CS154

Finishing up DFA Minimization, The Myhill-Nerode Theorem, and Streaming Algorithms

Theorem

For every regular language L, there is a unique (up to re-labeling of the states) minimal-state DFA M* such that L = L(M*).

Furthermore, there is an efficient algorithm which, given any DFA M, will output this unique M*.

Extending the transition function δ

Given DFA $M = (Q, \Sigma, \delta, q_0, F)$, we extend δ to a function $\Delta : Q \times \Sigma^* \rightarrow Q$ as follows:

$$\Delta(q, \varepsilon) = q$$

$$\Delta(q, \sigma) = \delta(q, \sigma)$$

$$\Delta(q, \sigma_1 \ldots \sigma_{k+1}) = \delta(\Delta(q, \sigma_1 \ldots \sigma_k), \sigma_{k+1})$$

Note: $\Delta(q_0, w) \in F \Leftrightarrow$ **M accepts w**

Def. $w \in \Sigma^*$ **distinguishes states** q_1 **and** q_2 **iff**

$$\Delta(q_1, w) \in F \Leftrightarrow \Delta(q_2, w) \notin F$$

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Fix $M = (Q, \Sigma, \delta, q_0, F)$ and let $p, q \in Q$

Definition:

State p is ***distinguishable*** from state q
iff there is $w \in \Sigma^*$ that distinguishes p and q
iff there is $w \in \Sigma^*$ so that
exactly *one* of $\Delta(p, w)$, $\Delta(q, w)$ is a final state

State p is ***indistinguishable*** from state q
iff p is **not** distinguishable from q
iff for all $w \in \Sigma^*$, $\Delta(p, w) \in F \Leftrightarrow \Delta(q, w) \in F$

Pairs of indistinguishable states are redundant…

Fix $M = (Q, \Sigma, \delta, q_0, F)$ and let $p, q, r \in Q$

Define a binary relation $\sim$ on the states of M:

$p \sim q$ iff p is **indistinguishable** from q

$p \not\sim q$ iff p is distinguishable from q

Proposition: $\sim$ is an **equivalence relation**

$p \sim p$ (reflexive)

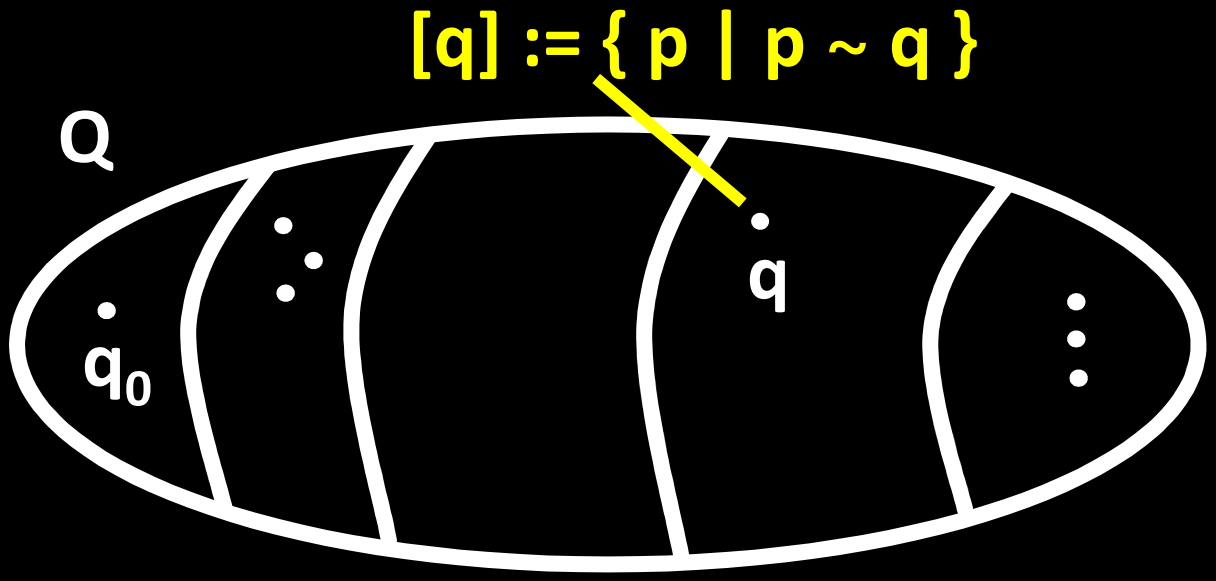
$p \sim q \Rightarrow q \sim p$ (symmetric)

$p \sim q$ and $q \sim r \Rightarrow p \sim r$ (transitive)

Fix $M = (Q, \Sigma, \delta, q_0, F)$ and let $p, q, r \in Q$

Proposition: ~ is an **equivalence relation**

As a consequence, the relation ~ partitions Q into disjoint equivalence classes



Algorithm: MINIMIZE-DFA

Input: DFA M

Output: DFA M_{MIN} such that:

