CS154

Lecture 6: Streaming Algorithms and Communication Complexity

Streaming Algorithms

Streaming Algorithms

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L = {x | x has more 1's than 0's}

Initialize C := 0 and B := 0

Read the next bit x from the stream

If (C = 0) then B := x, C := 1

If ($C \neq 0$) and (B = x) then C := C + 1

If ($C \neq 0$) and ($B \neq x$) then C := C – 1

When the stream stops,

accept if B=1 and C > 0, else *reject*

B = the majority bit
C = how many more times that B appears

On all strings of length n, the algorithm uses ($1+\log_2 n$) bits of space *(to store B and C)*

Streaming Algorithms

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Can recognize non-regular languages

Streaming algorithms differ from DFAs in several significant ways:

1. Streaming algorithms can output more than one bit

2. The “memory” or “space” of a streaming algorithm can (slowly) increase as it reads longer strings

3. Could also make multiple passes over the data, could be randomized

DFAs and Streaming

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Theorem: **Suppose a language L can be recognized by a DFA with ≤ 2^p states. Then L is computable by a streaming algorithm A using ≤ p bits of space.**

Proof Idea: **Algorithm A stores the DFA's current state in memory, beginning with the start state. Alg. A makes decisions based on DFA transitions. When the string ends, A outputs *accept* if the DFA state is accepting, *reject* otherwise.**

DFAs and Streaming

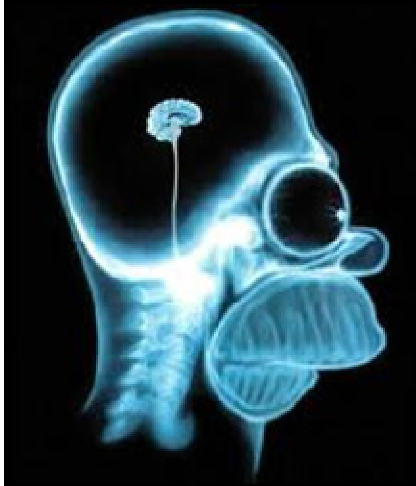
For any $L \subseteq \Sigma^*$ define $L_n = L \cap \Sigma^n$

Theorem: Suppose L is computable by a streaming algorithm A using $f(n)$ bits of space, on all strings of length n

Then for all n, L_n is recognized by a DFA with $\leq 2^{f(n)}$ states.

Proof Idea: The new DFA will have a state for each of the $2^{f(n)}$ possible configurations of A's memory. When A sees a symbol, its memory will update; the transition function of the DFA can simulate that.

L = {x | x has more 1's than 0's}



Is there a streaming algorithm for L using much *less than* ($\log_2 n$) space?

Theorem: Every streaming algorithm for L needs at least ($\log_2 n$)-1 bits of space

We will use:

- **Myhill-Nerode Theorem**
- **The connection between DFAs and streaming**

$L = \{x \mid x \text{ has more 1's than 0's}\}$

Theorem: Every streaming algorithm for L requires at least $(\log_2 n)-1$ bits of space

Proof Idea: Let n be even, and $L_n \subseteq \{0,1\}^n \cap L$

We will give a set S_n of $n/2+1$ strings such that each pair in S_n is *distinguishable* in L_n

