Mathematical Logic

Part Three

Outline for Today

- **Recap from Last Time**
- **More First-Order Translations**
- **First-Order Negations**

Recap from Last Time

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects, and
  - ***functions*** that map objects to one another,
  - ***quantifiers*** that allow us to reason about multiple objects simultaneously.

Quantifiers

- The biggest change from propositional logic to first-order logic is the use of ***quantifiers***.
- A ***quantifier*** is a statement that expresses that some property is true for some or all choices that could be made.
- Useful for statements like "for every action, there is an equal and opposite reaction."

The Universal Quantifier

- A statement of the form $\forall x.\ \psi$ asserts that for ***every*** choice of x, ψ is true.
- Examples:

$$\forall v.\ (Puppy(v) \to Cute(v))$$

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow \neg Odd(n)))$$

$$Tallest(SK) \to \forall x.\ (SK \neq x \to ShorterThan(x, SK))$$

- Note the use of the $\to$ connective.

The Existential Quantifier

- A statement of the form $\exists x.\ \psi$ asserts that for ***some*** choice of x, ψ is true.
- Examples:

$$\exists x.\ (Even(x) \land Prime(x))$$

$$\exists x.\ (TallerThan(x, me) \land LighterThan(x, me))$$

$$(\exists x.\ Appreciates(x, me)) \to Happy(me)$$

- Note the use of the $\land$ connective.

Checking a Translation

There's a tall tree that's a sequoia.

$$\exists t.\ (Tree(t) \land (Tall(t) \to Sequoia(t)))$$

Checking a Translation

There's a tall tree that's a sequoia.

$$\exists t.\ (Tree(t) \land (Tall(t) \to Sequoia(t)))$$

What if we pick t to be a short tree?

Checking a Translation

There's a tall tree that's a sequoia.

$$\exists t.\ (\mathit{Tree}(t) \land (\mathit{Tall}(t) \to \mathit{Sequoia}(t)))$$

What if we pick *t* to be a short tree?

Checking a Translation

There's a tall tree that's a sequoia.

$$\exists t.\ (Tree(t) \land (\cancel{Tall(t)} \to Sequoia(t)))$$

What if we pick t to be a short tree?

Checking a Translation

There's a tall tree that's a sequoia.

$$\exists t.\ (Tree(t) \land (Tall(t) \to Sequoia(t)))$$

What if we pick t to be a short tree?

Checking a Translation

There's a tall tree that's a sequoia.

$$\exists t.\ (Tree(t) \land (Tall(t) \to Sequoia(t)))$$

What if we pick t to be a short tree?

Checking a Translation

There's a tall tree that's a sequoia.

$$\exists t.\ (Tree(t) \land (Tall(t) \to Sequoia(t)))$$

What if we pick *t* to be a short tree?

Checking a Translation

There's a tall tree that's a sequoia.

$$\exists t.\ (Tree(t) \land (Tall(t) \to Sequoia(t)))$$

This statement can be true even if no tall sequoias exist.

Checking a Translation

There's a tall tree that's a sequoia.

$$\exists t.\ (Tree(t) \land Tall(t) \land Sequoia(t))$$

Checking a Translation

There's a tall tree that's a sequoia.

$$\exists t.\ (\mathit{Tree}(t) \land \mathit{Tall}(t) \land \mathit{Sequoia}(t))$$

Do you see why this statement doesn't have this problem?

Checking a Translation

Every tall tree is a sequoia.

$$\forall t.\ (Tree(t) \land Tall(t) \rightarrow Sequoia(t))$$

Checking a Translation

Every tall tree is a sequoia.

$$\forall t.\ (Tree(t) \land Tall(t) \to Sequoia(t))$$

Let's add parentheses to show operator precedence.

Checking a Translation

Every tall tree is a sequoia.

$$\forall t. \; ((Tree(t) \land Tall(t)) \to Sequoia(t))$$

Let's add parentheses to show operator precedence.

