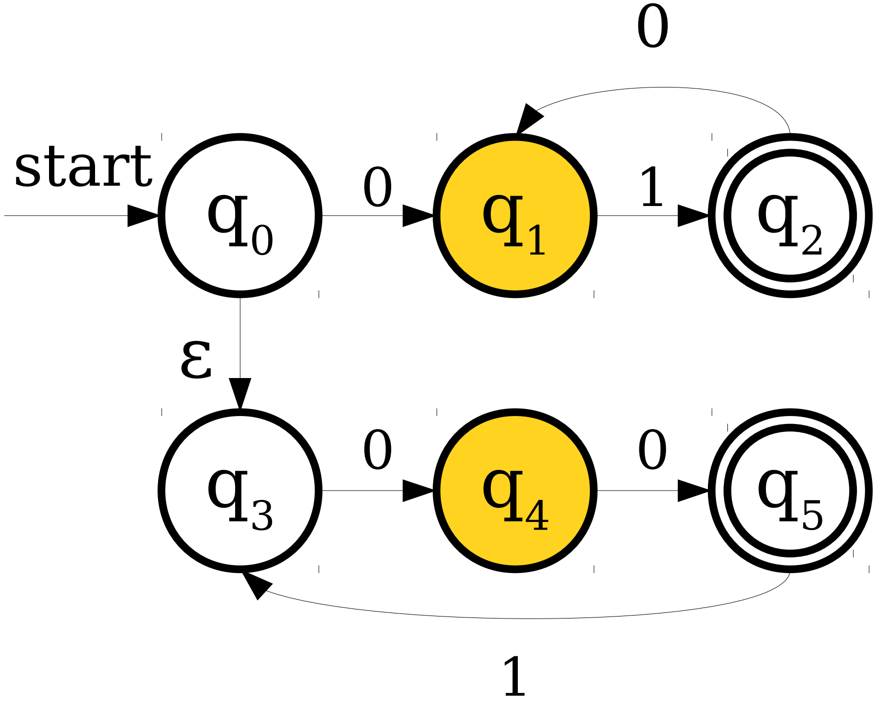

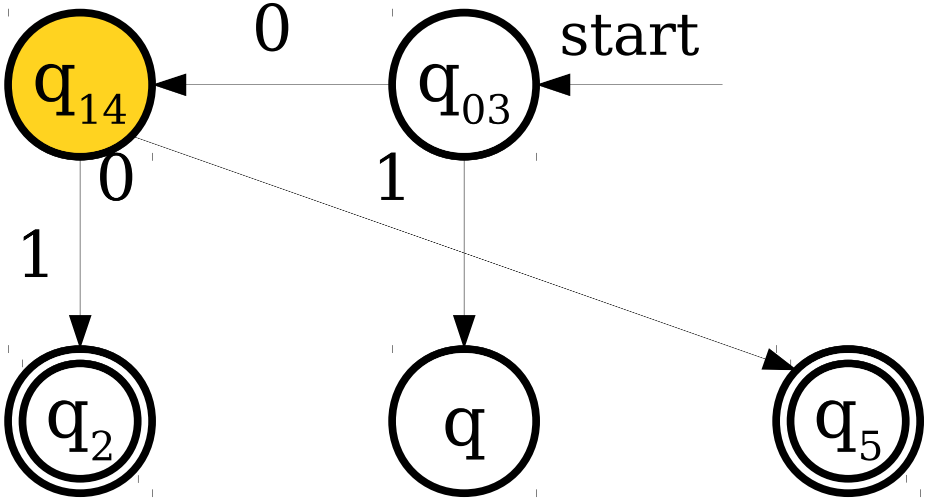

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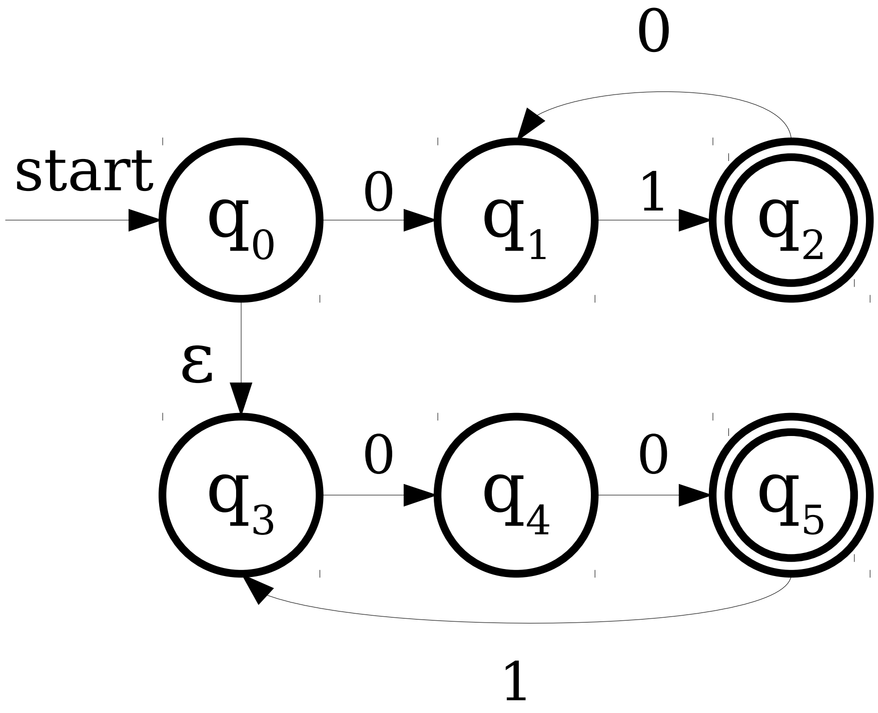

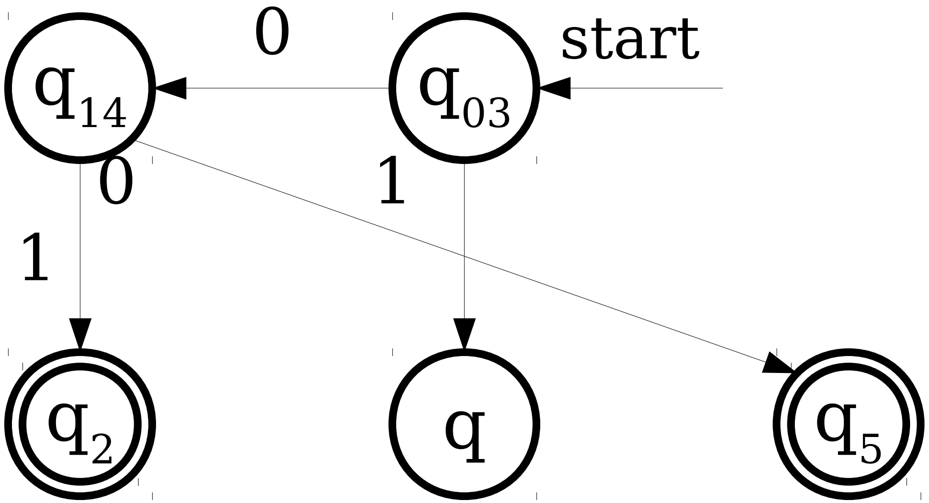

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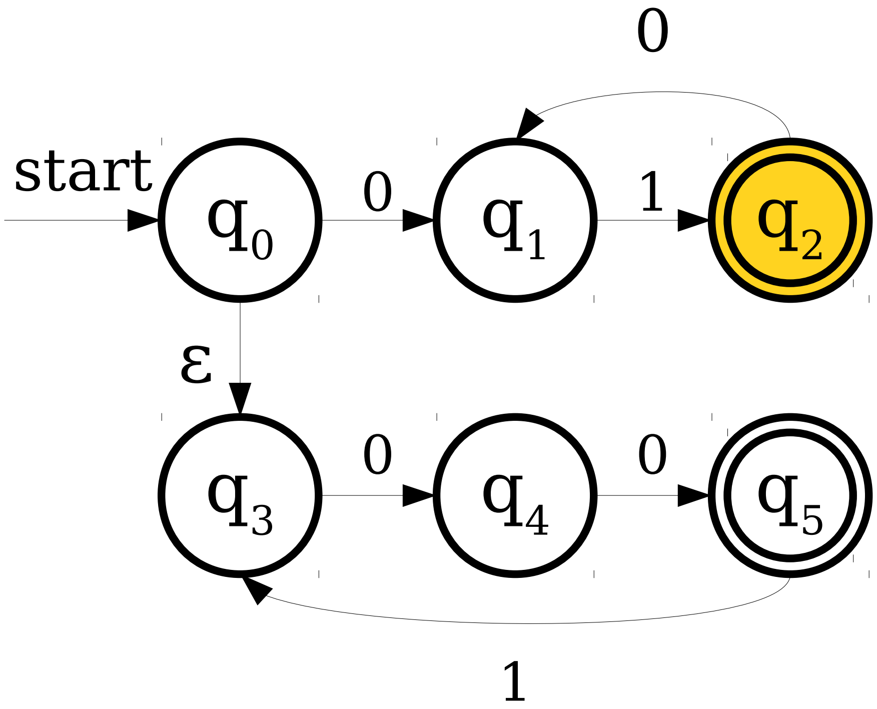

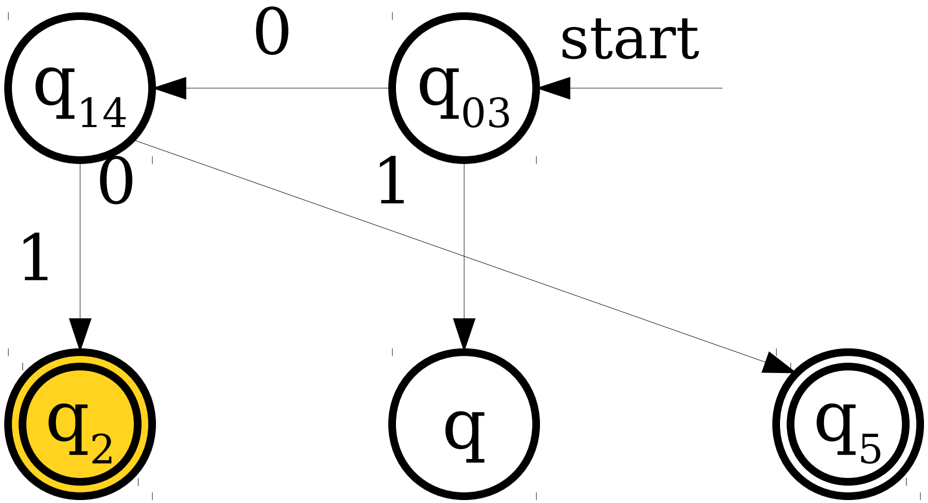

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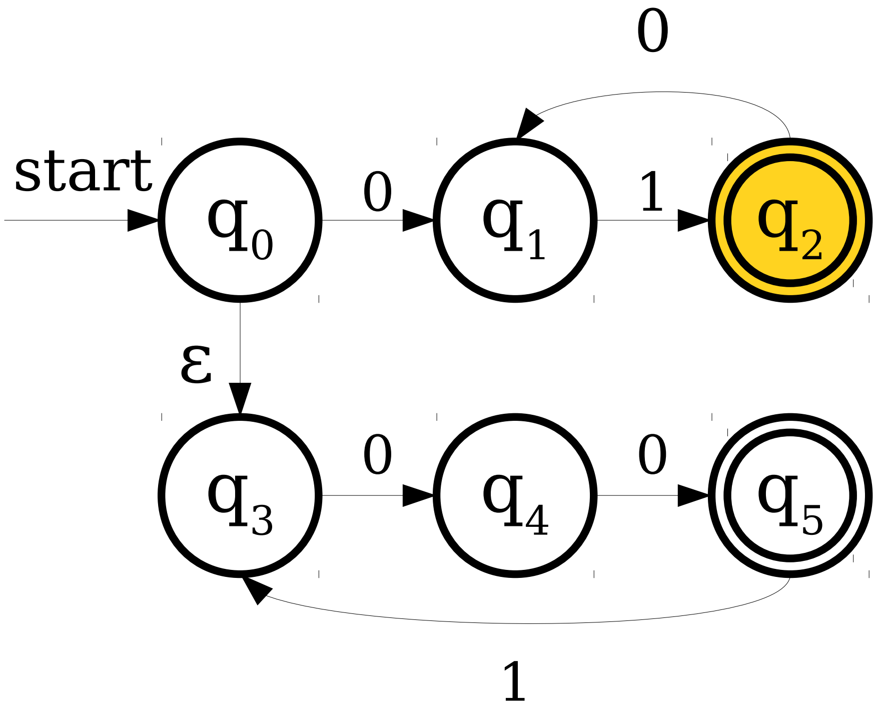

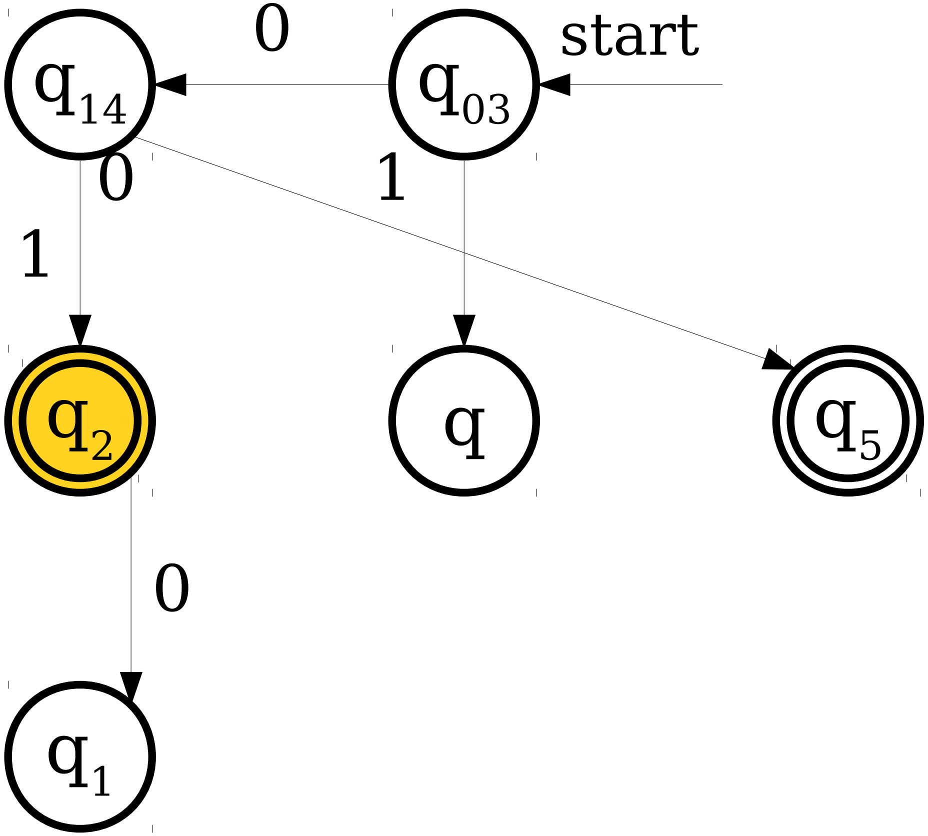

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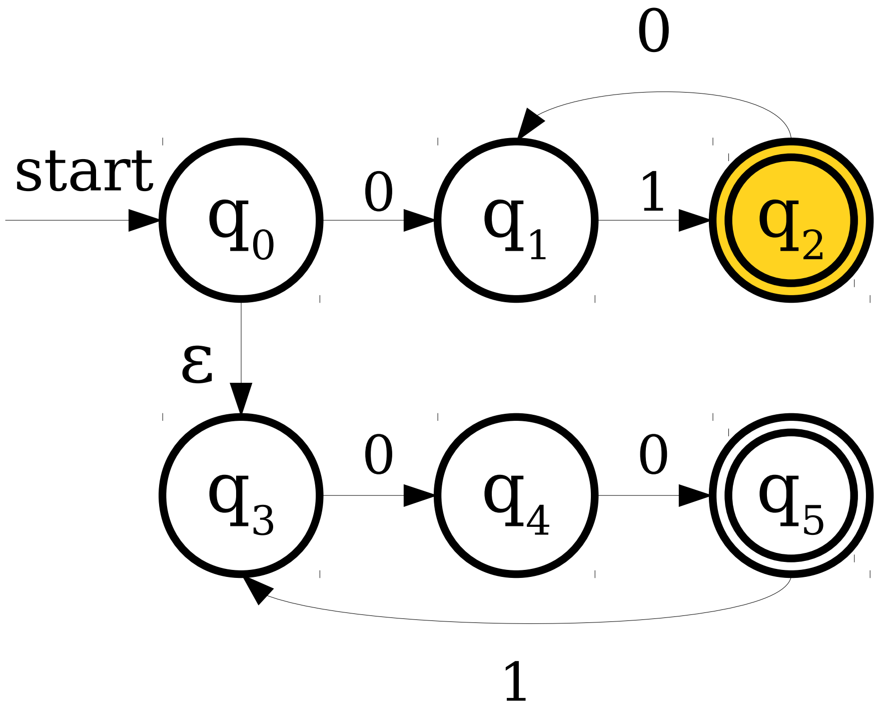

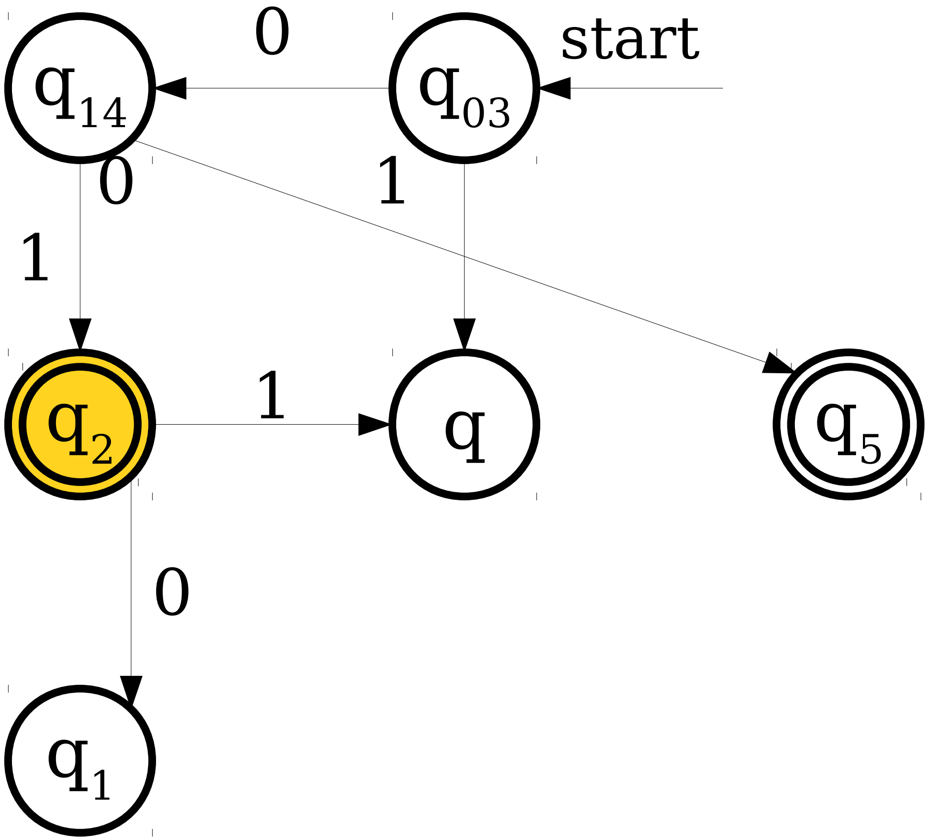

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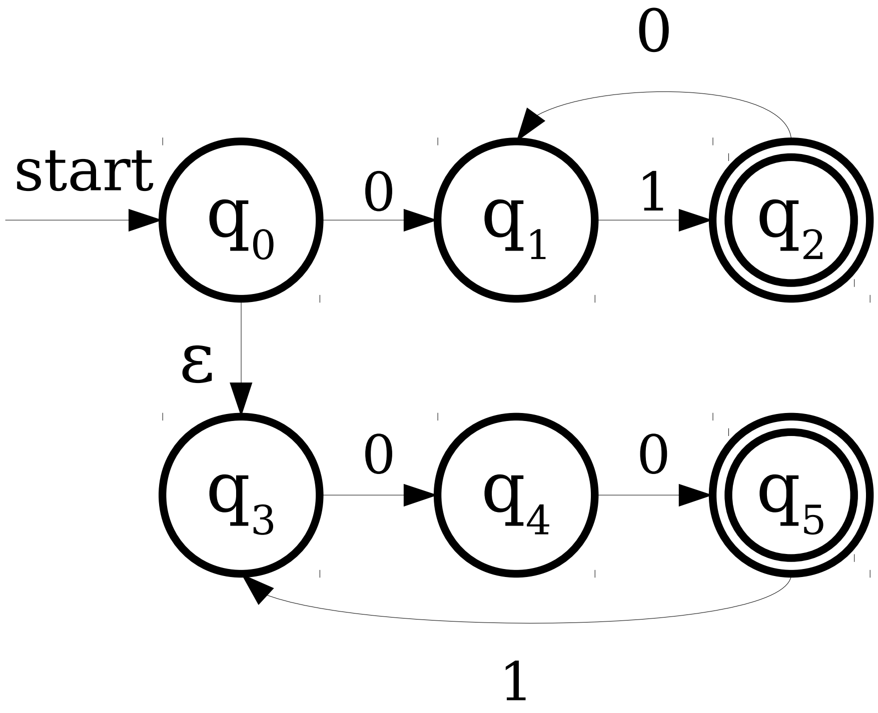

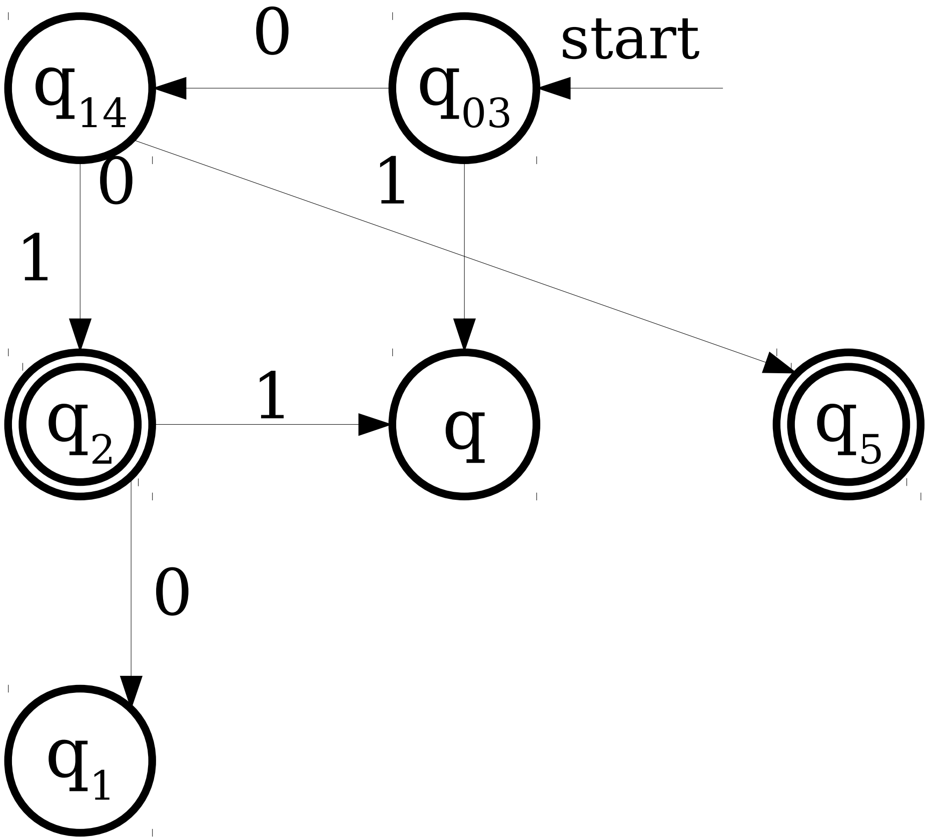

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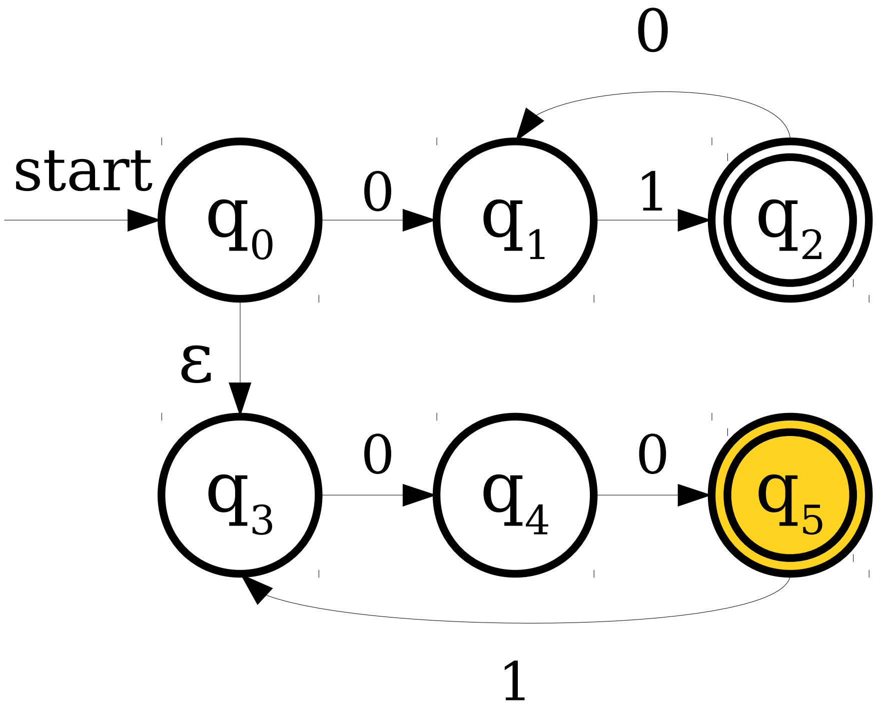

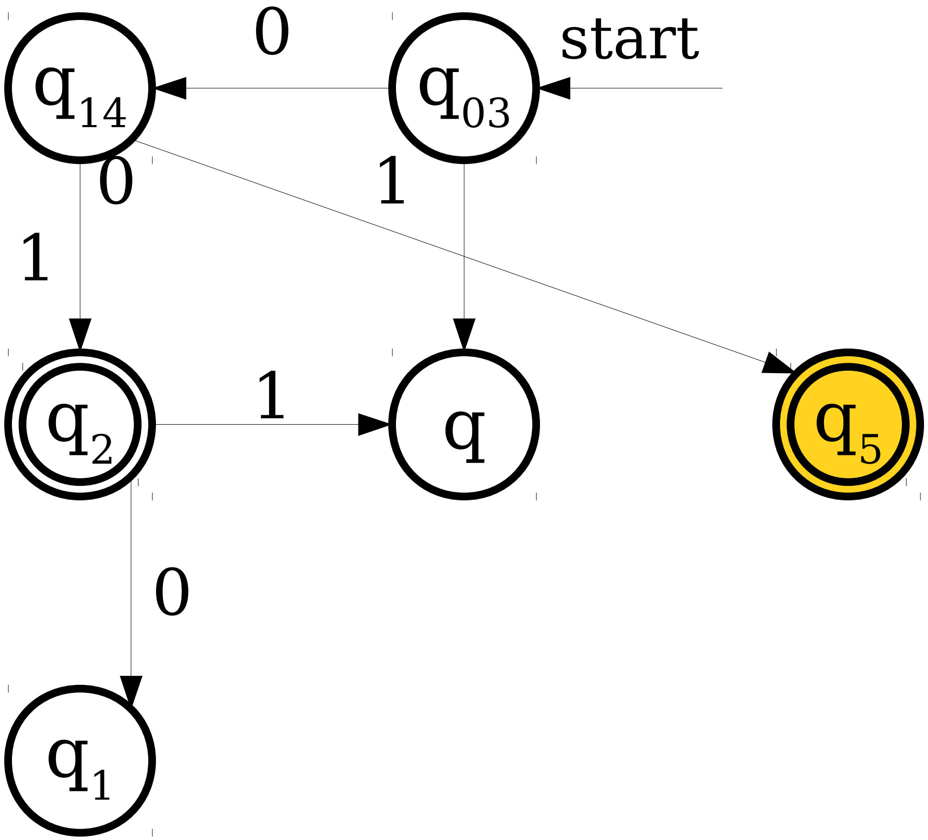

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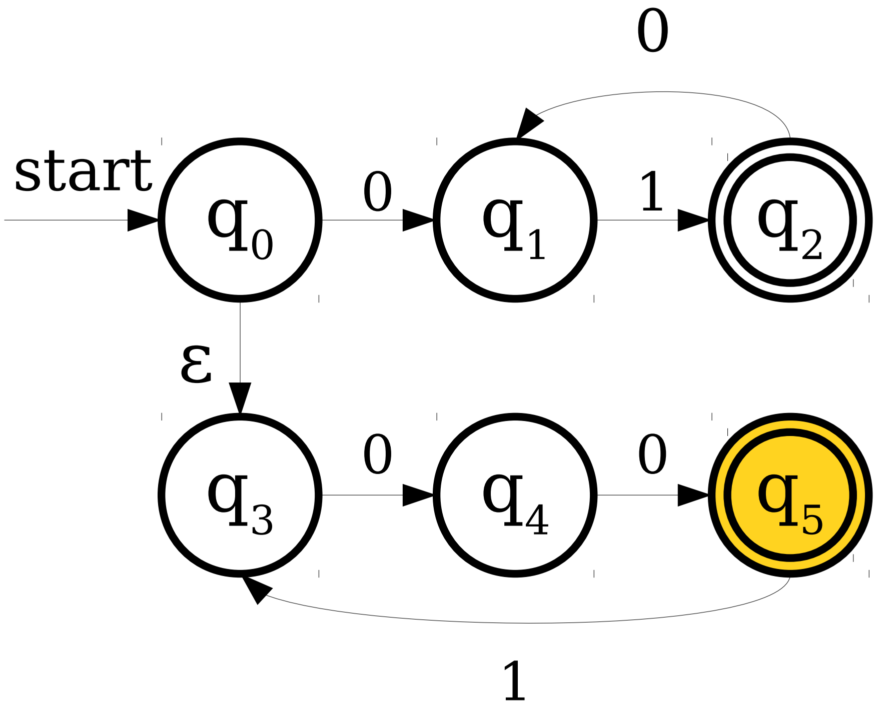

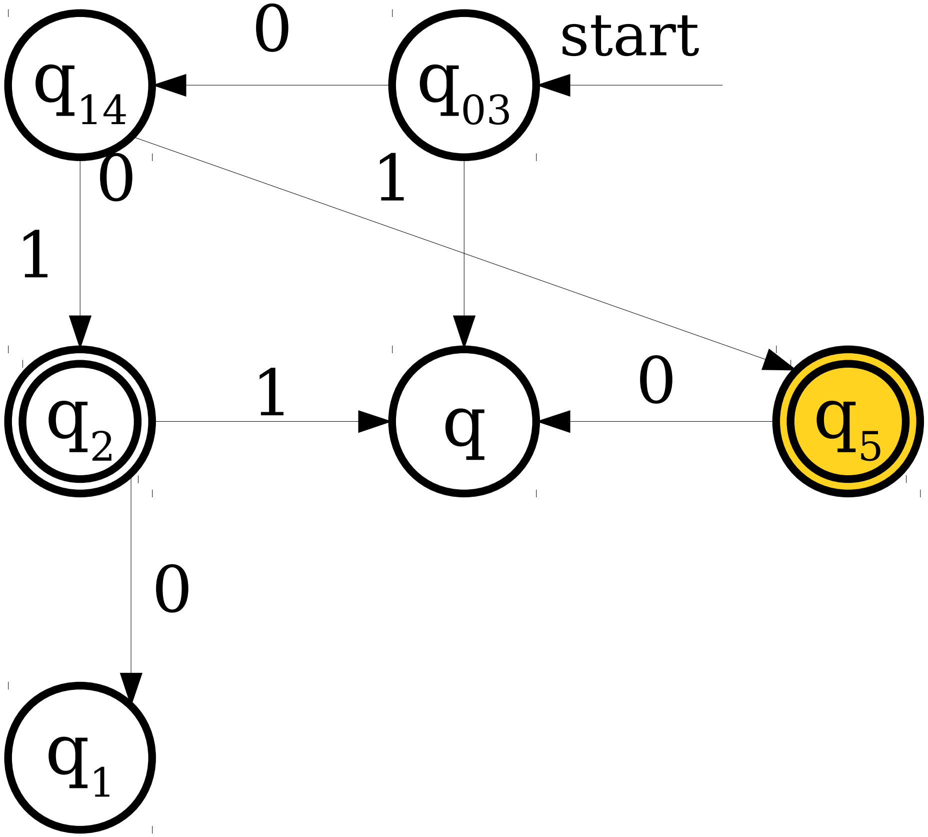

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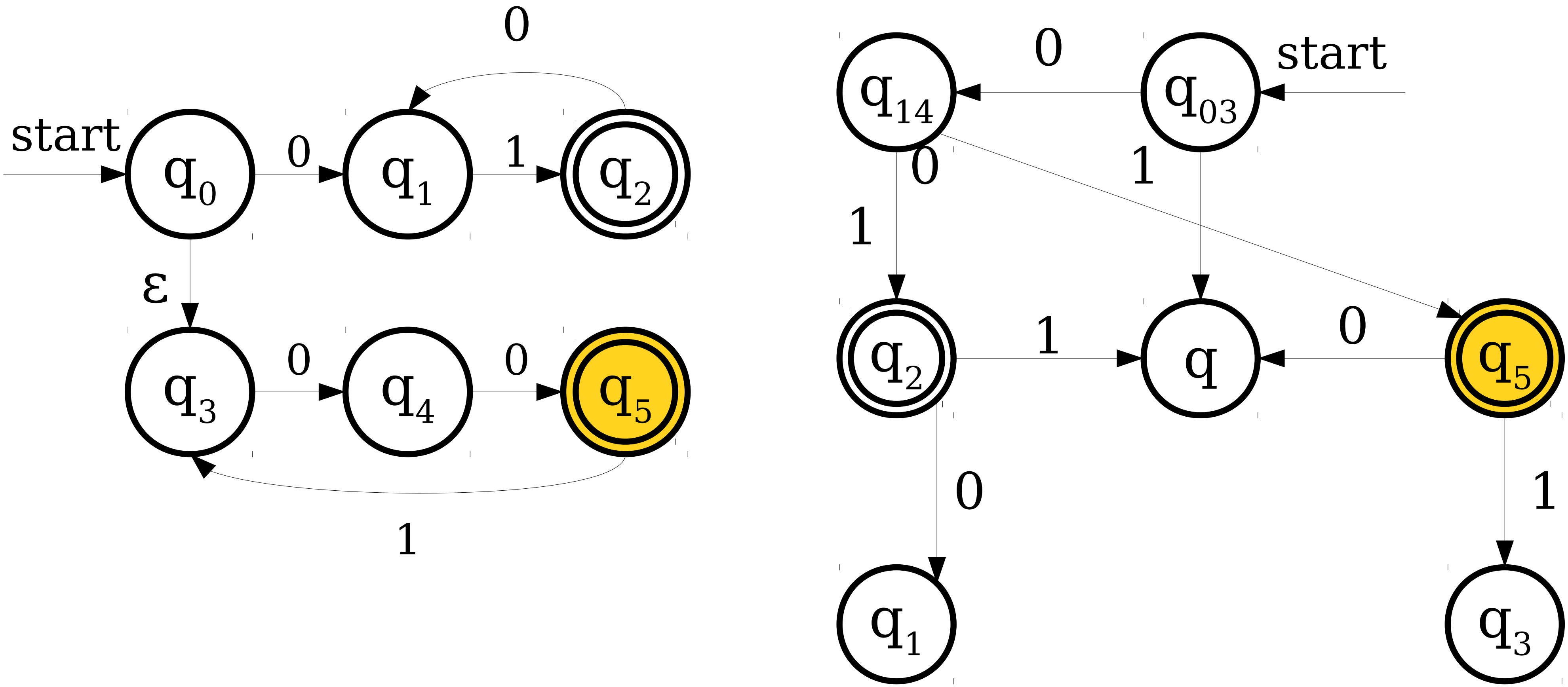

