MDP and RL

CS221: Section 4

Policy Iteration vs Value Iteration

SARSA vs Q-Learning

Function Approximation and Q-Learning

Policy Iteration vs Value Iteration

SARSA vs Q-Learning

Function Approximation and Q-Learning

Markov Decision Process (MDP)

$\boldsymbol{s} \in States$

$\boldsymbol{Actions}(\boldsymbol{s})$: $actions\ from\ state\ s$

$\boldsymbol{T}(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{s}')$: $probability\ next\ state\ is\ s',\ given\ action\ a\ in\ state\ s$

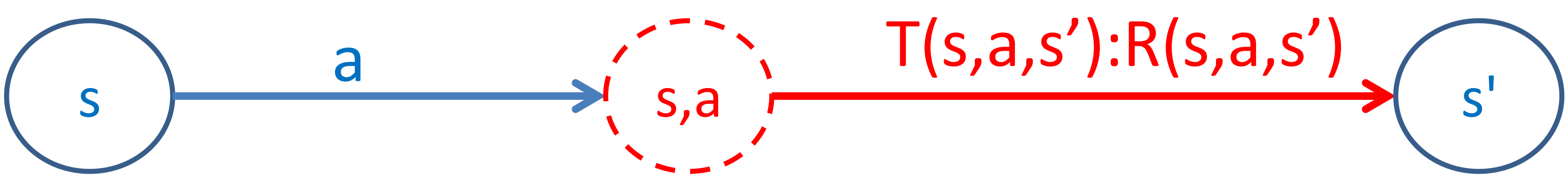
$$Note\ that\ \sum_{s'} T(s, a, s') = 1\ for\ all\ s \in States, a \in Actions(s)$$

$\boldsymbol{R}(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{s}')$: $reward\ for\ transition\ (s, a, s')$

$\boldsymbol{IsEnd}(\boldsymbol{s})$: $whether\ state\ marks\ end\ of\ game$

$$Alternatively, T(s, a, s) = 1\ and\ R(s, a, s) = 0$$

$\boldsymbol{0} \leq \boldsymbol{\gamma} \leq \boldsymbol{1}$: $discount\ factor$



Markov Decision Process (MDP)

Given MDP

$\boldsymbol{\pi}$: $policy$

$take\ action\ \pi(s) in\ state\ s$

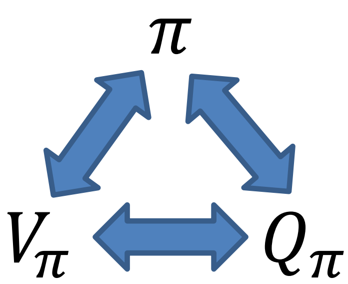
$\boldsymbol{V_\pi}$: $value\ function\ for\ policy\ \pi$

$V_\pi(s)\ is\ expected\ utility\ of\ following\ \pi\ starting\ in\ s$

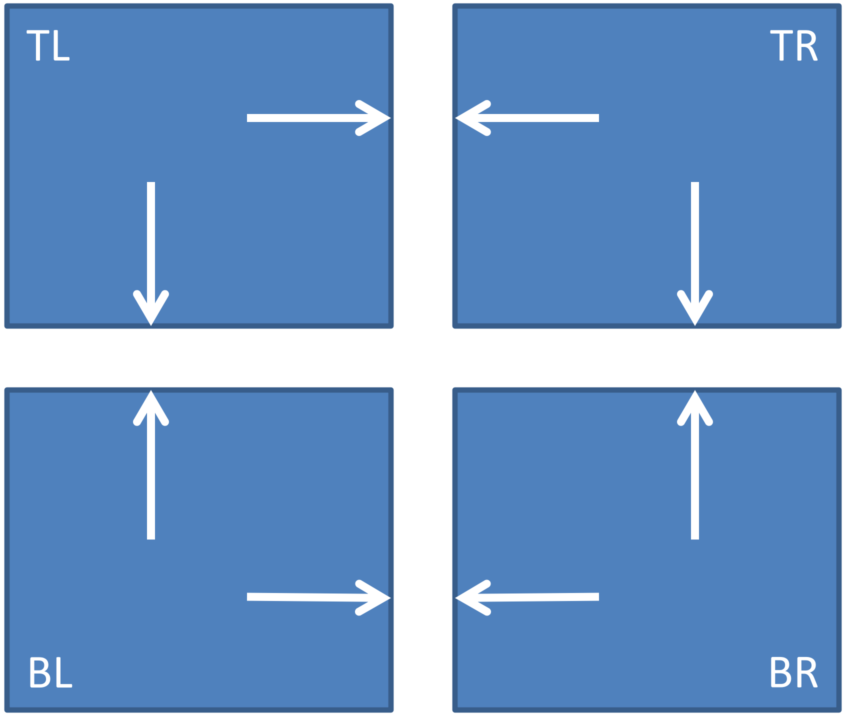
$\boldsymbol{Q_\pi}$: $Q - value\ of\ policy\ \pi$

$Q_\pi(s, a)\ is\ expected\ utility\ of\ taking\ action\ a\ in\ s, then\ following\ \pi$

Want to find optimal policy (that maximizes expected sum of discounted rewards)



Running Example



$States = \{TL, TR, NL, BR\}$
$EndState = BR$

$Actions(s) = \{H, V\}, \forall s$

$$\mathrm{T}(s, a, s') = \begin{cases} s'(s, a), & w.p. 3/4 \\ s'(s, a^c), & w.p. 1/4 \end{cases}$$
$except\ T(BR, a, BR) = 1$

$Reward(s, a, s') = 4\ if\ s' = BR\ else\ -1$

Simple $policy\ \pi(s) = H, \forall s$

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute V_π $\qquad \pi \Rightarrow V_\pi$

2. **Policy improvement**: given V_π , compute improved policy π' $\qquad V_\pi \Rightarrow \pi'$

3. **Repeat** until policy converges/does not change

$$\pi \Rightarrow V_\pi \Rightarrow \pi' \Rightarrow V_{\pi'} \Rightarrow \pi''$$

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute V_π
   a) Choose any V

$$\pi \longrightarrow V_\pi$$

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute V_π
   a) Choose any V
   b) Update

$$\pi \Longrightarrow V_\pi$$

$$V(s) \coloneqq \sum_{s'} T(s, \pi(s), s')\,[\, R(s, \pi(s), s') + \gamma V(\, s')], \qquad \forall s$$

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute V_π

$$\pi \Longrightarrow V_\pi$$

   a) Choose any V
   b) Update

$$V(s) \coloneqq \sum_{s'} T(s,\pi(s),s')\,[\,R(s,\pi(s),s') + \gamma V(\,s')],\qquad \forall s$$

or,
$$V \coloneqq F_\pi V \quad (compact)$$

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute V_π
   a) Choose any V
   b) Update $V \coloneqq F_\pi V$

$$\pi \longrightarrow V_\pi$$

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute V_π
    a) Choose any V
    b) Update $V \coloneqq F_\pi V$
    c) Repeat b) until convergence to obtain V_π
    We know $V_\pi = F_\pi F_\pi \ldots F_\pi V = F_\pi^\infty V$
    Practically, update t_{PE} times to get reasonable approximation

$$\pi \longrightarrow V_\pi$$

Policy Evaluation Example

V_0

TL 0	TR 0
BL 0	BR 0

Simple policy $\pi(s) = H, \forall s$
Evaluate V_π
$V_0(s) = 0, \forall s$

Policy Evaluation Example

V_0

TL 0	TR 0
BL 0	BR 0

V_1

TL -	TR -
BL -	BR -

$Simple\ policy\ \pi(s) = H, \forall s$
$Evaluate\ V_\pi$
$V_0(s) = 0, \forall s$

$$\begin{aligned} V_1(TL) &= \frac{3}{4} \times [-1 + V_0(TR)] + \frac{1}{4} \times [-1 + V_1(BL)] \\ &= \frac{3}{4} \times [-1 + \quad 0 \quad] + \frac{1}{4} \times [-1 + \quad 0 \quad] \\ &= -1 \end{aligned}$$

Policy Evaluation Example

V_0

TL 0	TR 0
BL 0	BR 0

V_1

TL -1	TR -
BL -	BR -

$Simple\ policy\ \pi(s) = H, \forall s$
$Evaluate\ V_\pi$
$V_0(s) = 0, \forall s$

$$
\begin{aligned}
V_1(TR) &= \frac{3}{4} \times [-1 + V_0(TL)] + \frac{1}{4} \times [4 + V_0(BR)] \\
&= \frac{3}{4} \times [-1 + \quad 0 \quad] + \frac{1}{4} \times [4 + \quad 0 \quad] \\
&= \frac{1}{4}
\end{aligned}
$$

Policy Evaluation Example

V_0

TL 0	TR 0
BL 0	BR 0

V_1

TL -1	TR 1/4
BL 1/4	BR 0

Policy Evaluation Example

V_0

TL 0	TR 0
0 BL	0 BR

V_1

TL -1	TR 1/4
1/4 BL	0 BR

V_2

TL -1/8	TR -1/2
5/2 BL	0 BR

…

V_π obtained

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute $V_\pi = F_\pi^\infty\ V$

$$\pi \Longrightarrow V_\pi$$

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute $V_\pi = F_\pi^\infty\, V$

$$\pi \Longrightarrow V_\pi$$

2. **Policy improvement**: given V_π , compute improved policy π'

$$V_\pi \Longrightarrow \pi'$$

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute $V_\pi = F_\pi^\infty V$

$$\pi \Rightarrow V_\pi$$

2. **Policy improvement**: given V_π , compute improved policy π'
   a) Compute

$$V_\pi \Rightarrow \pi'$$

$$Q_\pi(s,a) = \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_\pi(s')], \qquad \forall s,a$$

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute $V_\pi = F_\pi^\infty\ V$

$$\pi \Rightarrow V_\pi$$

2. **Policy improvement**: given V_π , compute improved policy π'

$$V_\pi \Rightarrow \pi'$$

   a) Compute

$$Q_\pi(s,a) = \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_\pi(s')], \qquad \forall s,a$$

   b) Compute $\pi'(s) = \arg \max_{a \in Actions(s)} Q_\pi(s,a)\,, \forall s$

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute $V_\pi = F_\pi^\infty\ V$

