Mathematical Induction

Part Two

Let P be some property. The ***principle of mathematical induction*** states that if

P is true for 0

If it starts true…

…and it stays true…

and

For any $k \in \mathbb{N}$, if P is true for k, then P is true for $k + 1$

then

P is true for every $n \in \mathbb{N}$.

…then it's always true.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: Let $P(n)$ be the statement "the sum of the first n powers of two is $2^n - 1$." We will prove, by induction, that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

For our base case, we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $2^0 - 1$. Since the sum of the first zero powers of two is zero and $2^0 - 1$ is zero as well, we see that $P(0)$ is true.

For the inductive step, assume that for some $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

$$2^0 + 2^1 + \ldots + 2^{k-1} = 2^k - 1. \qquad (1)$$

We need to show that $P(k + 1)$ holds, meaning that the sum of the first $k + 1$ powers of two is $2^{k+1} - 1$. To see this, notice that

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \qquad \textit{(via (1))} \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

Therefore, $P(k + 1)$ is true, completing the induction. ■

Induction in Practice

- Typically, a proof by induction will not explicitly state $P(n)$.
- Rather, the proof will describe $P(n)$ implicitly and leave it to the reader to fill in the details.
- Provided that there is sufficient detail to determine
  - what $P(n)$ is,
  - that $P(0)$ is true, and that
  - whenever $P(k)$ is true, $P(k+1)$ is true,

  the proof is usually valid.

Theorem: The sum of the first n powers of two is $2^n - 1$.

Proof: By induction.

For our base case, we'll prove the theorem is true when $n = 0$. The sum of the first zero powers of two is zero, and $2^0 - 1 = 0$, so the theorem is true in this case.

For the inductive step, assume the theorem holds when $n = k$ for some arbitrary $k \in \mathbb{N}$. Then

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^{k-1} + 2^k &= (2^0 + 2^1 + \ldots + 2^{k-1}) + 2^k \\ &= 2^k - 1 + 2^k \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1. \end{aligned}$$

So the theorem is true when $n = k+1$, completing the induction. ■

Variations on Induction: **Starting Later**

Induction Starting at 0

- To prove that $P(n)$ is true for all natural numbers greater than or equal to 0:
  - Show that $P(0)$ is true.
  - Show that for any $k \geq 0$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to 0.

Induction Starting at $\boldsymbol{m}$

- To prove that $P(n)$ is true for all natural numbers greater than or equal to $\boldsymbol{m}$:
  - Show that $P(\boldsymbol{m})$ is true.
  - Show that for any $k \geq \boldsymbol{m}$, that if $P(k)$ is true, then $P(k+1)$ is true.
  - Conclude $P(n)$ holds for all natural numbers greater than or equal to $\boldsymbol{m}$.

n^2 versus 2^n

$0^2 = 0$	$2^0 = 1$
$1^2 = 1$	$2^1 = 2$
$2^2 = 4$	$2^2 = 4$
$3^2 = 9$	$2^3 = 8$
$4^2 = 16$	$2^4 = 16$
$5^2 = 25$	$2^5 = 32$
$6^2 = 36$	$2^6 = 64$
$7^2 = 49$	$2^7 = 128$
$8^2 = 64$	$2^8 = 256$
$9^2 = 81$	$2^9 = 512$

n^2 versus 2^n

$0^2 = 0$	$<$	$2^0 = 1$
$1^2 = 1$	$<$	$2^1 = 2$
$2^2 = 4$	$=$	$2^2 = 4$
$3^2 = 9$	$>$	$2^3 = 8$
$4^2 = 16$	$=$	$2^4 = 16$
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2^n is <u>much</u> bigger here. Does the trend continue?

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Theorem: If n is a natural number and $n \geq 5$, then $n^2 < 2^n$.

Proof: Let $P(n)$ be the statement "$n^2 < 2^n$." We will prove by induction that $P(n)$ is true for all $n \in \mathbb{N}$ where $n \geq 5$, from which the theorem follows.

As our base case, we prove $P(5)$, that $5^2 < 2^5$. Note that $5^2 = 25$ and that $2^5 = 32$, so $5^2 < 2^5$.

For the inductive step, assume that for some $k \geq 5$ that $P(k)$ is true, meaning that $k^2 < 2^k$. We will prove $P(k+1)$, that $(k+1)^2 < 2^{k+1}$.

First, recall that $(k + 1)^2 = k^2 + 2k + 1$. Starting with this equality and using the fact that $k \geq 5$ (that is, that $5 \leq k$), we derive the following:

$$
\begin{aligned}
(k + 1)^2 &= k^2 + 2k + 1 \\
&< k^2 + 2k + k && \textit{(since } 1 < 5 \leq k\textit{)} \\
&= k^2 + 3k \\
&< k^2 + k \cdot k && \textit{(since } 3 < 5 \leq k\textit{)} \\
&= k^2 + k^2 \\
&= 2k^2
\end{aligned}
$$

So we see that $(k + 1)^2 < 2k^2$. Now, by our inductive hypothesis, we know $k^2 < 2^k$. Some further algebra gives us the following:

$$
\begin{aligned}
(k + 1)^2 &< 2k^2 && \textit{(from above)} \\
&< 2(2^k) && \textit{(from the inductive hypothesis)} \\
&= 2^{k+1}
\end{aligned}
$$

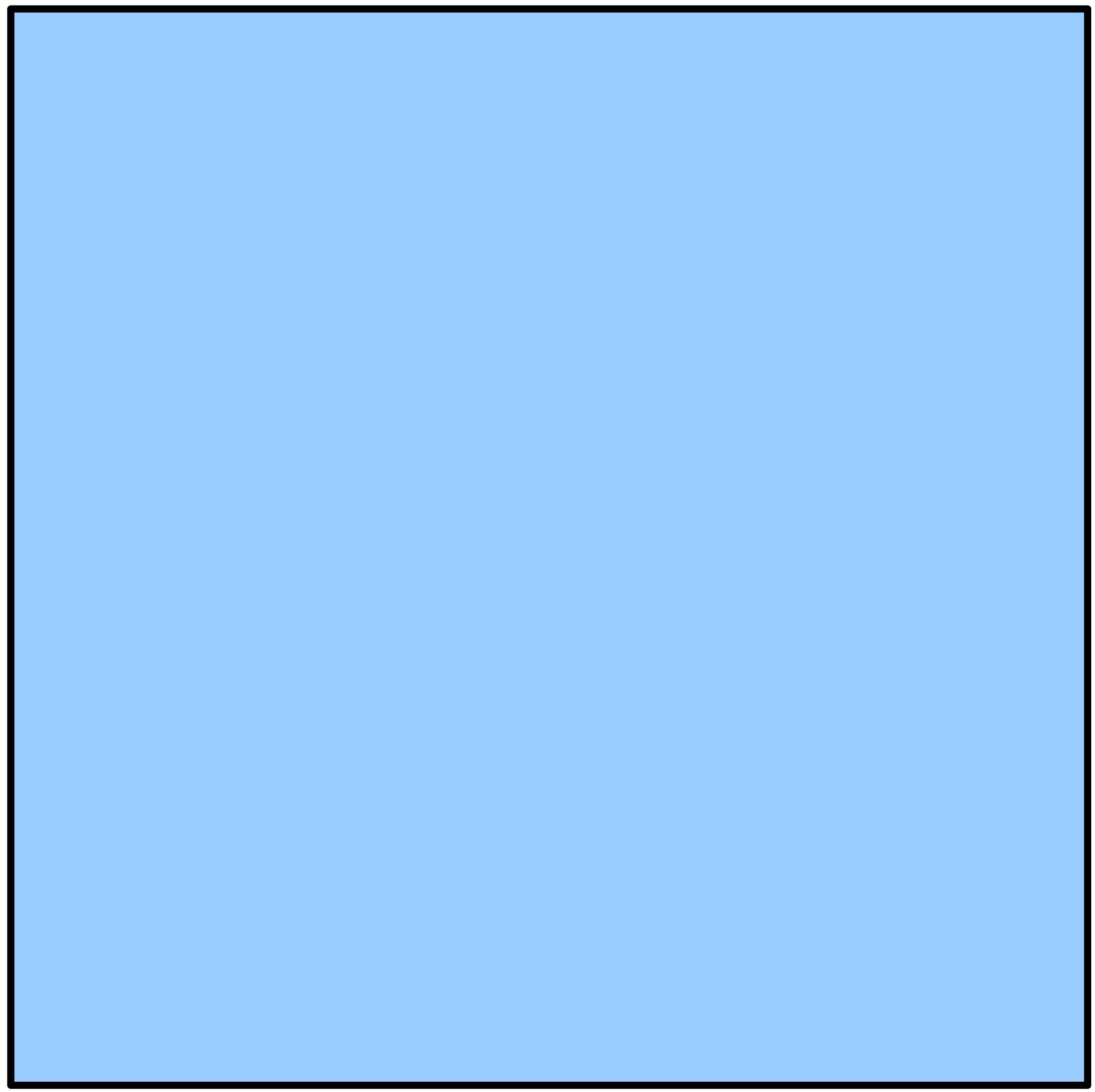
So $(k+1)^2 < 2^{k+1}$. Therefore, $P(k+1)$ is true, completing the induction. ■

