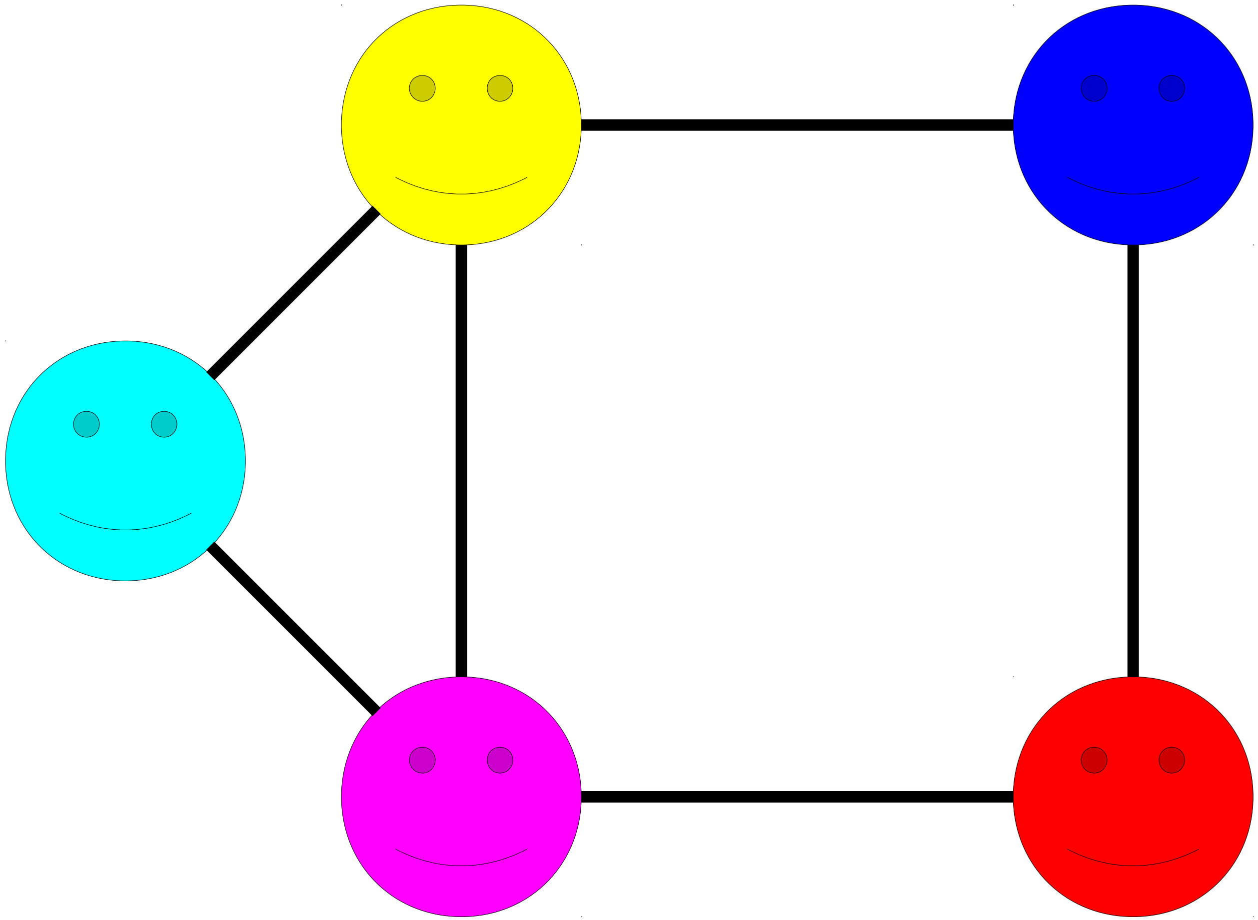
# Graphs

# Outline for Today

- **Navigating a Graph**
- **Undirected Connectivity**
- **Planar Graphs**
- **Graph Coloring**
- **An Overarching Question**
  - How exactly do you “do” math?

A ***graph*** is a mathematical structure for representing relationships.



A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

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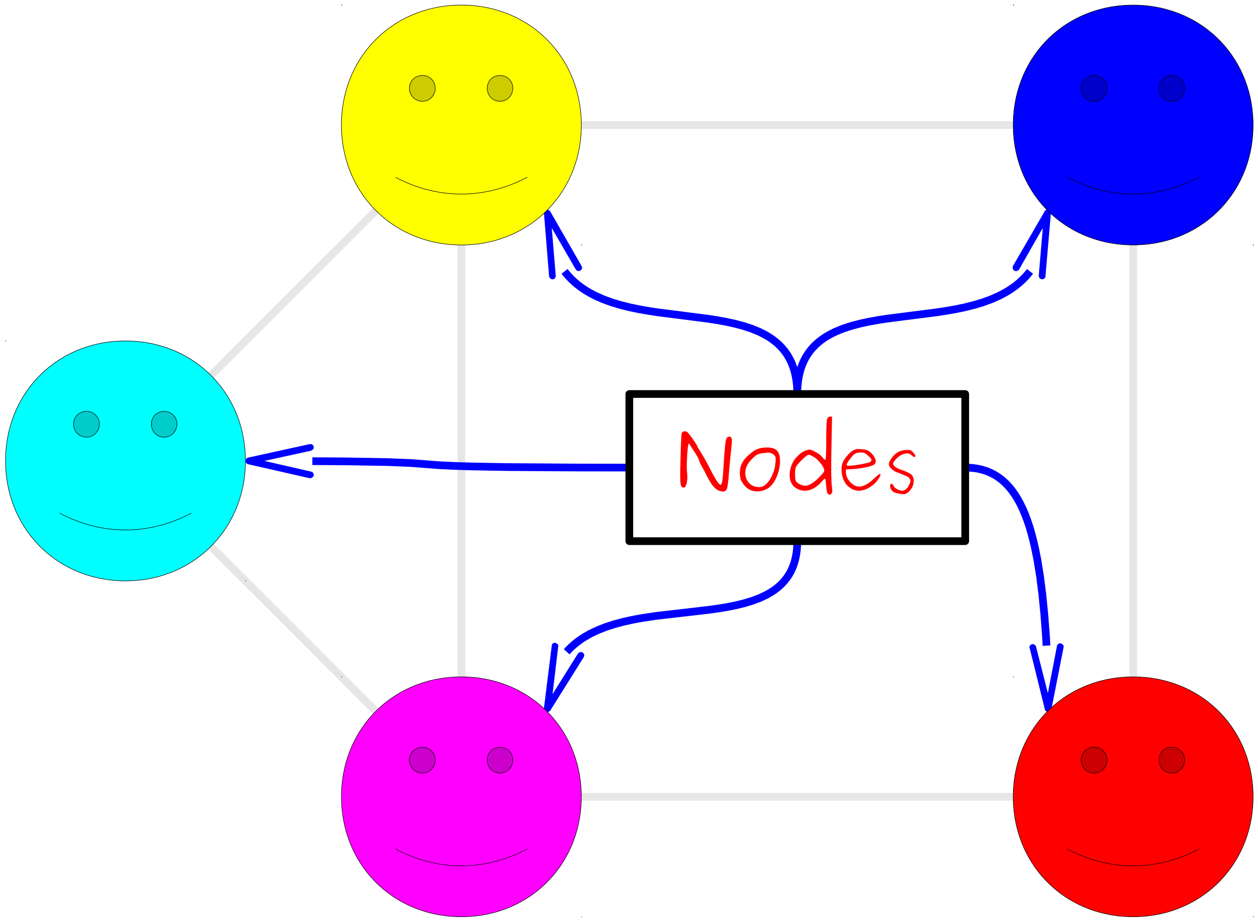


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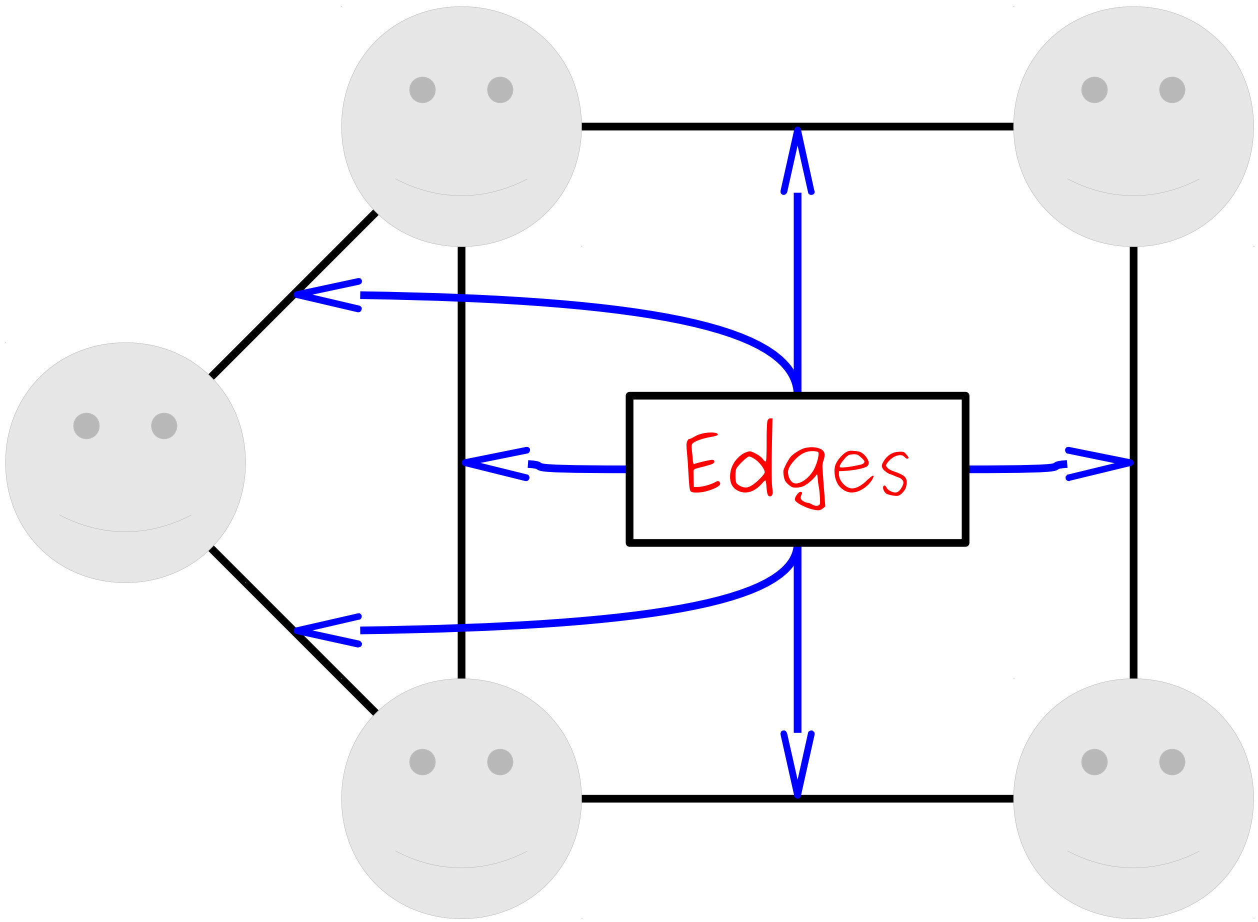


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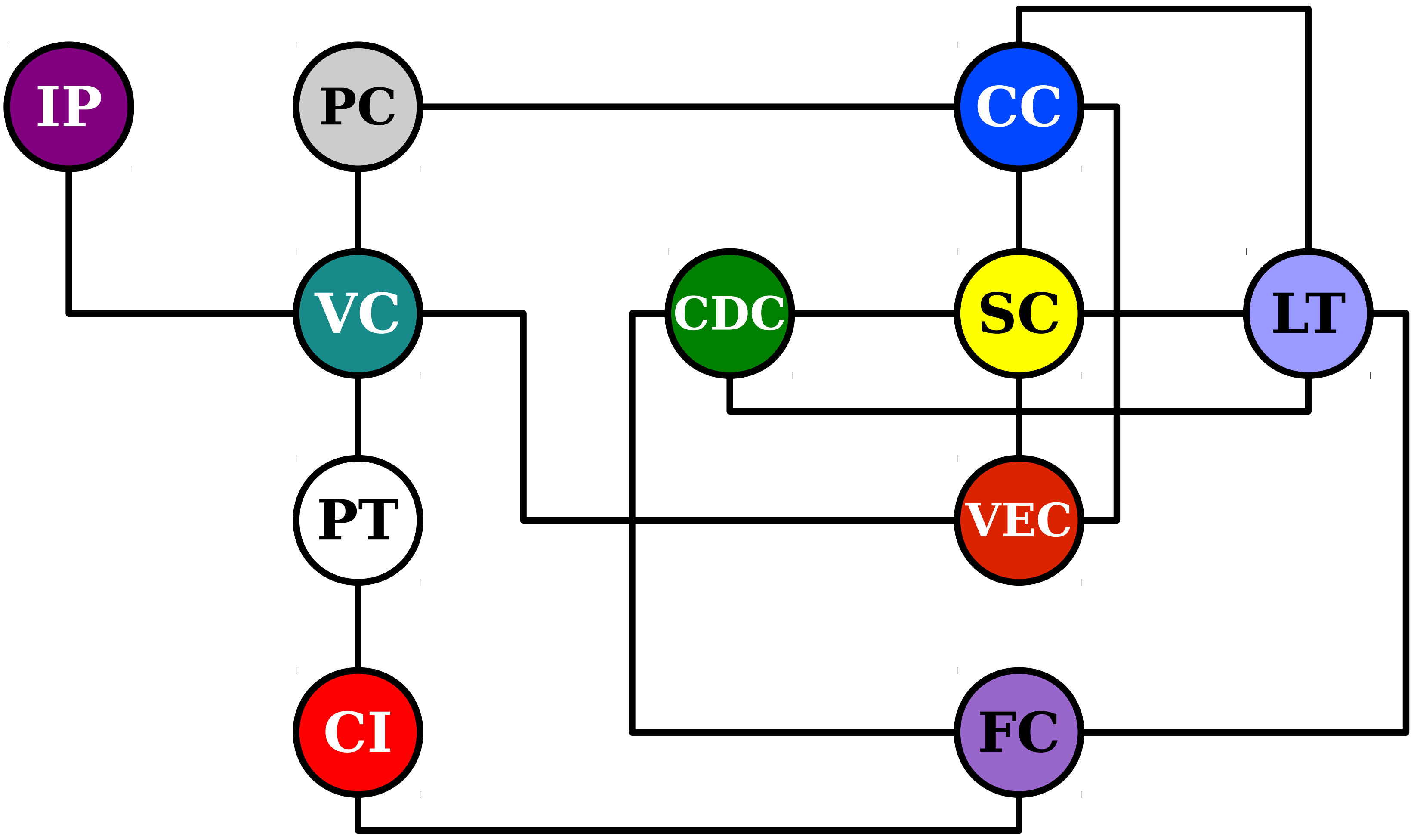
# Formalizing Graphs

- Formally, a ***graph*** is an ordered pair $G = (V, E)$, where
  - $V$ is a set of nodes.
  - $E$ is a set of edges, which are either ordered pairs or unordered pairs of elements from $V$.

# Undirected Connectivity

# Navigating a Graph



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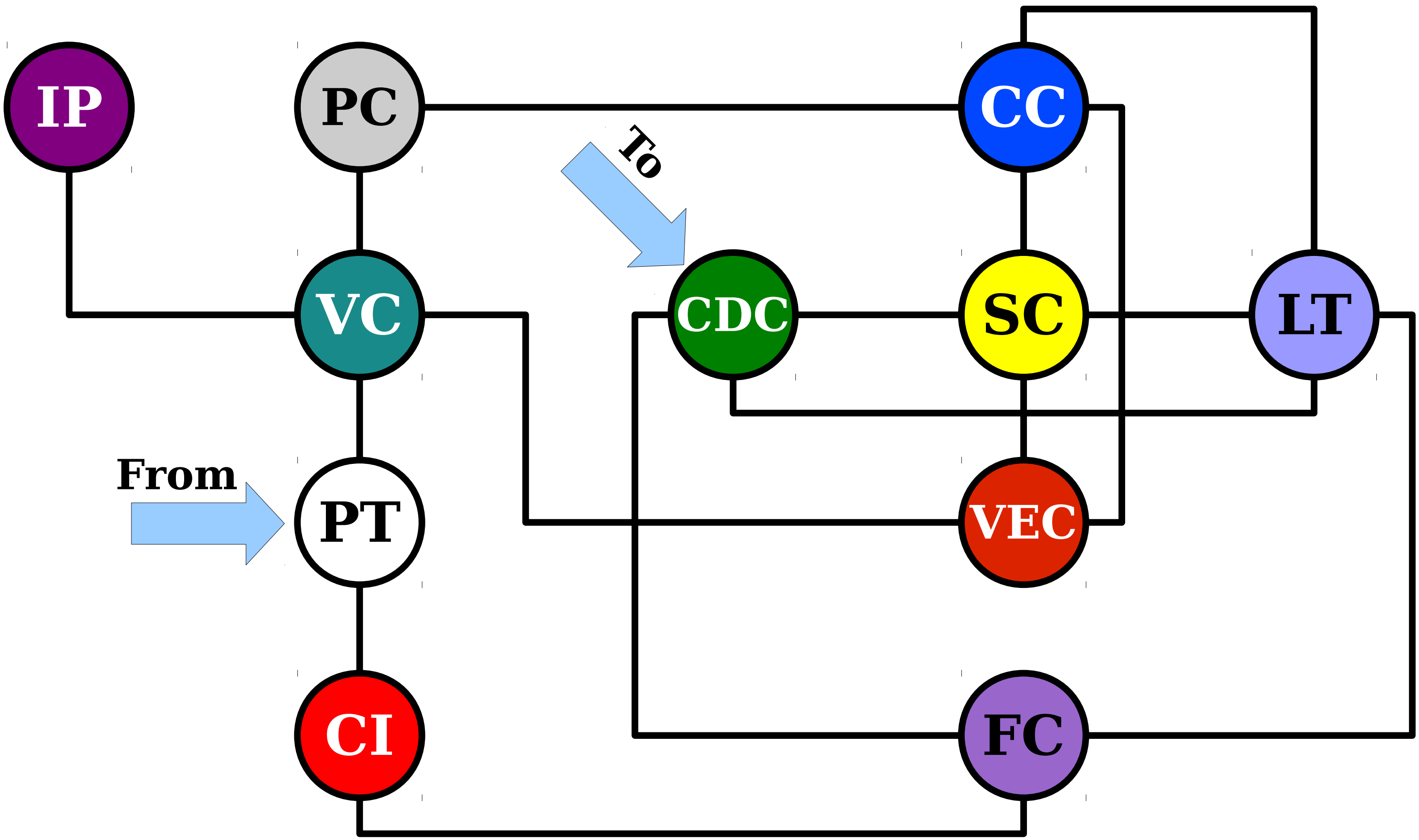

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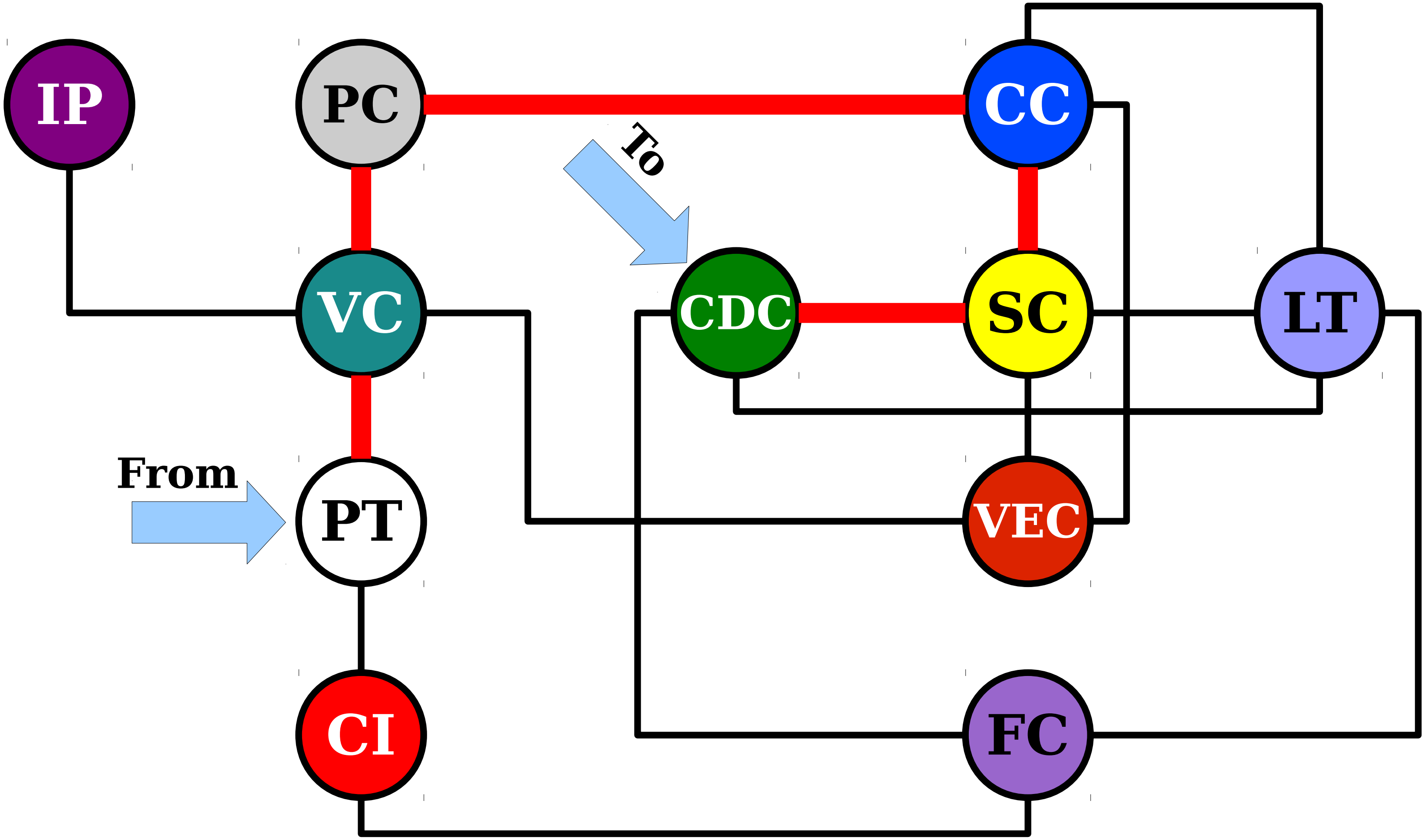

