CS154

Pumping Lemma, Minimizing DFAs

CS154

Homework 1 is due!

(11:59pm tonight…)

Homework 2 will appear this afternoon

The Pumping Lemma: Structure in Regular Languages

Let L be a regular language

Then there is a positive integer P s.t.

for all strings $w \in L$ with $|w| \geq P$
there is a way to write $w = xyz$, where:

1. $|y| > 0$ (that is, $y \neq \varepsilon$)
2. $|xy| \leq P$
3. For ***all*** $i \geq 0$, $xy^iz \in L$

Why is it called the pumping lemma? The word w gets *pumped* into longer and longer strings…

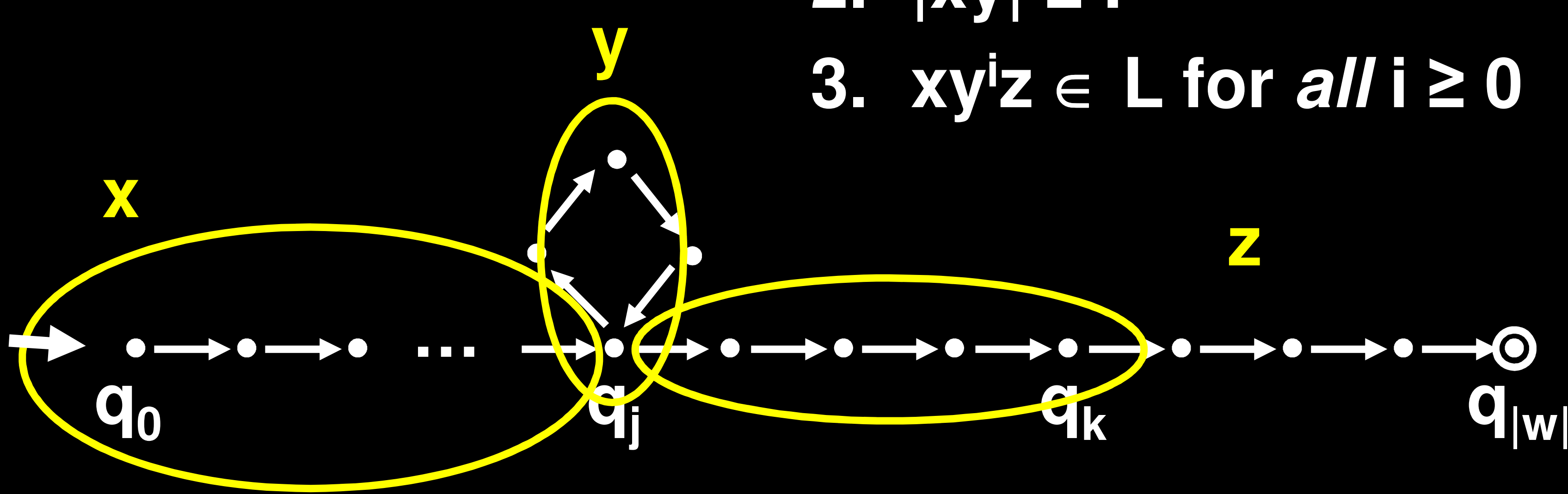
Proof: Let M be a DFA that recognizes L

Let P be the number of states in M

Let w be a string where $w \in L$ and $|w| \geq P$

We show: $w = xyz$

1. $|y| > 0$
2. $|xy| \leq P$
3. $xy^iz \in L$ for *all* $i \geq 0$



There must exist j and k such that $0 \leq j < k \leq P$, and $q_j = q_k$

Applying the Pumping Lemma

Let's prove that

$B = \{0^n 1^n \mid n \geq 0\}$ is not regular

By contradiction. Assume B is regular.

Let P be the number of states in a DFA for B.

Let $w = 0^P 1^P$

By the pumping lemma, there is a way to write w as $w = xyz$, $|y| > 0$, $|xy| \leq P$, and for all $i \geq 0$, xy^iz is *also* in B

Claim: The string y must be all zeroes.

Why? Because $|xy| \leq P$ and $w = xyz = 0^P 1^P$

But then xyyz has more 0s than 1s *Contradiction!*

Applying the Pumping Lemma

C = { w | w has equal number of 1s and 0s}

is not regular

Assume C is regular. Let $w = 0^P1^P$ (w is in C!)

By the pumping lemma, can write $w = xyz$, $|y| > 0$, $|xy| \leq P$, where for any $i \geq 0$, xy^iz is *also* in C

Note that $|xy| \leq P$, $w = xyz$, and $w = 0^P1^P$

Therefore y must be all zeroes.

But then xyyz has more 0s than 1s

Contradiction!

Applying the Pumping Lemma

Theorem:

$B = \{0^{n^2} \mid n \geq 0\}$ is not regular

Assume B is regular. Let $w = 0^{P^2}$

By the pumping lemma, we can write $w = xyz$, $|y| > 0$, $|xy| \leq P$, and for any $i \geq 0$, xy^iz is *also* in B

So we have $xyyz \in B$. Note that $xyyz = 0^{P^2+|y|}$

Observe that $0 < |y| \leq P$

therefore $P^2 + |y| \leq P^2 + P < P^2 + 2P + 1 = (P+1)^2$

$$P^2 < P^2 + |y| < (P+1)^2$$

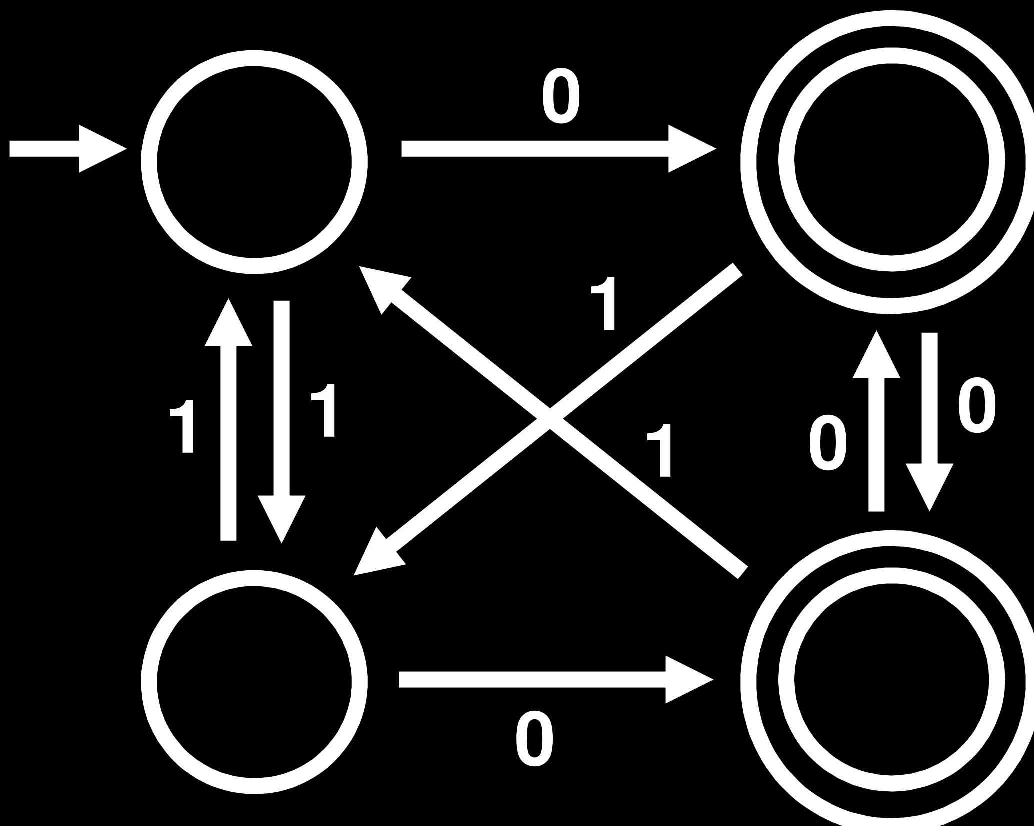
therefore $P^2 + |y|$ is *not* a perfect square!