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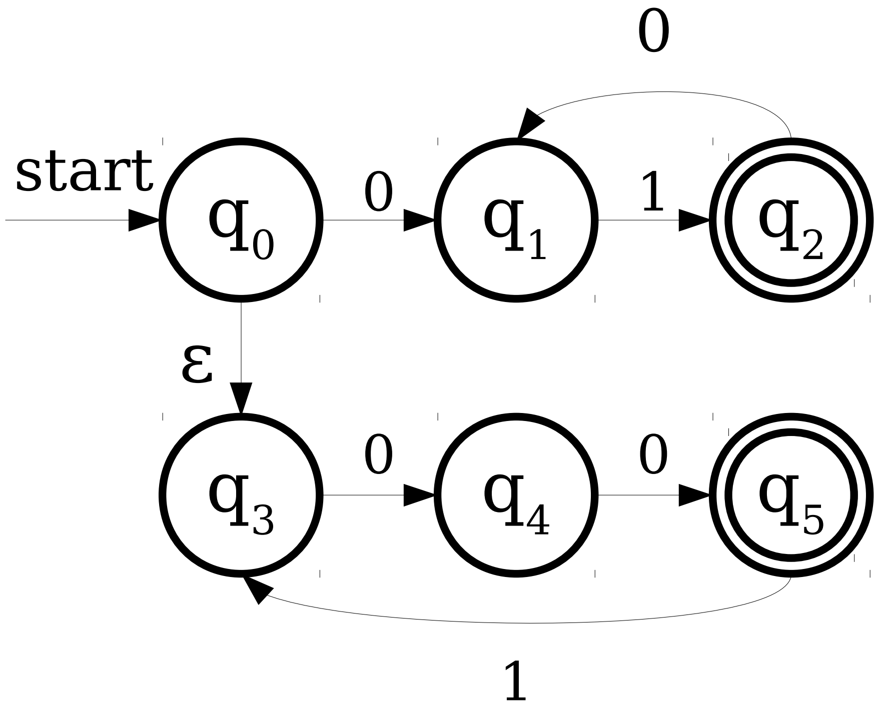

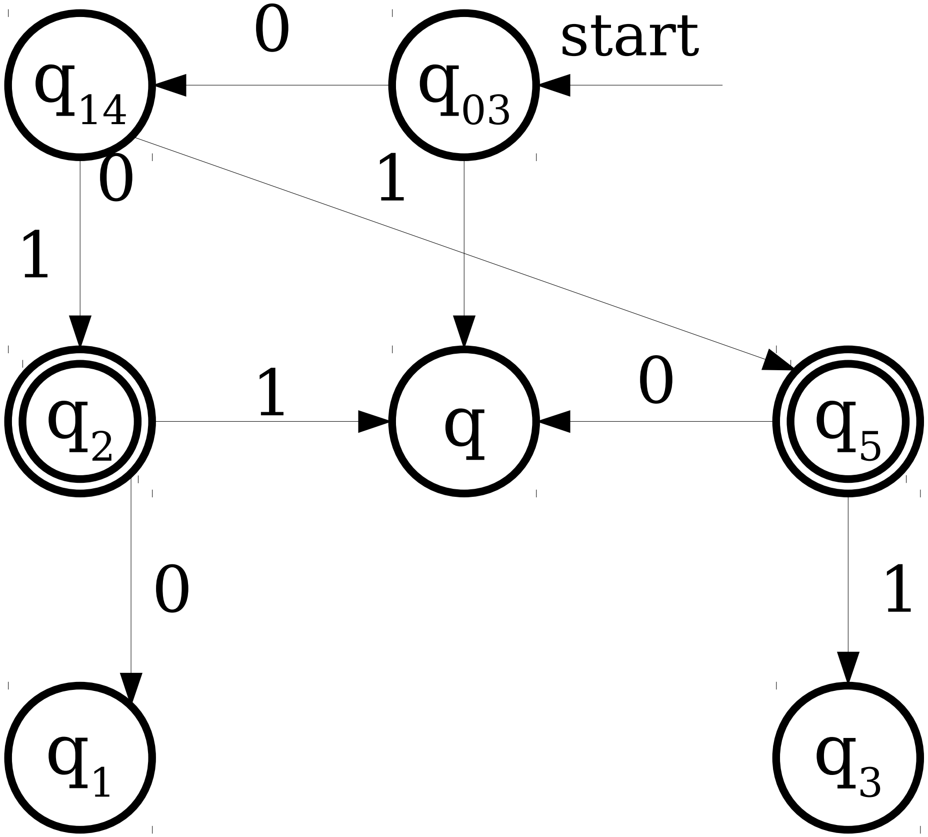

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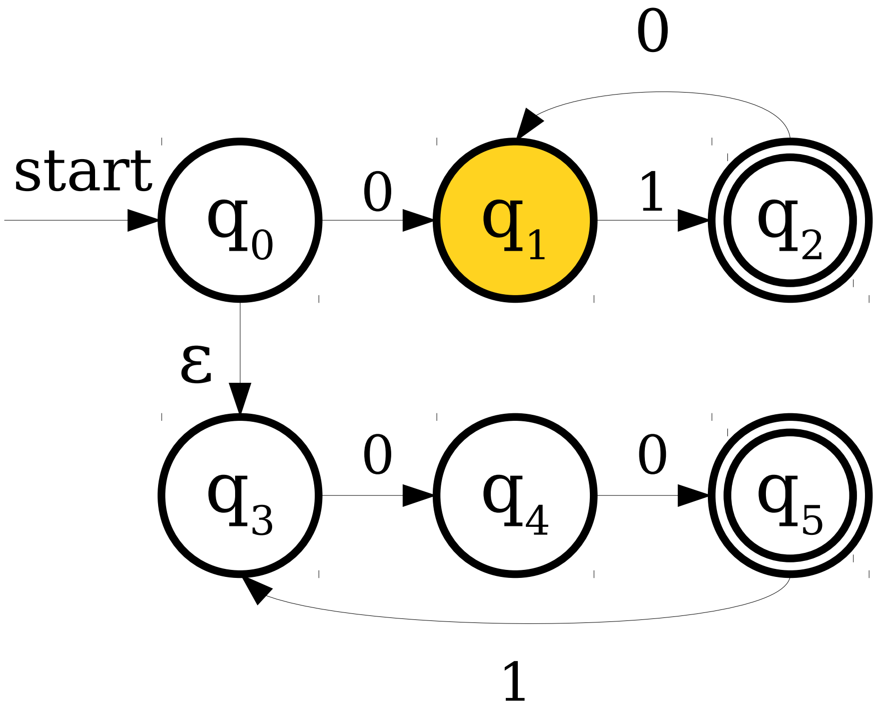

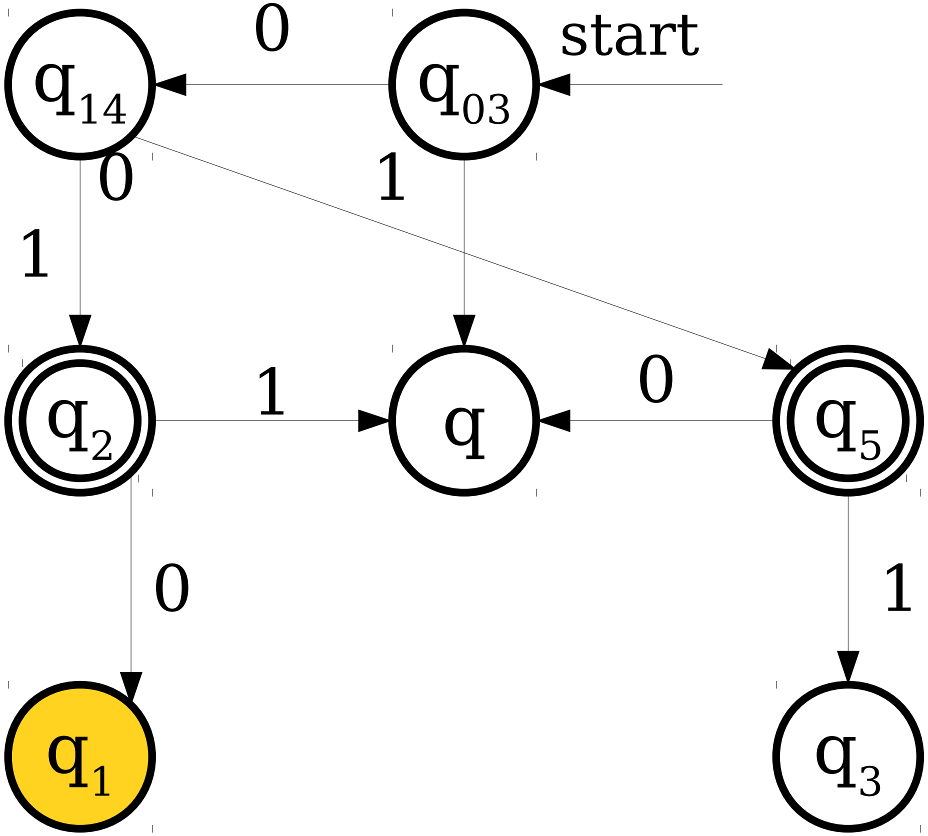

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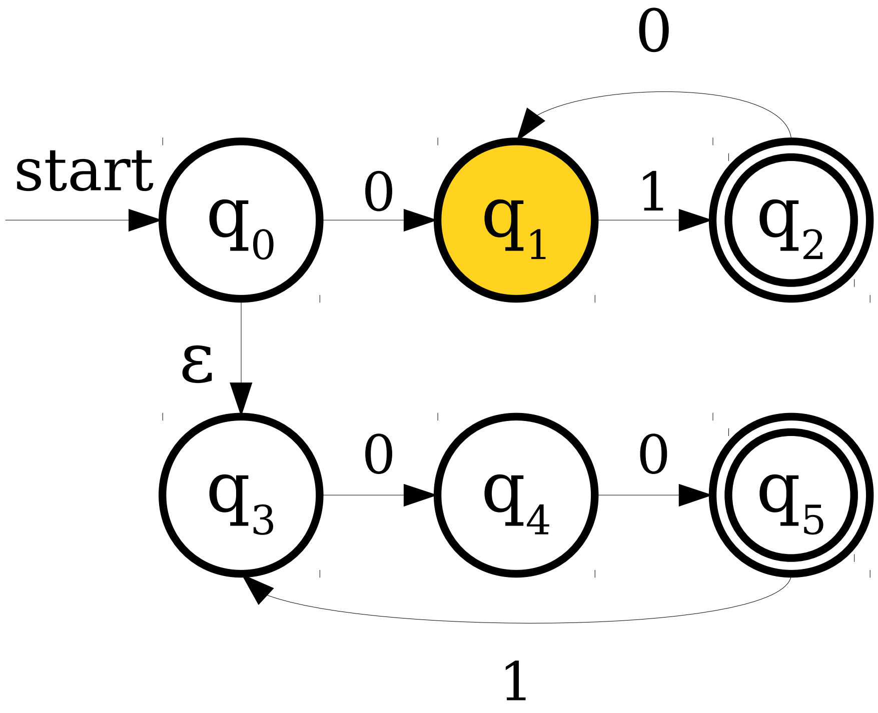

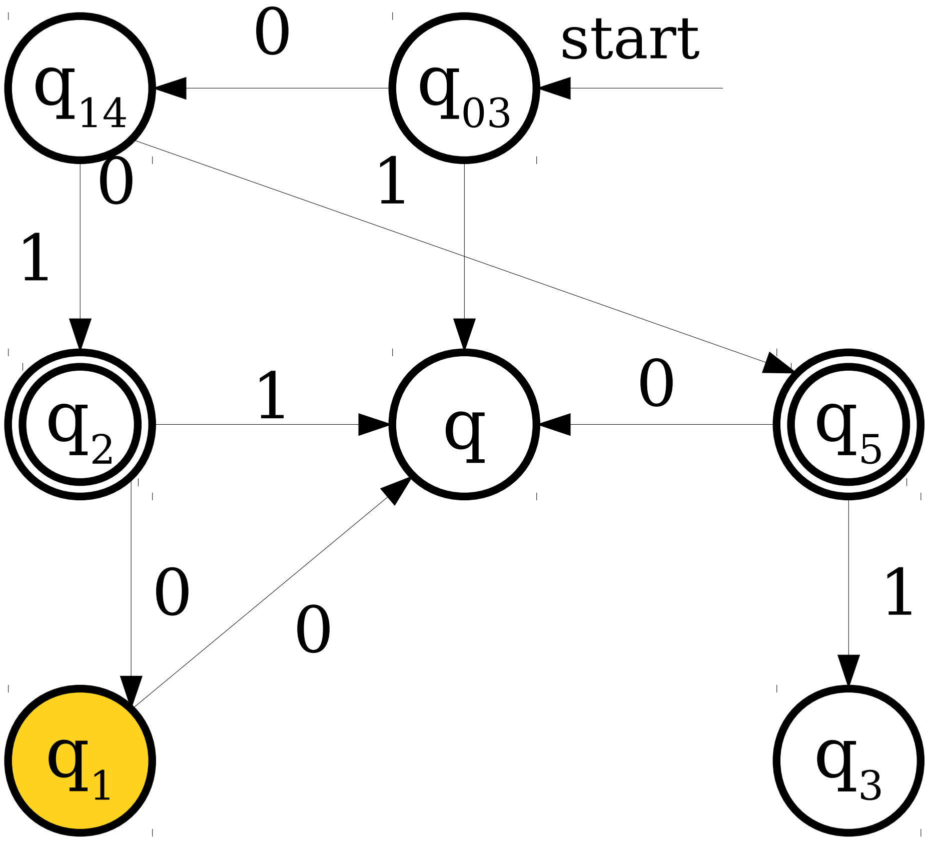

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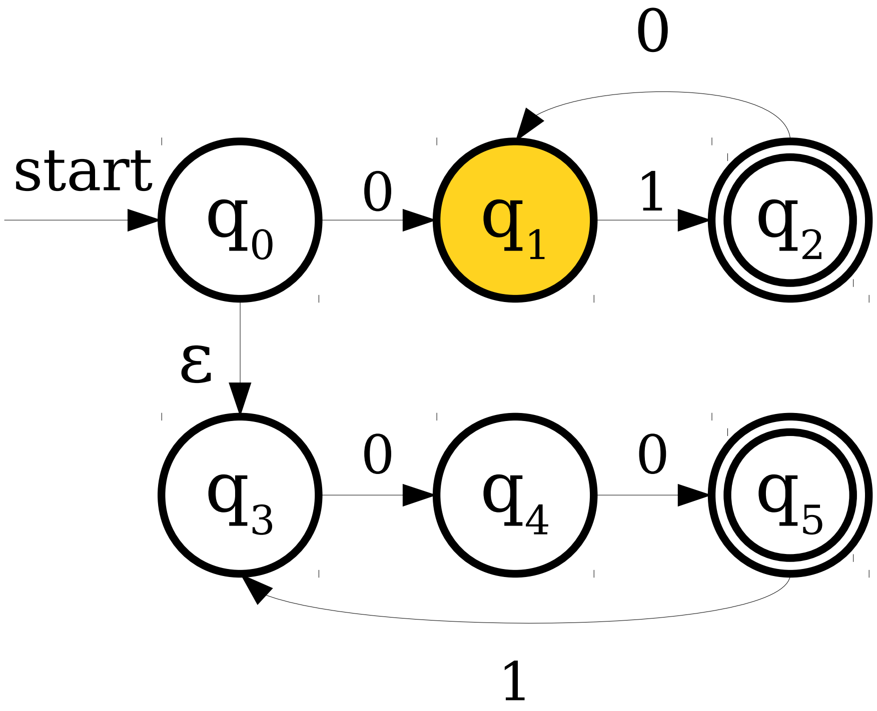

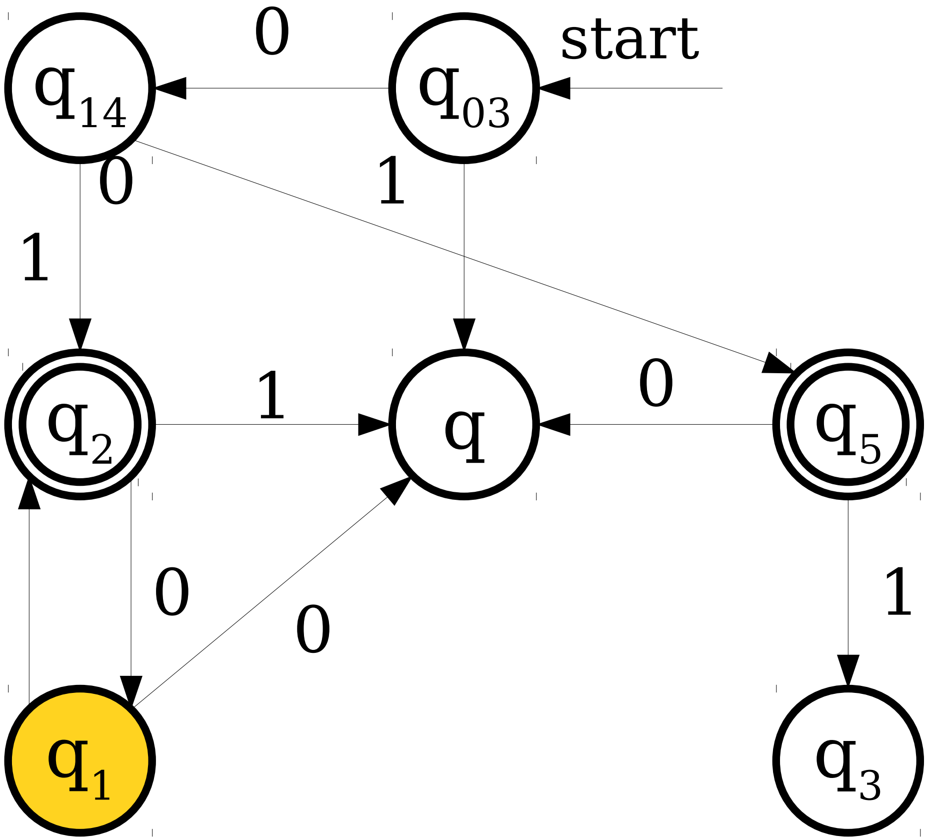

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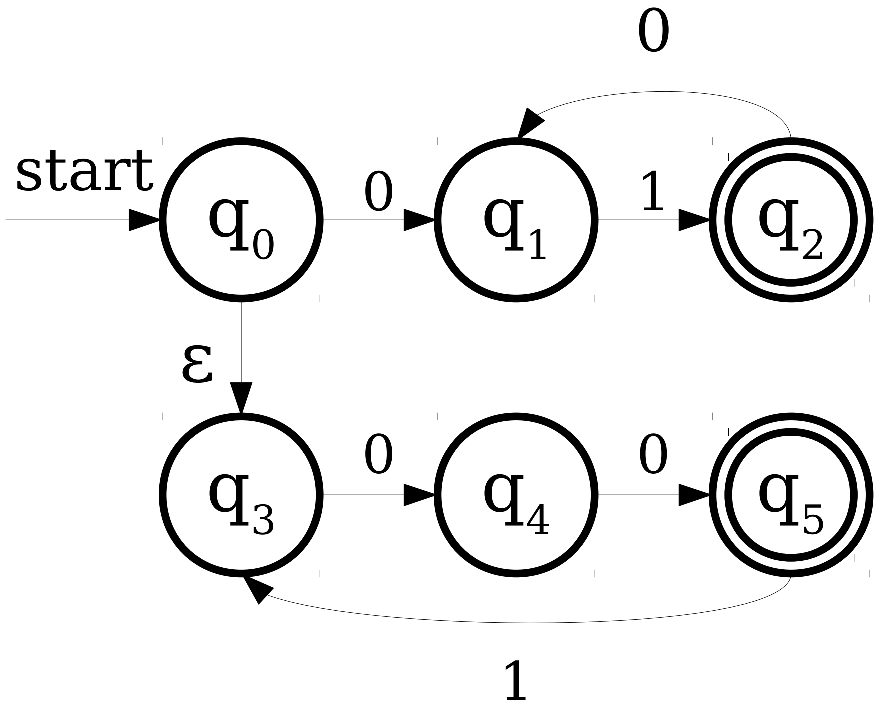

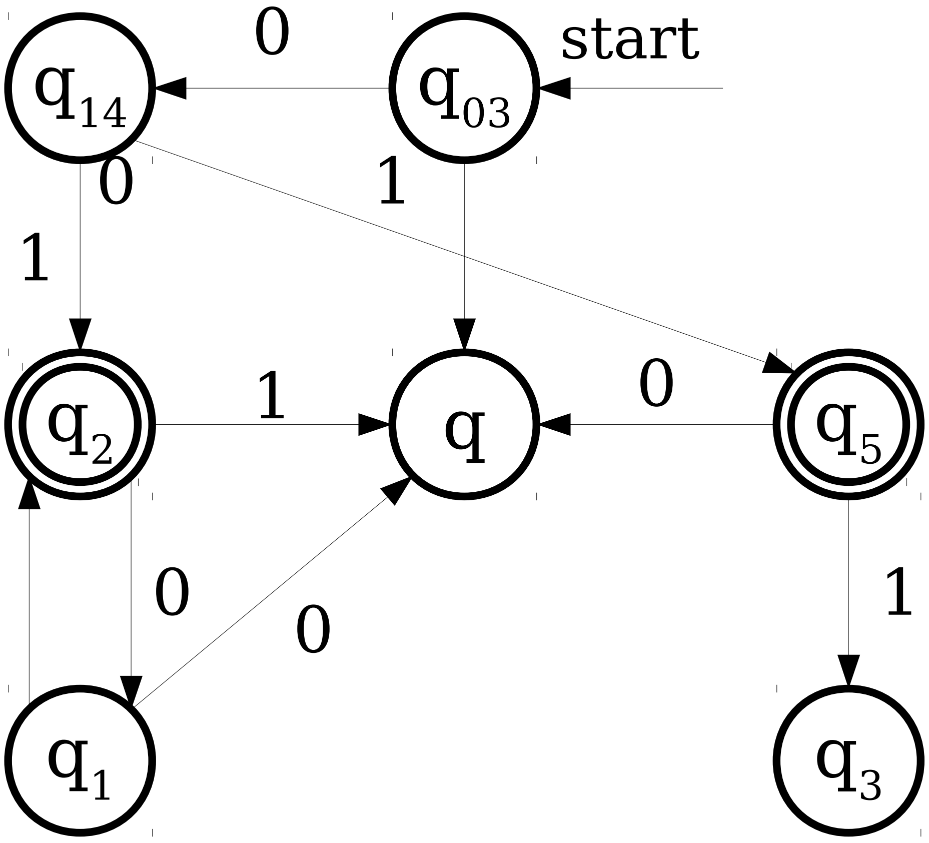

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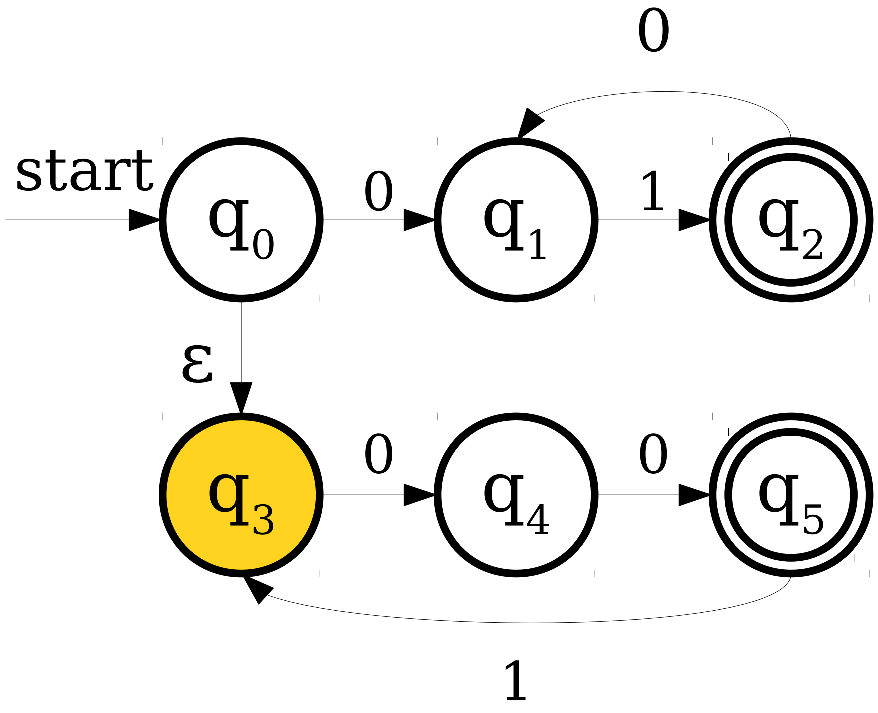

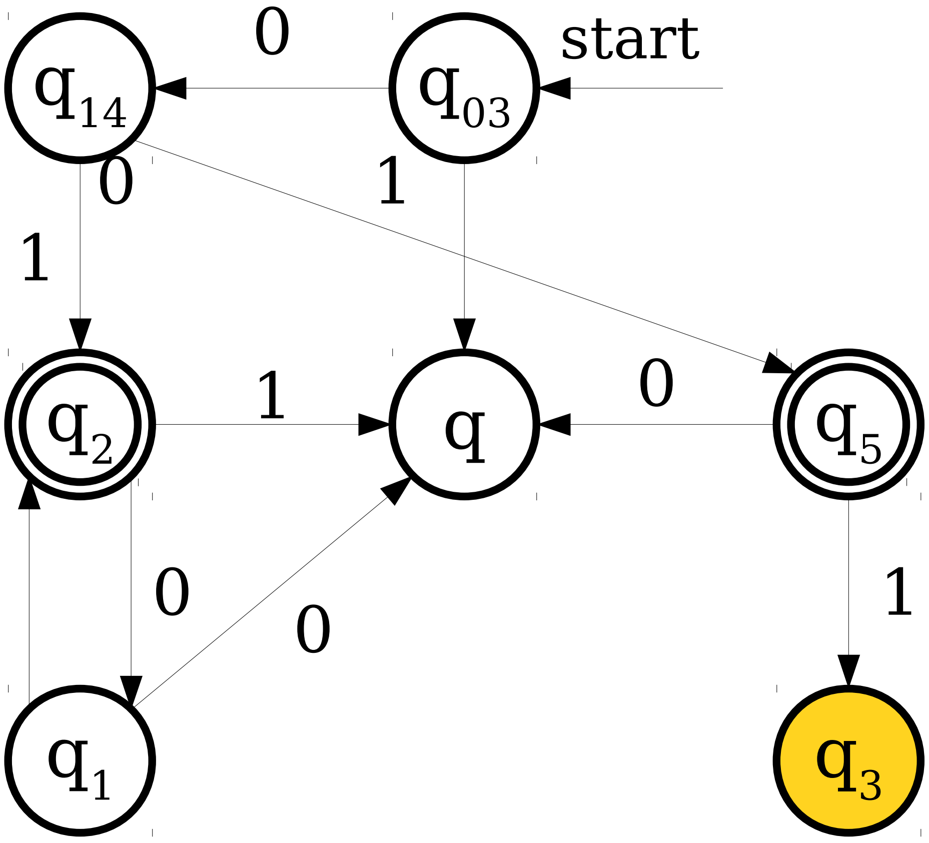

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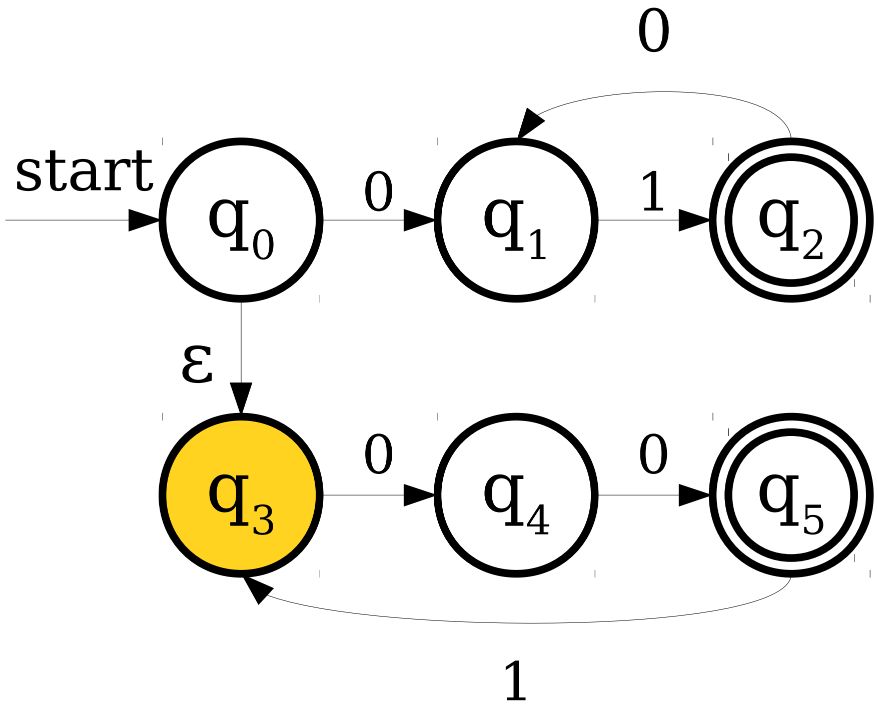

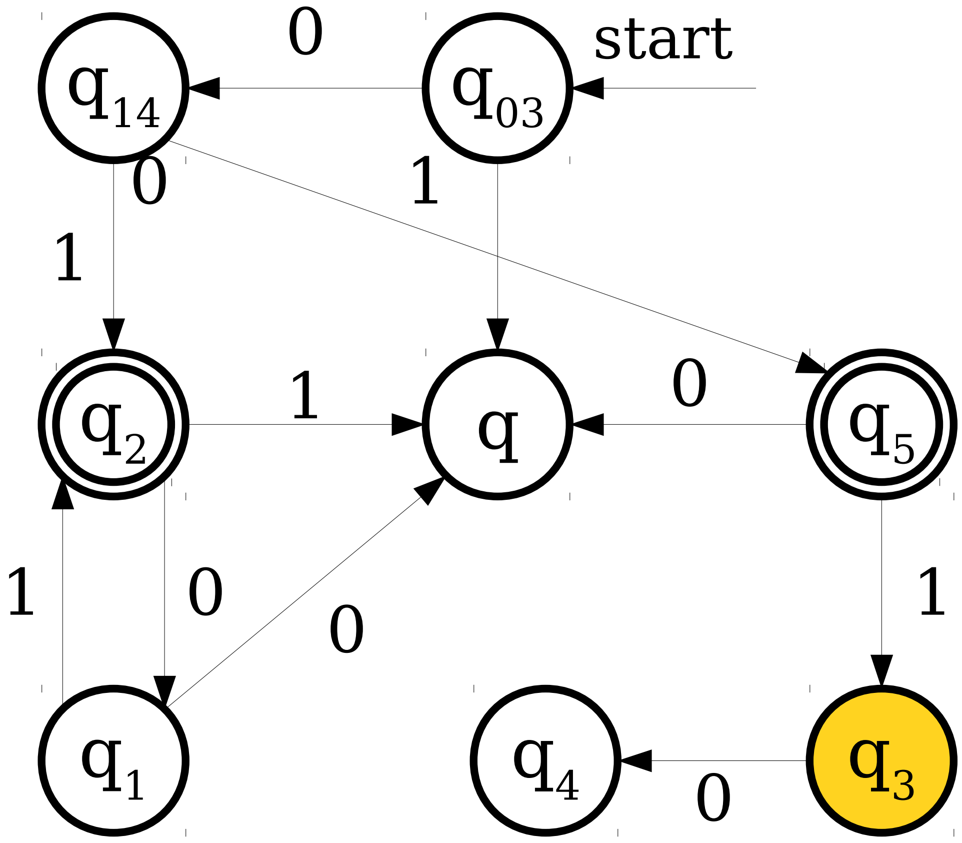

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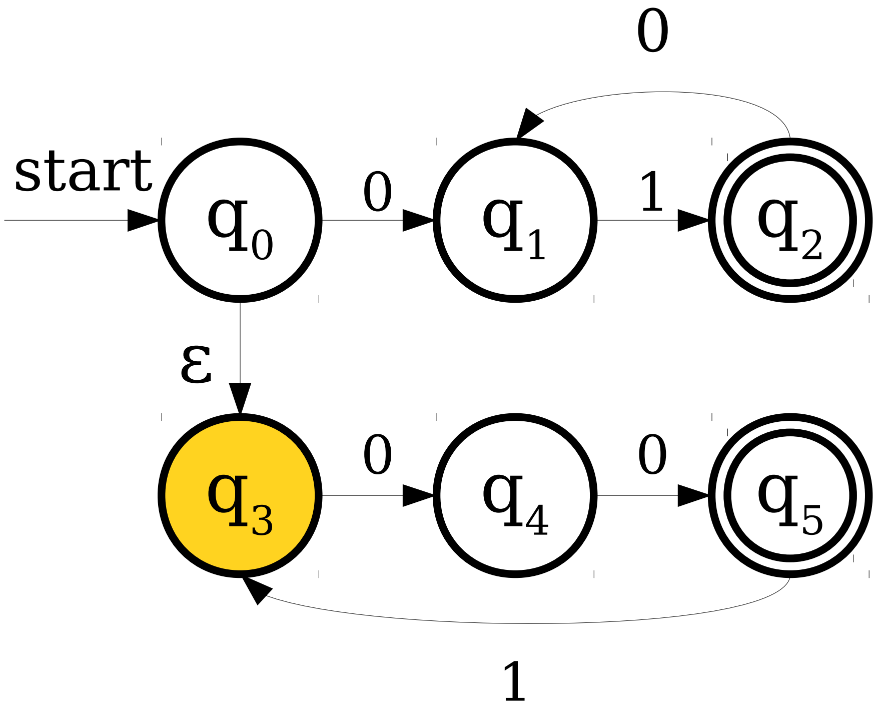