Checking a Translation

Every tall tree is a sequoia.

$$\forall t.\ ((Tree(t) \land Tall(t)) \to Sequoia(t))$$

Checking a Translation

Every tall tree is a sequoia.

$$\forall t.\ ((Tree(t) \land Tall(t)) \to Sequoia(t))$$

Do you see why this is a faithful translation?

Recap: Translating into Logic

Using the predicates

- *Person*(p), which states that p is a person, and
- *Loves*(x, y), which states that x loves y,

write a sentence in first-order logic that means "everybody loves someone else."

Everybody loves someone else

Every person loves some other person

Every person p loves some other person

$\forall p.\ (Person(p) \rightarrow$

p loves some other person

$)$

$\forall p.\ (Person(p) \rightarrow$

there is some other person that p loves

$)$

$\forall p. \; (Person(p) \to$

there is a person other than p that p loves

$)$

$\forall p. \ (Person(p) \rightarrow$

there is a person q other than p where p loves q

$)$

$\forall p. \ (Person(p) \rightarrow$

there is a person q other than p where p loves q

$)$

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q) \land p \neq q \land$

$\quad\quad$ *p loves q*

$\quad)$

$)$

$$
\begin{aligned}
&\forall p.\ (Person(p) \to \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad) \\
&)
\end{aligned}
$$

New Stuff!

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means "the empty set exists."

Using the predicates

- $Set(S)$, which states that S is a set, and
- $x \in y$, which states that x is an element of y,

write a sentence in first-order logic that means “the empty set exists.”

First-order logic doesn't have set operators or symbols “built in.” If we only have the predicates given above, how might we describe this?

The empty set exists.

There is some set S that is empty.

$\exists S.\ (Set(S)\ \wedge$

S is empty.

$)$

$\exists S.\ (Set(S)\ \wedge$

S does not contain any elements.

$)$

$\exists S.\ (Set(S) \wedge$

for every x, x is not an element of S.

$)$

$$
\begin{aligned}
&\exists S.\ (Set(S)\ \wedge \\
&\quad \forall x.\ \neg(x \in S) \\
&)
\end{aligned}
$$

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

Using the predicates

- *Tournament*(T), which states that T is a tournament;
- $p \in T$, which states that p is a player in tournament T; and
- *Beat*(p_1, p_2), which states that p_1 beat p_2,

write a sentence in first-order logic that means “every tournament has a tournament winner.”

Every tournament has a tournament winner

Every tournament T has a tournament winner

$$\forall T.\ (Tournament(T) \to$$

T has a tournament winner

$$)$$

$\forall T.\ (Tournament(T) \to$

some player in T is a tournament winner

$)$

$\forall T.\ (Tournament(T) \rightarrow$

some player w in T is a tournament winner

$)$

$\forall T.\ (Tournament(T) \rightarrow$

$\exists w.\ (w \in T\ \land$

w is a tournament winner

$)$

$)$

$\forall T. \ (Tournament(T) \to$

$\quad \exists w. \ (w \in T \ \land$

for each other player p, either w beat p or w beat someone who beat p

$\quad)$

$)$

$\forall T.\ (Tournament(T) \rightarrow$

$\quad \exists w.\ (w \in T\ \land$

for each other player p,

either w beat p or

w beat someone who beat p

$\quad)$

$)$

$\forall T.\ (Tournament(T) \rightarrow$

$\quad \exists w.\ (w \in T\ \land$

$\quad\quad \forall p.\ (p \in T \land p \neq w \rightarrow$

either w beat p or

w beat someone who beat p

$\quad\quad)$

$\quad)$

$)$

$$
\begin{aligned}
&\forall T.\ (\mathit{Tournament}(T) \rightarrow \\
&\quad \exists w.\ (w \in T\ \land \\
&\qquad \forall p.\ (p \in T \land\ p \neq\ w \rightarrow \\
&\qquad\quad \mathit{Beat}(w,\ p)\ \lor \\
&\qquad\quad \textit{w beat someone who beat p} \\
&\qquad) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{array}{l}
\forall T.\ (Tournament(T) \rightarrow \\
\quad \exists w.\ (w \in T \land \\
\quad\quad \forall p.\ (p \in T \land p \neq w \rightarrow \\
\quad\quad\quad Beat(w, p) \lor \\
\quad\quad\quad \textit{w beat some player q who beat p} \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