Hence $0^{P^2+|y|} = xyyz \notin B$, so our assumption must be false.

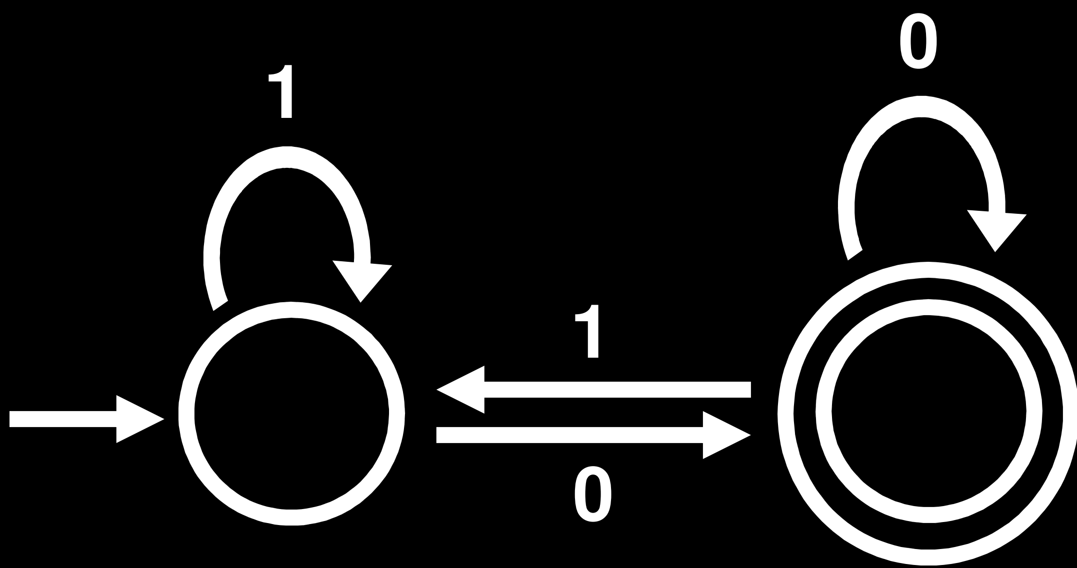
That is, B is not regular!

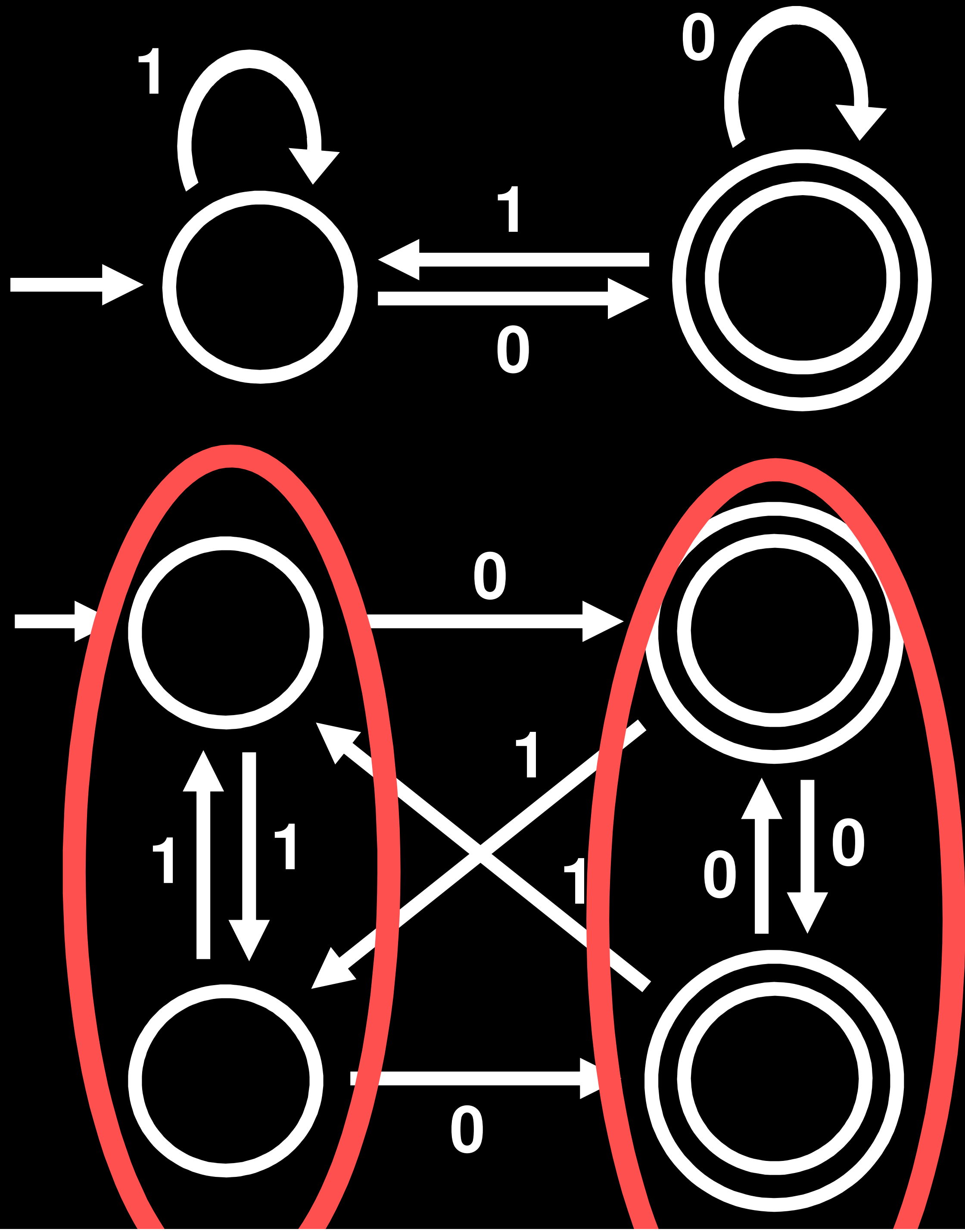
Does this DFA have a minimal number of states?

NO



Is this minimal?