$L(M) = L(M_{MIN})$

M_{MIN} has no *inaccessible* states

M_{MIN} is *irreducible*

||

For all states $p \neq q$ of M_{MIN}, p and q are distinguishable

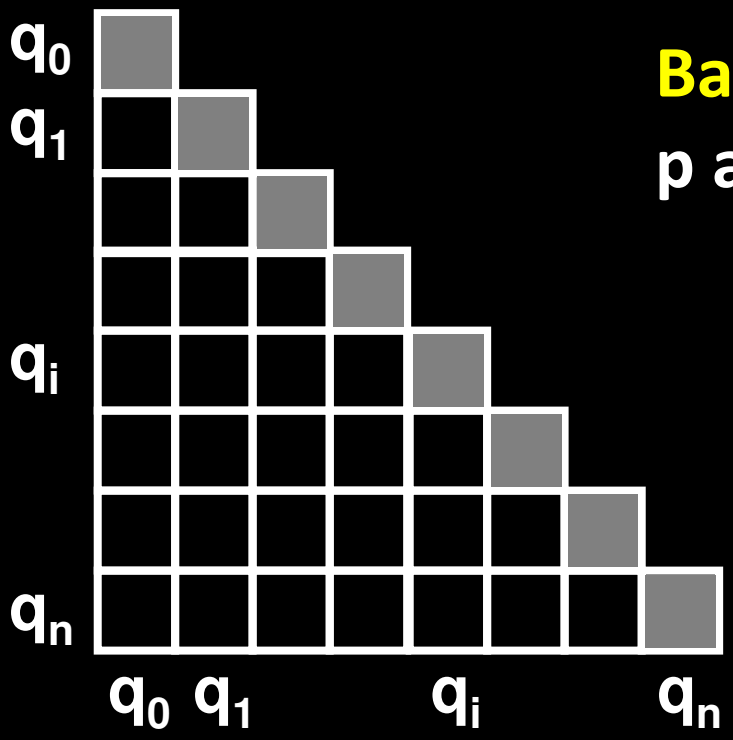
Theorem: M_{MIN} is the unique minimal DFA that is equivalent to M

The Table-Filling Algorithm

Input: DFA $M = (Q, \Sigma, \delta, q_0, F)$

Output: (1) $D_M = \{ (p, q) \mid p, q \in Q \text{ and } p \not\equiv q \}$

(2) $EQUIV_M = \{ [q] \mid q \in Q \}$



Base Case: For all (p, q) such that p accepts and q rejects $\Rightarrow p \not\equiv q$

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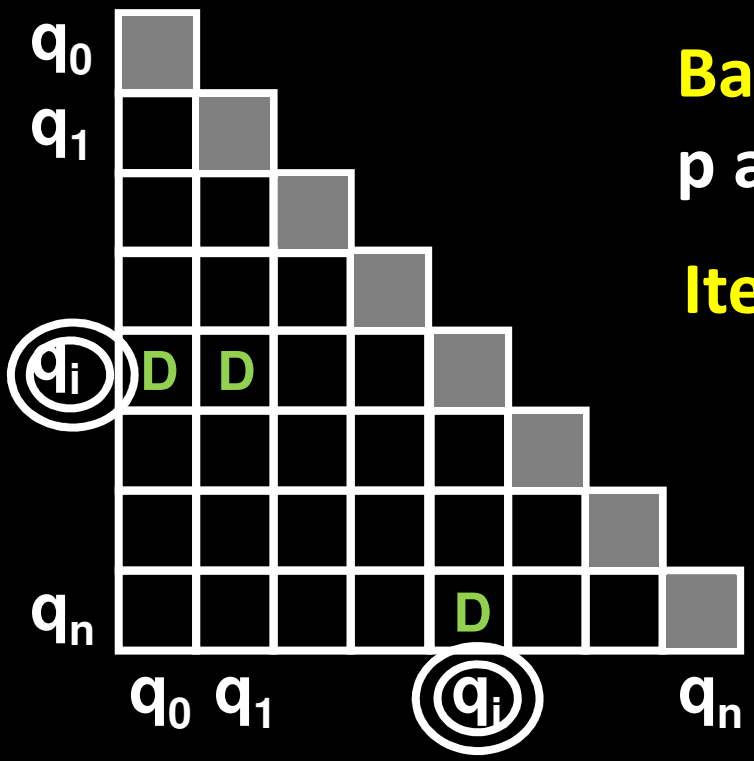


Base Case: For all (p, q) such that p accepts and q rejects $\Rightarrow p \not\equiv q$

Iterate: If there are states p, q and symbol $\sigma \in \Sigma$ satisfying:

$\delta(p, \sigma) = p'$

$p' \not\equiv q' \Rightarrow p \not\equiv q$

$\delta(q, \sigma) = q'$

Repeat until no more D's can be added

Claim: If (p, q) is marked D by the Table-Filling algorithm, then $p \not\sim q$

Claim: If (p, q) is not marked D by the Table-Filling algorithm, then $p \sim q$

Proof (by contradiction):

Suppose the pair (p, q) is not marked D by the algorithm, yet $p \not\sim q$ (call this a "bad pair")

Of all such bad pairs, let p, q be a pair with the *shortest* distinguishing string w

$\Delta(p, w) \in F$ and $\Delta(q, w) \notin F$ (Why is $|w| > 0$?)