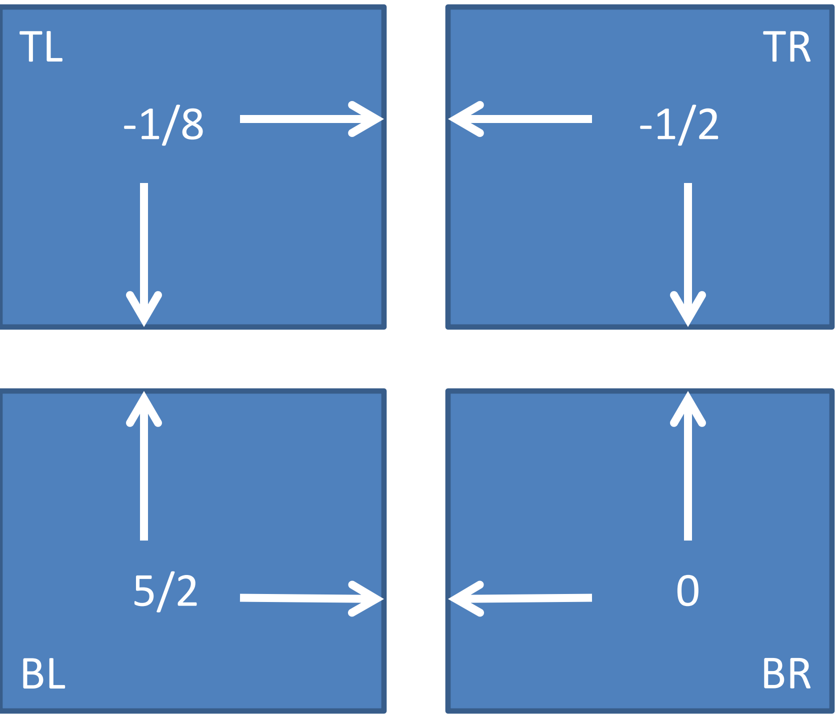
$$\pi \longrightarrow V_\pi$$

2. **Policy improvement**: given V_π , compute improved policy π'

$$V_\pi \longrightarrow \pi'$$

$$\pi'(s) = \arg \max_{a \in Actions(s)} \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_\pi(s')], \qquad \forall s$$

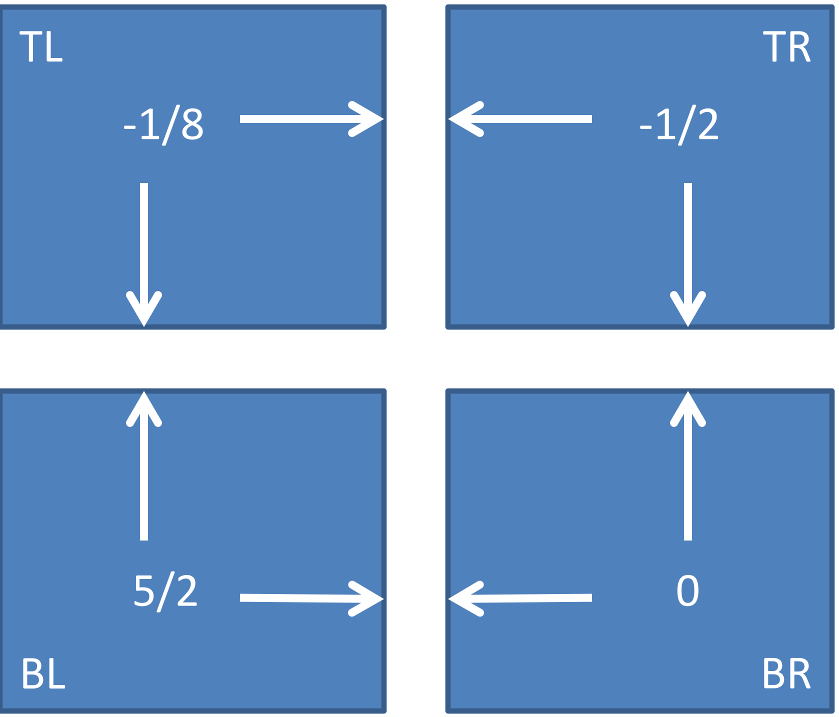
Policy Improvement Example



$Given\ V_\pi, evaluate\ \pi'\ (improved\ policy)$

$Find\ Q_\pi(s,a) for\ all\ states, choose\ best\ action$

Policy Improvement Example

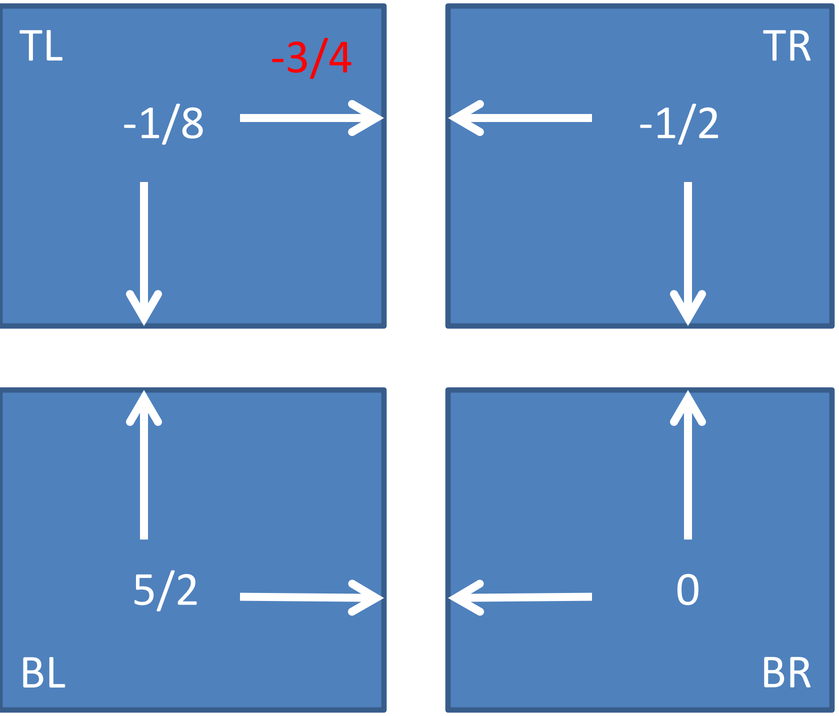


$$Given\ V_\pi, evaluate\ \pi'\ (improved\ policy)$$

$$Find\ Q_\pi(s,a) for\ all\ states, choose\ best\ action$$

$$Q_\pi(TL,H) = \frac{3}{4} \times \left[-1 + \frac{-1}{2}\right] + \frac{1}{4} \times \left[-1 + \frac{5}{2}\right] = \frac{-3}{4}$$

Policy Improvement Example

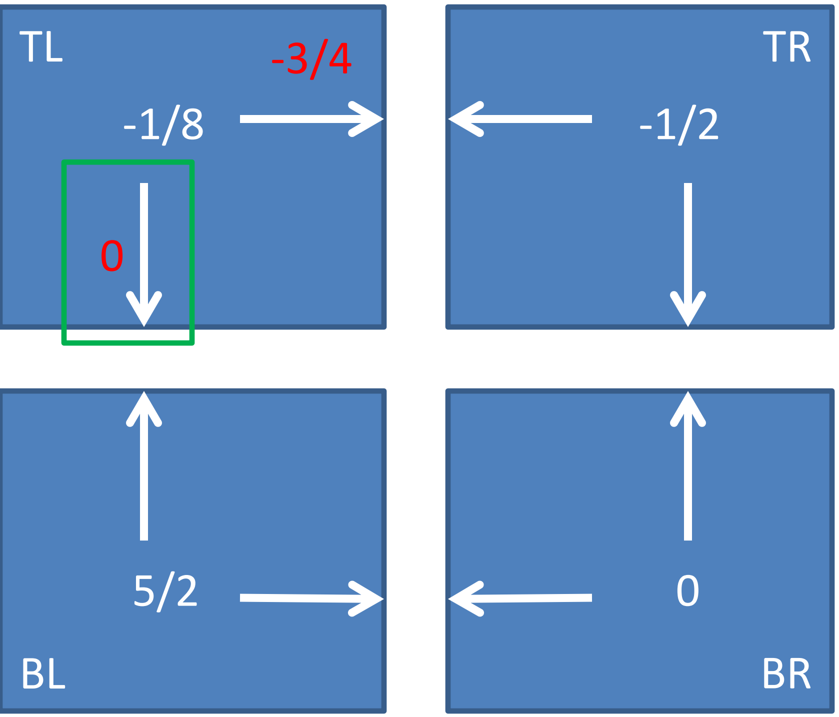


$Given\ V_\pi, evaluate\ \pi'\ (improved\ policy)$

$Find\ Q_\pi(s,a) for\ all\ states, choose\ best\ action$

$$Q_\pi(TL, V) = \frac{1}{4} \times \left[-1 + \frac{-1}{2}\right] + \frac{3}{4} \times \left[-1 + \frac{5}{2}\right] = 0$$