$$
\begin{aligned}
&\forall T.\ (\mathit{Tournament}(T) \rightarrow \\
&\quad \exists w.\ (w \in T\ \land \\
&\quad\quad \forall p.\ (p \in T \land p \neq w \rightarrow \\
&\quad\quad\quad \mathit{Beat}(w, p) \lor \\
&\quad\quad\quad \exists q.\ (q \in T\ \land \\
&\quad\quad\quad\quad \mathit{Beat}(w, q) \land \mathit{Beat}(q, p) \\
&\quad\quad\quad) \\
&\quad\quad) \\
&\quad) \\
&)
\end{aligned}
$$

Time-Out for Announcements!

Problem Sets

- Problem Set Two due right now if you're using a late period.
- Solutions will be released after lecture in hardcopy; feel free to pick them up here or in the filing cabinet.
  - SCPD students – solutions should arrive soon.
- Problem Set Three is due on Friday.
- Want more practice? Stop by a discussion section and work with TA guidance!

First Midterm Exam

- The first midterm exam is next ***Thursday, October 23*** at 7PM.
  - Full logistics next time.
- Covers material up through and including today's lecture.
- Focus is on topics also exercised in PS1 – PS3.
- Need to take the exam at an alternate time? Let us know no later than Wednesday at the start of class.

Reminder: Casual CS Dinner

- WiCS is holding its first Casual CS Dinner of the quarter this Wednesday from 6PM – 8PM on the fifth floor of Gates.
- Fantastic event; everyone is welcome and I highly recommend it.

The Blocks World Tool

- We have just uploaded a tool called *Blocks World* to the CS103 website.

  http://web.stanford.edu/class/cs103/tools/blocks-world/

- Useful for exploring first-order logic:
  - Drag and drop objects to create a world.
  - Write formulas in first-order logic.
  - Get feedback on whether your statements are true or false.
- Uses a slightly different syntax for FOL than what we've seen; check out the “Help” and “Examples” links for more information!

Your Questions

“Do we have/should we begin inductive proofs with vacuous cases? It seems like if we do so we are sidestepping the actual problem.”

“Why is the syllabus for CS103 so different from equivalent courses at other schools? MIT goes into much more detail about number theory and graphs, and barely touches anything about context free grammars etc. Are we missing out on anything?”

“Have you ever struggled with a math class? Why are you so good at it: is it just because you're super smart, or did you actually have to work at it? If we're not super gifted, will it ever just click, or is it an uphill battle the whole way?”

“Would you share something you learned while at the Grace Hopper conference that you found particularly useful or interesting?”

Back to CS103!

Mechanics: Negating Statements

Negating Quantifiers

- We spent much of Wednesday's lecture discussing how to negate propositional constructs.
- How do we negate quantifiers?

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	For some choice of x, $\neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	**For some choice of x, $\neg P(x)$**
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	**For some choice of x, $\neg P(x)$**	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	**For some choice of x, $\neg P(x)$**	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	For any choice of x, $\neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	**For any choice of x, $\neg P(x)$**
$\forall x.\ \neg P(x)$	**For any choice of x, $\neg P(x)$**	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	**For any choice of x, $\neg P(x)$**	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\boldsymbol{\exists x.\ \neg P(x)}$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\boldsymbol{\forall x.\ \neg P(x)}$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	For some choice of x, $P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\boldsymbol{\exists x.\ \neg P(x)}$
$\exists x.\ P(x)$	**For some choice of x, $P(x)$**	$\boldsymbol{\forall x.\ \neg P(x)}$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	**For some choice of x, $P(x)$**
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	**For some choice of x, $P(x)$**	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	For any choice of x, $P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For any choice of *x*, $P(x)$**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	**For any choice of *x*, $P(x)$**