We have $w = \sigma w'$, for some string w' and some $\sigma \in \Sigma$

Let $p' = \delta(p,\sigma)$ and $q' = \delta(q,\sigma)$

Then (p′, q′) is also a bad pair, but with a SHORTER w′ !

Algorithm MINIMIZE

Input: DFA M

Output: Equivalent minimal-state DFA M_{MIN}

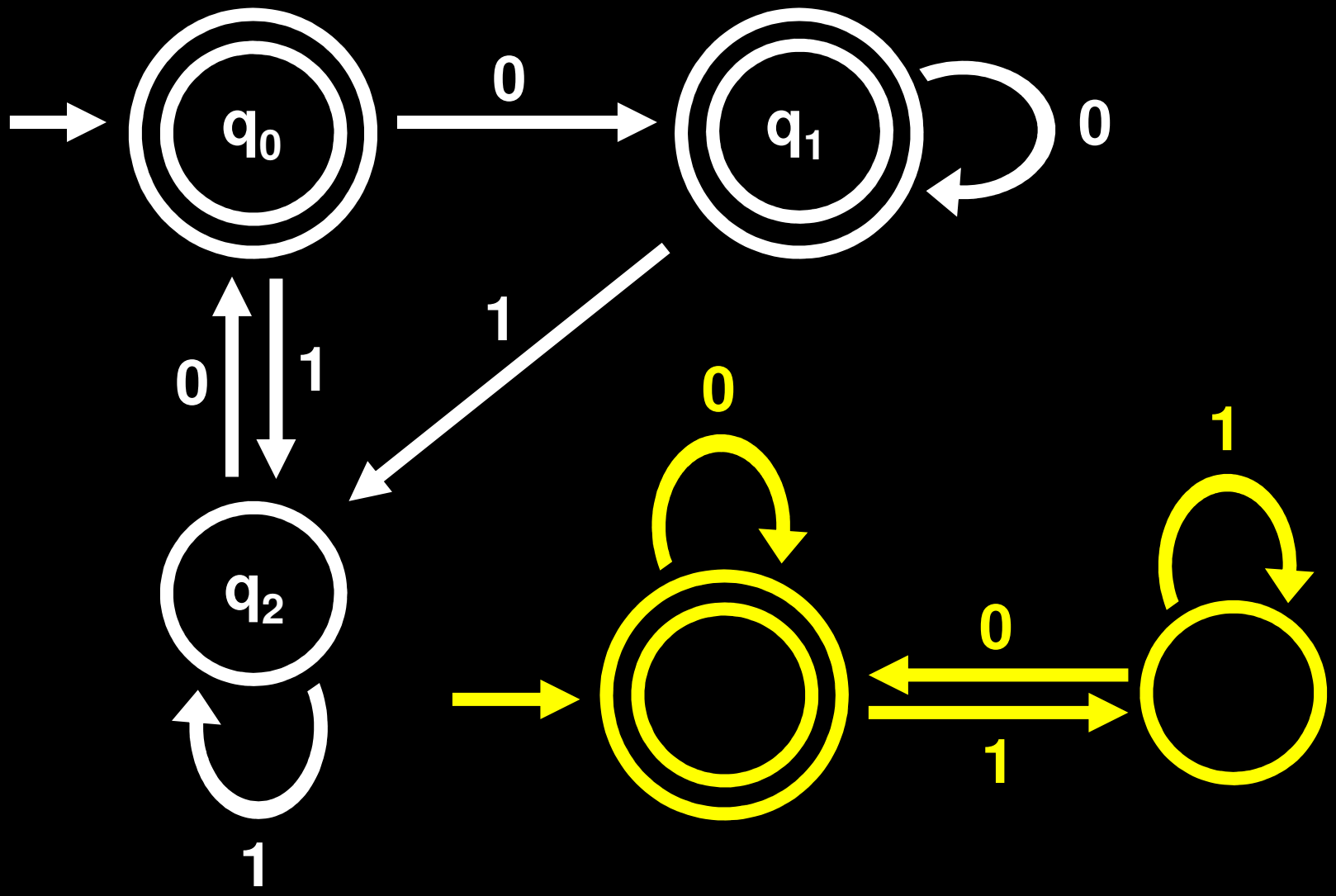
1. Remove all inaccessible states from M
2. Run Table-Filling algorithm on M to get:
   $EQUIV_M = \{ [q] \mid q \text{ is an accessible state of } M \}$
3. **Define:** $M_{MIN} = (Q_{MIN}, \Sigma, \delta_{MIN}, q_{0\,MIN}, F_{MIN})$

