CS 154

Lecture 10: Lots of Reductions, Rice’s Theorem

Next Tuesday (2/17)

Your Midterm: At 12:50pm,

in Bishop Auditorium

(see website for more information)

Today: instead of a new homework,

you'll get a practice midterm

Don't panic!

Practice midterm will be harder than midterm

Next Tuesday (2/17)

Your Midterm: At 12:50pm,
in Bishop Auditorium

FAQ: What is fair game for the midterm?

Everything up to this lecture

FAQ: Can I bring notes?

Yes, one single-sided sheet of notes, letter paper

Definition: A **decidable predicate $R(x,y)$** is a proposition about the input strings x and y, such that some TM M implements R. That is,

for all x, y, $R(x,y)$ is TRUE $\Rightarrow$ $M(x,y)$ accepts
$R(x,y)$ is FALSE $\Rightarrow$ $M(x,y)$ rejects

Can think of R as a function from $\Sigma^* \times \Sigma^* \rightarrow \{T,F\}$

EXAMPLES: $R(x,y)$ = "xy has at most 100 zeroes"
$R(N,y)$ = "TM N halts on y in at most 99 steps"

Proposition: A is decidable if and only if there is some decidable predicate R such that
$A = \{x \mid R(x,\varepsilon)\}$

Theorem: A language A is *recognizable* if and only if there is a decidable predicate $R(x, y)$ such that:

$$A = \{ x \mid \exists y\, R(x, y) \}$$

Proof: (1) If $A = \{ x \mid \exists y\, R(x,y) \}$ then A is recognizable

A TM can enumerate over all y's and try them in R.
If there is a y s.t. R(x,y) accepts, it will be found

(2) If A is recognizable, then $A = \{ x \mid \exists y\, R(x,y) \}$

Let M recognize A.
Let $R(x,y)$ be TRUE iff M accepts x in $|y|$ steps
M accepts x $\Leftrightarrow \exists y\, R(x,y)$

Mapping Reductions

$f : \Sigma^* \rightarrow \Sigma^*$ is a **computable function** if
there is a Turing machine M that halts with
just f(w) written on its tape, for every input w

A language A is ***mapping reducible*** to language B,
written as $A \leq_m B$, if there is a computable
$f : \Sigma^* \rightarrow \Sigma^*$ such that for every w,

$$w \in A \Leftrightarrow f(w) \in B$$

***f* is called a mapping reduction**
(or many-one reduction) from A to B

Theorem: If $A \leq_m B$ and B is decidable, then A is decidable

Corollary: If $A \leq_m B$ and A is undecidable, then B is undecidable

Theorem: If $A \leq_m B$ and B is recognizable, then A is recognizable

Corollary: If $A \leq_m B$ and A is unrecognizable, then B is unrecognizable

Theorem: $A_{TM} \leq_m HALT_{TM}$

f(z) := Decode z into a pair (M, w)

Construct M' with the specification:

"M'(w) = Simulate M on w.

if M(w) accepts then *accept*

else *loop forever*"

Output (M', w)

We have $z \in A_{TM} \Leftrightarrow (M', w) \in HALT_{TM}$

Theorem: $A_{TM} \leq_m HALT_{TM}$

Corollary: $\neg A_{TM} \leq_m \neg HALT_{TM}$

Proof?

Corollary: $\neg HALT_{TM}$ is unrecognizable!