Policy Improvement Example

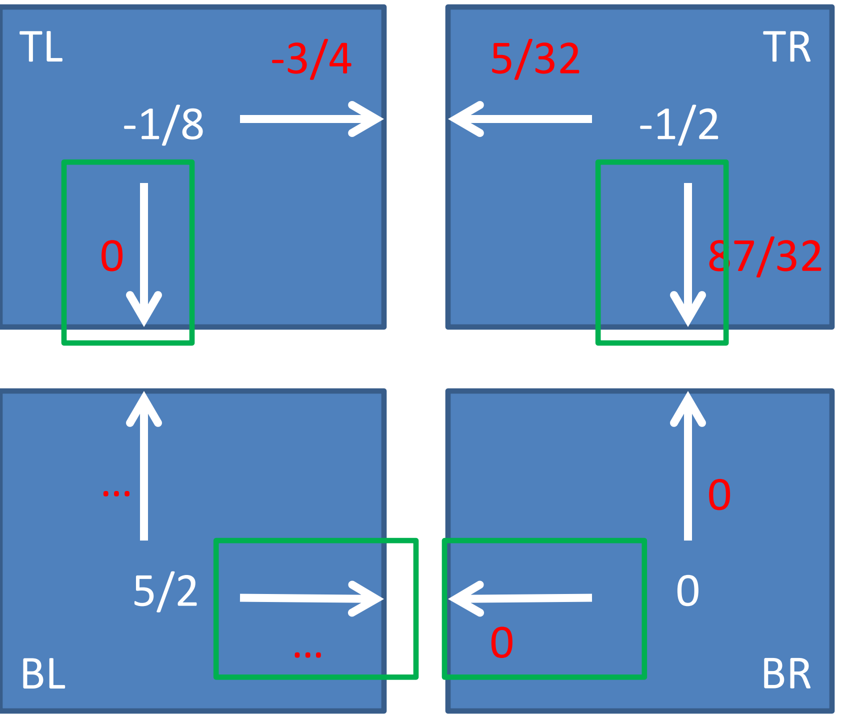


$Given\ V_\pi, evaluate\ \pi'\ (improved\ policy)$

$Find\ Q_\pi(s,a) for\ all\ states, choose\ best\ action$

$Optimal\ action$
$\pi'(TL) = V$

Policy Improvement Example



$Given\ V_{\pi}, evaluate\ \pi'\ (improved\ policy)$

$Find\ Q_{\pi}(s,a) for\ all\ states, choose\ best\ action$

$Optimal\ action$
$\pi'(TL) = V$
$\pi'(TR) = V$
$\pi'(BL) = \mathrm{H}$

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute $V_\pi = F_\pi^\infty V$

$$\pi \Longrightarrow V_\pi$$

2. **Policy improvement**: given V_π , compute improved policy π'

$$V_\pi \Longrightarrow \pi'$$

$$\pi'(s) = \arg \max_{a \in Actions(s)} \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_\pi(s')], \qquad \forall s$$

3. **Repeat** until policy converges/does not change

$$\pi \Longrightarrow V_\pi \Longrightarrow \pi' \Longrightarrow V_{\pi'} \Longrightarrow \pi''$$

Policy Iteration Example

π_0

→	→
→	→

V_0

TL 0	TR 0
BL 0	BR 0

V_1

TL -1	TR 1/4
BL 1/4	BR 0

V_2

TL -1/8	TR -1/2
BL 5/2	BR 0

…

π_1

↓	↓
→	→

V_0

TL 0	TR 0
BL 0	BR 0

…

Policy Iteration Example

π_0

→	→
→	→

V_0

TL 0	TR 0
BL 0	BR 0

V_1

TL -1	TR 1/4
BL 1/4	BR 0

V_2

TL -1/8	TR -1/2
BL 5/2	BR 0

…

π_1

↓	↓
→	→

V_0

TL -1/8	TR -1/2
BL 5/2	BR 0

…

Warm start!

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute $V_\pi = F_\pi^\infty\ V$

$$\pi \Longrightarrow V_\pi$$

2. **Policy improvement**: given V_π , compute improved policy π'

$$V_\pi \Longrightarrow \pi'$$

$$\pi'(s) = \arg\max_{a \in Actions(s)} \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_\pi(s')], \qquad \forall s$$

3. **Repeat** until policy converges/does not change

$$\pi \Longrightarrow V_\pi = F_\pi^\infty V \Longrightarrow \pi' \Longrightarrow V_{\pi'} = F_\pi^\infty V \Longrightarrow \pi''$$

Policy Iteration

Start with any π

1. **Policy evaluation**: given π, compute $V_\pi = F_\pi^\infty V$

$$\pi \Rightarrow V_\pi$$

2. **Policy improvement**: given V_π , compute improved policy π'

$$V_\pi \Rightarrow \pi'$$

$$\pi'(s) = \arg \max_{a \in Actions(s)} \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_\pi(s')], \qquad \forall s$$

3. **Repeat** until policy converges/does not change

$$\pi \Rightarrow V_\pi = F_\pi^\infty V \Rightarrow \pi' \Rightarrow V_{\pi'} = F_\pi^\infty V_\pi \Rightarrow \pi''$$

Warm start!

Value Iteration

Start with any V

1. **Compute Q-values**: given V, compute Q

$$Q(s,a) = \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V(s')], \qquad \forall s,a$$

Value Iteration

Start with any V

1. **Compute Q-values**: given V, compute Q

$$Q(s,a) = \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V(s')], \qquad \forall s,a$$

2. **Compute V':** given Q, compute V'

$$V'(s) = \max_{a \in Actions(s)} Q(s,a), \qquad \forall s$$

Value Iteration

Start with any V

1. **Compute Q-values**: given V, compute Q

$$Q(s,a) = \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V(s')], \qquad \forall s,a$$

2. **Compute V':** given Q, compute V'

$$V'(s) = \max_{a \in Actions(s)} Q(s,a), \qquad \forall s$$

3. **Repeat** until value function converges

Value Iteration

Start with any V

1. **Compute V'**: given V, compute V'

$$V'(s) = \max_{a \in Actions(s)} \sum_{s'} T(s, a, s')[R(s, a, s') + \gamma V(s')], \qquad \forall s$$

2. **Repeat** until value function converges

Value Iteration

Start with any V

1. **Compute V'**: given V, compute π', then compute V'
   a) Compute $\pi'(s) = \underset{a \in Actions(s)}{\arg\max} \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V(s')]$, $\forall s$
   b) Compute $V' = F_{\pi'} V$
2. **Repeat** until value function converges

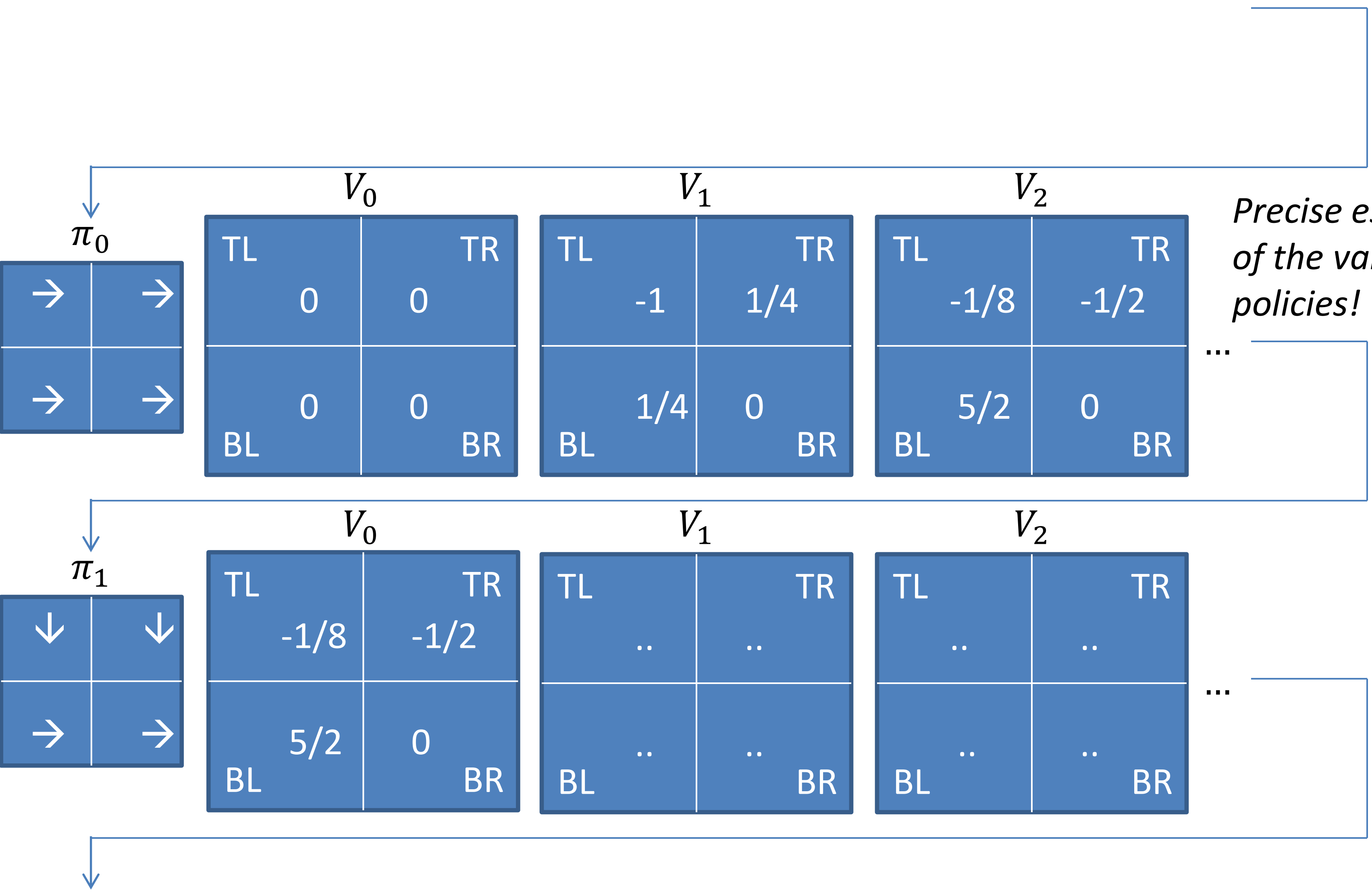
Value Iteration

Start with any V

1. **Compute V'**: given V, compute π', then compute V'
   a) Compute $\pi'(s) = \underset{a \in Actions(s)}{\arg\max} \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V(s')]$, $\forall s$
   b) Compute $V' = F_{\pi'} V$
2. **Repeat** until value function converges