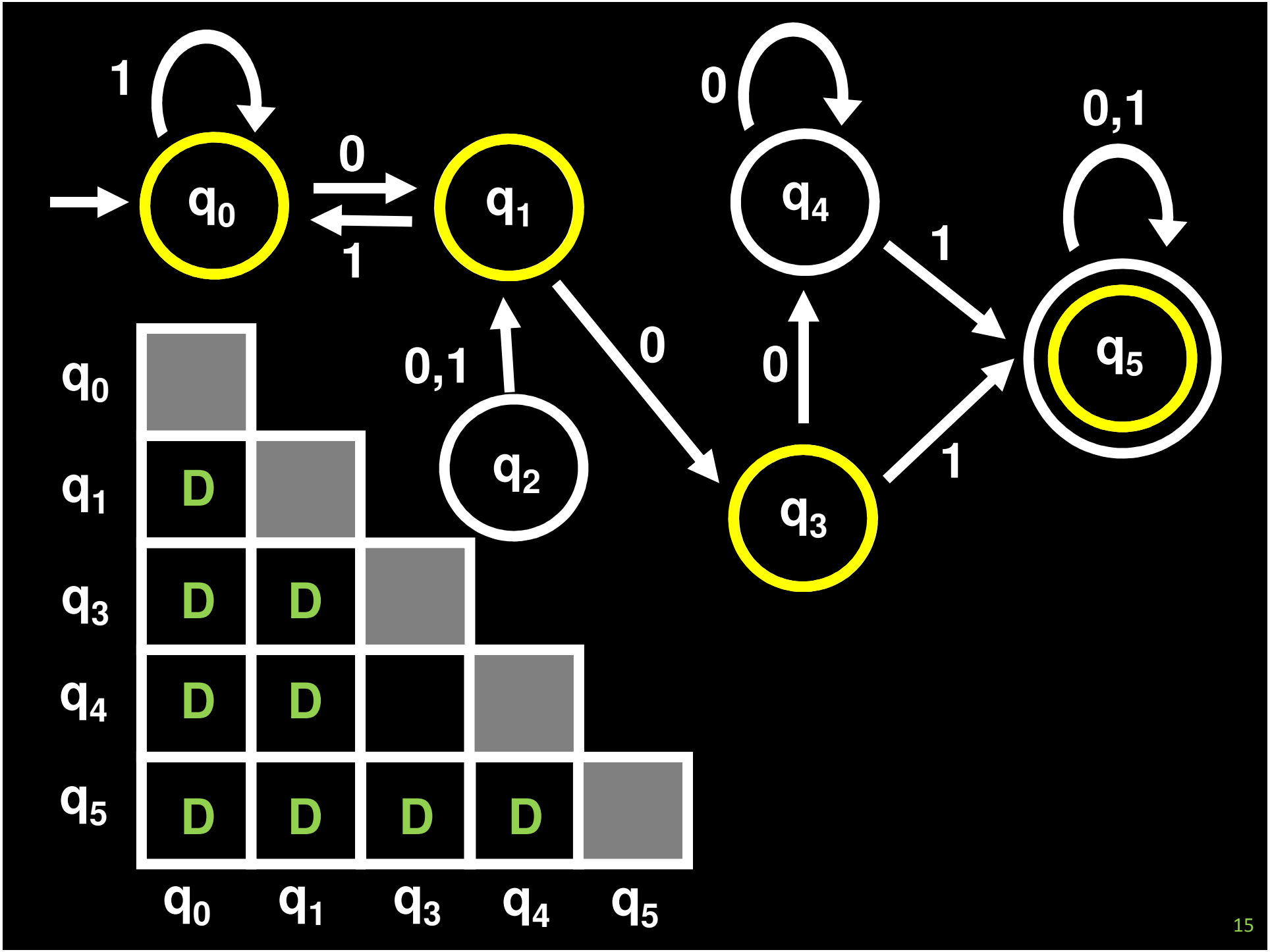
$Q_{MIN} = EQUIV_M$, $q_{0\,MIN} = [q_0]$, $F_{MIN} = \{ [q] \mid q \in F \}$