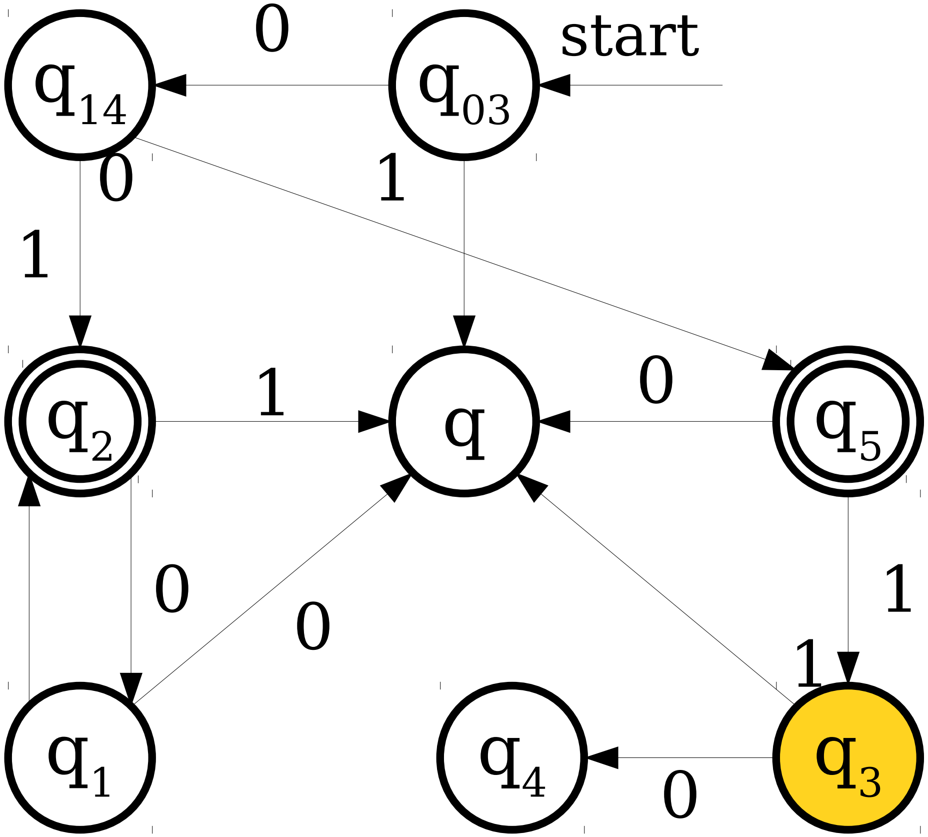

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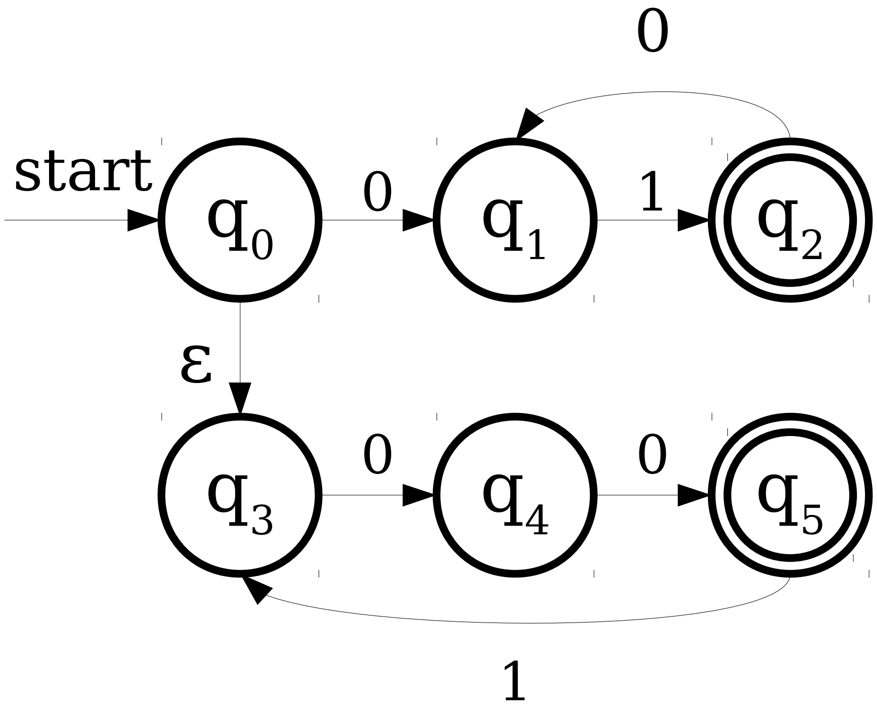

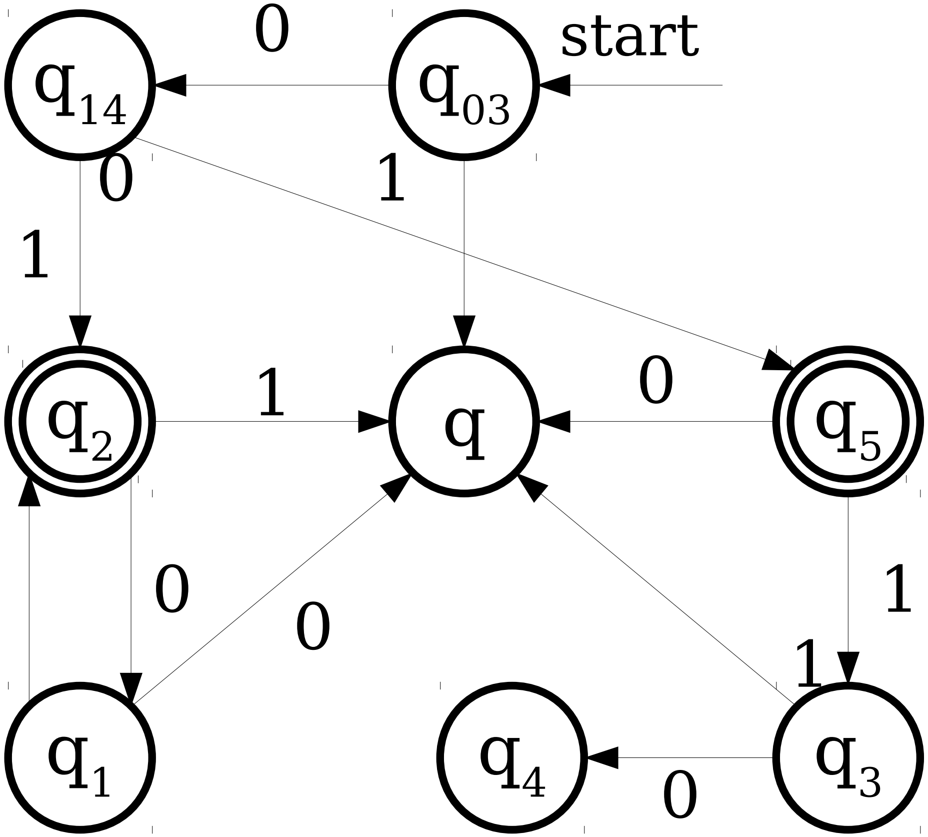

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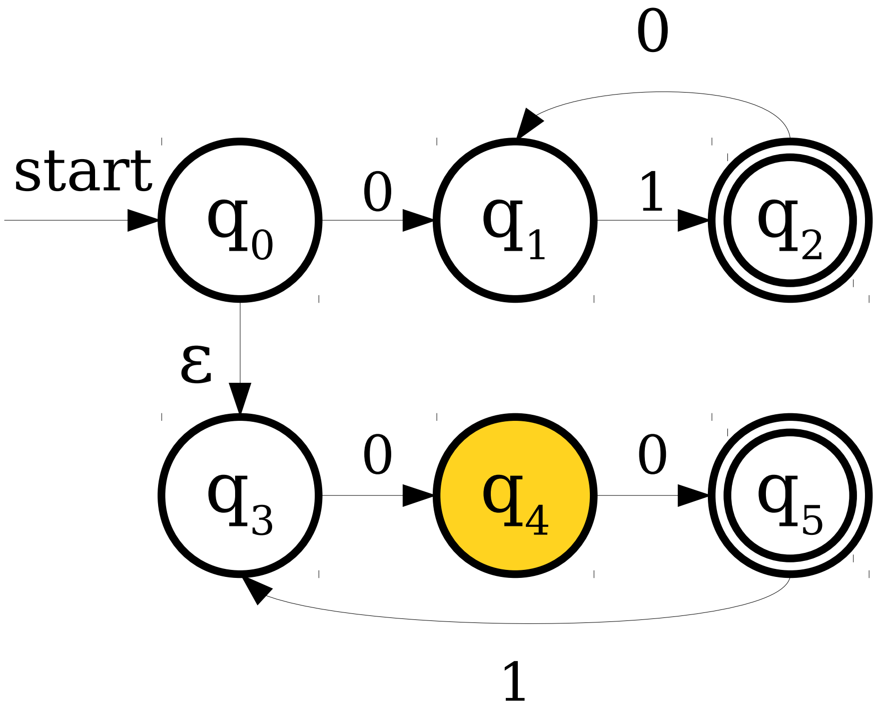

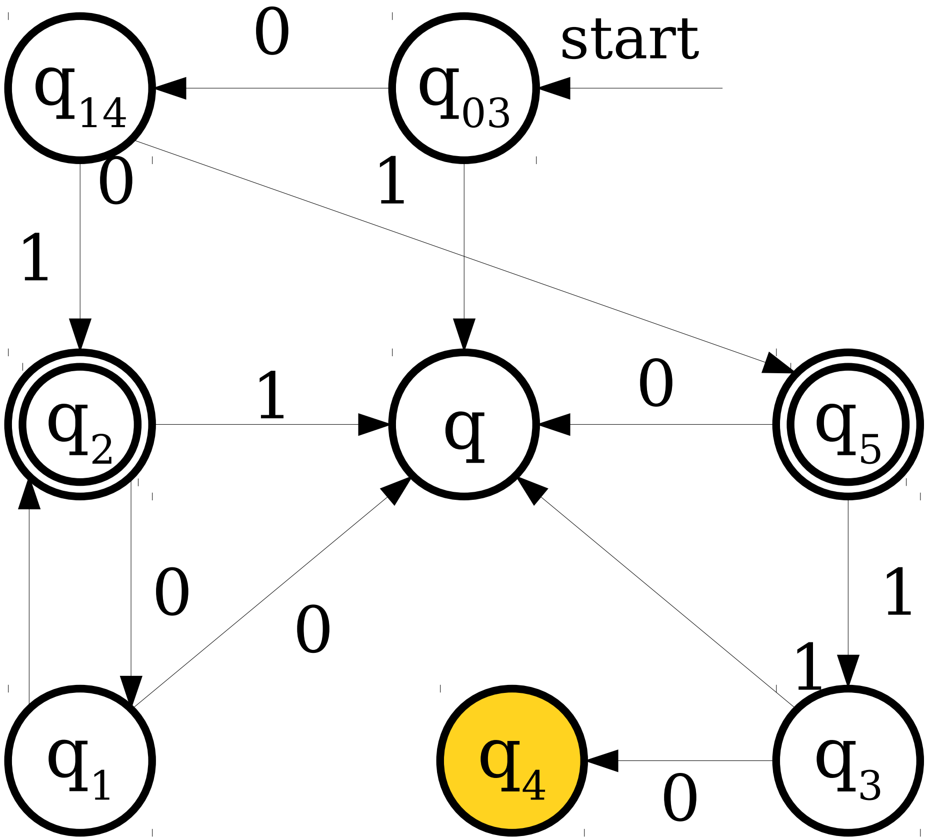

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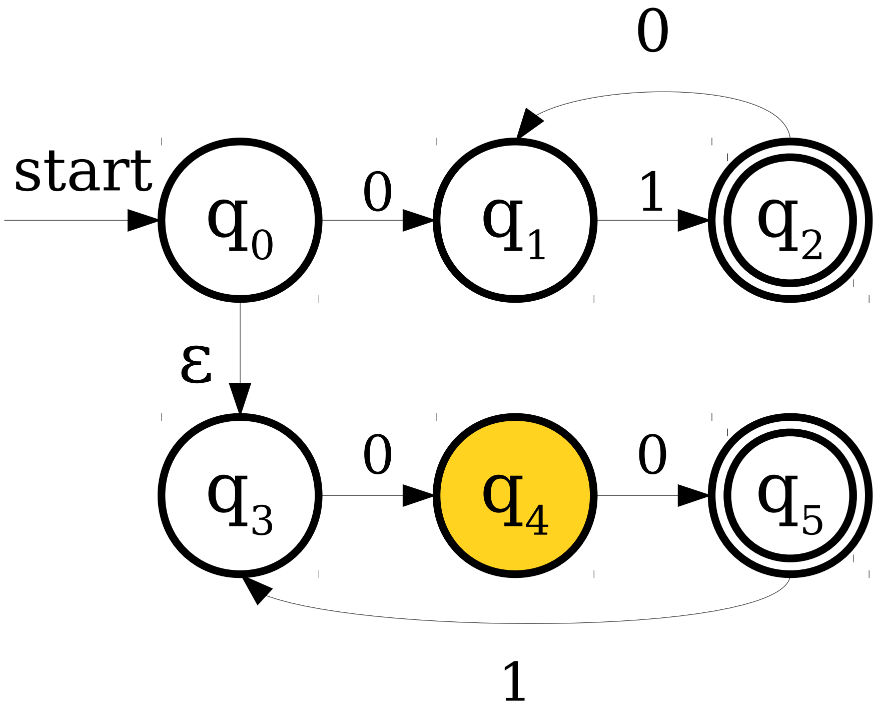

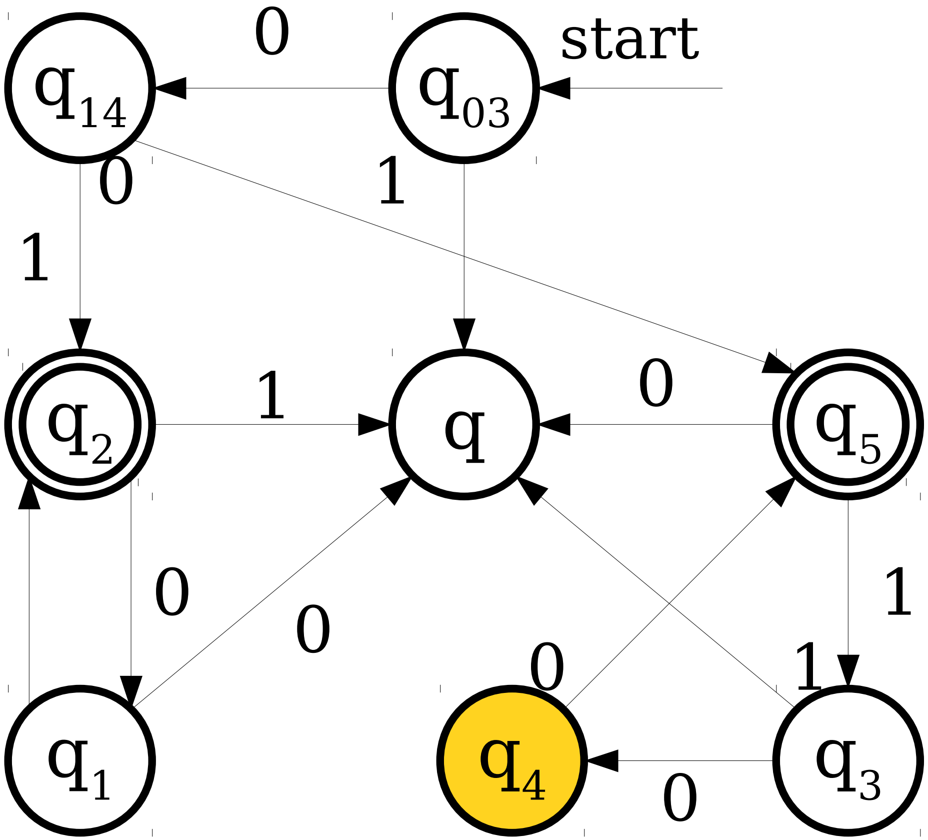

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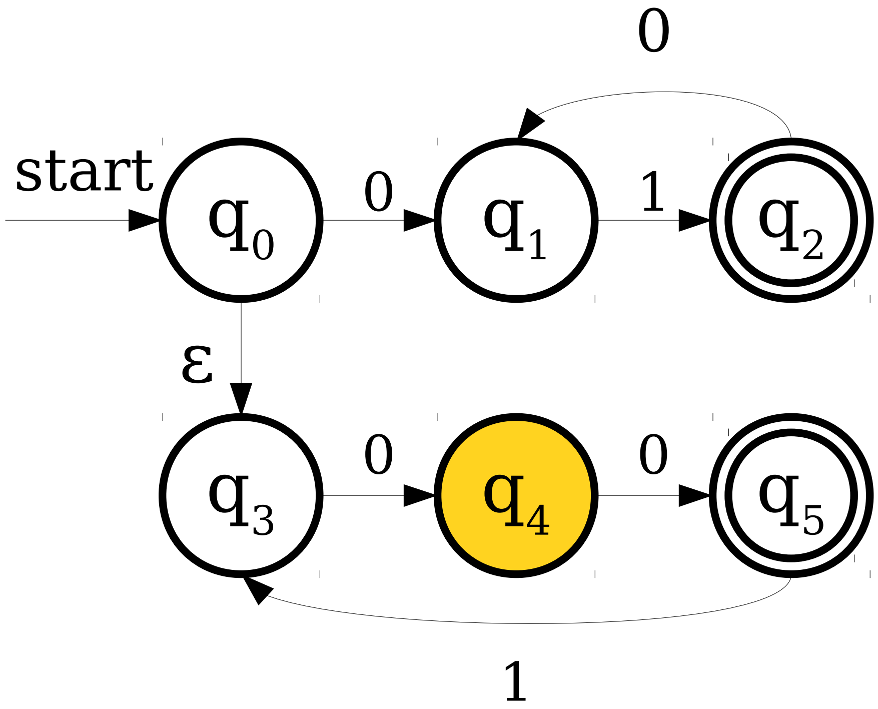

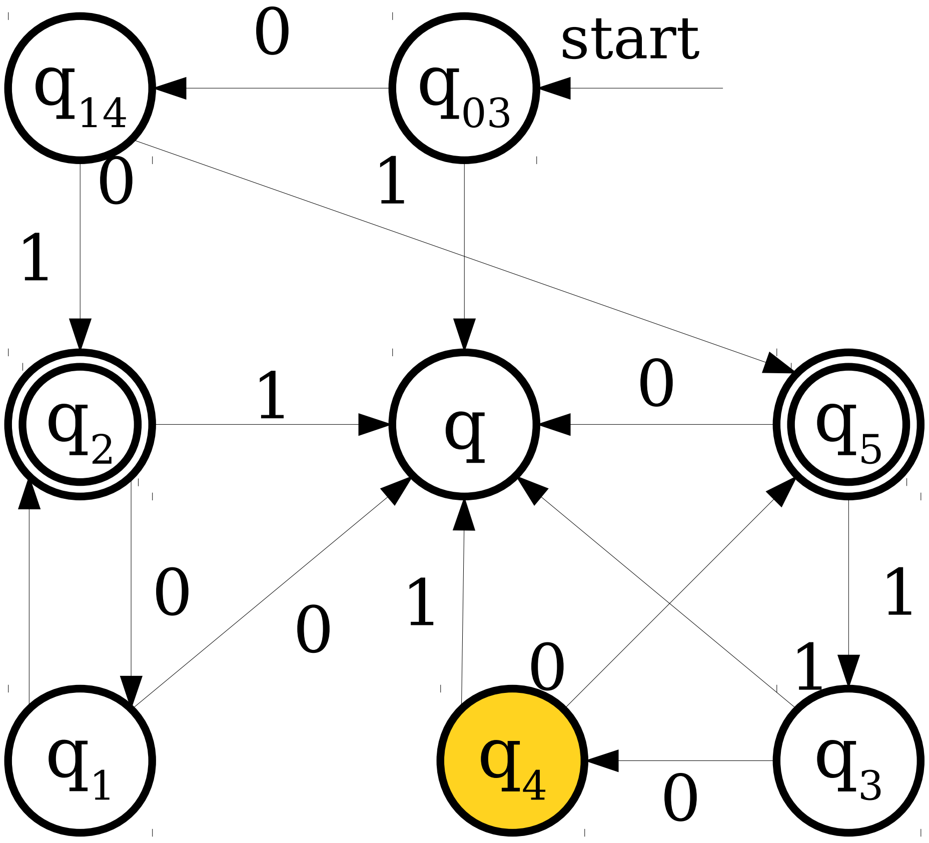

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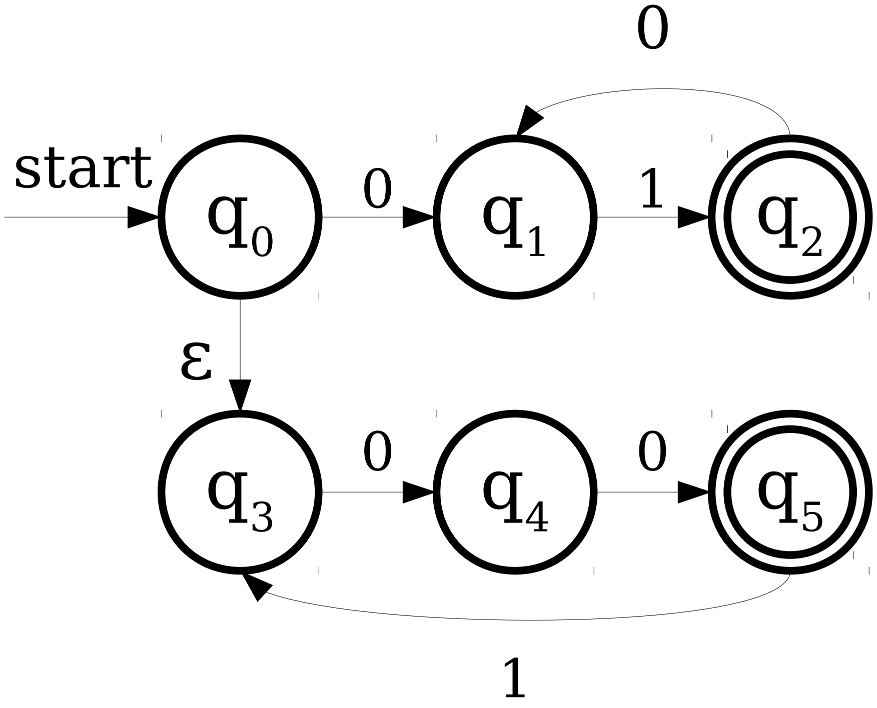

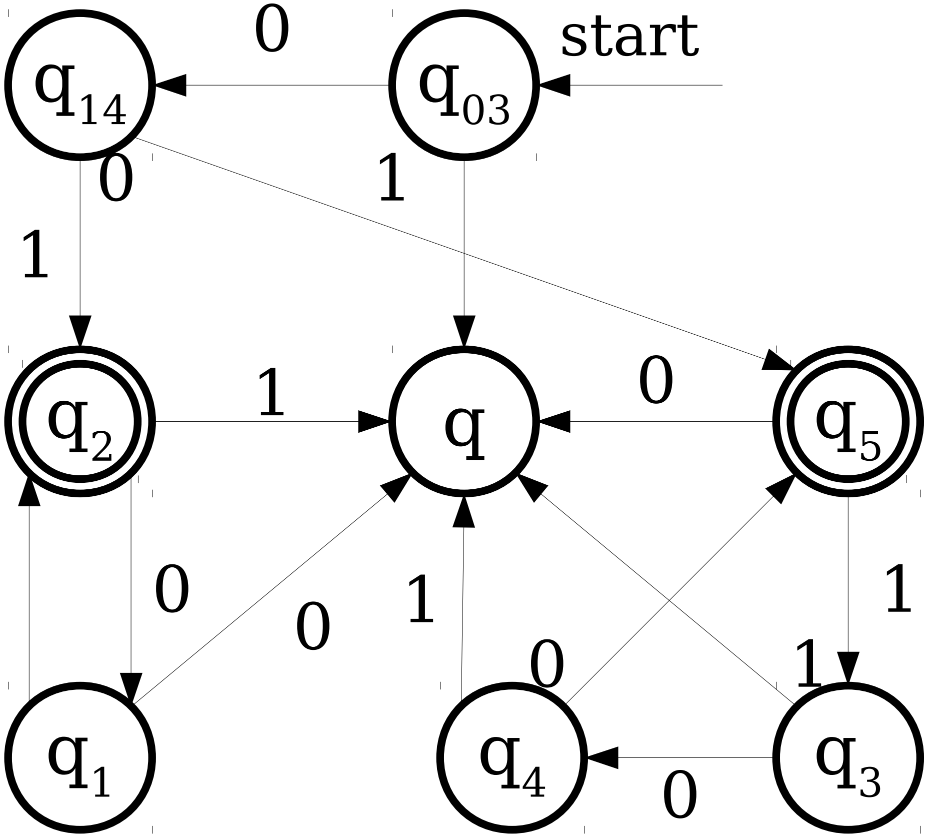

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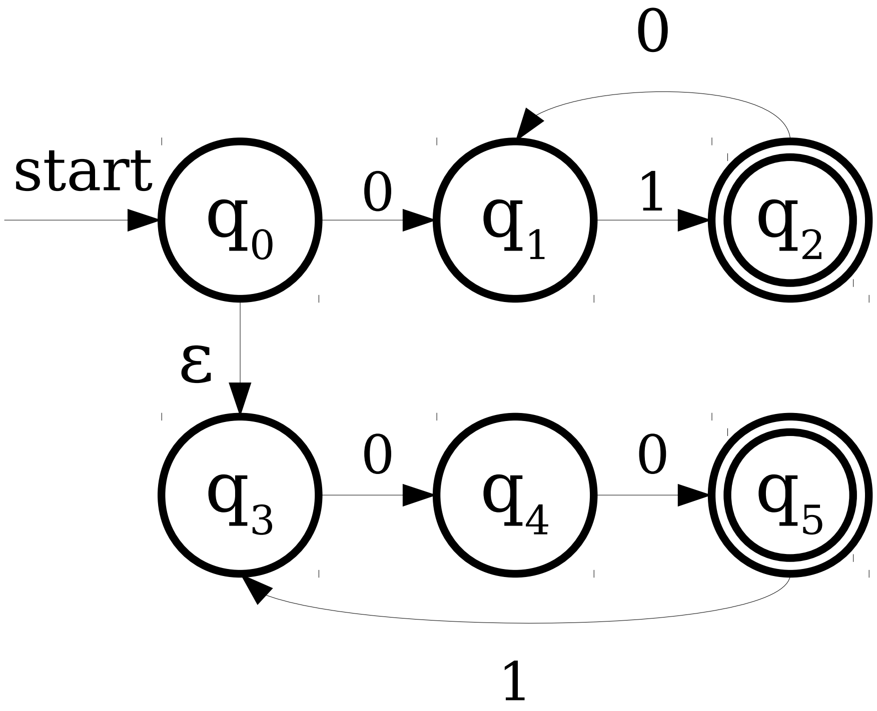

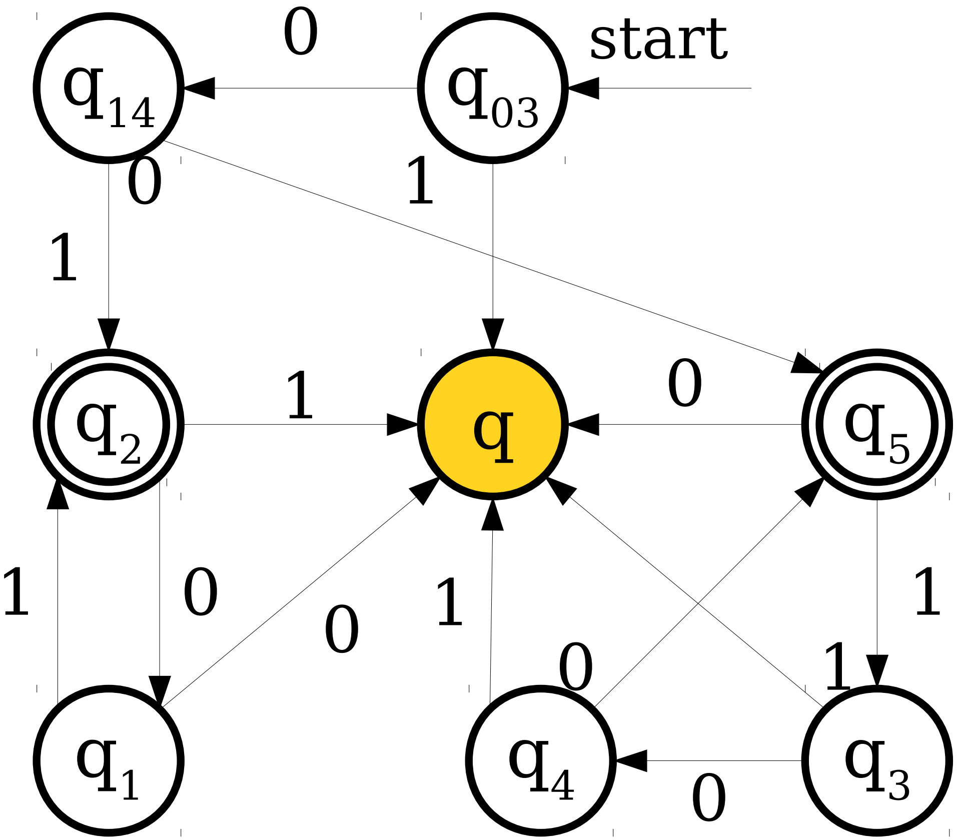

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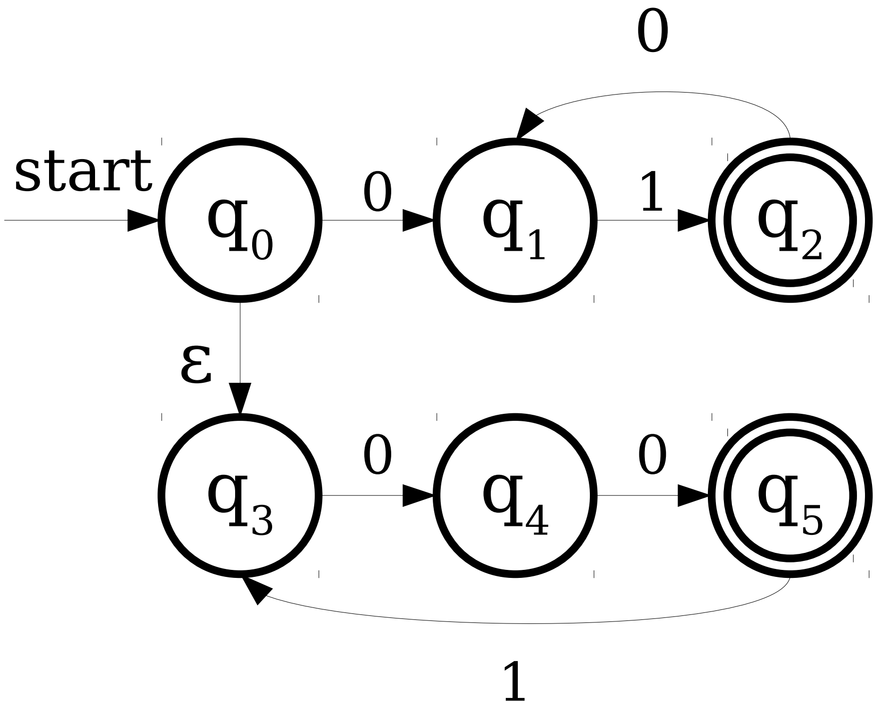