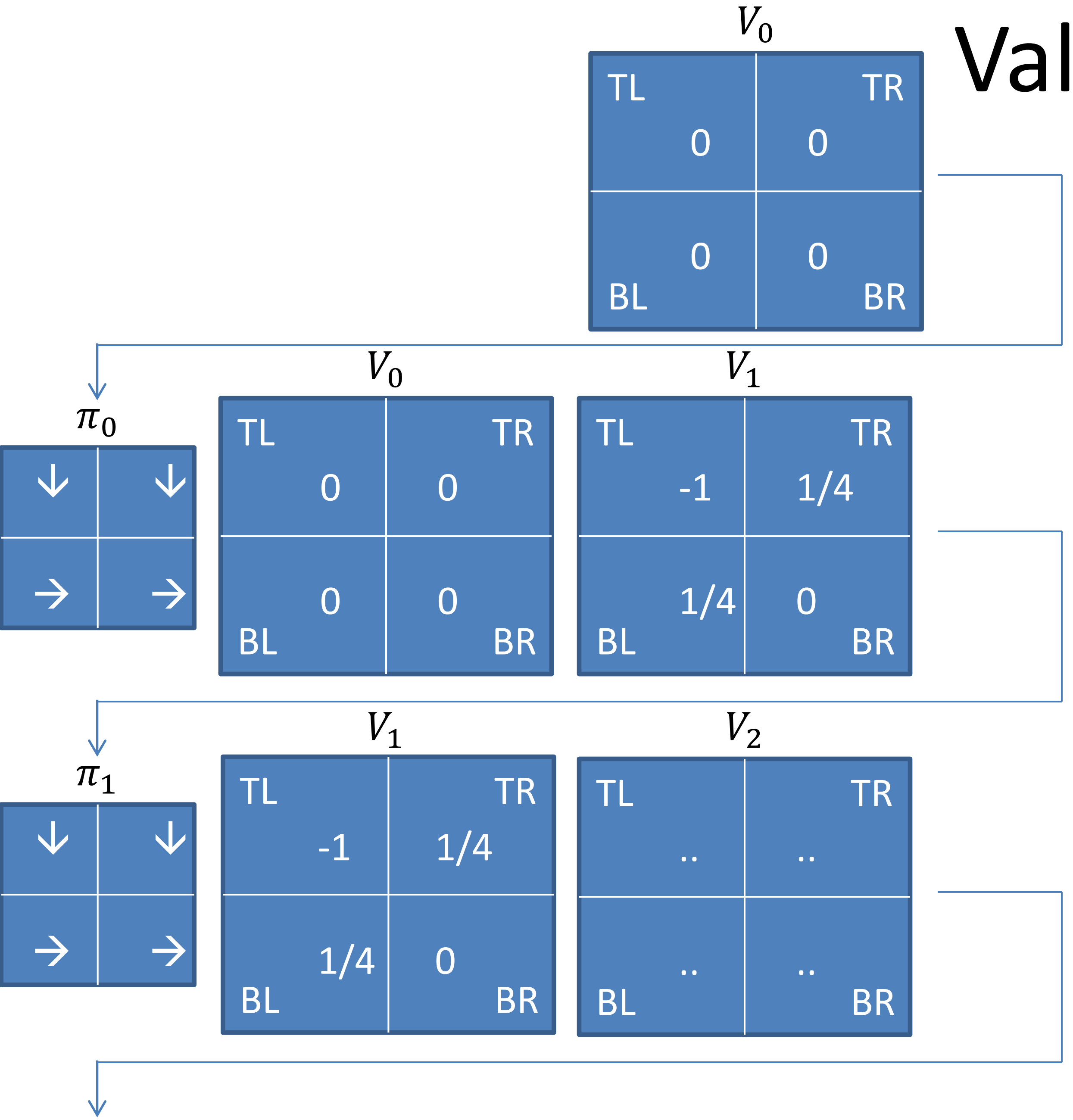
$$V \Rightarrow \pi' \Rightarrow V' = F_{\pi'} V \Rightarrow \pi'' \Rightarrow V'' = F_{\pi''} V'$$

Policy Iteration



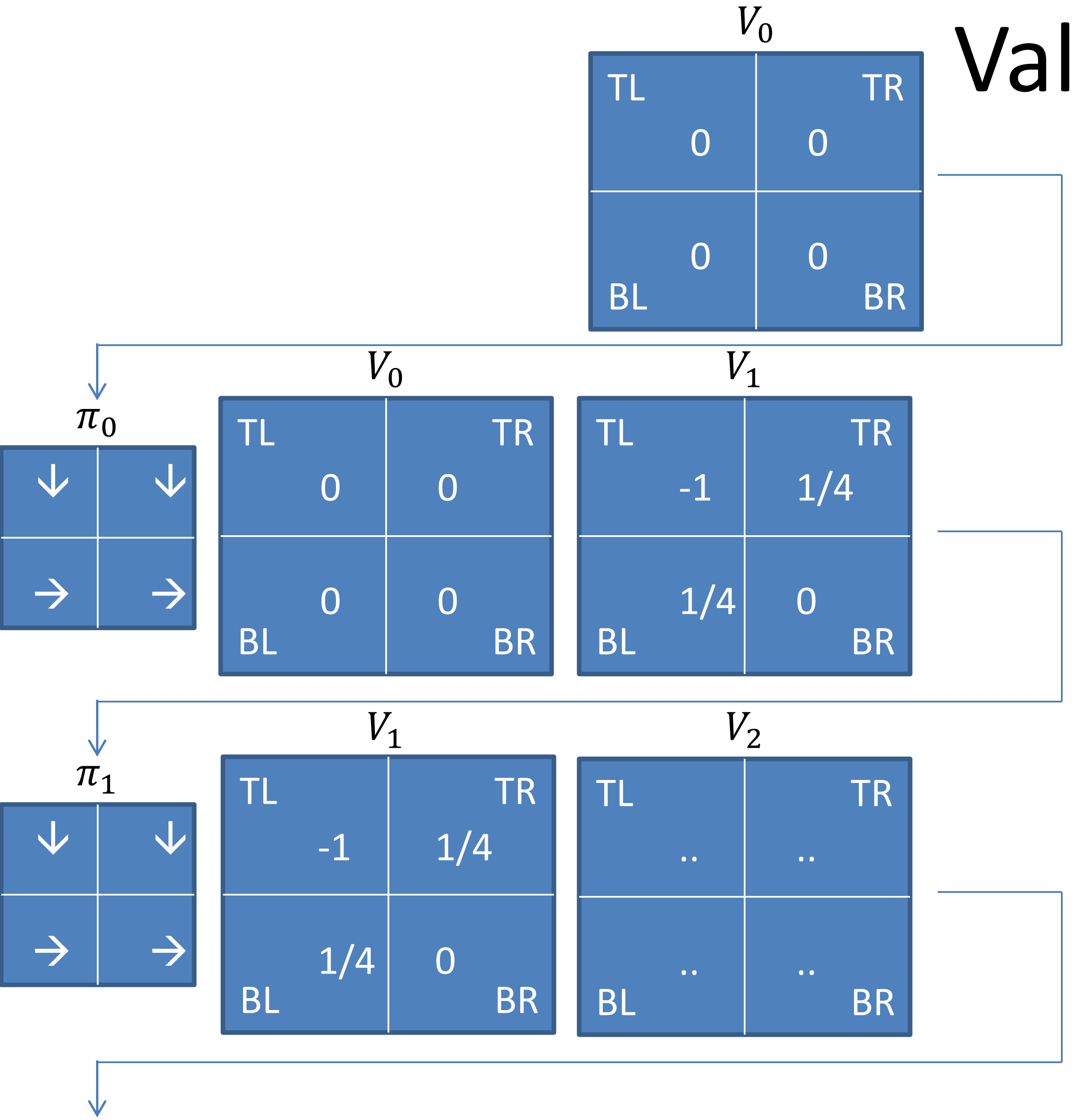
Precise estimate of the value of policies!

Value Iteration



Crude estimate of the value of policies!

Value Iteration



Crude estimate of the value of policies!

Exact numbers on slide may not be correct

Policy Iteration vs Value Iteration

Policy Iteration

$$\pi \Rightarrow V_\pi = F_\pi^\infty V \Rightarrow \pi' \Rightarrow V_{\pi'} = F_\pi^\infty V$$

$$V \Rightarrow \pi \Rightarrow V' = F_\pi V \Rightarrow \pi' \Rightarrow V'' = F_{\pi'} V'$$

Value Iteration

Policy Iteration vs Value Iteration

Policy Iteration

Precise Policy Evaluation + Policy Improvement

$$\pi \Longrightarrow V_\pi = F_\pi^\infty V \Longrightarrow \pi' \Longrightarrow V_{\pi'} = F_\pi^\infty V$$

$$V \Longrightarrow \pi \Longrightarrow V' = F_\pi V \Longrightarrow \pi' \Longrightarrow V'' = F_{\pi'} V'$$

Crude Policy Evaluation + Policy Improvement

Value Iteration

Policy Iteration vs Value Iteration

SARSA vs Q-Learning

Function Approximation and Q-Learning

Reinforcement Learning (RL)

$\boldsymbol{s} \in States$

$\boldsymbol{Actions}(\boldsymbol{s})$: $actions\ from\ state\ s$

$\boldsymbol{T}(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{s}')$: $probability\ next\ state\ is\ s',\ given\ action\ a\ in\ state\ s$

$$Note\ that \sum_{s'} T(s, a, s') = 1\ for\ all\ s \in States, a \in Actions(s)$$

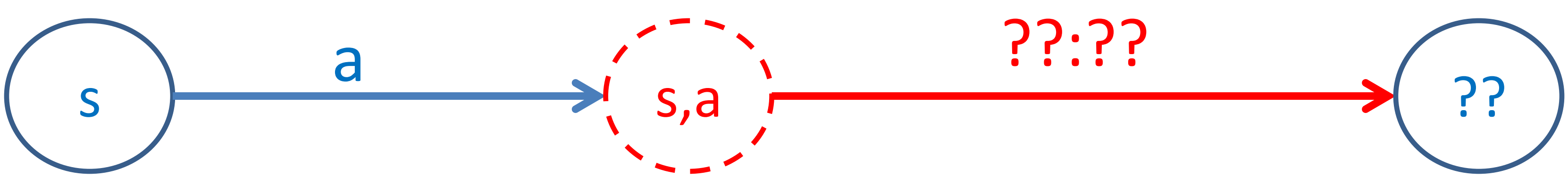
UNKNOWN

$\boldsymbol{R}(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{s}')$: $reward\ for\ transition\ (s, a, s')$