$$\delta_{MIN}([q], \sigma) = [\delta(q, \sigma)]$$

Claim: $L(M_{MIN}) = L(M)$

MINIMIZE

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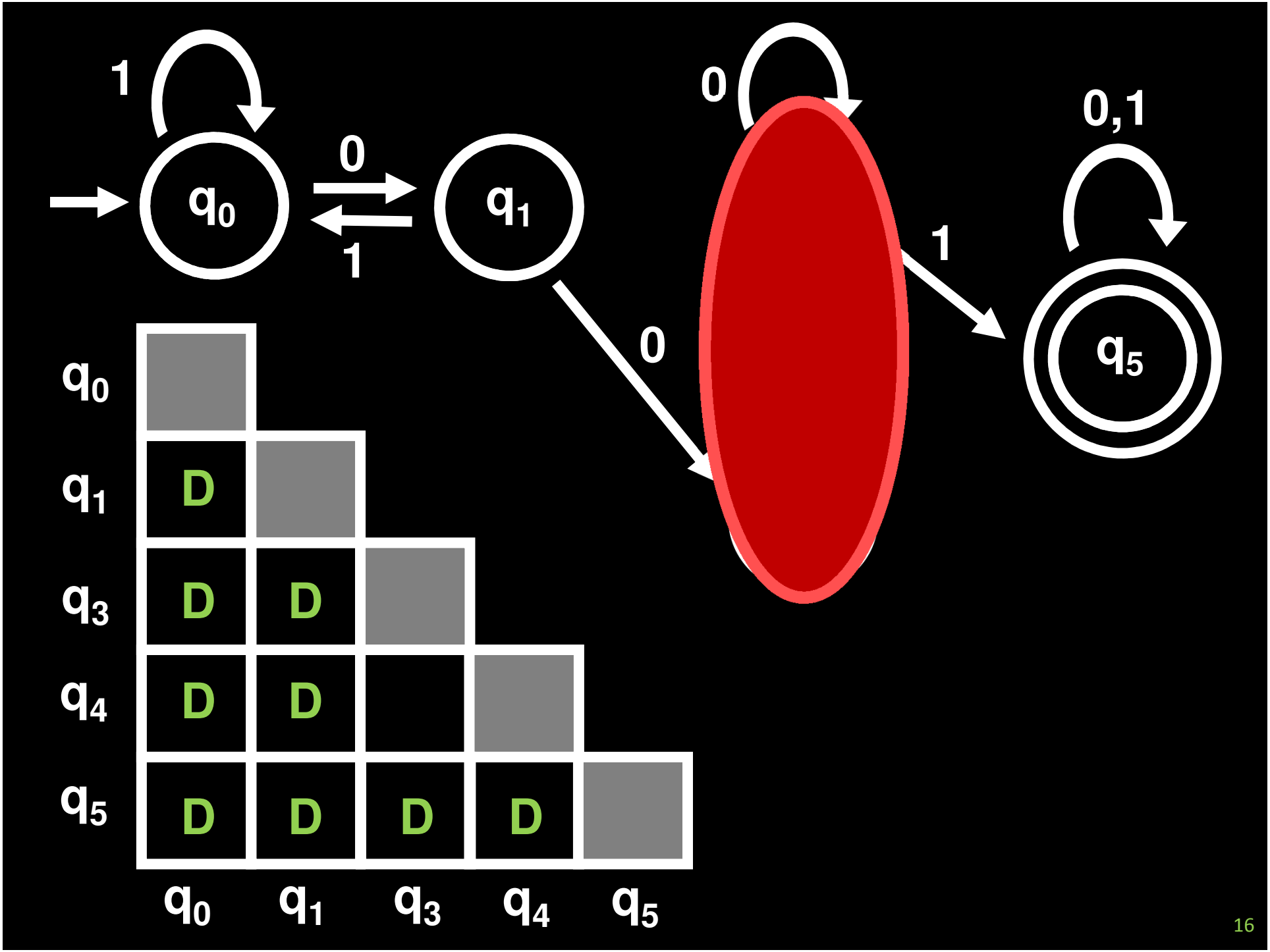
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Thm: M_{MIN} is the unique minimal DFA equivalent to M

Claim: Suppose $L(M') = L(M_{MIN})$ and M' has no inaccessible states and M' is irreducible.
Then there is an *isomorphism* between M' and M_{MIN}

Suppose for now the Claim is true.
If M' is a minimal DFA, then M' has no inaccessible states and is irreducible *(why?)*
So the Claim implies:

If M' is a minimal DFA for M, then there is an isomorphism between M' and M_{MIN}.
Therefore the Thm holds!

Thm: M_{MIN} **is the unique minimal DFA equivalent to M**

Claim: Suppose $L(M') = L(M_{MIN})$ **and** M' has no inaccessible states **and** M' is irreducible.
Then there is an *isomorphism* between M' and M_{MIN}

Proof: We recursively construct a map from the states of M_{MIN} to the states of M'

Base Case: $q_{0\,MIN} \mapsto q_0'$

Recursive Step: If $p \mapsto p'$

$p \xrightarrow{\sigma} q$ and $p' \xrightarrow{\sigma} q'$ Then $q \mapsto q'$

Base Case: $q_{0\ MIN} \mapsto q_0'$

Recursive Step: If $p \mapsto p'$

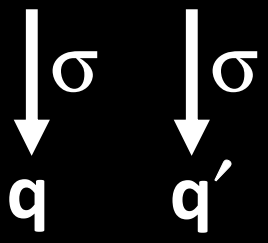
$\downarrow \sigma$ $\downarrow \sigma$

q q'