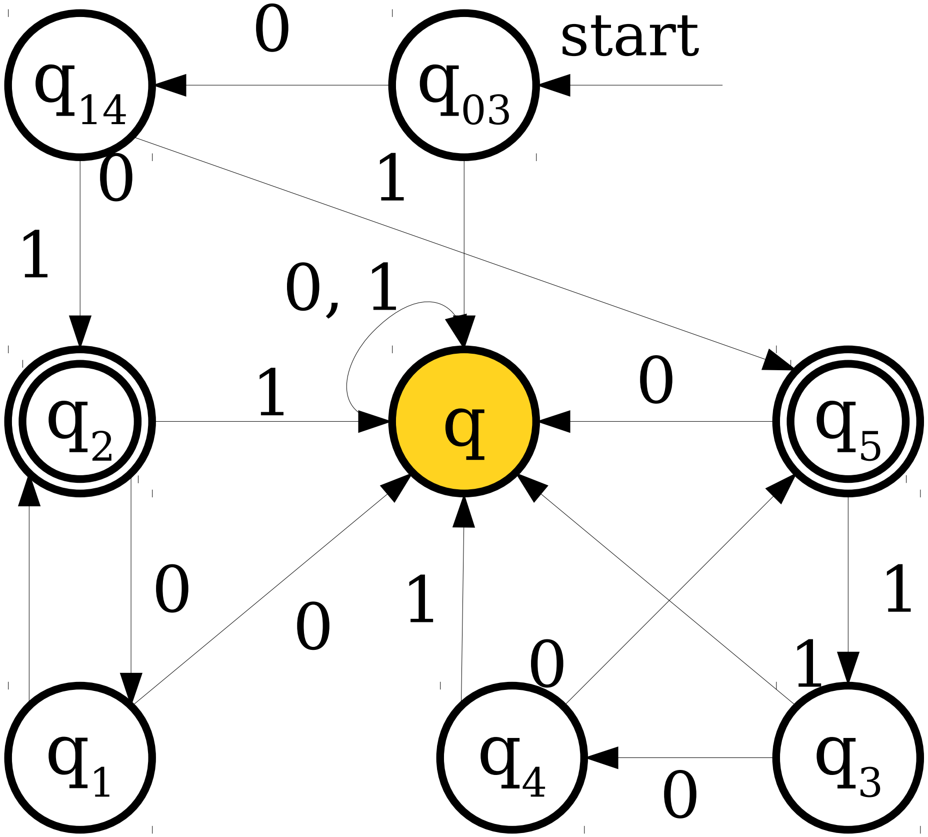

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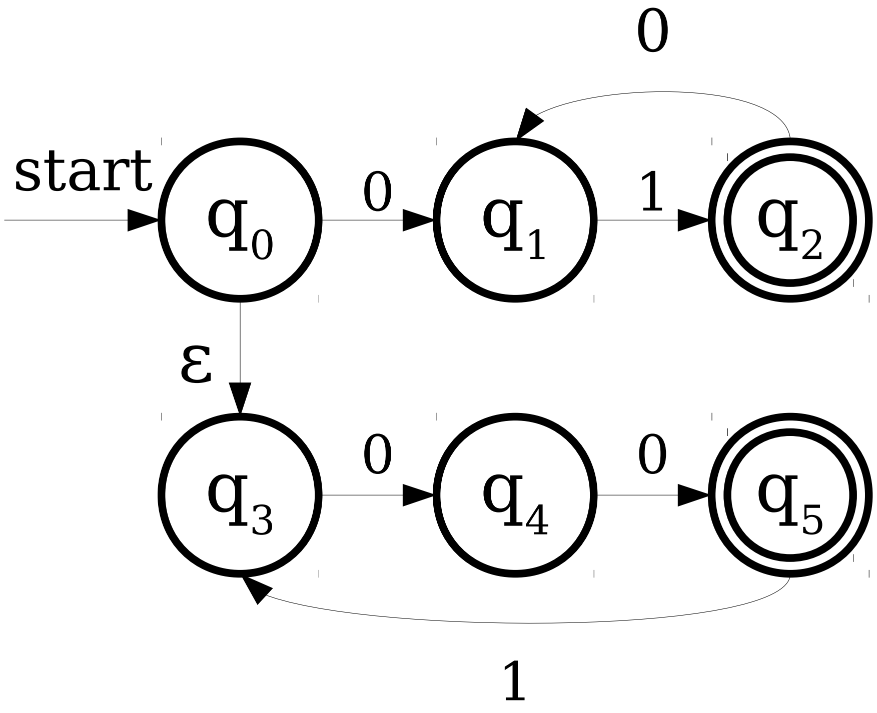

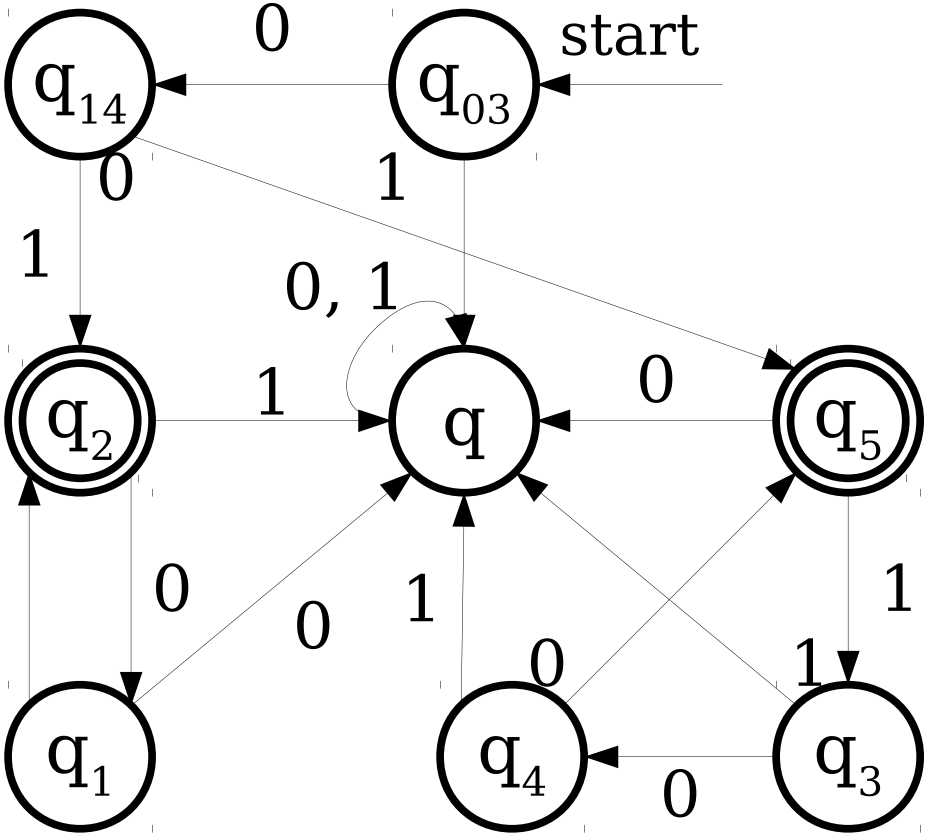

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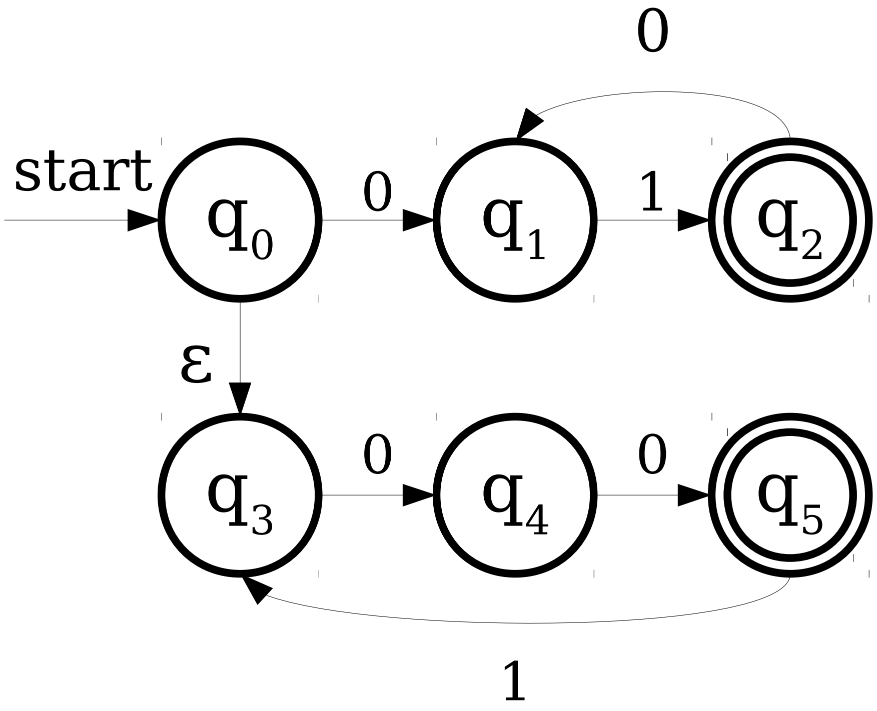

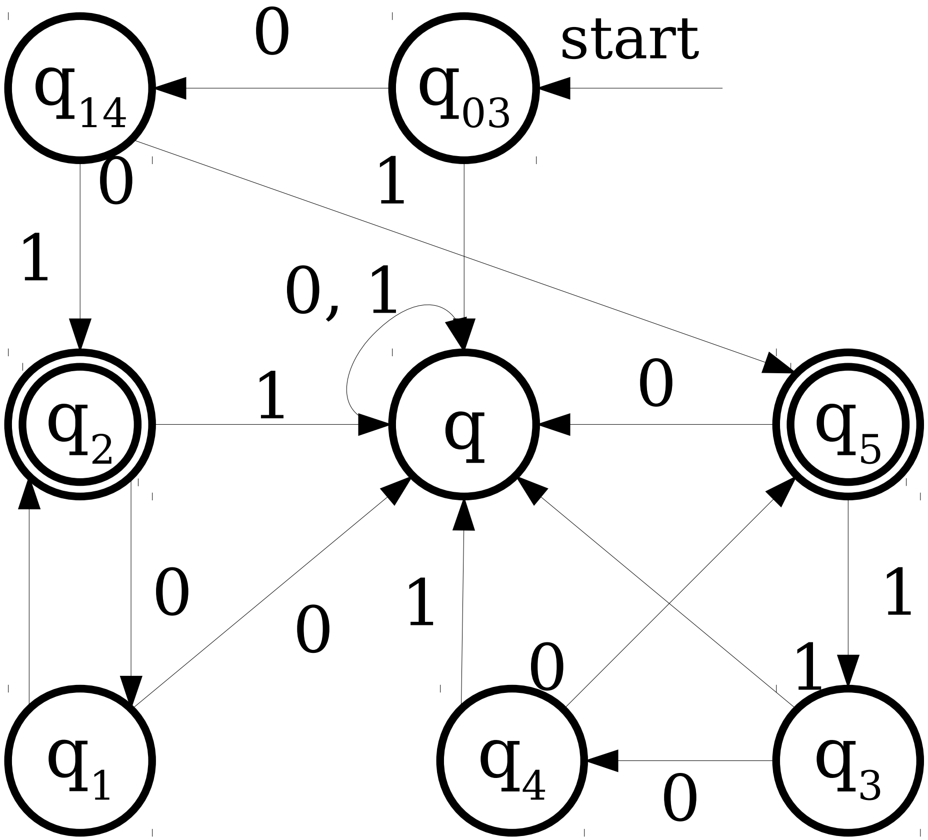

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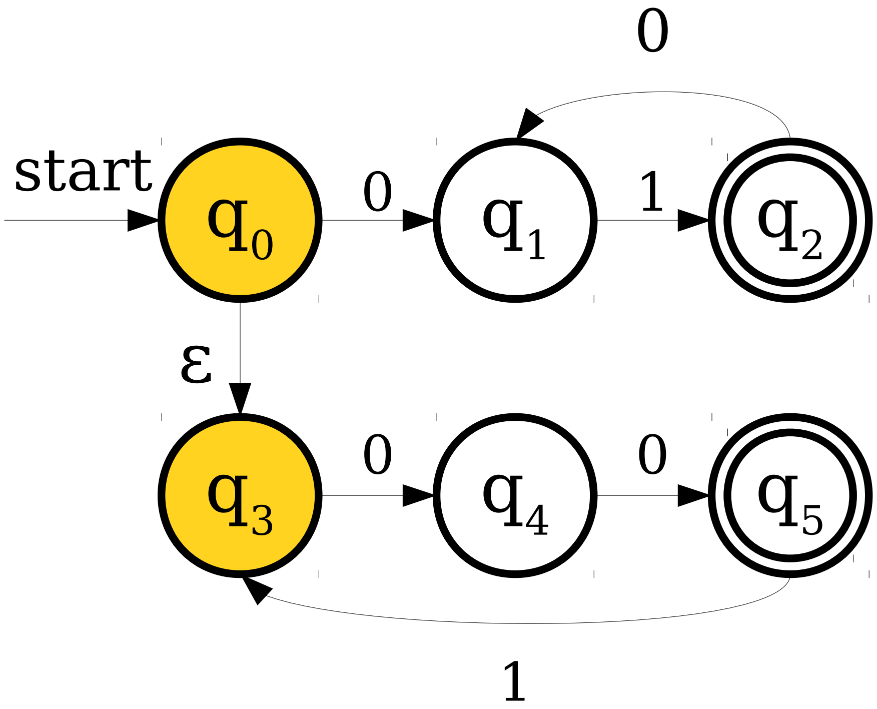

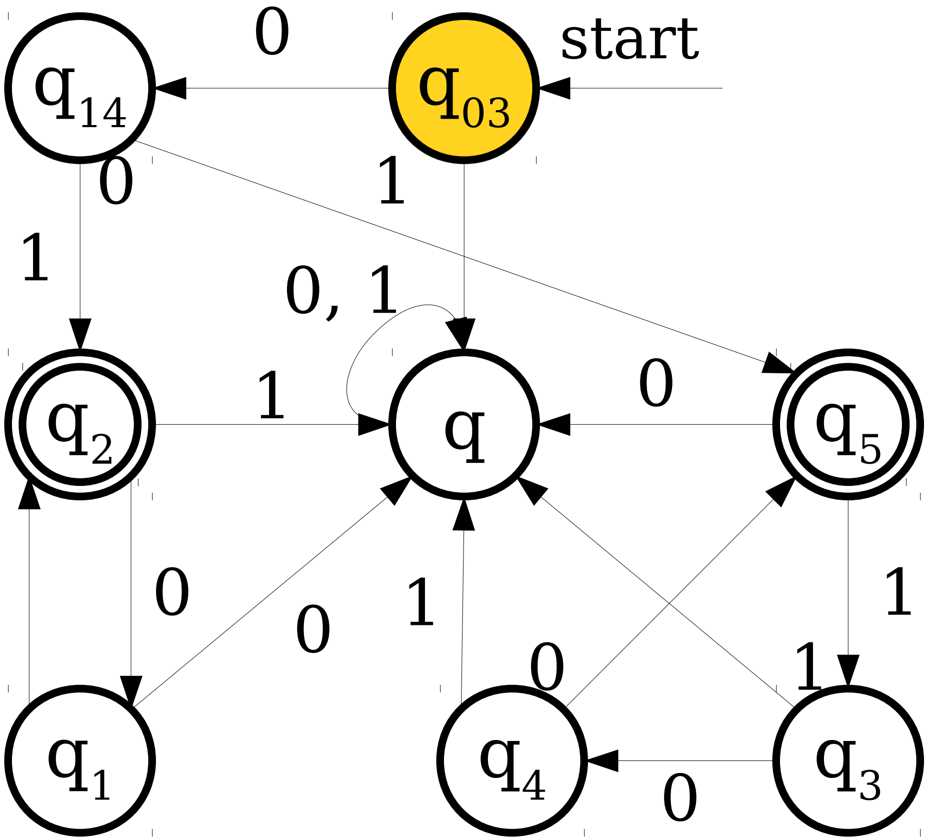

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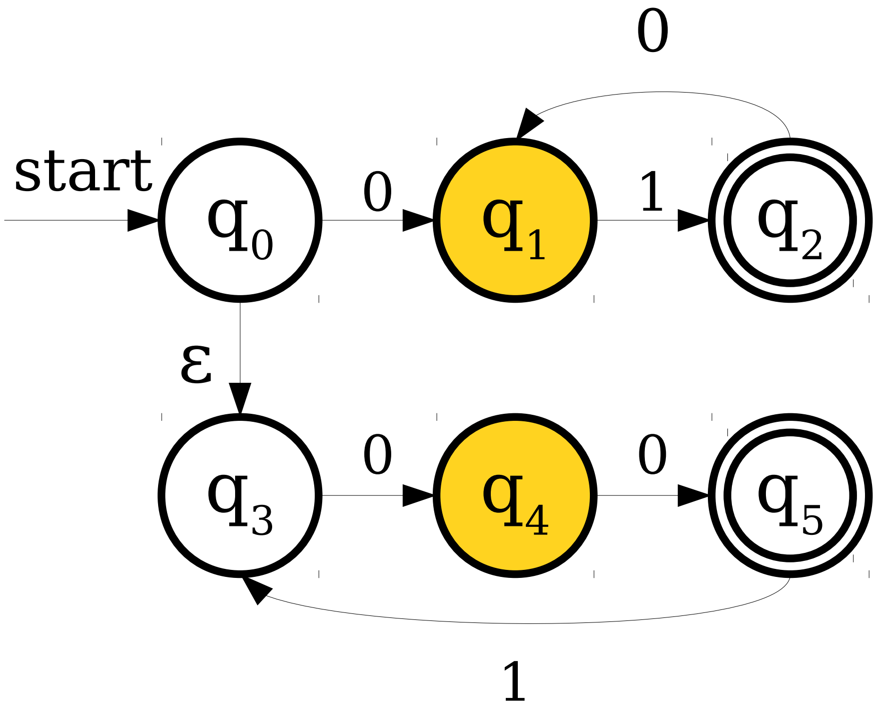

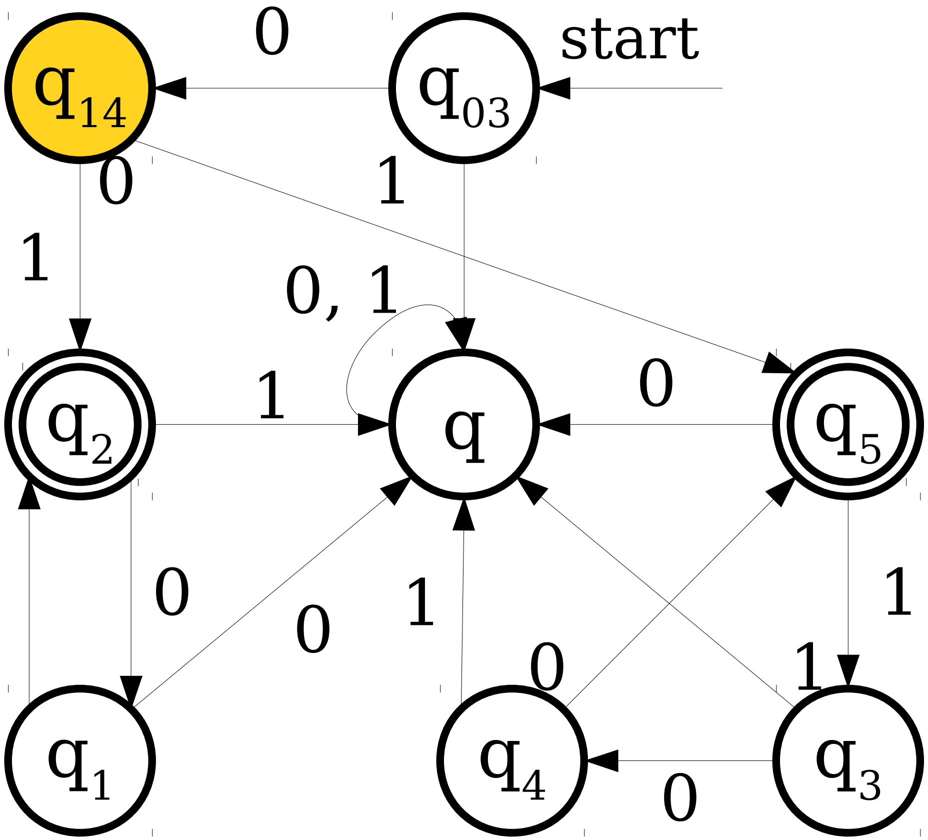

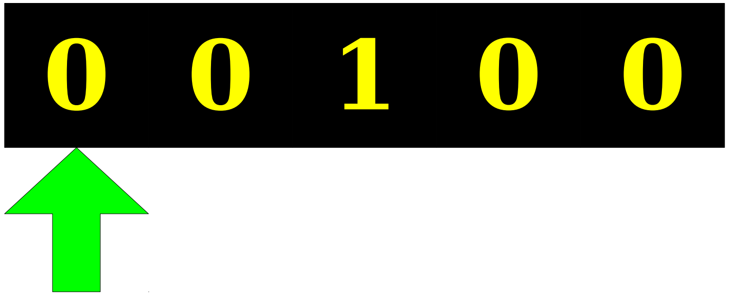

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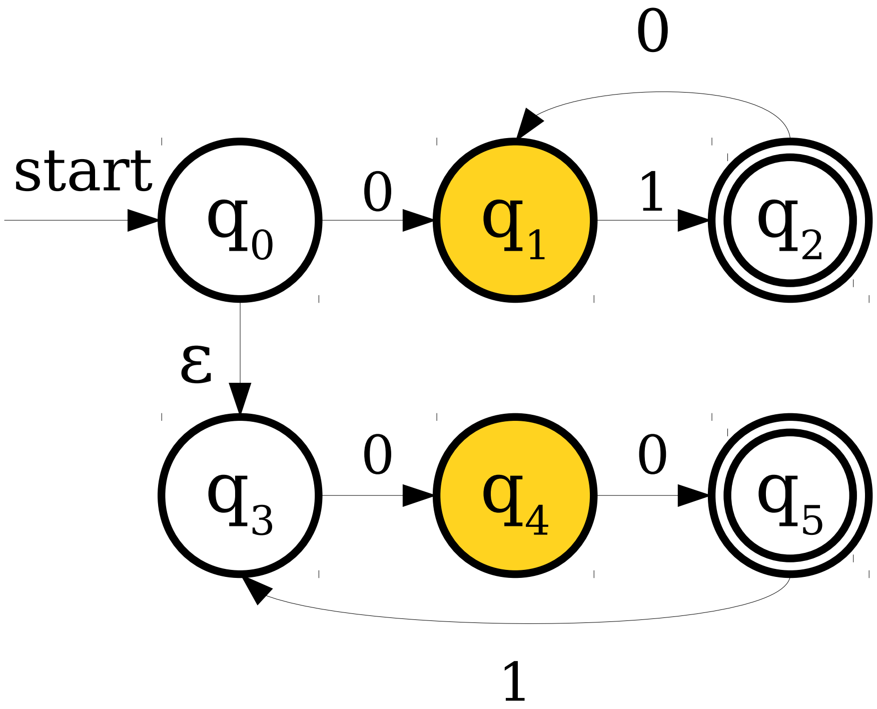

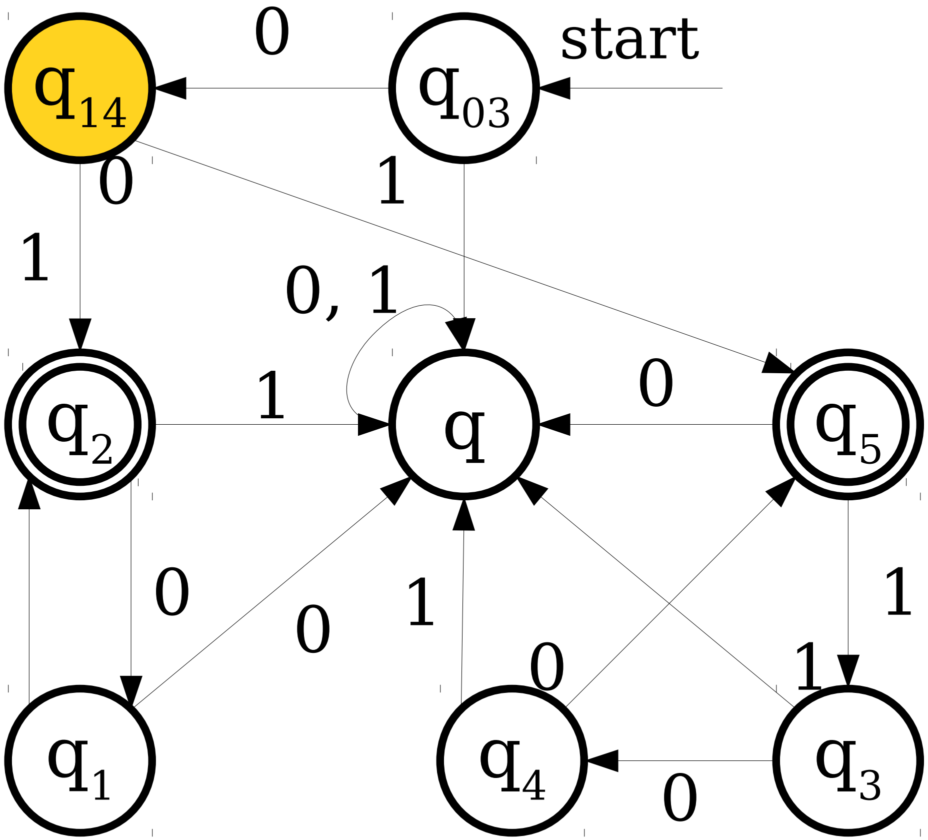

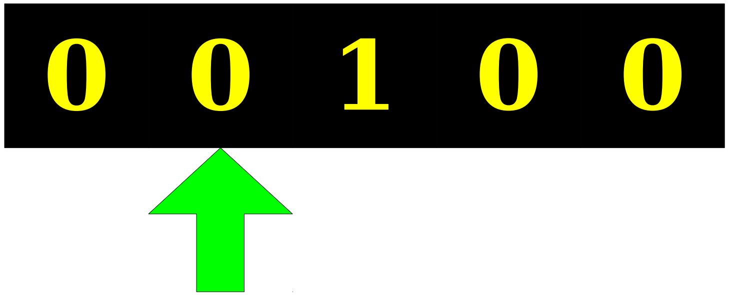

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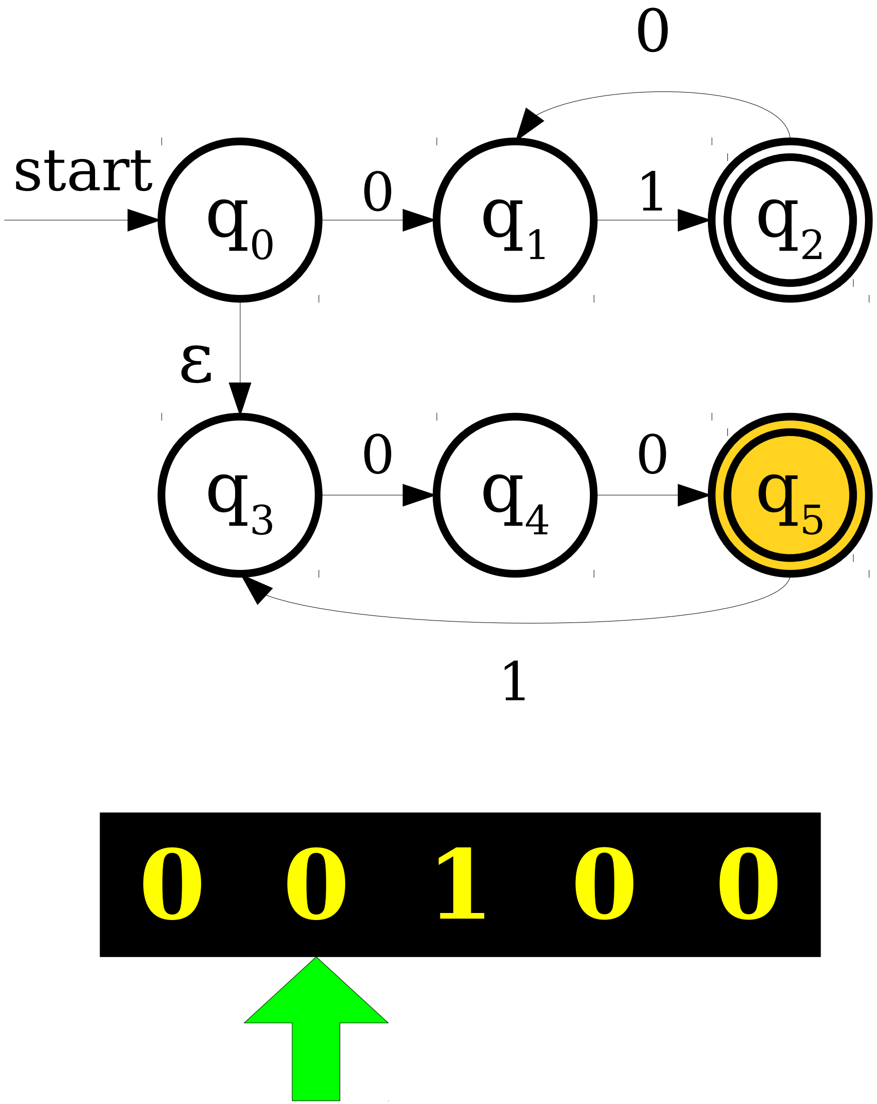

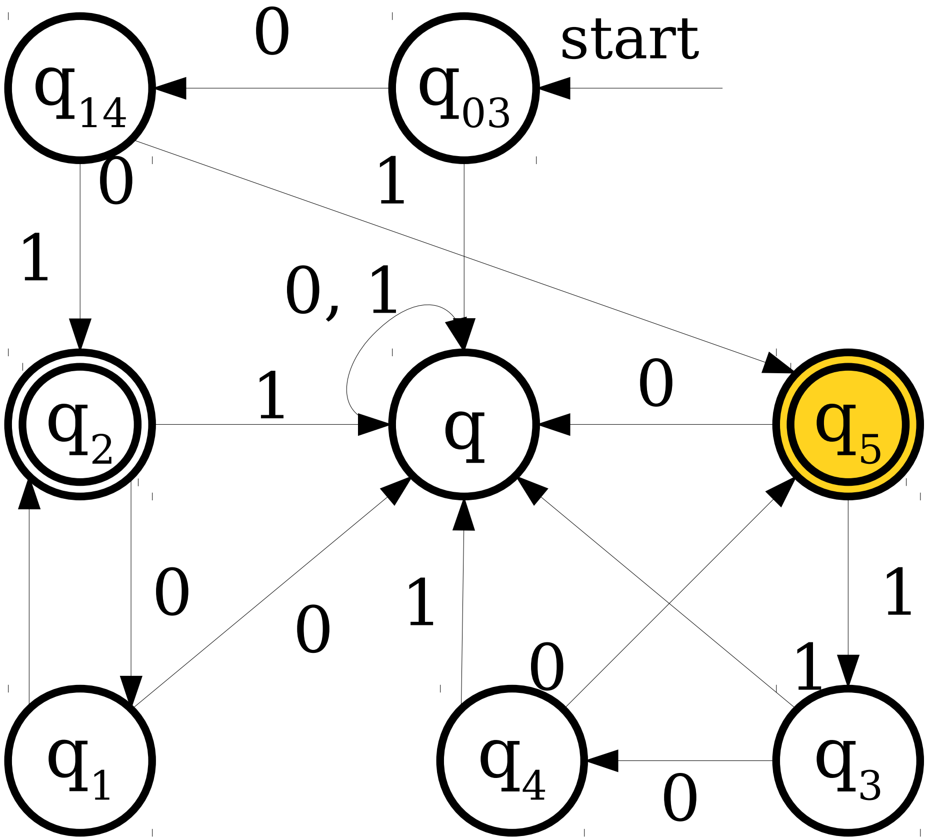

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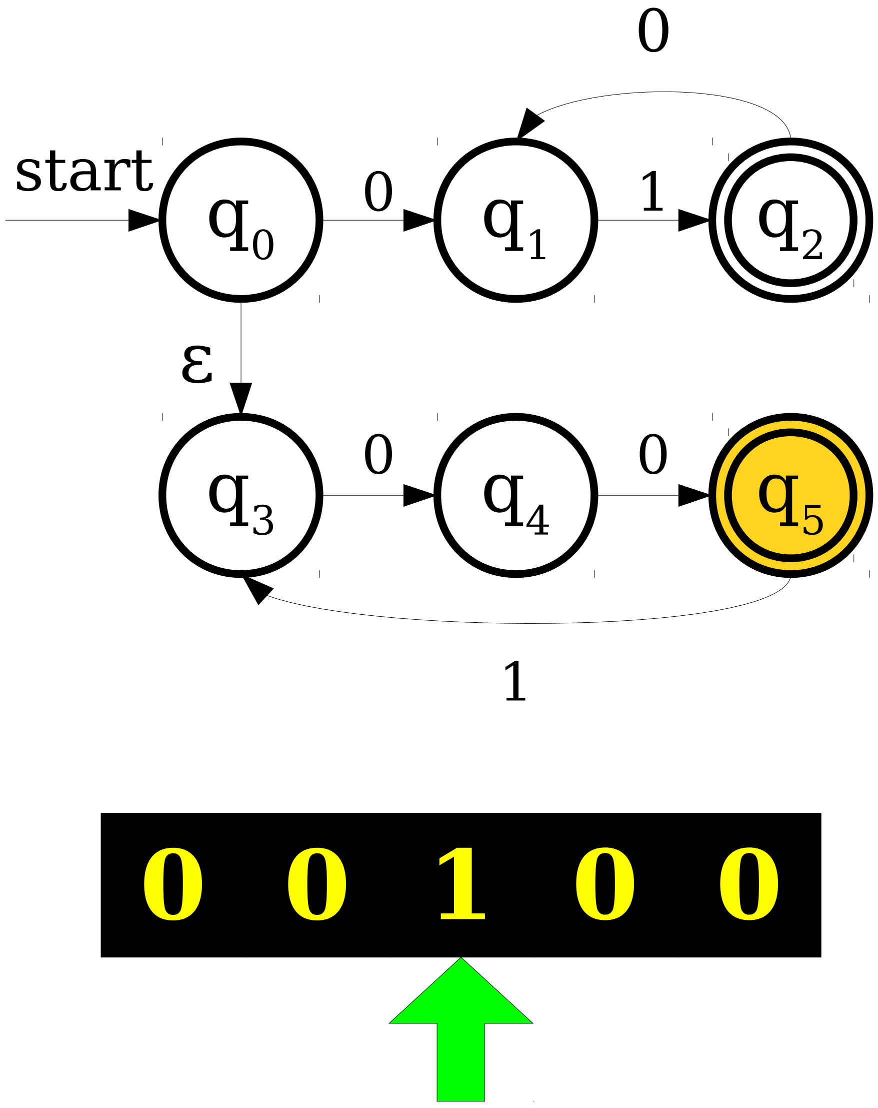