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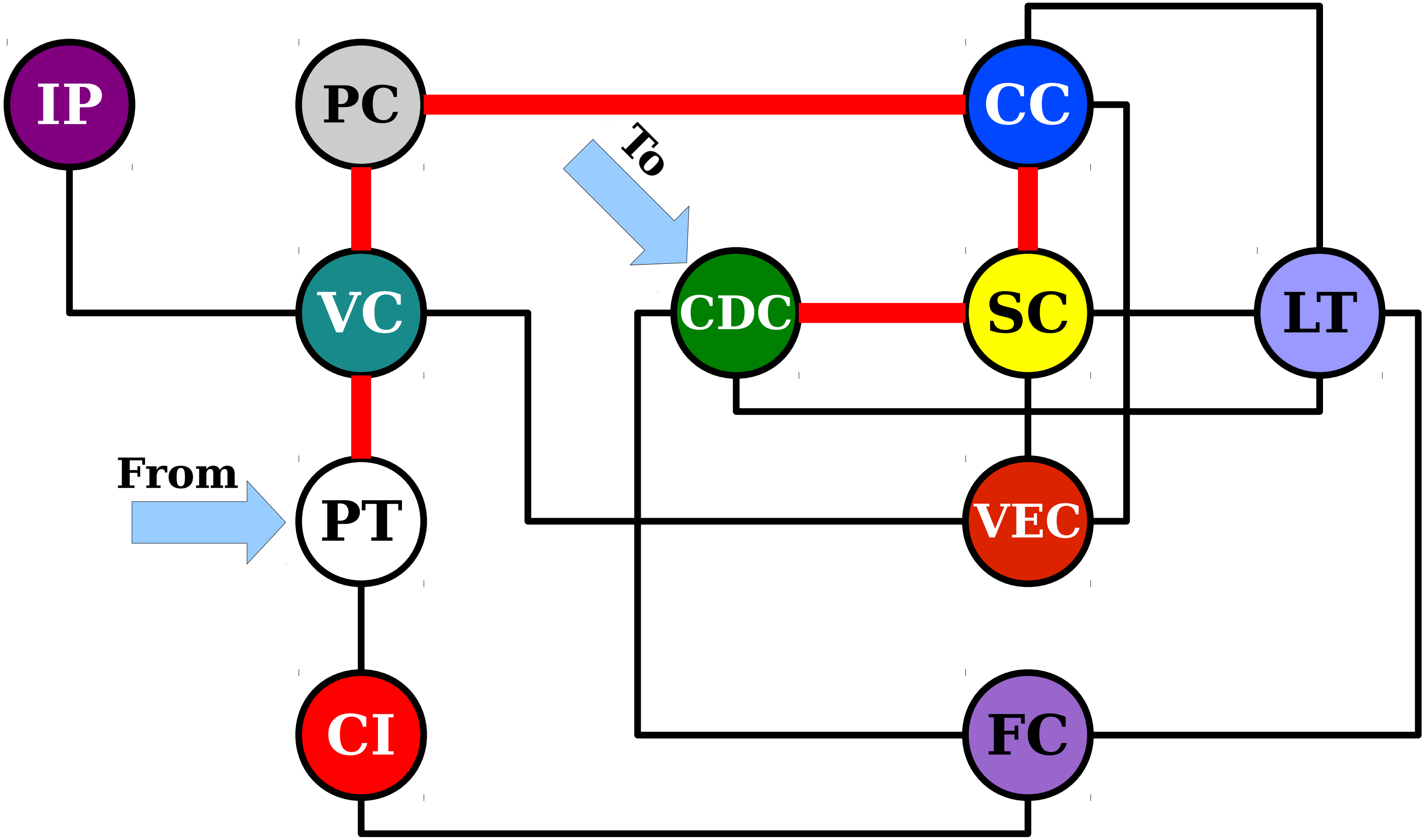
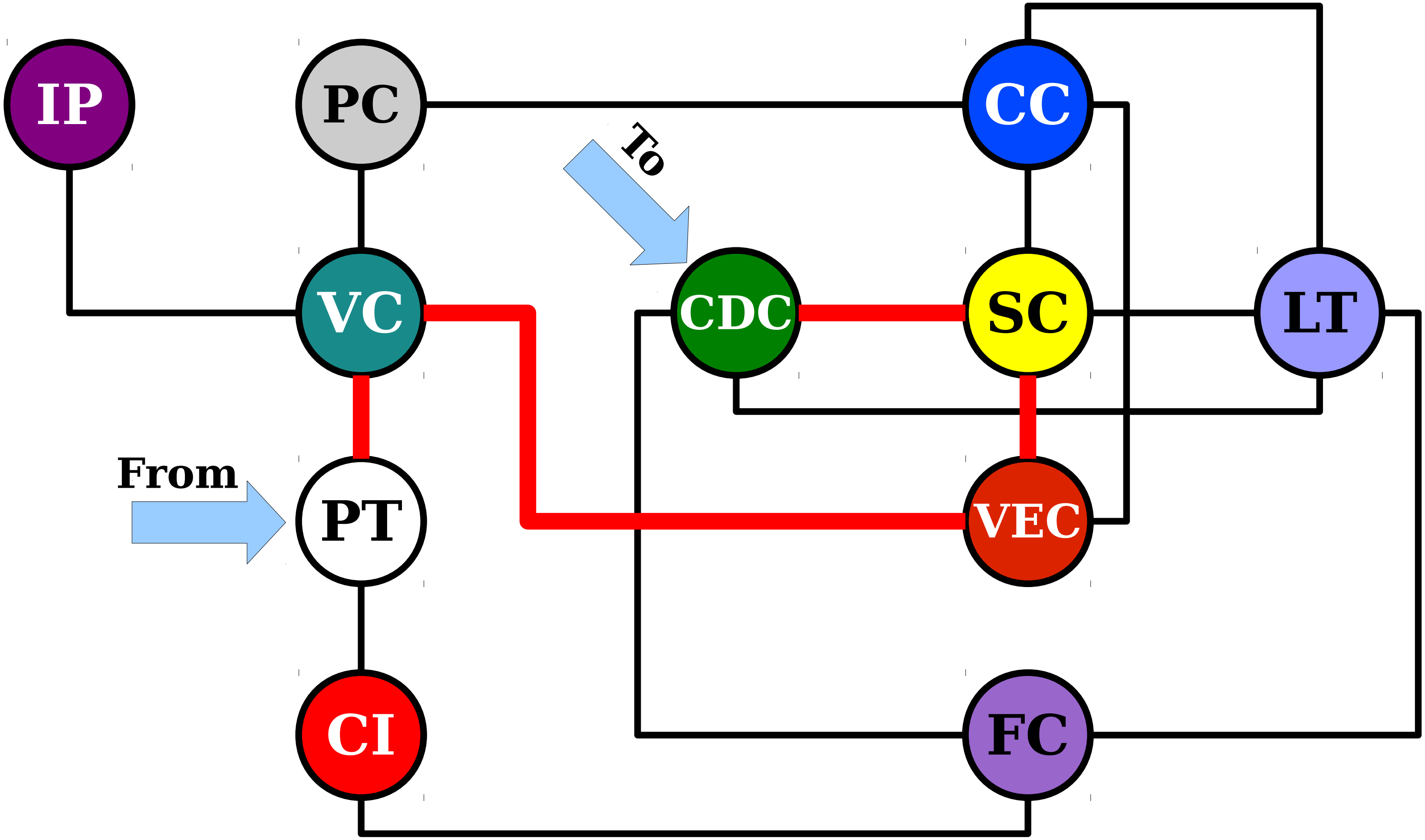



PT → VC → PC → CC → SC → CDC

# Navigating a Graph



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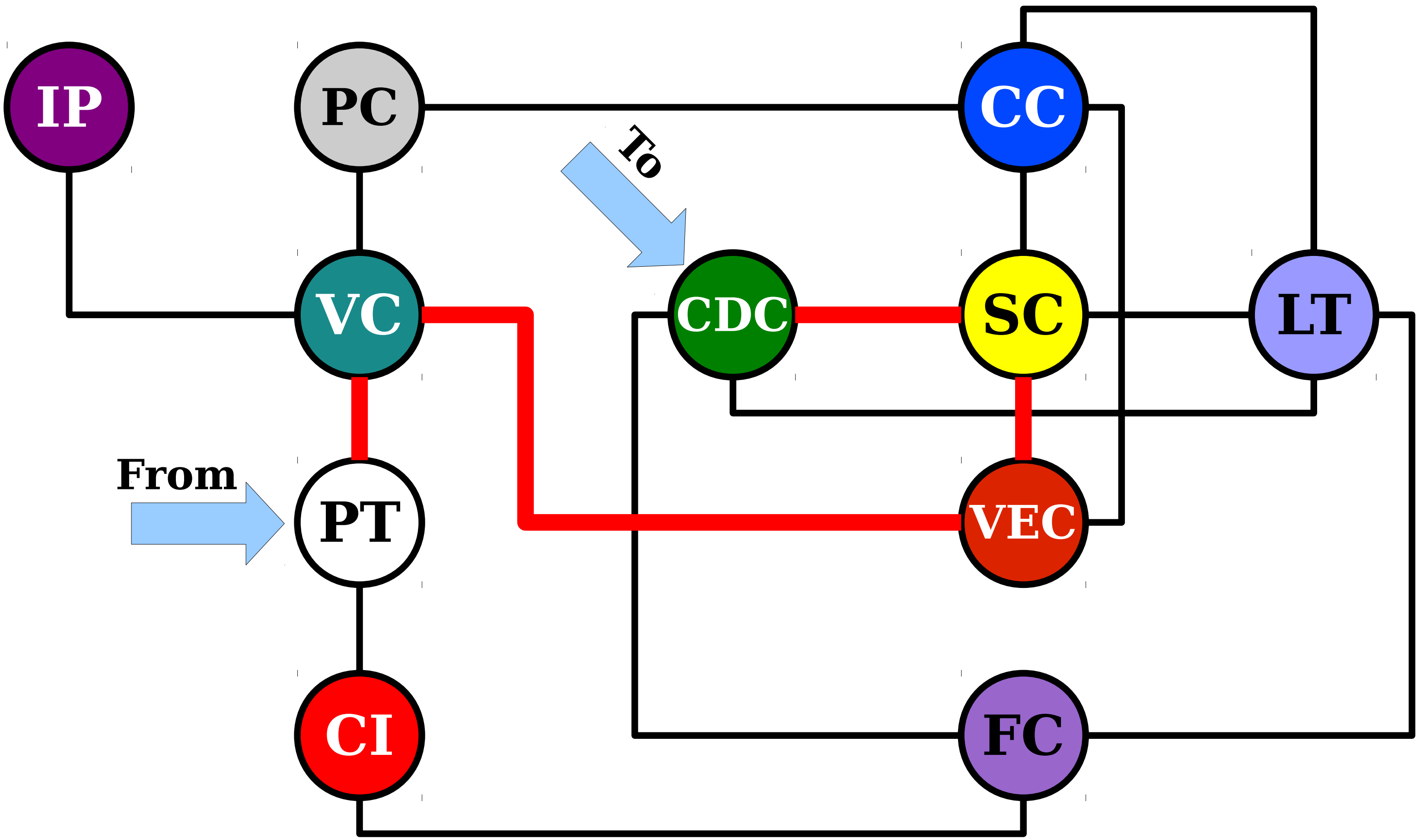


PT → VC → VEC → SC → CDC

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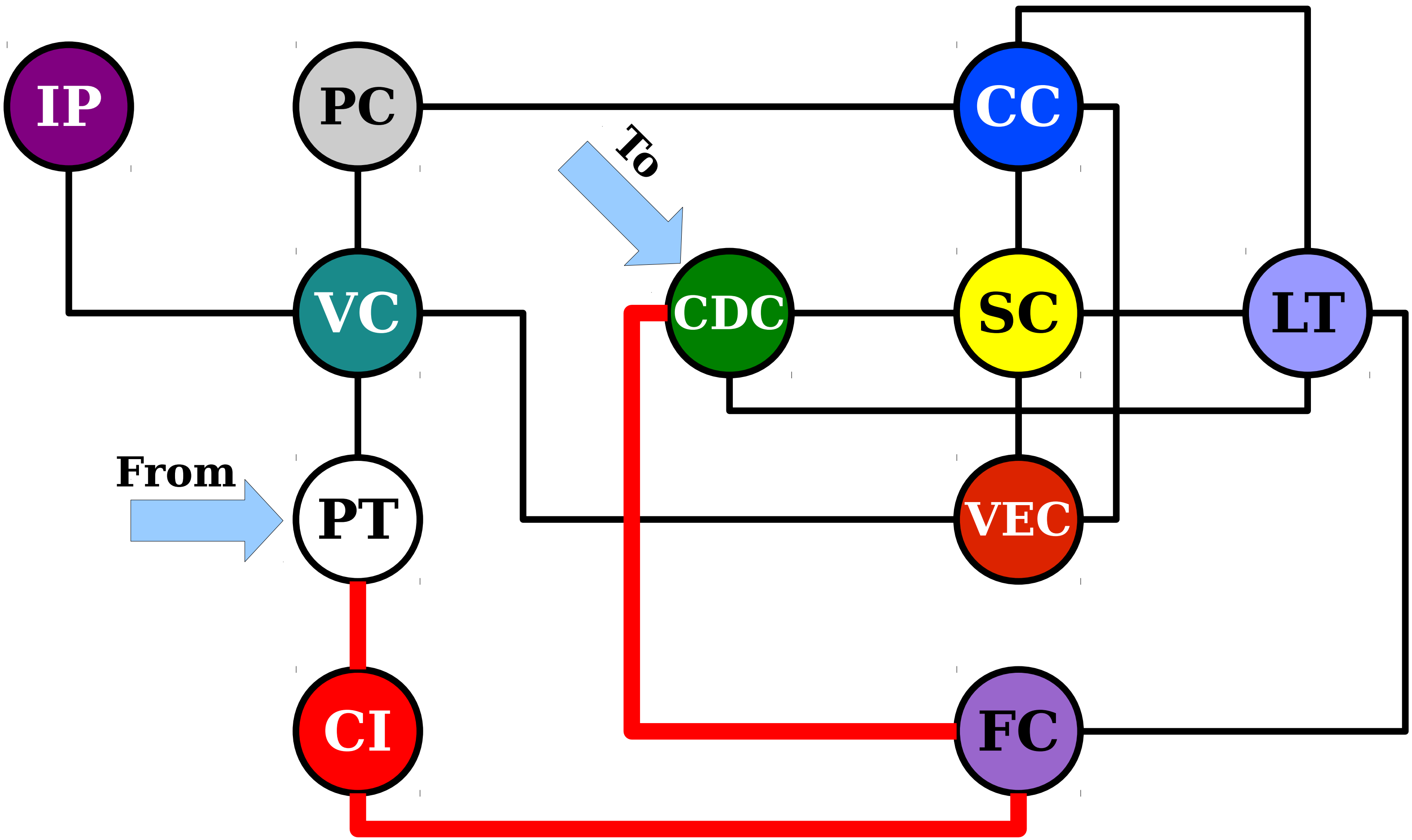

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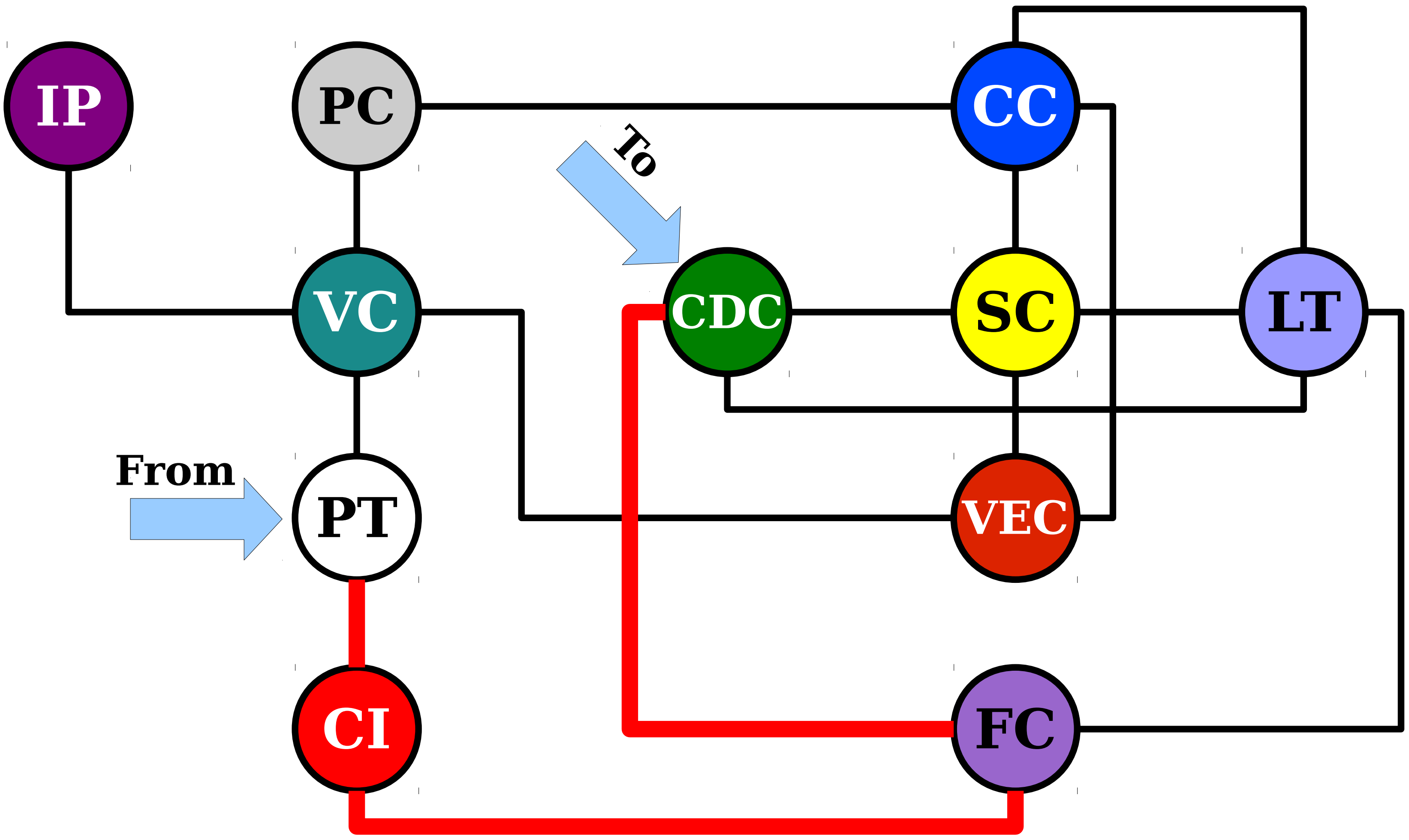


PT → CI → FC → CDC

A ***path*** from $v_1$ to $v_n$ is a sequence of nodes $v_1, v_2, \ldots, v_n$ where $(v_k, v_{k+1}) \in E$ for all natural numbers in the range $1 \leq k \leq n - 1$.

The ***length*** of a path is the number of edges it contains, which is one less than the number of nodes in the path.

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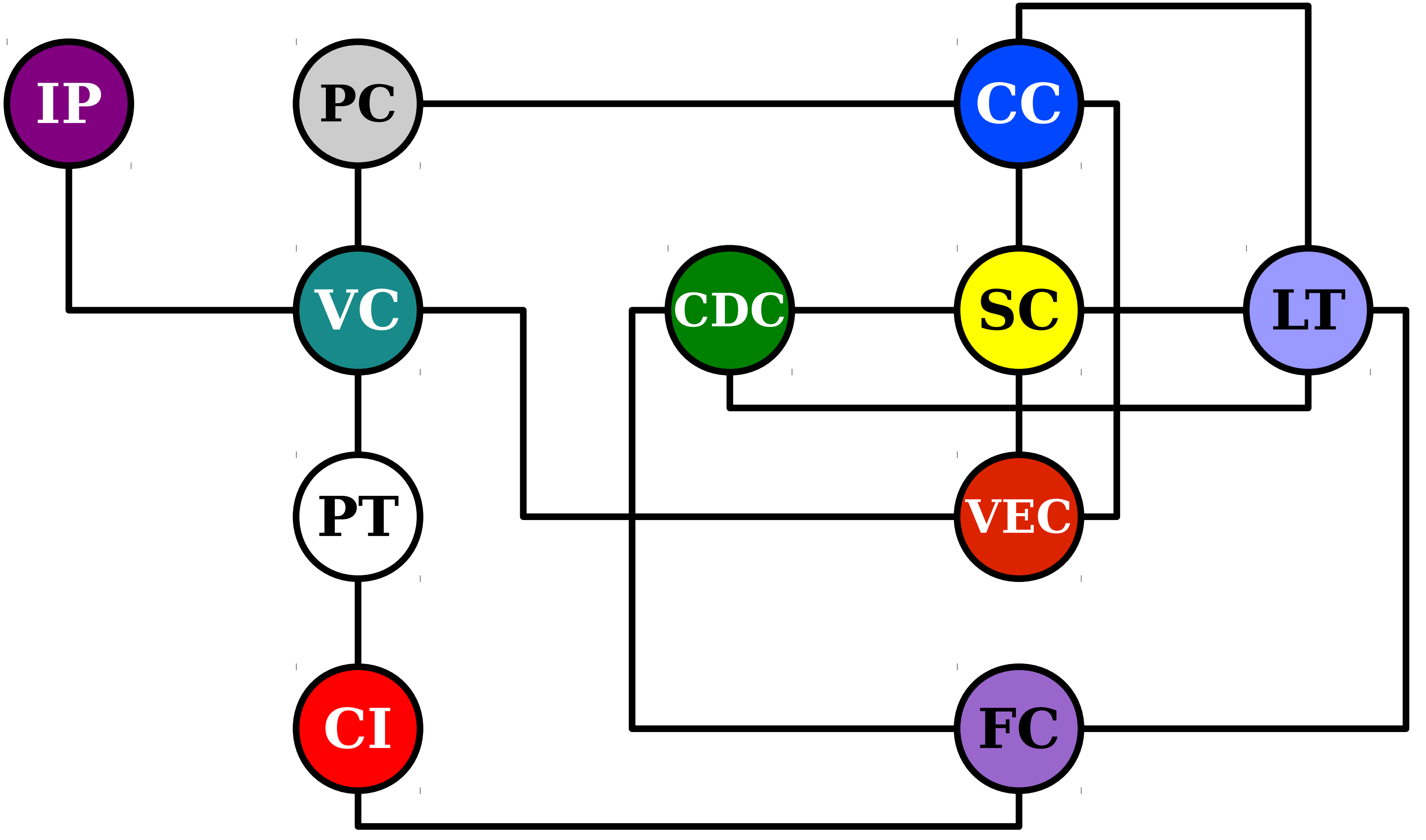
# Navigating a Graph



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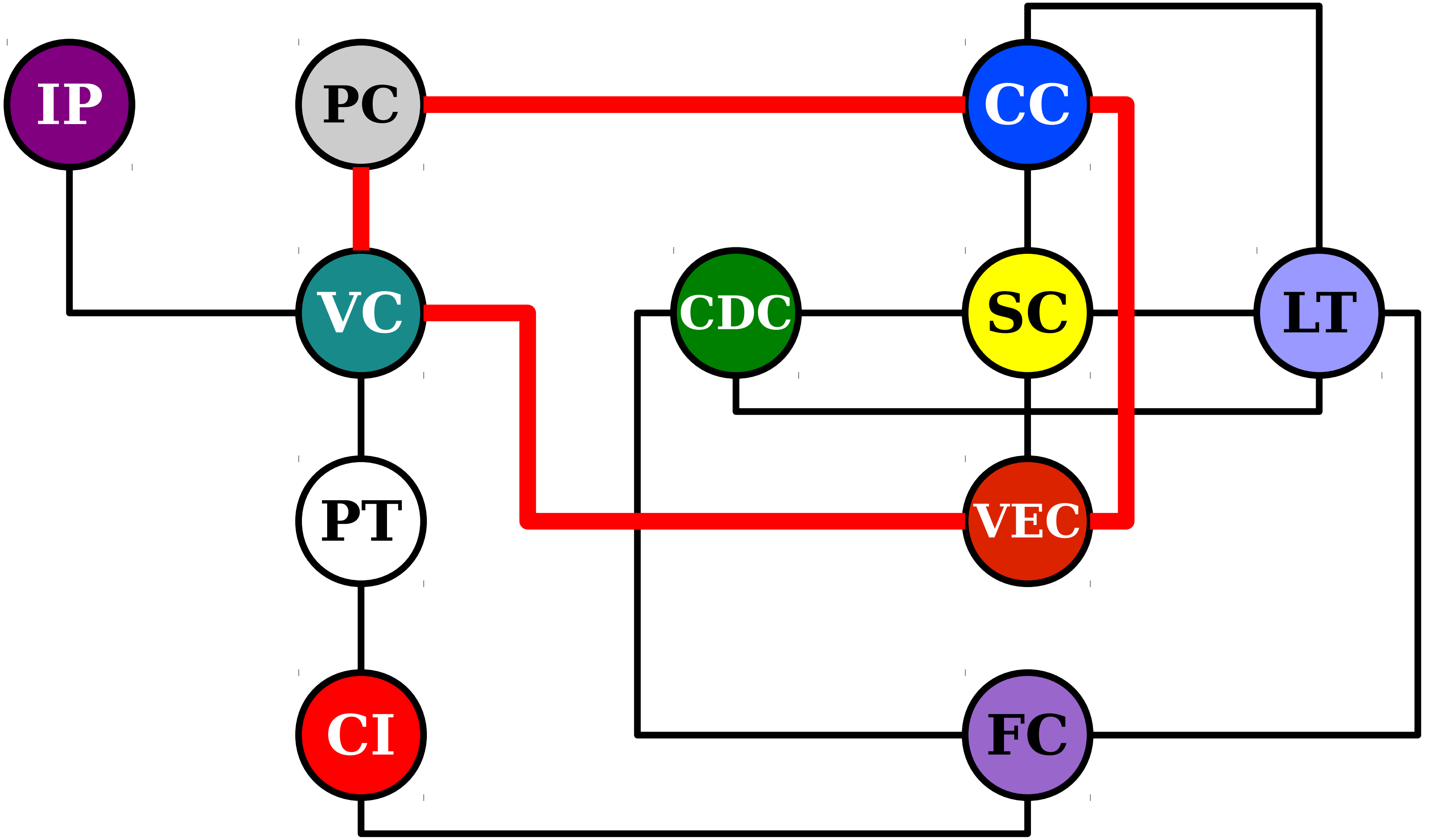

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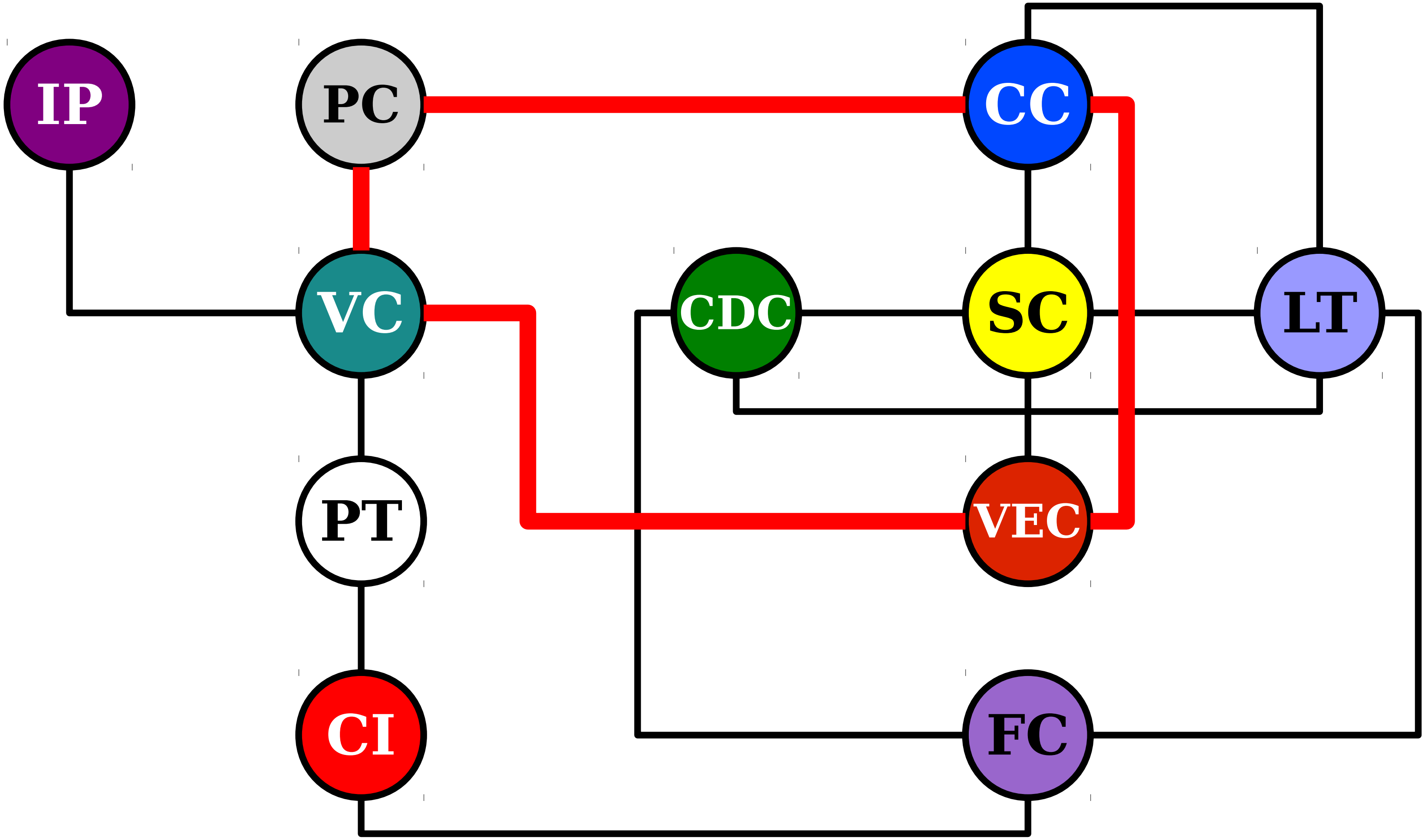