Then $q \mapsto q'$

Base Case: $q_{0\,MIN} \mapsto q_0'$

Recursive Step: If $p \mapsto p'$



Then $q \mapsto q'$

We need to prove:

The map is **defined** everywhere

The map is **well defined**

The map is a **bijection**

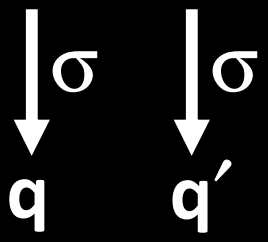
The map **preserves all transitions:**

If $p \mapsto p'$ then $\delta_{MIN}(p, \sigma) \mapsto \delta'(p', \sigma)$

(this follows from the definition of the map!)

Base Case: $q_{0\,MIN} \mapsto q_0'$

Recursive Step: If $p \mapsto p'$



Then $q \mapsto q'$

The map is defined everywhere

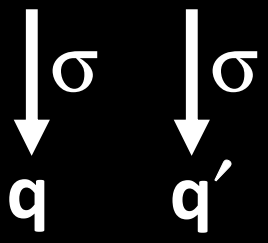
That is, for all states q of M_{MIN}
there is some state q′ of M′ such that $q \mapsto q'$

If $q \in M_{MIN}$, there is a string w such that
$\Delta_{MIN}(q_{0\,MIN}, w) = q$ (Why?)