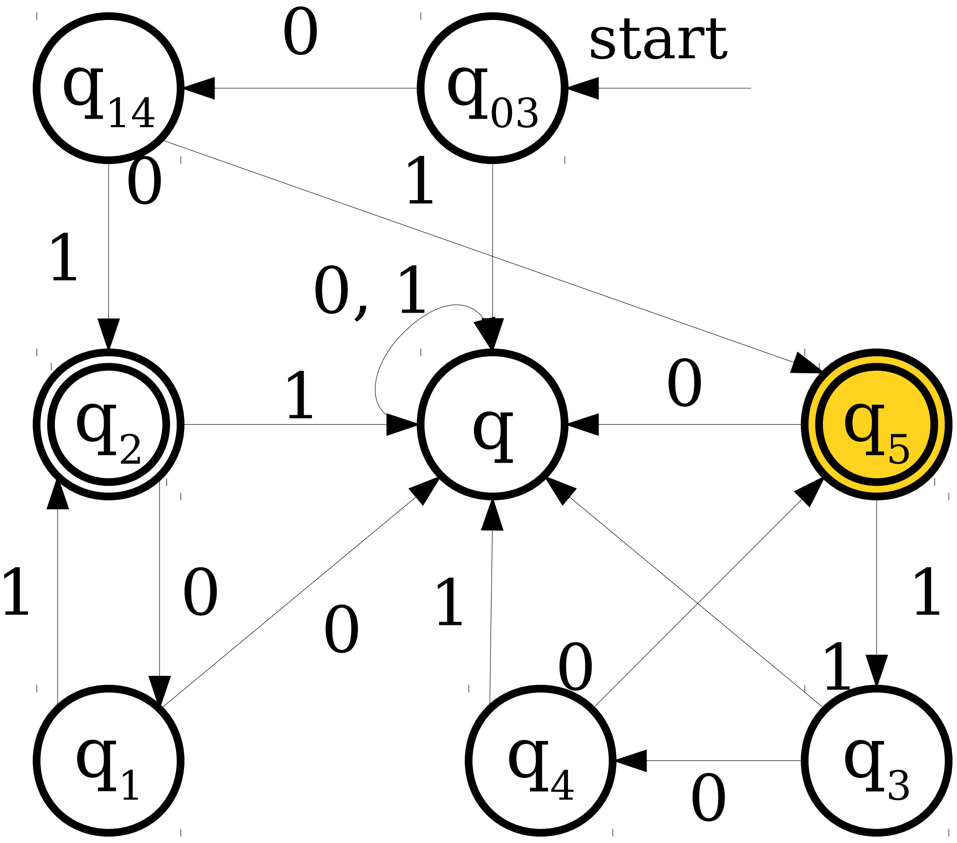

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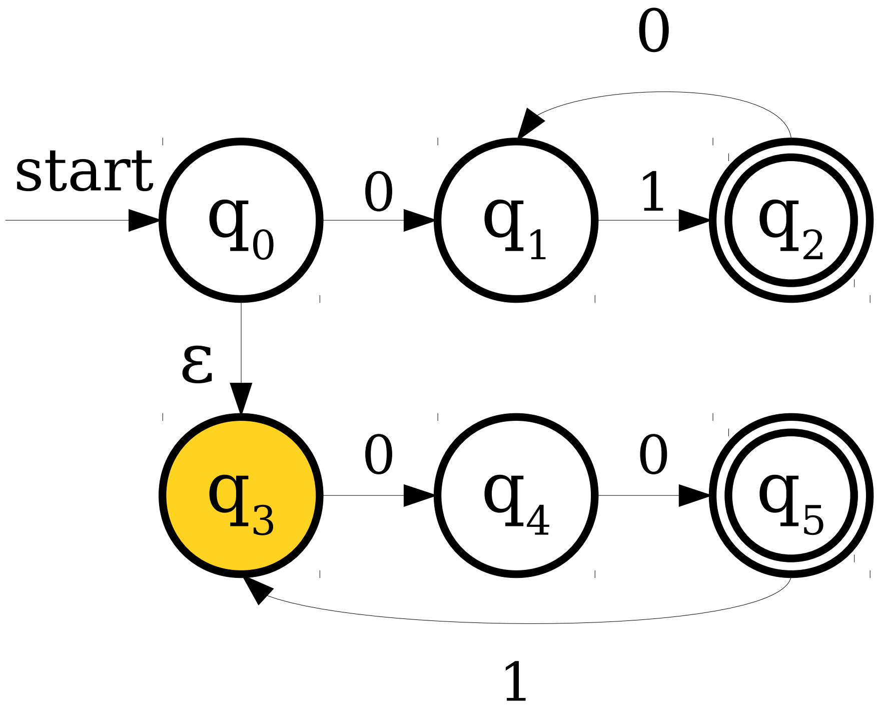

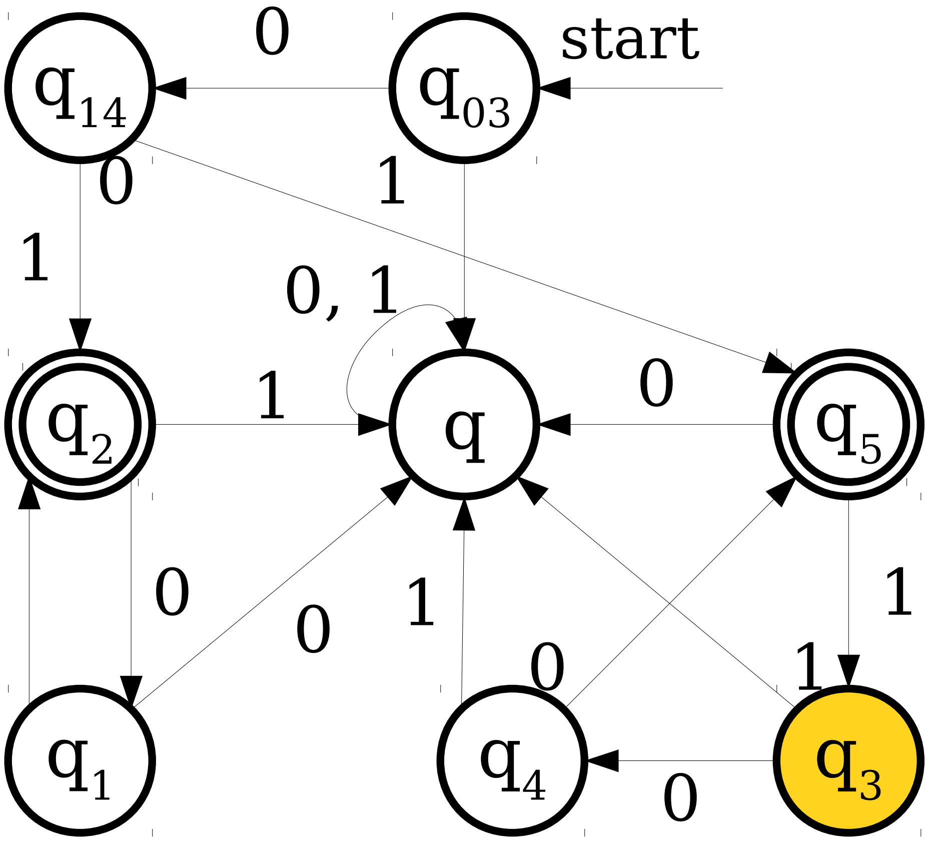

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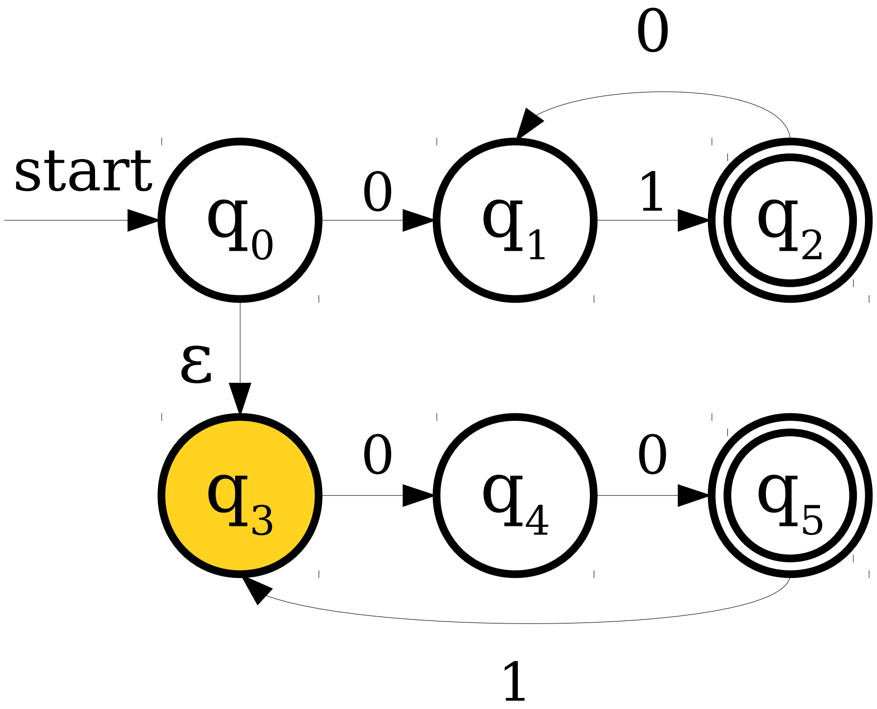

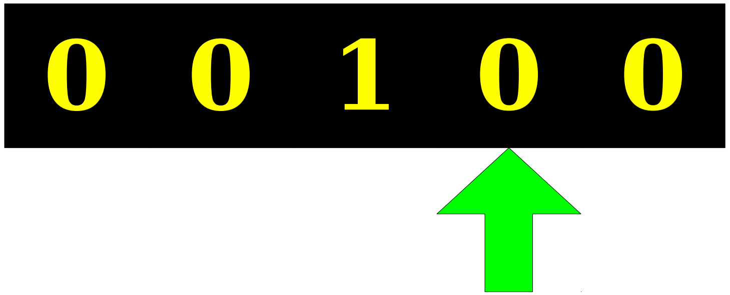

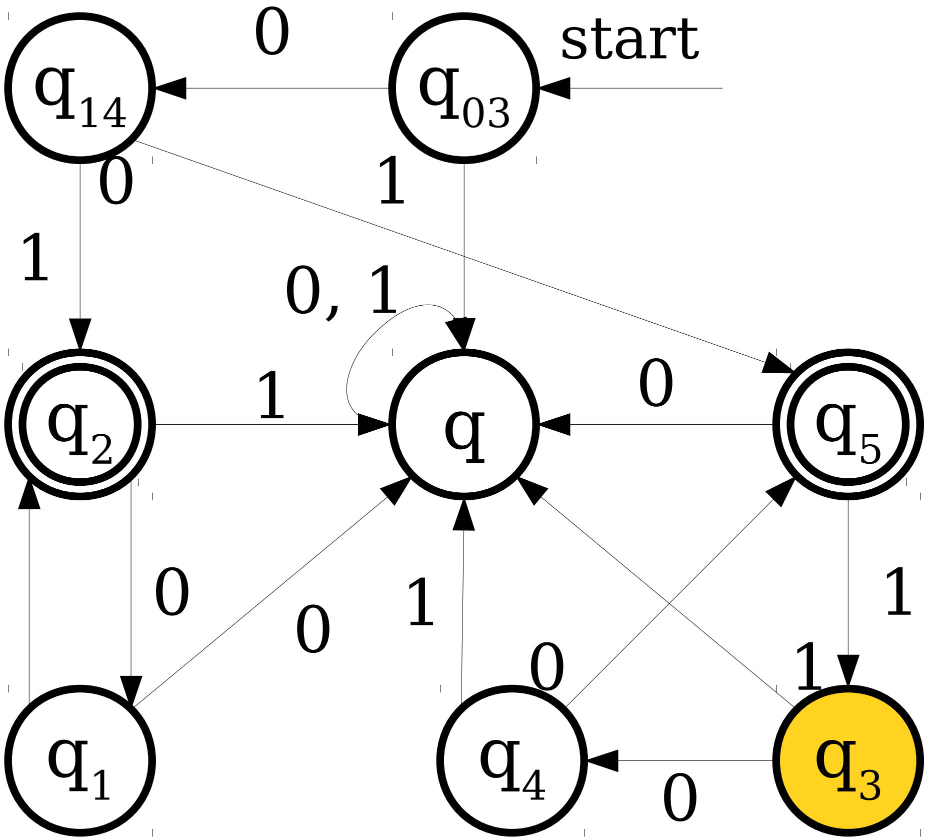

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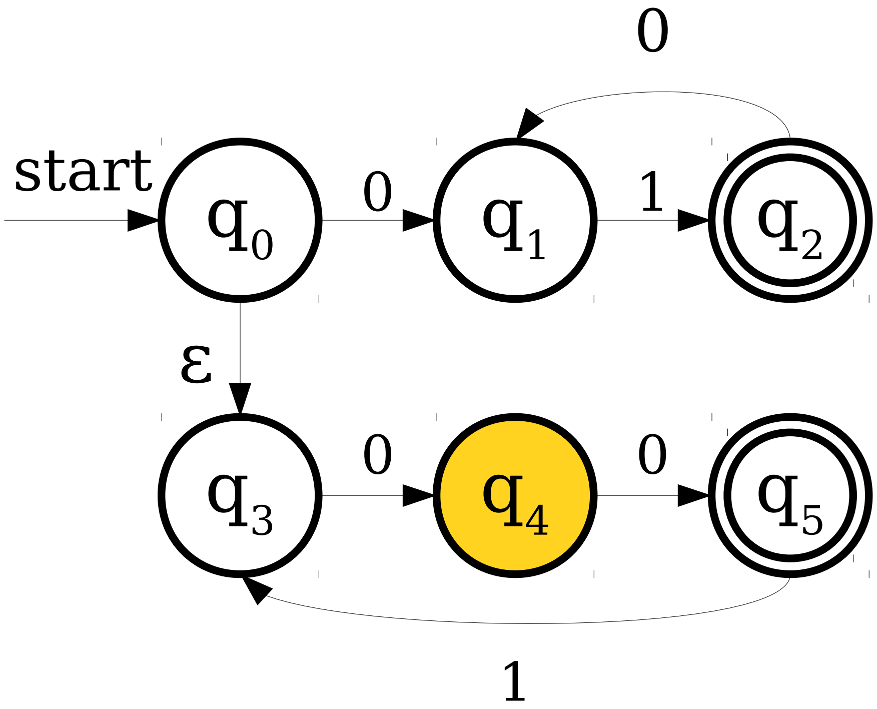

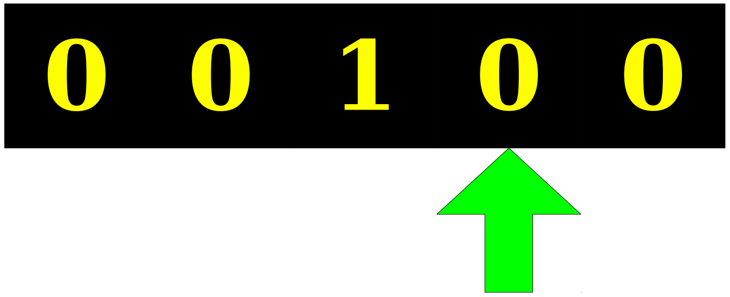

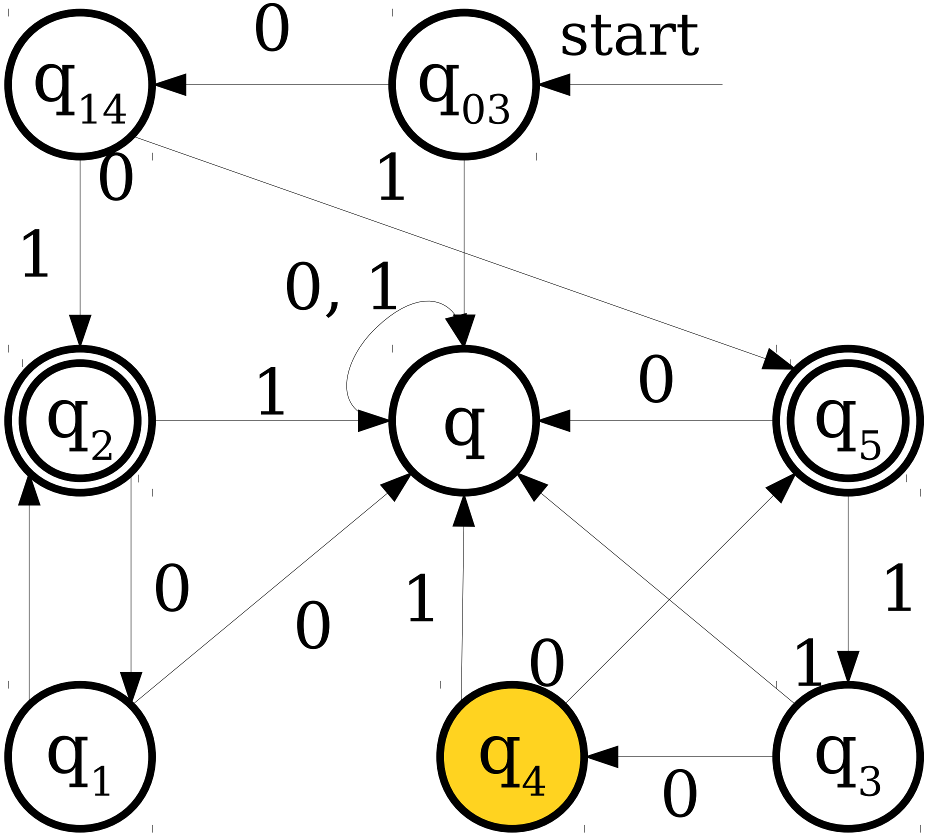

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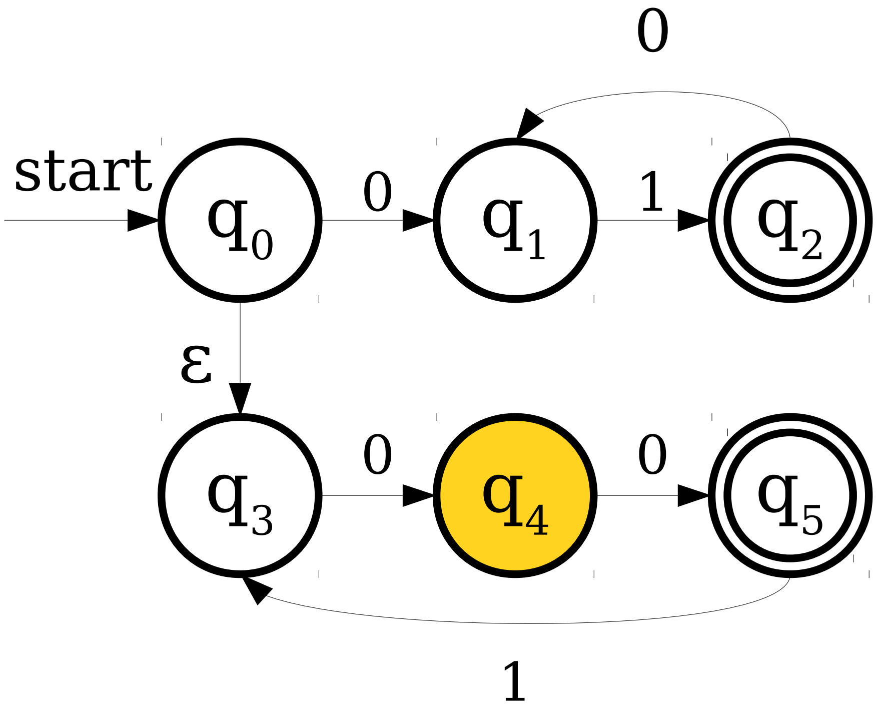

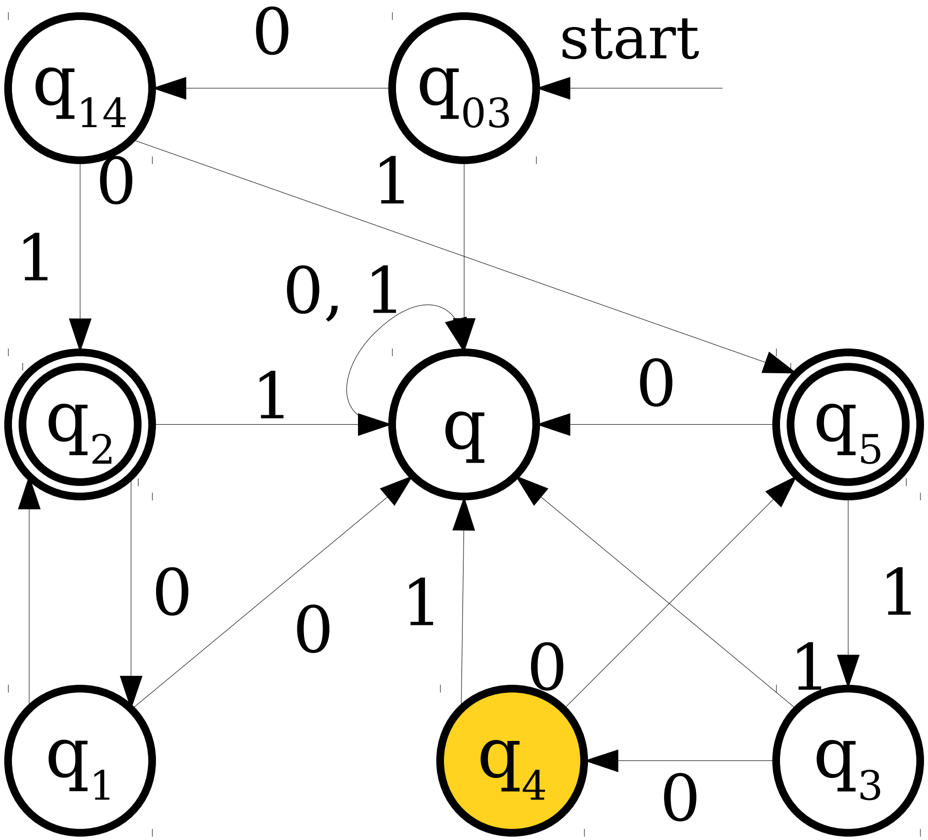

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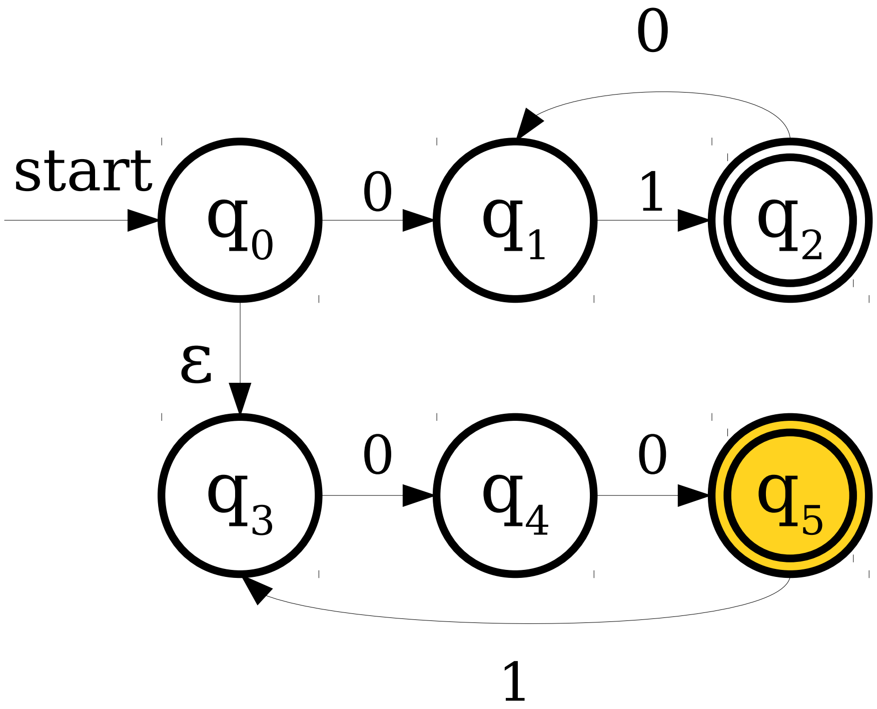

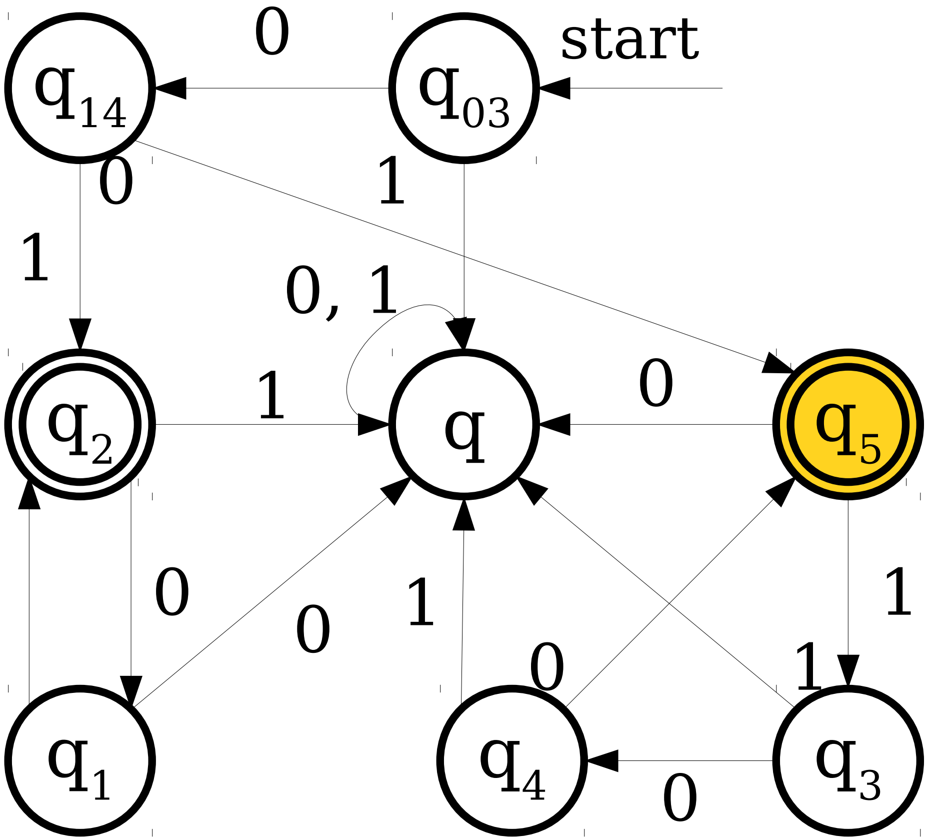

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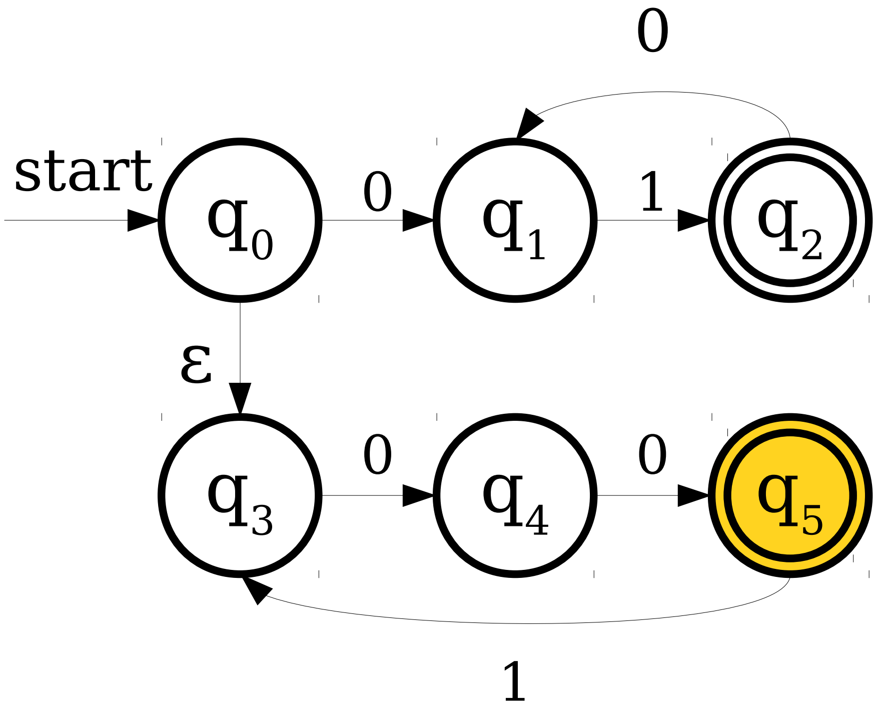

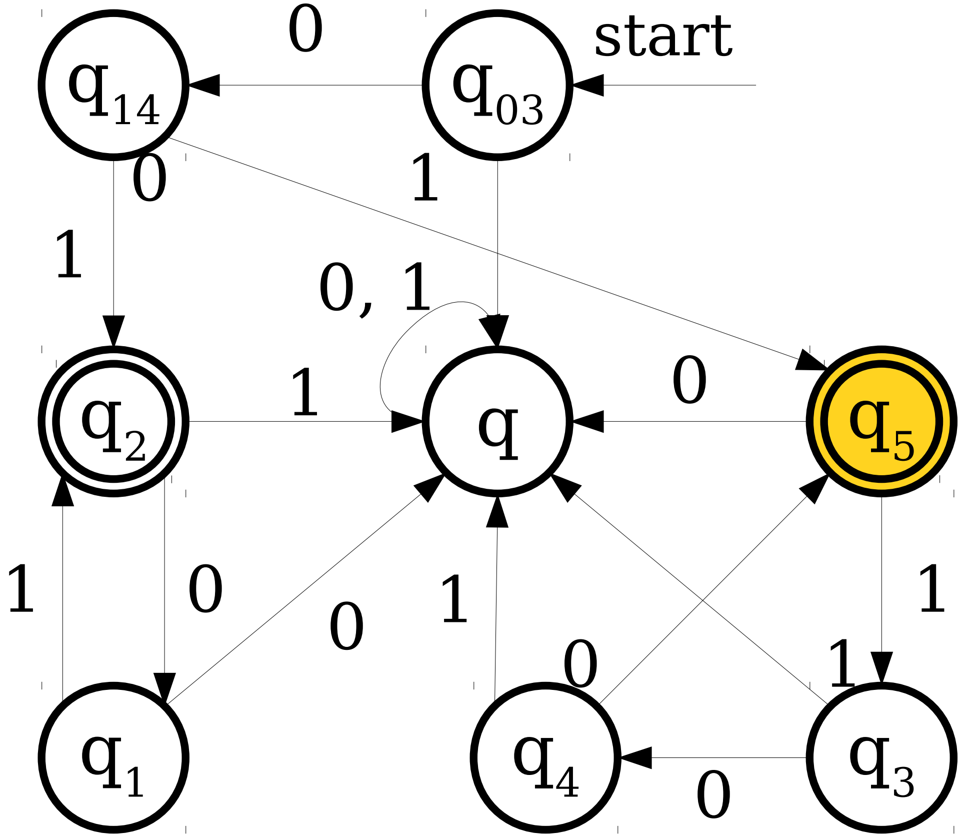

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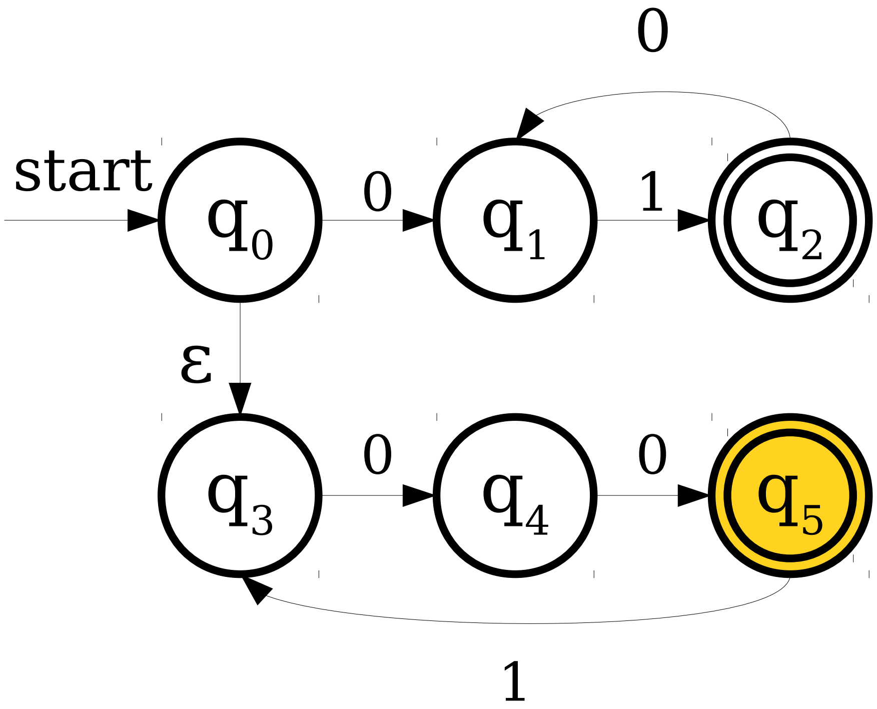

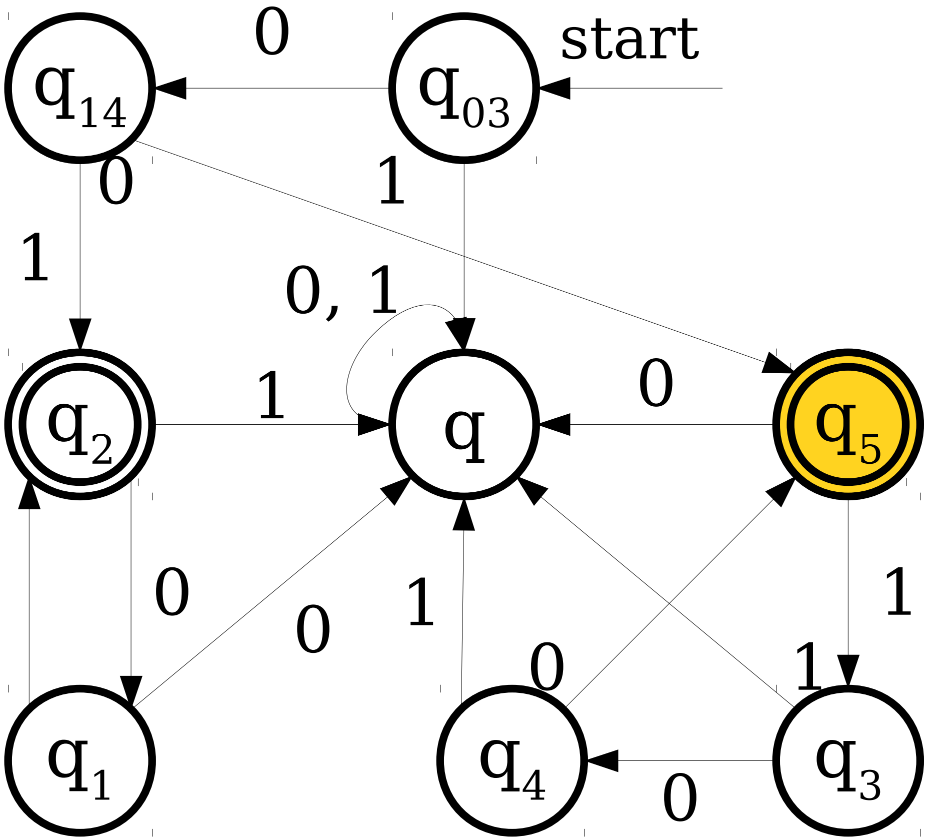

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The Subset Construction

- This construction for transforming an NFA into a DFA is called the ***subset construction*** (or sometimes the ***powerset construction***).
  - States of the new DFA correspond to *sets of states* of the NFA.
  - The initial state is the start state, plus all states reachable from the start state via ε-transitions.
  - Transition on state *S* on character **`a`** is found by following all possible transitions on **`a`** for each state in *S*, then taking the set of states reachable from there by ε-transitions.
  - Accepting states are any set of states where *some* state in the set is an accepting state.
- ***Read Sipser for a formal account.***

The Subset Construction

- In converting an NFA to a DFA, the DFA's states correspond to sets of NFA states.
- ***Recall:*** $|\wp(S)| = 2^{|S|}$ for any finite set S.
- In the worst-case, the construction can result in a DFA that is *exponentially larger* than the original NFA.
- Interesting challenge: Find a language for which this worst-case behavior occurs (there are infinitely many of them!)