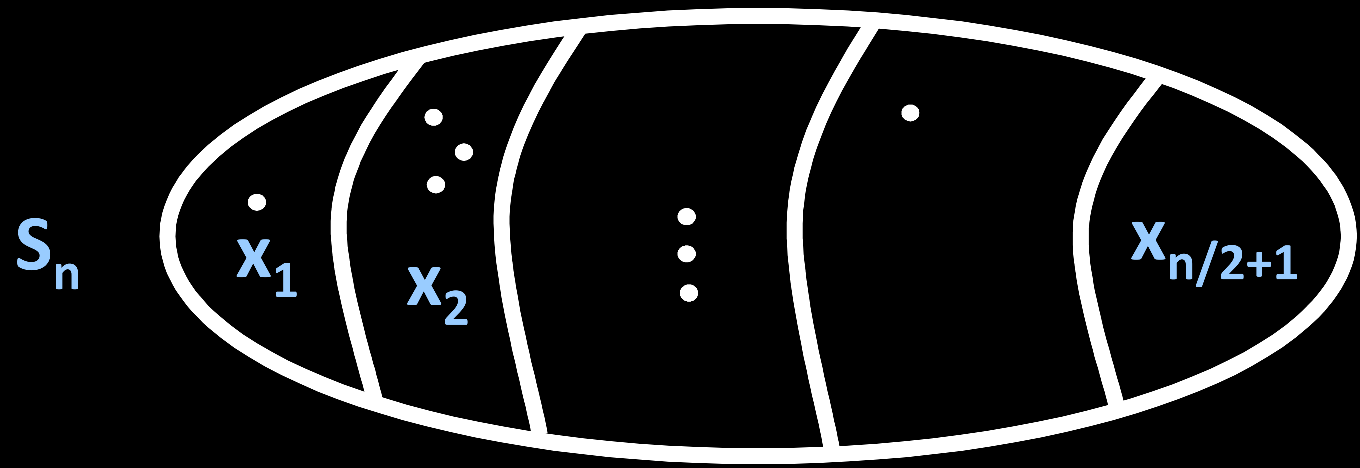
Myhill-Nerode $\Rightarrow$ Every DFA recognizing L_n needs at least $n/2+1$ states

$\Rightarrow$ Every streaming algorithm for L requires at least $(\log n)-1$ bits of memory

$L = \{x \mid x \text{ has more 1's than 0's}\}$

Theorem: Every streaming algorithm for L requires at least $(\log_2 n)-1$ bits of space

Suppose we partition all strings into their equivalence classes under $\equiv_{Ln}$



But the number of states in every DFA recognizing L_n is *at least* the number of equivalence classes under $\equiv_{Ln}$

$L = \{x \mid x \text{ has more 1's than 0's}\}$

Theorem: Every streaming algorithm for L requires at least $(\log_2 n)-1$ bits of space

Proof (Slide 1): Let $S_n = \{0^{n/2-i}1^i \mid i = 0,\ldots,n/2\}$

Let $x=0^{n/2-k}1^k$ and $y=0^{n/2-j}1^j$ be from S_n, $k > j$

Claim: $z = 0^{k-1}1^{n/2-(k-1)}$ distinguishes x and y in L_n

xz has n/2-1 zeroes and n/2+1 ones $\Rightarrow xz \in L_n$

yz has n/2+(k-j-1) zeroes and n/2-(k-j-1) ones

But $k-j-1 \geq 0$ … so $yz \notin L_n$

So $x \not\equiv_{L_n} y$, because z distinguishes x and y

$L = \{x \mid x$ has more 1's than 0's$\}$

Theorem: Every streaming algorithm for L requires at least $(\log_2 n)-1$ bits of space

Proof (Slide 2):

All pairs of strings in S_n are distinguishable in L_n

$\Rightarrow$ There are at least $|S_n|$ equiv classes of $\equiv_{L_n}$

Then, from the Myhill-Nerode Theorem:

$\Rightarrow$ All DFAs recognizing L_n need $\geq |S_n|$ states

$\Rightarrow$ Every streaming algorithm for L requires at least $(\log_2 |S_n|)$ bits of space.

Recall $|S_n|=n/2+1$ and we're done!

Number of Distinct Elements

The DE problem

Input: $x \in \{0,1,\ldots,2^k\}^*$, $2^k > |x|^2$

Output: The number of distinct elements appearing in x

Note: There is a streaming algorithm for DE using $O(k\ n)$ space

Theorem: Every streaming algorithm for DE requires $\Omega(k\ n)$ space

Randomized Algorithms Help!

The DE problem

Input: $x \in \{0,1,\ldots,2^k\}^*$, $2^k > |x|^2$

Output: The number of distinct elements appearing in x

Theorem: There is a *randomized* streaming algorithm that can approximate DE to within 0.1% error, using $O(k + \log n)$ space!