Let $q' = \Delta'(q_0', w)$. Then $q \mapsto q'$

Base Case: $q_{0\,MIN} \mapsto q_0'$

Recursive Step: If $p \mapsto p'$



Then $q \mapsto q'$

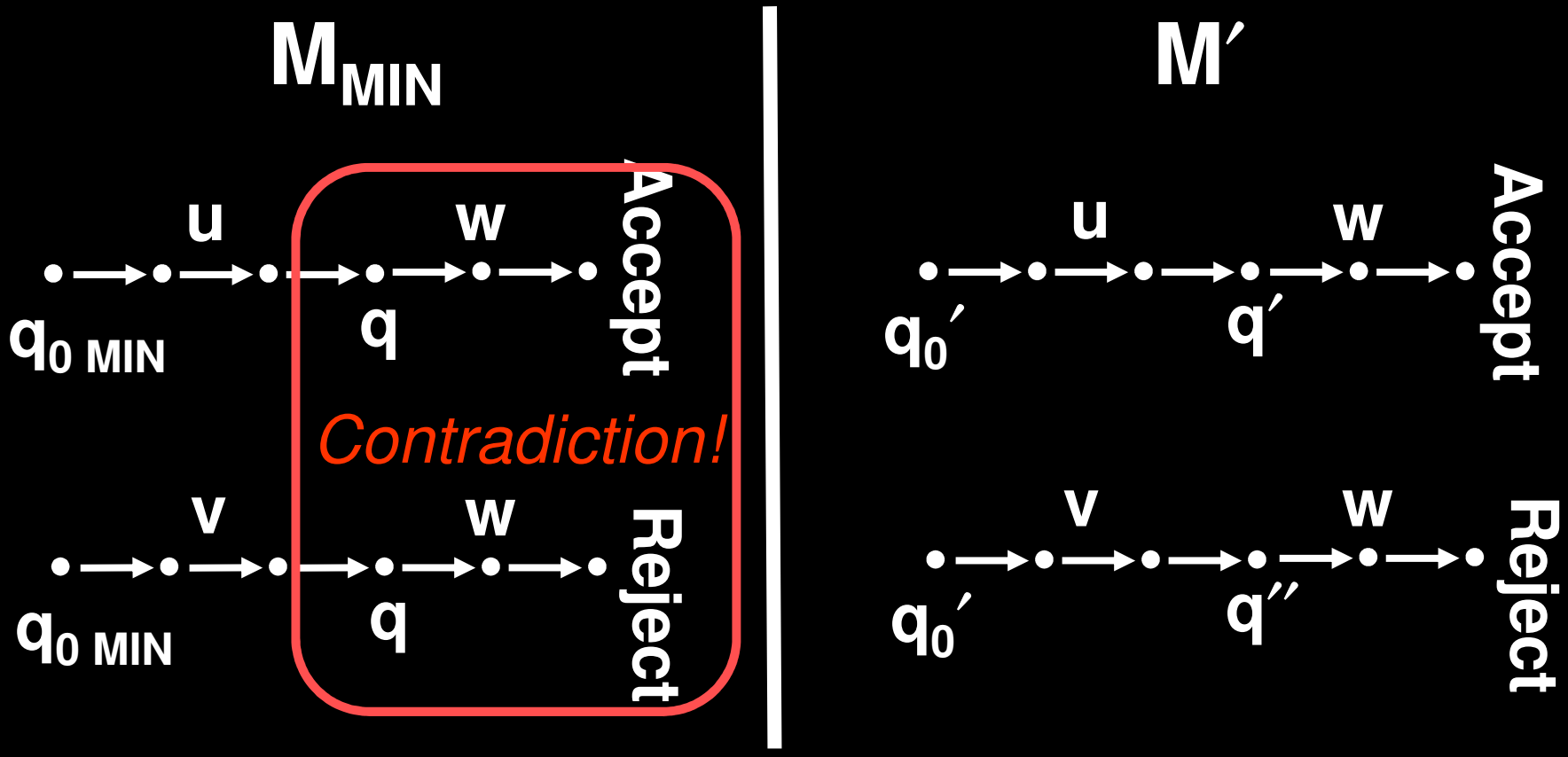
The map is well defined

Suppose there are states q' and q'' such that $q \mapsto q'$ and $q \mapsto q''$

We show that q' and q'' are ***indistinguishable***, so it must be that $q' = q''$

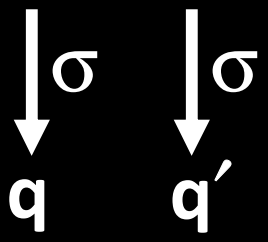
Suppose there are states q' and q'' such that $q \mapsto q'$ and $q \mapsto q''$

Now suppose q' and q'' are **distinguishable…**



Base Case: $q_{0\,MIN} \mapsto q_0'$

Recursive Step: If $p \mapsto p'$



Then $q \mapsto q'$

The map is **onto**

Want to show: For all states q′ of M′ there is a state q of M_{MIN} such that $q \mapsto q'$

For every q′ there is a string w such that M' reaches state q' after reading in w

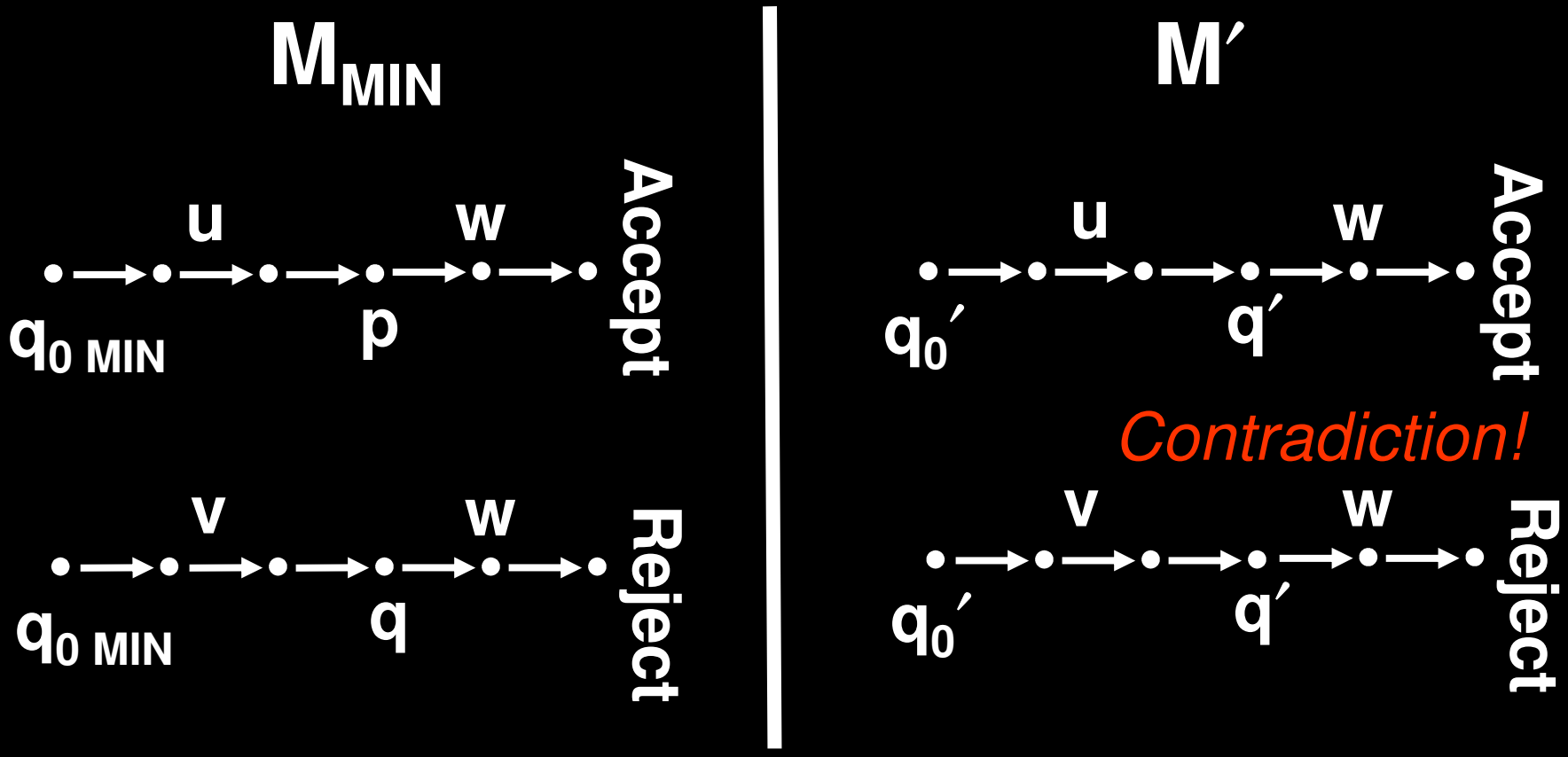
Let q be the state of M_{MIN} after reading in w

