Constraint Satisfaction Problems (CSPs)

CS 221 Section – 10/30/15

Agenda

- Quick Review
- Problem Modeling
- N-ary Constraints
- Elimination Example
- Code Example

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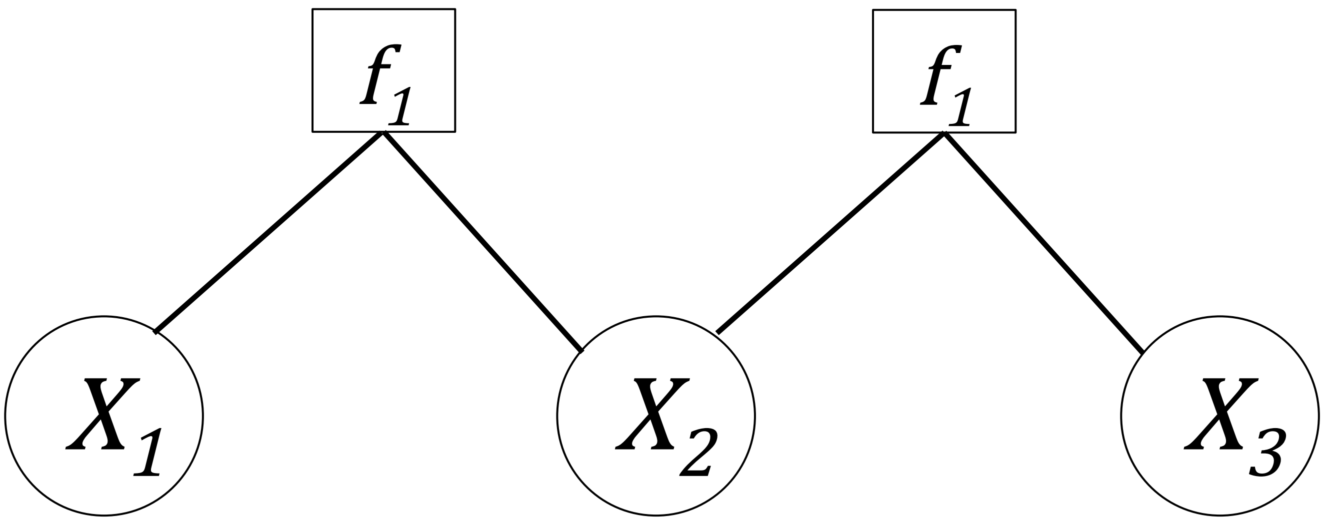
Definition: Factor Graph

Variables:

$X = (X_1,\dots,X_n)$, where $X_i \in Domain_i$

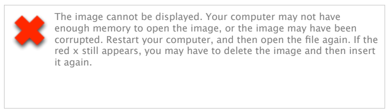
Factors:

$f_1,\dots,f_m,$ with each $f_j(X) \geq 0$



Definition: Constraint Satisfaction Problem (CSP)

A CSP is a factor graph where all factors are **constraints**:



for all $j = 1, \ldots, m$.

The constraint is satisfied iff $f_j(x) = 1$.

Definition: Consistent Assignments

An assignment x if $Weight(x) = 1$ (i.e., all constraints are satisfied.)

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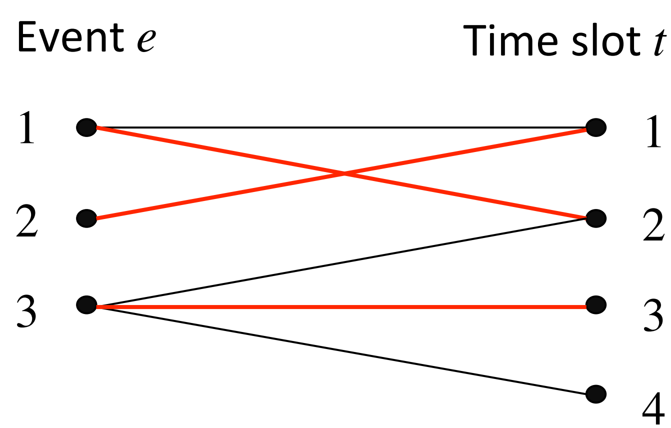
Event Scheduling

Setup:

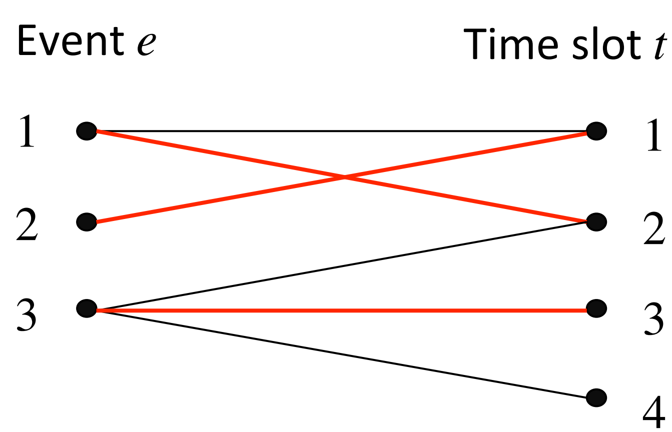
- Have E events and T time slots
- Each event e must be put in **exactly one** time slot
- Each time slot t can have **at most one** event
- Event e only allowed at time slot e if (e, t) in A



Event Scheduling

Formulation 1a:

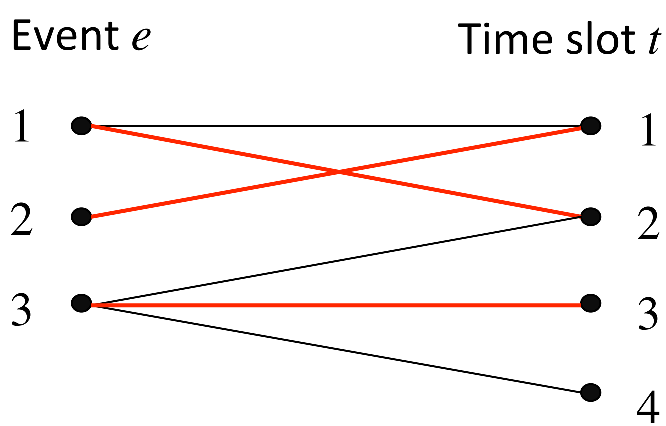
- Variables for each event e, $X_e \in \{1, \ldots, T\}$



Event Scheduling

Formulation 1a:

- Variables for each event e, $X_e \in \{1,\ldots,T\}$
- Constraints (only one event per time slot): for each pair of events $e \neq e'$, enforce $[X_e \neq X_{e'}]$



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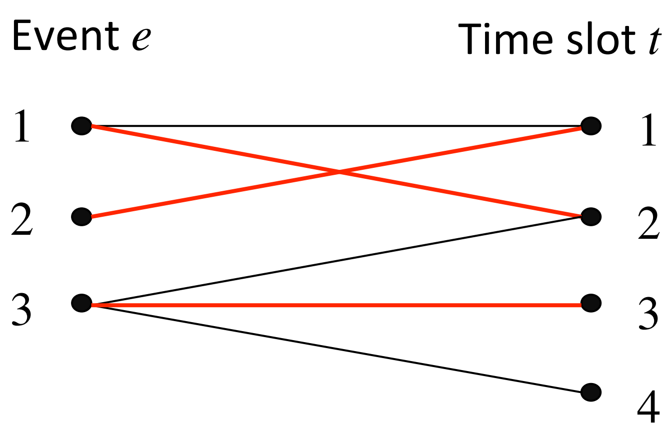

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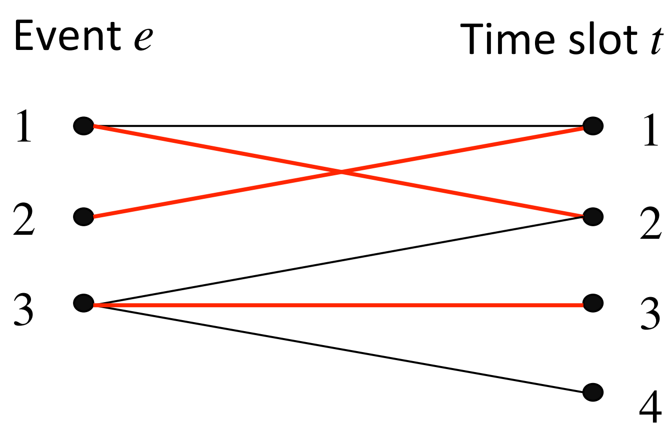

$\{X_1 : 1, X_2 : 1, X_3 : 3\}$

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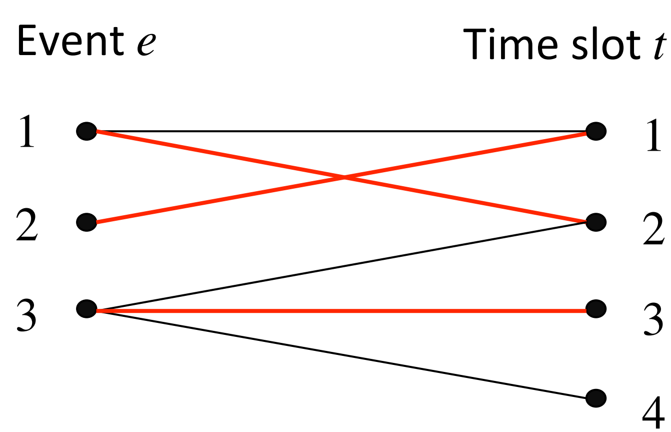