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	**For any choice of *x*, $P(x)$**	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of *x*, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of *x*, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of *x*, $\neg P(x)$	$\forall x.\ P(x)$

An Extremely Important Table

	When is this true?	When is this false?
$\forall x.\ P(x)$	For any choice of x, $P(x)$	$\exists x.\ \neg P(x)$
$\exists x.\ P(x)$	For some choice of x, $P(x)$	$\forall x.\ \neg P(x)$
$\forall x.\ \neg P(x)$	For any choice of x, $\neg P(x)$	$\exists x.\ P(x)$
$\exists x.\ \neg P(x)$	For some choice of x, $\neg P(x)$	$\forall x.\ P(x)$

Negating First-Order Statements

- Use the equivalences

$$\neg \forall x.\ \varphi \quad \equiv \quad \exists x.\ \neg \varphi$$

$$\neg \exists x.\ \varphi \quad \equiv \quad \forall x.\ \neg \varphi$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

Taking a Negation

$$\forall x.\ \exists y.\ Loves(x, y)$$

("Everyone loves someone.")

$$\neg \forall x.\ \exists y.\ Loves(x, y)$$
$$\exists x.\ \neg \exists y.\ Loves(x, y)$$
$$\exists x.\ \forall y.\ \neg Loves(x, y)$$

("There's someone who doesn't love anyone.")

Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \equiv p \to \neg q$$

$$\neg(p \to q) \equiv p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

Negating Quantifiers

- What is the negation of the following statement, which says “there is a cute puppy”?

$$\boldsymbol{\exists x.\ (Puppy(x) \land Cute(x))}$$

- We can obtain it as follows:

$$\neg\exists x.\ (Puppy(x) \land Cute(x))$$

$$\forall x.\ \neg(Puppy(x) \land Cute(x))$$

$$\forall x.\ (Puppy(x) \to \neg Cute(x))$$

- This says “every puppy is not cute.”
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

("There is a set that doesn't contain anything")

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

("Every set contains at least one element")

These two statements are *not* negations of one another. Can you explain why?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

$$\forall S.\ (Set(S) \land \exists x.\ (x \in S))$$

These two statements are *not* negations of one another. Can you explain why?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

("There is a set that doesn't contain anything")

$$\forall S.\ (Set(S) \land \exists x.\ (x \in S))$$

These two statements are *not* negations of one another. Can you explain why?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

(*“There is a set that doesn't contain anything”*)

$$\forall S.\ (Set(S) \land \exists x.\ (x \in S))$$

(*“Everything is a set that contains something”*)

These two statements are *not* negations of one another. Can you explain why?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

(*"There is a set that doesn't contain anything"*)

$$\forall S.\ (Set(S) \land \exists x.\ (x \in S))$$

(*"Everything is a set that contains something"*)

Remember: ∀ usually goes with →, not ∧

$$
\begin{array}{l}
\forall T.\ (\mathit{Tournament}(T) \rightarrow \\
\quad \exists w.\ (w \in T \wedge \\
\quad\quad \forall p.\ (p \in T \wedge p \neq w \rightarrow \\
\quad\quad\quad (\mathit{Beat}(w, p) \vee \\
\quad\quad\quad\quad \exists q.\ (q \in T \wedge \mathit{Beat}(w, q) \wedge \\
\quad\quad\quad\quad\quad \mathit{Beat}(q, p) \\
\quad\quad\quad\quad) \\
\quad\quad\quad) \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