Claim: $q \mapsto q'$

The map is **one-to-one**

Proof by contradiction. Suppose there are states $p \neq q$ such that $p \mapsto q'$ and $q \mapsto q'$

If $p \neq q$, then p and q are **distinguishable**



How can we prove that two regular expressions are equivalent?

The Myhill-Nerode Theorem

We can also define a similar equivalence relation over ***strings*** and ***languages***:

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$

$x \equiv_L y$ iff for all $z \in \Sigma^*$, $[xz \in L \Leftrightarrow yz \in L]$

Define: x and y are **indistinguishable to L iff** $x \equiv_L y$

Claim: $\equiv_L$ **is an equivalence relation**

Proof?

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$

$x \equiv_L y$ iff for all $z \in \Sigma^*$, $[xz \in L \Leftrightarrow yz \in L]$

The Myhill-Nerode Theorem:
A language L is regular *if and only if* the number of equivalence classes of $\equiv_L$ is *finite*.

Proof ($\Rightarrow$) Let $M = (Q, \Sigma, \delta, q_0, F)$ be a min DFA for L.
Define the relation: $x \sim_M y \Leftrightarrow \Delta(q_0,x) = \Delta(q_0,y)$

Claim: $\sim_M$ is an equivalence relation with $|Q|$ classes

Claim: If $x \sim_M y$ then $x \equiv_L y$

Proof: $x \sim_M y$ implies for all $z \in \Sigma^*$, xz and yz reach the *same state* of M. So $xz \in L \Leftrightarrow yz \in L$, and $x \equiv_L y$

Corollary: Number of equiv. classes of $\equiv_L$ is at most the number of equiv. classes of $\sim_M$ (which is $|Q|$)

Let $L \subseteq \Sigma^*$ and $x, y \in \Sigma^*$

$x \equiv_L y$ iff for all $z \in \Sigma^*$, $[xz \in L \Leftrightarrow yz \in L]$

($\Leftarrow$) If the number of equivalence classes of $\equiv_L$ is *k* then there is a DFA for L with *k* states

Idea: Build a DFA with these equivalence classes!

Define a DFA M where

Q is the set of equivalence classes of $\equiv_L$

$q_0 = [\varepsilon] = \{y \mid y \equiv_L \varepsilon\}$

$\delta([x], \sigma) = [x\,\sigma]$

$F = \{[x] \mid x \in L\}$

Claim: M accepts x if and only if $x \in L$

The **Myhill-Nerode Theorem** gives us a ***new*** way to prove that a given language is not regular:

L is not regular
if and only if
there are infinitely many equiv. classes of $\equiv_L$

L is not regular
if and only if
There are infinitely many strings w_1, w_2, … so that
for all $w_i \neq w_j$, *w_i and w_j are distinguishable to L:*
there is a $z \in \Sigma^*$ such that
exactly one **of $w_i z$ and $w_j z$ is in L**

Distinguishing set for L

The Myhill-Nerode Theorem gives us a *new* way to prove that a given language is not regular:

Theorem: $L = \{0^n 1^n \mid n \geq 0\}$ is not regular.

Proof: Consider the infinite set of strings

$$S = \{0, 0^1, 0^2, \ldots, 0^n, \ldots\}$$

Take any pair $(0^m, 0^n)$ of distinct strings in S

Let $z = 1^m$

Then $0^m 1^m$ is in L, but $0^n 1^m$ is *not* in L

That is, all pairs of strings in S are distinguishable

Hence there are infinitely many equivalence classes of $\equiv_L$, and L is not regular.

Streaming Algorithms

Streaming Algorithms

Q

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L = {x | x has more 1's than 0's}

Initialize C := 0 and B := 0
Read the next bit x from the stream
If (C = 0) then B := x, C := 1
If (C ≠ 0) and (B = x) then C := C + 1
If (C ≠ 0) and (B ≠ x) then C := C – 1
When the stream stops, *accept*
if and only if B=1 and C > 0

B = the majority bit
C = how many more times that B appears

On all strings of length n, the algorithm uses $(1+\log_2 n)$ bits of space *(to store B and C)*