PC → CC → VEC → VC → PC

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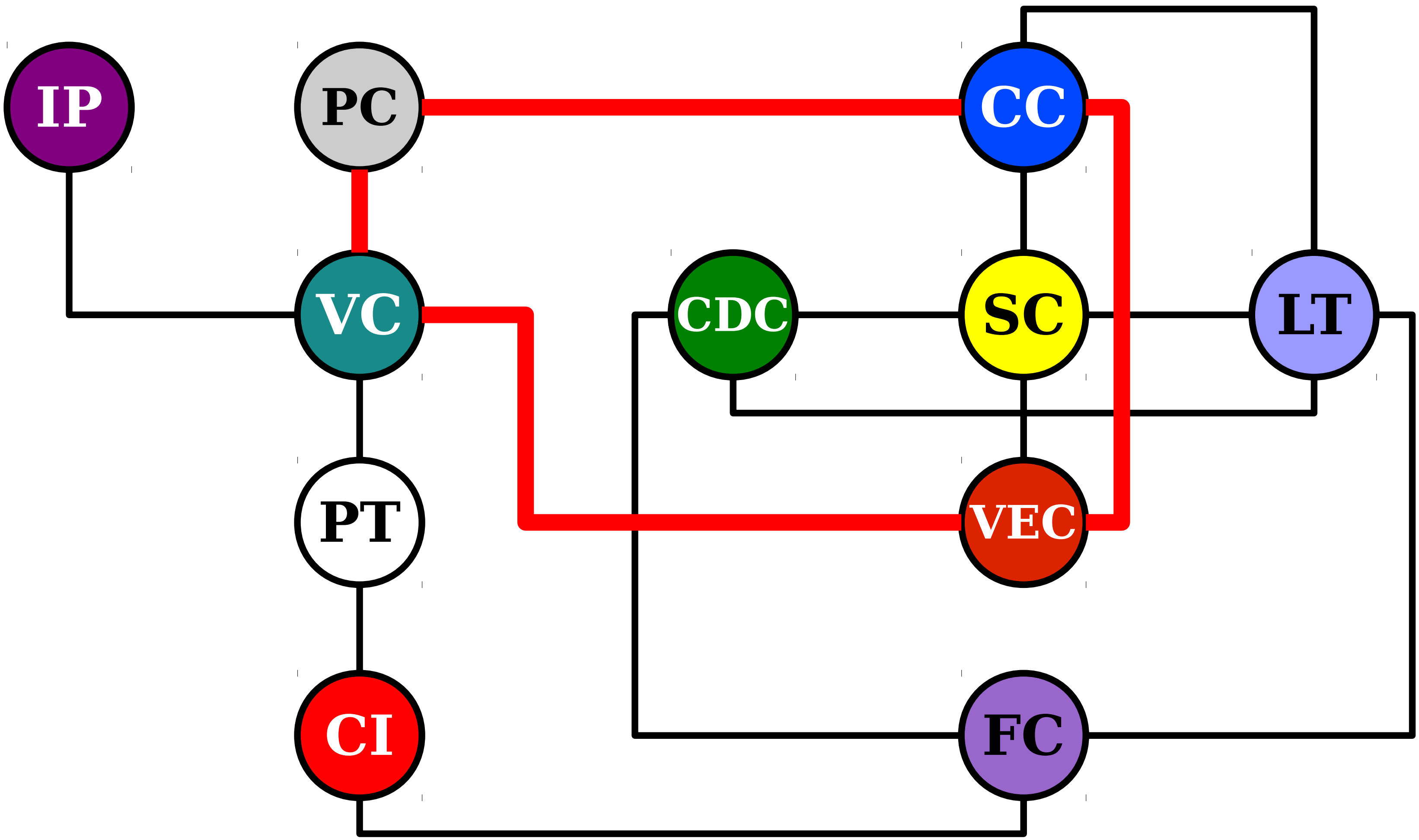


PC → CC → VEC → VC → PC → CC → VEC → VC → PC

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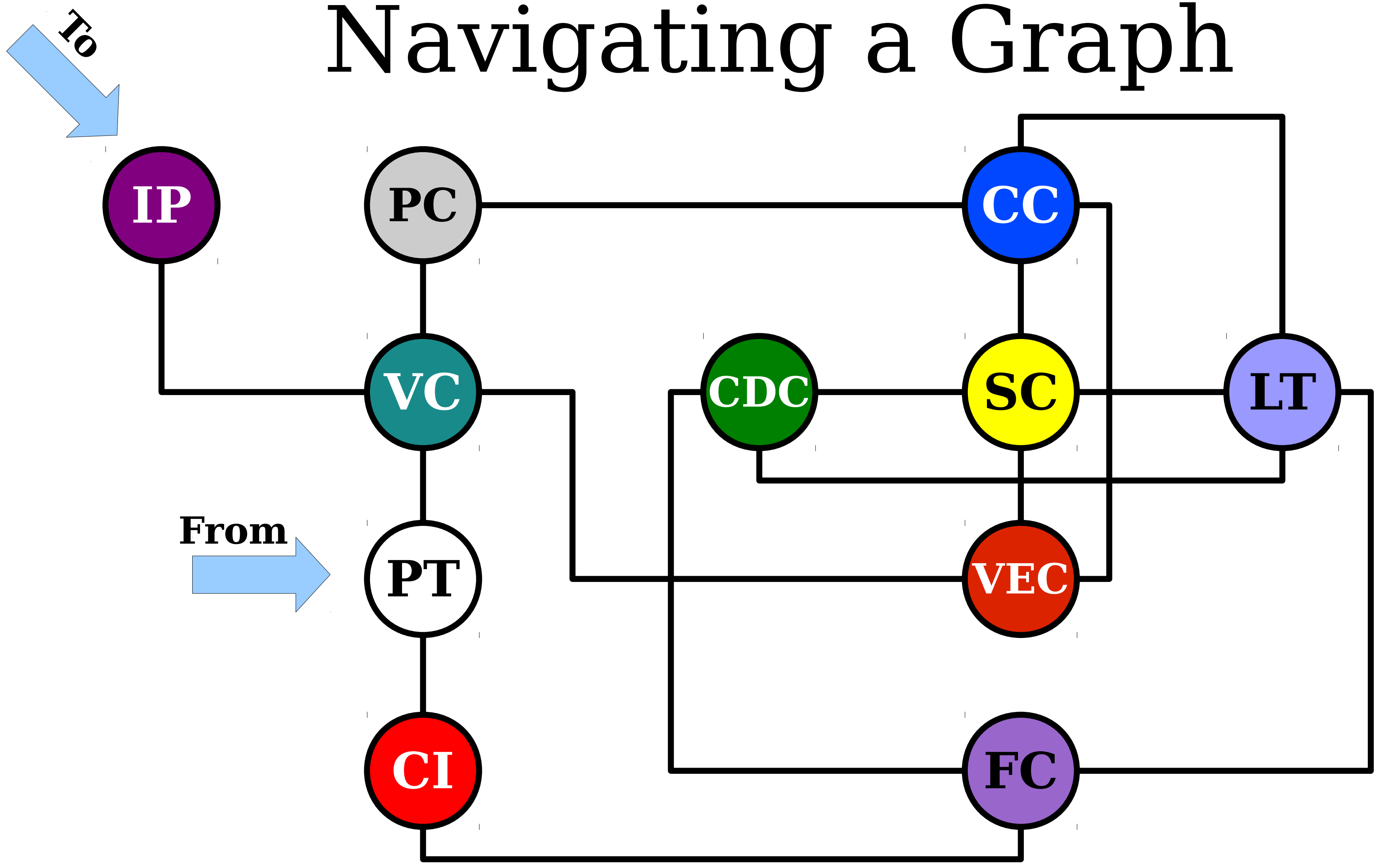

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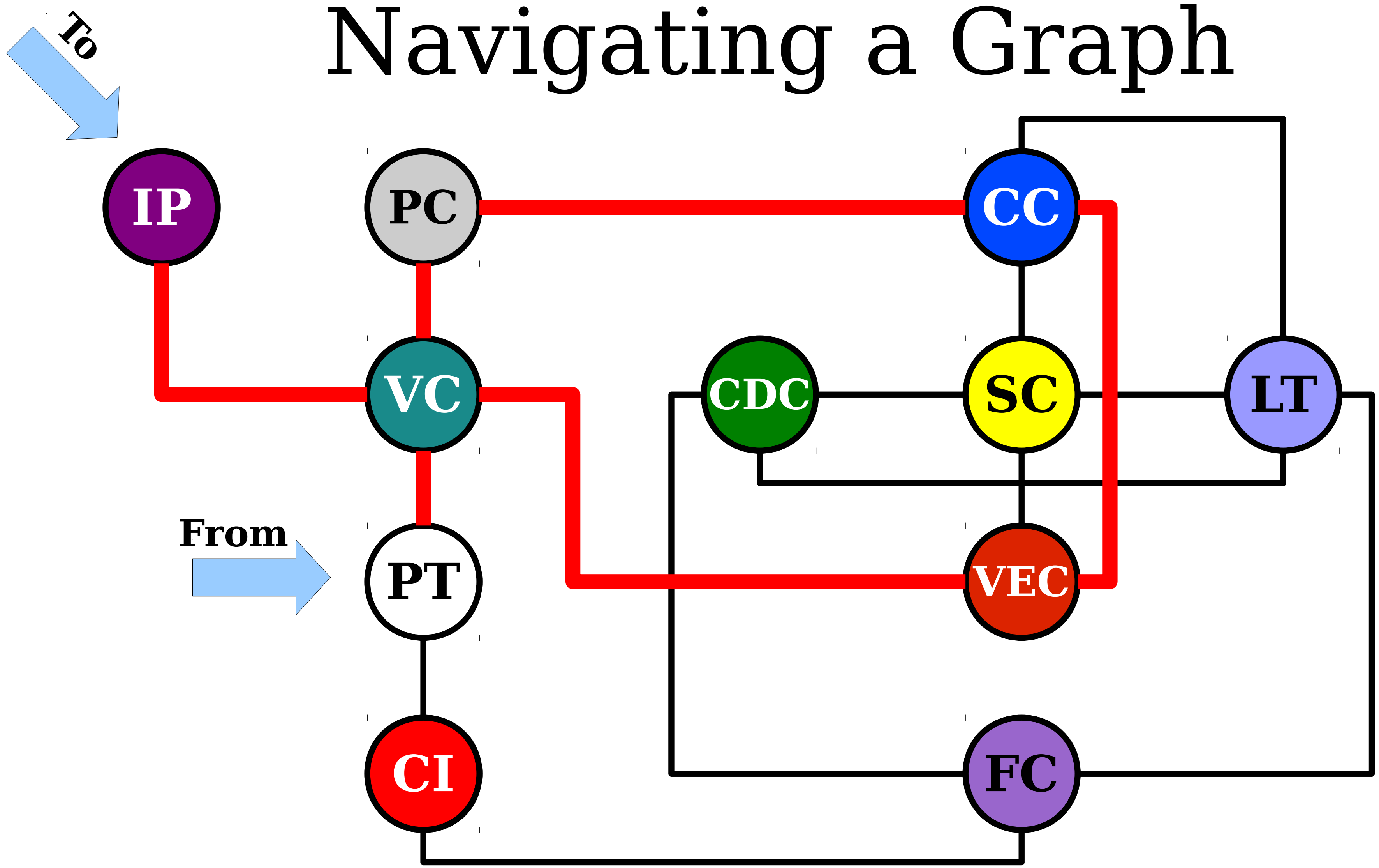

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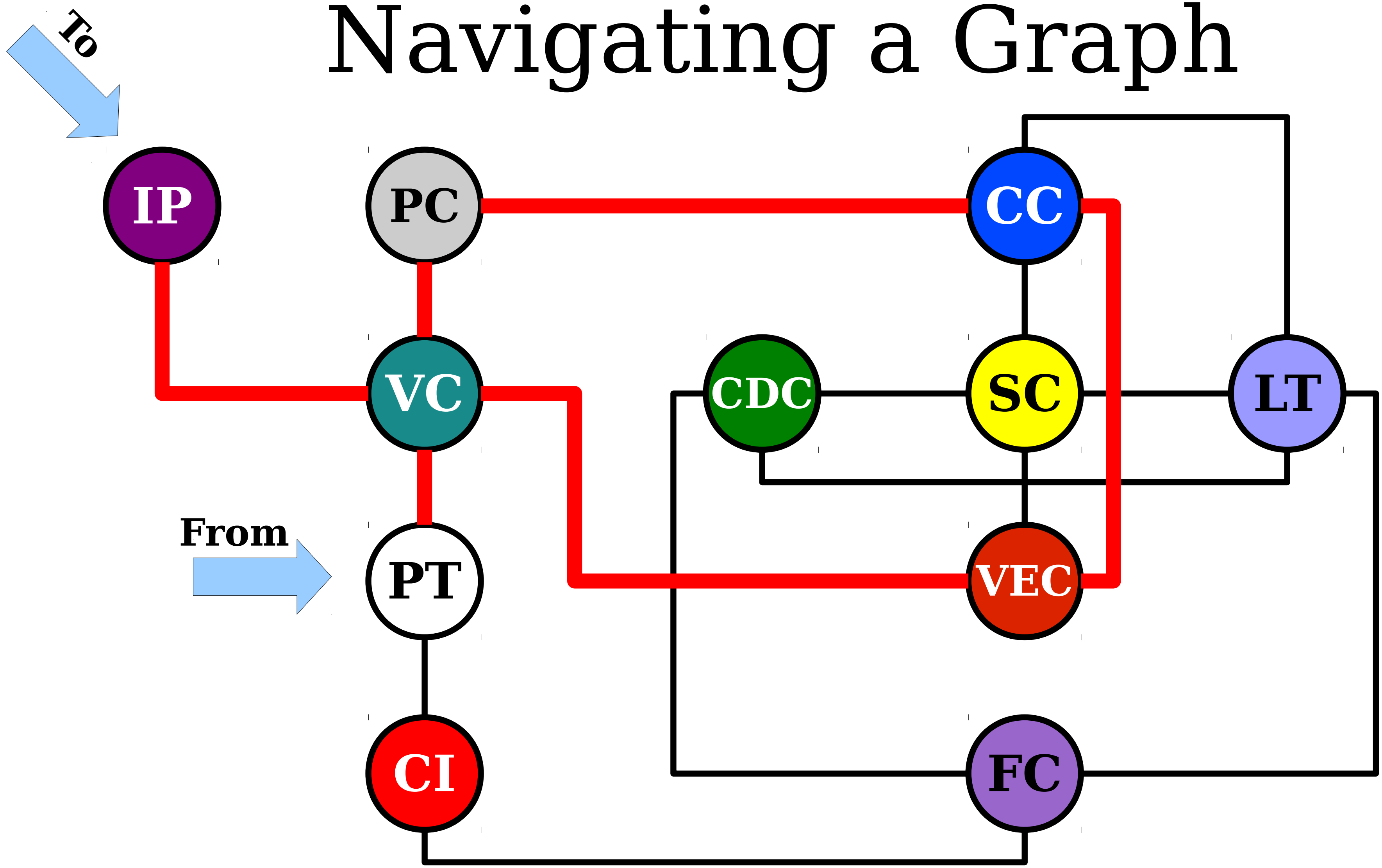



PT → VC → PC → CC → VEC → VC → IP

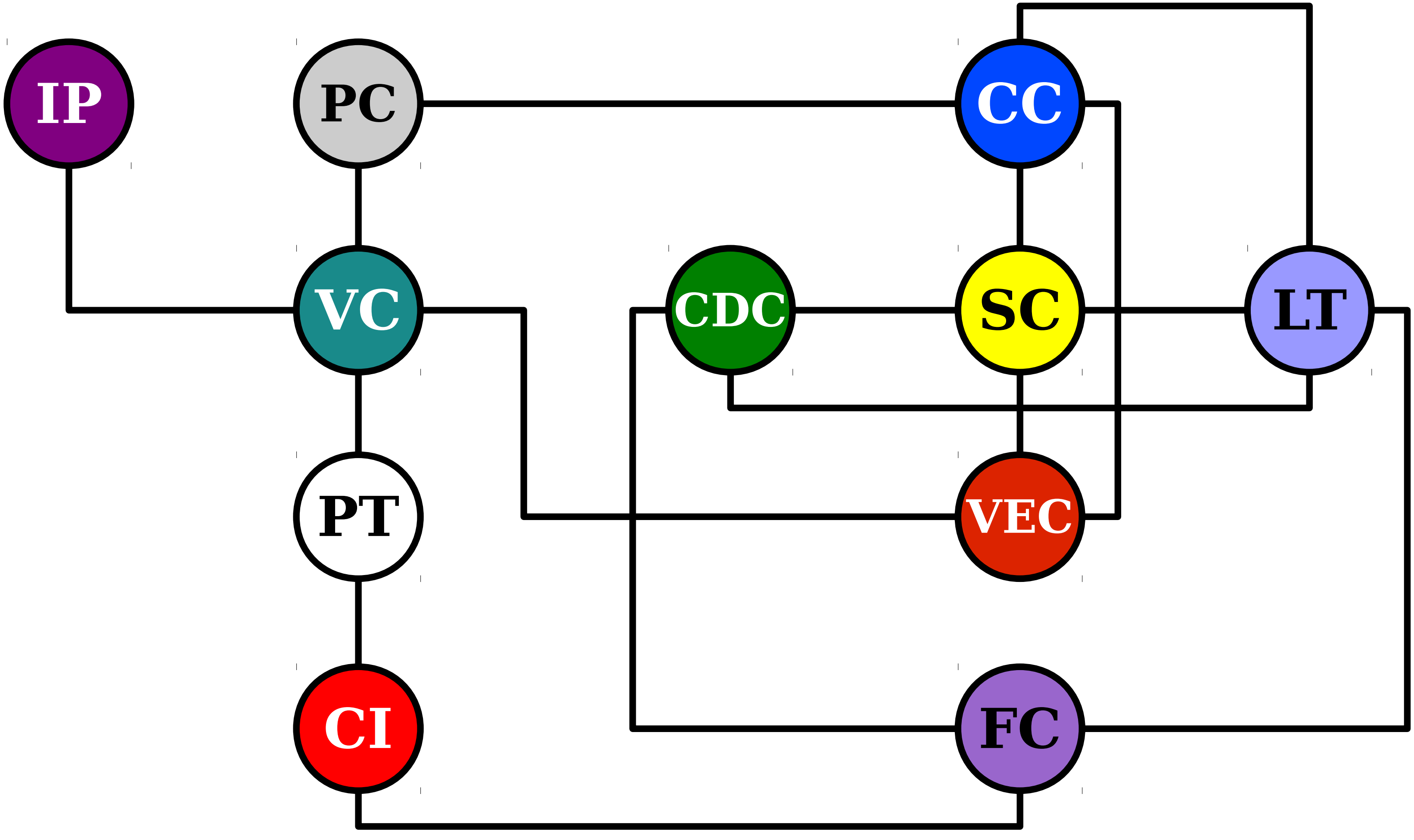
A ***cycle*** in a graph is a path from a node to itself.

The ***length*** of a cycle is the number of edges in that cycle.

A ***simple path*** in a graph is a path that does not revisit any nodes or edges.

A ***simple cycle*** in a graph is a cycle that does not revisit any nodes or edges (except the start/end node).

# Navigating a Graph



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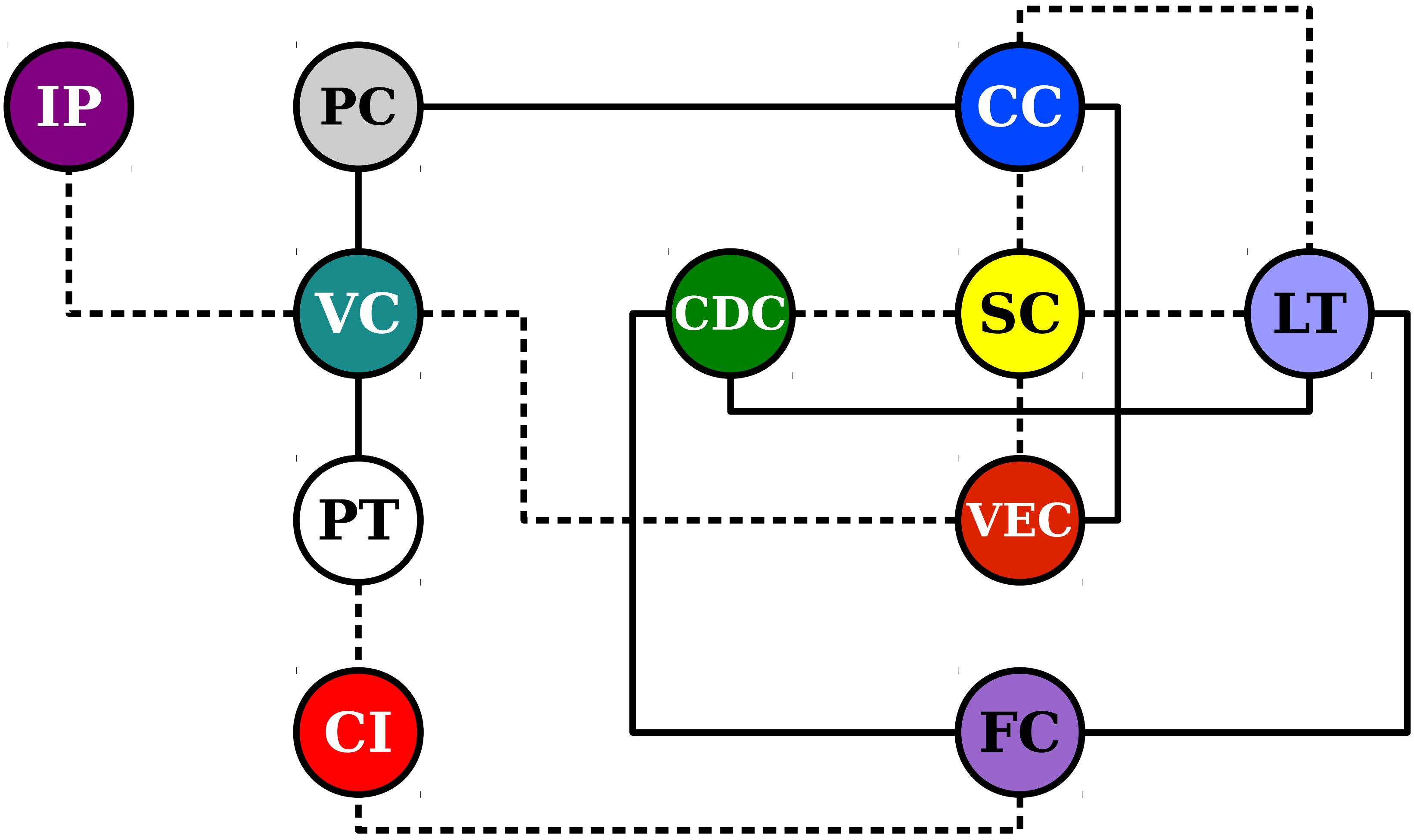

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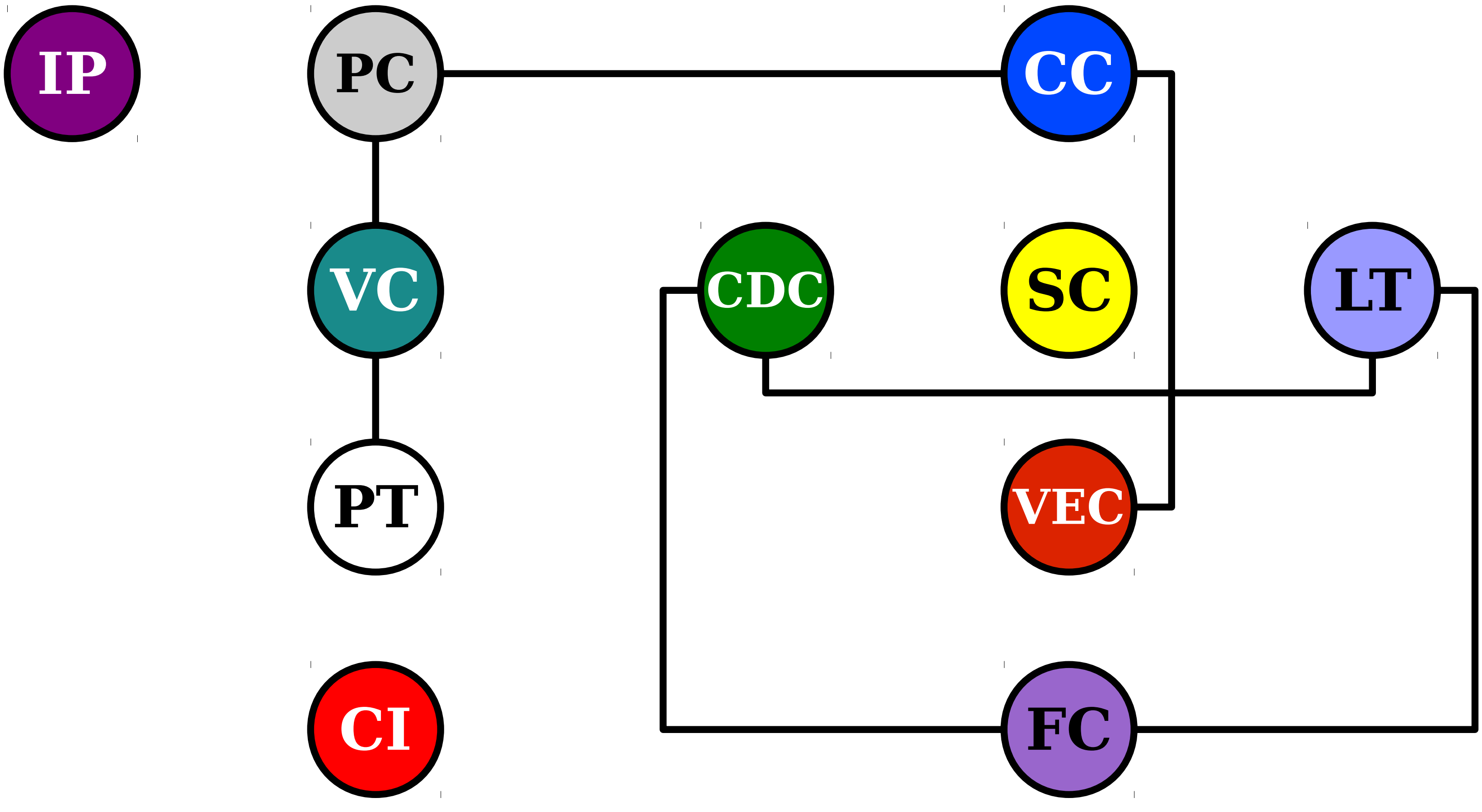