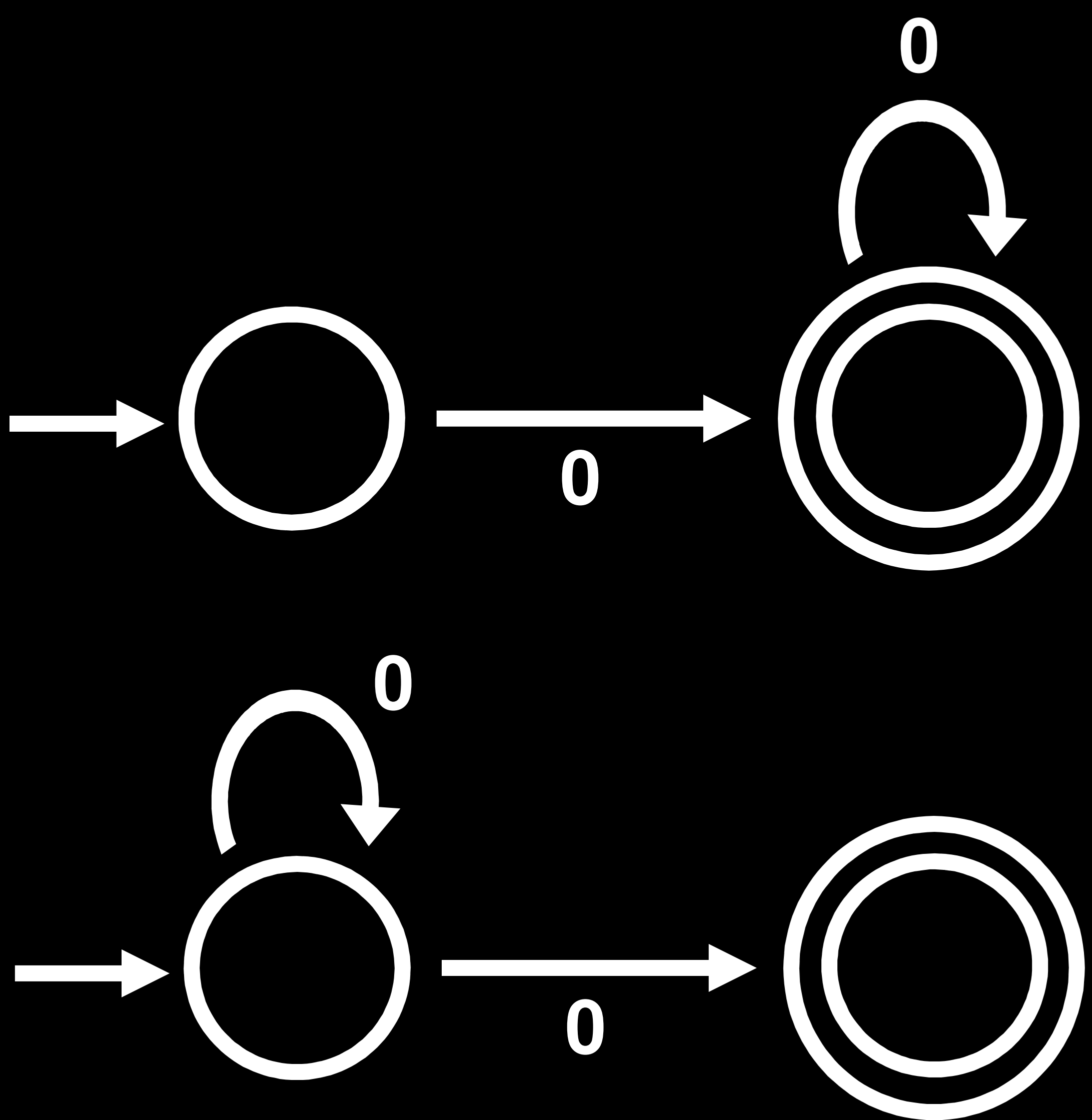

1
0
1
0
0
1
1
1
1
0
0
0

Theorem

For every regular language L, there is a unique (up to re-labeling of the states) minimal-state DFA M* such that L = L(M*).

Furthermore, there is an *efficient algorithm* which, given any DFA M, will output this unique M*.

There isn't a uniquely minimal NFA



Extending the transition function δ

Given DFA $M = (Q, \Sigma, \delta, q_0, F)$, we extend δ to a function $\Delta : Q \times \Sigma^* \to Q$ as follows:

$$\Delta(q, \varepsilon) = q$$

$$\Delta(q, \sigma) = \delta(q, \sigma)$$

$$\Delta(q, \sigma_1 \ldots \sigma_{k+1}) = \delta(\Delta(q, \sigma_1 \ldots \sigma_k), \sigma_{k+1})$$

$\Delta(q, w)$ = *the state of M reached after starting in state q and reading in w*

Note: $\Delta(q_0, w) \in F \Leftrightarrow M$ accepts w

Def. $w \in \Sigma^*$ **distinguishes** states q_1 and q_2 iff

$$\Delta(q_1, w) \in F \Leftrightarrow \Delta(q_2, w) \notin F$$

Extending the transition function δ

Given DFA $M = (Q, \Sigma, \delta, q_0, F)$, we extend δ to a function $\Delta : Q \times \Sigma^* \to Q$ as follows:

$$\Delta(q, \varepsilon) = q$$

$$\Delta(q, \sigma) = \delta(q, \sigma)$$

$$\Delta(q, \sigma_1 \ldots \sigma_{k+1}) = \delta(\Delta(q, \sigma_1 \ldots \sigma_k), \sigma_{k+1})$$

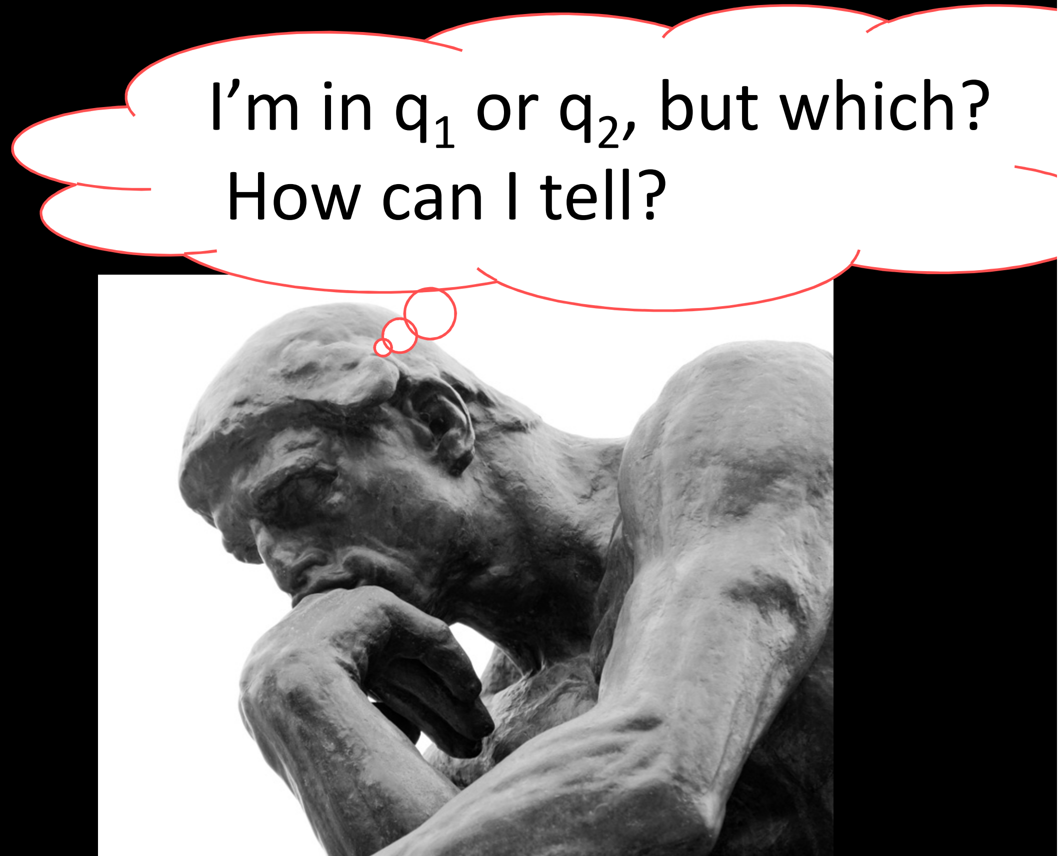
$\Delta(q, w)$ = *the state of M reached after starting in state q and reading in w*