$\boldsymbol{IsEnd}(\boldsymbol{s})$: $whether\ state\ marks\ end\ of\ game$

$$Alternatively, T(s, a, s) = 1\ and\ R(s, a, s) = 0$$

$\boldsymbol{0 \leq \gamma \leq 1}$: $discount\ factor$



Reinforcement Learning (RL)

Algorithm	Estimate	Comments
Model-based Monte Carlo	$T(s,a,s'), R(s,a,s')$	Find Q_{opt}, π_{opt} afterward
Model-free Monte Carlo	Q_π using utility u	No optimal policy (on-policy)
SARSA	Q_π using utility r, Q_π	No optimal policy (on-policy)
Q-Learning	Q_{opt} using utility r, Q_{opt}	Optimal policy (off-policy)

Reinforcement Learning (RL)

Algorithm	Estimate	Comments
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Starting in state s_0, apply policy π s_0;
s_0

Reinforcement Learning (RL)

Algorithm	Estimate	Comments
Model-based Monte Carlo	$T(s,a,s'), R(s,a,s')$	Find Q_{opt}, π_{opt} afterward
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Starting in state s_0, apply policy π

s_0

$a_1 \sim \pi(s_0)$

$s_0; a_1$

Reinforcement Learning (RL)

Algorithm	Estimate	Comments
Model-based Monte Carlo	$T(s,a,s'), R(s,a,s')$	Find Q_{opt}, π_{opt} afterward
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Starting in state s_0, apply policy π

s_0

$a_1 \sim \pi(s_0)$

$s_1 \sim T(s_0, a_1, .)$

$r_1 \sim P_R(s_0, a_1, s_1)$

$s_0; a_1, r_1, s_1$

Reinforcement Learning (RL)

Algorithm	Estimate	Comments
Model-based Monte Carlo	$T(s,a,s'), R(s,a,s')$	Find Q_{opt}, π_{opt} afterward
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Starting in state s_0, apply policy π

s_0

$a_1 \sim \pi(s_0)$

$s_1 \sim T(s_0, a_1, .)$

$r_1 \sim P_R(s_0, a_1, s_1)$

$a_2 \sim \pi(s_1)$

$s_2 \sim T(s_1, a_2, .)$

$r_2 \sim P_R(s_1, a_2, s_2)$

$\vdots$

$s_0; a_1, r_1, s_1; a_2, r_2, s_2; a_3, r_3, s_3$

SARSA vs Q-Learning

SARSA: Update $\hat{Q}_\pi(s,a)$ based on $(\mathrm{s}, \mathrm{a}, \mathrm{r}, s', a')$

$$\hat{Q}_\pi(s,a) \leftarrow (1-\eta)\hat{Q}_\pi(s,a) + \eta\left[\, r + \gamma \hat{Q}_\pi(s',a') \,\right]$$

Helps evaluate policy π by sampling transitions and rewards from policy π
$\Rightarrow$ **On-policy** algorithm

SARSA evaluates policy π. Policy Improvement comes next. Iterate.

SARSA vs Q-Learning

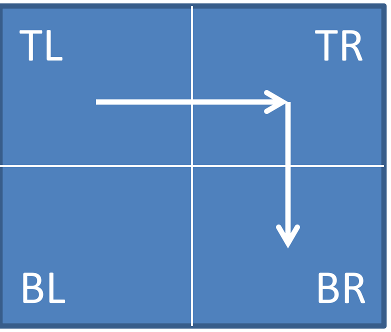
SARSA: Update $\hat{Q}_\pi(s, a)$ based on $(\mathrm{s}, \mathrm{a}, \mathrm{r}, s', a')$

$$\hat{Q}_\pi(s, a) \leftarrow \hat{Q}_\pi(s, a) - \eta\left[\hat{Q}_\pi(s, a) - (r + \gamma \hat{Q}_\pi(s', a'))\right]$$

Helps evaluate policy π by sampling transitions and rewards from policy π
⇒ **On-policy** algorithm

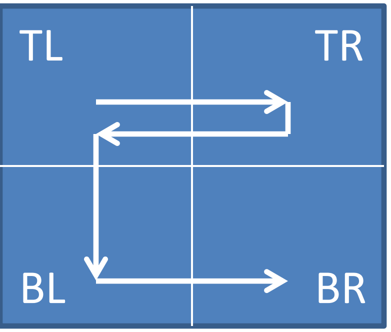
SARSA evaluates policy π. Policy Improvement comes next. Iterate.

SARSA Example



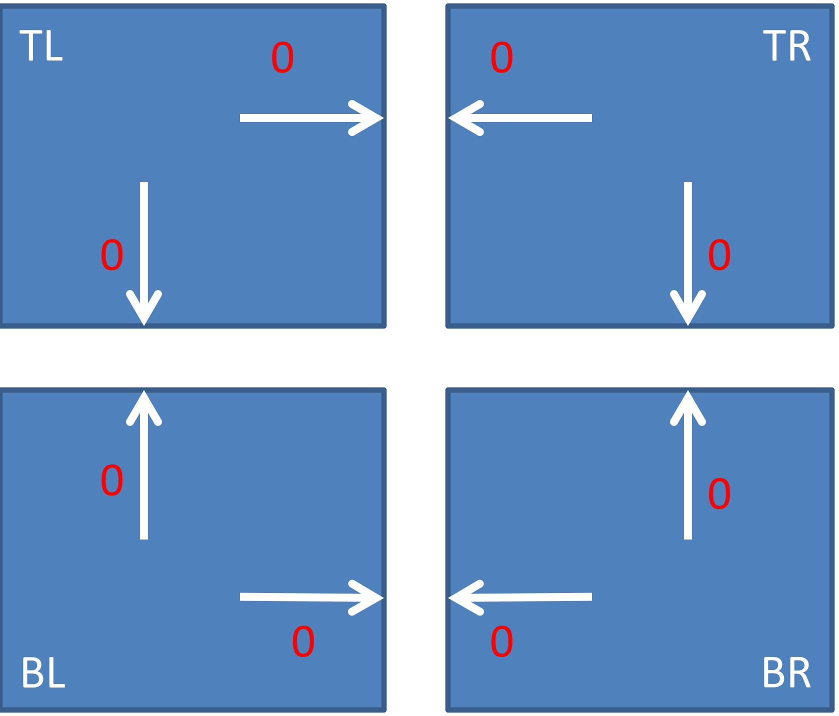
$Fixed\ policy\ \pi(s) = H, \forall s$
$Generate\ sample\ paths$
$1. TL; H, -1, TR; H, 4, BR$

SARSA Example



$Fixed\ policy\ \pi(s) = H, \forall s$
$Generate\ sample\ paths$
1. $TL; H, -1, TR; H, 4, BR$
2. $TL; H, -1, TR; H, -1, TL; H, -1, BL; H, 4, BR$

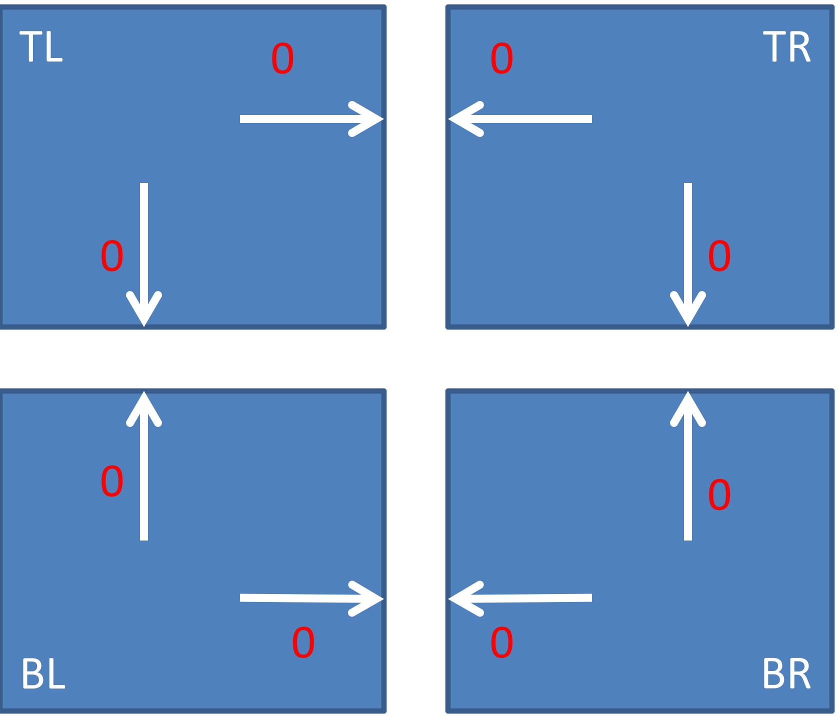
SARSA Example



$Fixed\ policy\ \pi(s) = H, \forall s$
$Generate\ sample\ paths$
$1. TL; H, -1, TR; H, 4, BR$
$2. TL; H, -1, TR; H, -1, TL; H, -1, BL; H, 4, BR$

$\boldsymbol{SARSA}$
$Set\ Q_{\pi}(s, a) = 0$

SARSA Example



Fixed policy $\pi(s) = H, \forall s$

Generate sample paths

1. $\boldsymbol{TL}; \boldsymbol{H}, \boldsymbol{-1}, \boldsymbol{TR}; \boldsymbol{H}, 4, BR$
2. $TL; H, -1, TR; H, -1, TL; H, -1, BL; H, 4, BR$