See the lecture notes for more details.

Communication Complexity

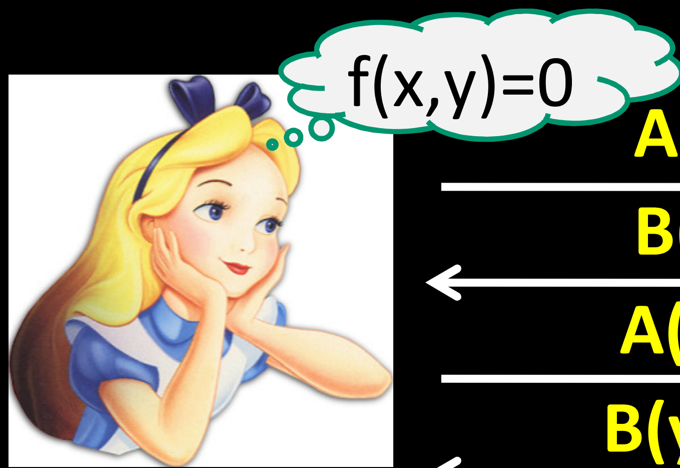
Communication Complexity

A theoretical model of distributed computing

- **Function $f : \{0,1\}^* \times \{0,1\}^* \to \{0,1\}$**
  - Two inputs, $x \in \{0,1\}^*$ and $y \in \{0,1\}^*$
  - **We assume $|x|=|y|=n$. Think of n as HUGE**
- **Two computers: Alice** and **Bob**
  - **Alice** *only* knows x, **Bob** *only* knows y
- **Goal: Compute $f(x, y)$ by communicating as few bits as possible between Alice and Bob**

We do not count computation cost. We *only* care about the number of bits communicated.

Alice and Bob Have a Conversation

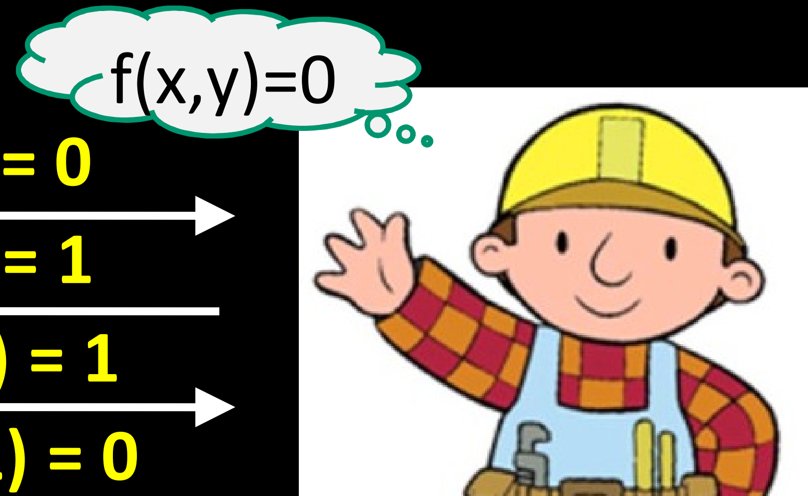


A(x,ε) = 0 →

← B(y,0) = 1

A(x,01) = 1 →

← B(y,011) = 0

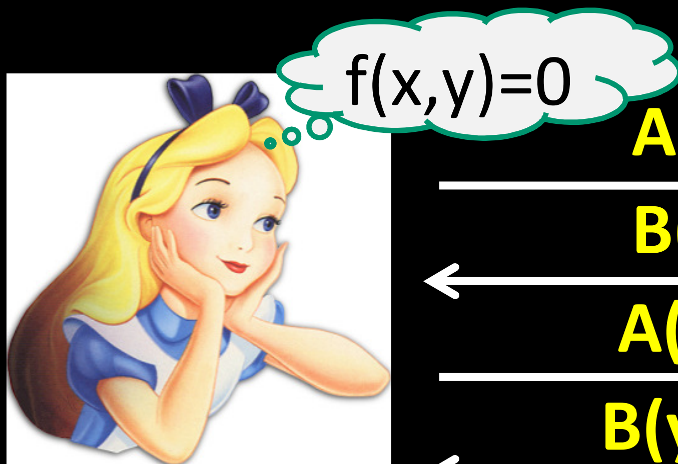


x **y**

In every step: Each bit sent is a function of the party's input and all the bits communicated so far in the conversation.