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Bad! ($X_1 = X_2$)

Event Scheduling

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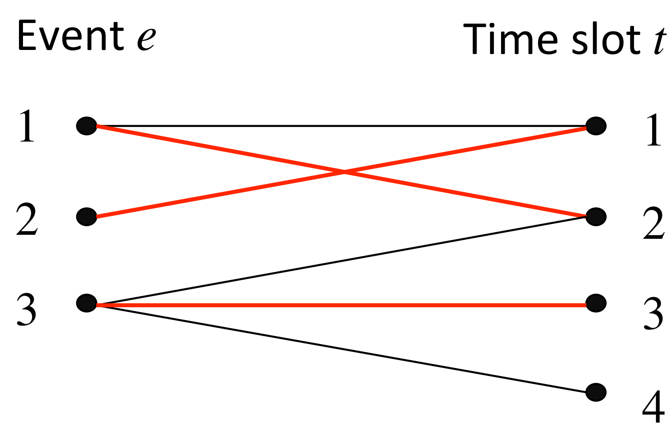

$\{X_1 : 1, X_2 : 1, X_3 : 3\}$

Bad! ($X_1 = X_2$)

$\{X_1 : 1, X_2 : 4, X_3 : 3\}$

Event Scheduling

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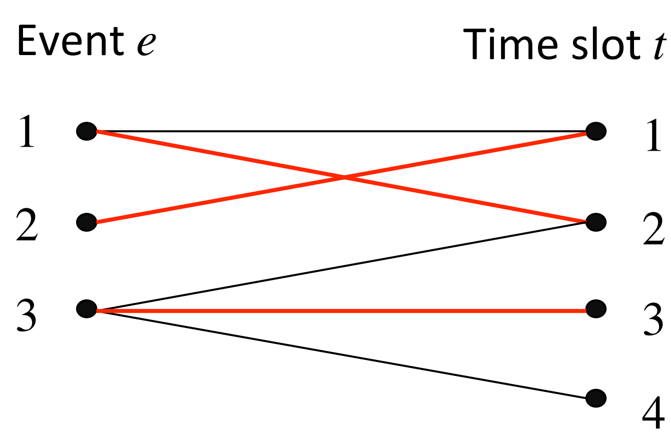

$\{X_1 : 1, X_2 : 1, X_3 : 3\}$

Bad! ($X_1 = X_2$)

$\{X_1 : 1, X_2 : 4, X_3 : 3\}$

Bad! ($(2, 4)$ not in A)

Event Scheduling

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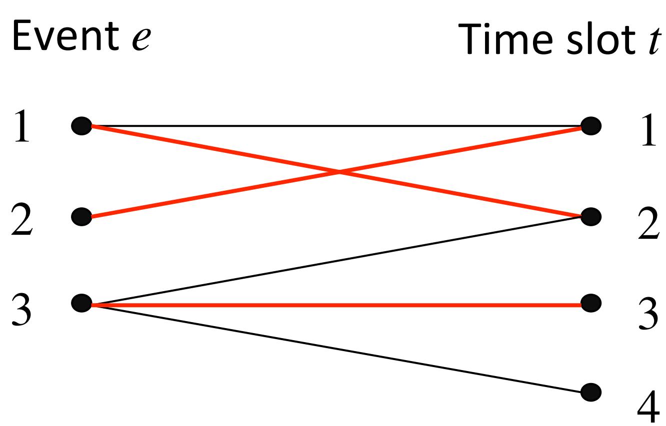
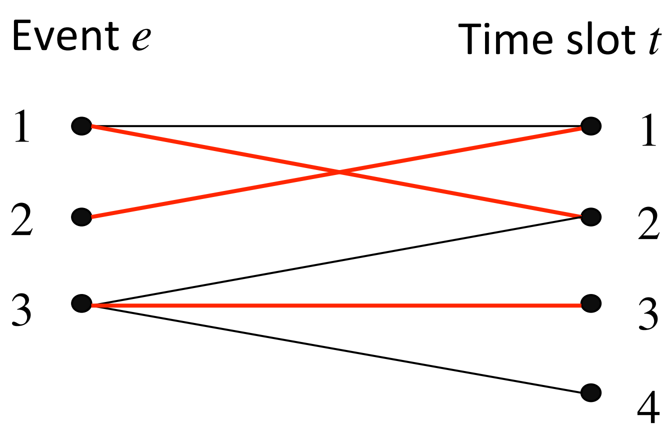


Event Scheduling

Formulation 1b:

- Variables for each event e, $X_1,\dots,X_E$

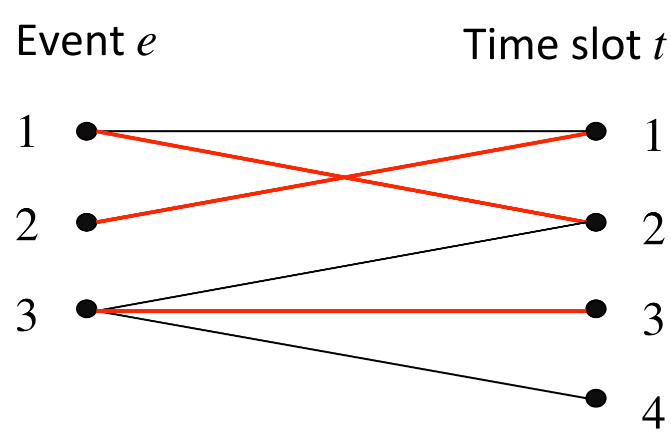


Event Scheduling

Formulation 1b:

- Variables for each event e, $X_1, \ldots, X_E$

$$Domain_i = \{t : (i,t) \in A\}$$



Event Scheduling

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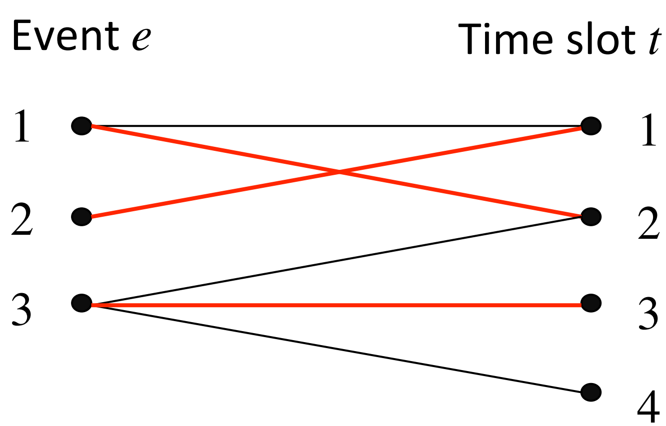
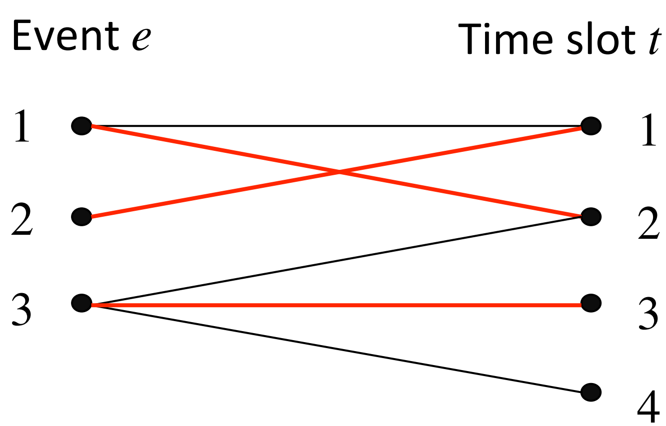


Event Scheduling

Formulation 2a:

- Variables for each time slot t: $Y_t \in \{1,\ldots,E\} \cup \{\varnothing\}$



Event Scheduling

Formulation 2a:

- Variables for each time slot t: $Y_t \in \{1,\ldots,E\} \cup \{\varnothing\}$
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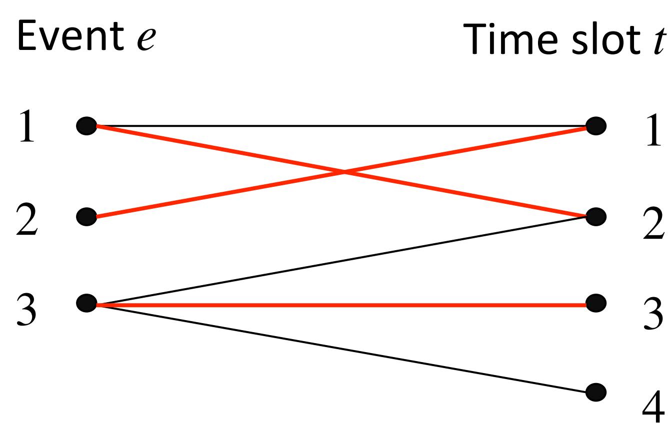

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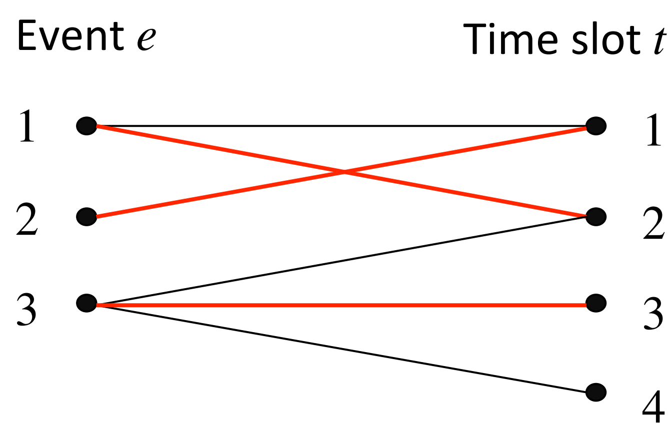