A language L is called a ***regular language*** if there exists a DFA D such that $\mathscr{L}(D) = L$.

An Important Result

Theorem: A language L is regular iff there is some NFA N such that $\mathscr{L}(N) = L$.

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Why This Matters

- We now have two perspectives on regular languages:
  - Regular languages are languages accepted by DFAs.
  - Regular languages are languages accepted by NFAs.
- We can now reason about the regular languages in two different ways.

Time-Out for Announcements!

Problem Set Five

- Problem Set Four was due at the start of today's lecture.
  - Due Wednesday with a late period.
- Problem Set Five goes out now, due next Monday at 2:15PM.
  - Play around with DFAs, NFAs, regular languages, and regular expressions!
  - No checkpoint problem.

Midterms Graded

- Midterm exams have been graded; will be distributed with solutions after today's lecture.
- SCPD students – exams will be sent back through the SCPD office.
- Didn't pick up exams today? They'll be available right outside of Keith's office (Gates 178).

Your Questions

“What's your favorite class to teach?”

“What CS classes do you recommend next quarter?”

“Do I need to get a summer internship to be a 'successful' CS major? If yes, what're some good companies to work at? If no, what else can I do over the summer?”

“I've noticed that you're very passionate about diversity in CS. What advice would you have for someone who is part of a minority and is about to have a job/internship in an environment that is not likely to be diverse?”

“I am not quite convinced by the redefinition of cardinality- not that using a bijective function doesn't make sense, but that you used the point that we simply can't define cardinality in any other substantive form. But we did just that when we first learned it – it's the number of elements in a set. Why do we need this new definition?”

“I think CS is so cool. I love learning it. However it doesn't come naturally, which is fine--I just try my hardest, work hard. However, with things like coterming, etc, I worry 'finding it interesting' isn't enough, especially with grades. Thoughts?”

Back to CS103!

Properties of Regular Languages

The Union of Two Languages

- If L_1 and L_2 are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?

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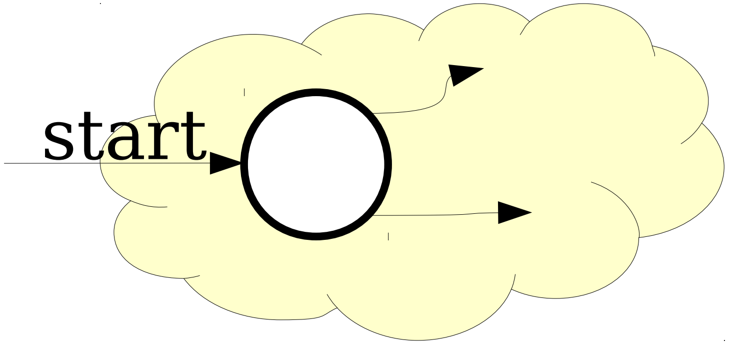
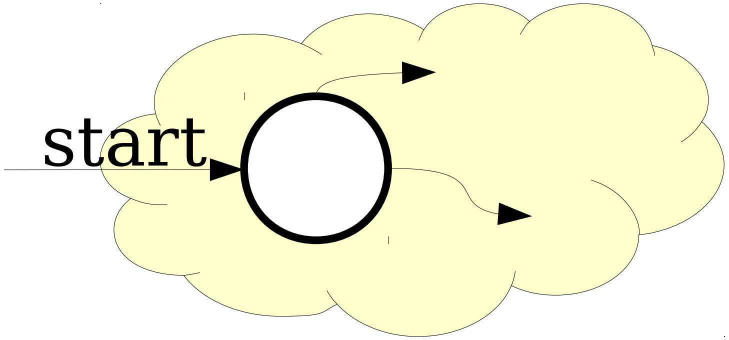


Machine for L_1



Machine for L_2

The Union of Two Languages

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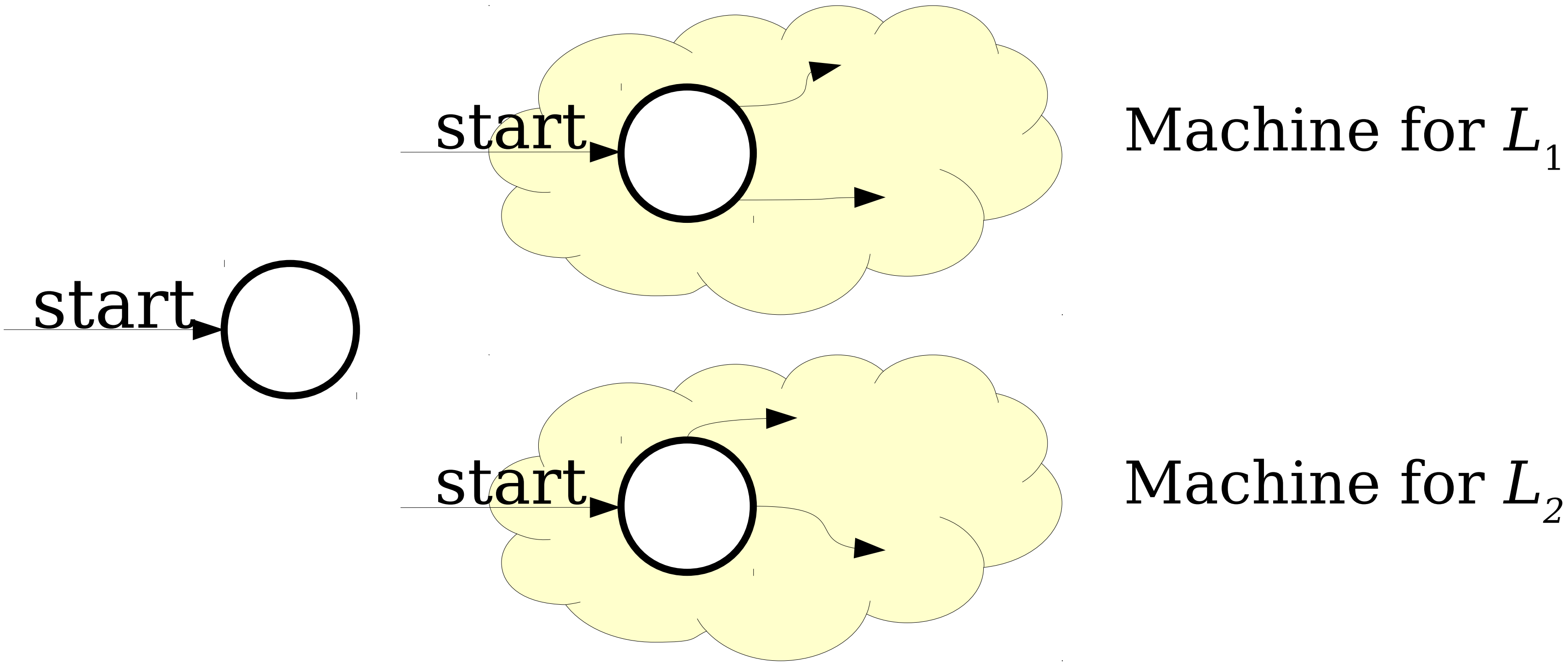
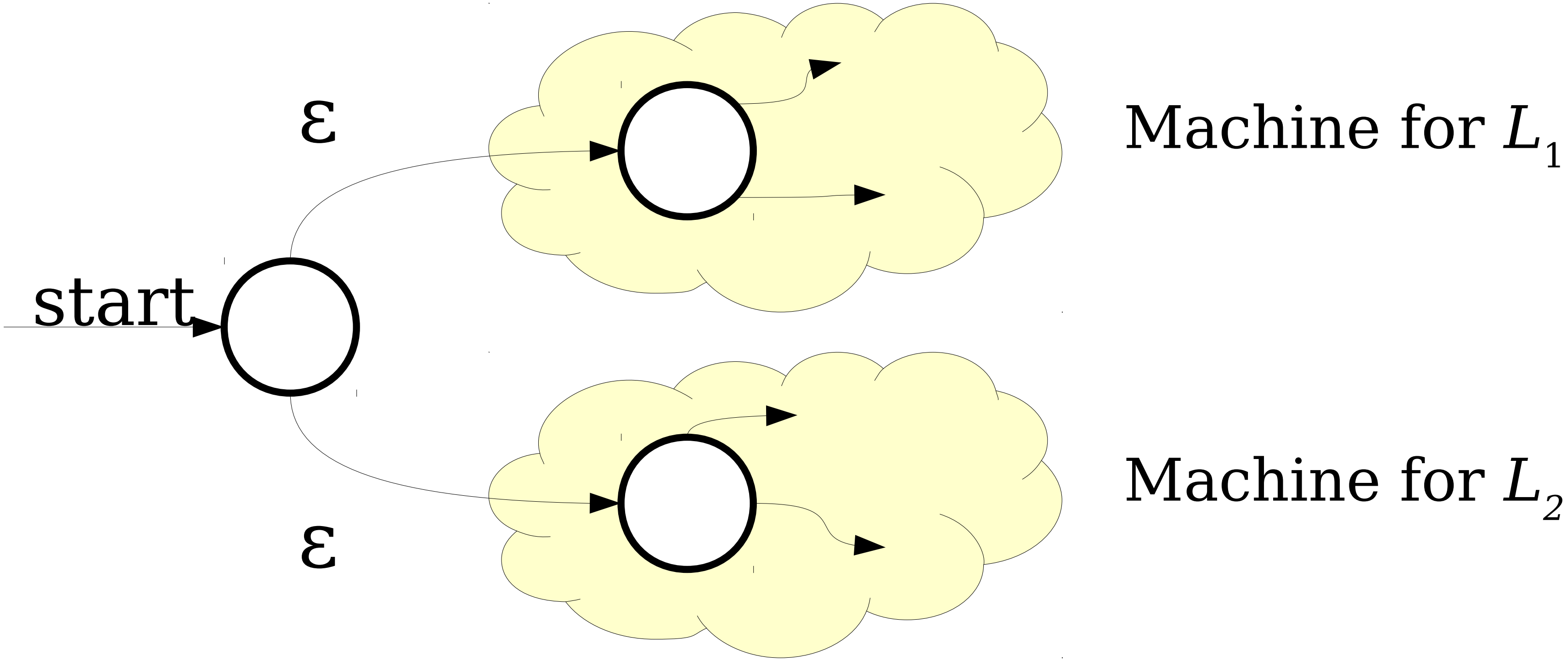


The Union of Two Languages

- If L_1 and L_2 are languages over the alphabet Σ, the language $L_1 \cup L_2$ is the language of all strings in at least one of the two languages.
- If L_1 and L_2 are regular languages, is $L_1 \cup L_2$?



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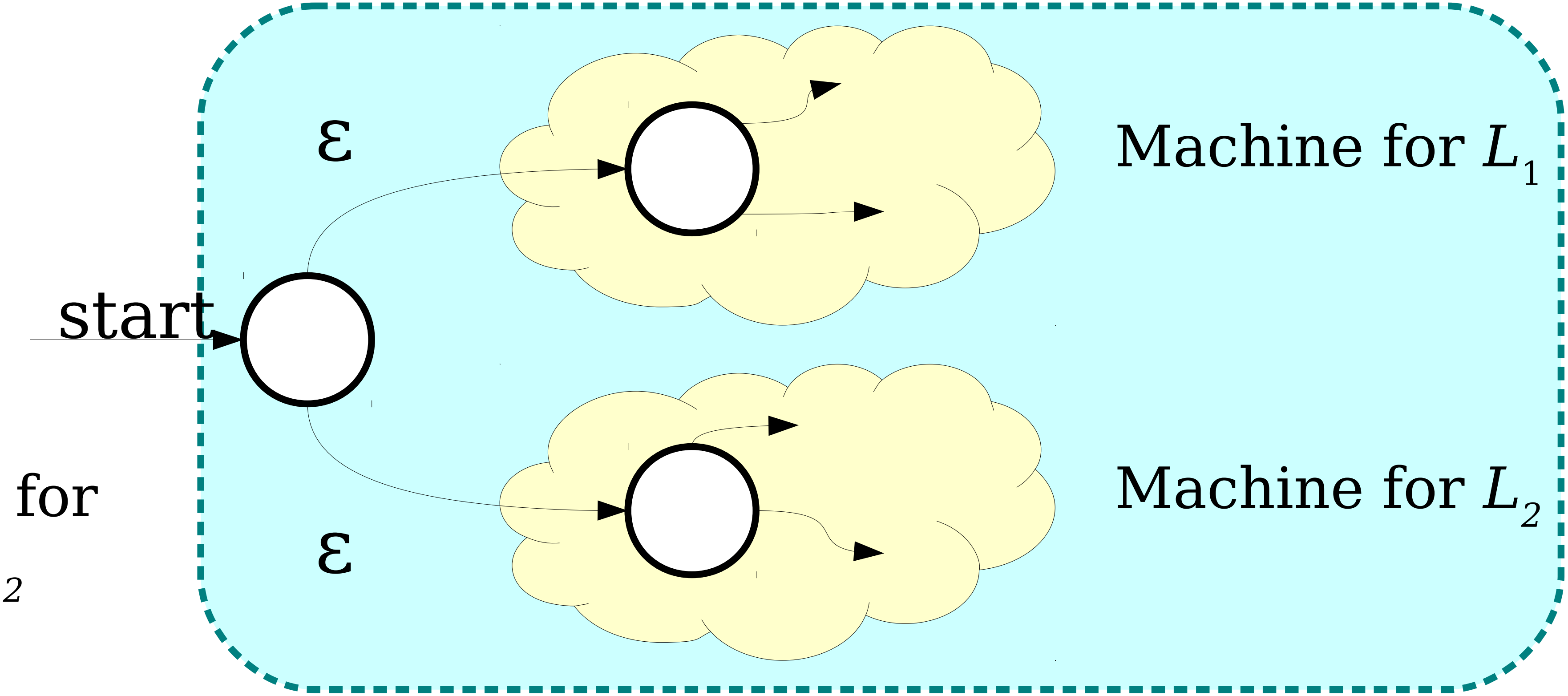

Machine for $L_1 \cup L_2$

The Intersection of Two Languages

- If L_1 and L_2 are languages over Σ, then $L_1 \cap L_2$ is the language of strings in both L_1 and L_2.
- Question: If L_1 and L_2 are regular, is $L_1 \cap L_2$ regular as well?

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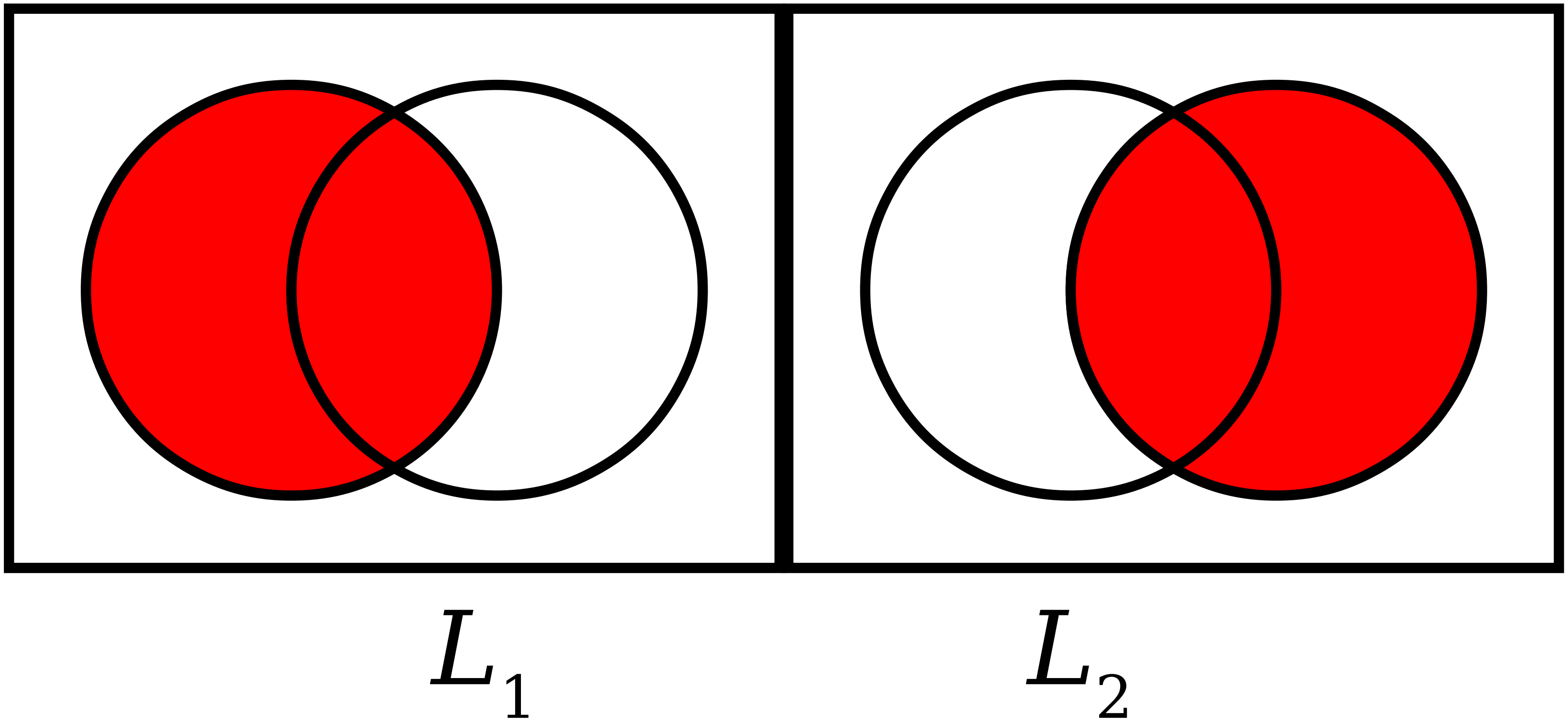

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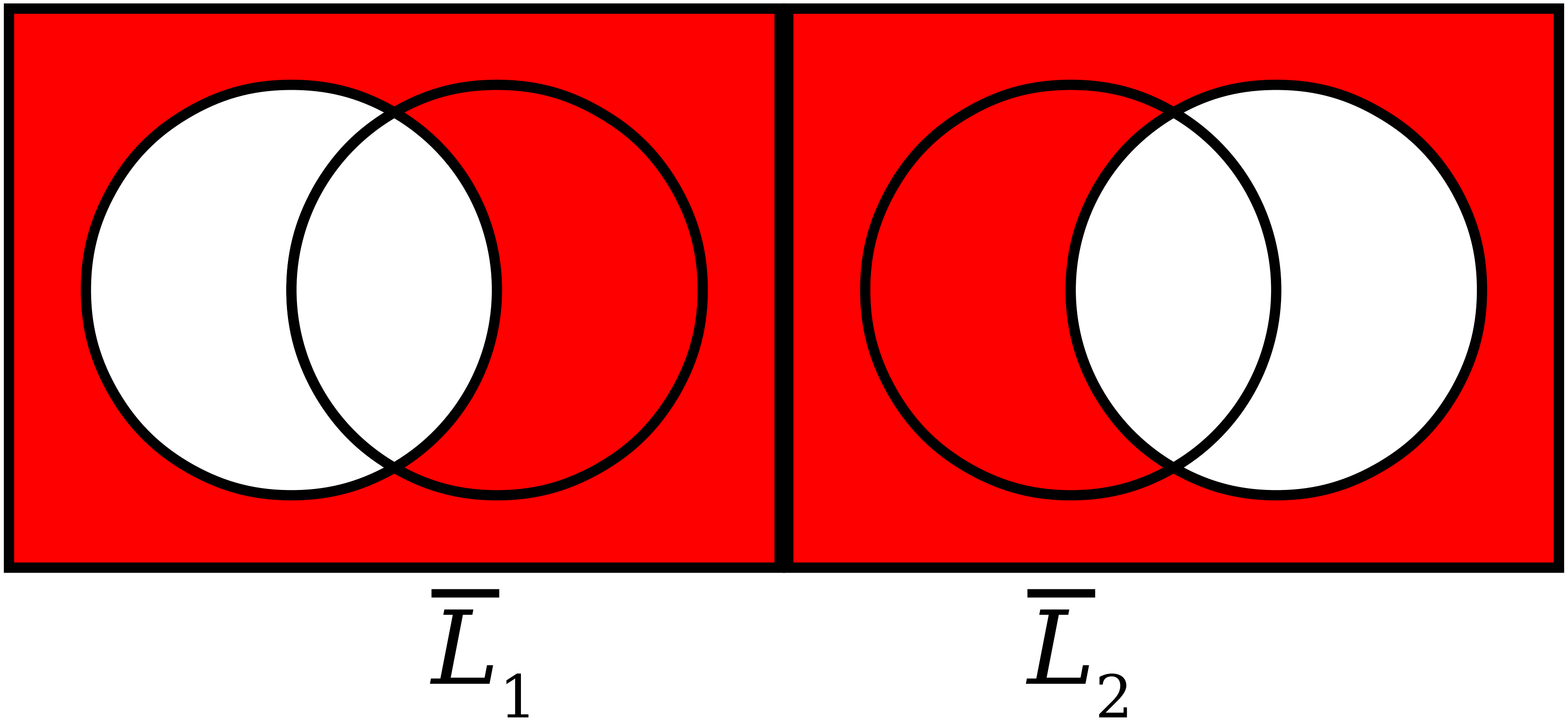

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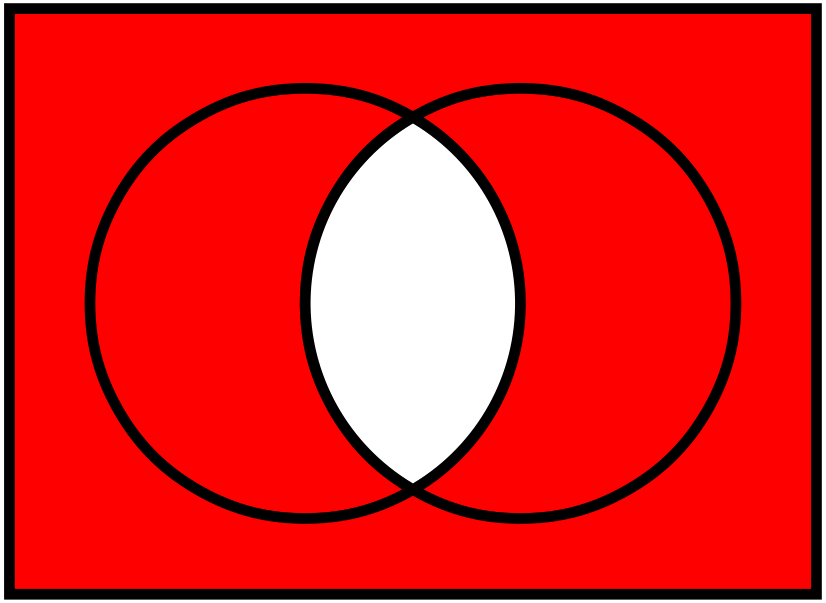

$$\overline{L}_1 \cup \overline{L}_2$$

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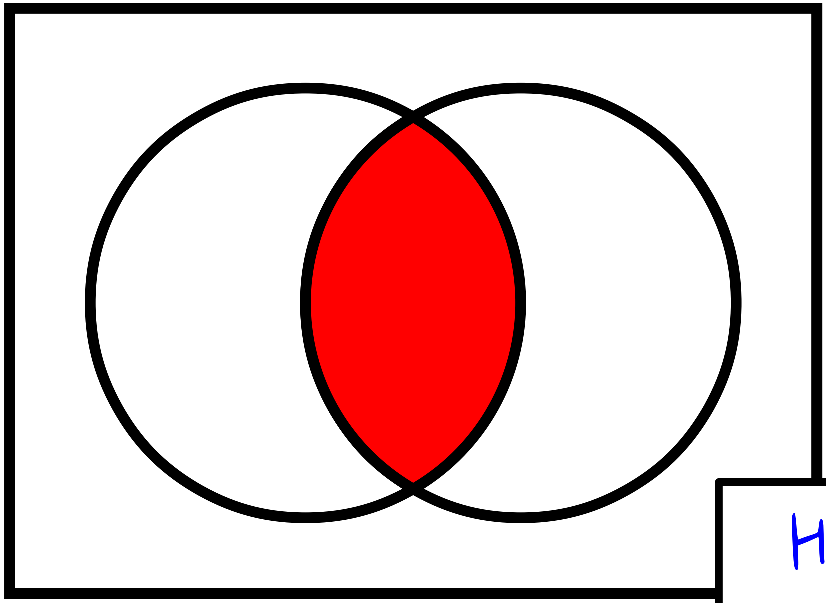

$$\overline{\overline{L_1} \cup \overline{L_2}}$$

Hey, it's De Morgan's laws!

Concatenation

- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

$$L_1L_2 = \{ wx \in \Sigma^* \mid w \in L_1 \land x \in L_2 \}$$

- The set of strings that can be split into two pieces: a piece from L_1 and a piece from L_2.
- Conceptually similar to the Cartesian product of two sets, only with strings.

Concatenation Example

- Let Σ = { `a`, `b`, …, `z`, `A`, `B`, …, `Z` } and consider these languages over Σ:
  - ***Noun*** = { `Puppy`, `Rainbow`, `Whale`, … }
  - ***Verb*** = { `Hugs`, `Juggles`, `Loves`, … }
  - ***The*** = { `The` }
- The language ***TheNounVerbTheNoun*** is

  { `ThePuppyHugsTheWhale`,
  `TheWhaleLovesTheRainbow`,
  `TheRainbowJugglesTheRainbow`, … }

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?

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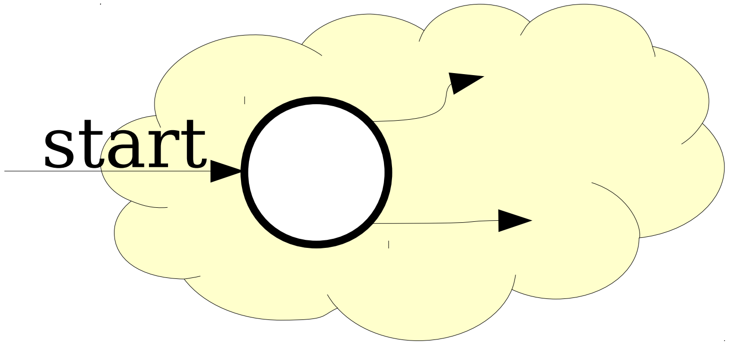
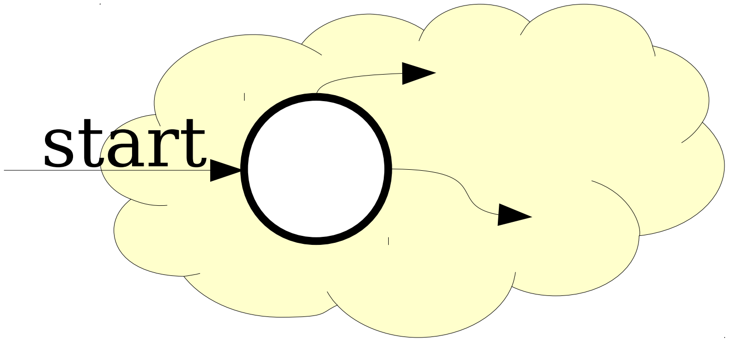


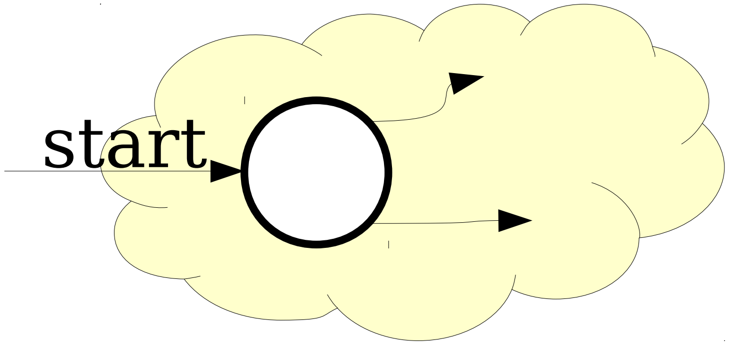
Machine for L_1



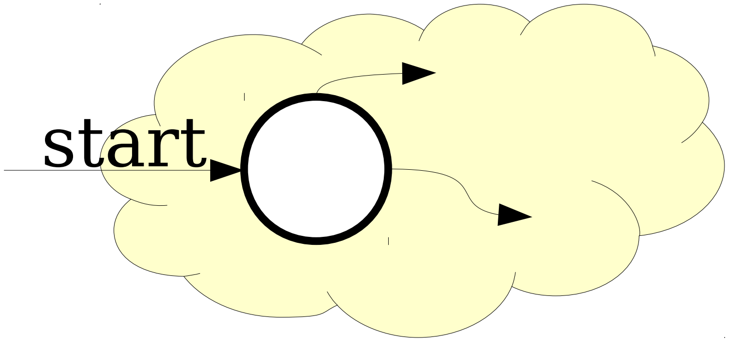
Machine for L_2

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?



Machine for L_1



Machine for L_2

b	**o**	**o**	**k**	**k**	**e**	**e**	**p**	**e**	**r**

Concatenating Regular Languages

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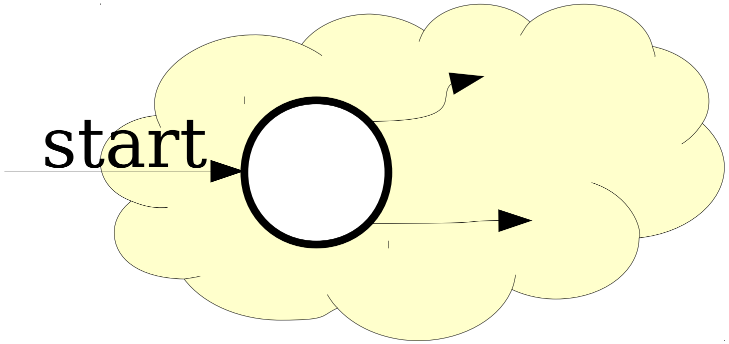
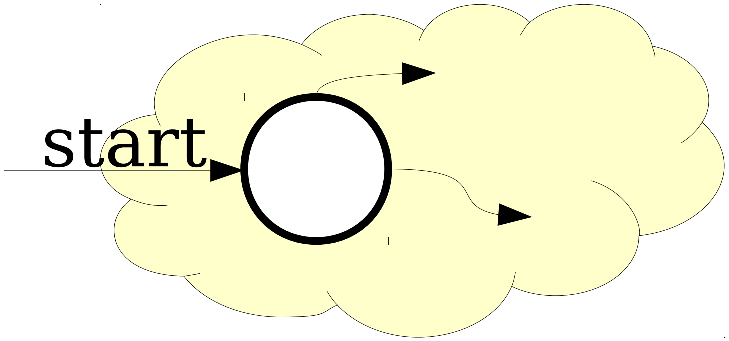


Machine for L_1



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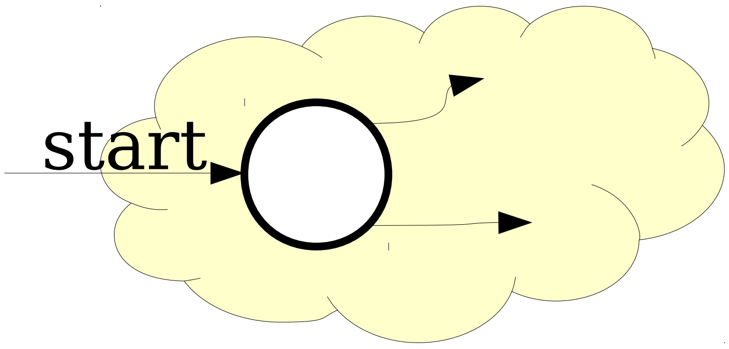
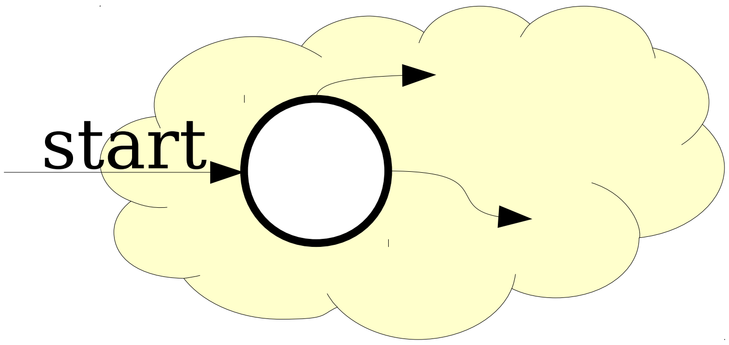


Machine for L_1



Machine for L_2

b	**o**	**o**	**k**

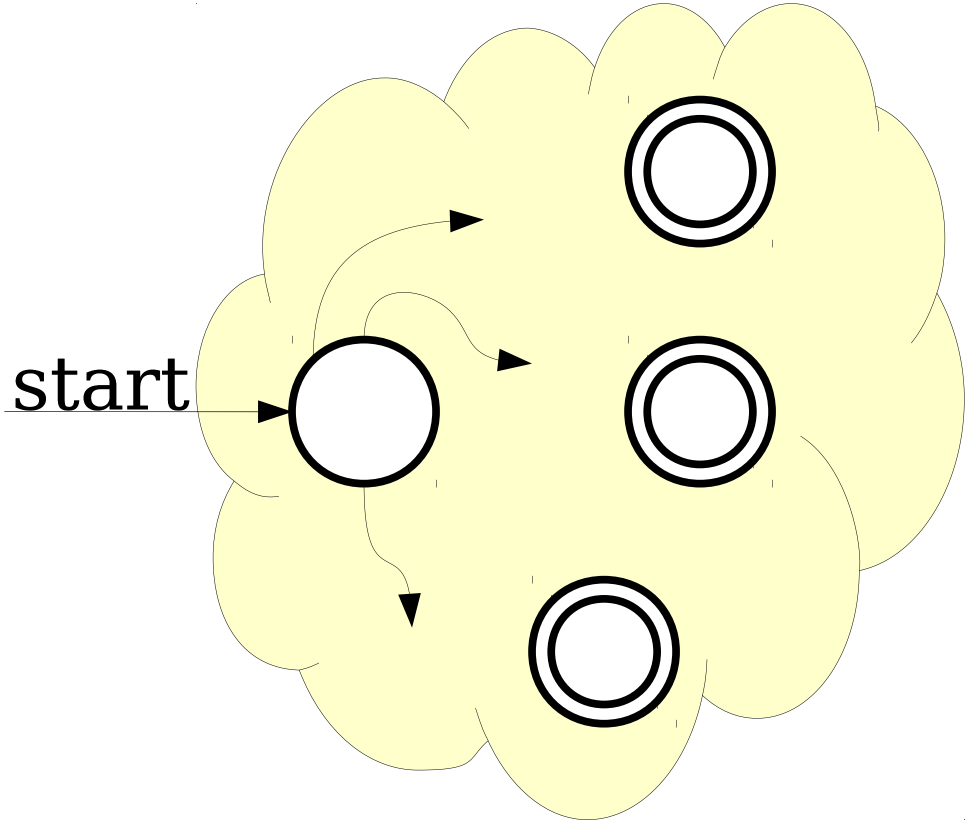
k	**e**	**e**	**p**	**e**	**r**

Concatenating Regular Languages

- If L_1 and L_2 are regular languages, is L_1L_2?
- Intuition – can we split a string w into two strings xy such that $x \in L_1$ and $y \in L_2$?
- **Idea**: Run the automaton for L_1 on w, and whenever L_1 reaches an accepting state, optionally hand the rest off w to L_2.
  - If L_2 accepts the remainder, then L_1 accepted the first part and the string is in L_1L_2.
  - If L_2 rejects the remainder, then the split was incorrect.