Proof: If $\neg HALT_{TM}$ were recognizable, then $\neg A_{TM}$ would be recognizable…

Theorem: $\text{HALT}_{\text{TM}} \leq_m A_{\text{TM}}$

Proof: Define the computable function:

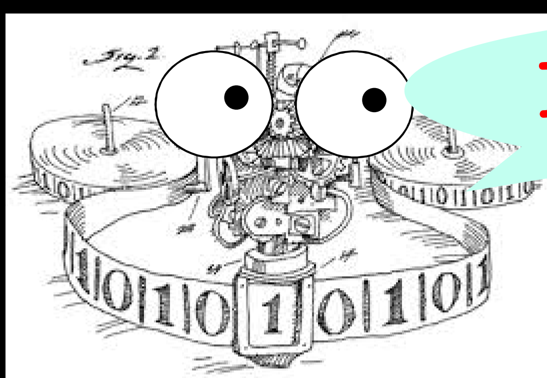
f(z) := Decode z into a pair (M, w)
Construct M' with the specification:
"M'(w) = Simulate M on w.
If M(w) halts then *accept*
else *loop forever*"
Output (M', w)

Observe $(M, w) \in \text{HALT}_{\text{TM}} \Leftrightarrow (M', w) \in A_{\text{TM}}$

Corollary: $HALT_{TM} \equiv_m A_{TM}$

Yo, T.M.! I can give you the magical power to either compute the halting problem, or the acceptance problem. Which do you want?

Wow, hm, so hard to choose…



I can't decide!

The Emptiness Problem

$EMPTY_{DFA}$ = { M | M is a DFA such that L(M) = ∅}

Given a DFA, does it reject every input?

Theorem: $EMPTY_{DFA}$ is decidable

Why?

$EMPTY_{NFA}$ = { M | M is a NFA such that L(M) = ∅}

$EMPTY_{REX}$ = { R | R is a regexp such that L(R) = ∅}

The Emptiness Problem for TMs

$EMPTY_{TM}$ = { M | M is a TM such that $L(M) = \varnothing$}

Given a program, does it reject every input?

Theorem: $EMPTY_{TM}$ is *not recognizable*

Proof: Show that $\neg A_{TM} \leq_m EMPTY_{TM}$

f(z) := Decode z into a pair (M, w).

Output a TM M' with the behavior:

"M'(x) := if (x = w) then

run M(w) and output its answer, else *reject*"

$z \in A_{TM} \Leftrightarrow L(M') \neq \varnothing$

$\Leftrightarrow M' \notin EMPTY_{TM}$

$\Leftrightarrow f(z) \notin EMPTY_{TM}$

The Equivalence Problem

$EQ_{TM} = \{(M, N) \mid M, N \text{ are TMs and } L(M) = L(N)\}$

Do two programs compute the same function?

Theorem: EQ_{TM} is *unrecognizable*

Proof: Reduce $EMPTY_{TM}$ to EQ_{TM}

Let $M_{\varnothing}$ be a "dummy" TM
with no path from start state to accept state

Define $f(M) := (M, M_{\varnothing})$

$$\begin{aligned} M \in EMPTY_{TM} &\Leftrightarrow L(M) = L(M_{\varnothing}) = \varnothing \\ &\Leftrightarrow (M, M_{\varnothing}) \in EQ_{TM} \end{aligned}$$

Problem 1

REVERSE = { M | M is a TM with the property:
for all w, M(w) accepts $\Leftrightarrow$ M(w^R) accepts}.

Decidable or not?

REVERSE is undecidable.

x → M_w If x = 01, accept.
If x = 10, run M(w).
Otherwise, reject. →

Given a machine D for deciding the language REVERSE, we show how to decide A_{TM}

N → D →

N accepts w^R iff
?
N accepts w

M(w) accepts → $L(M_w) = \{01,10\}$

M(w) doesn't acc → $L(M_w) = \{01\}$

Problem 2 Undecidable

{ (M, w) | M is a TM that on input w, tries to move its head past the left end of the input }

Problem 3 Decidable

{ (M, w) | M is a TM that on input w, moves its head left at least once, at some point}

Problem 2 Undecidable

L' = { (M, w) | M is a TM that on input w, tries to move its head past the left end of the input }

Proof: Reduce A_{TM} to L'

On input (M,w), make a TM N that shifts w over one cell, marks a special symbol $ on the leftmost cell, then simulates M(w) on the tape.