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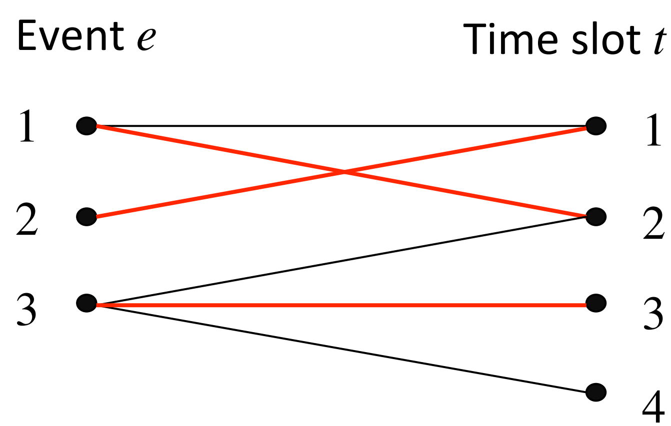

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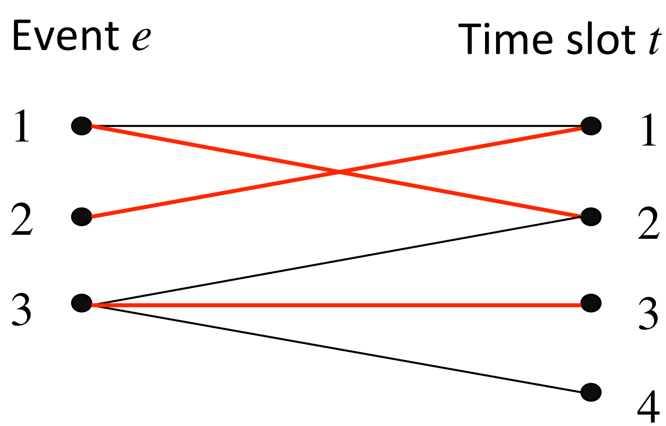
- Variables for each time slot t: $Y_1,\ldots,Y_T$



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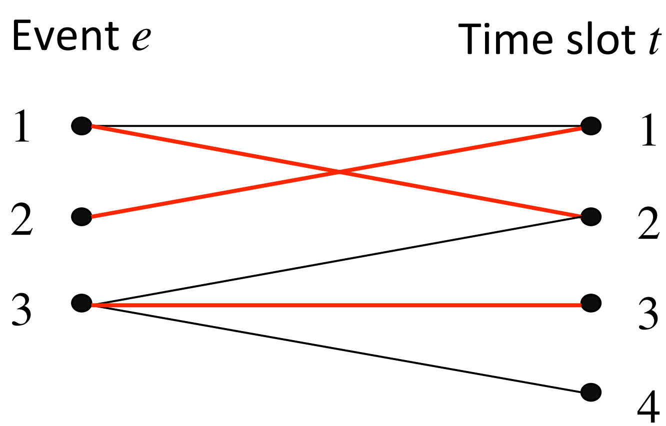

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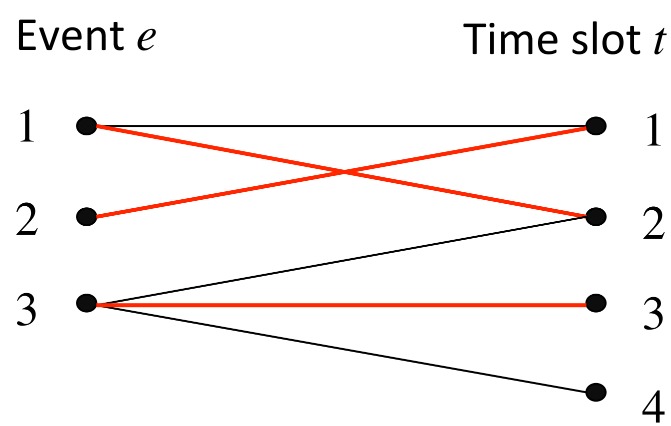

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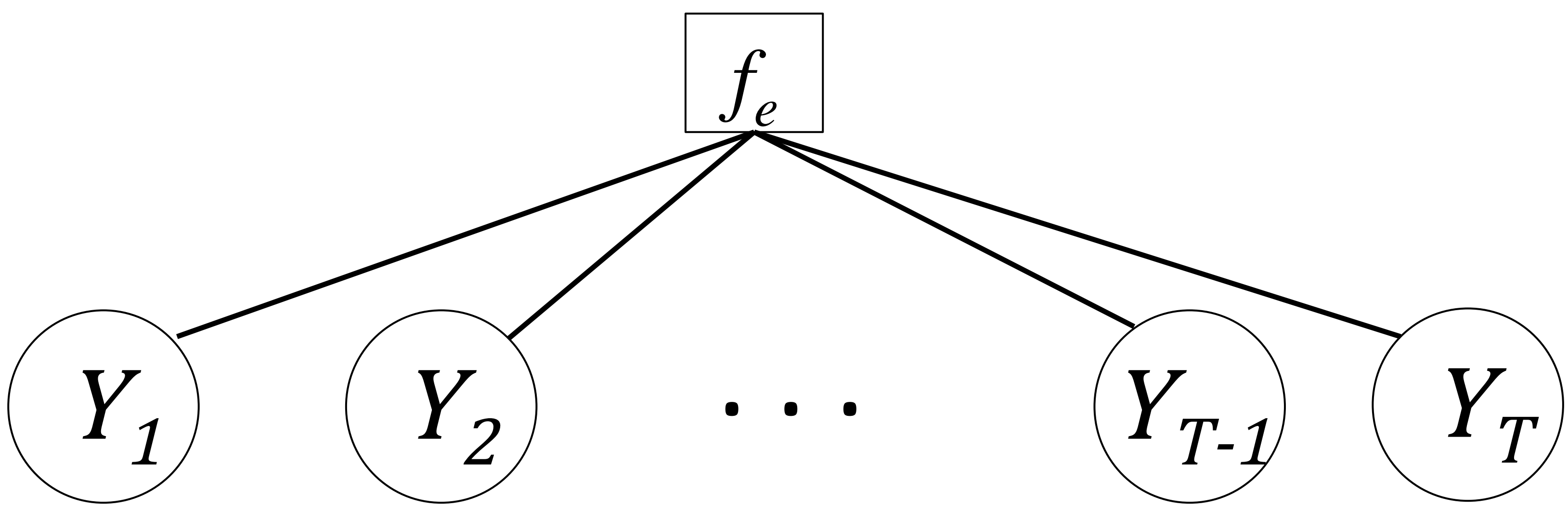
N-ary Constraints

- From event scheduling:
  - Constraints (each event is scheduled exactly once): for each event e, enforce

$$[Y_t = e \text{ for exactly one } t]$$



N-ary Constraints

Key Idea: Auxiliary Variables

Auxiliary Variables hold intermediate computation.

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Factors:

Initialization: $[A_0 = 0]$

$\boldsymbol{i}$	**0**	**1**	**2**	**3**	**4**
Y_i		3	1	2	1
A_i	0				

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Factors:

Initialization: $[A_0 = 0]$

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Factors:

- Initialization: $[A_0 = 0]$
- Processing: $[A_i = \min(A_{i-1} + 1[Y_i = e], 2)]$
- Final Output: $[A_T = 1]$

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i	**0**	**1**	**2**	**3**	**4**
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Still have factors with three variables…

N-ary Constraints

Key idea: Combine A_{i-1} and A_i into one variable B_i

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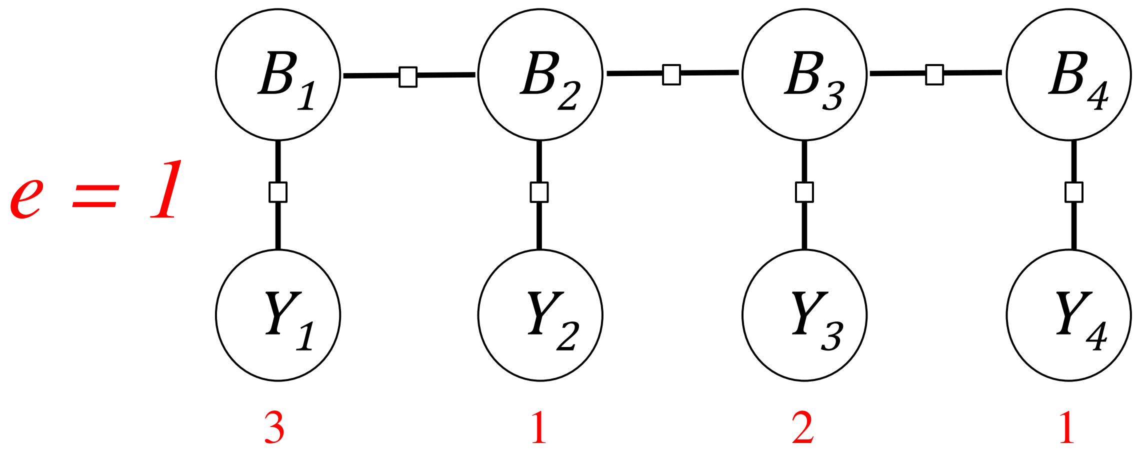