$$
\begin{aligned}
&\neg\forall T.\ (Tournament(T) \rightarrow \\
&\quad \exists w.\ (w \in T \wedge \\
&\qquad \forall p.\ (p \in T \wedge p \neq w \rightarrow \\
&\qquad\quad (Beat(w, p) \vee \\
&\qquad\qquad \exists q.\ (q \in T \wedge Beat(w, q) \wedge \\
&\qquad\qquad\quad Beat(q, p) \\
&\qquad\qquad) \\
&\qquad\quad) \\
&\qquad) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{array}{l}
\exists T.\ \neg(Tournament(T) \rightarrow \\
\quad \exists w.\ (w \in T\ \wedge \\
\quad\quad \forall p.\ (p \in T\ \wedge\ p \neq w \rightarrow \\
\quad\quad\quad (Beat(w, p)\ \vee \\
\quad\quad\quad\quad \exists q.\ (q \in T\ \wedge\ Beat(w, q)\ \wedge \\
\quad\quad\quad\quad\quad Beat(q, p) \\
\quad\quad\quad\quad) \\
\quad\quad\quad) \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

$$
\begin{array}{l}
\exists T.\ (Tournament(T) \land \\
\quad \neg\exists w.\ (w \in T \land \\
\quad\quad \forall p.\ (p \in T \land p \neq w \rightarrow \\
\quad\quad\quad (Beat(w, p) \lor \\
\quad\quad\quad\quad \exists q.\ (q \in T \land Beat(w, q) \land \\
\quad\quad\quad\quad\quad Beat(q, p) \\
\quad\quad\quad\quad) \\
\quad\quad\quad) \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

$$
\begin{array}{l}
\exists T.\ (\mathit{Tournament}(T) \land \\
\quad \forall w.\ \neg(w \in T \land \\
\quad\quad \forall p.\ (p \in T \land p \neq w \rightarrow \\
\quad\quad\quad (\mathit{Beat}(w, p) \lor \\
\quad\quad\quad\quad \exists q.\ (q \in T \land \mathit{Beat}(w, q) \land \\
\quad\quad\quad\quad\quad \mathit{Beat}(q, p) \\
\quad\quad\quad\quad) \\
\quad\quad\quad) \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

$$
\begin{array}{l}
\exists T.\ (\mathit{Tournament}(T)\ \land \\
\quad \forall w.\ (w \in T \rightarrow \\
\quad\quad \neg\forall p.\ (p \in T \land p \neq w \rightarrow \\
\quad\quad\quad (\mathit{Beat}(w, p) \lor \\
\quad\quad\quad\quad \exists q.\ (q \in T \land \mathit{Beat}(w, q) \land \\
\quad\quad\quad\quad\quad \mathit{Beat}(q, p) \\
\quad\quad\quad\quad) \\
\quad\quad\quad) \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

$$
\begin{array}{l}
\exists T.\ (Tournament(T) \land \\
\quad \forall w.\ (w \in T \rightarrow \\
\quad\quad \exists p.\ \neg(p \in T \land p \neq w \rightarrow \\
\quad\quad\quad (Beat(w, p) \lor \\
\quad\quad\quad\quad \exists q.\ (q \in T \land Beat(w, q) \land \\
\quad\quad\quad\quad\quad Beat(q, p) \\
\quad\quad\quad\quad) \\
\quad\quad\quad) \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

$$
\begin{aligned}
&\exists T.\ (Tournament(T) \land \\
&\quad \forall w.\ (w \in T \rightarrow \\
&\quad\quad \exists p.\ (p \in T \land p \neq w \land \\
&\quad\quad\quad \neg(Beat(w, p) \lor \\
&\quad\quad\quad\quad \exists q.\ (q \in T \land Beat(w, q) \land \\
&\quad\quad\quad\quad\quad Beat(q, p) \\
&\quad\quad\quad\quad) \\
&\quad\quad\quad) \\
&\quad\quad) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{array}{l}
\exists T.\ (\mathit{Tournament}(T) \land \\
\quad \forall w.\ (w \in T \rightarrow \\
\quad\quad \exists p.\ (p \in T \land p \neq w \land \\
\quad\quad\quad (\neg \mathit{Beat}(w, p) \land \\
\quad\quad\quad\quad \neg \exists q.\ (q \in T \land \mathit{Beat}(w, q) \land \\
\quad\quad\quad\quad\quad \mathit{Beat}(q, p) \\
\quad\quad\quad\quad) \\
\quad\quad\quad) \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