Note: $\Delta(q_0, w) \in F \Leftrightarrow$ M accepts w

Def. $w \in \Sigma^*$ **distinguishes** states q_1 and q_2 iff exactly *one* of $\Delta(q_1, w)$, $\Delta(q_2, w)$ is a final state

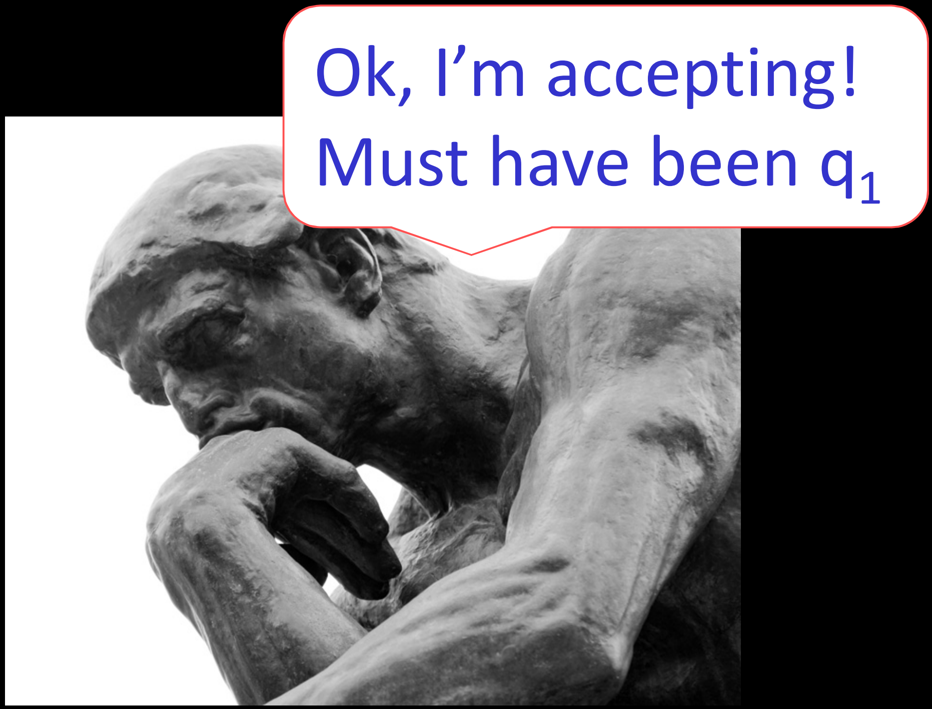
Distinguishing two states

Def. $w \in \Sigma^*$ **distinguishes** states q_1 and q_2 iff exactly *one* of $\Delta(q_1, w)$, $\Delta(q_2, w)$ is a final state



Distinguishing two states

Def. $w \in \Sigma^*$ **distinguishes** states q_1 and q_2 iff exactly *one* of $\Delta(q_1, w)$, $\Delta(q_2, w)$ is a final state



Fix $M = (Q, \Sigma, \delta, q_0, F)$ and let $p, q \in Q$

Definition:

State p is *distinguishable* from state q

iff some $w \in \Sigma^*$ distinguishes p and q

iff there is a string $w \in \Sigma^*$ so that

exactly *one* of $\Delta(p, w)$, $\Delta(q, w)$ is a final state

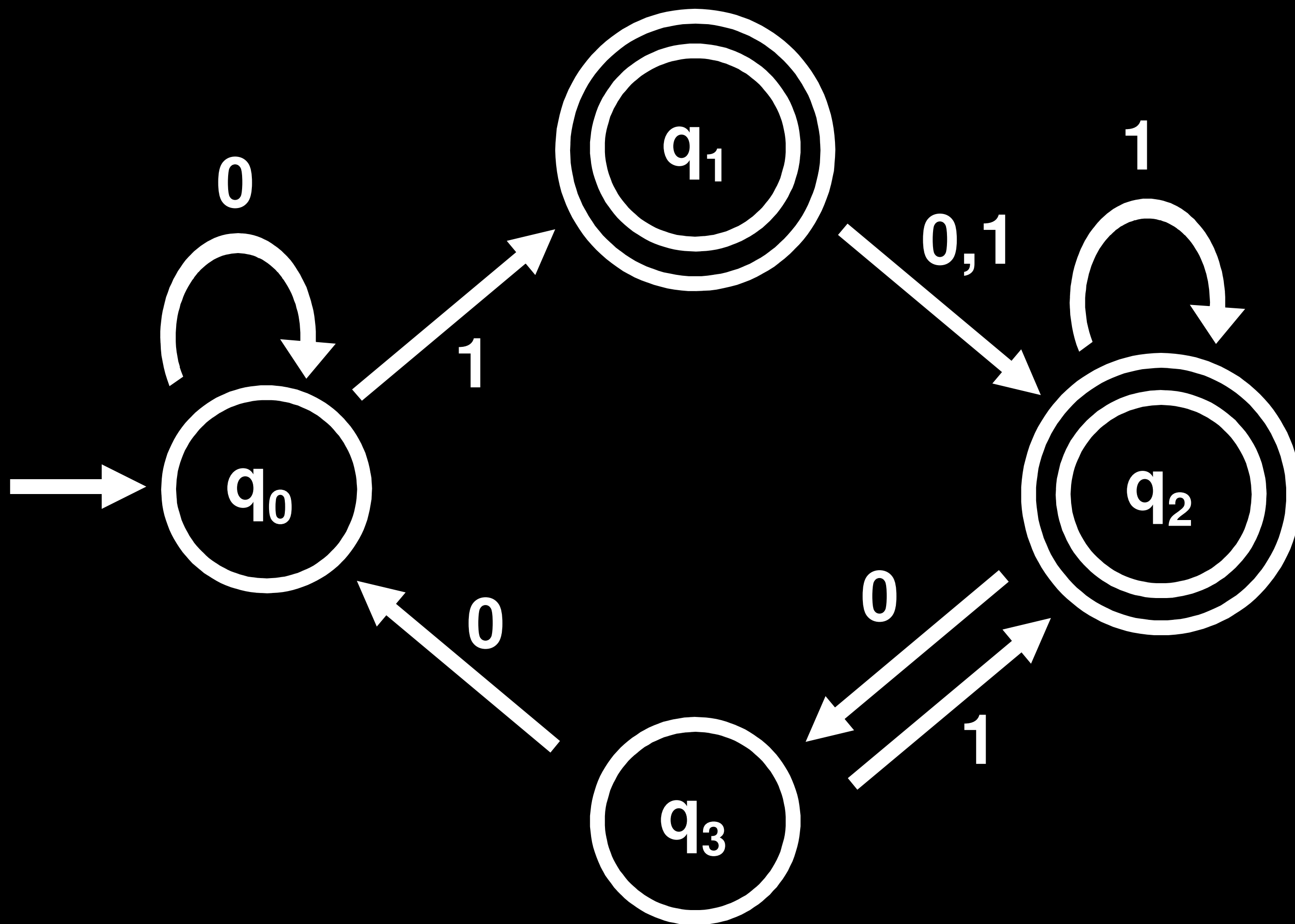
State p is *indistinguishable* from state q