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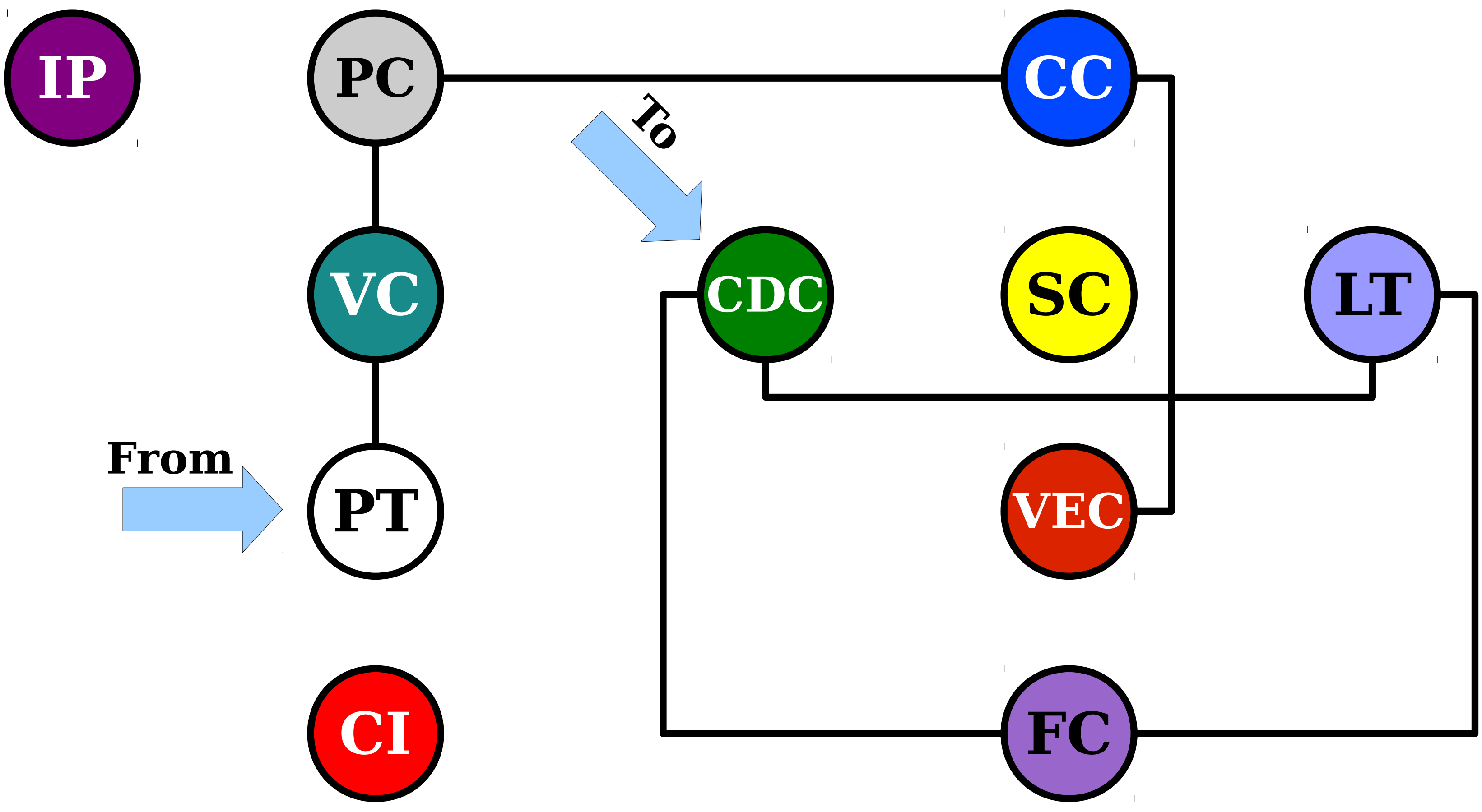

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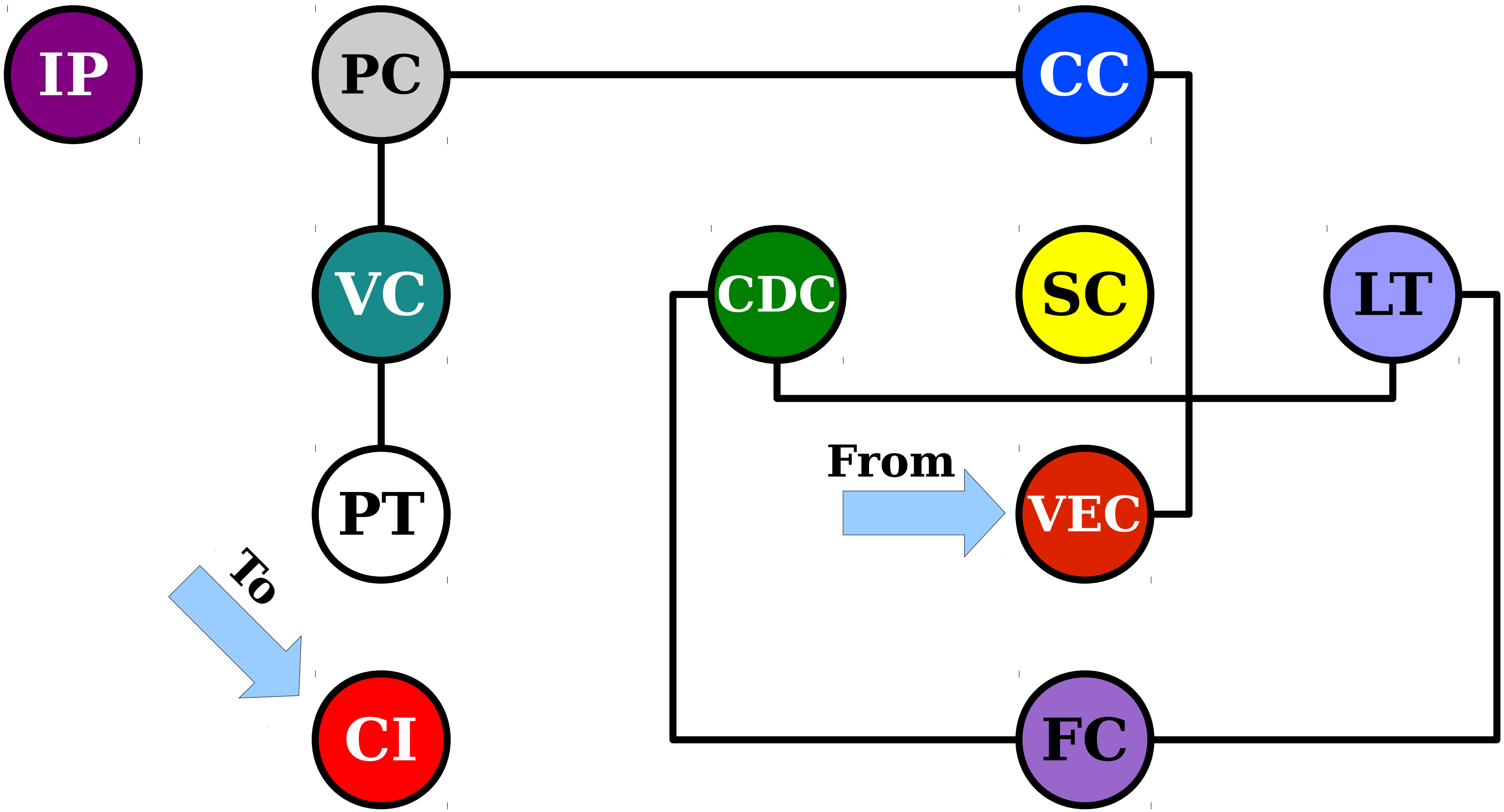


In an undirected graph, two nodes $u$ and $v$ are called ***connected*** if there is a path from $u$ to $v$.

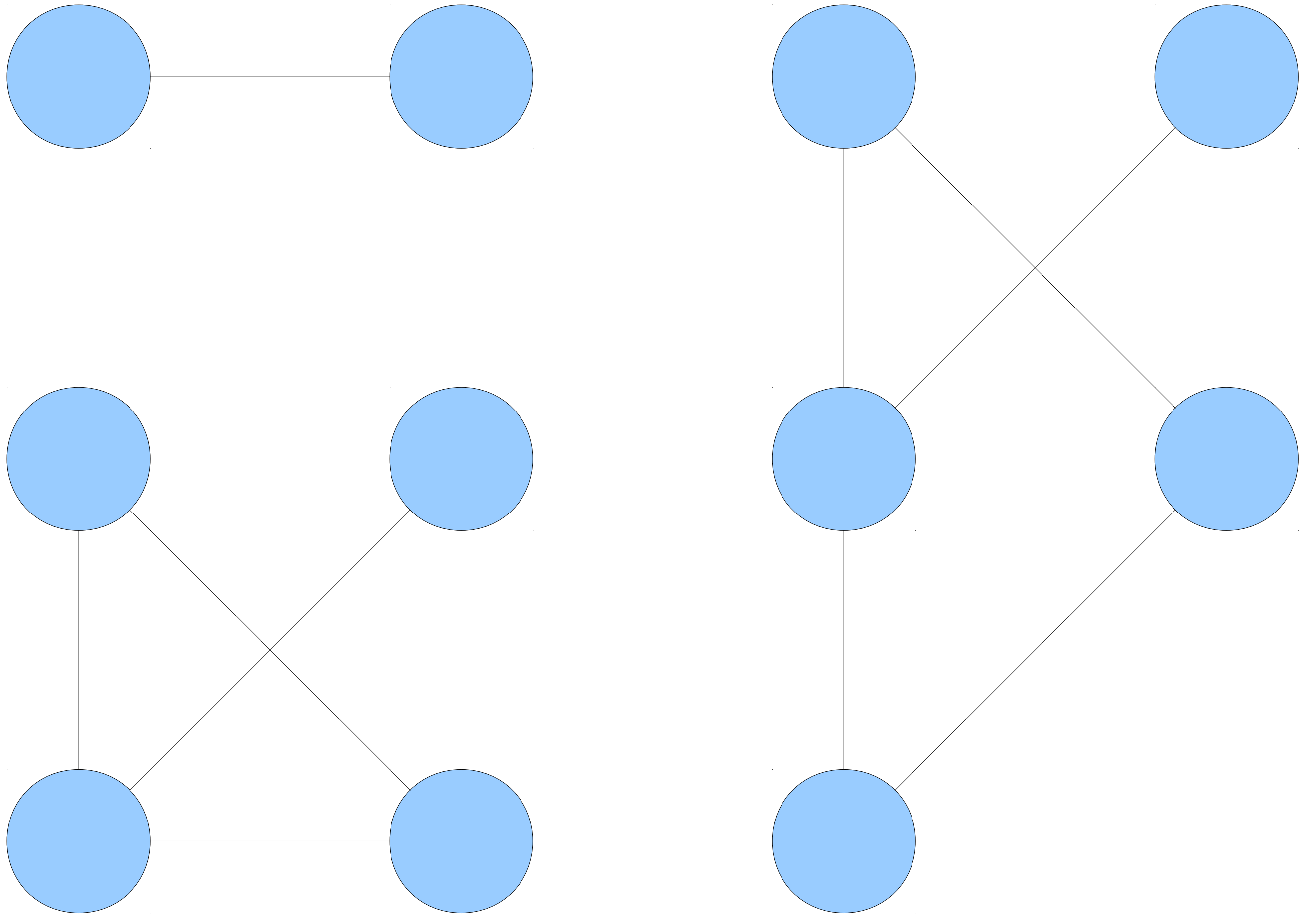
We denote this as $\boldsymbol{u \leftrightarrow v}$.

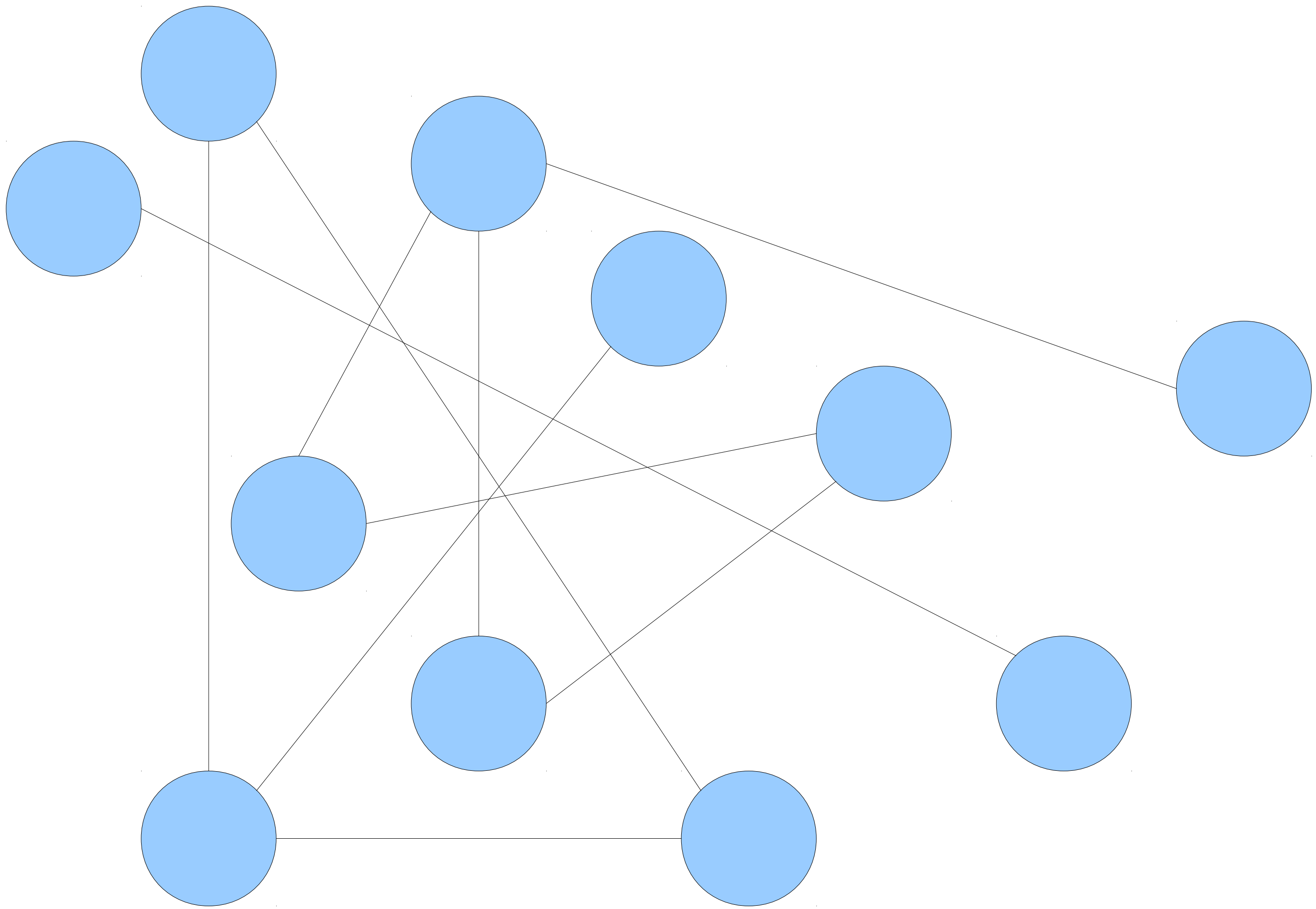
If $u$ is not connected to $v$, we write $\boldsymbol{u \nleftrightarrow v}$.

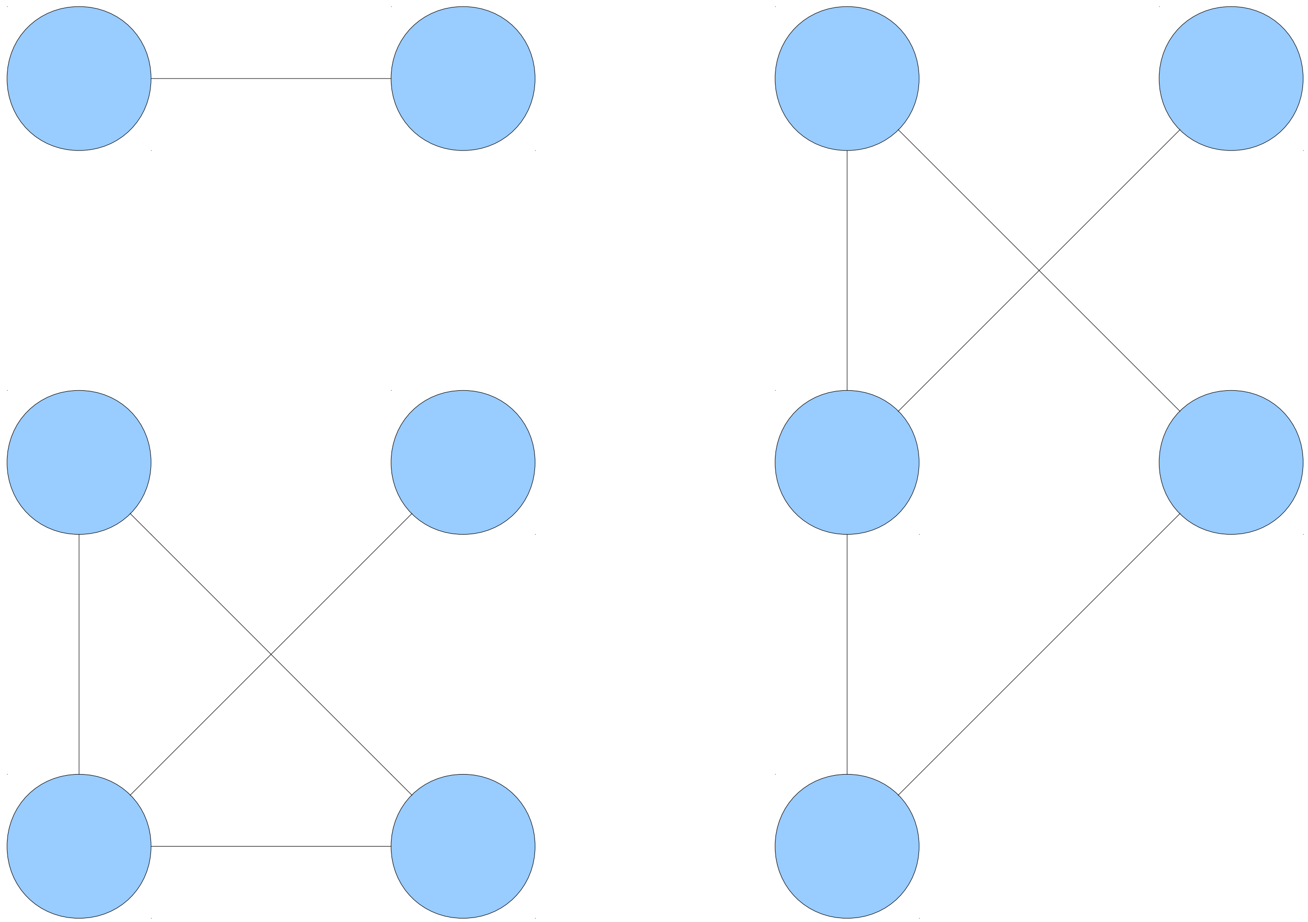
# Properties of Connectivity

- ***Theorem:*** The following properties hold for the connectivity relation $\leftrightarrow$:
  - For any node $v \in V$, we have $v \leftrightarrow v$.
  - For any nodes $u, v \in V$, if $u \leftrightarrow v$, then $v \leftrightarrow u$.
  - For any nodes $u, v, w \in V$, if $u \leftrightarrow v$ and $v \leftrightarrow w$, then $u \leftrightarrow w$.
- Can prove by thinking about the paths that are implied by each.