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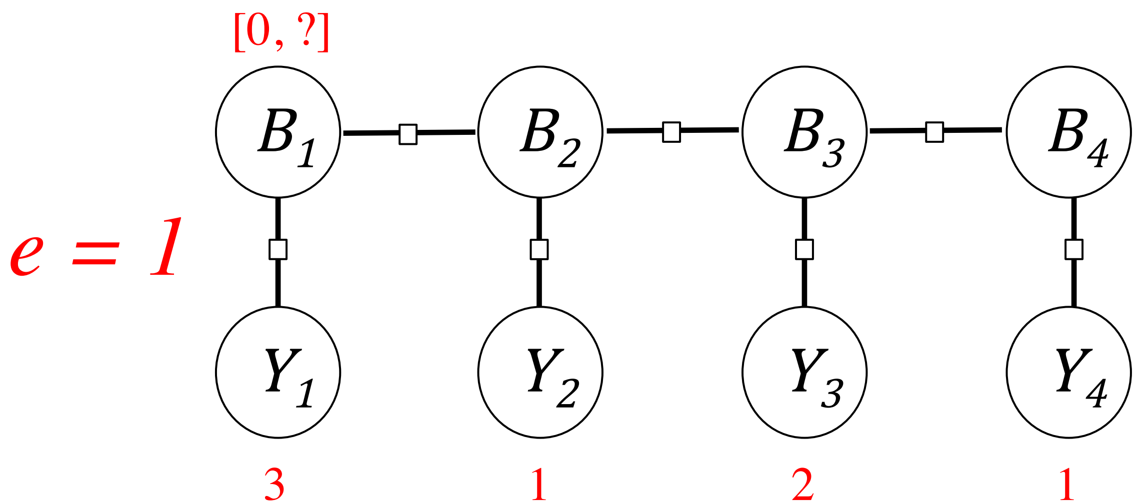


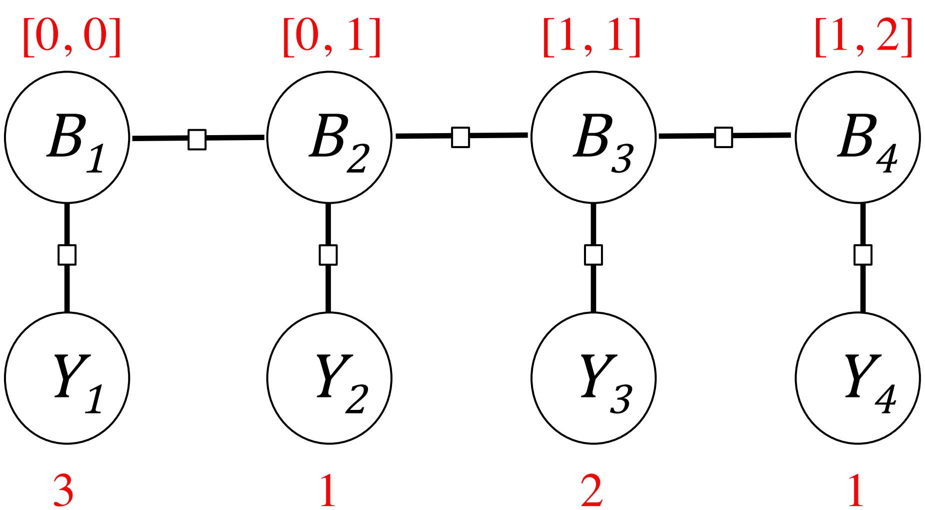
Factors:

Initialization: $[B_1[0] = 0]$

N-ary Constraints

Key idea: Combine A_{i-1} and A_i into one variable B_i

$e = 1$



Factors:

Initialization: $[B_1[0] = 0]$

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N-ary Constraints

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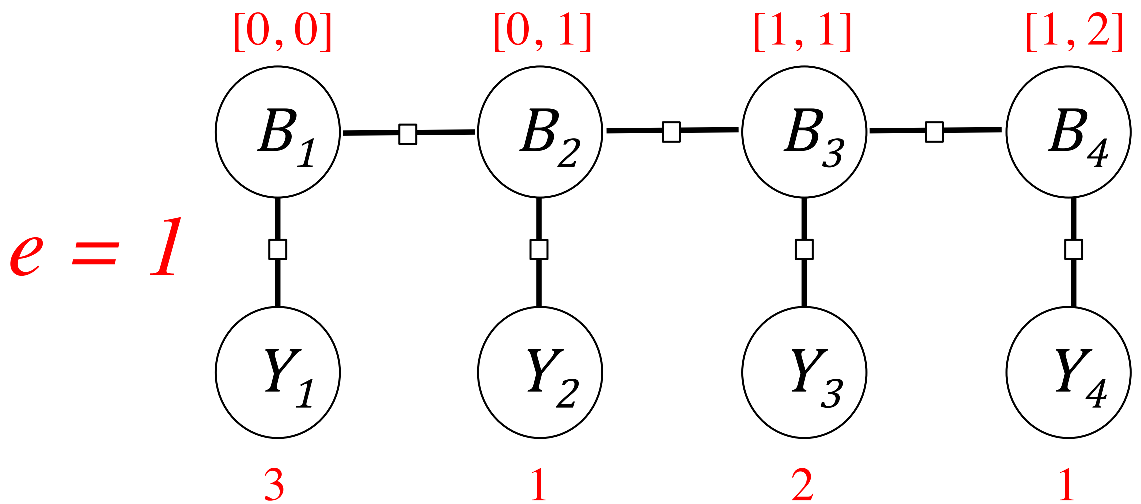

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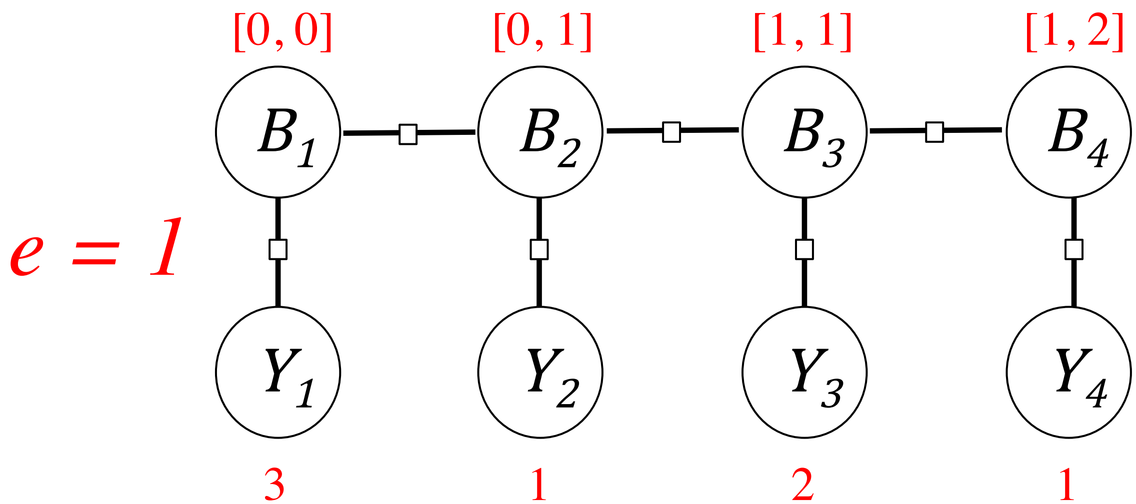

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Final Output: $[B_T[1] = 1]$

Consistency: $[B_{i-1}[1] = B_i[0]]$

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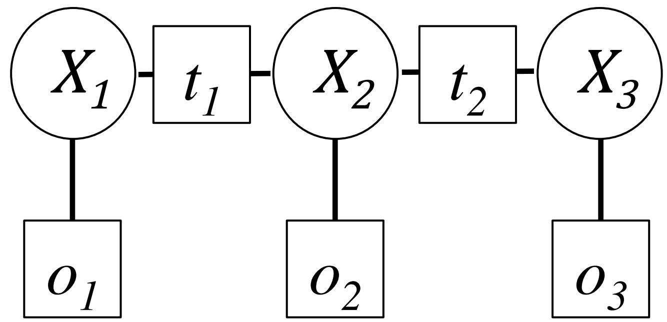
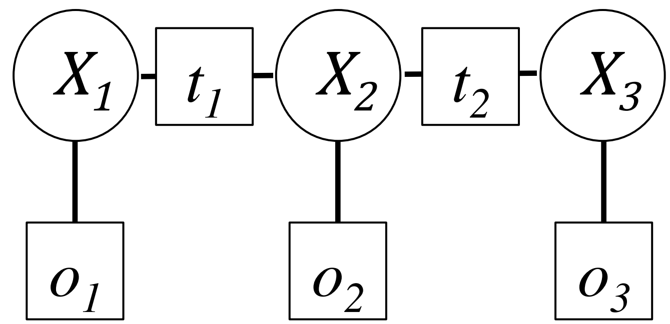
Person Tracking Example



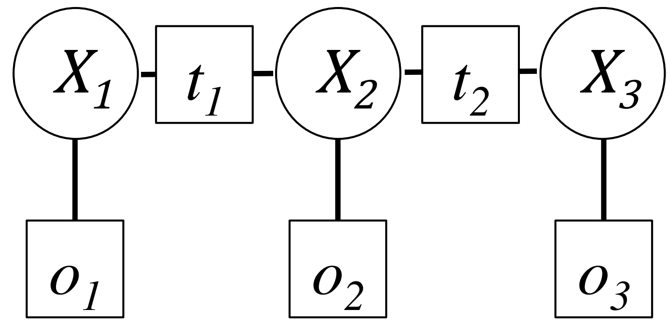
- Variables X_i: Location of object at position i