iff p is not distinguishable from q

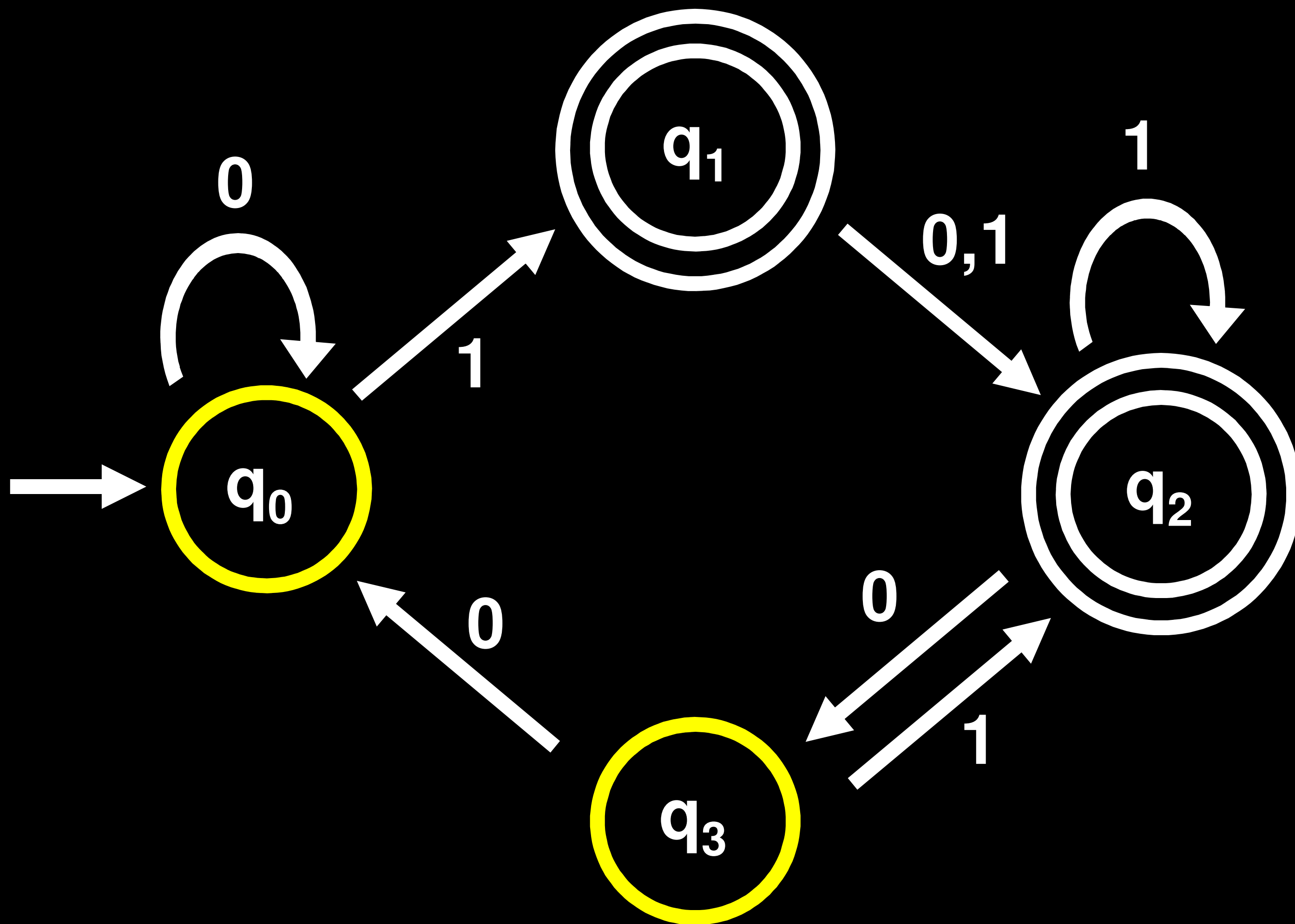
iff for all $w \in \Sigma^*$, $\Delta(p, w) \in F \Leftrightarrow \Delta(q, w) \in F$

Pairs of indistinguishable states are redundant…

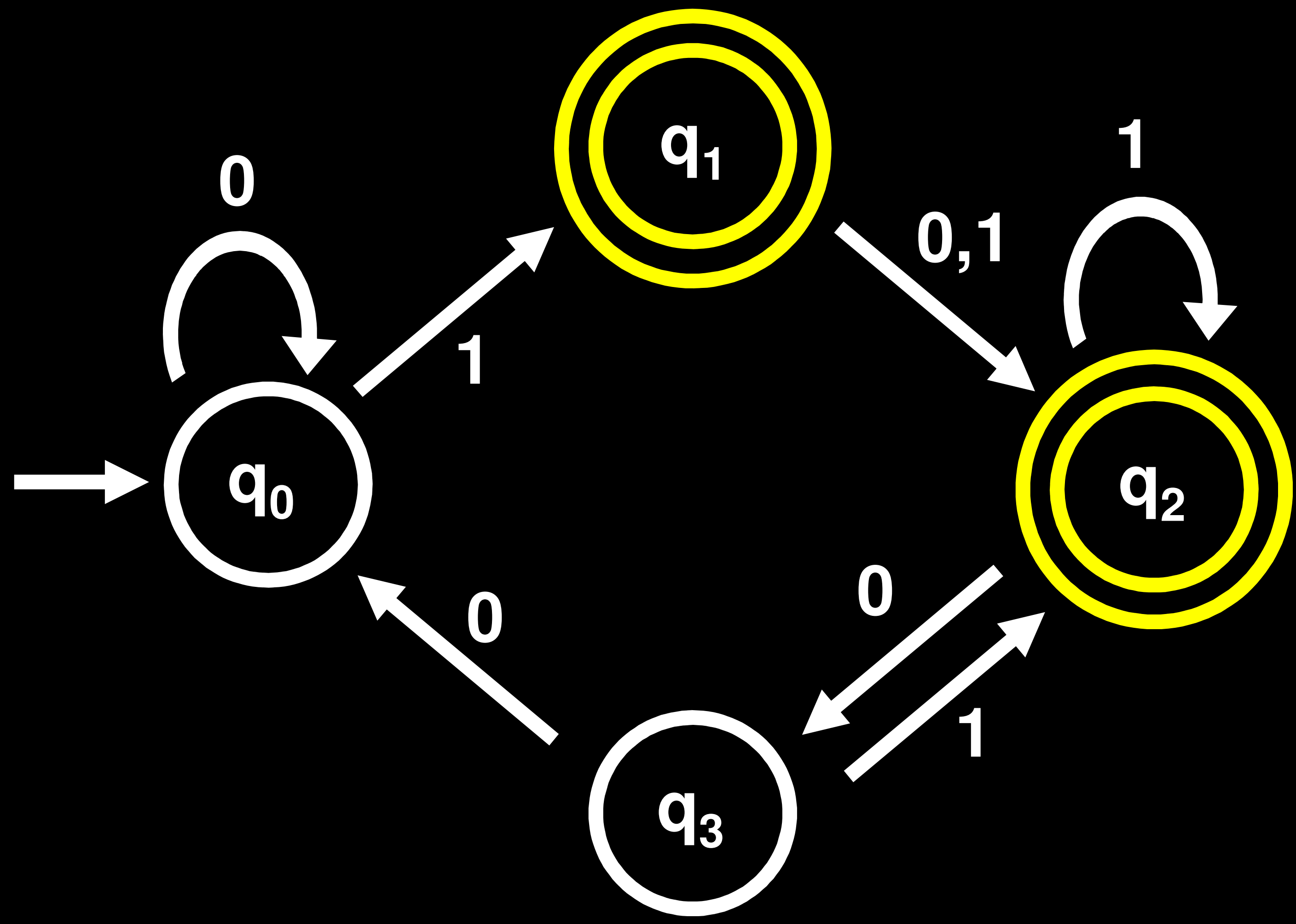
they lead to the same accept/reject behavior!