# Connected Components



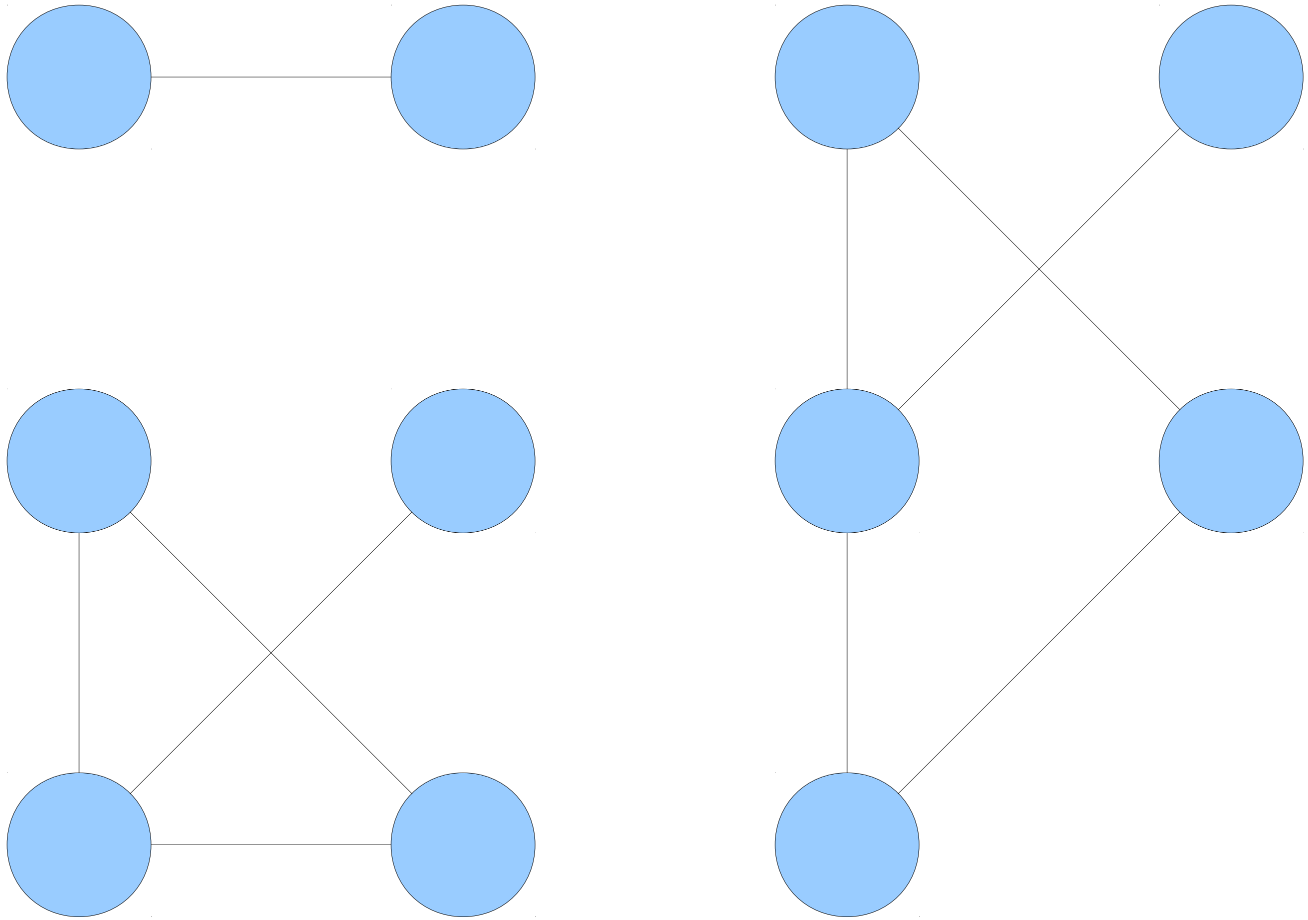


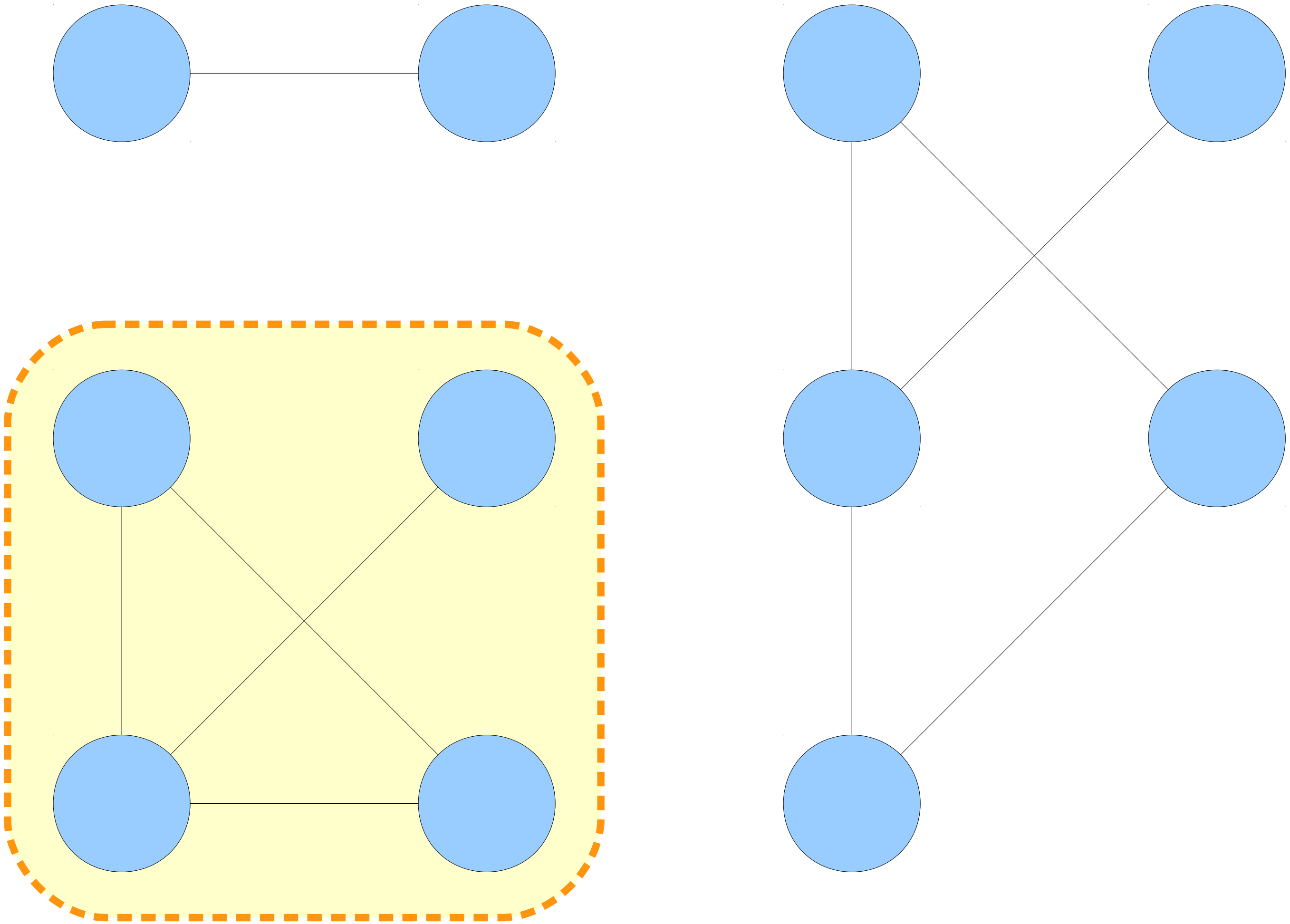


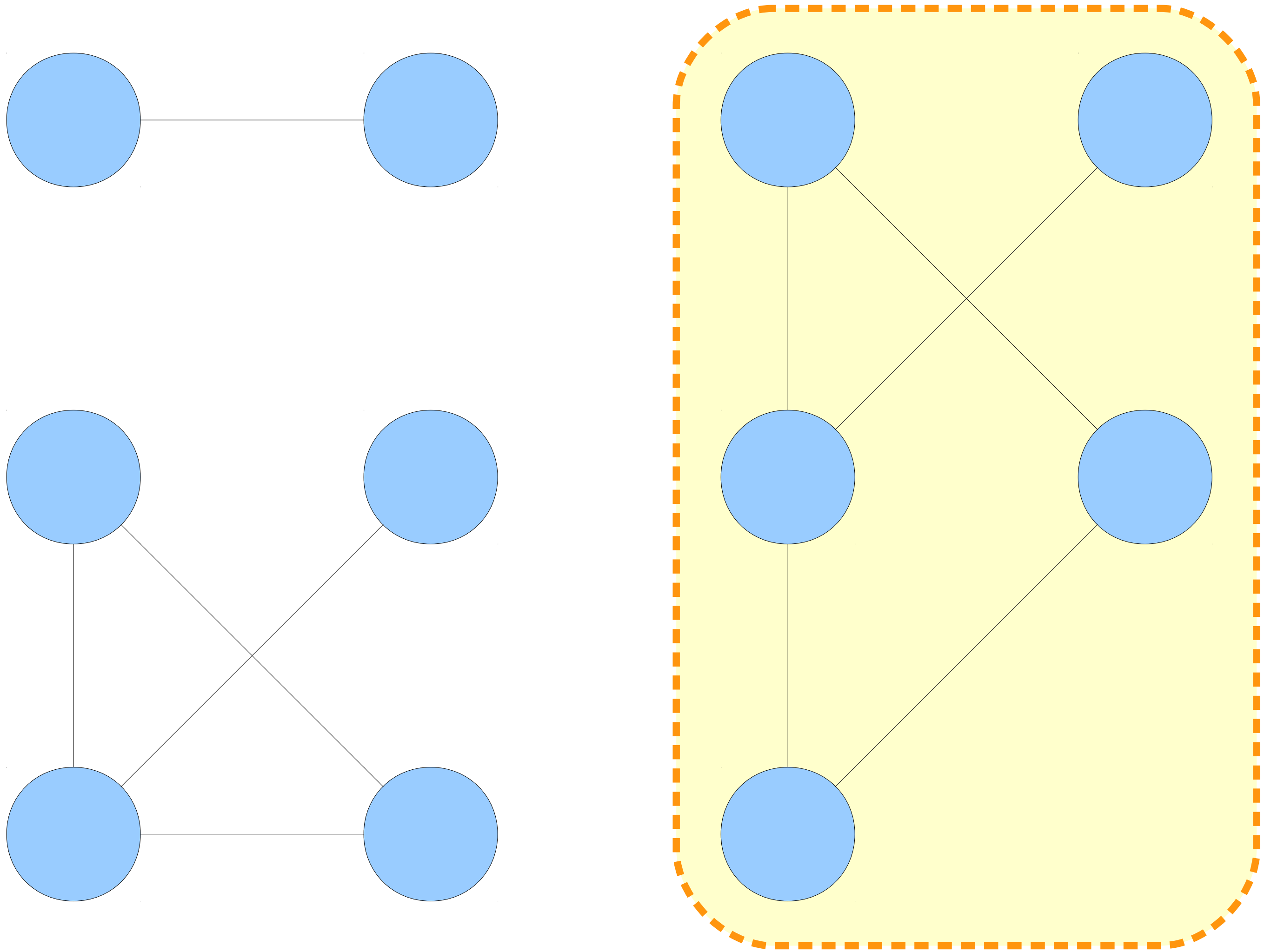
# An Initial Definition

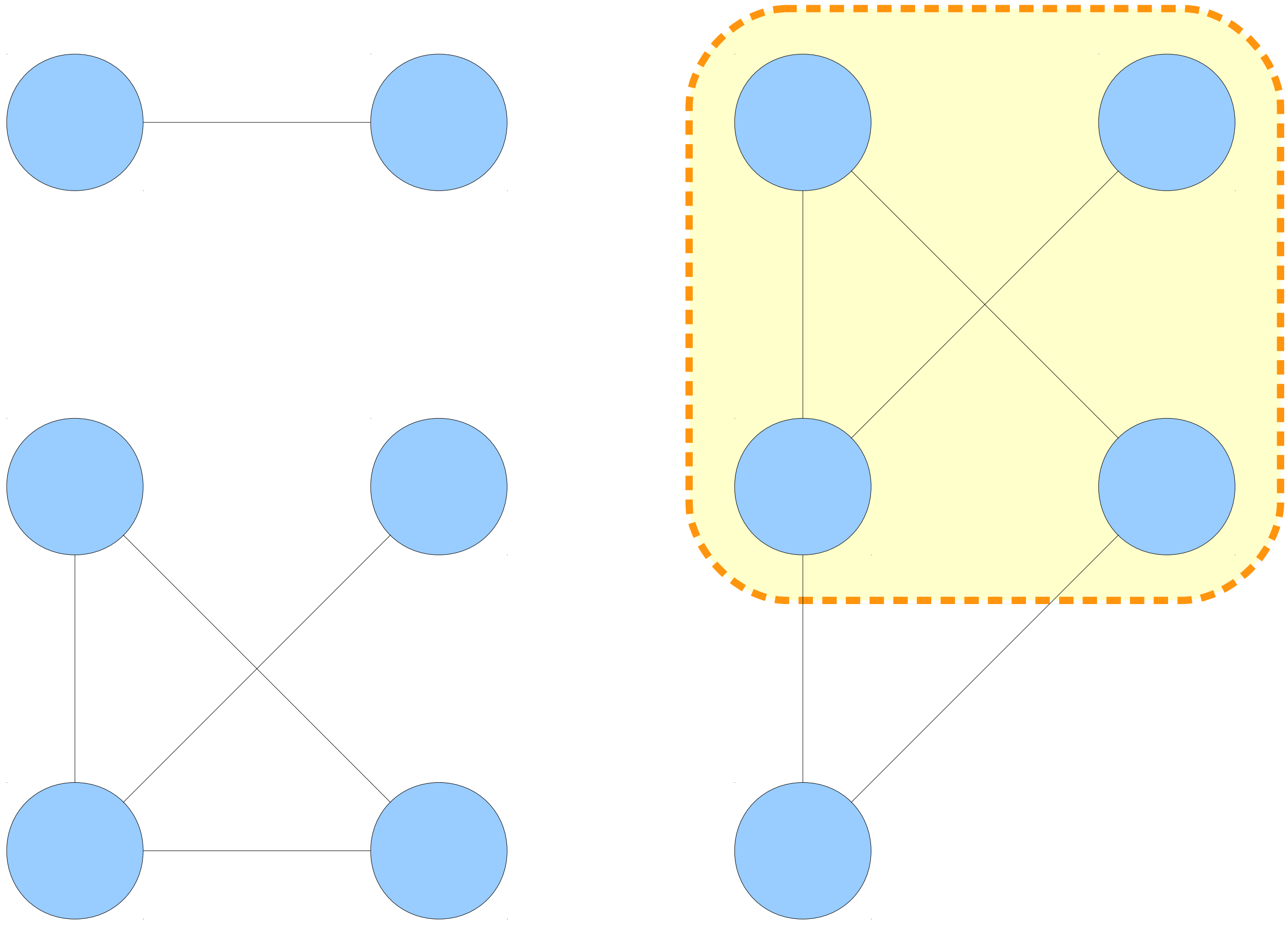
- **Attempted Definition #1:** A *piece* of an undirected graph $G = (V, E)$ is a set $C \subseteq V$ such that for any nodes $u, v \in C$, the relation $u \leftrightarrow v$ holds.
- Intuition: a piece of a graph is a set of nodes that are all connected to one another.

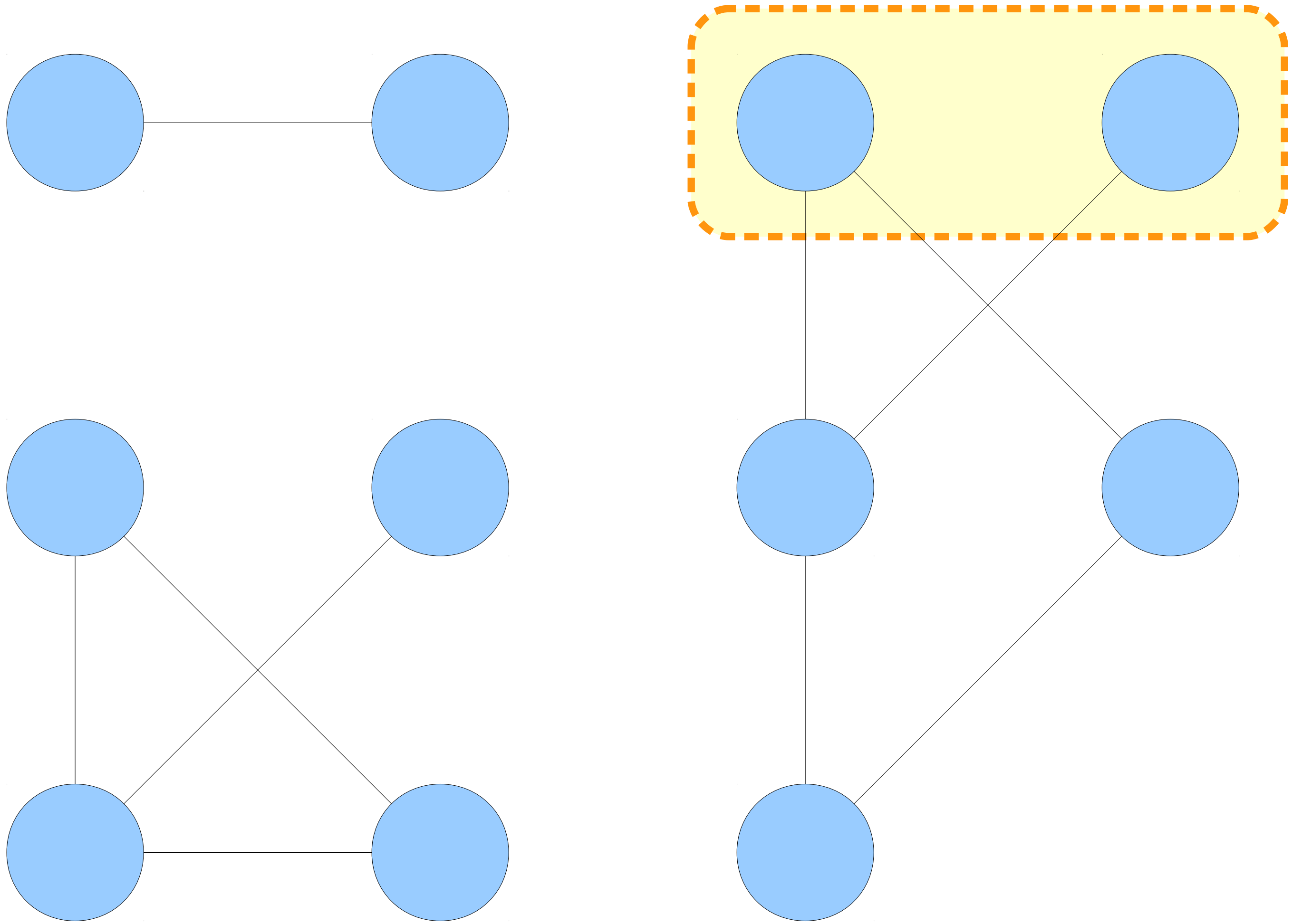
*This definition has some problems; please don't use it as a reference.*