A Note on the Previous Proof

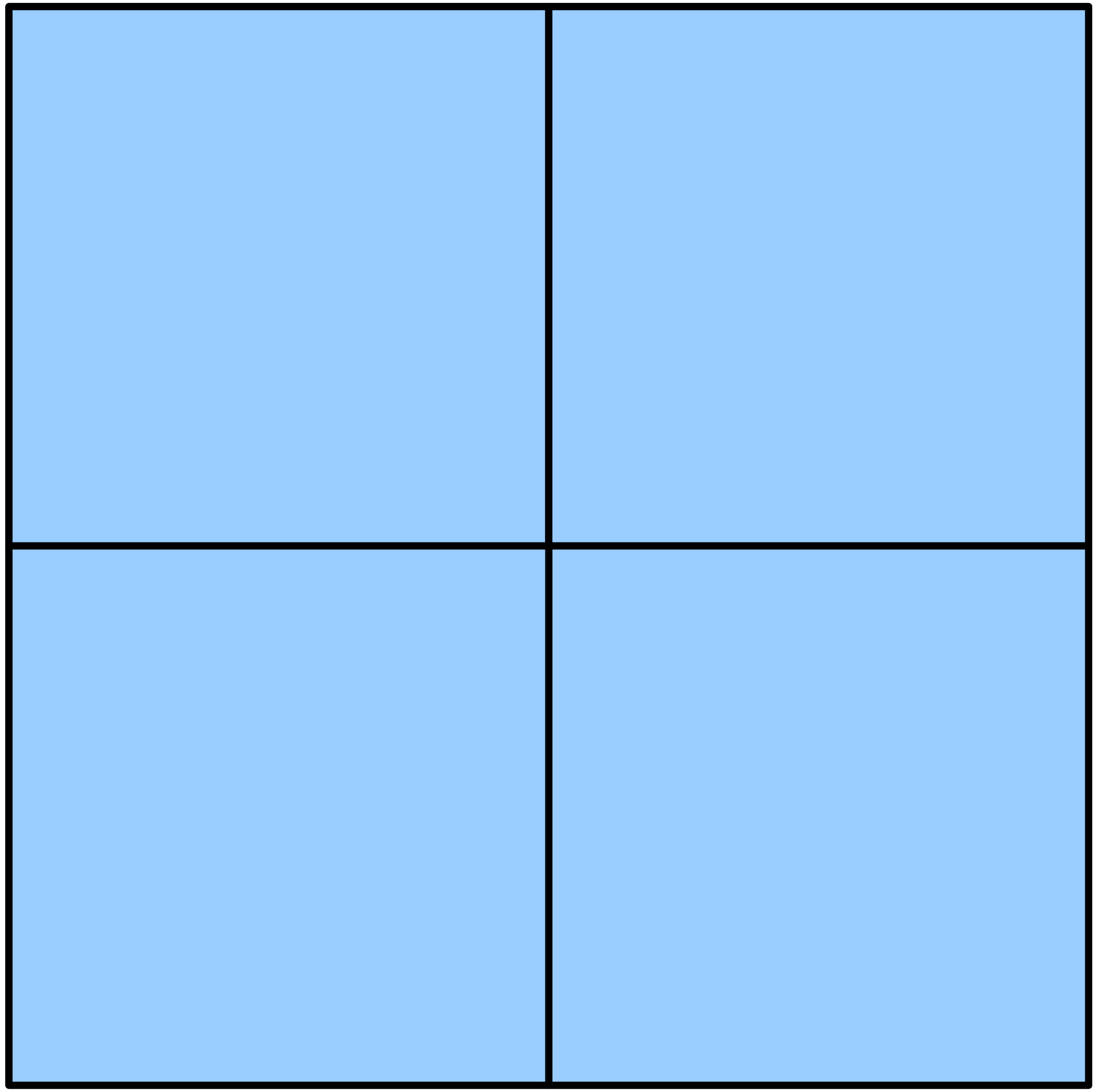
- I chose this proof for a few reasons:
  - It uses induction to prove a result that is false for small n, but true for all sufficiently large n.
  - It doesn't require us to introduce any new terms and definitions.
- That said, I'm not a fan of it for a few reasons:
  - This result is not particularly deep or interesting.
  - There's a lot of algebra.
  - It's not at all obvious how to come up with this line of reasoning.
- **Challenge:** Find an inductive proof with the two “good” qualities, but without the three “bad” qualities.

Variations on Induction: **Bigger Steps**

Subdividing a Square



Subdividing a Square



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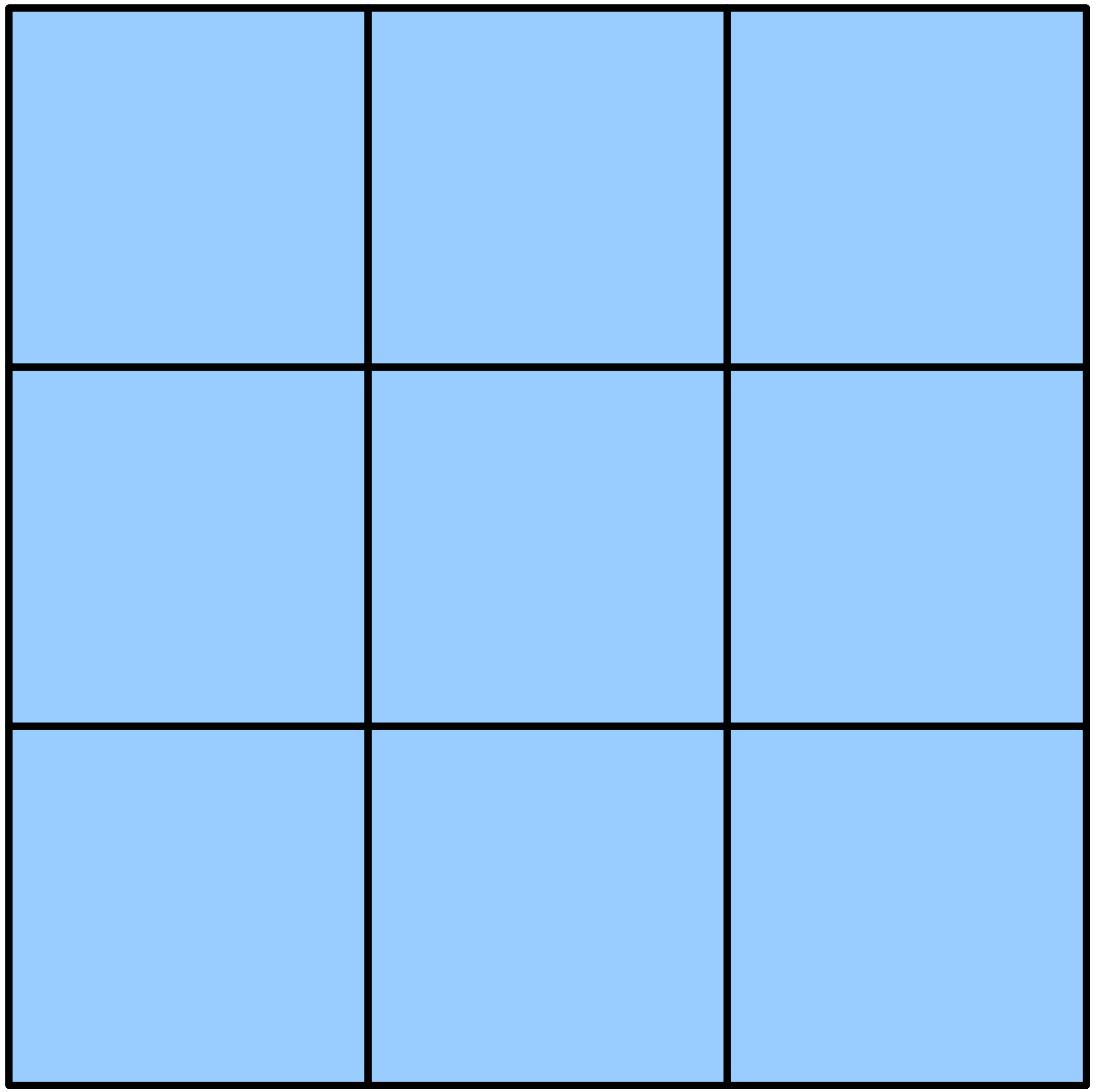