$$
\begin{aligned}
&\exists T.\ (Tournament(T)\ \land \\
&\quad \forall w.\ (w \in T \rightarrow \\
&\quad\quad \exists p.\ (p \in T \land p \neq w \land \\
&\quad\quad\quad (\neg Beat(w, p) \land \\
&\quad\quad\quad\quad \forall q.\ \neg(q \in T \land Beat(w, q) \land \\
&\quad\quad\quad\quad\quad Beat(q, p) \\
&\quad\quad\quad\quad) \\
&\quad\quad\quad) \\
&\quad\quad) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{array}{l}
\exists T.\ (Tournament(T) \land \\
\quad \forall w.\ (w \in T \rightarrow \\
\quad\quad \exists p.\ (p \in T \land p \neq w \land \\
\quad\quad\quad (\neg Beat(w, p) \land \\
\quad\quad\quad\quad \forall q.\ \neg((q \in T \land Beat(w, q)) \land \\
\quad\quad\quad\quad\quad Beat(q, p) \\
\quad\quad\quad\quad) \\
\quad\quad\quad) \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

$$
\begin{array}{l}
\exists T.\ (Tournament(T) \land \\
\quad \forall w.\ (w \in T \rightarrow \\
\quad\quad \exists p.\ (p \in T \land p \neq w \land \\
\quad\quad\quad (\neg Beat(w, p) \land \\
\quad\quad\quad\quad \forall q.\ ((q \in T \land Beat(w, q)) \rightarrow \\
\quad\quad\quad\quad\quad \neg Beat(q, p) \\
\quad\quad\quad\quad) \\
\quad\quad\quad) \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

$$
\begin{aligned}
&\exists T.\ (Tournament(T) \land \\
&\quad \forall w.\ (w \in T \to \\
&\quad\quad \exists p.\ (p \in T \land p \neq w \land \\
&\quad\quad\quad (\neg Beat(w, p) \land \\
&\quad\quad\quad\quad \forall q.\ (q \in T \land Beat(w, q) \to \\
&\quad\quad\quad\quad\quad \neg Beat(q, p) \\
&\quad\quad\quad\quad) \\
&\quad\quad\quad) \\
&\quad\quad) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{array}{l}
\exists T.\ (Tournament(T)\ \land \\
\quad \forall w.\ (w \in T \to \\
\quad\quad \exists p.\ (p \in T \land p \neq w \land \\
\quad\quad\quad (\neg Beat(w, p)\ \land \\
\quad\quad\quad\quad \forall q.\ (q \in T \land Beat(w, q) \to \\
\quad\quad\quad\quad\quad \neg Beat(q, p)) \\
\quad\quad\quad\quad) \\
\quad\quad\quad) \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

$$
\begin{aligned}
&\exists T.\ (Tournament(T) \land \\
&\quad \forall w.\ (w \in T \rightarrow \\
&\quad\quad \exists p.\ (p \in T \land p \neq w \land \\
&\quad\quad\quad (\neg Beat(w, p) \land \\
&\quad\quad\quad\quad \forall q.\ (q \in T \land Beat(w, q) \rightarrow \\
&\quad\quad\quad\quad\quad \neg Beat(q, p)) \\
&\quad\quad\quad\quad) \\
&\quad\quad\quad) \\
&\quad\quad) \\
&\quad) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\exists T.\ (Tournament(T)\ \land \\
&\quad \forall w.\ (w \in T \to \\
&\quad\quad \exists p.\ (p \in T \land p \neq w \land \\
&\quad\quad\quad (\neg Beat(w, p) \land \\
&\quad\quad\quad\quad \forall q.\ (q \in T \land Beat(w, q) \to \\
&\quad\quad\quad\quad\quad \neg Beat(q, p)) \\
&\quad\quad\quad\quad) \\
&\quad\quad\quad) \\
&\quad\quad) \\
&\quad) \\
&)
\end{aligned}
$$