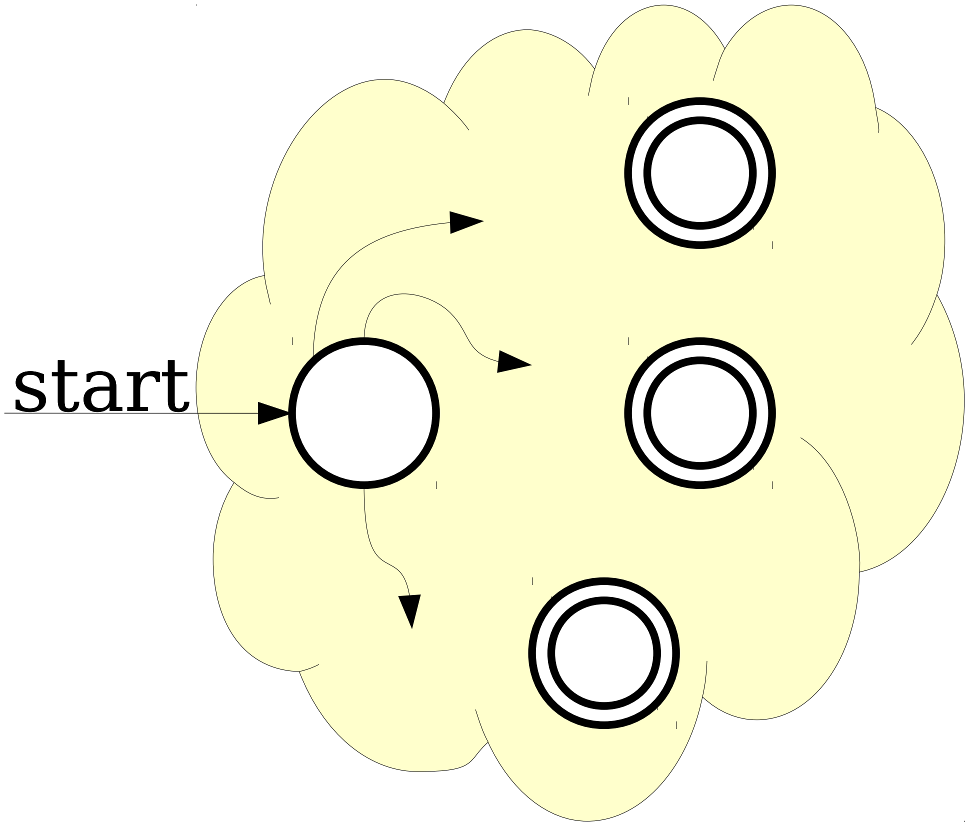
Concatenating Regular Languages

Concatenating Regular Languages



Machine for L_1

Concatenating Regular Languages



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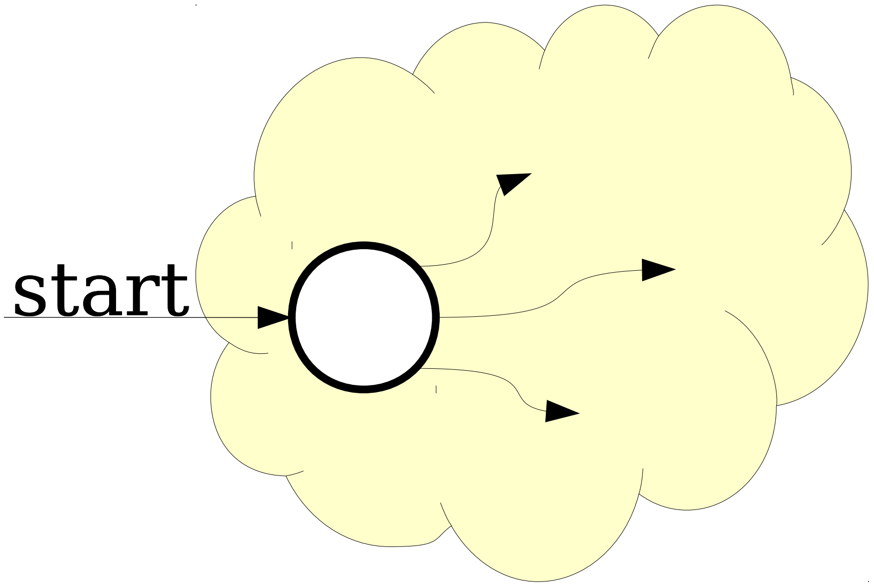
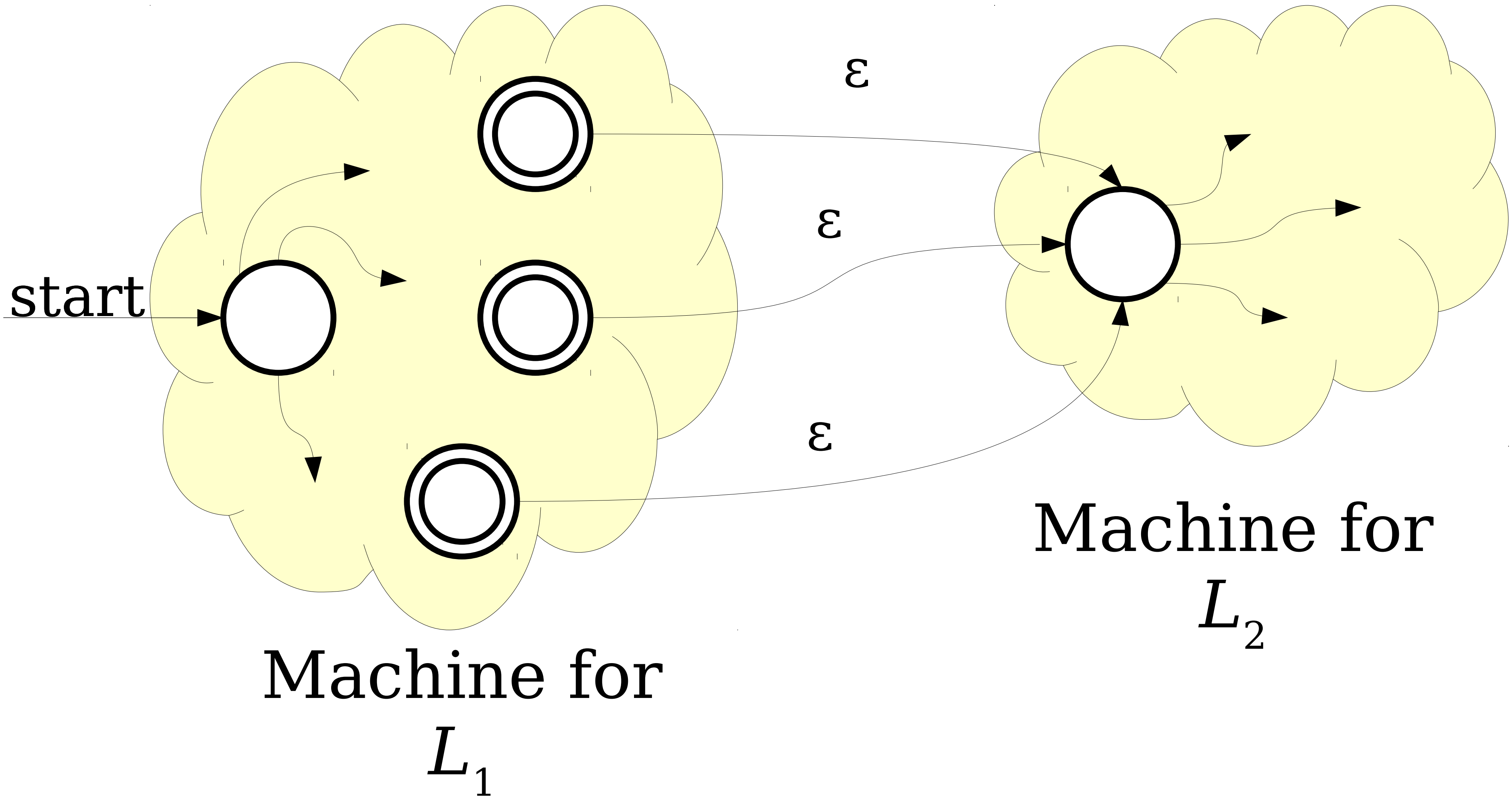
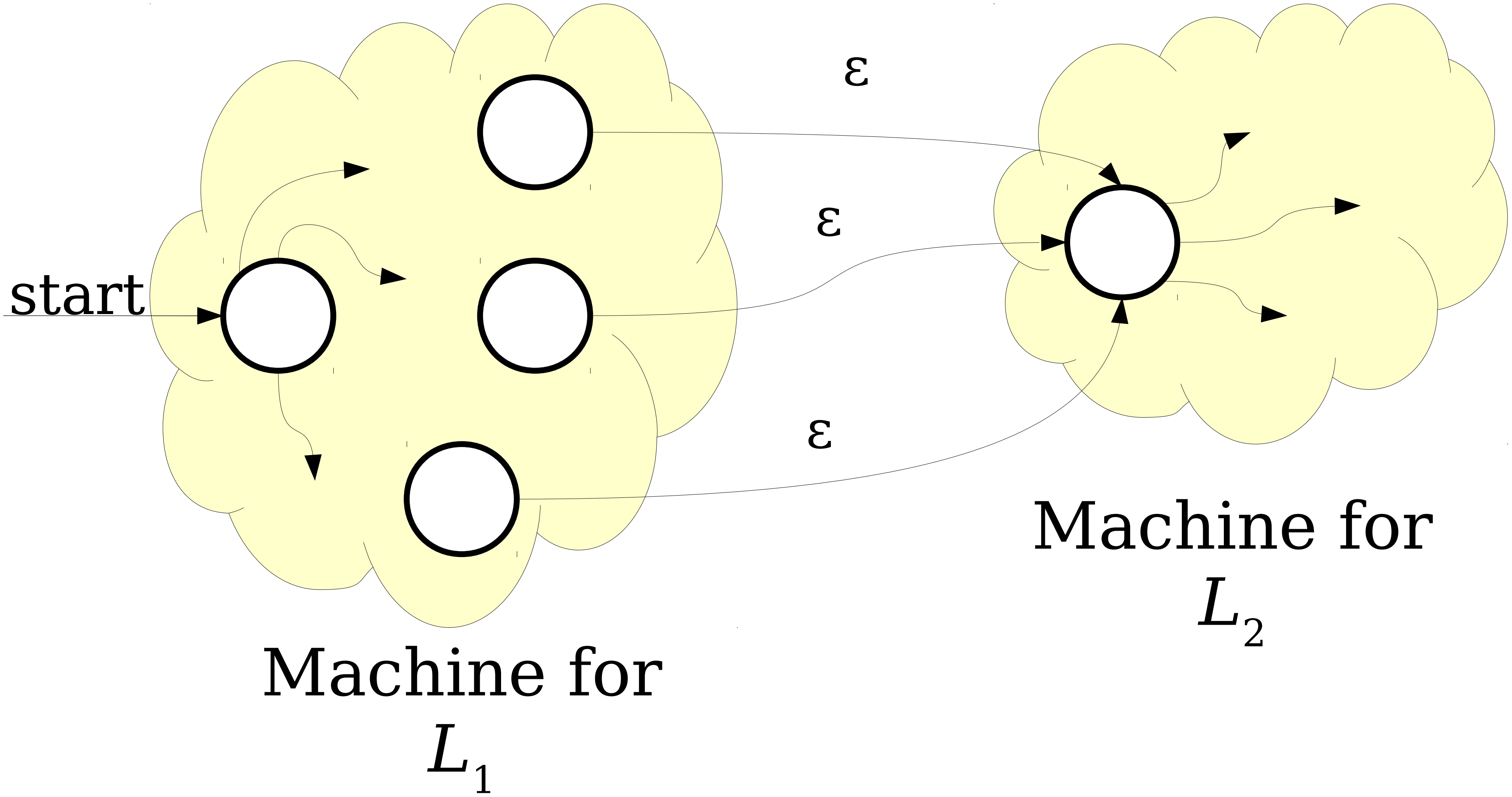


Machine for
L_2

Concatenating Regular Languages



Concatenating Regular Languages



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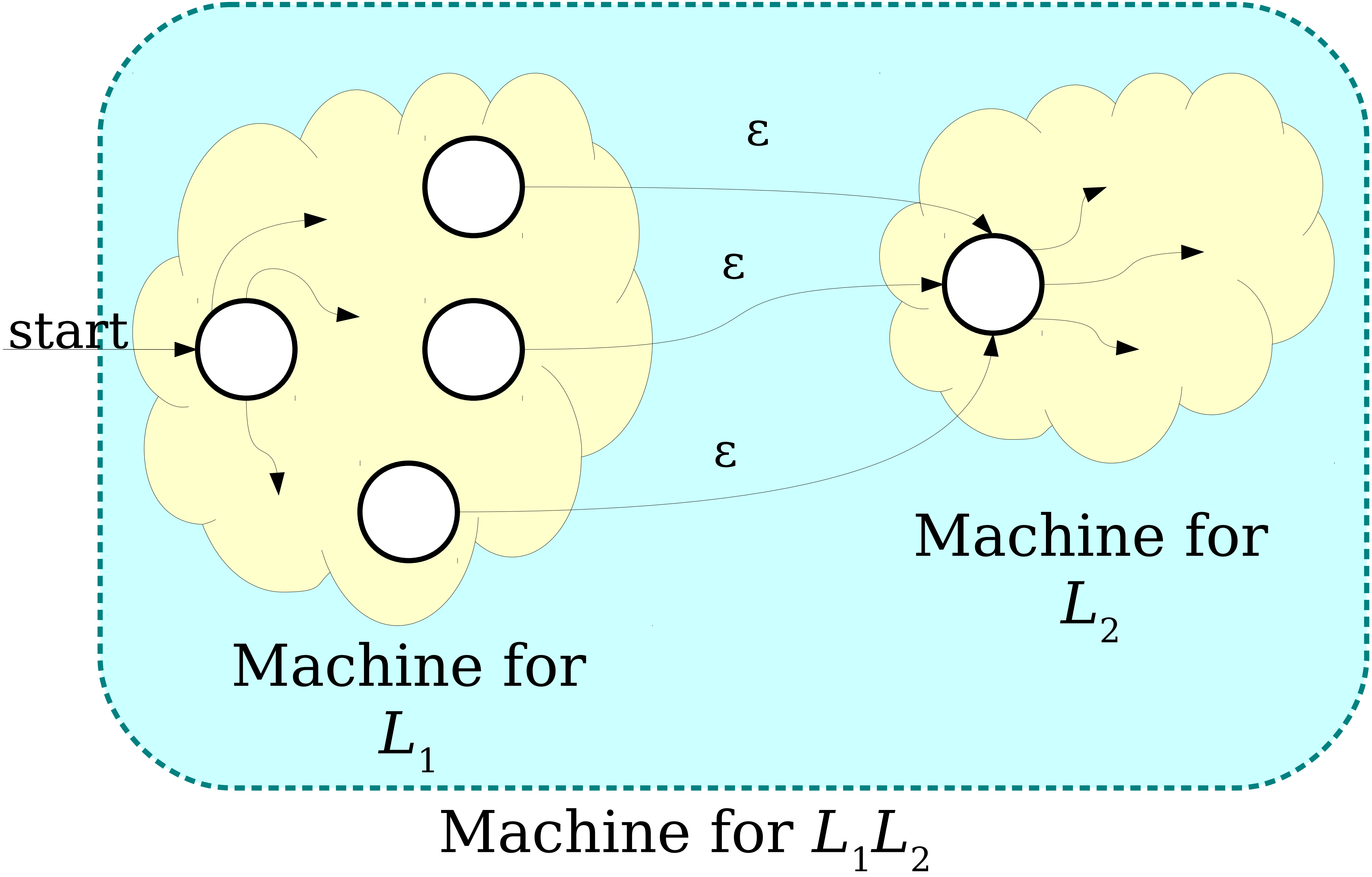


Lots and Lots of Concatenation

- Consider the language L = { **aa**, **b** }
- LL is the set of strings formed by concatenating pairs of strings in L.

  { **aaaa**, **aab**, **baa**, **bb** }

- LLL is the set of strings formed by concatenating triples of strings in L.

  { **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

  { **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**, **aabaab**, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**, **baabb**, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

Language Exponentiation

- We can define what it means to “exponentiate” a language as follows:
- $L^0 = \{ \varepsilon \}$
  - The set containing just the empty string.
  - Idea: Any string formed by concatenating zero strings together is the empty string.
- $L^{n+1} = LL^n$
  - Idea: Concatenating (n+1) strings together works by concatenating n strings, then concatenating one more.

The Kleene Closure

- An important operation on languages is the ***Kleene Closure***, which is defined as

$$L^* = \bigcup_{i=0}^{\infty} L^i$$

- Mathematically:

$$\boldsymbol{w \in L^*} \quad \textbf{iff} \quad \boldsymbol{\exists n \in \mathbb{N}.\ w \in L^n}$$

- Intuitively, all possible ways of concatenating any number of copies of strings in L together.

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

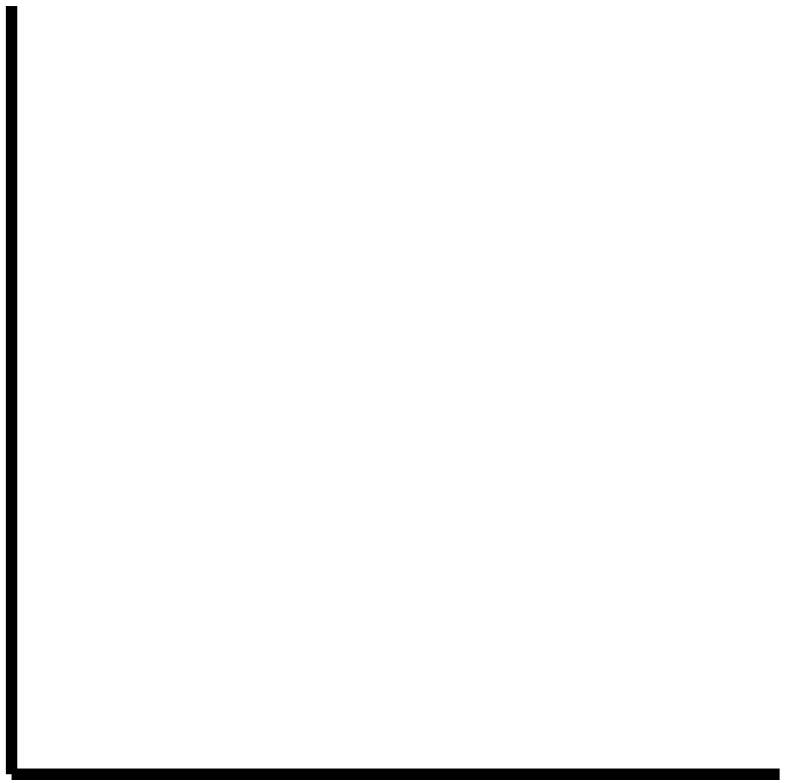
…

}

Reasoning about Infinity

- If L is regular, is L^* necessarily regular?
- **A Bad Line of Reasoning**:
  - $L^0 = \{ \varepsilon \}$ is regular.
  - $L^1 = L$ is regular.
  - $L^2 = LL$ is regular
  - $L^3 = L(LL)$ is regular
  - …
  - Regular languages are closed under union.
  - So the union of all these languages is regular.

Reasoning about Infinity



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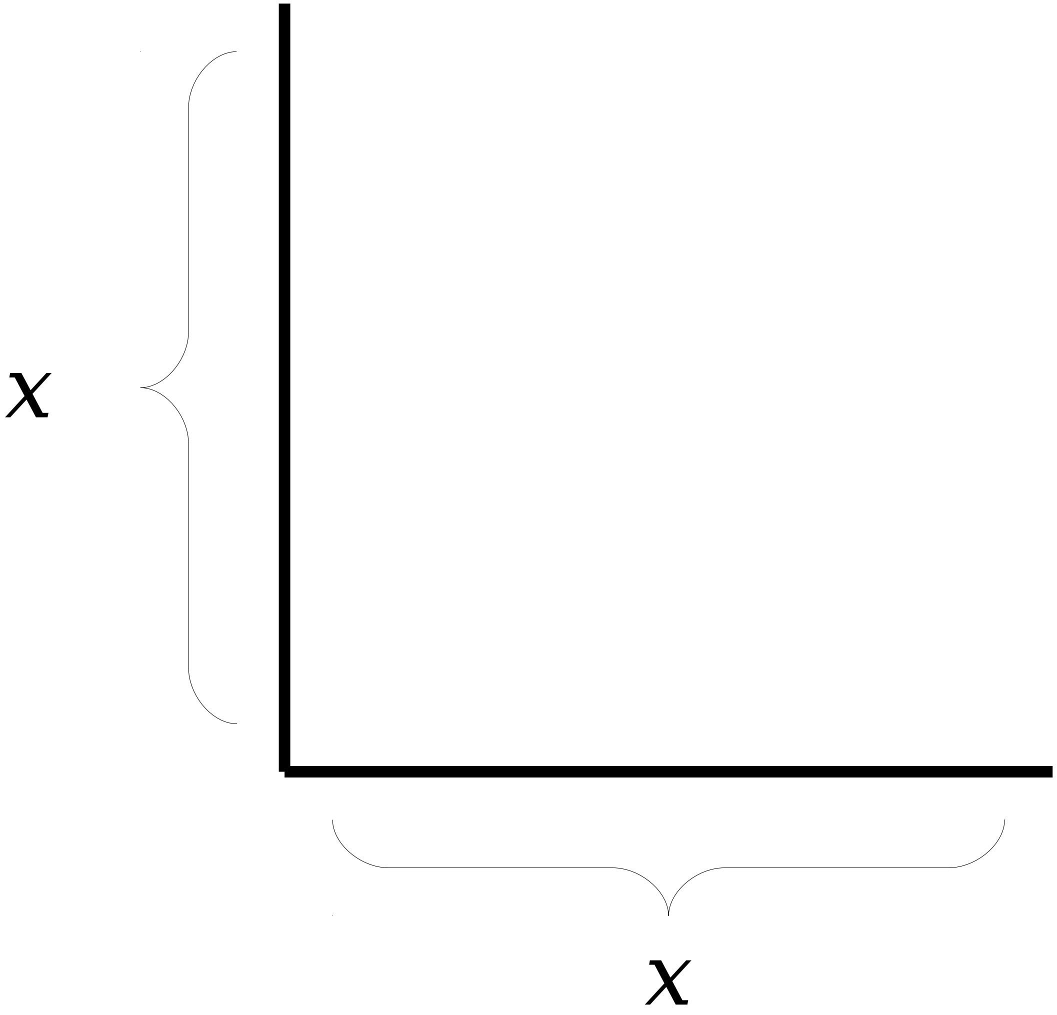

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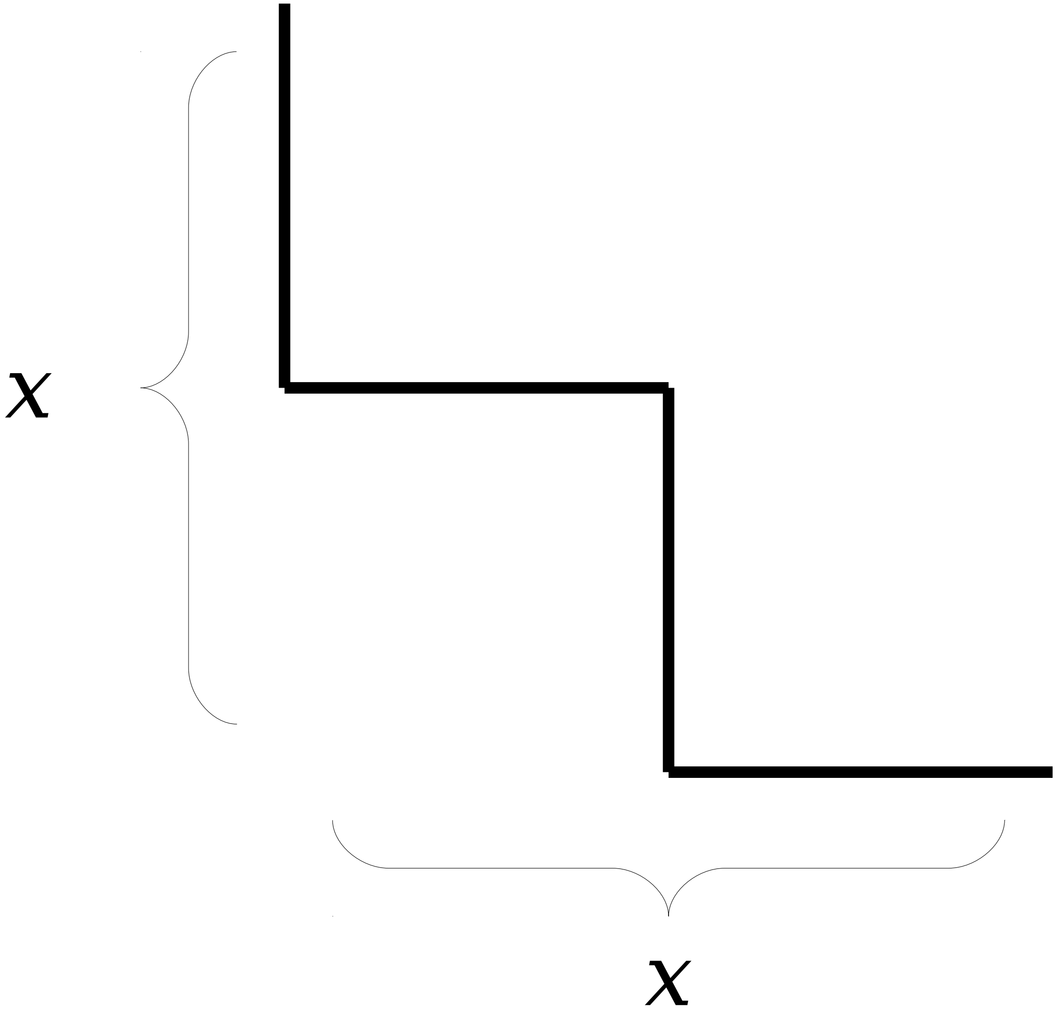

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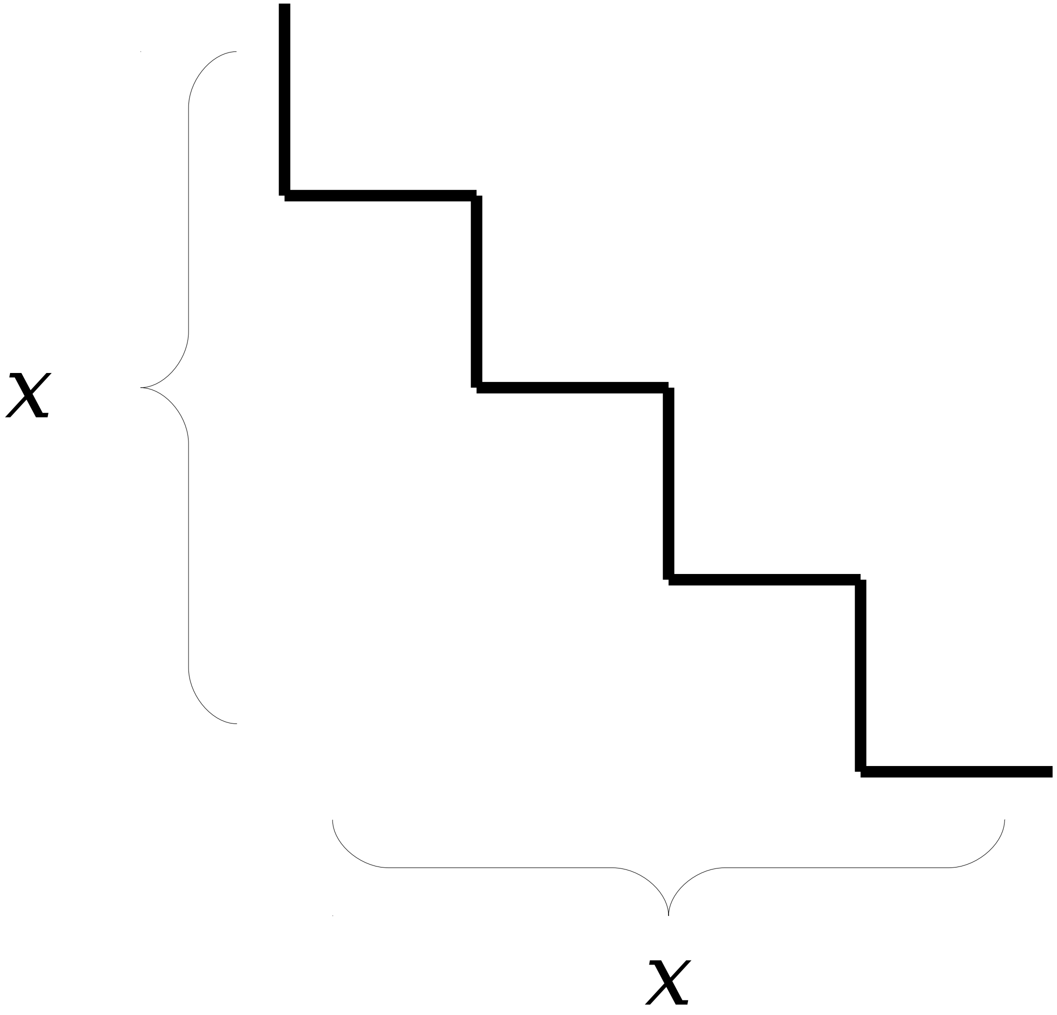

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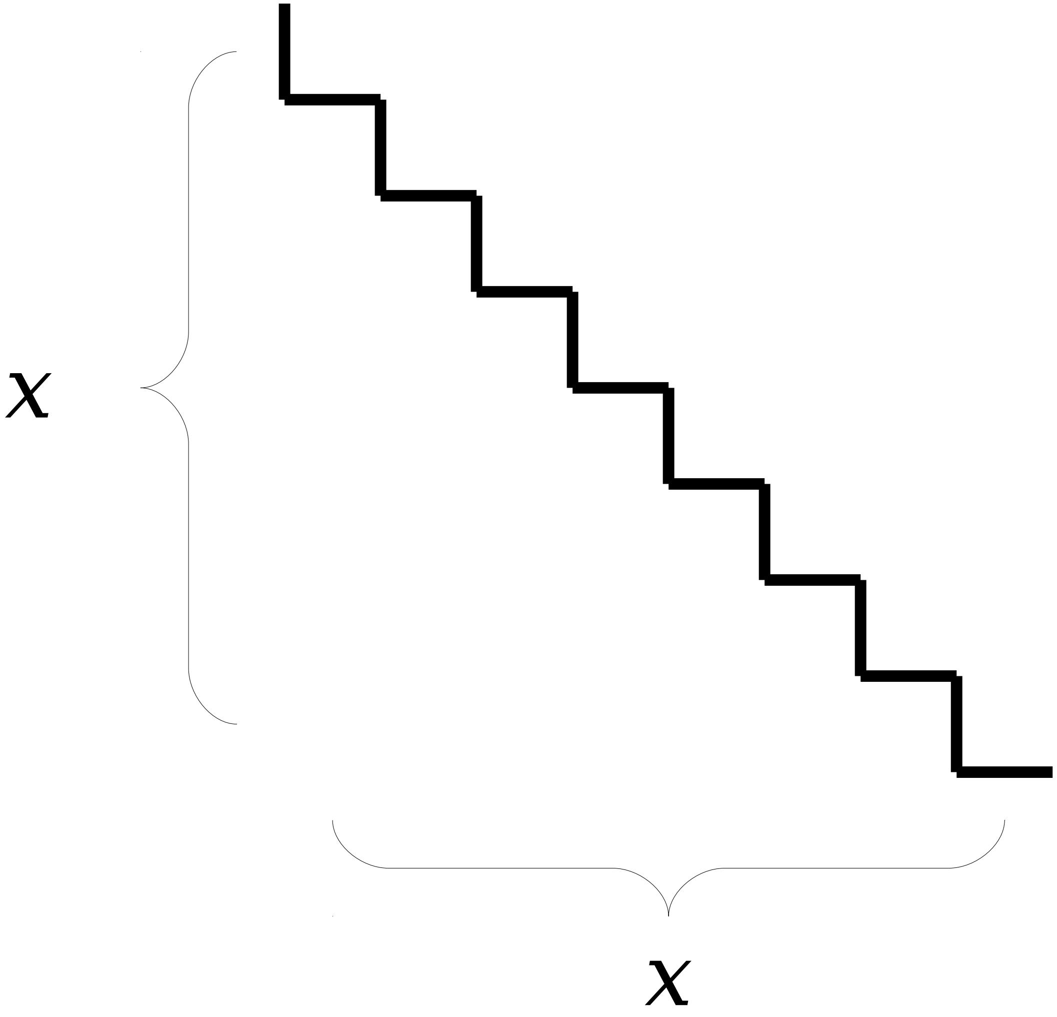