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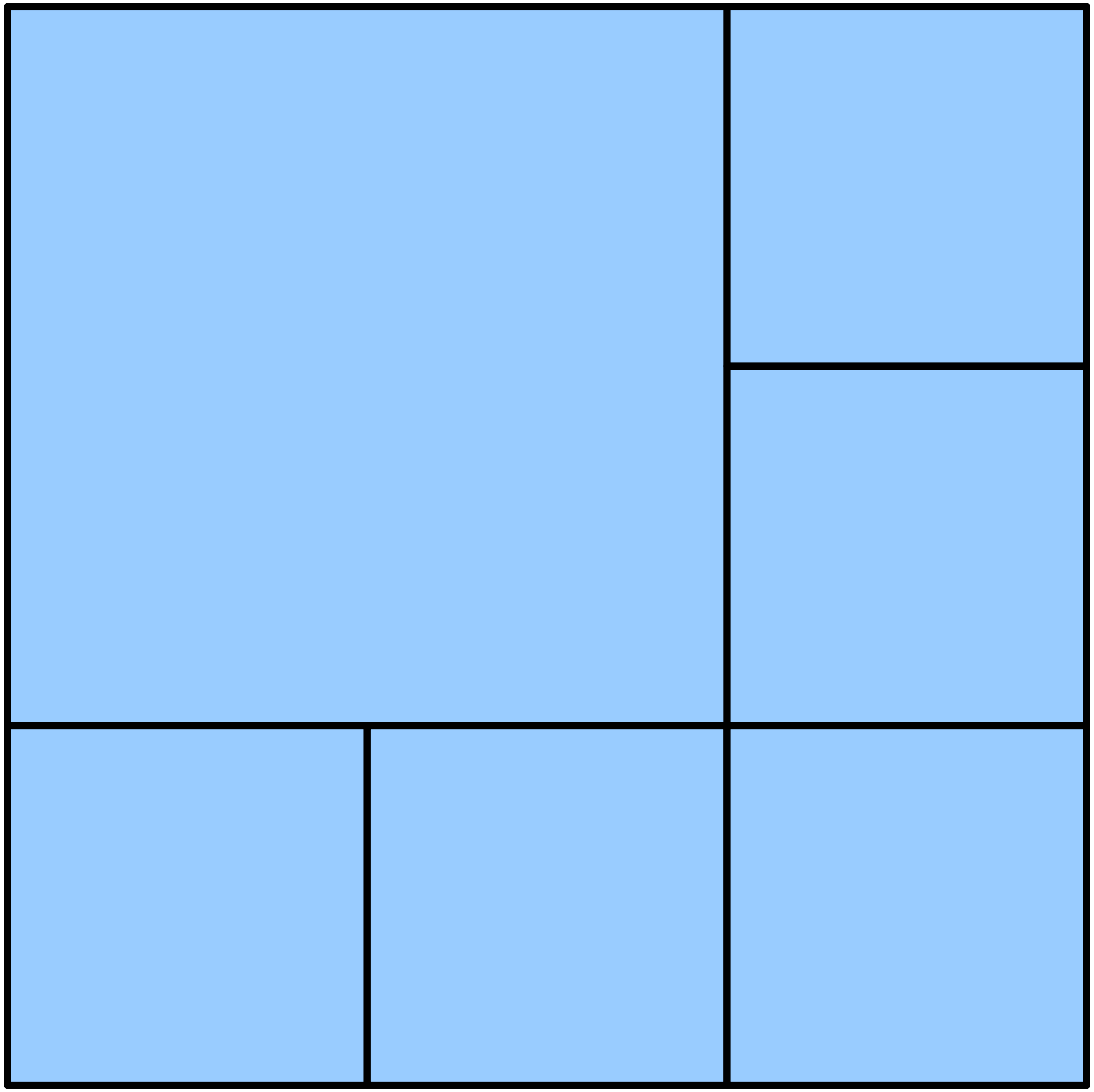

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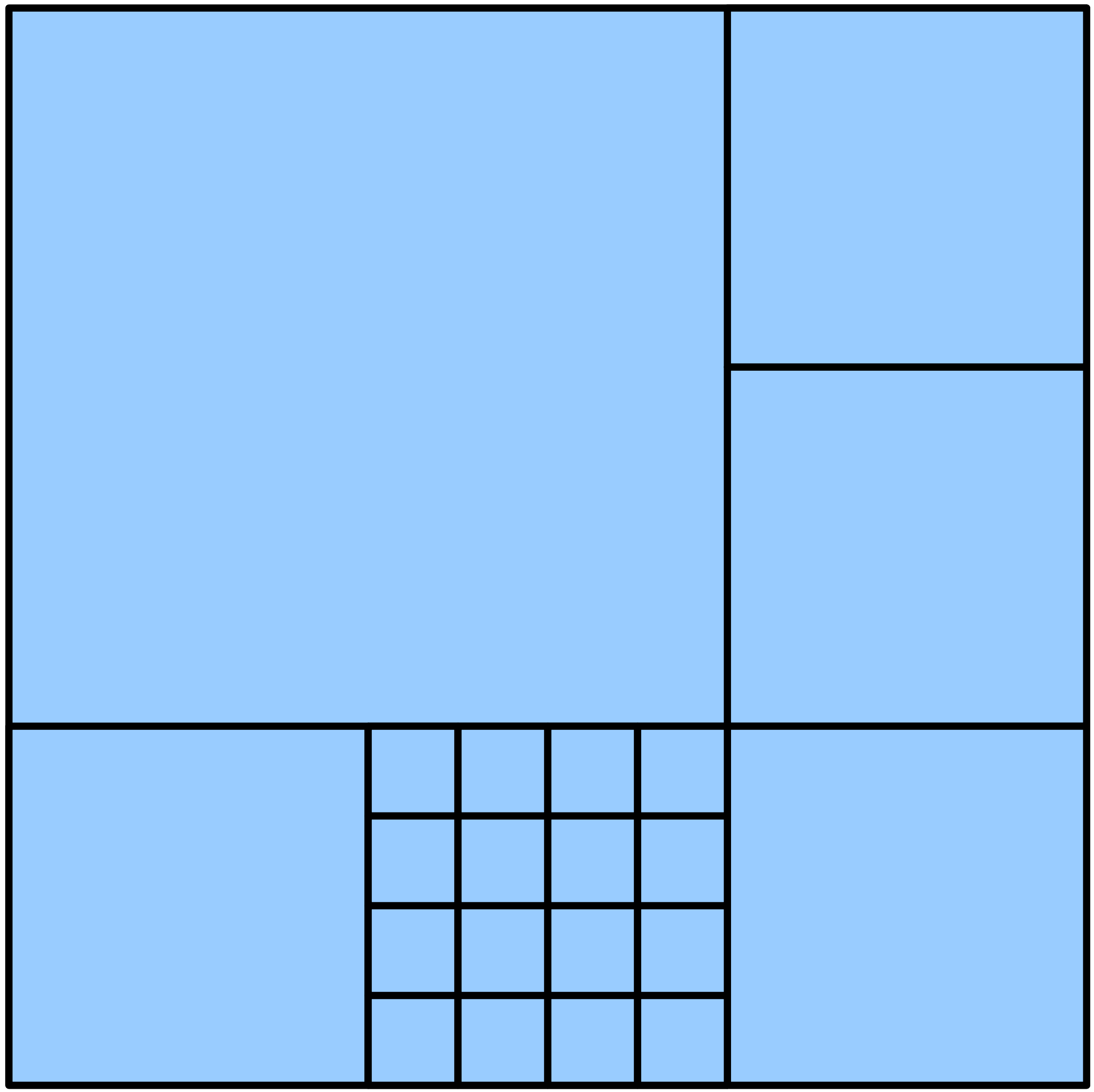


For what values of n can a square be subdivided into n squares?

1 2 3 4 5 6 7 8 9 10 11 12

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

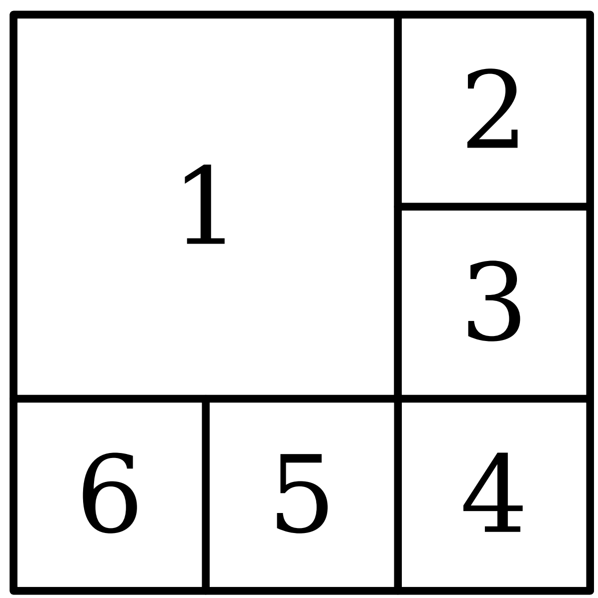
1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

$\boxed{1}$

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1	2
4	3

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12



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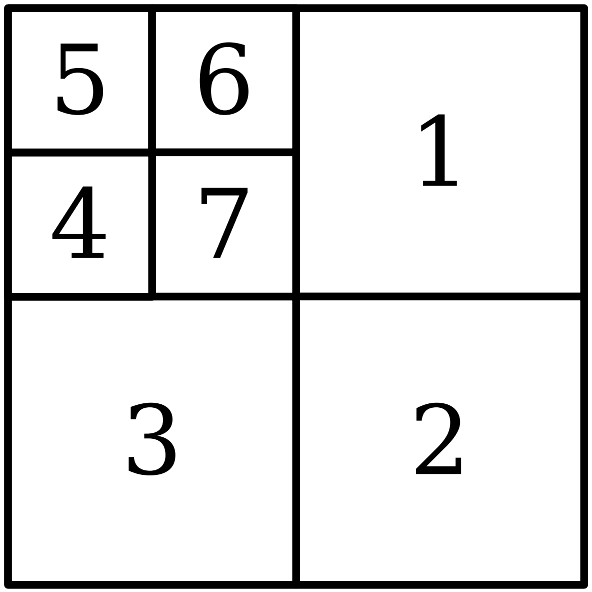

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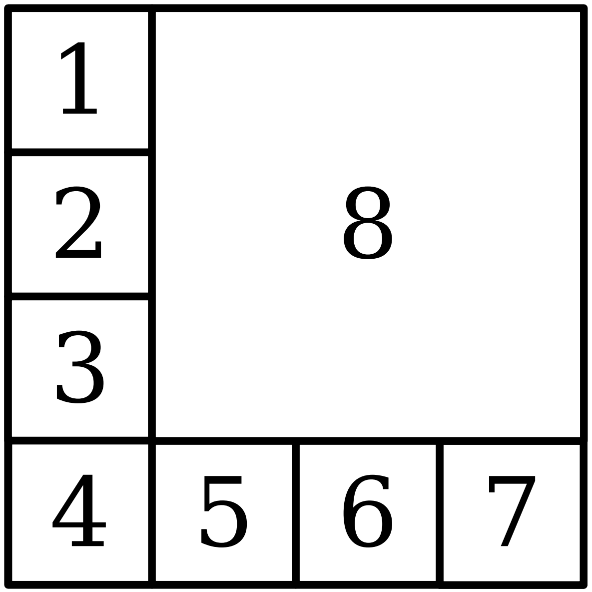