If M's head moves to the cell with $ but has not yet accepted, N moves the head back to the right.

If M accepts, N tries to move its head past the $.

(M,w) is in A_{TM} if and only if (N,w) is in L'

Problem 3 Decidable

{ (M, w) | M is a TM that on input w, moves its head left at least once, at some point}

On input (M,w), run M on w for $|Q_M| + |w| + 1$ steps.

Accept If M's head moved left at all
Reject Otherwise

(Why does this work?)

Rice's Theorem

Suppose L is a language that satisfies two conditions:

1. (Nontrivial) There are TMs M_{YES} and M_{NO}, where $M_{YES} \in L$ and $M_{NO} \notin L$
2. (Semantic) For all TMs M_1 and M_2 such that $L(M_1) = L(M_2)$, $M_1 \in L$ if and only if $M_2 \in L$

Then, L is undecidable.

A Huge Hammer for Undecidability!

Examples and Non-Examples

Semantic Properties *P(M)*

- M accepts 0
- for all w, M(w) accepts iff M(w^R) accepts
- L(M) = {0}
- L(M) is empty
- L(M) = Σ^*
- M accepts 154 strings

Not Semantic!

- M halts and rejects 0
- M tries to move its head off the left end of the tape, on input 0
- M never moves its head left on input 0
- M has exactly 154 states
- M halts on all inputs

L = {M | *P*(M) is true} is undecidable

Rice's Theorem: Any nontrivial semantic L over Turing machines is undecidable.

Proof: We'll reduce A_{TM} to the language L

Define $M_{\varnothing}$ to be a TM that never halts

Suppose first that $M_{\varnothing} \notin L$

Let $M_{YES} \in L$ (such M_{YES} exists, by assumption)

Reduction from A_{TM} On input (M,w), output:

"$M_w(x)$:= If (M acc. w) & (M_{YES} acc. x) then ACCEPT else REJECT"

If $M_{\varnothing} \in L$ instead, we can reduce $\neg A_{TM}$ to L. Output:

"$M_w(x)$:= If (M acc. w) & (M_{NO} acc. x), then ACCEPT else REJECT"

The Regularity Problem for Turing Machines

REGULAR_{TM} = { M | M is a TM and L(M) is regular}

Given a program, is it equivalent to some DFA?

Theorem: REGULAR_{TM} is *not recognizable*

Proof 1: Show that $\neg A_{TM} \leq_m \text{REGULAR}_{TM}$

f(z) := Decode z into (M, w). Output a TM M':
"M'(x) := if ($x = 0^n1^n$) then run M(w)
else reject"

$z \in A_{TM} \Rightarrow$ f(z) = M' such that M' accepts $\{0^n1^n\}$
$z \notin A_{TM} \Rightarrow$ f(z) = M' such that M' accepts nothing
$z \notin A_{TM} \Leftrightarrow f(M, w) \in \text{REGULAR}_{TM}$

The Regularity Problem for Turing Machines

$\text{REGULAR}_{\text{TM}}$ = { M | M is a TM and L(M) is regular}

Given a program, is it equivalent to some DFA?

Theorem: $\text{REGULAR}_{\text{TM}}$ is *not recognizable*

Proof 2: Use Rice's Theorem!

$\text{REGULAR}_{\text{TM}}$ is nontrivial:

- there's an $M_\varnothing$ which never halts: $M_\varnothing \in \text{REGULAR}_{\text{TM}}$
- there's M' deciding $\{0^n1^n \mid n \geq 0\}$: $M' \notin \text{REGULAR}_{\text{TM}}$

$\text{REGULAR}_{\text{TM}}$ is semantic:

If L(M) = L(M') then L(M) is regular iff L(M') is regular, therefore $M \in \text{REGULAR}_{\text{TM}}$ iff $M' \in \text{REGULAR}_{\text{TM}}$