ε distinguishes accept states
and non-accept states



The string 10 distinguishes q0 and q3



The string 0 distinguishes q1 and q2

Fix $M = (Q, \Sigma, \delta, q_0, F)$ and let $p, q, r \in Q$

Define a binary relation $\sim$ on the states of M:

$p \sim q$ iff p is **indistinguishable** from q

$p \nsim q$ iff p is distinguishable from q

Proposition: $\sim$ is an **equivalence relation**

$p \sim p$ (reflexive)

$p \sim q \Rightarrow q \sim p$ (symmetric)

$p \sim q$ and $q \sim r \Rightarrow p \sim r$ (transitive)

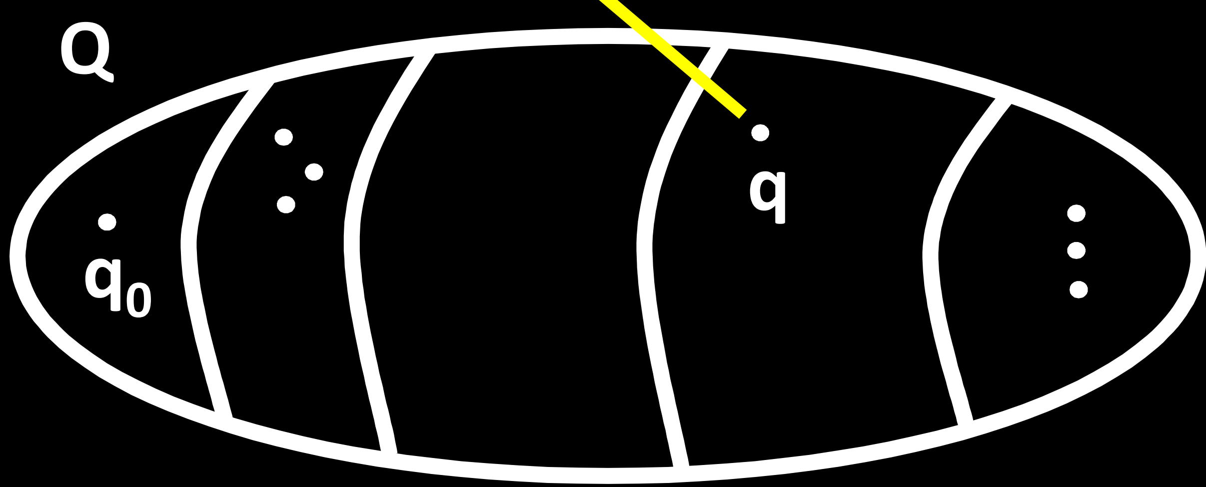
Proof?

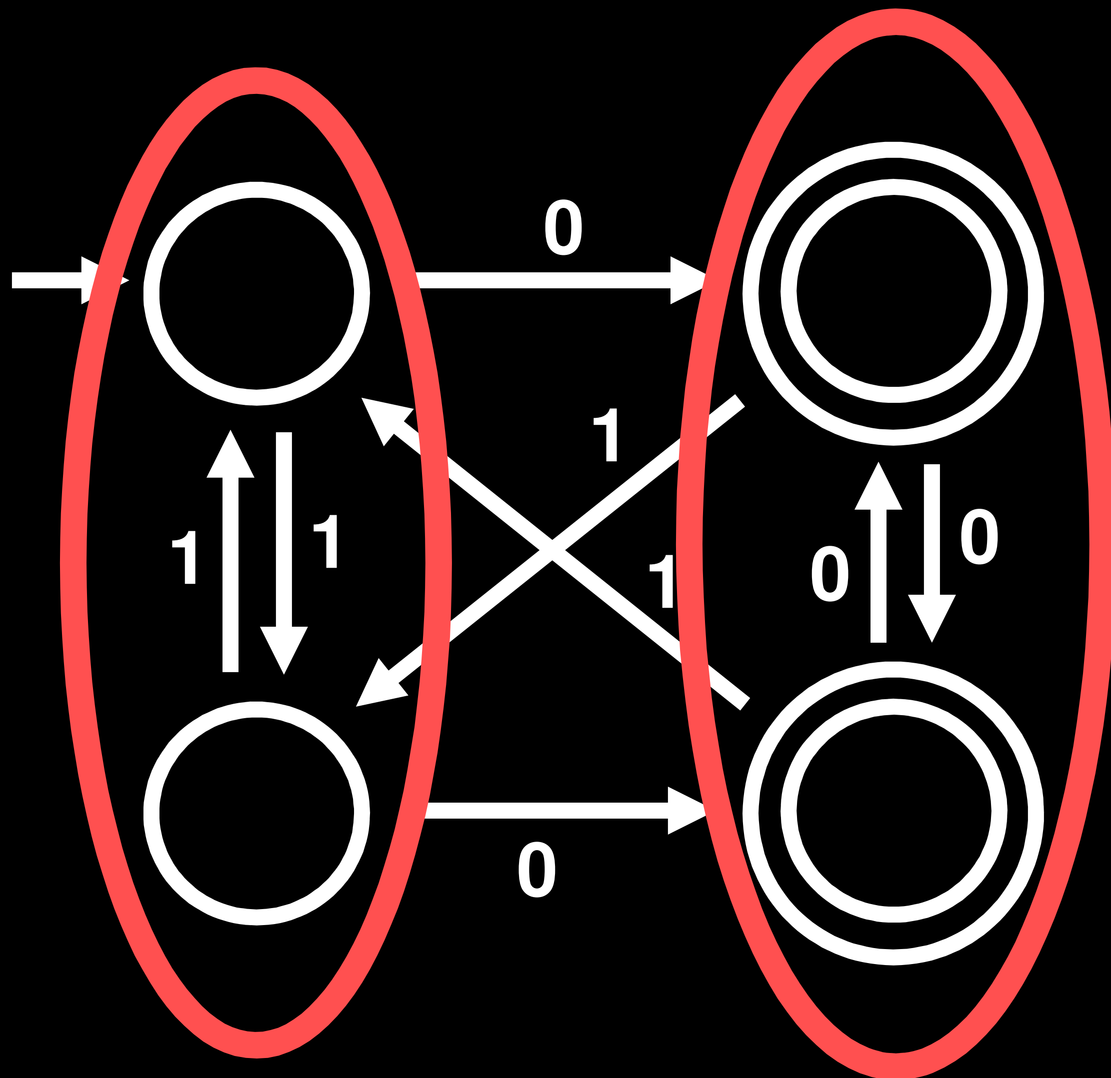
Fix $M = (Q, \Sigma, \delta, q_0, F)$ and let $p, q, r \in Q$