1 ~~2~~ ~~3~~ 4 ~~5~~ 6 7 8 9 10 11 12

1	2	3
8	9	4
7	6	5

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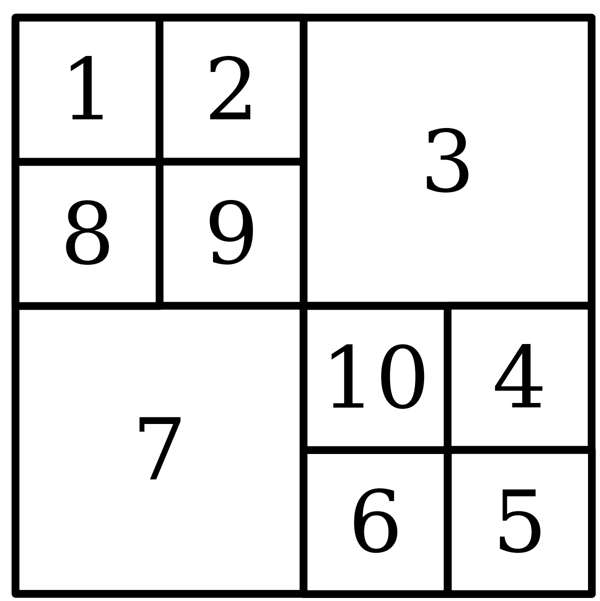

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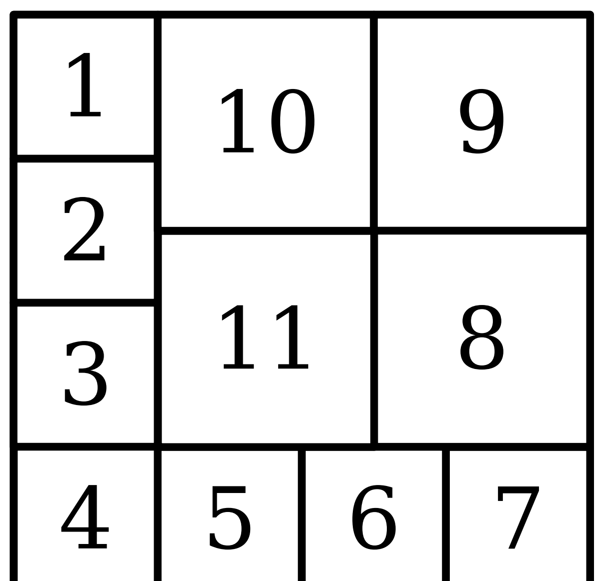

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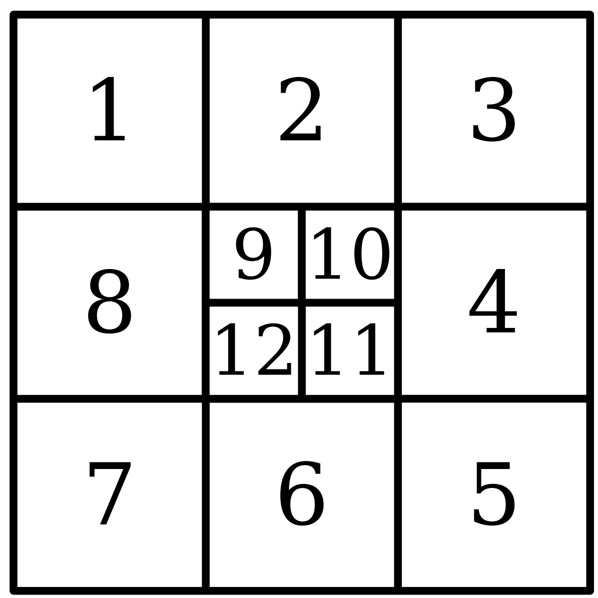

The Key Insight

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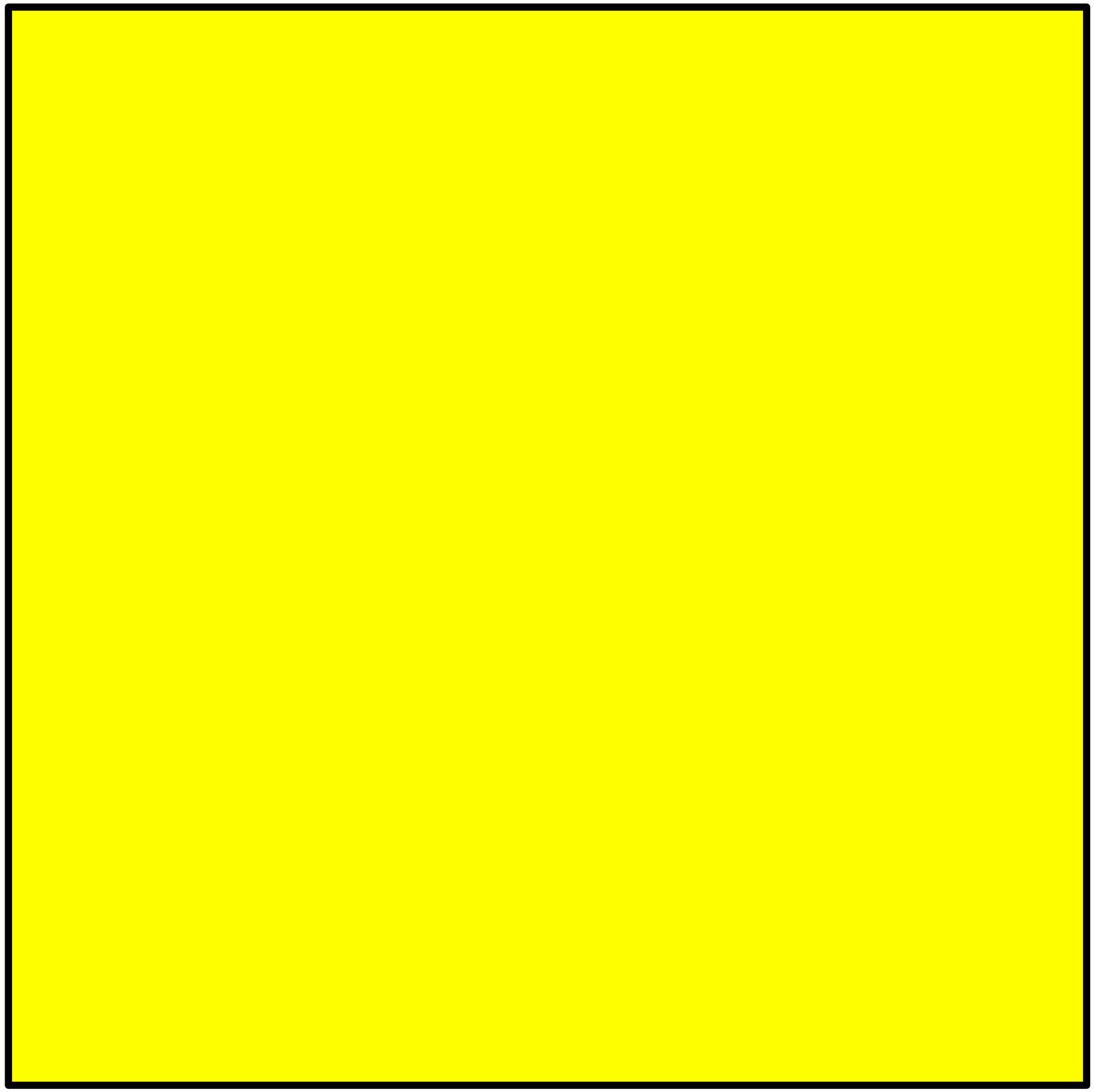

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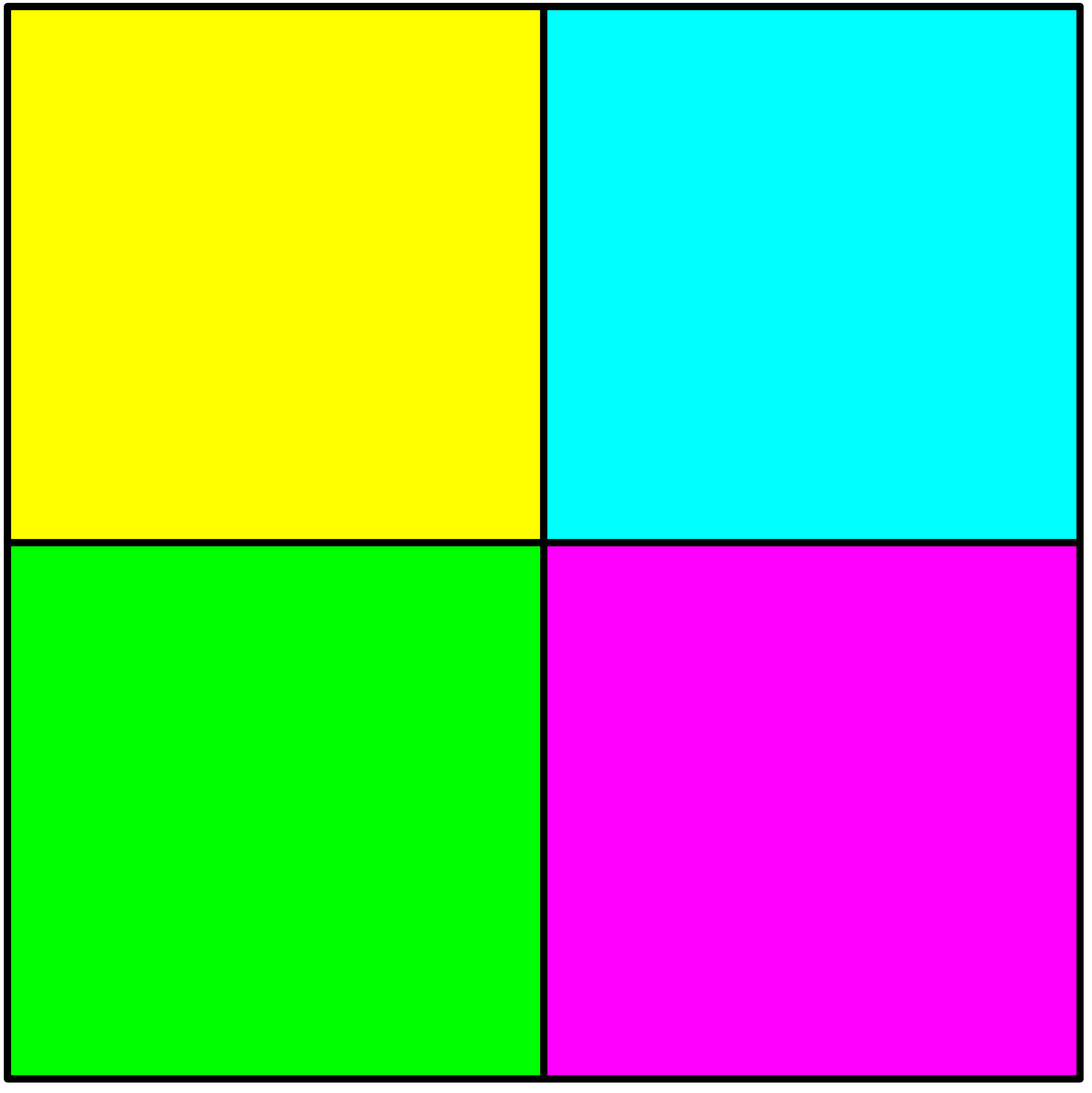