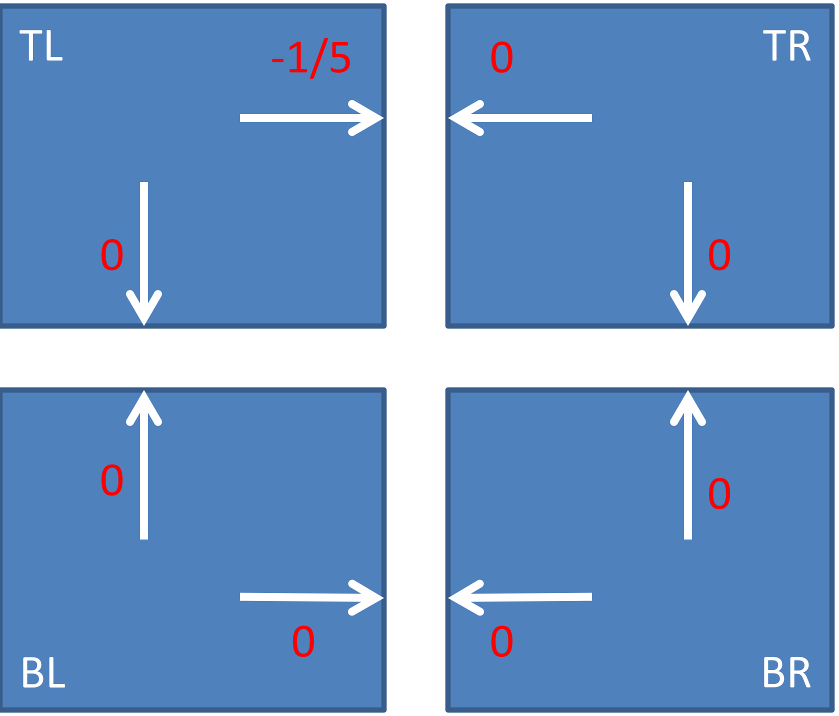
$\boldsymbol{SARSA}$

Set $Q_\pi(s, a) = 0$

Consider $\underset{s}{\boldsymbol{TL}}, \underset{a}{\boldsymbol{H}}, \underset{r}{\boldsymbol{-1}}, \underset{s'}{\boldsymbol{TR}}, \underset{a'}{\boldsymbol{H}}$

$$\begin{aligned} Q_\pi(TL, H) &:= \frac{4}{5} \times Q_\pi(TL, H) + \frac{1}{5} \times [-1 + Q_\pi(TR, H)] \\ &= \frac{4}{5} \times \quad 0 \quad + \frac{1}{5} \times [-1 + \quad 0 \quad] = \frac{-1}{5} \end{aligned}$$

SARSA Example



Fixed policy $\pi(s) = H, \forall s$

Generate sample paths

1. $TL; H, -1, \boldsymbol{TR}; \boldsymbol{H}, \boldsymbol{4}, \boldsymbol{BR}$
2. $TL; H, -1, TR; H, -1, TL; H, -1, BL; H, 4, BR$

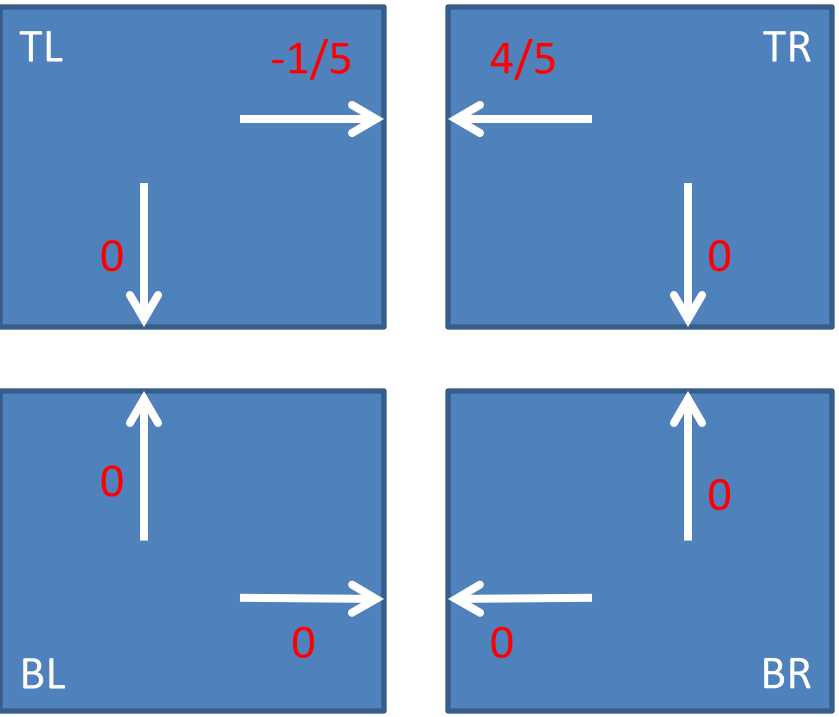
SARSA

Set $Q_\pi(s, a) = 0$

Consider $\underset{s}{\boldsymbol{TR}}, \underset{a}{\boldsymbol{H}}, \underset{r}{\boldsymbol{4}}, \underset{s'}{\boldsymbol{BR}}, \underset{a'}{-}$

$$Q_\pi(TR, H) := \frac{4}{5} \times Q_\pi(TR, H) + \frac{1}{5} \times [4 + Q_\pi(BR, -)]$$
$$= \frac{4}{5} \times 0 + \frac{1}{5} \times [4 + 0] = \frac{4}{5}$$

SARSA Example



Fixed policy $\pi(s) = H, \forall s$

Generate sample paths

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2. $\boldsymbol{TL}; \boldsymbol{H}, \boldsymbol{-1}, \boldsymbol{TR}; \boldsymbol{H}, -1, TL; H, -1, BL; H, 4, BR$

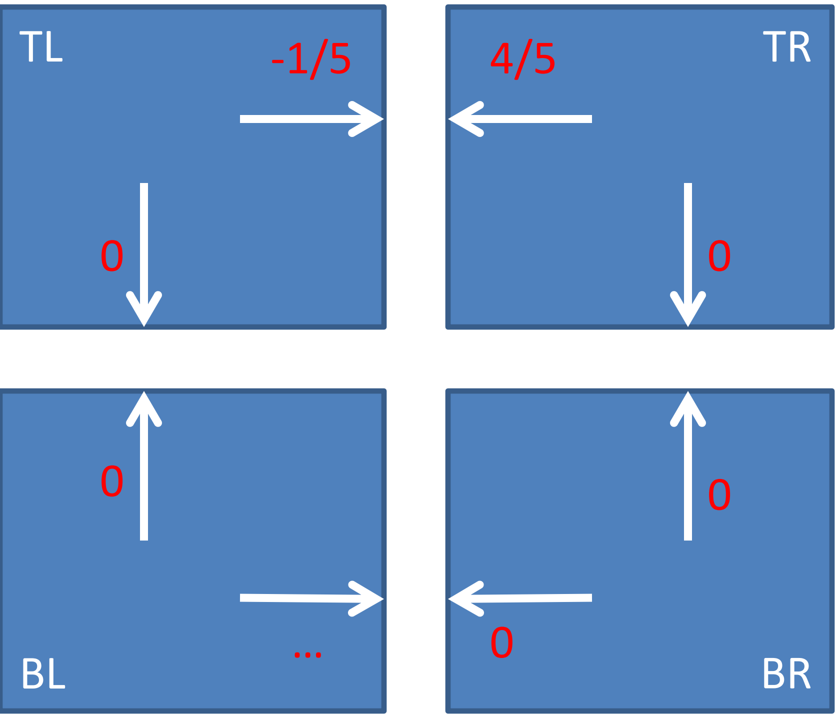
SARSA

Set $Q_\pi(s, a) = 0$

Consider $\underset{s}{\boldsymbol{TL}}, \underset{a}{\boldsymbol{H}}, \underset{r}{\boldsymbol{-1}}, \underset{s'}{\boldsymbol{TR}}, \underset{a'}{\boldsymbol{H}}$

$Q_\pi(TL, H) \coloneqq \cdots$

SARSA Example



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$\mathit{Set}\ Q_\pi(s,a) = 0$

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$$Q_\pi(TL, H) \coloneqq \cdots$$

Like Policy Evaluation – given enough iterations, get precise estimates

SARSA vs Q-Learning

SARSA: Update $\hat{Q}_\pi(s,a)$ based on $(\mathrm{s}, \mathrm{a}, \mathrm{r}, s', a')$

$$\hat{Q}_\pi(s,a) \leftarrow (1-\eta)\hat{Q}_\pi(s,a) + \eta\big[\, r + \gamma \hat{Q}_\pi(s',a') \,\big]$$

Helps evaluate policy π by sampling transitions and rewards from policy π
⇒ **On-policy** algorithm

SARSA evaluates policy π. Policy Improvement comes next. Iterate.

Q-Learning: Update $\hat{Q}_{opt}(s,a)$ based on $(\mathrm{s}, \mathrm{a}, \mathrm{r}, s')$

$$\hat{Q}_{opt}(s,a) \leftarrow (1-\eta)\hat{Q}_{opt}(s,a) + \eta\big[\, r + \gamma \hat{V}_{opt}(s')\big]$$

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Use (randomized) policy to explore π to explore all (s,a), find greedy policy!