Therefore, the relation $\sim$ partitions Q into disjoint equivalence classes

Proposition: $\sim$ is an equivalence relation

$[q] := \{ p \mid p \sim q \}$




0
1
1
1
1
0
0
0

Algorithm: MINIMIZE-DFA

Input: DFA M

Output: DFA M_{MIN} such that:

$L(M) = L(M_{MIN})$

M_{MIN} has no *inaccessible* states

M_{MIN} is *irreducible*

||

For all states $p \neq q$ of M_{MIN}, p and q are distinguishable

Theorem: M_{MIN} is the unique minimal DFA that is equivalent to M

Intuition: States of M_{MIN} will be the *equivalence classes* of states of M

We'll uncover these equivalent states with a *dynamic programming* algorithm

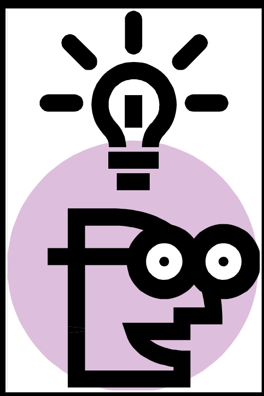
The Table-Filling Algorithm

Input: DFA $M = (Q, \Sigma, \delta, q_0, F)$

Output: (1) $D_M = \{ (p, q) \mid p, q \in Q \text{ and } p \not\sim q \}$

(2) $EQUIV_M = \{ [q] \mid q \in Q \}$

High-Level Idea:



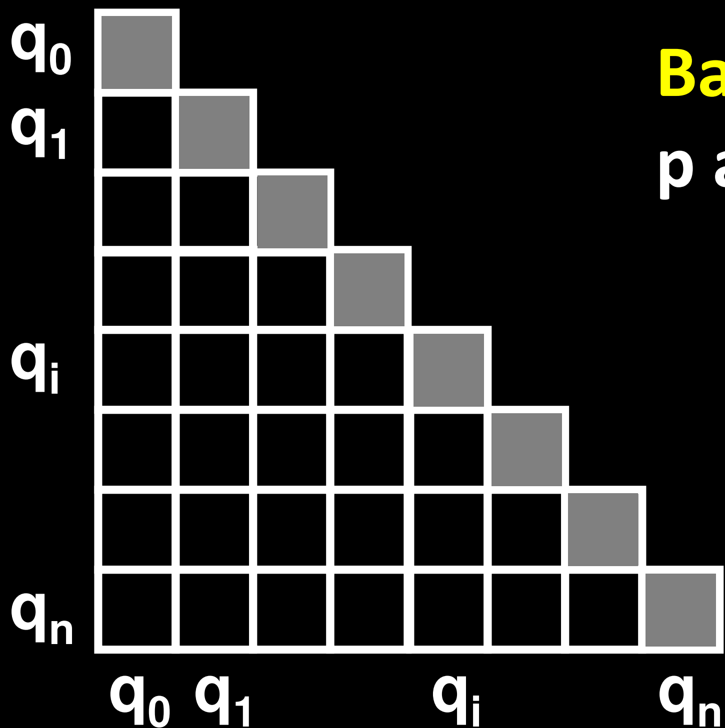
- We know how to find those pairs of states that the string ε distinguishes…
- Use this and *iteration* to find those pairs distinguishable with *longer* strings
- The pairs of states left over will be indistinguishable

The Table-Filling Algorithm

Input: DFA $M = (Q, \Sigma, \delta, q_0, F)$

Output: (1) $D_M = \{ (p, q) \mid p, q \in Q \text{ and } p \nsim q \}$

(2) $EQUIV_M = \{ [q] \mid q \in Q \}$



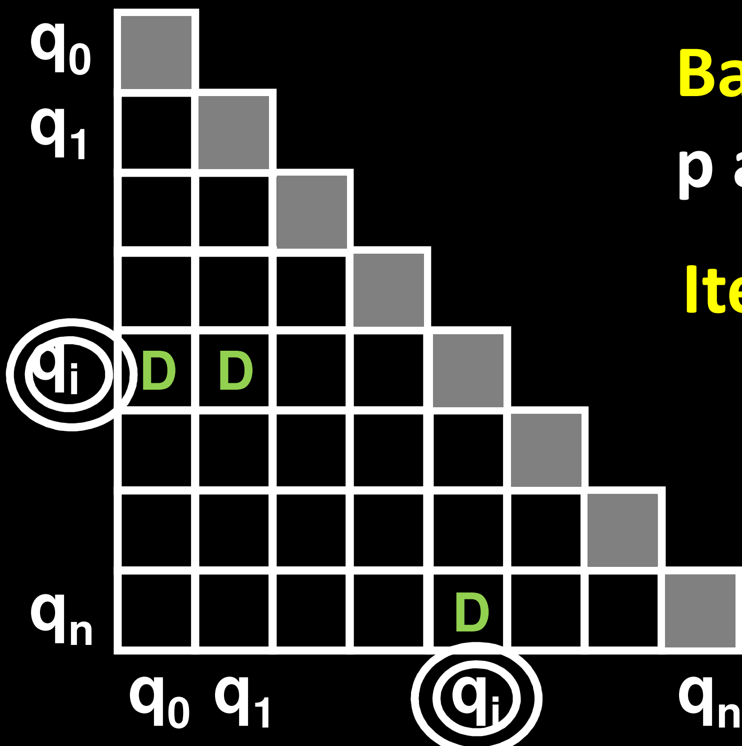
Base Case: For all (p, q) such that p accepts and q rejects $\Rightarrow p \nsim q$