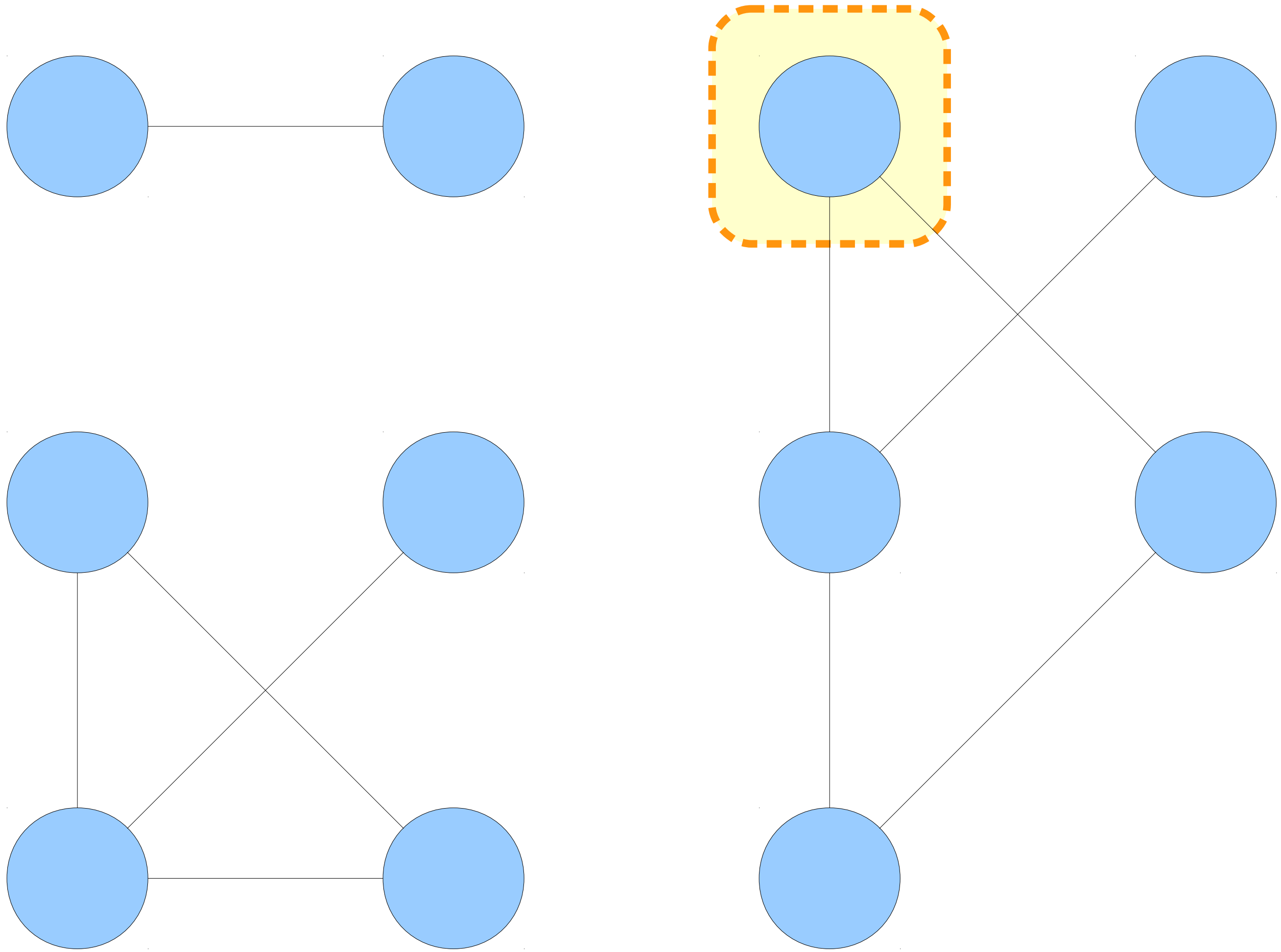








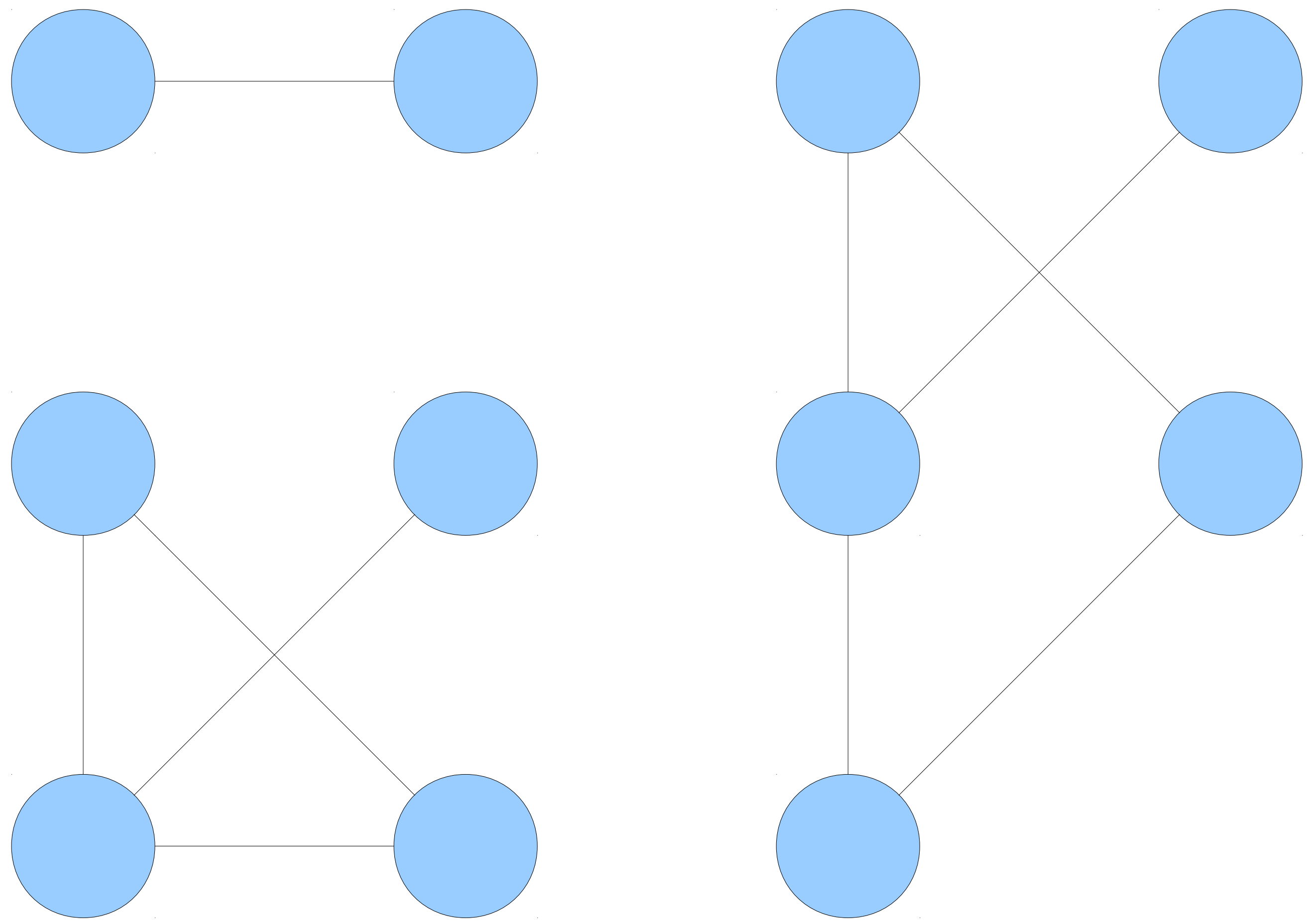


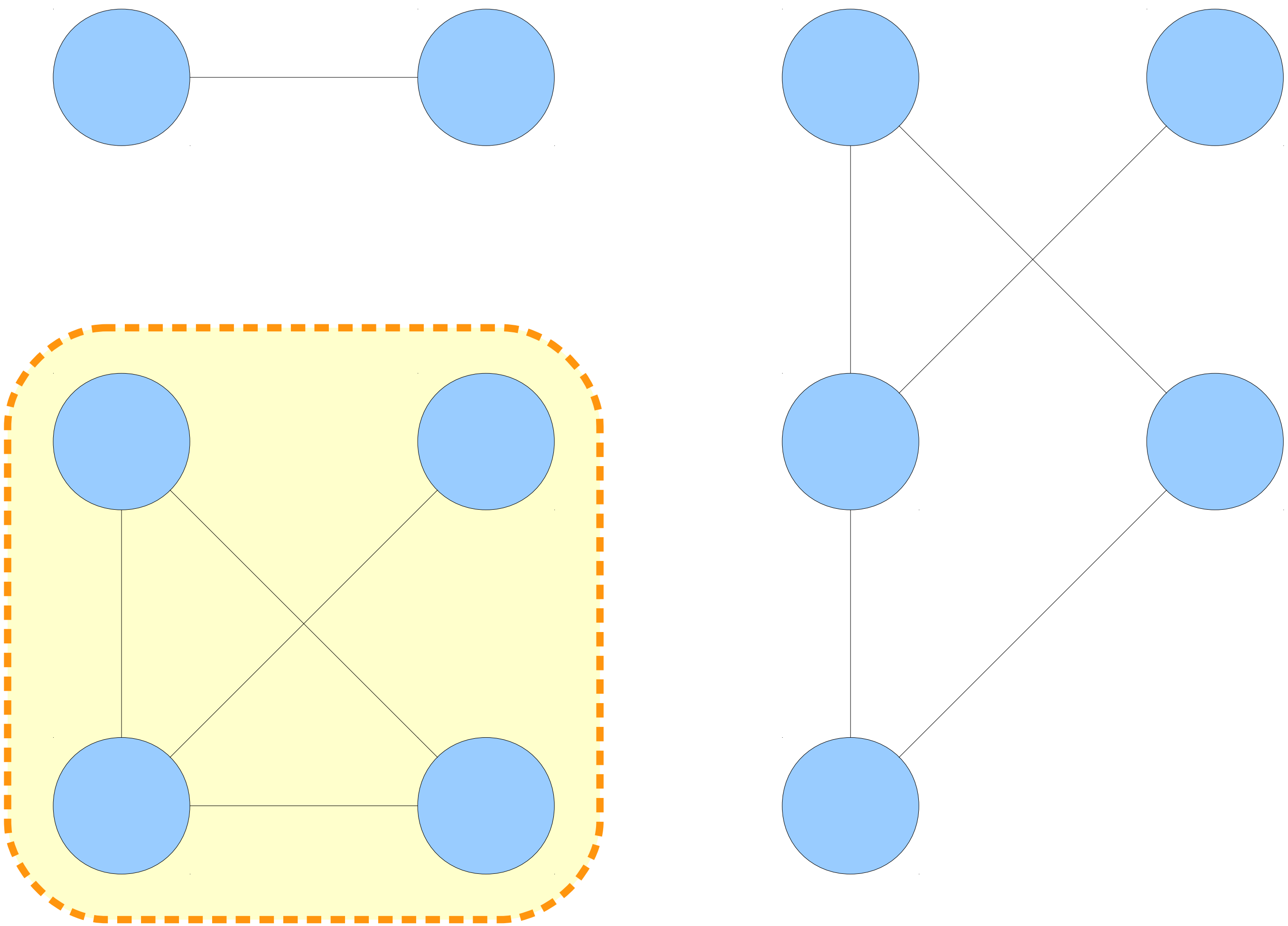


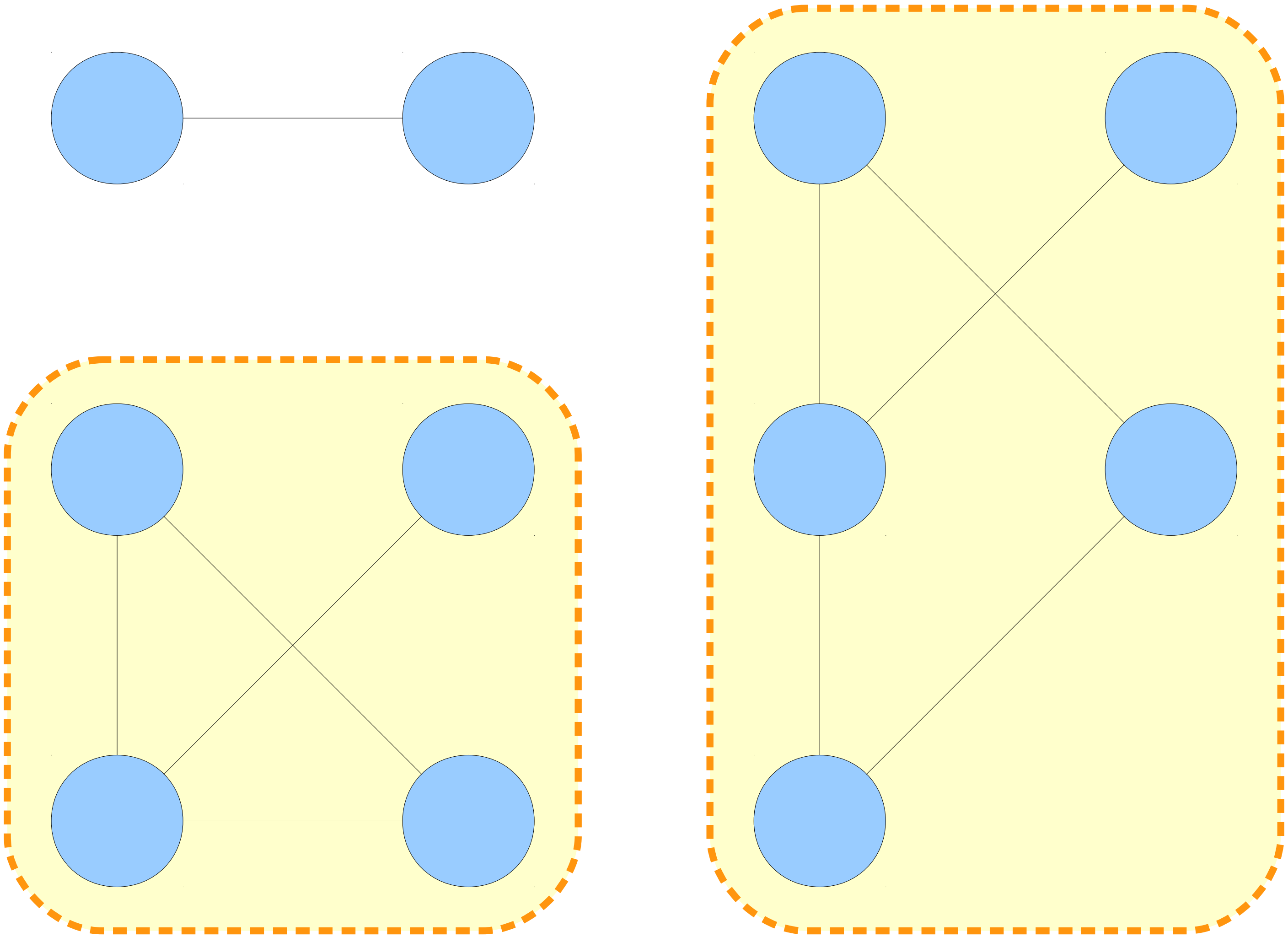
# An Updated Definition

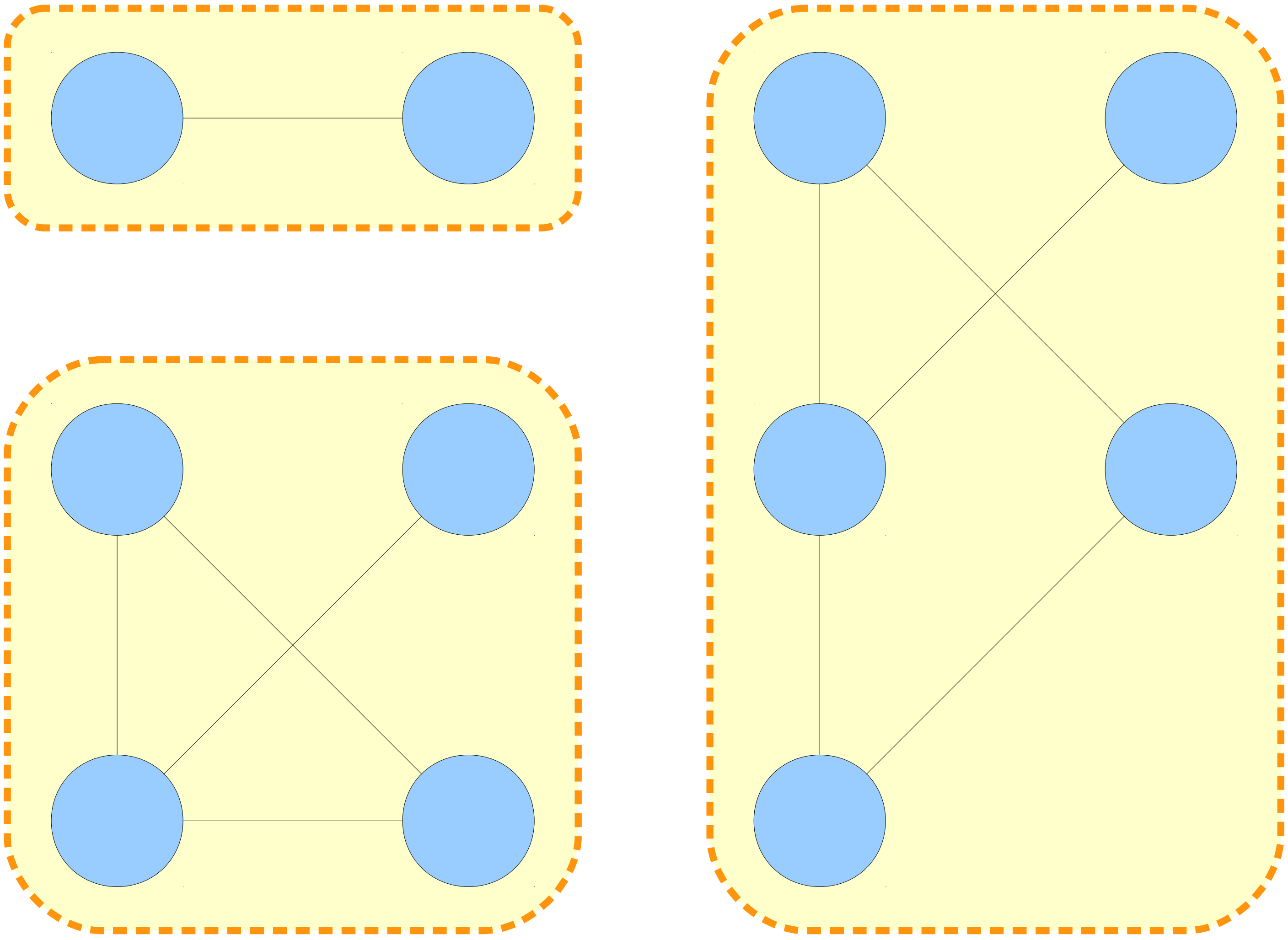
- **Attempted Definition #2:** A *piece* of an undirected graph $G = (V, E)$ is a set $C \subseteq V$ where
  - For any nodes $u, v \in C$, the relation $u \leftrightarrow v$ holds.
  - For any nodes $u \in C$ and $v \in V - C$, the relation $u \not\leftrightarrow v$ holds.
- Intuition: a piece of a graph is a set of nodes that are all connected to one another that doesn't “miss” any nodes.

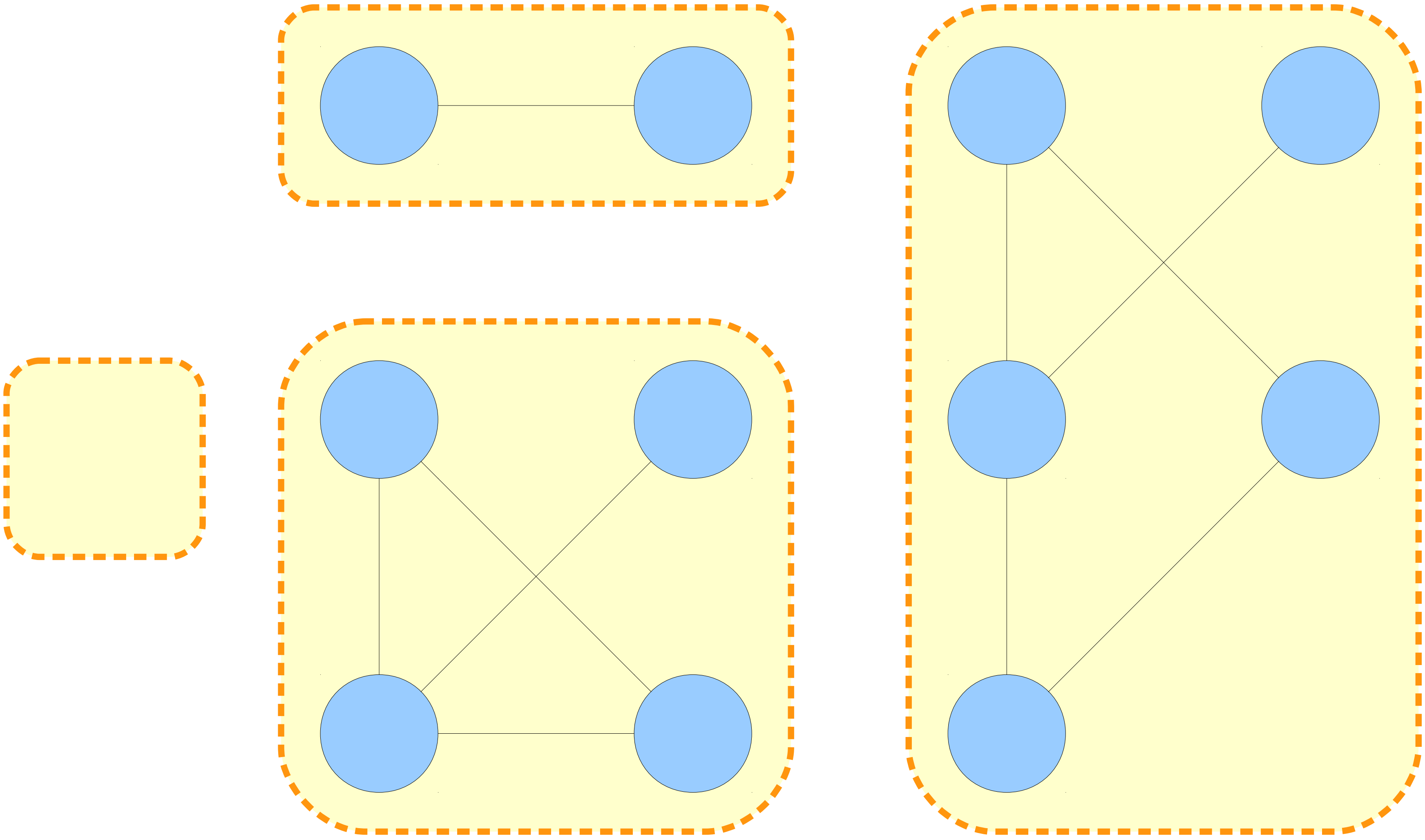
*This definition still has problems; please don't use it as a reference.*

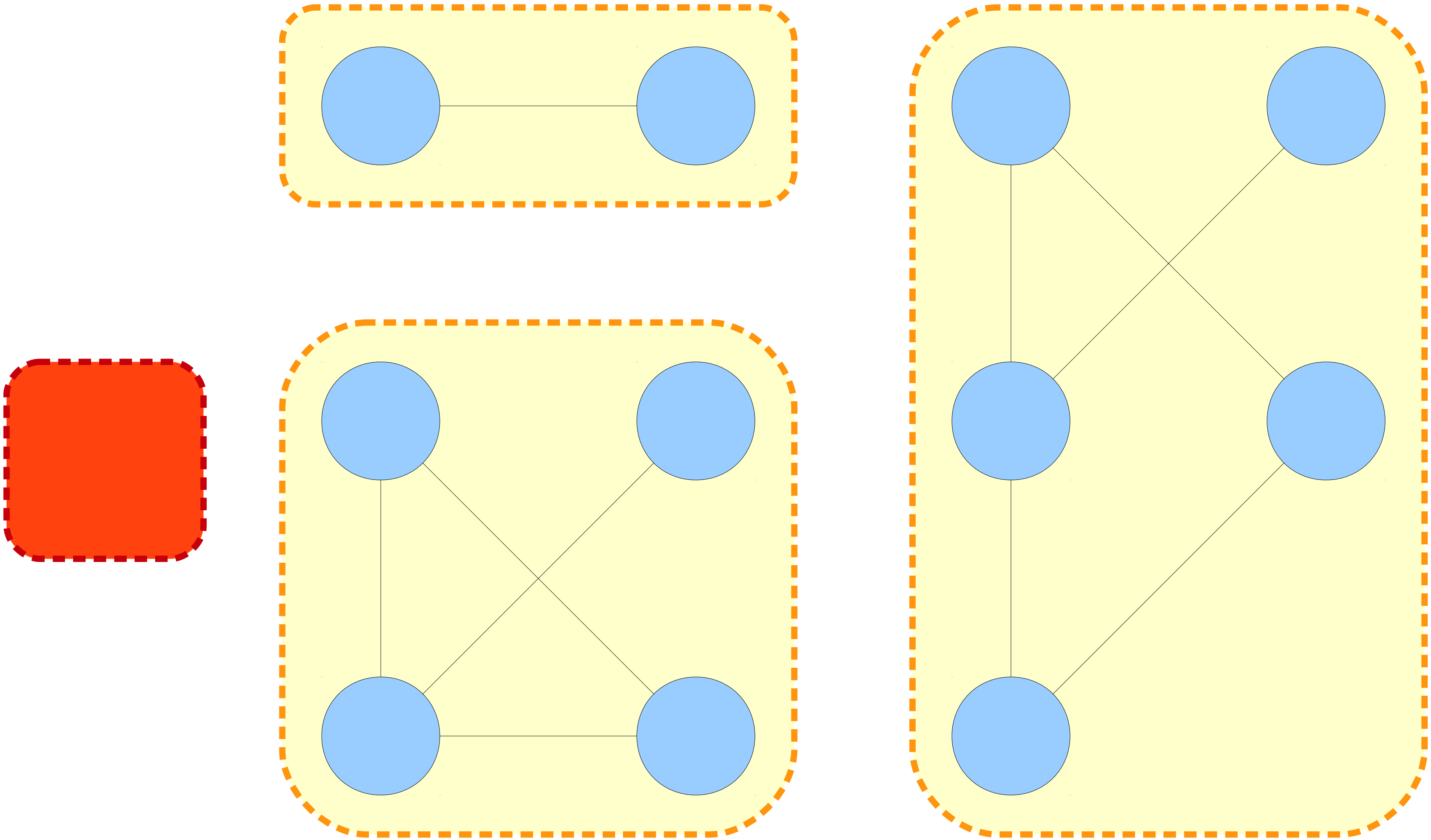












# A Final Definition

- **Definition:** A ***connected component*** of an undirected graph $G = (V, E)$ is a nonempty set $C \subseteq V$ where
  - For any nodes $u, v \in C$, the relation $u \leftrightarrow v$ holds.
  - For any nodes $u \in C$ and $v \in V - C$, the relation $u \nleftrightarrow v$ holds.
- Intuition: a connected component is a nonempty set of nodes that are all connected to one another that includes as many nodes as possible.

Time-Out for Announcements!

# Announcements

- Problem Set 1 solutions released at end of today's lecture.
  - Aiming to return problem sets no later than Wednesday.
- Problem Set 2 out, due Friday at the start of lecture.
  - Checkpoints due at the start of this lecture, will be returned by Wednesday.
  - Have questions? Ask on Piazza, stop by office hours, or email the staff list at cs103-aut1415-staff@lists.stanford.edu.

# Scoryst Signup

- We will be retroactively grouping PS1 submissions so that everyone in the group can view feedback.
- Please do not resubmit PS1 as a group. The people who run Scoryst will automatically reassign everyone.
- *Please register for Scoryst as soon as possible*. Click the “Assignment Submissions” link on the CS103 website to get into the system.

# Logistical Updates

- For this week only, I'll be moving my office hours to two one-hour blocks on Tuesday:
  - 4:00 – 5:00 and 5:45 – 6:45, Gates 415.
- Maesen and I will be out of town later this week at the Grace Hopper Conference. Stephen Macke will be the acting head TA.
  - Going to GHC? Want to meet up there? Let us know!

# Your Questions

“How can programs be written to create proofs? For many of these problems, you've told us that solving them is a matter of developing an 'intuition,' but how can we add 'intuition' to a program rather than using brute force?”

“What is so special about the number 137?”

“How many questions per problem is too many questions? Sometimes I hesitate to ask a question on a problem if I've already asked one before on the same problem because I might not be doing enough to solve it on my own.”

“Is this statement false?”

“Why are most computer science majors socially un-developed? Does coding/starting at a computer on hours a day re-wire your brain to negatively impact this part of your life?”

Back to CS103!

# Manipulating our Definition

# Proving the Obvious

- ***Theorem:*** If $G = (V, E)$ is a graph, then every node $v \in V$ belongs to exactly one connected component.
- How exactly would we prove a statement like this one?
- Use an **existence and uniqueness proof**:
  - Prove there is *at least* one object of that type.
  - Prove there is *at most* one object of that type.
- These are usually separate proofs.

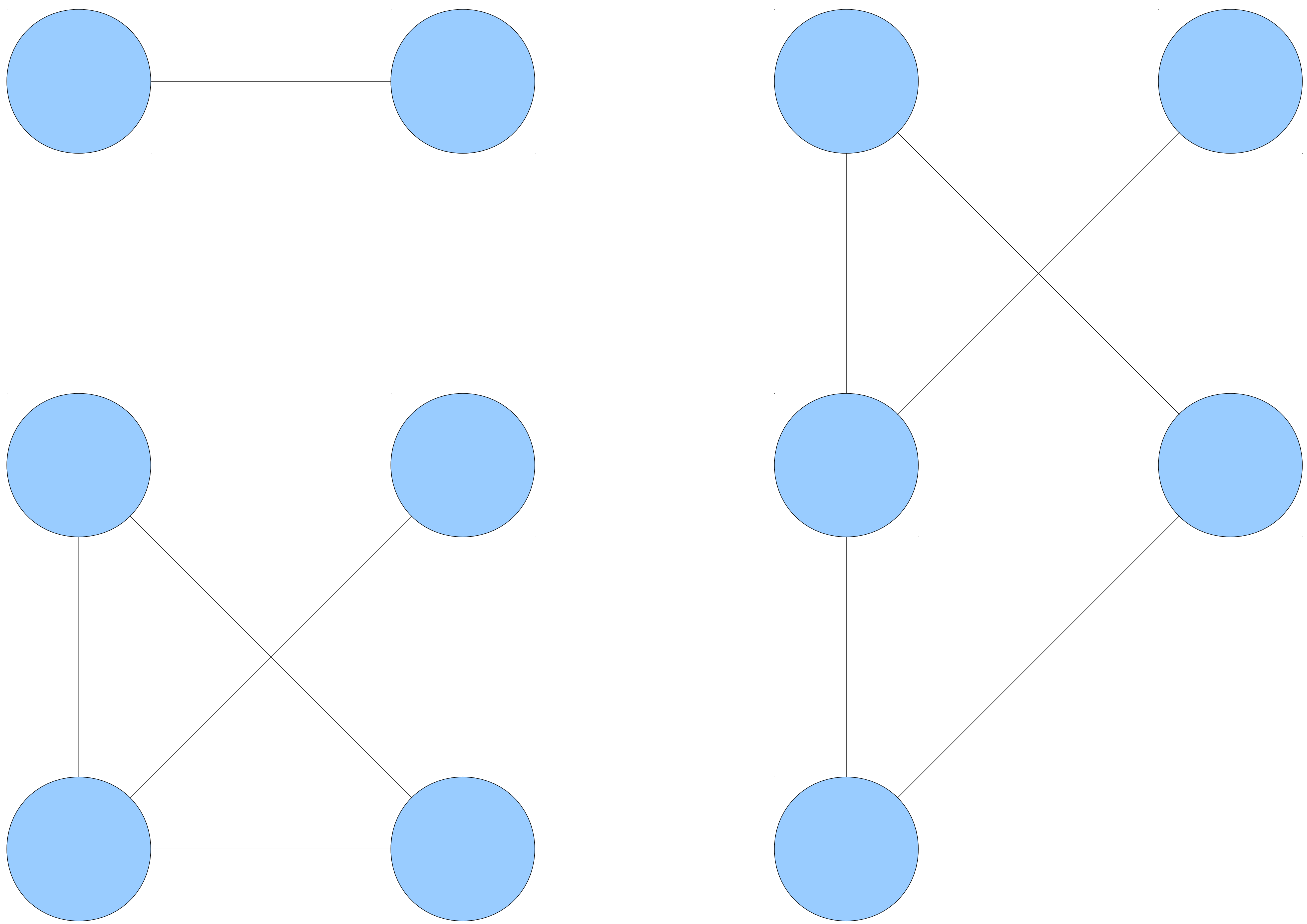
Part 1: **Every node belongs to at least one connected component.**

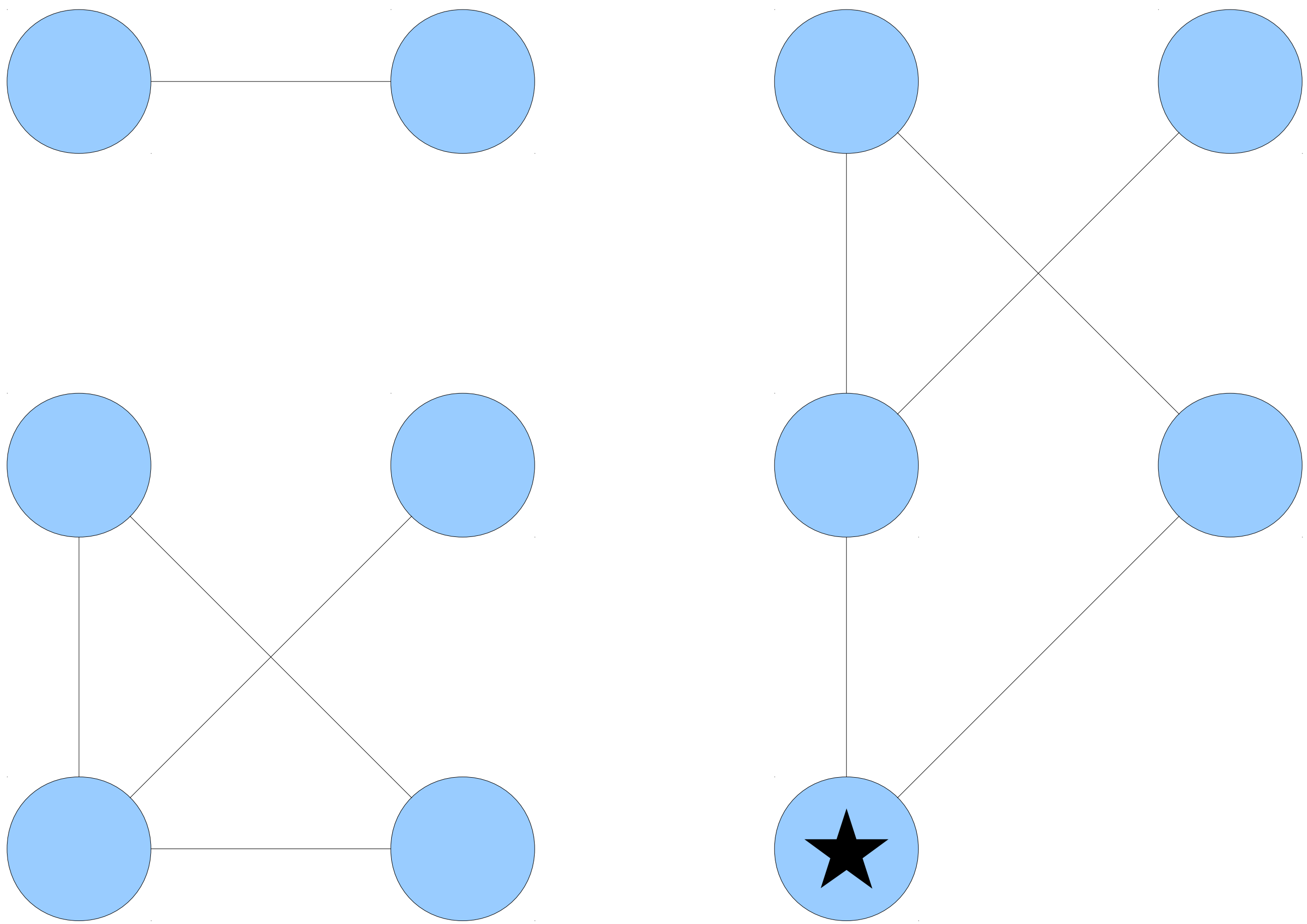
# Proving Existence

- Given an arbitrary graph $G = (V, E)$ and an arbitrary node $v \in V$, we need to show that there exists some connected component $C$ where $v \in C$.
- The key part of this is the existential statement

**There exists** a connected component $C$ such that $v \in C$.

- The challenge: how can we find the connected component that $v$ belongs to given that $v$ is an arbitrary node in an arbitrary graph?





# The Conjecture

- **Conjecture:** Let $G = (V, E)$ be an undirected graph. Then for any node $v \in V$, the set $\{ x \in V \mid v \leftrightarrow x \}$ is a connected component and it contains $v$.
- If we can prove this, we have shown *existence*: at least one connected component contains $v$.