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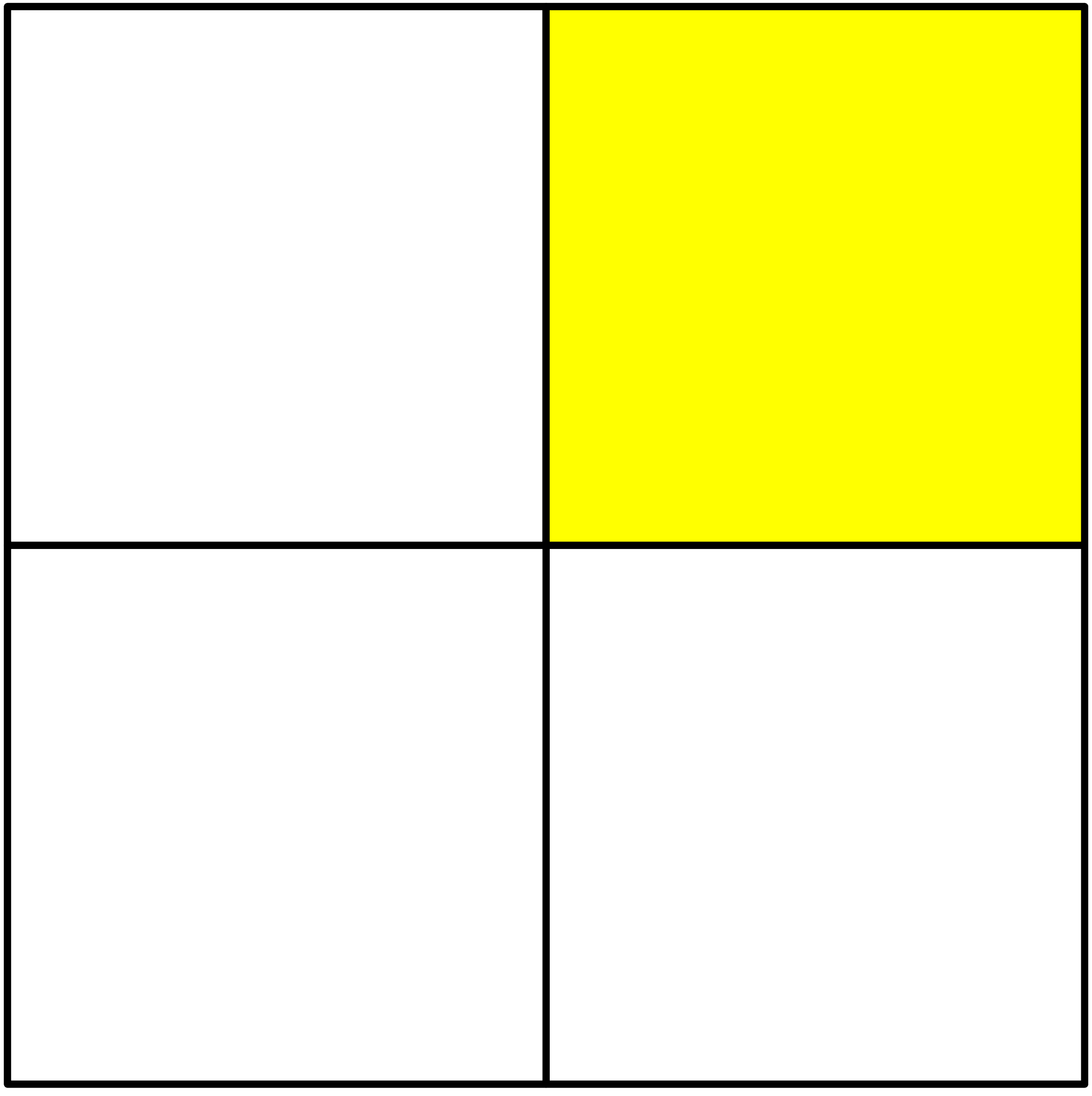

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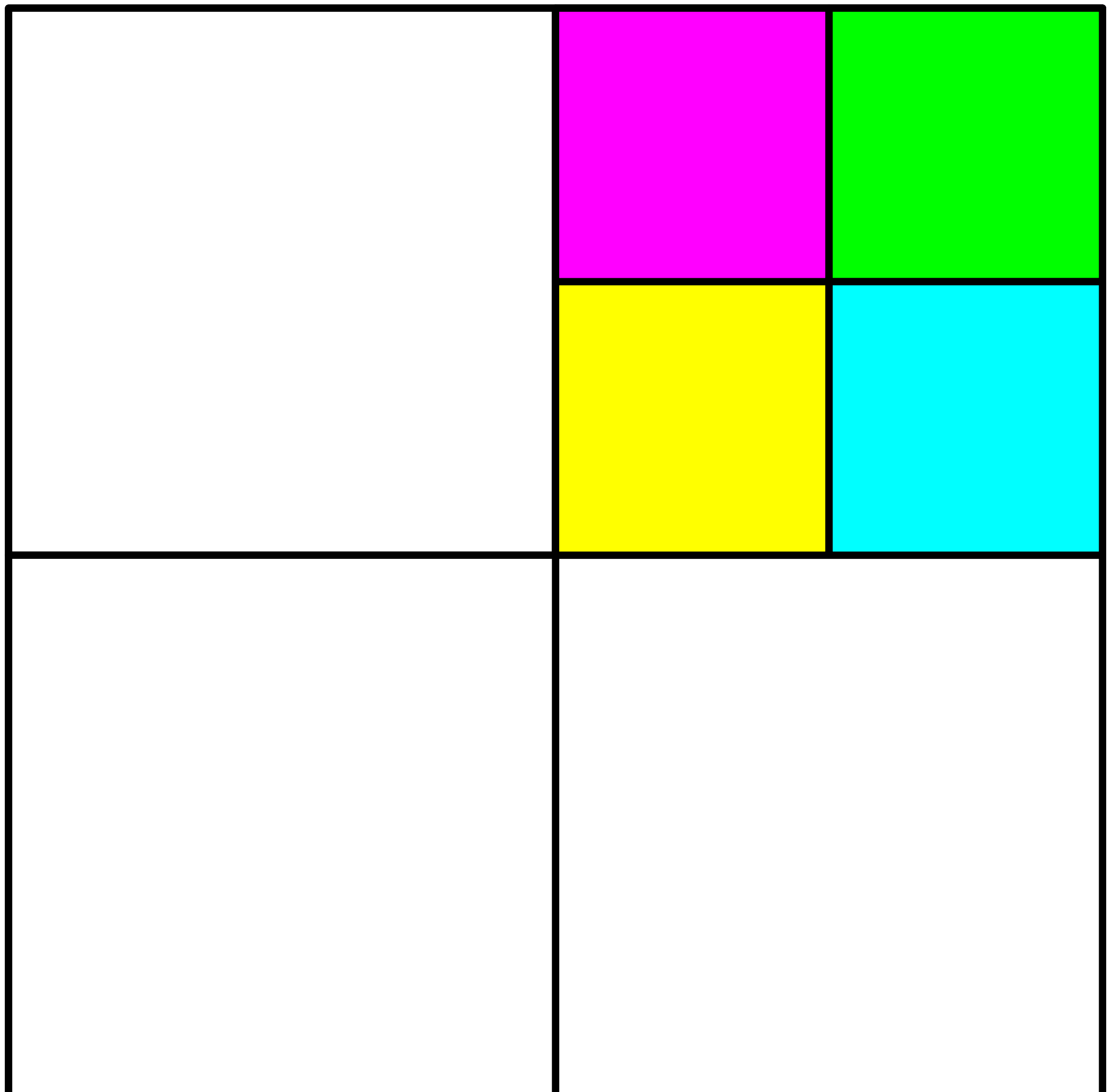