Communication cost = number of bits communicated
= 4 (in the example)

We assume Alice and Bob alternate in communicating, and the last bit sent is the value of f(x,y)



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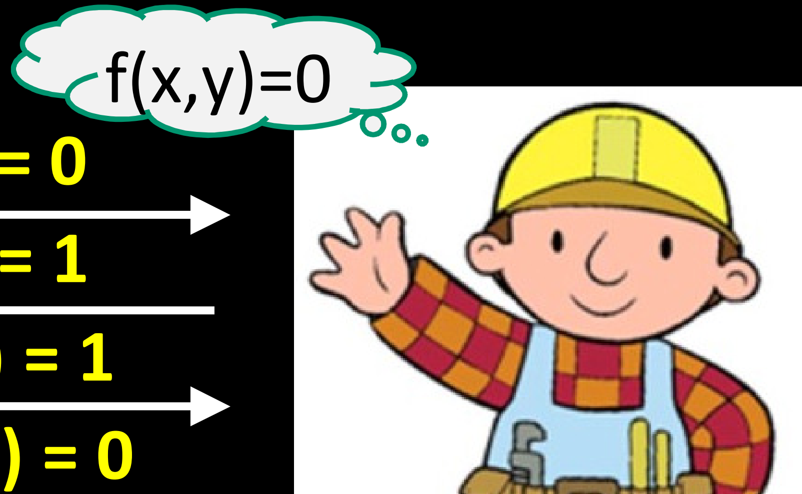


x y

Def. A *protocol* for a function ***f*** is a pair of functions

A, B : {0,1}* × {0,1}* → {0, 1, STOP} with the semantics:

On input (x, y), let r := 0, b_0 = ε.