Person Tracking Example



- Variables X_i: Location of object at position i
- Transition Factors $t_i(x_i, x_{i+1})$: object positions can't change too much

Person Tracking Example



- Variables X_i: Location of object at position i
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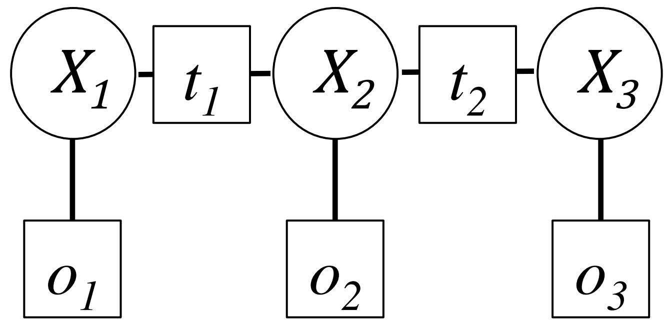
- Observation Factors $o_i(x_i)$: noisy information compatible with position

Person Tracking Example



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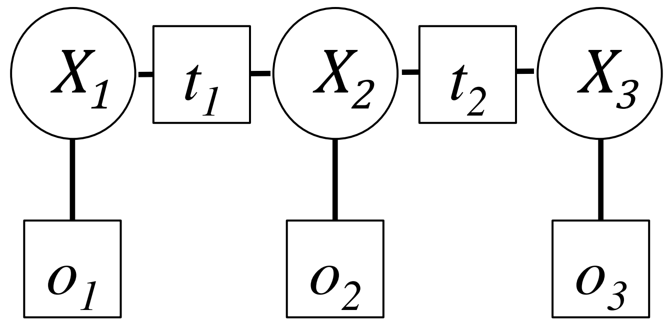
```
def t(x, y):
    if x == y: return 2
    if abs(x - y) == 1: return 1
    return 0
```