The Table-Filling Algorithm

Input: DFA $M = (Q, \Sigma, \delta, q_0, F)$

Output: (1) $D_M = \{ (p, q) \mid p, q \in Q \text{ and } p \not\sim q \}$

(2) $EQUIV_M = \{ [q] \mid q \in Q \}$



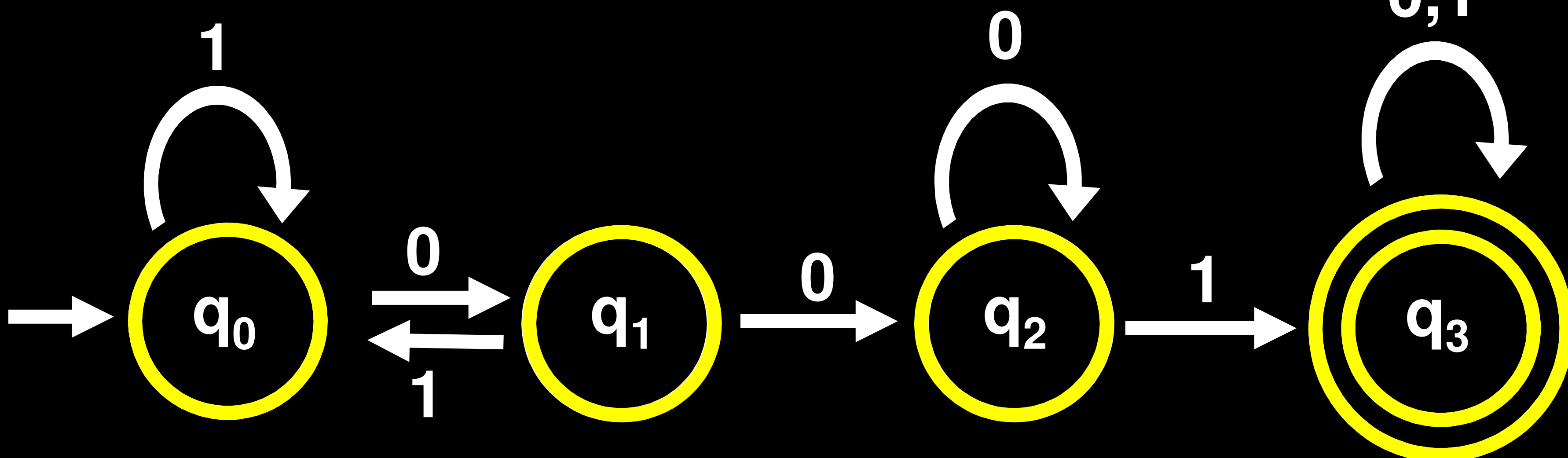
Base Case: For all (p, q) such that p accepts and q rejects $\Rightarrow p \not\sim q$

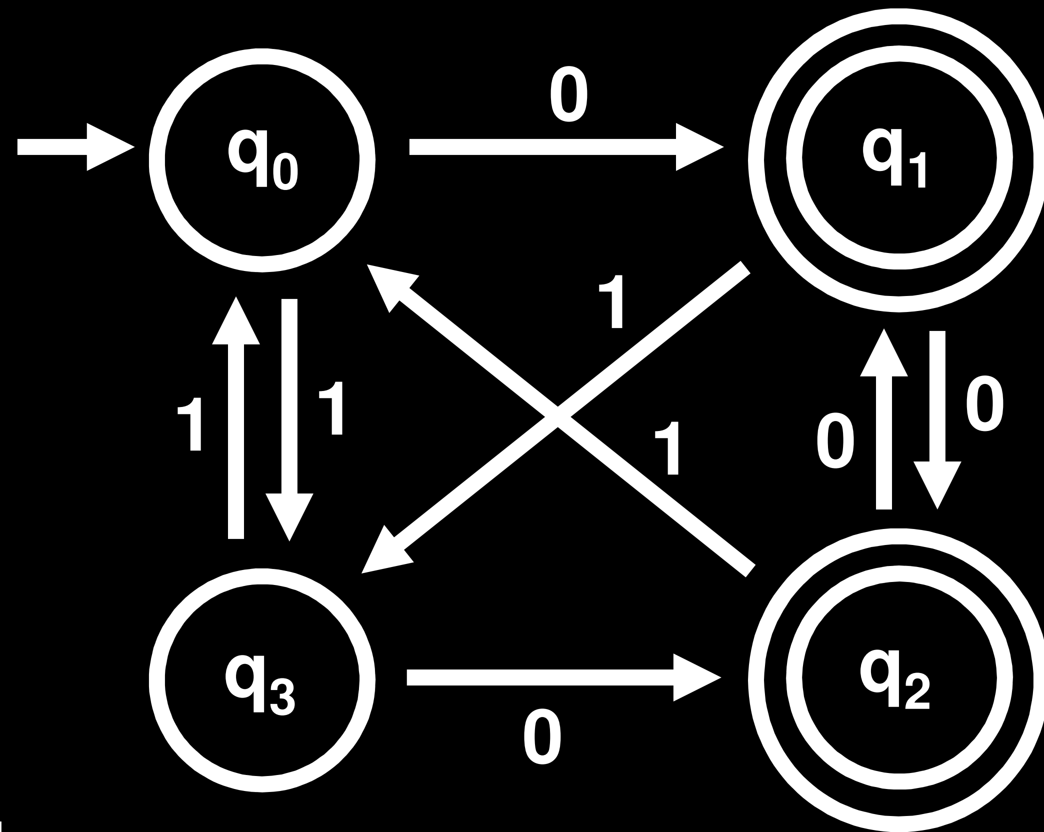
Iterate: If there are states p, q and symbol $\sigma \in \Sigma$ satisfying:

$$\left.\begin{array}{l} \delta(p, \sigma) = p' \\ \qquad\qquad\quad \not\sim \\ \delta(q, \sigma) = q' \end{array}\right\} \Rightarrow \text{Mark } p \not\sim q$$

Repeat until no more D's can be added

q_0				
q_1	D			
q_2	D	D		
q_3	D	D	D	
	q_0	q_1	q_2	q_3





	q_0	q_1	q_2	q_3
q_0				
q_1	D			
q_2	D			
q_3		D	D	

Claim: If (p, q) is marked D by the Table-Filling algorithm, then $p \nsim q$

Proof: By induction on the length of the string distinguishing p and q.

If (p, q) is marked D at the start, then one's in F and one isn't, so ε distinguishes p and q

Suppose (p, q) is marked D at a later point.

Then there are states p', q' such that:

1. (p', q') are marked D $\Rightarrow$ $p' \nsim q'$ (by induction)

$\Rightarrow$ There is a string w s.t. $\Delta(p', w) \in F \Leftrightarrow \Delta(q', w) \notin F$

2. $p' = \delta(p,\sigma)$ and $q' = \delta(q,\sigma)$, where $\sigma \in \Sigma$

The string σw distinguishes p and q!

Claim: If (p, q) is **not marked D** by the Table-Filling algorithm, then $p \sim q$

Proof (by contradiction):

Suppose the pair (p, q) is not marked D by the algorithm, yet $p \nsim q$ (call this a "bad pair")

Then there is a string w such that $|w| > 0$ and:

$$\Delta(p, w) \in F \text{ and } \Delta(q, w) \notin F$$ (Why is $|w| > 0$?)

Of all such bad pairs, let p, q be a pair with the *shortest* distinguishing string w

Claim: If (p, q) is not marked **D** by the Table-Filling algorithm, then $p \sim q$

Proof (by contradiction):

Suppose the pair (p, q) is not marked **D** by the algorithm, yet $p \nsim q$ (call this a "bad pair")

Of all such bad pairs, let p, q be a pair with the *shortest* distinguishing string **w**

$\Delta(p, w) \in F$ and $\Delta(q, w) \notin F$ (Why is $|w| > 0$?)

We have $w = \sigma w'$, for some string w' and some $\sigma \in \Sigma$

Let $p' = \delta(p,\sigma)$ and $q' = \delta(q,\sigma)$

Then (p', q') is also a bad pair, but with a SHORTER w' !