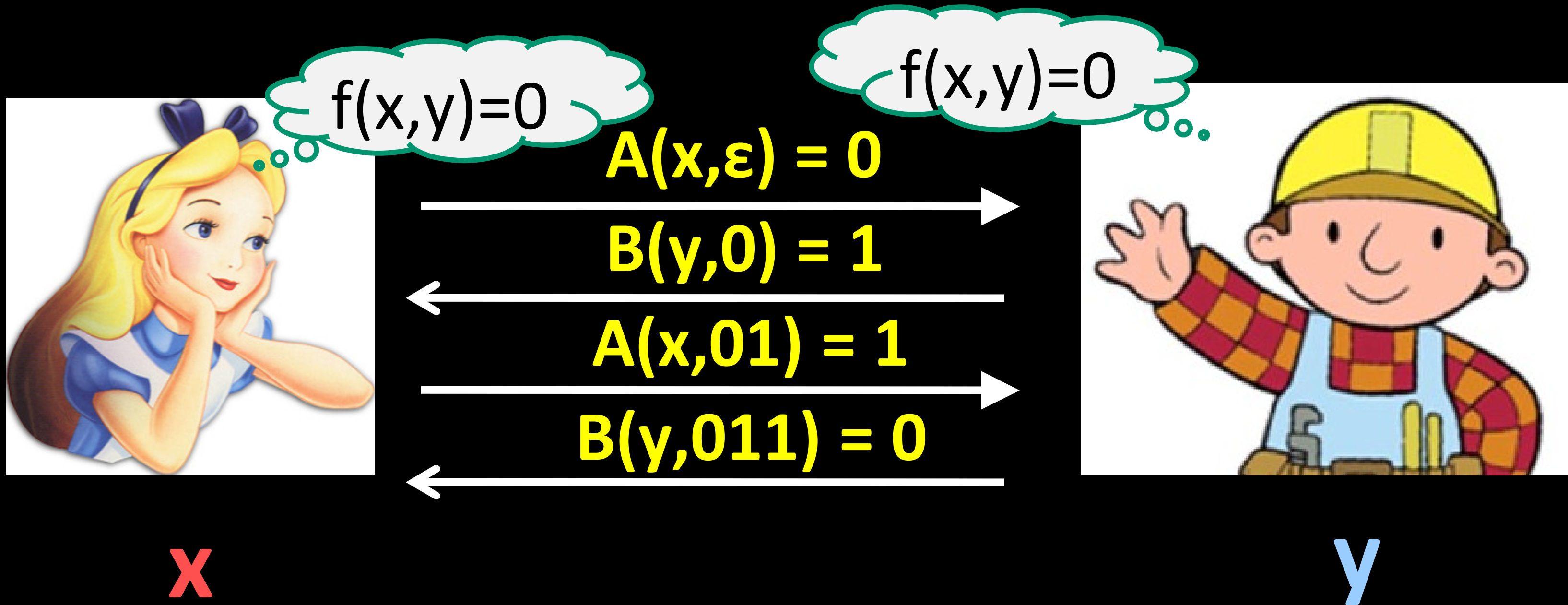
While ($b_r \neq$ STOP),

$r++$

If r is odd, Alice sends $b_r = A(x, b_1 \cdots b_{r-1})$

else Bob sends $b_r = B(y, b_1 \cdots b_{r-1})$

Output b_{r-1}. Number of *rounds* $= r - 1$

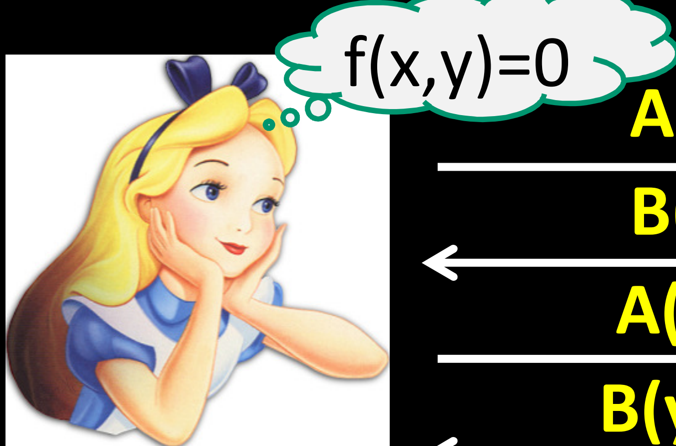


Def. The *cost of a protocol* P *for f on n-bit strings is*

$\max_{x,y \in\{0,1\}^n}$ [number of rounds in P to compute $f(x,y)$]

The *communication complexity* of ***f*** on n-bit strings is the ***minimum* cost** over all protocols for ***f*** on n-bit strings

= the minimum number of rounds used in any protocol for computing $f(x,y)$ over all n-bit x,y



A(x,ε) = 0 →

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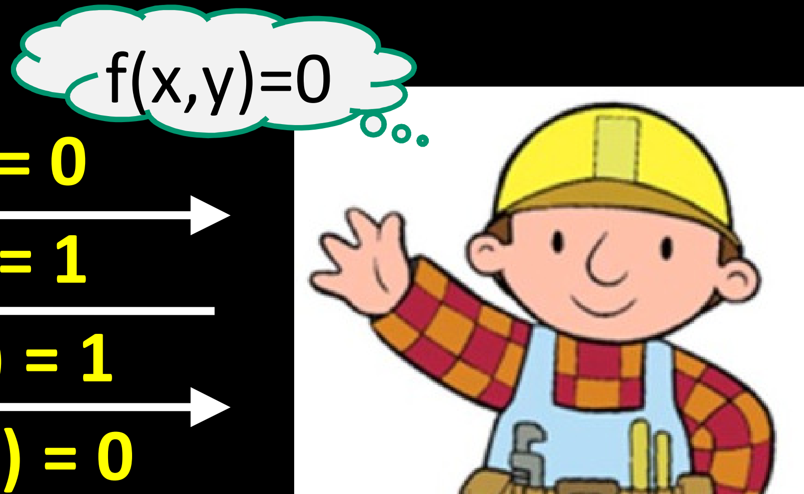


x　　　　　　　　　　y

Example. Let f : {0,1}* × {0,1}* → {0,1} be arbitrary

There is always a "trivial" protocol:

Alice sends bits of x in odd rounds

Bob sends bits of y in even rounds

After $2n$ rounds, they both know each other's input!

The communication complexity of every f is at most $2n$

A(x,ε) = 0 →

← B(y,0) = 1

A(x,01) = 1 →

← B(y,011) = 0

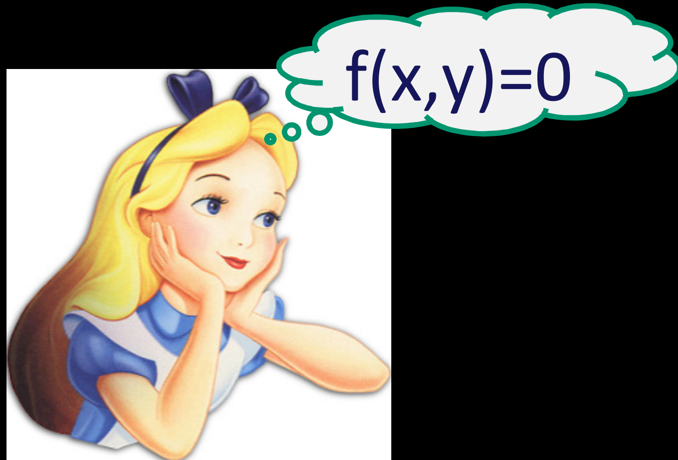
x **y**

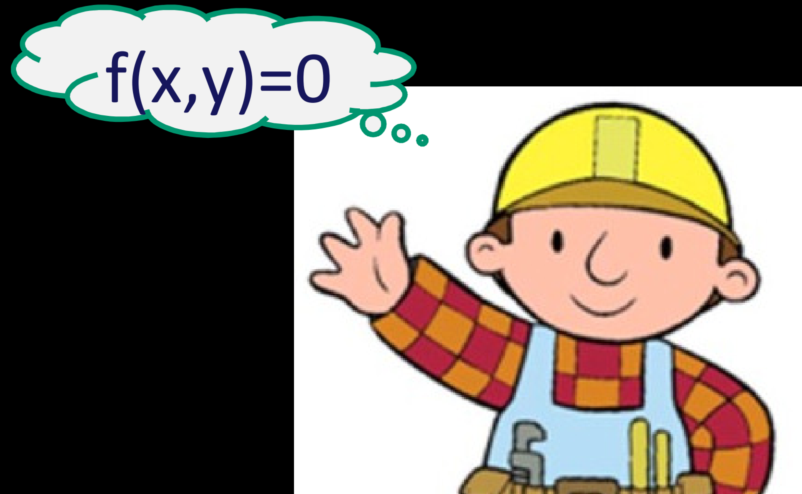
Example. **PARITY(x, y) = $\sum_i x_i$ + $\sum_i y_i$ mod 2.**

What's a good protocol for computing PARITY?

Alice sends b_1 = ($\sum_i x_i$ mod 2)
Bob sends b_2 = (b_1 + $\sum_i y_i$ mod 2). Alice stops.