Person Tracking Example



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    return 0
```

```
def o1(x): return t(x, 0)
def o2(x): return t(x, 2)
def o3(x): return t(x, 2)
```

Variable Elimination

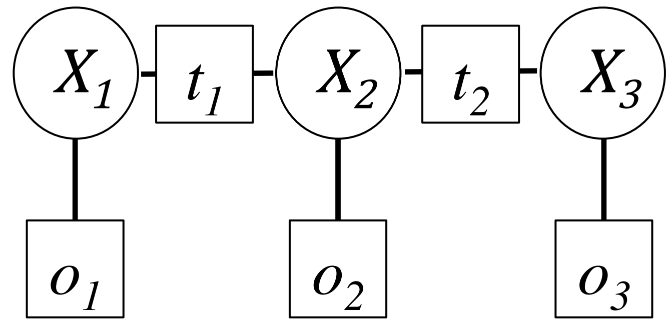
Backtracking search is bad because it is exponential. We have to consider every possible assignment of values.

But since most variables don't immediately depend on other variables, what if we locally maximized each variable?

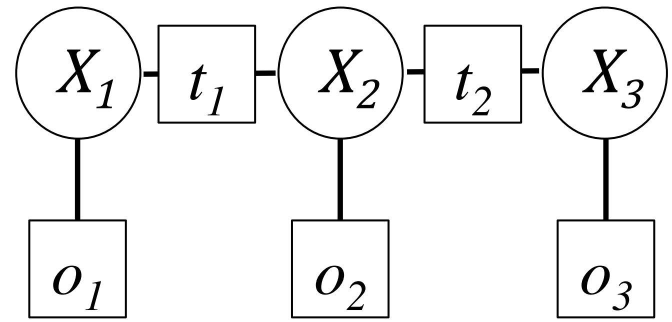
Definition: Elimination

- To **eliminate** a variable X_i, consider all factors $f_1, \ldots, f_k$, that depend on X_i
- Remove X_i and $f_1, \ldots, f_k$
- Add $f_{new}(x) = \max_{x_i} \prod_{j=1}^{k} f_j(x)$

- Eliminate X_1



- Eliminate X_1
- Factors that depend on X_1:
  - o_1, t_1



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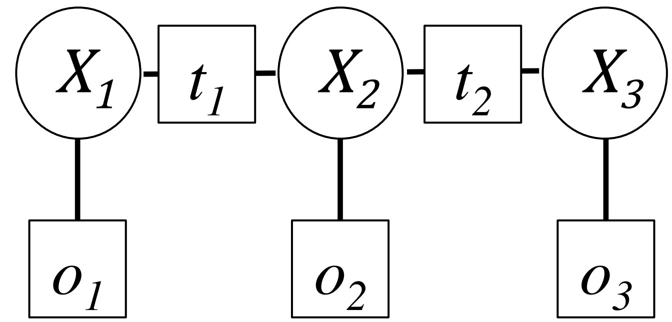

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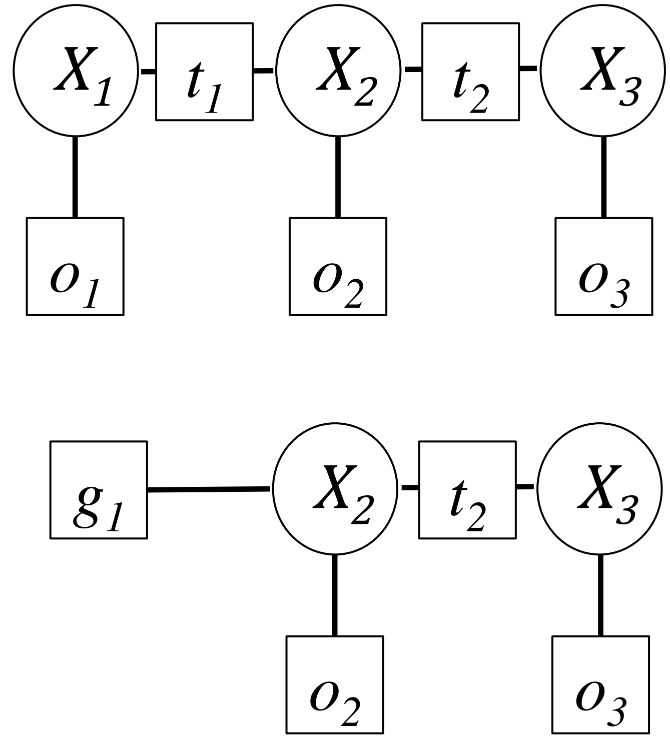
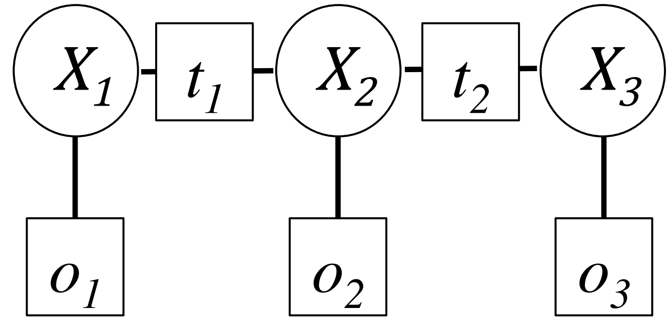
- $g_1(x_2) = \max_{x_1 \in \{0,1,2\}} o_1(x_1) \cdot t_1(x_1, x_2)$

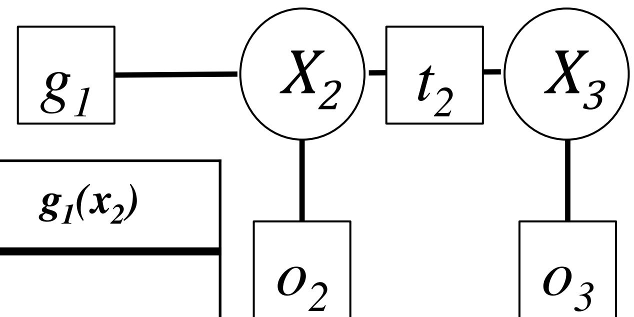


- Eliminate X_1
- Factors that depend on X_1:
  - o_1, t_1



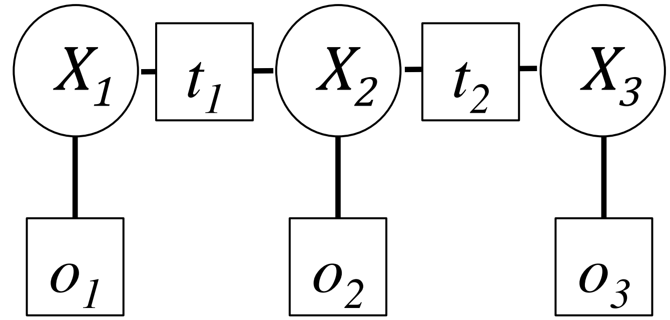
- Add $f_{new}(x) = \max_{x_i} \prod_{j=1}^{k} f_j(x)$

- $g_1(x_2) = \max_{x_1 \in \{0,1,2\}} o_1(x_1) \cdot t_1(x_1, x_2)$



x_2	x_1	$o_1(x_1)$	$t_1(x_1, x_2)$	$o_1(x_1)\ t_1(x_1, x_2)$	$g_1(x_2)$
0	0				
0	1				
0	2				
1	0				
1	1				
1	2				
2	0				
2	1				
2	2				

- Eliminate X_1
- Factors that depend on X_1:
  - o_1, t_1



- Add $f_{new}(x) = \max_{x_i} \prod_{j=1}^{k} f_j(x)$

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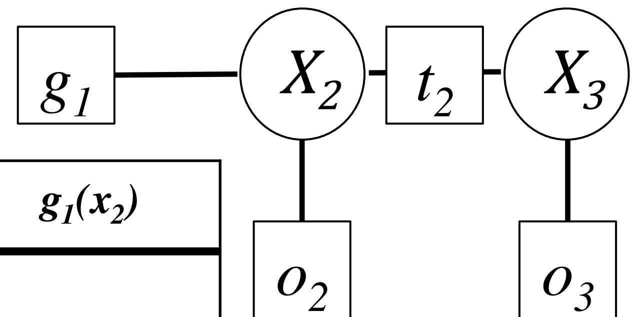

x_2	x_1	$o_1(x_1)$	$t_1(x_1, x_2)$	$o_1(x_1)\ t_1(x_1, x_2)$	$g_1(x_2)$
0	0	2			
0	1	1			
0	2	0			
1	0	2			
1	1	1			
1	2	0			
2	0	2			
2	1	1			
2	2	0			

- Eliminate X_1
- Factors that depend on X_1:
  - o_1, t_1
- Add $f_{new}(x) = \max_{x_i} \prod_{j=1}^{k} f_j(x)$
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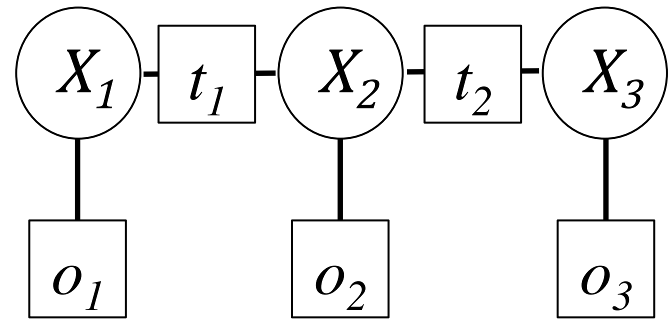
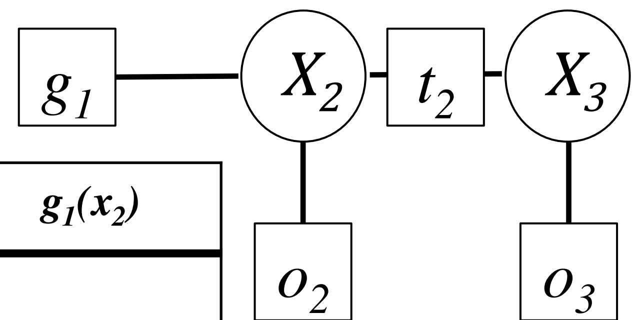
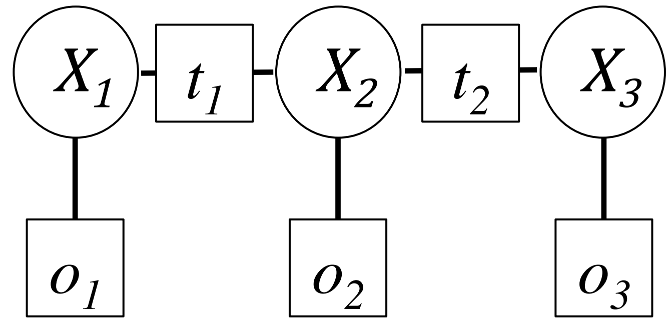




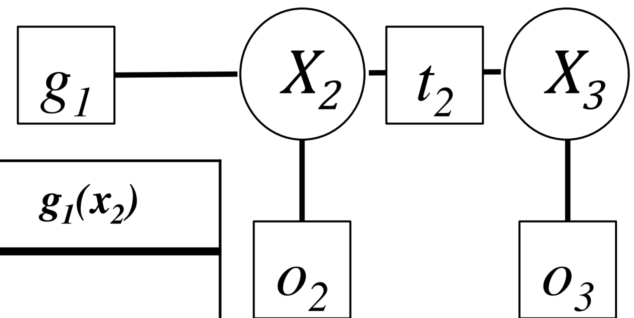
x_2	x_1	$o_1(x_1)$	$t_1(x_1, x_2)$	$o_1(x_1)\ t_1(x_1, x_2)$	$g_1(x_2)$
0	0	2	2		
0	1	1	1		
0	2	0	0		
1	0	2	1		
1	1	1	2		
1	2	0	1		
2	0	2	0		
2	1	1	1		
2	2	0	2		

- Eliminate X_1
- Factors that depend on X_1:
  - o_1, t_1



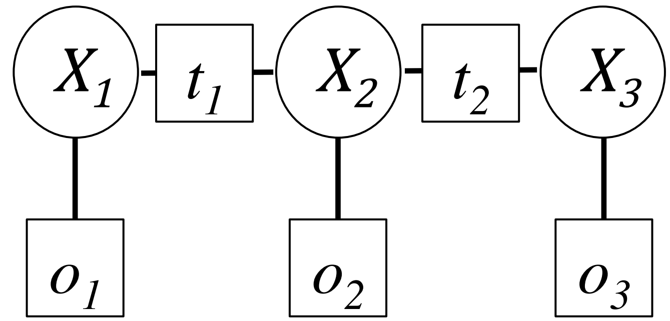
- Add $f_{new}(x) = \max_{x_i} \prod_{j=1}^{k} f_j(x)$

- $g_1(x_2) = \max_{x_1 \in \{0,1,2\}} o_1(x_1) \cdot t_1(x_1, x_2)$



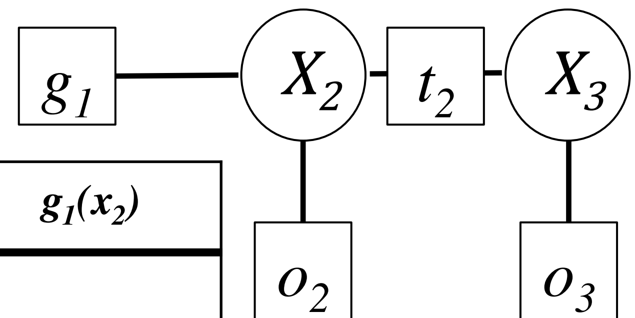
x_2	x_1	$o_1(x_1)$	$t_1(x_1, x_2)$	$o_1(x_1)\ t_1(x_1, x_2)$	$g_1(x_2)$
0	0	2	2	4	
0	1	1	1	1	
0	2	0	0	0	
1	0	2	1	2	
1	1	1	2	2	
1	2	0	1	0	
2	0	2	0	0	
2	1	1	1	1	
2	2	0	2	0	

- Eliminate X_1
- Factors that depend on X_1:
  - o_1, t_1