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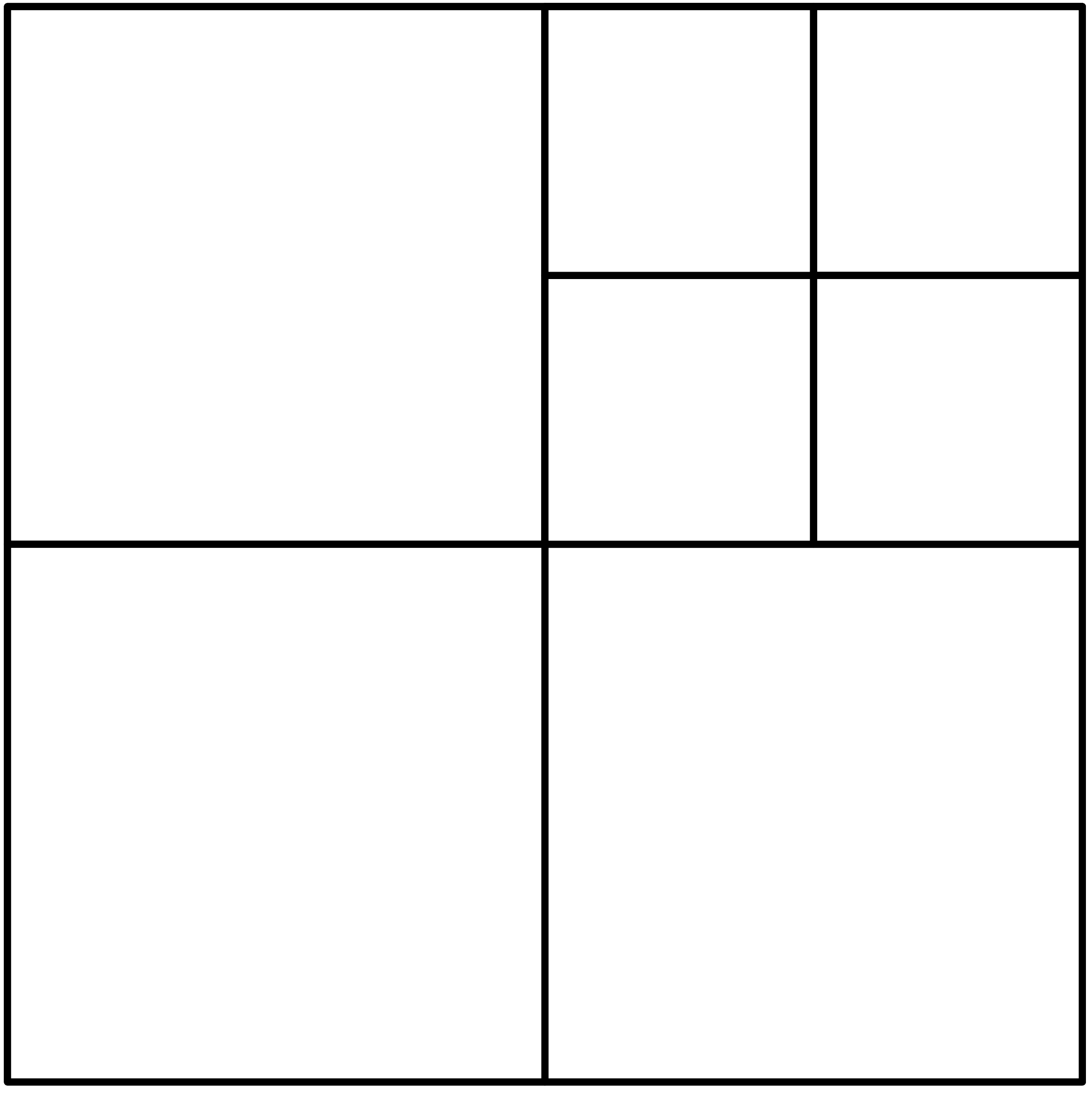

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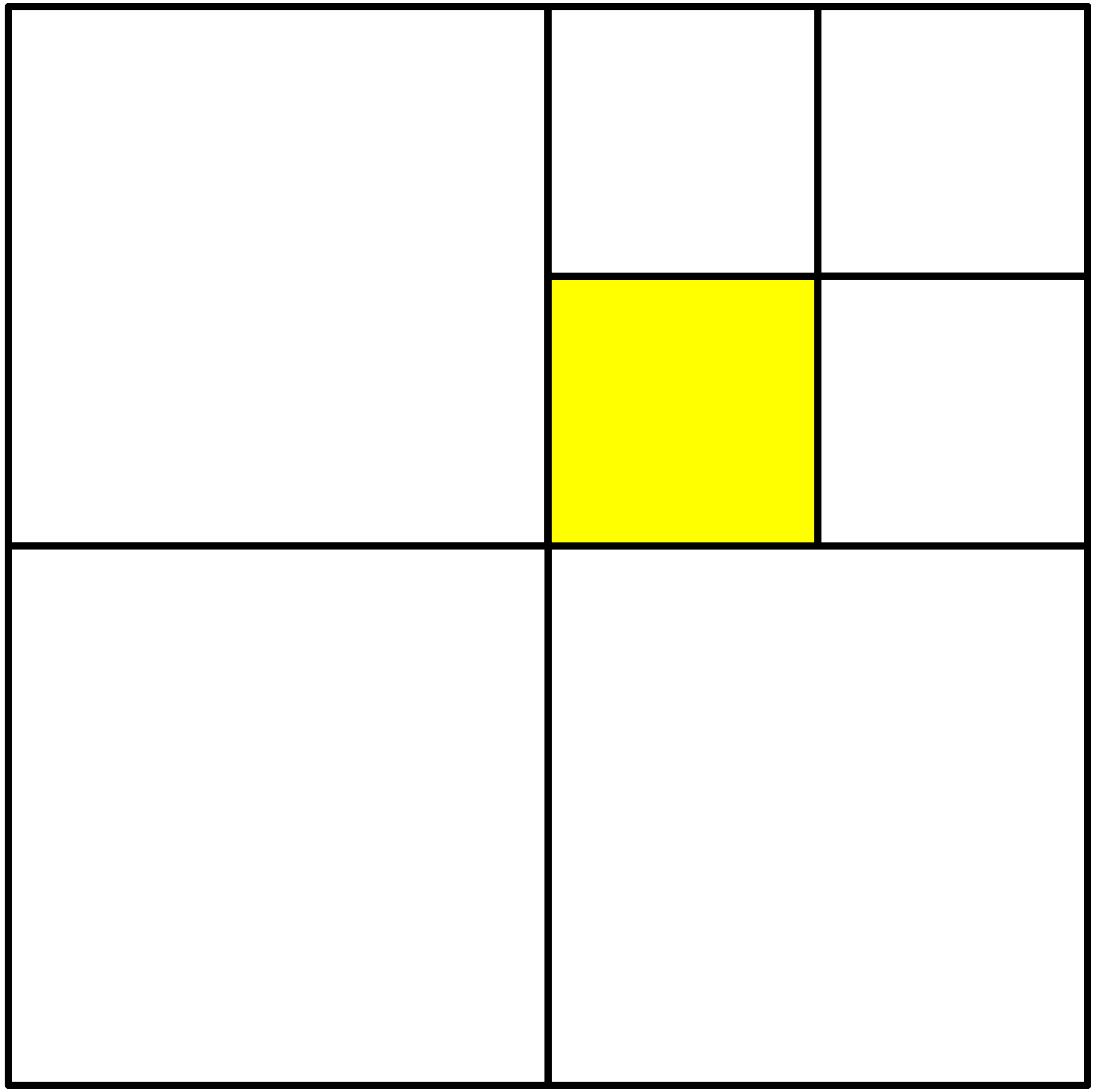

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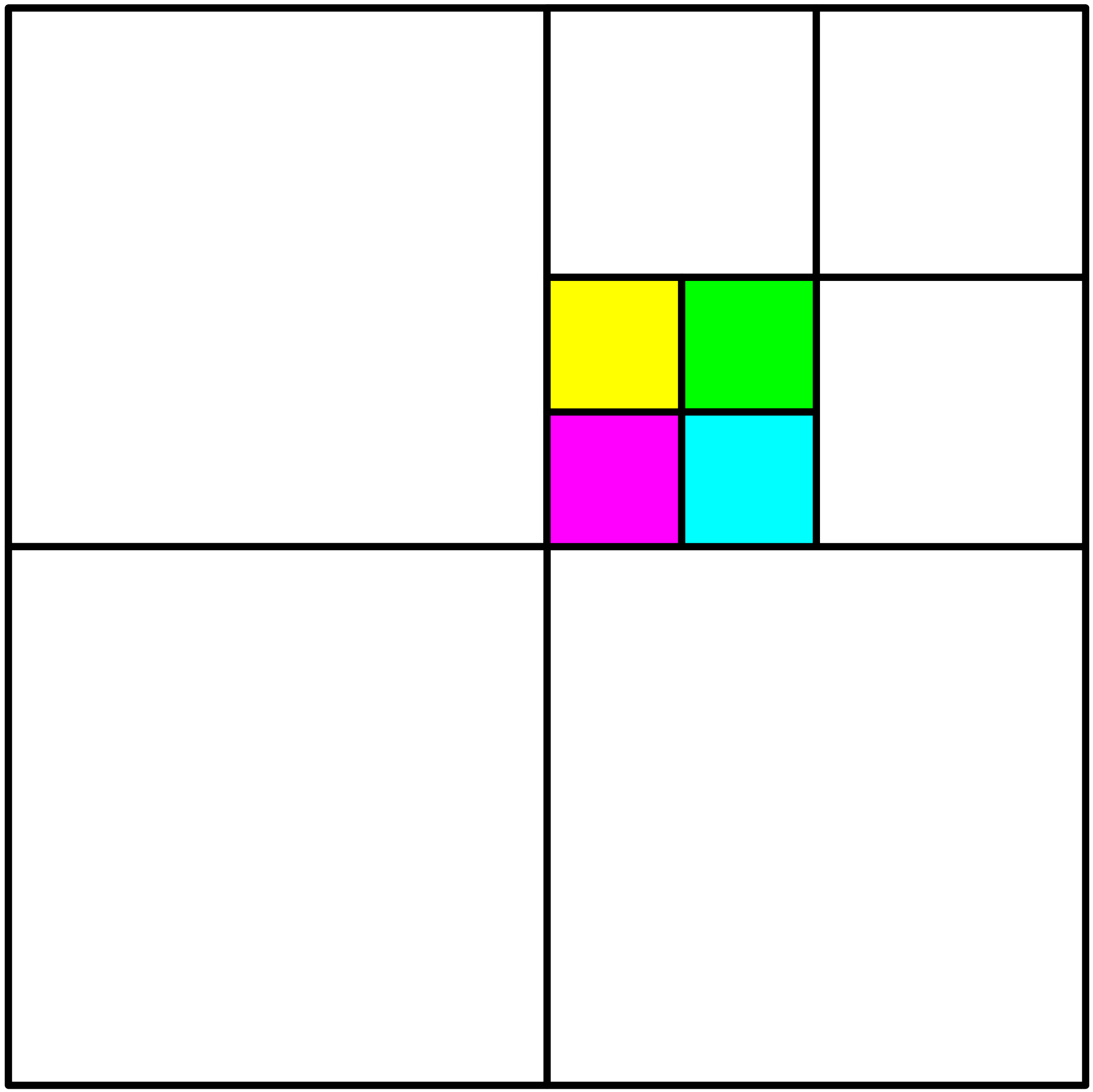