***Lemma 1:*** Let $G = (V, E)$ be an undirected graph. For any node $v \in V$, the set $C = \{ x \in V \mid v \leftrightarrow x \}$ contains $v$.

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***Proof:*** The relation $v \leftrightarrow v$ holds for any $v \in V$.

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***Proof:*** The relation $v \leftrightarrow v$ holds for any $v \in V$. Therefore, by definition of $C$, we see that $v \in C$.

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# The Tricky Part

- We need to show for any $v \in V$ that the set $C = \{ x \in V \mid v \leftrightarrow x \}$ is a connected component.
- Therefore, we need to show
  - $C \neq \emptyset$;
  - for any $x, y \in C$, the relation $x \leftrightarrow y$ holds; and
  - for any $x \in C$ and $y \notin C$, the relation $x \not\leftrightarrow y$ holds.

***Lemma 2:*** Let $G = (V, E)$ be an undirected graph. Choose some node $v \in V$ and let $C = \{ x \in V \mid v \leftrightarrow x \}$. Then for any nodes $x, y \in C$, we have $x \leftrightarrow y$.

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***Proof:*** By definition, since $x \in C$ and $y \in C$, we have $v \leftrightarrow x$ and $v \leftrightarrow y$.

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***Proof:*** By definition, since $x \in C$ and $y \in C$, we have $v \leftrightarrow x$ and $v \leftrightarrow y$. By our earlier theorem, since $v \leftrightarrow x$, we know $x \leftrightarrow v$.

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***Lemma 3:*** Let $G = (V, E)$ be an undirected graph. Choose some node $v \in V$ and let $C = \{ x \in V \mid v \leftrightarrow x \}$. Then for any nodes $x \in C$ and $y \in V - C$, we have $x \nleftrightarrow y$.

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***Proof:*** By contradiction; assume $x \in C$ and $y \in V - C$, but that $x \leftrightarrow y$.

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Part 2: **Every node belongs to at most one connected component.**

# Uniqueness Proofs

- To show there is at most one object with some property $P$, show the following:

**If $x$ has property $P$ and $y$ has property $P$, then $x = y$.**

- Rationale: $x$ and $y$ are just different names for the same thing; at most one object of the type can exist.

# Uniqueness Proofs

- Suppose that $C_1$ and $C_2$ are connected components containing $v$.
- We need to prove that $C_1 = C_2$.
- Idea: $C_1$ and $C_2$ are sets, so we can try to show that $C_1 \subseteq C_2$ and that $C_2 \subseteq C_1$.
  - Just because we're working at a higher level of abstraction doesn't mean our existing techniques aren't useful!

*Lemma:* Let $C$ be a connected component of an undirected graph $G = (V, E)$ and $v \in V$ a node contained in $C$. Then for any $x \in V$, we have $x \in C$ iff $v \leftrightarrow x$.

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($\Rightarrow$) First, we prove that if $x \in C$, then $v \leftrightarrow x$.

($\Leftarrow$) Next, we prove that if $v \leftrightarrow x$, then $x \in C$.

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When proving a biconditional, it is common to split the proof apart into two directions. The symbols ($\Rightarrow$) and ($\Leftarrow$) denote where in the proof the two directions can be found.

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By using a similar line of reasoning and interchanging the roles of $C_2$ and $C_1$, we also see that $C_2 \subseteq C_1$.

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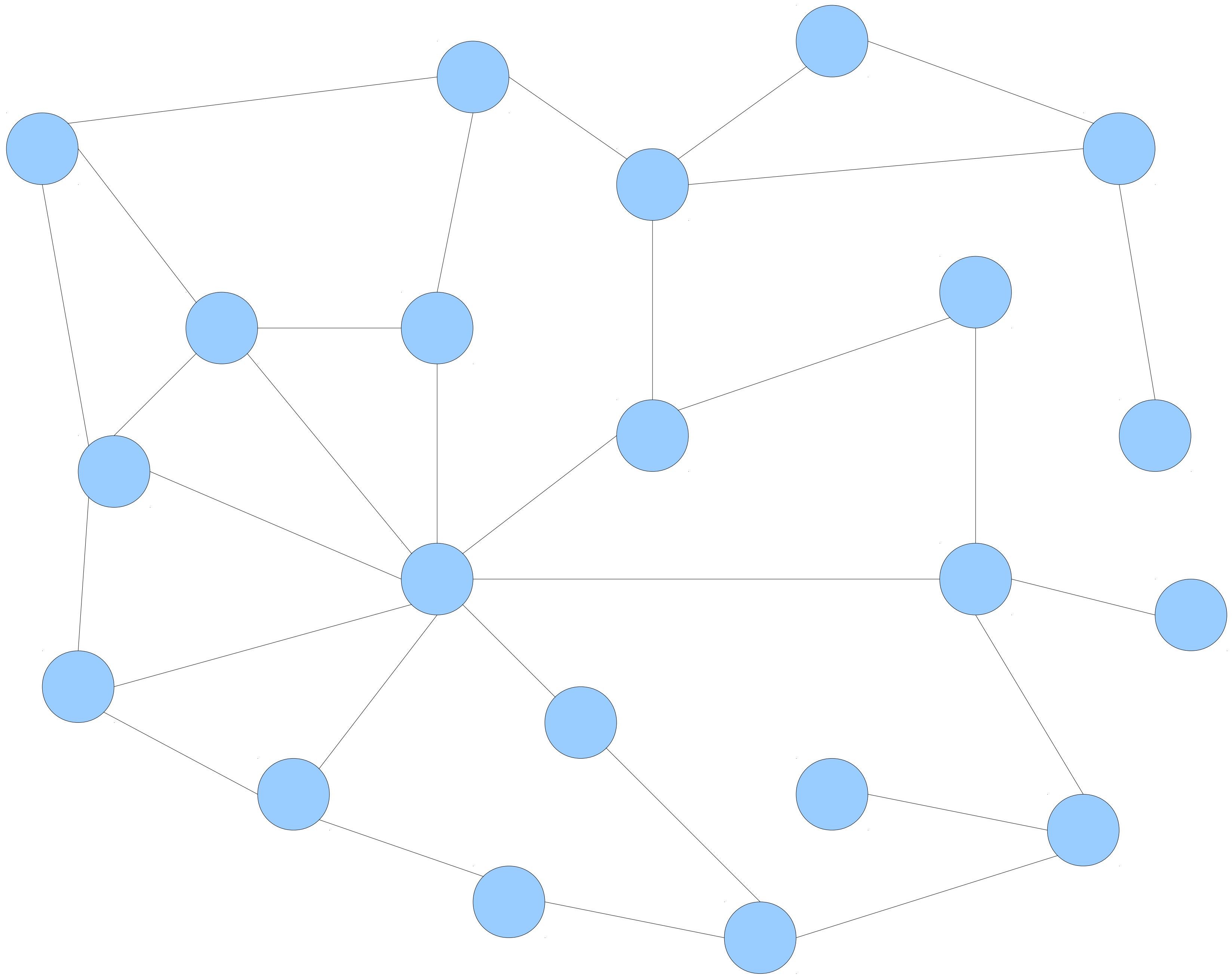
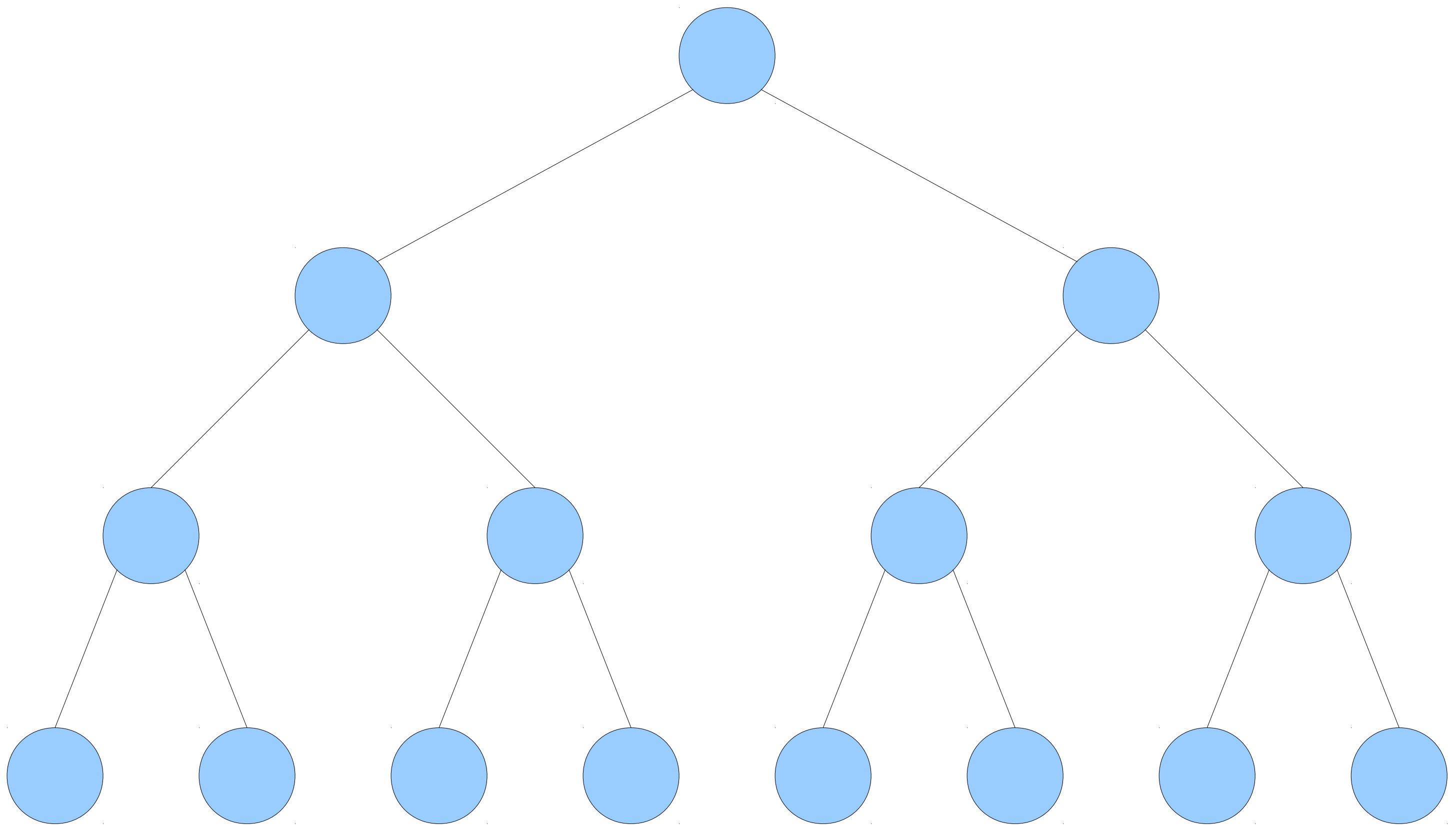
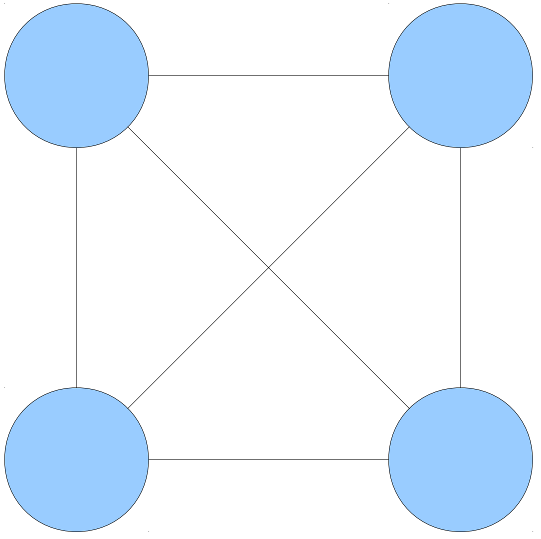
# Why All This Matters

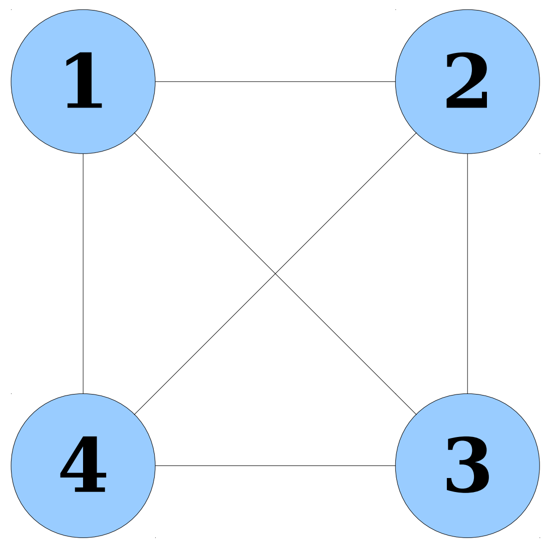
- I chose the example of connected components to
  - describe how to come up with a precise definition for intuitive terms;
  - see how to manipulate a definition once we've come up with one;
  - explore existence and uniqueness proofs, which we'll see more of later on; and
  - explore multipart proofs with several different lemmas.

# Planar Graphs





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2
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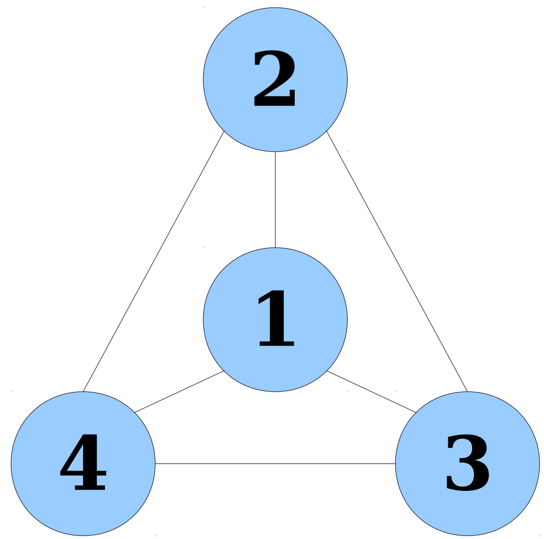
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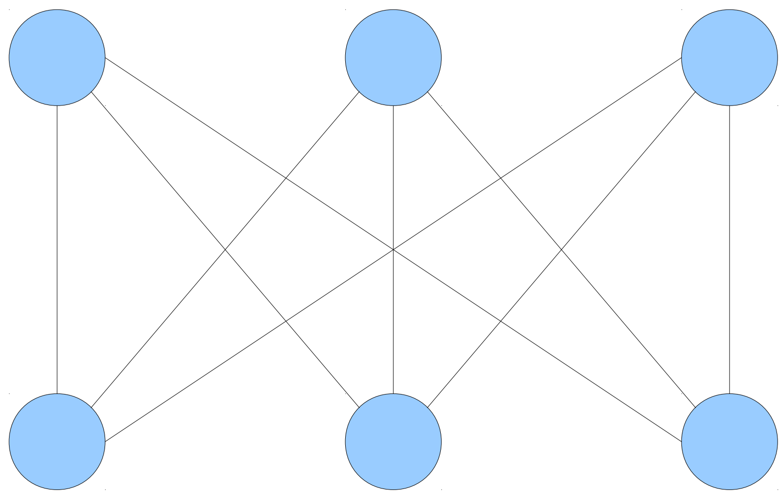
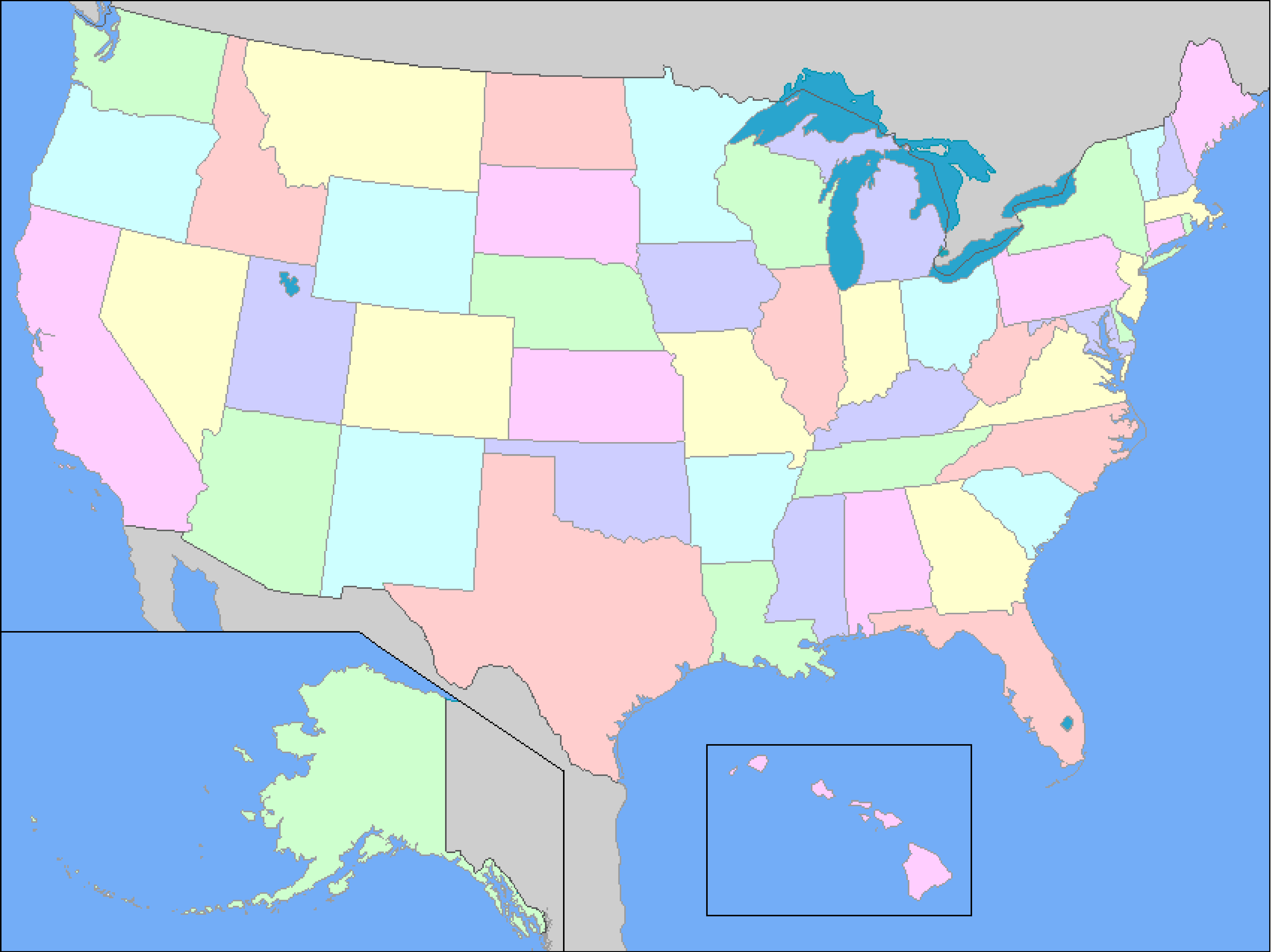
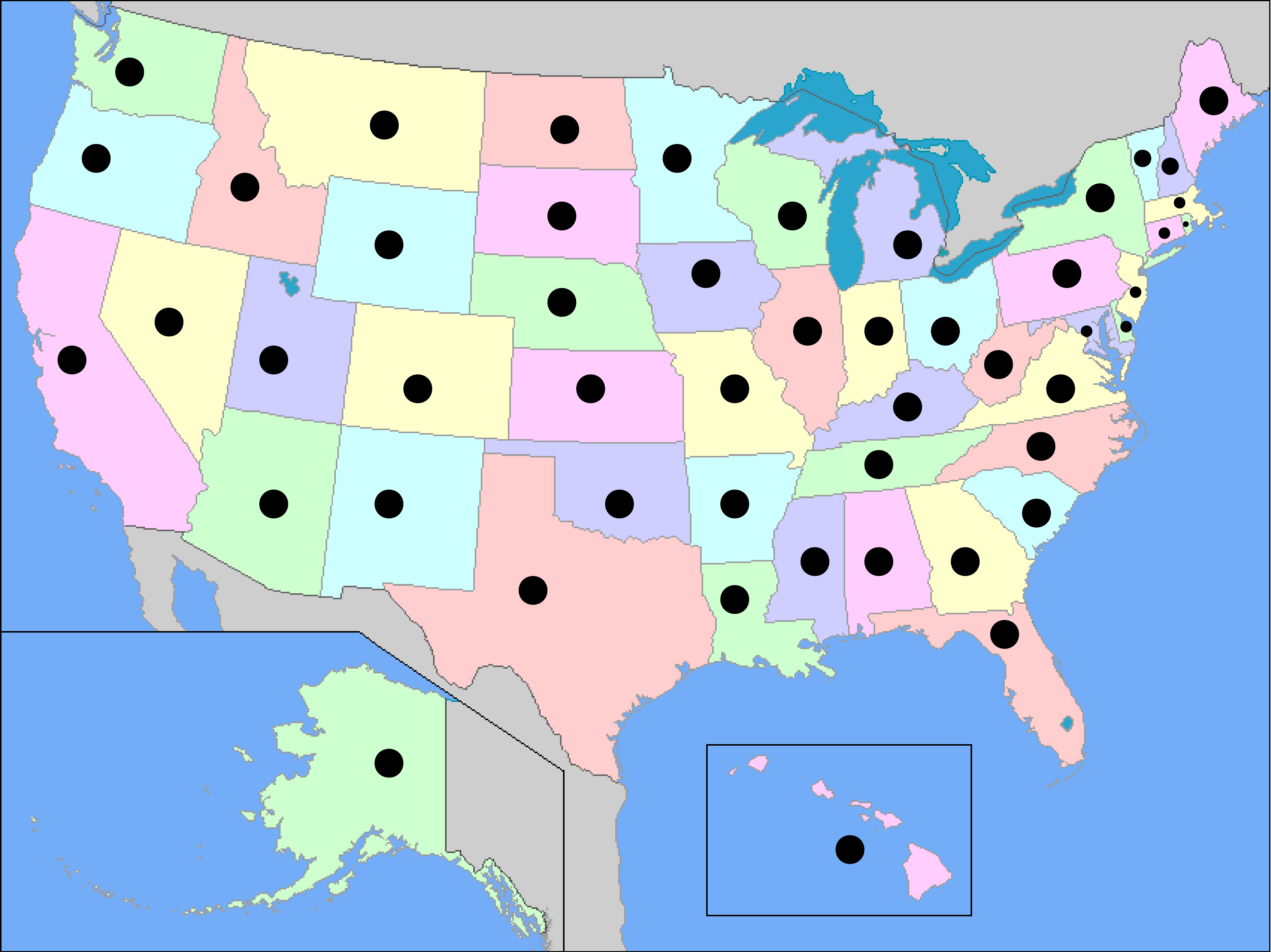
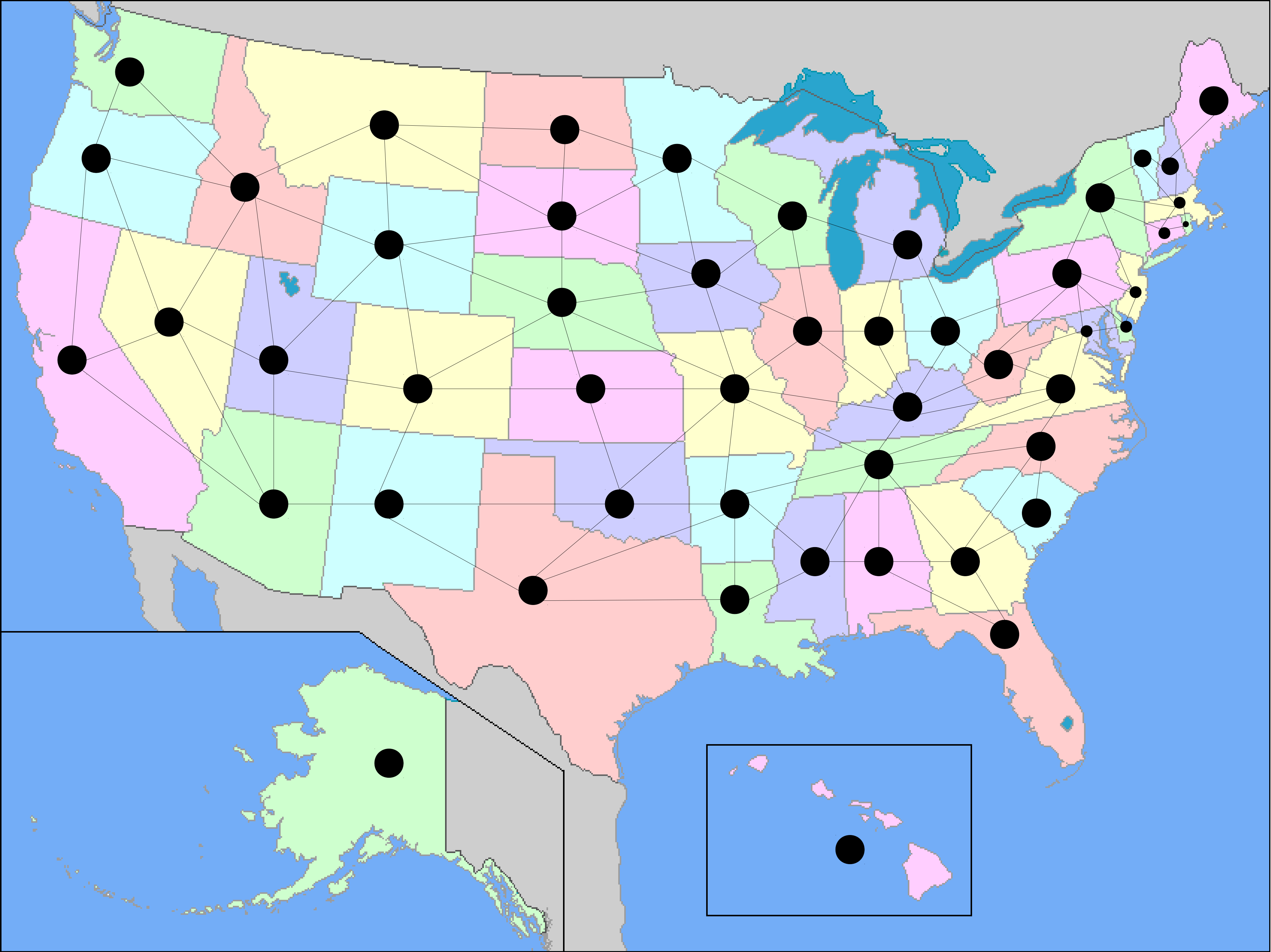