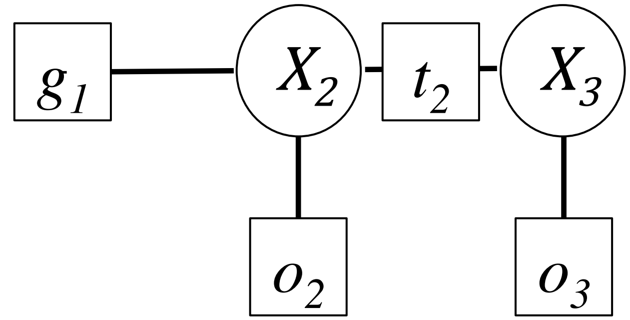
- Add $f_{new}(x) = \max_{x_i} \prod_{j=1}^{k} f_j(x)$

- $g_1(x_2) = \max_{x_1 \in \{0,1,2\}} o_1(x_1) \cdot t_1(x_1, x_2)$

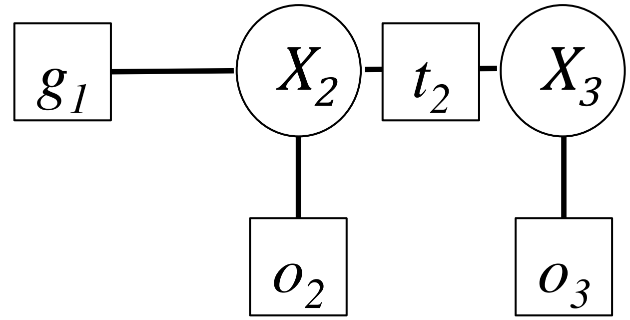


x_2	x_1	$o_1(x_1)$	$t_1(x_1, x_2)$	$o_1(x_1)\ t_1(x_1, x_2)$	$g_1(x_2)$
0	0	2	2	4	4: $\{x_1{:}0\}$
0	1	1	1	1	
0	2	0	0	0	
1	0	2	1	2	2: $\{x_1{:}\ 1\}$
1	1	1	2	2	
1	2	0	1	0	
2	0	2	0	0	1: $\{x_1{:}\ 1\}$
2	1	1	1	1	
2	2	0	2	0	

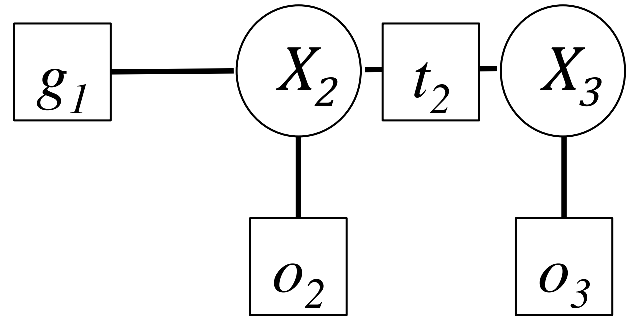
- Eliminate X_2



- Eliminate X_2
- Factors that depend on X_2:
    - o_2, t_2, g_1



- Eliminate X_2
- Factors that depend on X_2:
  - o_2, t_2, g_1
- Add $f_{new}(x) = \max_{x_i} \prod_{j=1}^{k} f_j(x)$



- Eliminate X_2
- Factors that depend on X_2:
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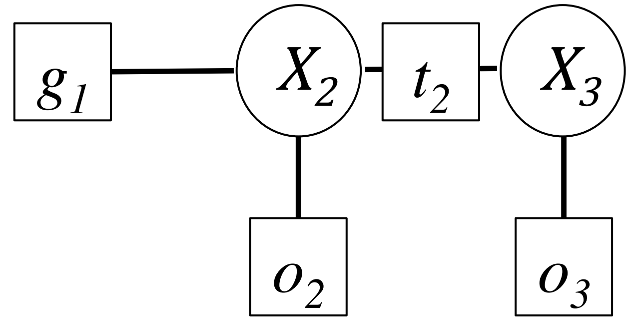
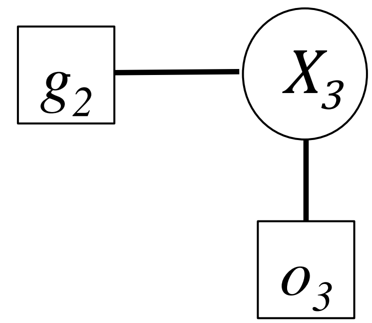
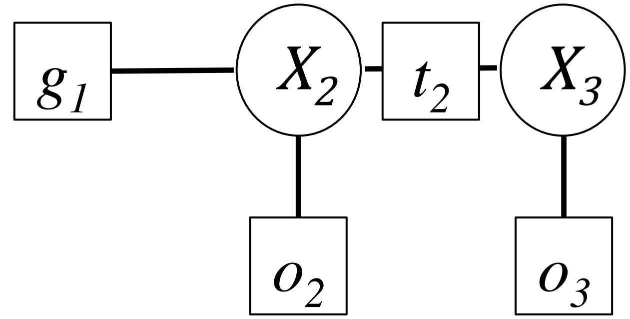
- Add $f_{new}(x) = \max_{x_i} \prod_{j=1}^{k} f_j(x)$
- $g_2(x_3) = \max_{x_2 \in \{0,1,2\}} g_1(x_2) \cdot o_2(x_2) \cdot t_2(x_2, x_3)$

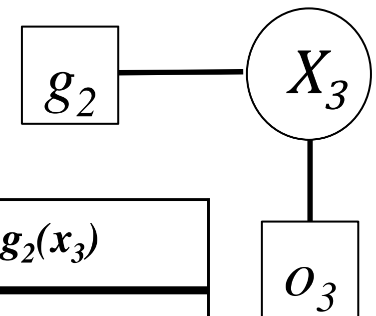




- Eliminate X_2
- Factors that depend on X_2:
  - o_2, t_2, g_1

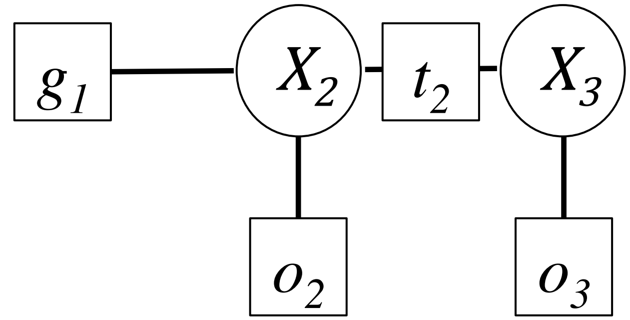


- Add $f_{new}(x) = \max_{x_i} \prod_{j=1}^{k} f_j(x)$

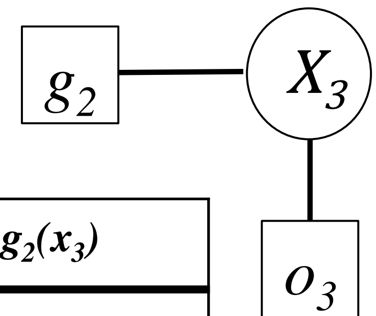


- $g_2(x_3) = \max_{x_2 \in \{0,1,2\}} g_1(x_2) \cdot o_2(x_2) \cdot t_2(x_2, x_3)$

x_3	x_2	$g_1(x_2)$	$o_2(x_2)$	$t_2(x_2, x_3)$	$g_1(x_2)\ o_2(x_2)\ t_2(x_2, x_3)$	$g_2(x_3)$
0	0					
0	1					
0	2					
1	0					
1	1					
1	2					
2	0					
2	1					
2	2					

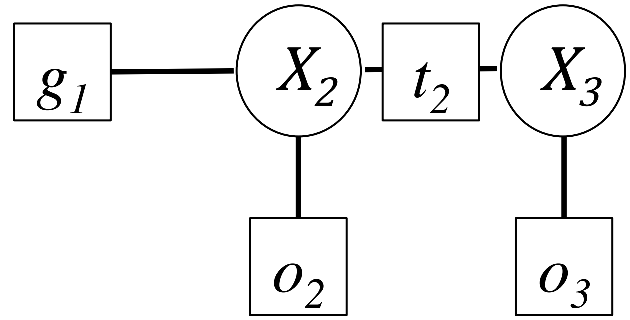


- Eliminate X_2
- Factors that depend on X_2:
  - o_2, t_2, g_1
- Add $f_{new}(x) = \max_{x_i} \prod_{j=1}^{k} f_j(x)$



- $g_2(x_3) = \max_{x_2 \in \{0,1,2\}} g_1(x_2) \cdot o_2(x_2) \cdot t_2(x_2, x_3)$

x_3	x_2	$g_1(x_2)$	$o_2(x_2)$	$t_2(x_2, x_3)$	$g_1(x_2)\ o_2(x_2)\ t_2(x_2, x_3)$	$g_2(x_3)$
0	0	4: $\{x_1{:}0\}$				
0	1	2: $\{x_1{:}\ 1\}$				
0	2	1: $\{x_1{:}\ 1\}$				
1	0	4: $\{x_1{:}0\}$				
1	1	2: $\{x_1{:}\ 1\}$				
1	2	1: $\{x_1{:}\ 1\}$				
2	0	4: $\{x_1{:}0\}$				
2	1	2: $\{x_1{:}\ 1\}$				
2	2	1: $\{x_1{:}\ 1\}$				



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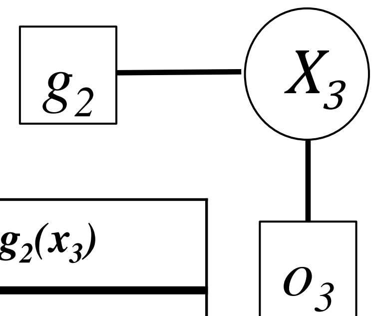
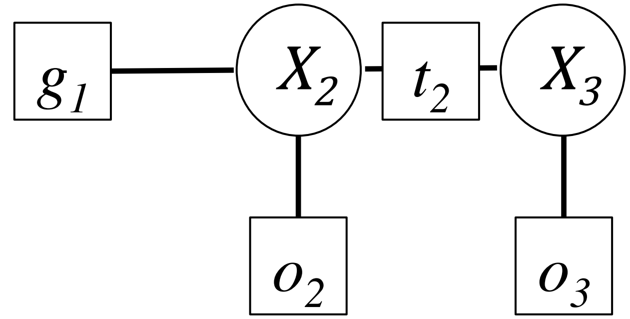


- $g_2(x_3) = \max_{x_2 \in \{0,1,2\}} g_1(x_2) \cdot o_2(x_2) \cdot t_2(x_2, x_3)$

x_3	x_2	$g_1(x_2)$	$o_2(x_2)$	$t_2(x_2, x_3)$	$g_1(x_2)\ o_2(x_2)\ t_2(x_2, x_3)$	$g_2(x_3)$
0	0	4: $\{x_1: 0\}$	0			
0	1	2: $\{x_1: 1\}$	1			
0	2	1: $\{x_1: 1\}$	2			
1	0	4: $\{x_1: 0\}$	0			
1	1	2: $\{x_1: 1\}$	1			
1	2	1: $\{x_1: 1\}$	2			
2	0	4: $\{x_1: 0\}$	0			
2	1	2: $\{x_1: 1\}$	1			
2	2	1: $\{x_1: 1\}$	2			



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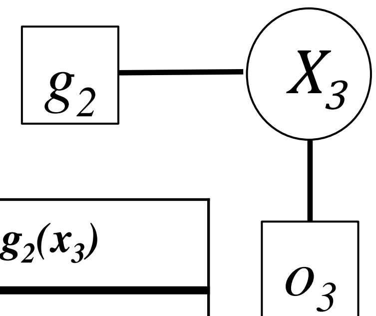
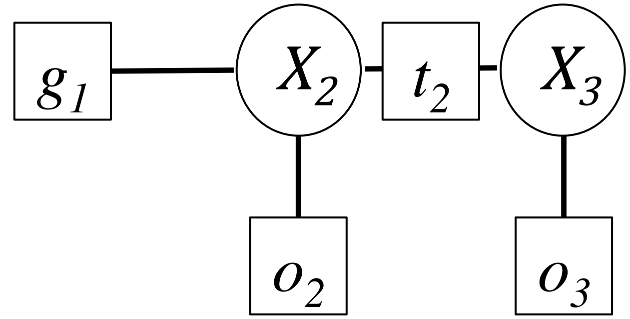
- Add $f_{new}(x) = \max_{x_i} \prod_{j=1}^{k} f_j(x)$