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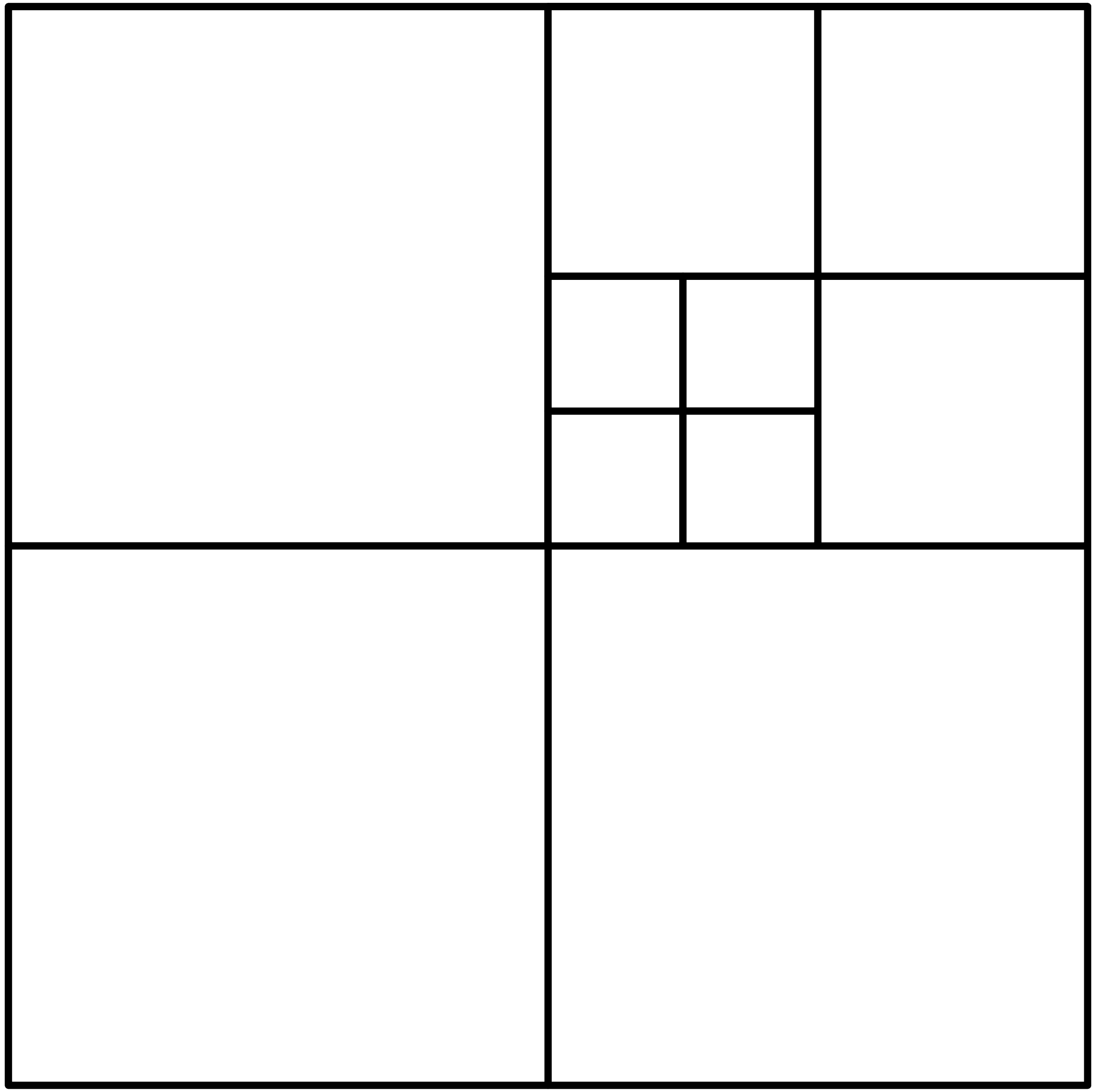

The Key Insight

- If we can subdivide a square into n squares, we can also subdivide it into $n + 3$ squares.
- Since we can subdivide a bigger square into 6, 7, and 8 squares, we can subdivide a square into n squares for any $n \geq 6$:
  - For multiples of three, start with 6 and keep adding three squares until n is reached.
  - For numbers congruent to one modulo three, start with 7 and keep adding three squares until n is reached.
  - For numbers congruent to two modulo three, start with 8 and keep adding three squares until n is reached.

Theorem: For any $n \geq 6$, it is possible to subdivide a square into n smaller squares.

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As our base cases, we prove $P(6)$, $P(7)$, and $P(8)$, that a square can be subdivided into 6, 7, and 8 squares.

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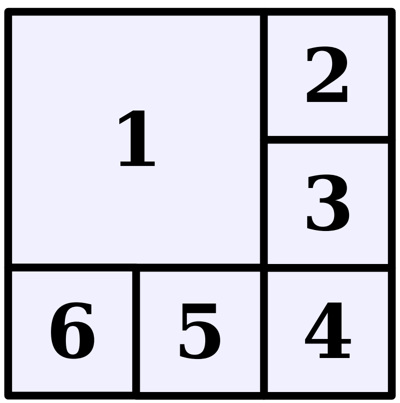

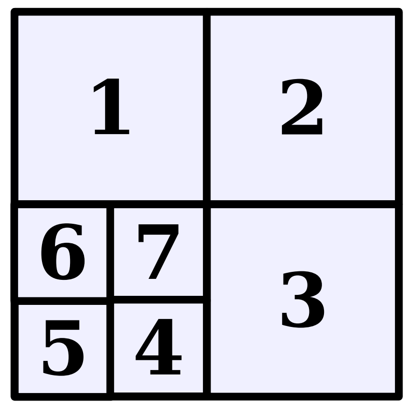

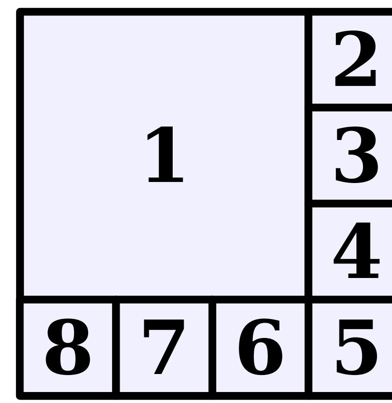

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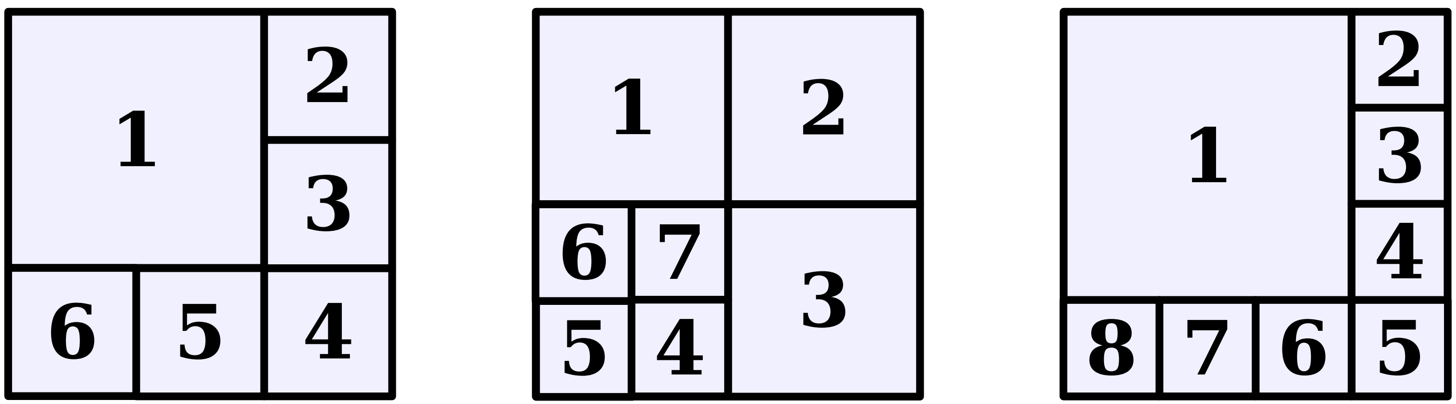

For the inductive step, assume that for some $k \geq 6$ that $P(k)$ is true and a square can be subdivided into k squares.