There is a tournament T where

$$
\begin{array}{l}
\exists T.\ (Tournament(T)\ \land \\
\quad \forall w.\ (w \in T \to \\
\quad\quad \exists p.\ (p \in T \land p \neq w\ \land \\
\quad\quad\quad (\neg Beat(w, p)\ \land \\
\quad\quad\quad\quad \forall q.\ (q \in T \land Beat(w, q) \to \\
\quad\quad\quad\quad\quad \neg Beat(q, p)) \\
\quad\quad\quad\quad) \\
\quad\quad\quad) \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

There is a tournament T where

For any player w in T,

$$
\begin{aligned}
&\exists T.\ (Tournament(T)\ \land \\
&\quad \forall w.\ (w \in T \to \\
&\quad\quad \exists p.\ (p \in T \land p \neq w \land \\
&\quad\quad\quad (\neg Beat(w, p)\ \land \\
&\quad\quad\quad\quad \forall q.\ (q \in T \land Beat(w, q) \to \\
&\quad\quad\quad\quad\quad \neg Beat(q, p)) \\
&\quad\quad\quad\quad) \\
&\quad\quad\quad) \\
&\quad\quad) \\
&\quad) \\
&)
\end{aligned}
$$

There is a tournament T where

For any player w in T,

There is some other player p where

$$
\begin{array}{l}
\exists T.\ (Tournament(T)\ \land \\
\quad \forall w.\ (w \in T \to \\
\quad\quad \exists p.\ (p \in T \land p \neq w \land \\
\quad\quad\quad (\neg Beat(w, p)\ \land \\
\quad\quad\quad\quad \forall q.\ (q \in T \land Beat(w, q) \to \\
\quad\quad\quad\quad\quad \neg Beat(q, p)) \\
\quad\quad\quad\quad) \\
\quad\quad\quad) \\
\quad\quad) \\
\quad) \\
)
\end{array}
$$

There is a tournament *T* where

For any player *w* in *T*,

There is some other player *p* where

w didn't beat *p* and

$$
\begin{aligned}
&\exists T.\ (Tournament(T)\ \land \\
&\quad \forall w.\ (w \in T \rightarrow \\
&\quad\quad \exists p.\ (p \in T \land p \neq w \land \\
&\quad\quad\quad (\neg Beat(w, p) \land \\
&\quad\quad\quad\quad \forall q.\ (q \in T \land Beat(w, q) \rightarrow \\
&\quad\quad\quad\quad\quad \neg Beat(q, p)) \\
&\quad\quad\quad\quad) \\
&\quad\quad\quad) \\
&\quad\quad) \\
&\quad) \\
&)
\end{aligned}
$$

There is a tournament T where
For any player w in T,
There is some other player p where
w didn't beat p and
For each player q, if w beat q,

$$
\begin{aligned}
&\exists T.\ (Tournament(T) \land \\
&\quad \forall w.\ (w \in T \rightarrow \\
&\qquad \exists p.\ (p \in T \land p \neq w \land \\
&\qquad\quad (\neg Beat(w, p) \land \\
&\qquad\qquad \forall q.\ (q \in T \land Beat(w, q) \rightarrow \\
&\qquad\qquad\quad \neg Beat(q, p)) \\
&\qquad\qquad) \\
&\qquad\quad) \\
&\qquad) \\
&\quad) \\
&)
\end{aligned}
$$

There is a tournament T where
    For any player w in T,
        There is some other player p where
            w didn't beat p and
                For each player q, if w beat q,
                    then q didn't beat p.

Why All This Matters

- Translating a statement into first-order logic forces you to see the underlying mathematical structure of that statement.
- Having a statement written in first-order logic permits you to transform and manipulate that statement in ways that can help guide you in a proof.

Next Time

- **Binary Relations**
  - Studying properties of $=$, $\subseteq$, $\neq$, $<$, $\equiv_k$, etc.
  - Using first-order logic in definitions.