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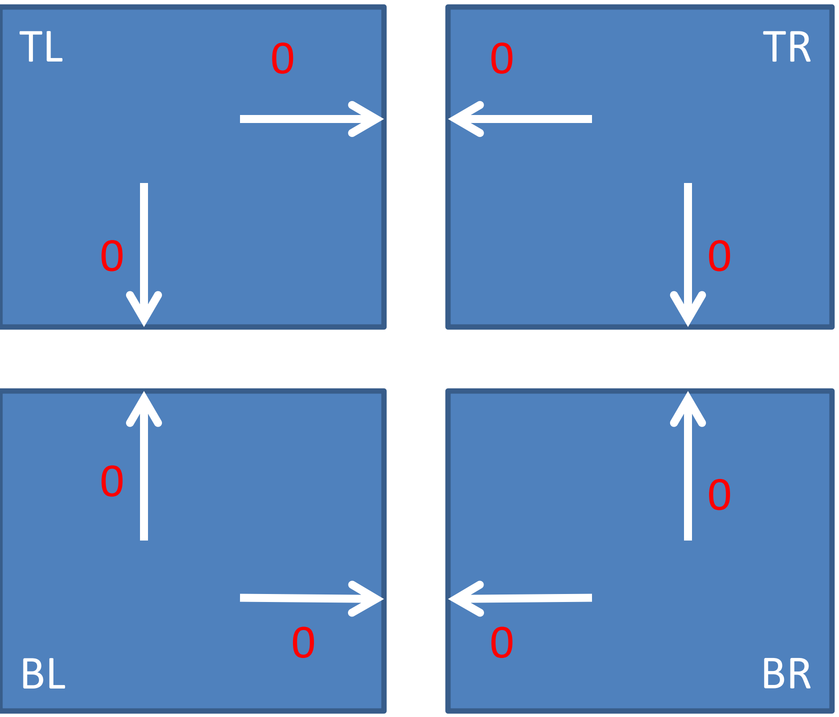
Example from notes:
http://web.stanford.edu/class/cs221/lectures/index.html#include=mdp2.js&slideIndex=46

Q-Learning Example



$Fixed\ policy\ \pi(s) = H, \forall s$

$Generate\ sample\ paths$

1. $\boldsymbol{TL}; \boldsymbol{H}, \boldsymbol{-1}, \boldsymbol{TR}; H, 4, BR$
2. $TL; H, -1, TR; H, -1, TL; H, -1, BL; H, 4, BR$

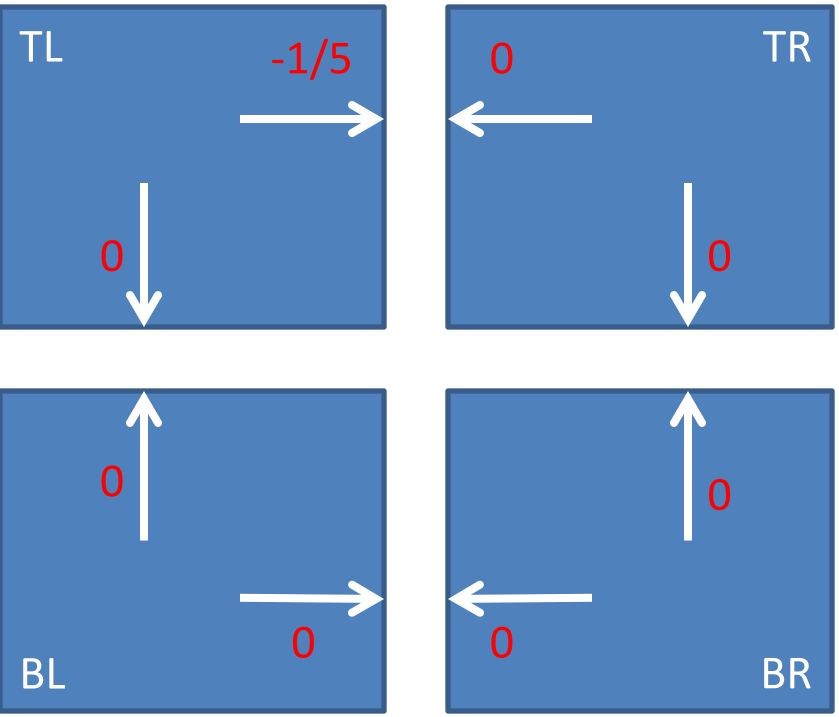
$\boldsymbol{Q - learning}$

$Set\ Q_{opt}(s,a) = 0$

$Consider\ \underset{s}{\boldsymbol{TL}}, \underset{a}{\boldsymbol{H}}, \underset{r}{\boldsymbol{-1}}, \underset{s'}{\boldsymbol{TR}}$

$$Q_{opt}(TL,H) := \frac{4}{5} \times Q_{opt}(TL,H) + \frac{1}{5} \times \left[-1 + \max_{a \in A(s)} Q_{opt}(TR,a)\right]$$

Q-Learning Example



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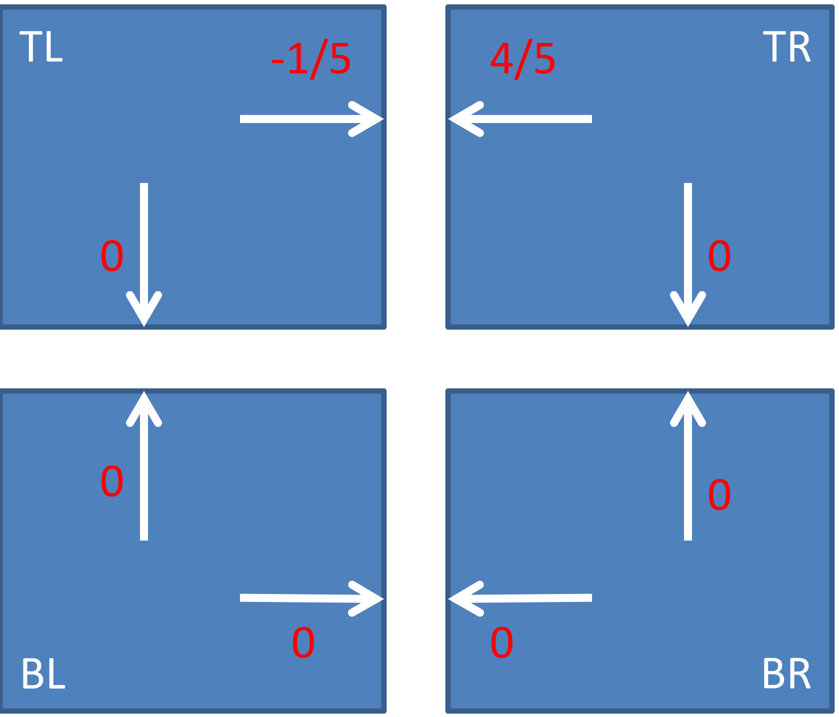
$$= \frac{4}{5} \times \quad 0 \qquad + \frac{1}{5} \times [-1 + \qquad 0 \qquad] = \frac{-1}{5}$$

Q-Learning Example



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$\boldsymbol{Q - learning}$

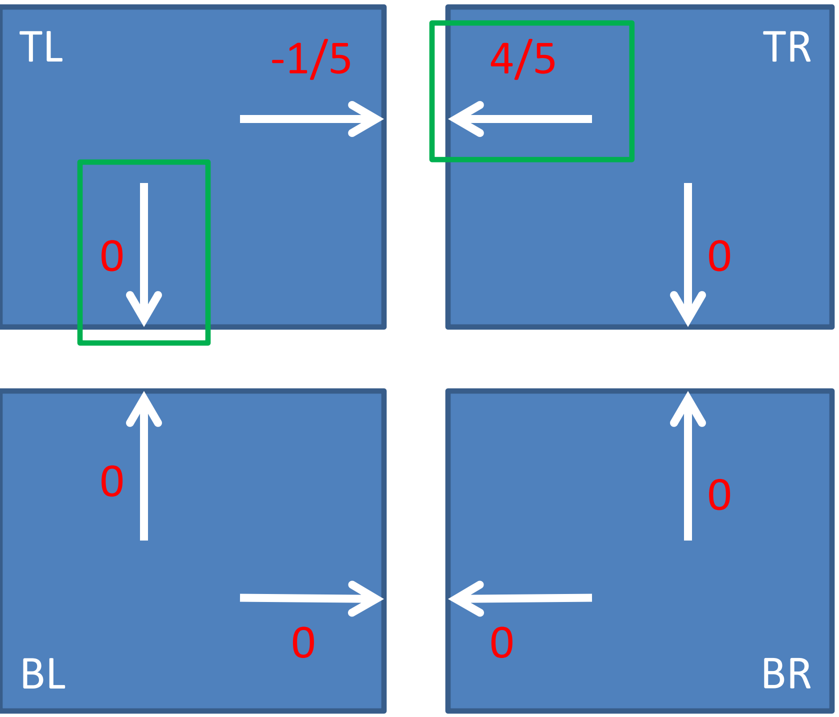
$Set\ Q_{opt}(s, a) = 0$

$Consider\ \underset{s}{\boldsymbol{TR}}, \underset{a}{\boldsymbol{H}}, \underset{r}{\boldsymbol{4}}, \underset{s'}{\boldsymbol{BR}}$

$$Q_{opt}(TR, H) := \frac{4}{5} \times Q_{opt}(TR, H) + \frac{1}{5} \times \left[4 + \max_{a \in A(s)} Q_{opt}(BR, a)\right]$$

$$= \frac{4}{5} \times \quad 0 \quad + \frac{1}{5} \times [4 + \quad 0 \quad] = \frac{4}{5}$$

Q-Learning Example



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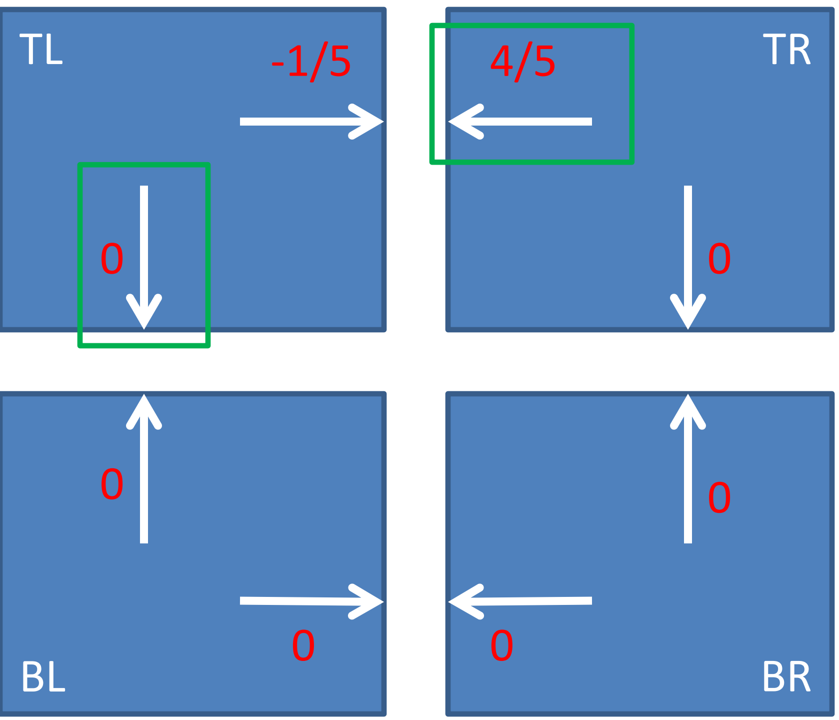
$\boldsymbol{Q - learning}$
$Set\ Q_{opt}(s, a) = 0$

$\boldsymbol{Optimal\ policy\ different\ from\ \pi}$
$\boldsymbol{\Rightarrow off - policy\ algorithm}$

Q-Learning Example



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$\boldsymbol{Q - learning}$

$Set\ Q_{opt}(s, a) = 0$

$\boldsymbol{Optimal\ policy\ different\ from\ \pi}$

$\boldsymbol{\Rightarrow off - policy\ algorithm}$

Problems: *Q(s,V)* never explored

Need to use exploration/randomized policy

Policy Iteration vs Value Iteration

SARSA vs Q-Learning

Function Approximation and Q-Learning

Function Approximation

For optimal policy, compute $Q_{opt}(s, a)$ for all states $s \in S, a \in A(s)$

Our current setup does not generalize to unseen states!