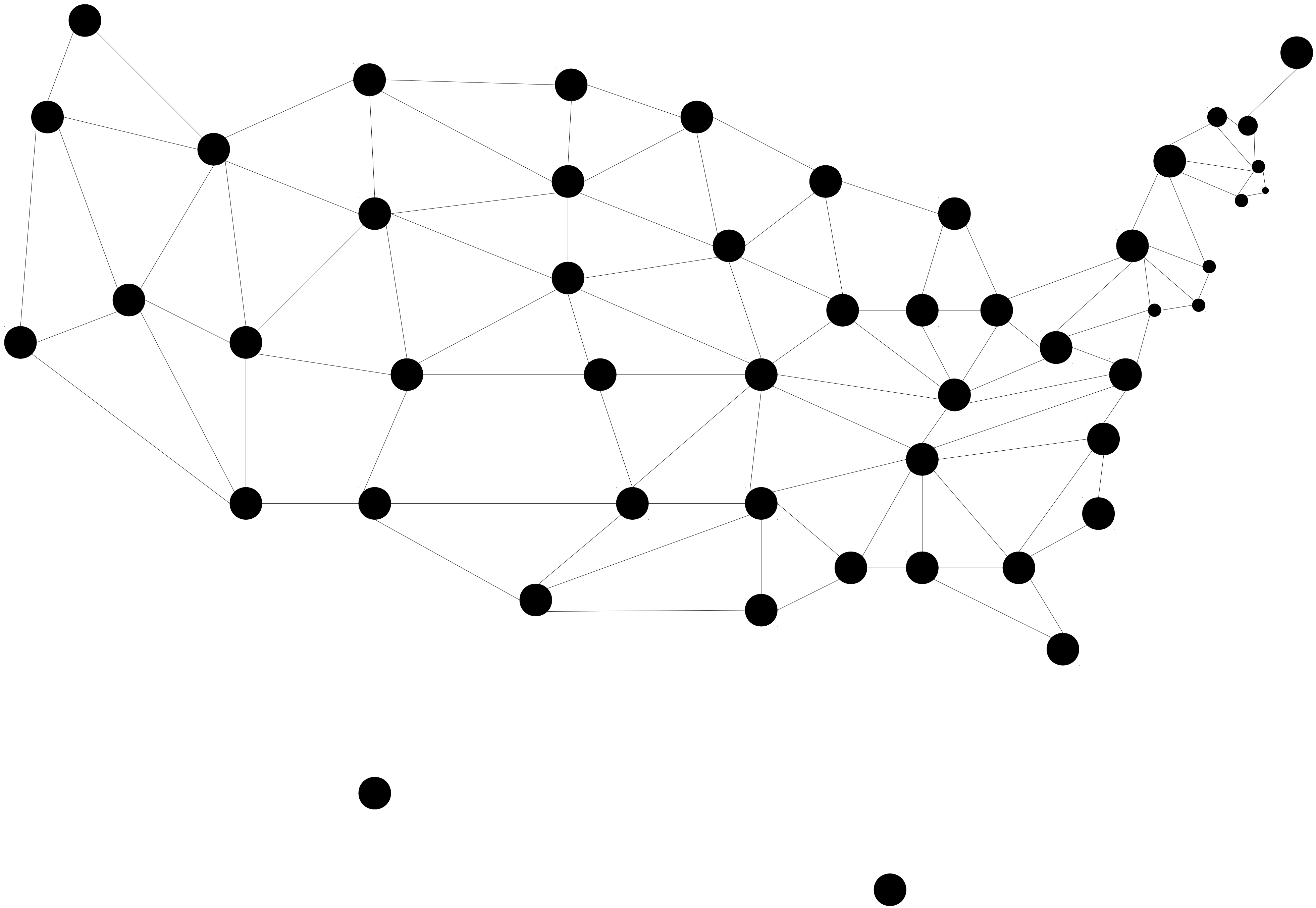
This graph is sometimes called the ***utility graph***.

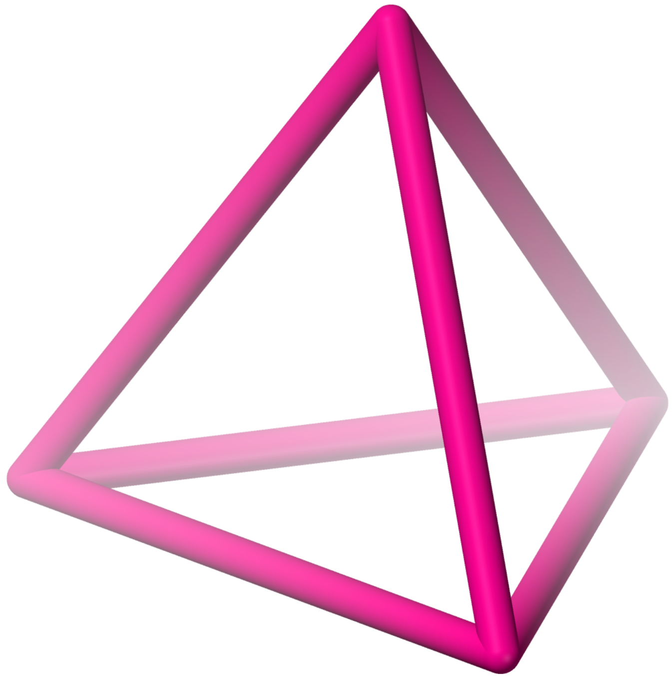
A graph is called a ***planar graph*** if there is some way to draw it in a 2D plane without any of the edges crossing.

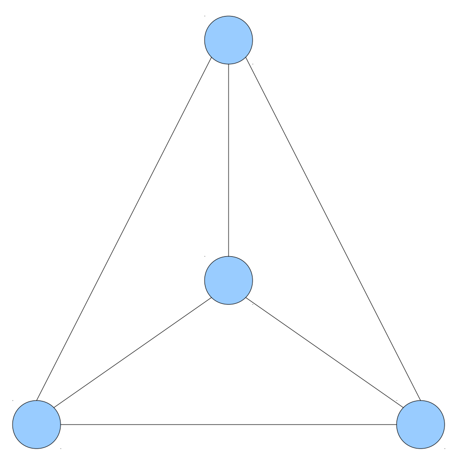


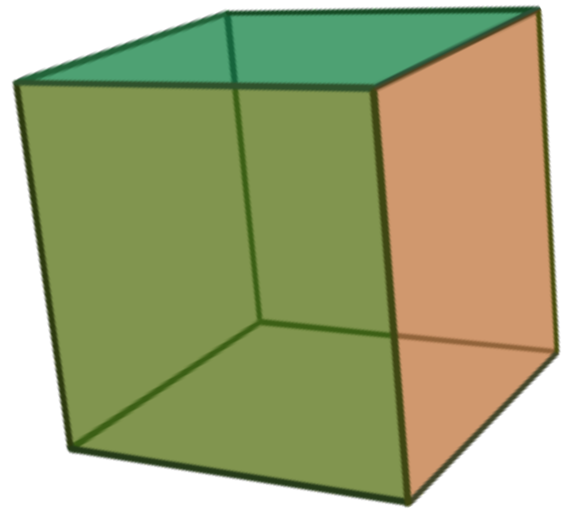


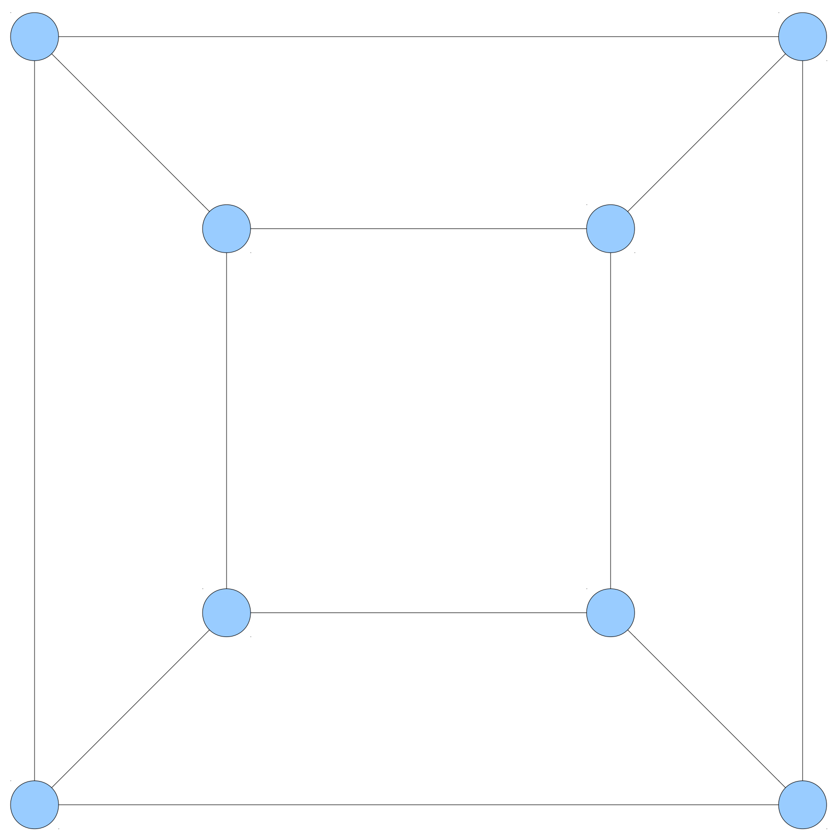


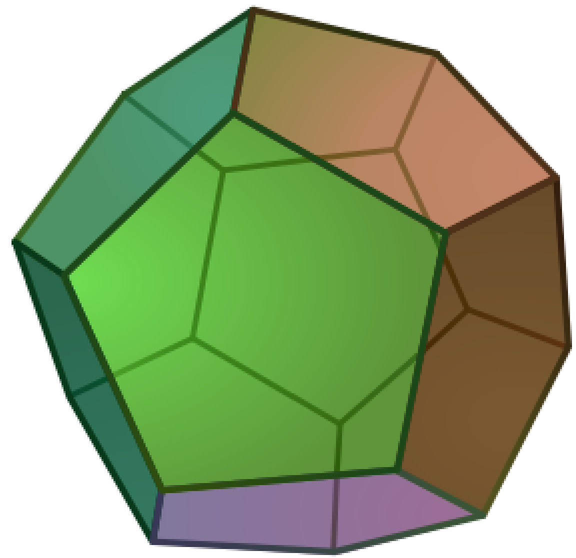


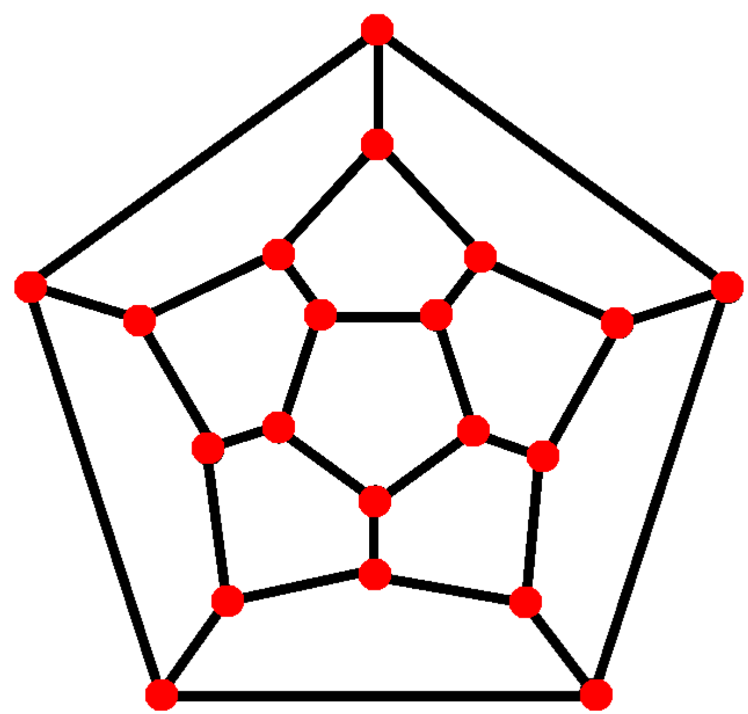




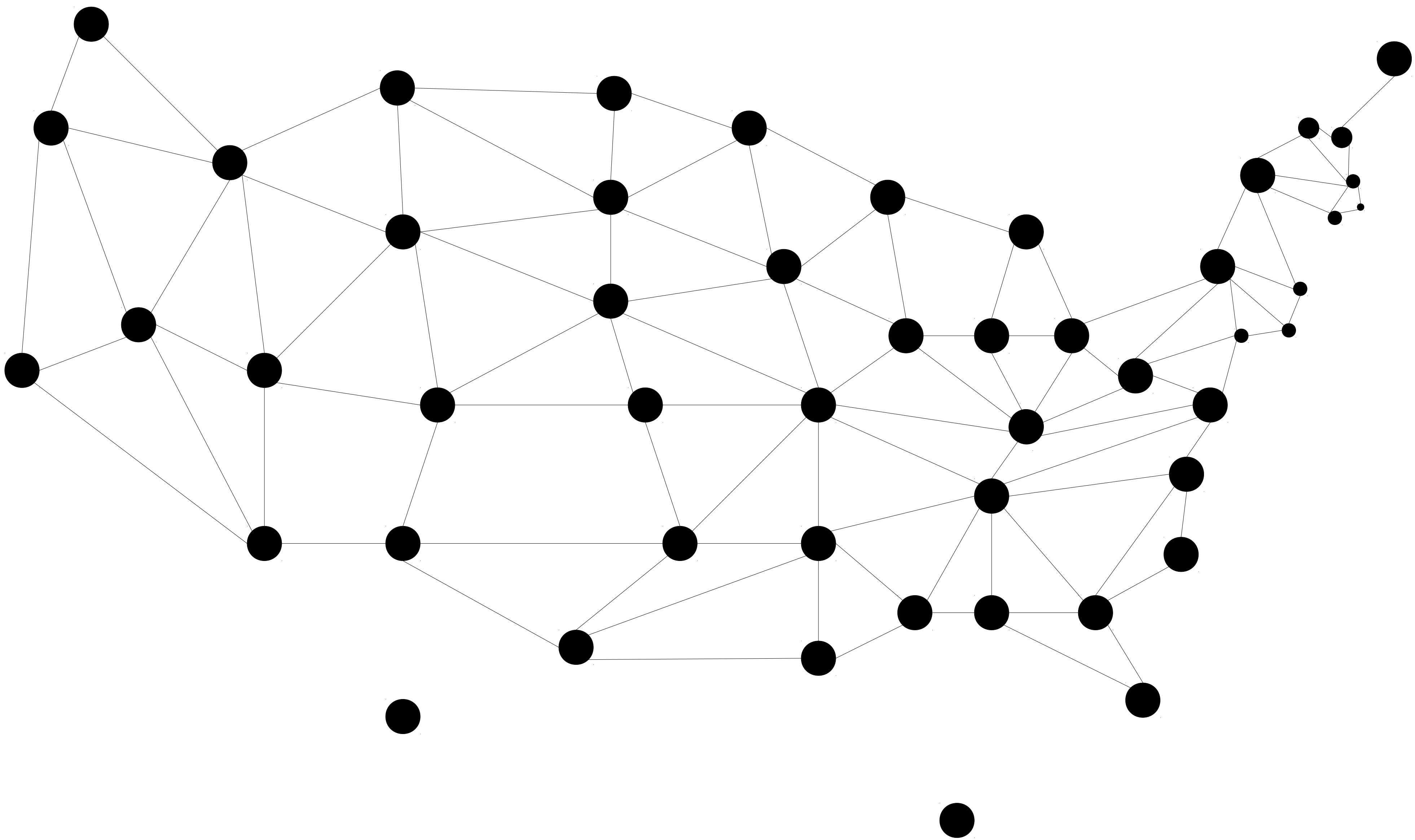








# Graph Coloring



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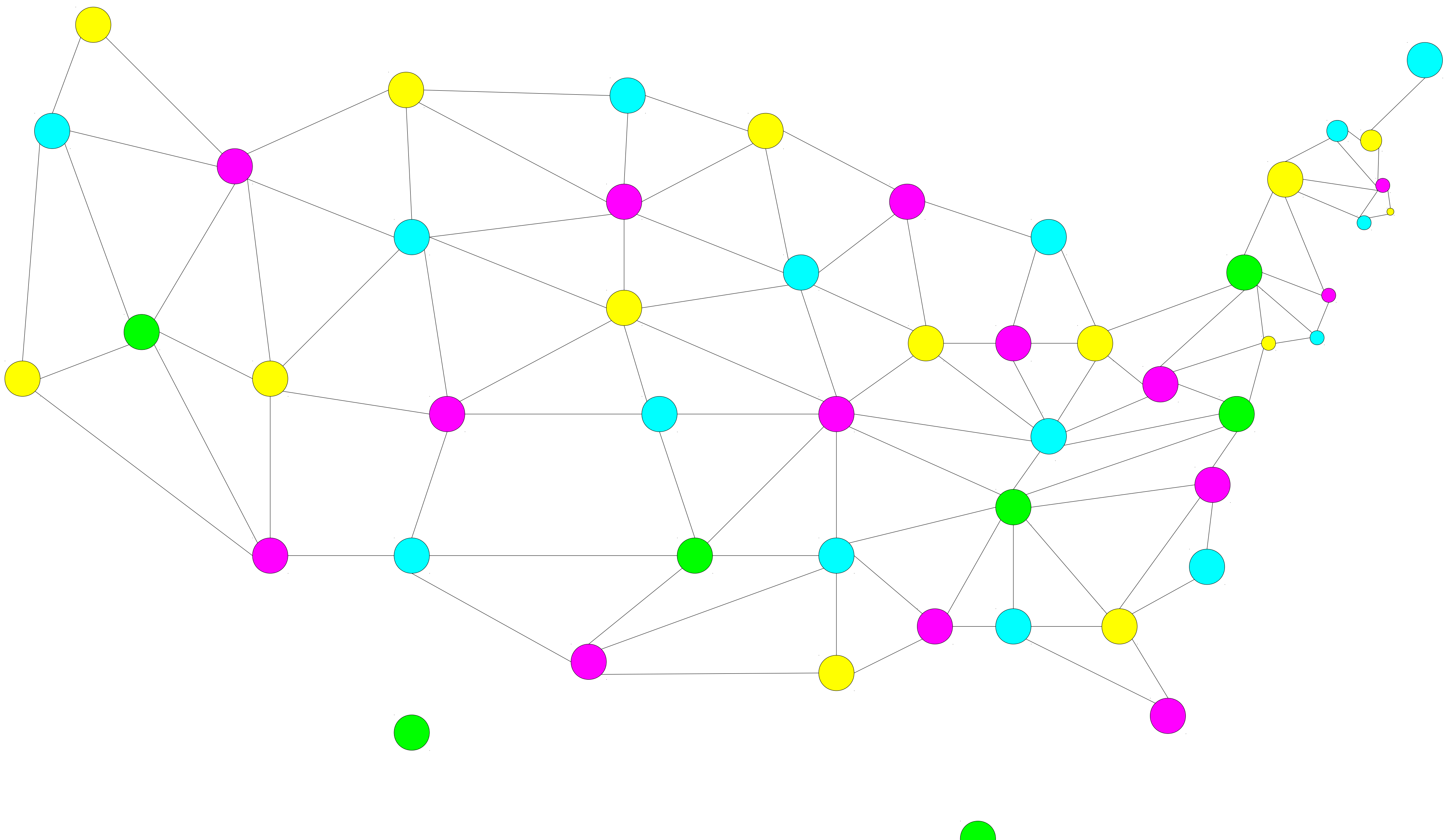

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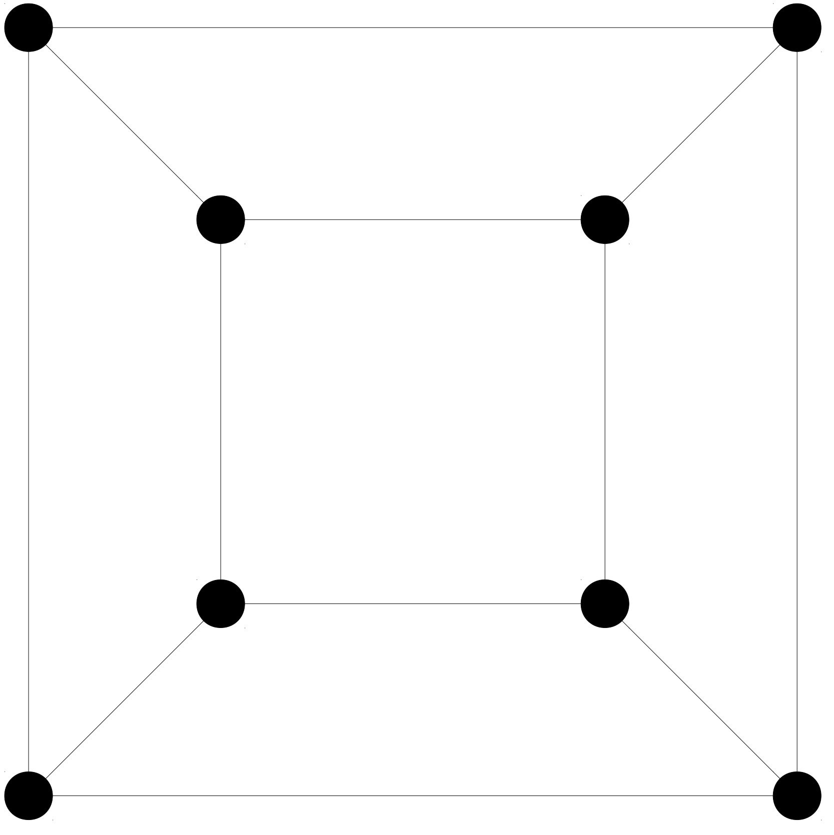

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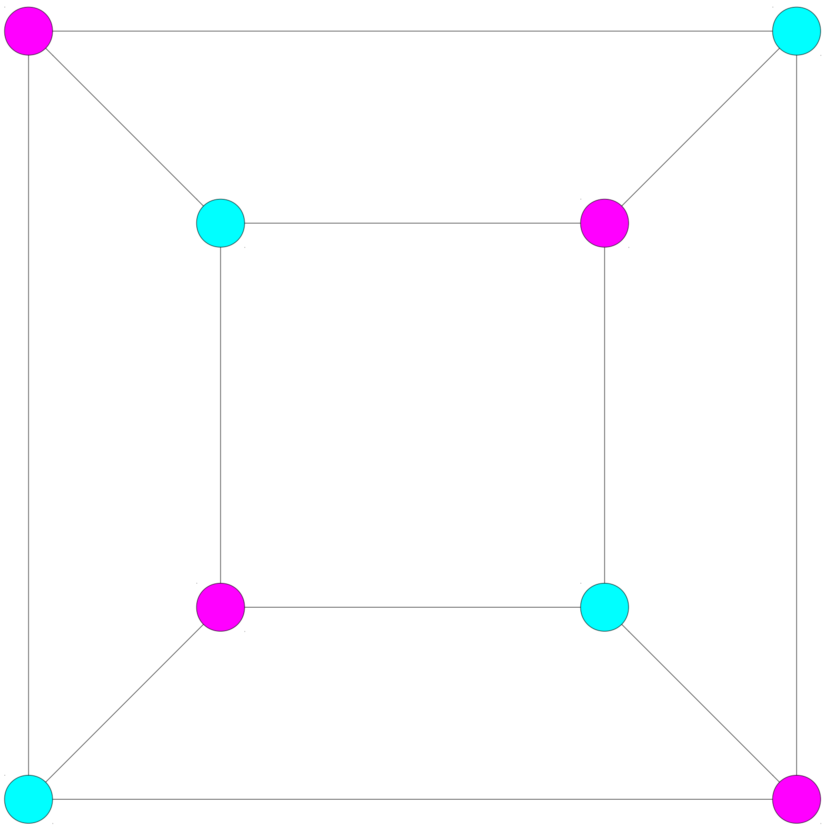

# Graph Coloring

- An undirected graph $G = (V, E)$ with no self-loops (edges from a node to itself) is called ***k-colorable*** iff the nodes in $V$ can be assigned one of $k$ different colors such that no two nodes of the same color are joined by an edge.
- The minimum number of colors needed to color a graph is called that graph's ***chromatic number***.

***Theorem (Four-Color Theorem):*** Every planar graph is 4-colorable.

- **1850s:** Four-Color Conjecture posed.
- **1879:** Kempe proves the Four-Color Theorem.
- **1890:** Heawood finds a flaw in Kempe's proof.
- **1976:** Appel and Haken design a computer program that proves the Four-Color Theorem. The program checked 1,936 specific cases that are “minimal counterexamples;” any counterexample to the theorem must contain one of the 1,936 specific cases.
- **1980s:** Doubts rise about the validity of the proof due to errors in the software.
- **1989:** Appel and Haken revise their proof and show it is indeed correct. They publish a book including a 400-page appendix of all the cases to check.
- **1996:** Roberts, Sanders, Seymour, and Thomas reduce the number of cases to check down to 633.
- **2005:** Werner and Gonthier repeat the proof using an established automatic theorem prover (Coq), improving confidence in the truth of the theorem.

# Next Time

- **Propositional Logic**
  - How do we formalize mathematical reasoning?
- **(ITA) First-Order Logic, Part One**
  - How do we reason about collections of objects?