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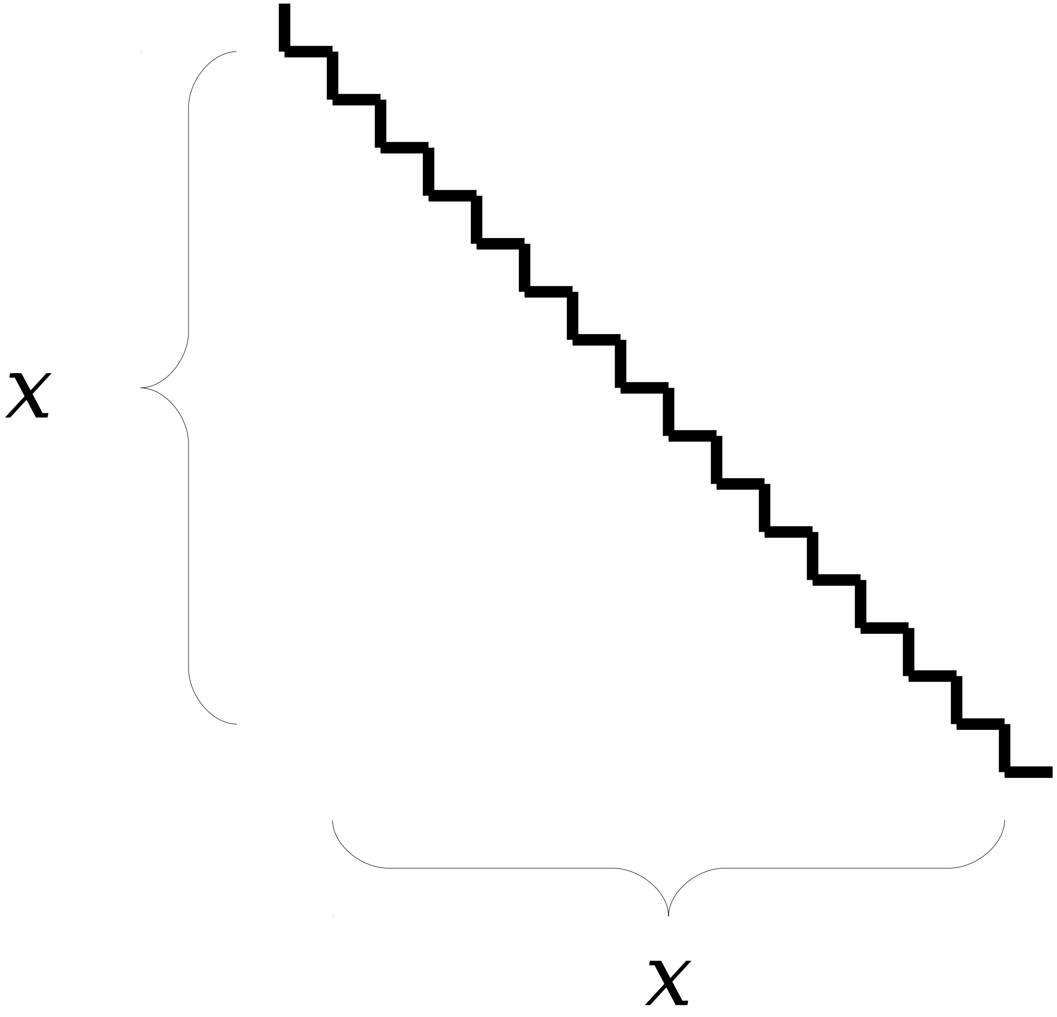

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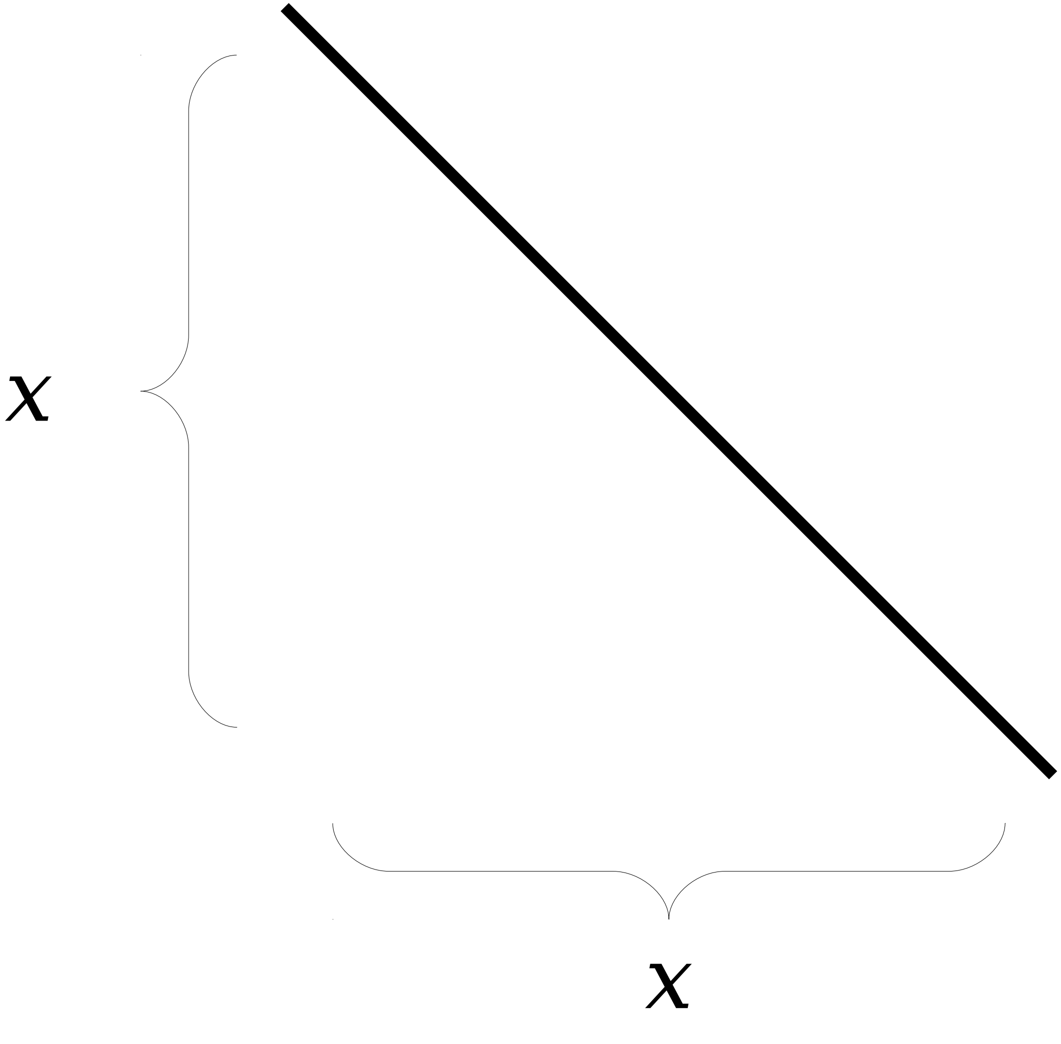
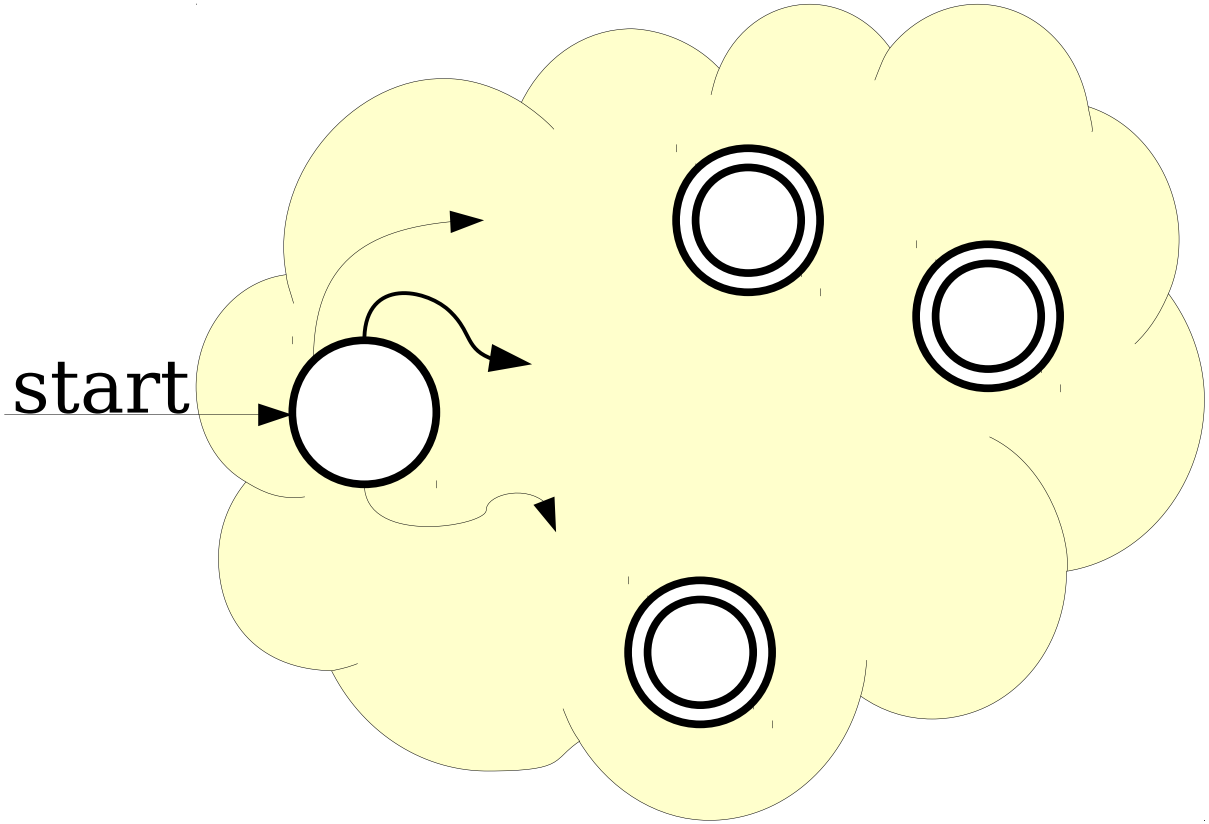


Reasoning About the Infinite

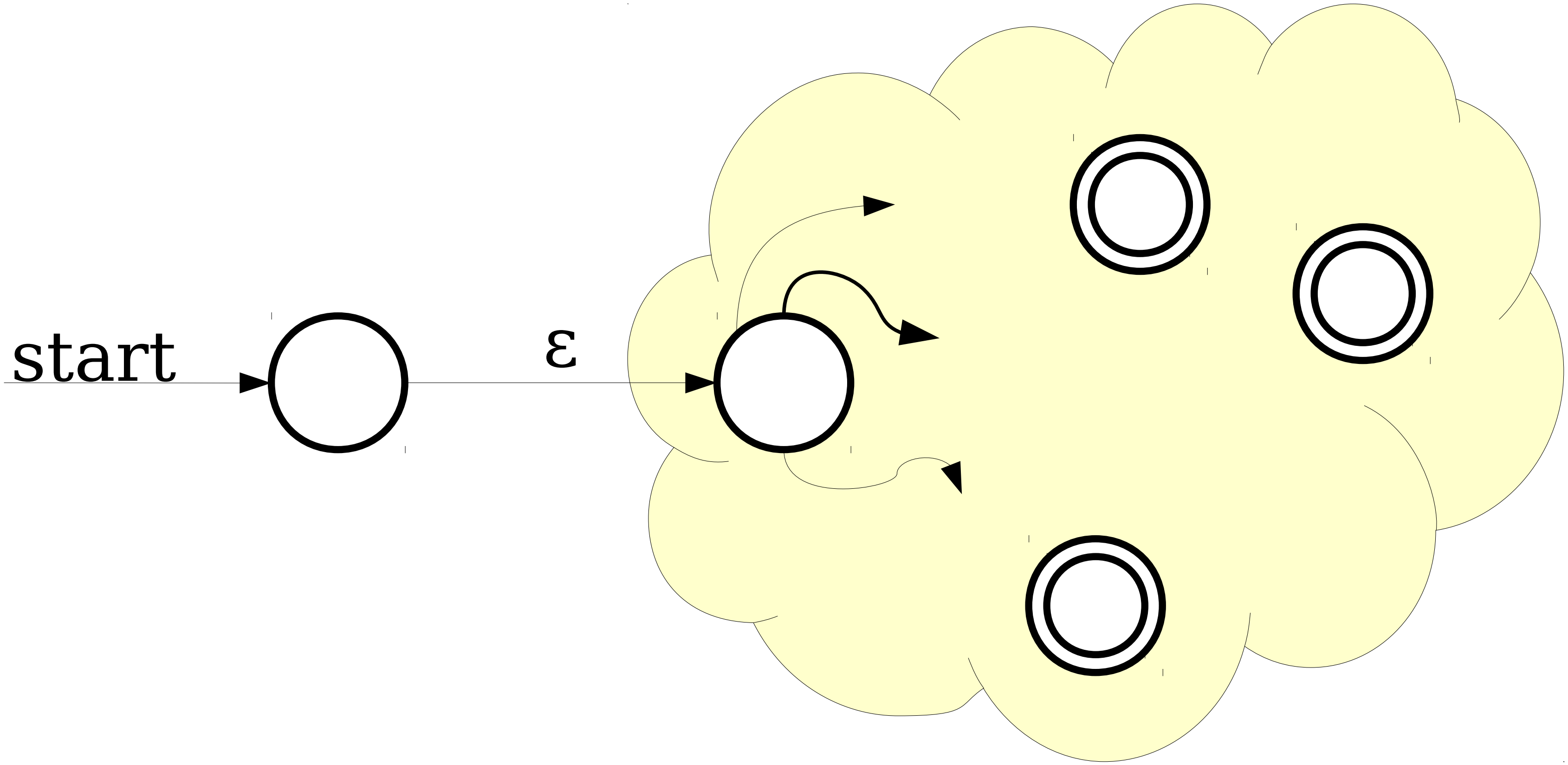
- If a series of finite objects all have some property, their infinite union ***does not*** necessarily have that property!
  - No matter how many times we zigzag that line, it's never straight.
  - Concluding that it must be equal “in the limit” is not mathematically valid (nor is it correct!)
  - (This is why calculus is interesting).
- **Better idea:** Can we convert an NFA for the language L to an NFA for the language L^*?

The Kleene Star



Machine for L

The Kleene Star



Machine for *L*

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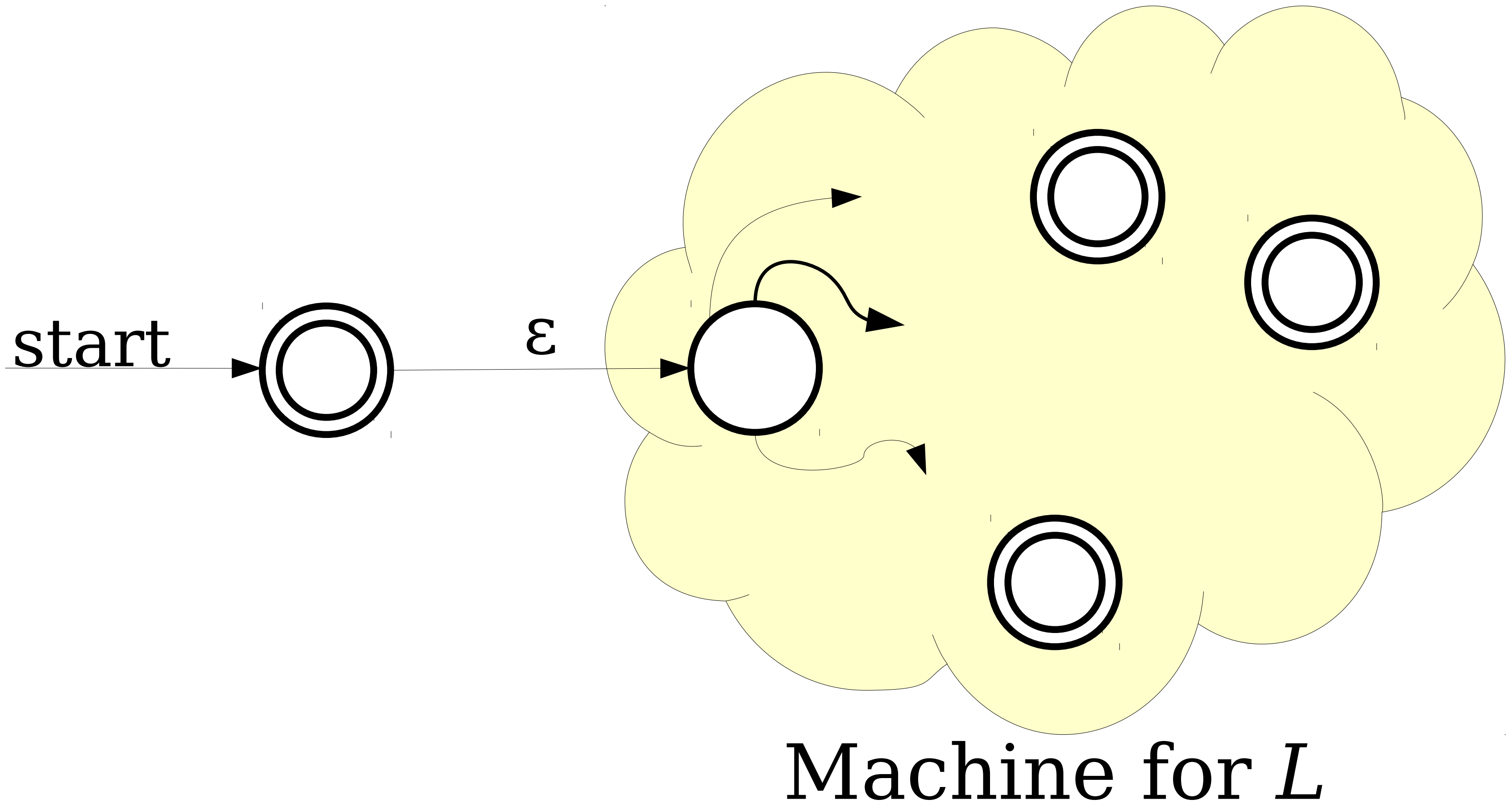

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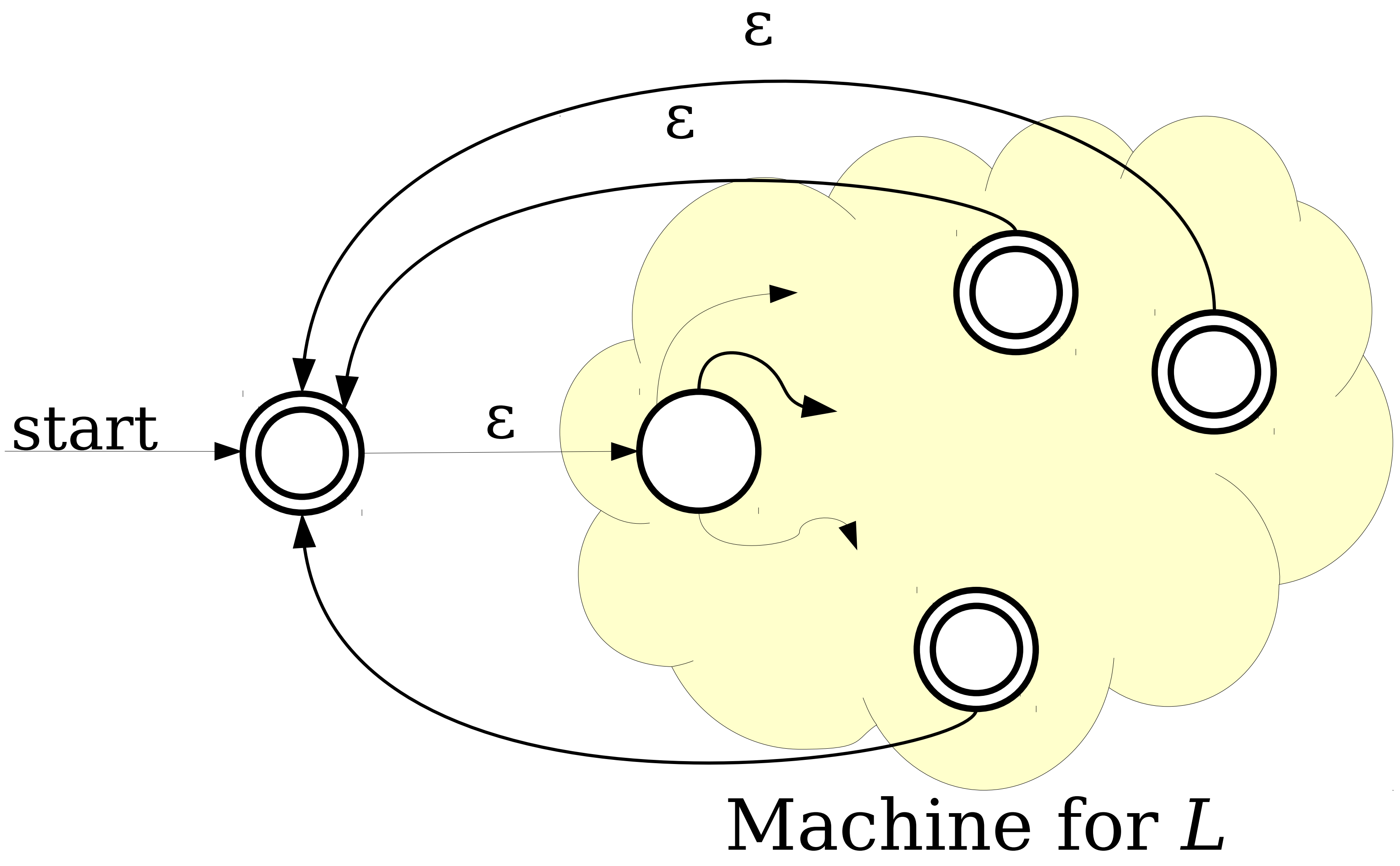

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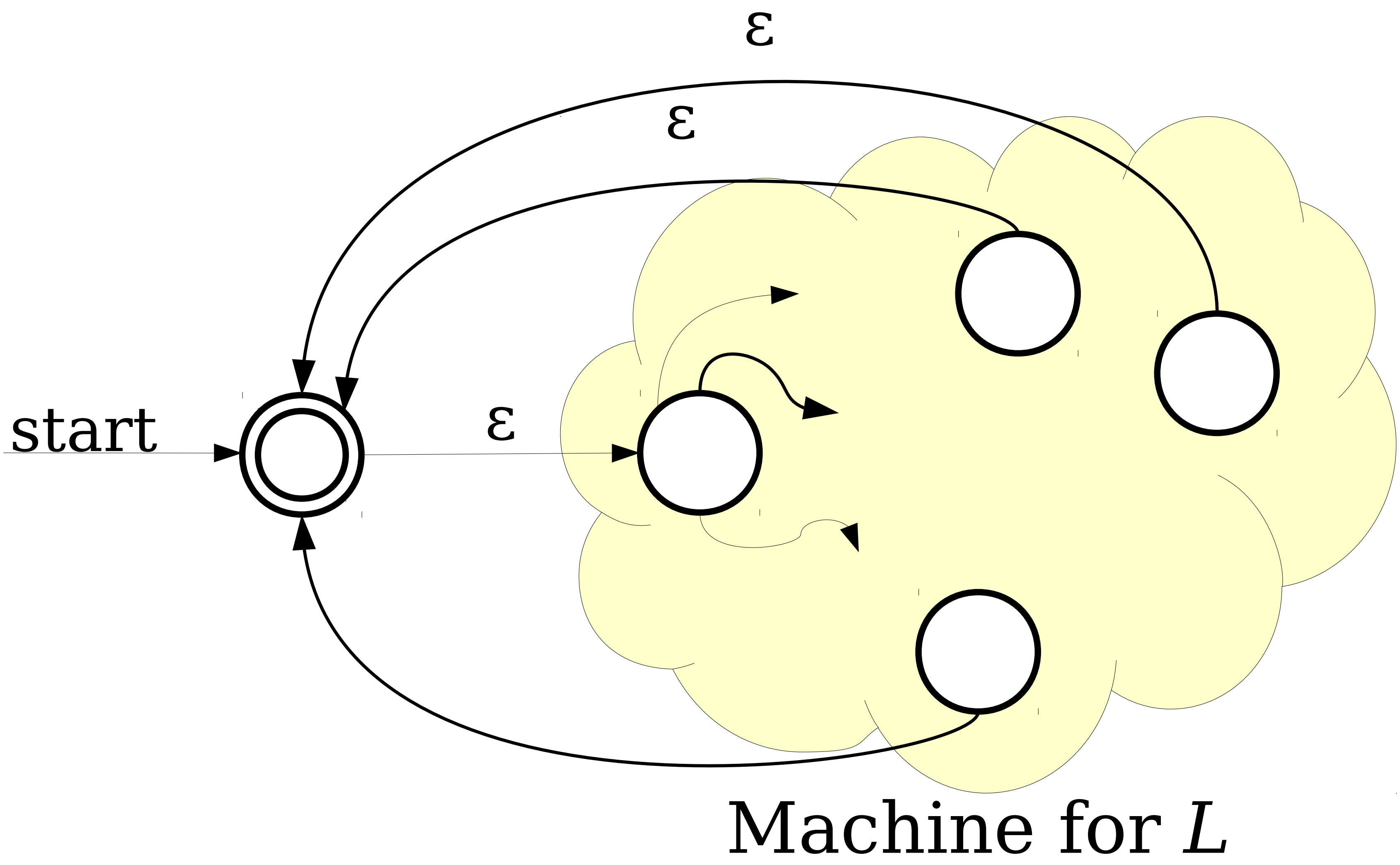

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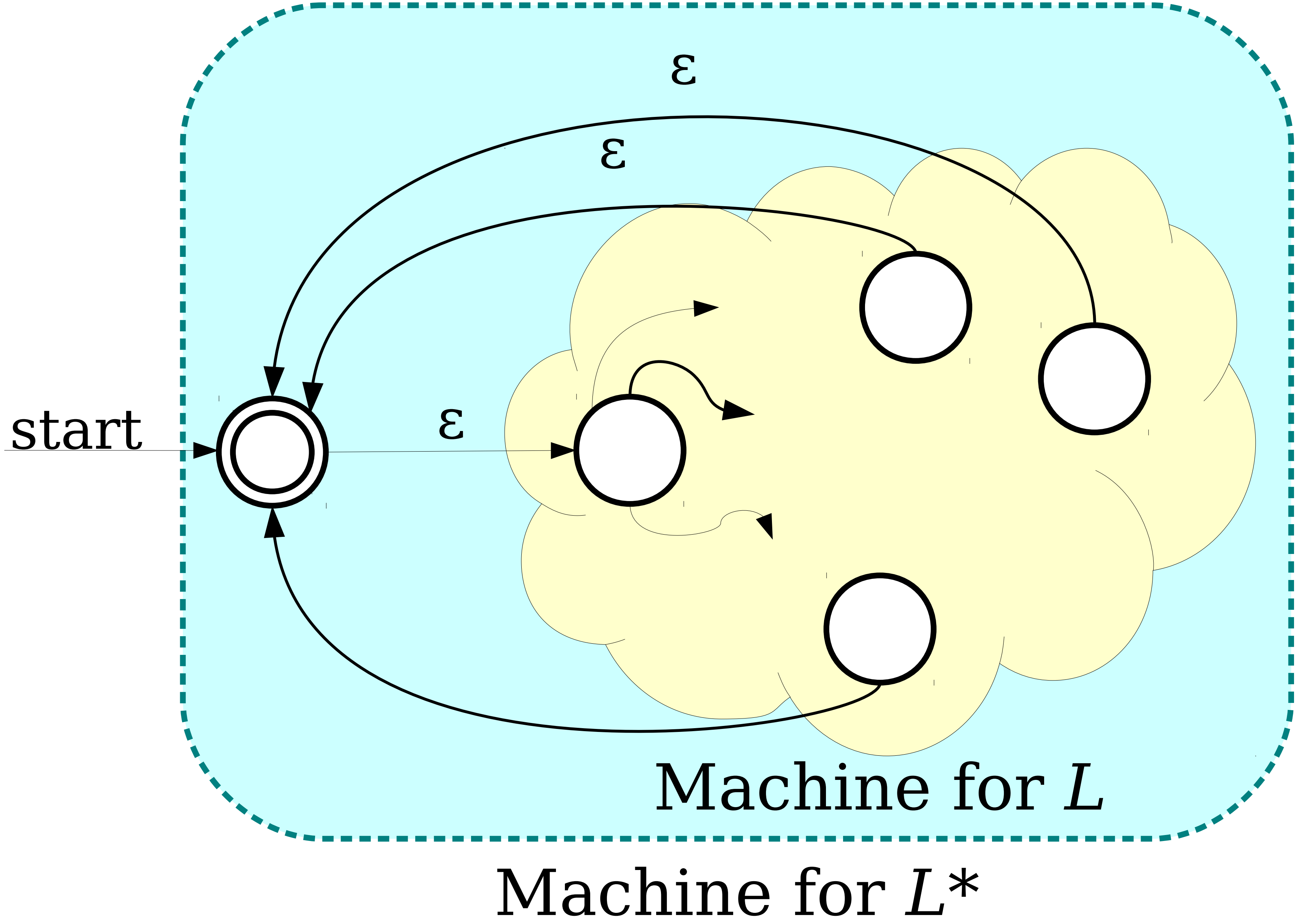

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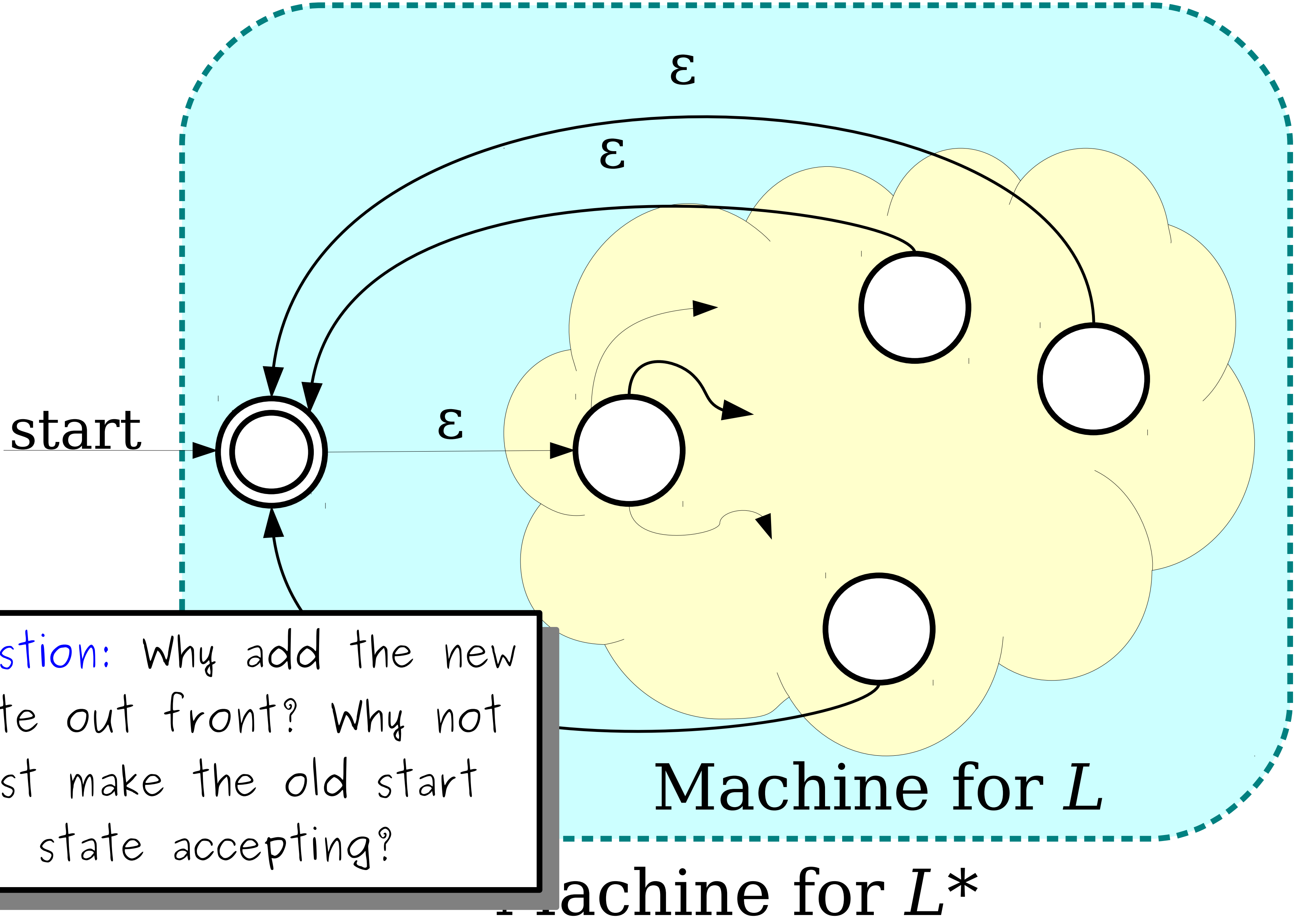

Question: Why add the new state out front? Why not just make the old start state accepting?

Summary

- NFAs are a powerful type of automaton that allows for ***nondeterministic*** choices.
- NFAs can also have ***ε-transitions*** that move from state to state without consuming any input.
- The ***subset construction*** shows that NFAs are not more powerful than DFAs, because any NFA can be converted into a DFA that accepts the same language.
- The union, intersection, complement, concatenation, and Kleene closure of regular languages are all regular languages.