- $g_2(x_3) = \max_{x_2 \in \{0,1,2\}} g_1(x_2) \cdot o_2(x_2) \cdot t_2(x_2, x_3)$

x_3	x_2	$g_1(x_2)$	$o_2(x_2)$	$t_2(x_2, x_3)$	$g_1(x_2)\ o_2(x_2)\ t_2(x_2, x_3)$	$g_2(x_3)$
0	0	4: $\{x_1{:}0\}$	0	2		
0	1	2: $\{x_1{:}\ 1\}$	1	1		
0	2	1: $\{x_1{:}\ 1\}$	2	0		
1	0	4: $\{x_1{:}0\}$	0	1		
1	1	2: $\{x_1{:}\ 1\}$	1	2		
1	2	1: $\{x_1{:}\ 1\}$	2	1		
2	0	4: $\{x_1{:}0\}$	0	0		
2	1	2: $\{x_1{:}\ 1\}$	1	1		
2	2	1: $\{x_1{:}\ 1\}$	2	2		



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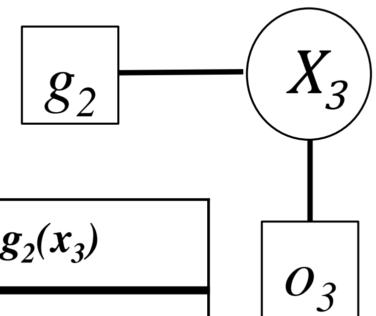
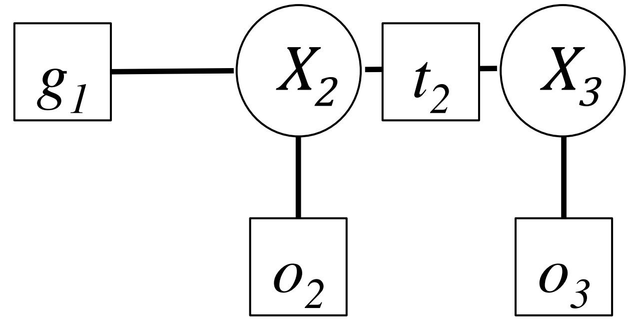
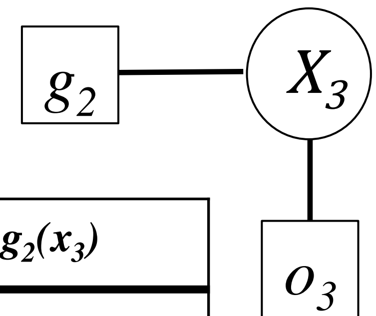


- $g_2(x_3) = \max_{x_2 \in \{0,1,2\}} g_1(x_2) \cdot o_2(x_2) \cdot t_2(x_2, x_3)$

x_3	x_2	$g_1(x_2)$	$o_2(x_2)$	$t_2(x_2, x_3)$	$g_1(x_2)\ o_2(x_2)\ t_2(x_2, x_3)$	$g_2(x_3)$
0	0	4: $\{x_1{:}0\}$	0	2	0	
0	1	2: $\{x_1{:}\ 1\}$	1	1	2	
0	2	1: $\{x_1{:}\ 1\}$	2	0	2	
1	0	4: $\{x_1{:}0\}$	0	1	4	
1	1	2: $\{x_1{:}\ 1\}$	1	2	4	
1	2	1: $\{x_1{:}\ 1\}$	2	1	2	
2	0	4: $\{x_1{:}0\}$	0	0	0	
2	1	2: $\{x_1{:}\ 1\}$	1	1	2	
2	2	1: $\{x_1{:}\ 1\}$	2	2	4	



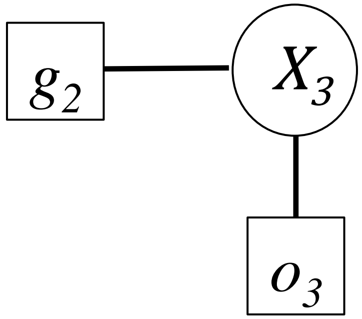
- Eliminate X_2
- Factors that depend on X_2:
  - o_2, t_2, g_1
- Add $f_{new}(x) = \max_{x_i} \prod_{j=1}^{k} f_j(x)$



- $g_2(x_3) = \max_{x_2 \in \{0,1,2\}} g_1(x_2) \cdot o_2(x_2) \cdot t_2(x_2, x_3)$

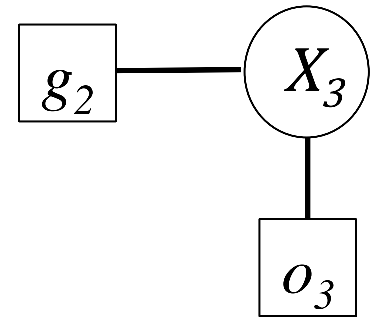
x_3	x_2	$g_1(x_2)$	$o_2(x_2)$	$t_2(x_2, x_3)$	$g_1(x_2)\ o_2(x_2)\ t_2(x_2, x_3)$	$g_2(x_3)$
0	0	4: $\{x_1{:}0\}$	0	2	0	2: $\{x_1 : 1, x_2 : 2\}$
0	1	2: $\{x_1{:}\ 1\}$	1	1	2	
0	2	1: $\{x_1{:}\ 1\}$	2	0	2	
1	0	4: $\{x_1{:}0\}$	0	1	4	4: $\{x_1 : 1, x_2 : 1\}$
1	1	2: $\{x_1{:}\ 1\}$	1	2	4	
1	2	1: $\{x_1{:}\ 1\}$	2	1	2	
2	0	4: $\{x_1{:}0\}$	0	0	0	4: $\{x_1 : 1, x_2 : 2\}$
2	1	2: $\{x_1{:}\ 1\}$	1	1	2	
2	2	1: $\{x_1{:}\ 1\}$	2	2	4	

- Maximum assignment is now:



$$\max_{x_3 \in \{0,1,2\}} g_2(x_3) \cdot o_3(x_3)$$

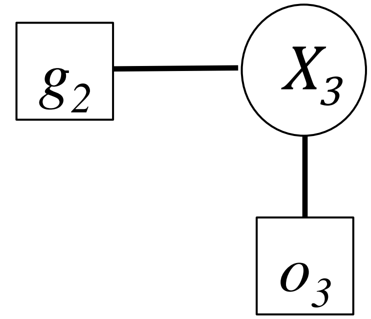
- Maximum assignment is now:



$$\max_{x_3 \in \{0,1,2\}} g_2(x_3) \cdot o_3(x_3)$$

x_3	$g_2(x_3)$	$o_3(x_3)$	$g_2(x_3)\ o_3(x_3)$	*Optimal Weight*
0				
1				
2				

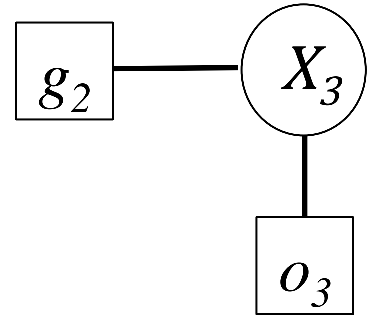
- Maximum assignment is now:



$$\max_{x_3 \in \{0,1,2\}} g_2(x_3) \cdot o_3(x_3)$$

x_3	$g_2(x_3)$	$o_3(x_3)$	$g_2(x_3)\ o_3(x_3)$	*Optimal Weight*
0	2: $\{x_1 : 1, x_2 : 2\}$	0		
1	4: $\{x_1 : 1, x_2 : 1\}$	1		
2	4: $\{x_1 : 1, x_2 : 2\}$	2		

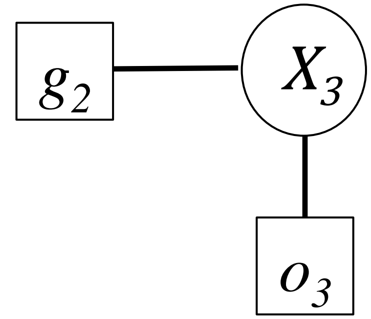
- Maximum assignment is now:



$$\max_{x_3 \in \{0,1,2\}} g_2(x_3) \cdot o_3(x_3)$$

x_3	$g_2(x_3)$	$o_3(x_3)$	$g_2(x_3)\ o_3(x_3)$	*Optimal Weight*
0	2: $\{x_1 : 1, x_2 : 2\}$	0	2	
1	4: $\{x_1 : 1, x_2 : 1\}$	1	4	
2	4: $\{x_1 : 1, x_2 : 2\}$	2	8	

- Maximum assignment is now:



$$\max_{x_3 \in \{0,1,2\}} g_2(x_3) \cdot o_3(x_3)$$

x_3	$g_2(x_3)$	$o_3(x_3)$	$g_2(x_3)\, o_3(x_3)$	*Optimal Weight*
0	2: $\{x_1 : 1, x_2 : 2\}$	0	2	8: $\{x_1 : 1, x_2 : 2, x_3 : 2\}$
1	4: $\{x_1 : 1, x_2 : 1\}$	1	4	
2	4: $\{x_1 : 1, x_2 : 2\}$	2	8	