The communication complexity of PARITY is 2





x **y**

Example. MAJORITY(x, y) = most frequent bit in xy

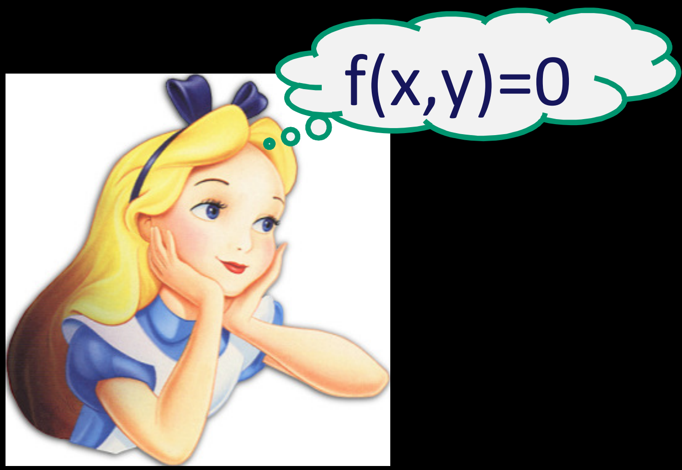
What's a good protocol for computing MAJORITY?

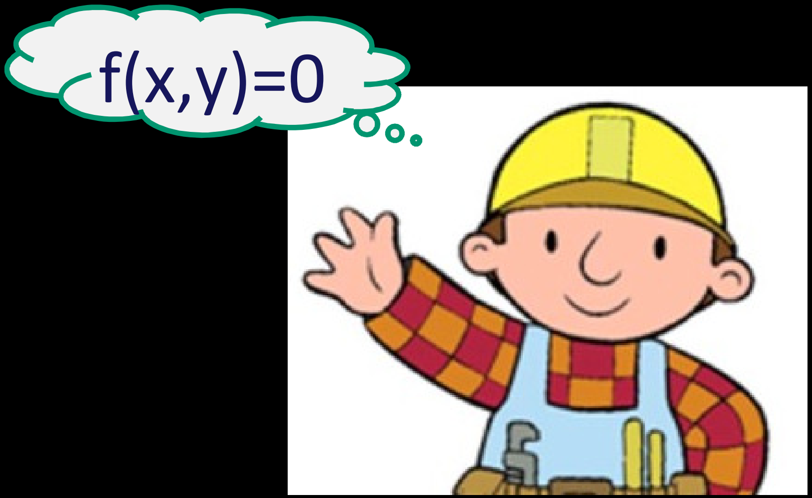
Alice sends b $=$ number of 1s in x

Bob computes c $=$ number of 1s in y,

sends 1 iff b + c is greater than (|x|+|y|)/2 = n

Communication complexity of MAJORITY is O(log n)





x

y

Example. EQUALS(x, y) = 1 $\Leftrightarrow$ x = y

What's a good protocol for computing EQUALS?

????

Communication complexity* *of EQUALS is at most $2n$

Connection to Streaming and DFAs

x

y

Let $L \subseteq \{0,1\}^*$

Def. f_L: $\{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$

for x, y with $|x| = |y|$ as:

$$f_L(x, y) = 1 \Leftrightarrow xy \in L$$

Examples:

L = { x | x has an odd number of 1s}

$\Rightarrow f_L(x, y)$ = PARITY(x,y) = $\sum_i x_i + \sum_i y_i$ mod 2

L = { x | x has more 1s than 0s}

$\Rightarrow f_L(x, y)$ = MAJORITY(x,y)

L = { xx | x $\in \{0,1\}^*$}

$\Rightarrow f_L(x, y)$ = EQUALS(x,y)

Connection to Streaming and DFAs

x

y

Let $L \subseteq \{0,1\}^*$

Def. $f_L: \{0,1\}^* \times \{0,1\}^* \rightarrow \{0,1\}$

for x, y with $|x|=|y|$ as:

$$f_L(x, y) = 1 \Leftrightarrow xy \in L$$

Theorem: If L has a streaming algorithm using $\leq s$ space, then the comm. complexity of f_L is at most $4s + 5$.

Proof: Alice runs streaming algorithm A on x.

Sends the *memory content* of A: this is s bits of space

Bob starts up A with that memory content, runs A on y.

Gets an output bit, sends to Alice.

(...why 4s+5 rounds? Can you do better?)