What if

- Number of states $|S|$ is huge?
    - Extreme: what if continuous state? Infinitely many of them…

- Number of actions $|A(s)|$ is huge?

- Majority of neighboring states should have similar Q-values

Function Approximation

IDEA:
Express $Q_{opt}(s, a)$ as a weighted combination of its features

$$Q_{opt}(s, a) \approx w \,.\, \Phi(s, a) = w_1\phi_1(s, a) + w_2\phi_2(s, a) + \ \ldots + w_n\phi_n(s, a)$$

$\Phi(s, a) =$ Feature vector
$w =$ Weights

Function Approximation

IDEA:
Express $Q_{opt}(s,a)$ as a weighted combination of its features

$$Q_{opt}(s,a) \approx w\,.\,\Phi(s,a) = \mathrm{Q_{opt}(s,a;w)}$$

Q-Learning with function approximation: Update $\hat{Q}_{opt}(s,a;w)$ based on $(\mathrm{s,a,r},s')$

$$w \leftarrow w - \eta\left[\hat{Q}_{opt}(s,a;w) - \left(r + \gamma \hat{V}_{opt}(s';w)\right)\right]\Phi(s,a)$$

Amounts to SGD on:

$$\min_{w} \sum_{(s,a,r,s')} \left(\hat{Q}_{opt}(s,a;w) - \left(r + \gamma \hat{V}_{opt}(s';w)\right)\right)^2$$

Function Approximation

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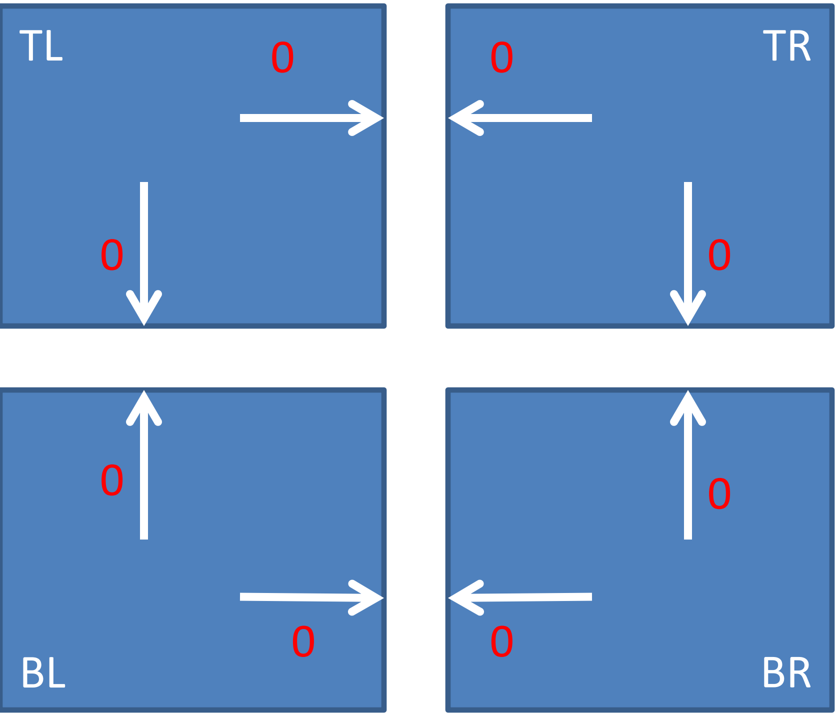
Q-Learning with Function Approximation Example



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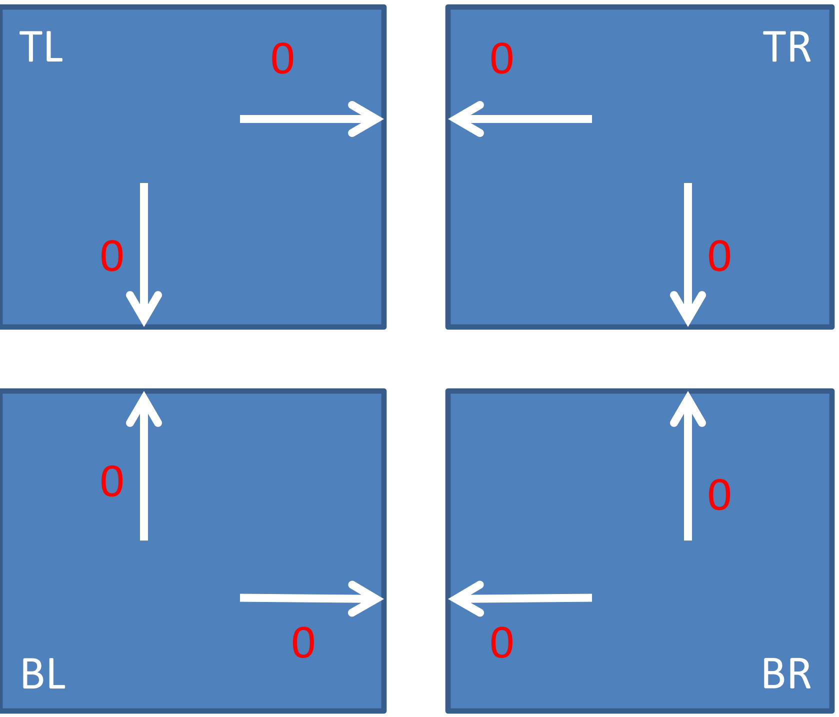
$\boldsymbol{Q - learning\ with\ function\ approximation}$

$$\Phi(s,a) = [1, \mathbf{1} = \{s =* L\}, \mathbf{1} = \{a = H\}, \mathbf{1} = \{s =* L\}\mathbf{1} = \{a = H\}]^T$$

$w = [w_1, w_2, w_3, w_4]^T$

$Set\ w = 0 \Rightarrow\ Q_{opt}(s,a;w) = 0$

Q-Learning with Function Approximation Example



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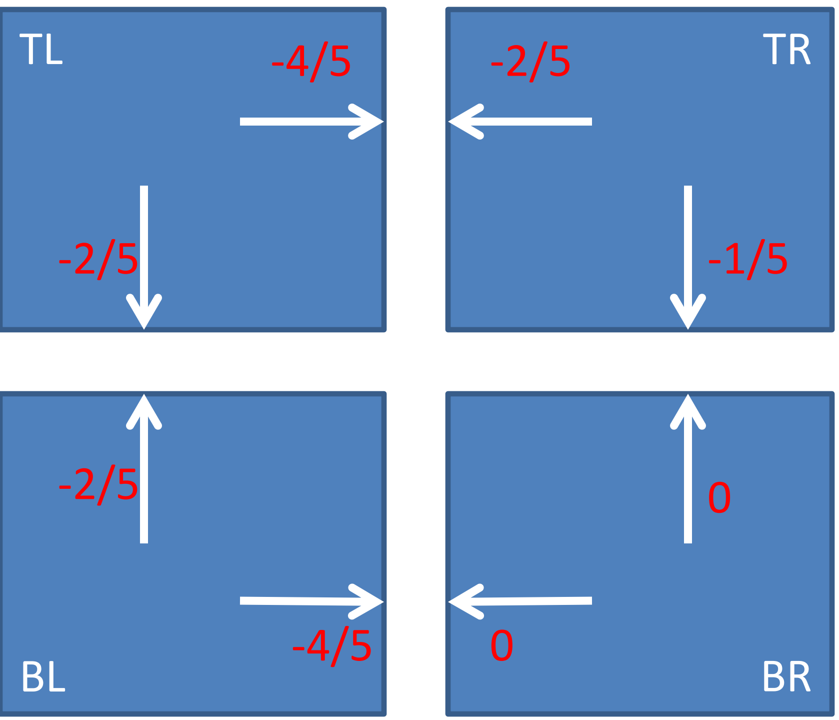
$$Set\ w = 0 \Rightarrow \ Q_{opt}(s, a; w) = 0$$

$Consider\ \underset{s}{\boldsymbol{TL}}, \underset{a}{\boldsymbol{H}}, \underset{r}{\boldsymbol{-1}}, \underset{s'}{\boldsymbol{TR}}$

$$w := [0\ 0\ 0\ 0]^T - \frac{1}{5}\left[Q_{opt}(TL, H; w)\ - \left(-1 + \max_{a' \in A(TR)} Q_{opt}(TR, a'; w)\right)\right]\Phi(TL, H)$$

$$= [0\ 0\ 0\ 0]^T - \frac{1}{5}[\qquad 0 \qquad - (-1 + \max_{a' \in A(TR)} \qquad 0 \qquad)][1\ 1\ 1\ 1]^T$$

Q-Learning with Function Approximation Example



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$$w := [0\ 0\ 0\ 0]^T - \frac{1}{5}\left[Q_{opt}(TL, H; w) - \left(-1 + \max_{a' \in A(TR)} Q_{opt}(TR, a'; w)\right)\right] \Phi(TL, H)$$

$$= \left[\frac{-1}{5}\ \frac{-1}{5}\ \frac{-1}{5}\ \frac{-1}{5}\right]^T$$