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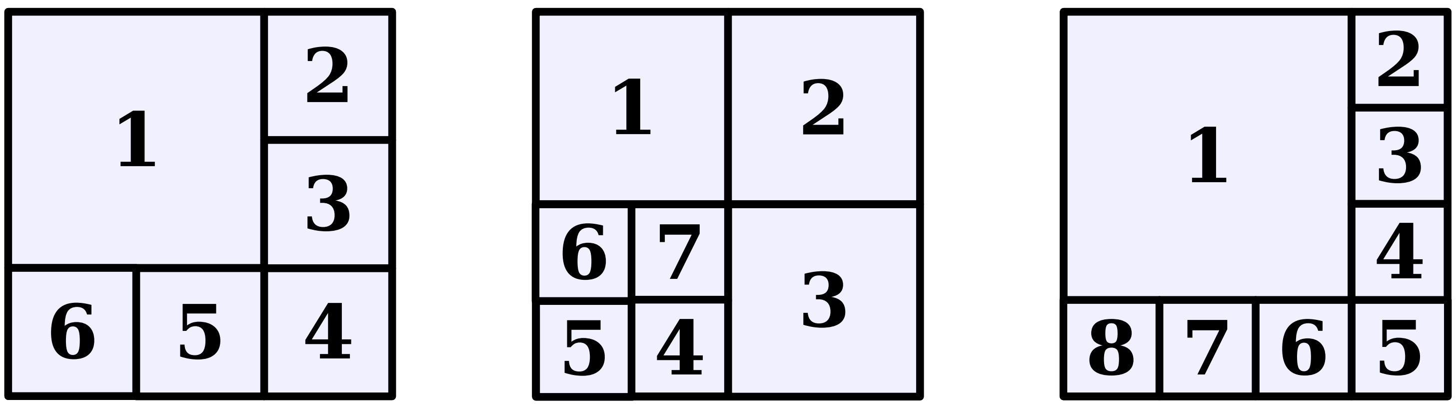

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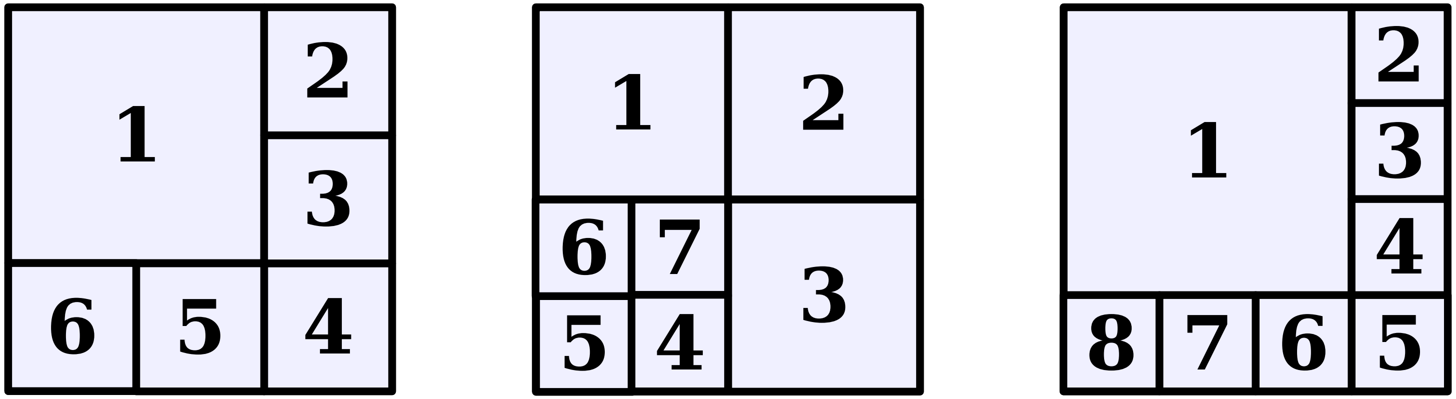

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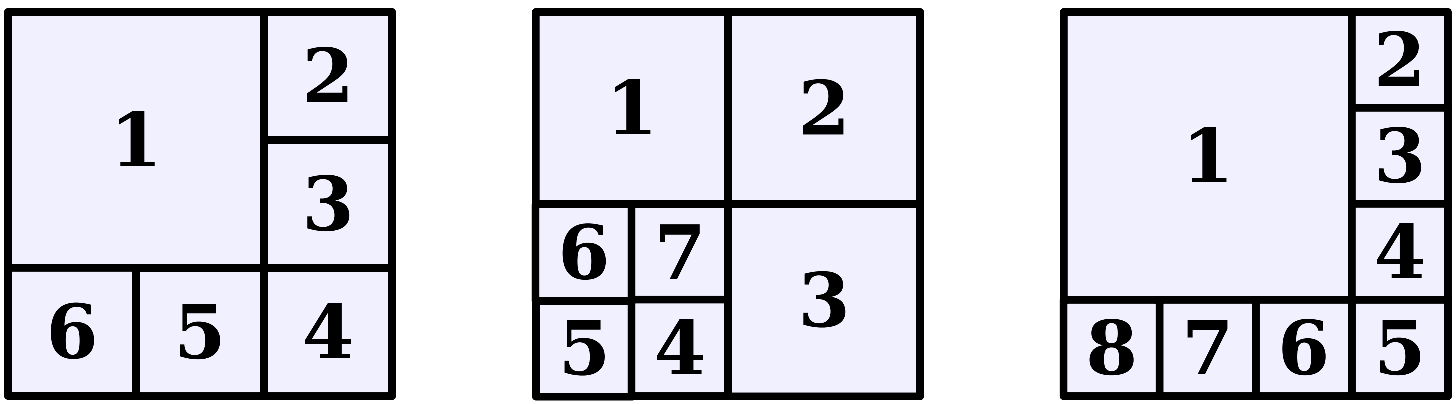

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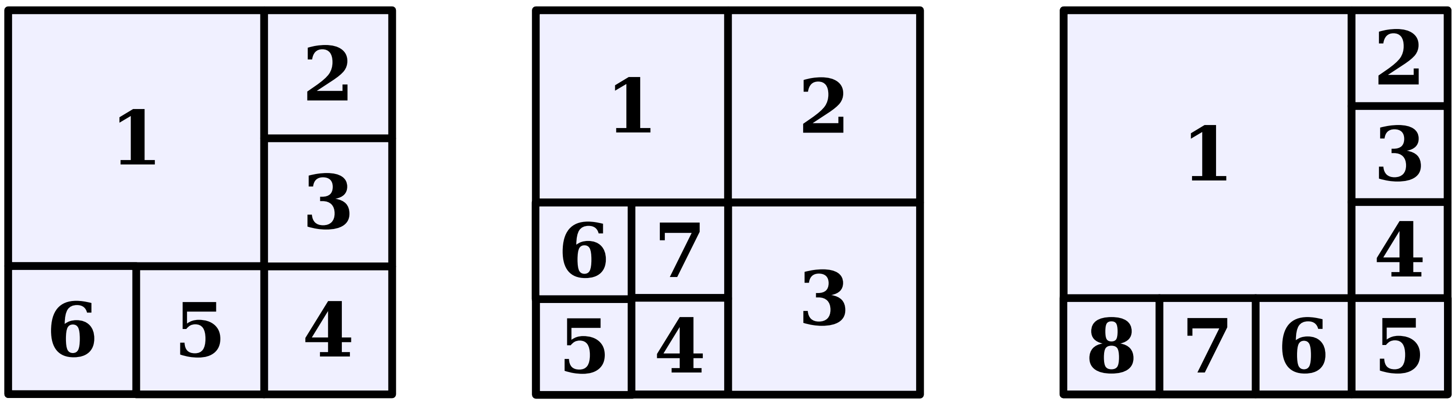

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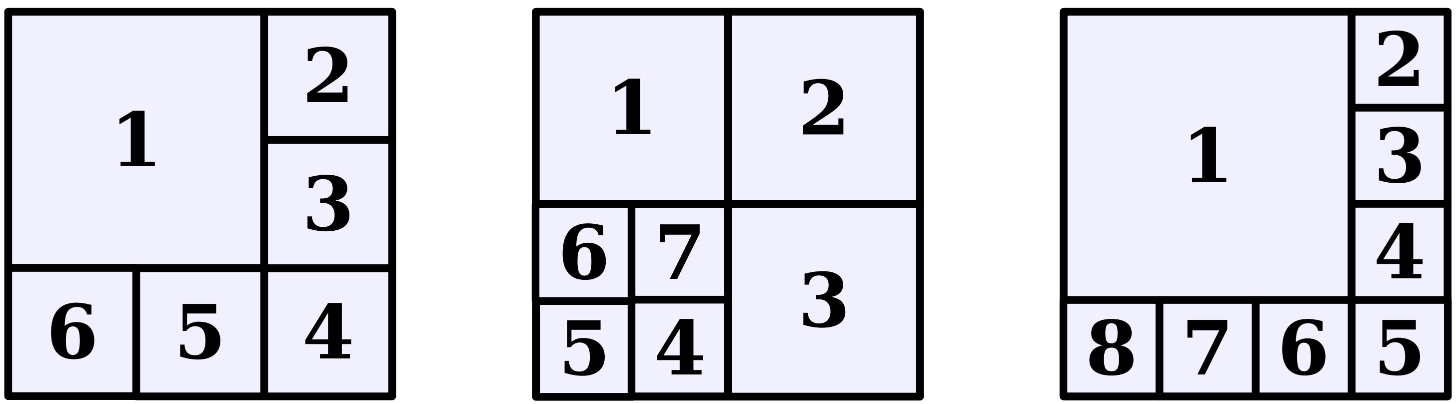

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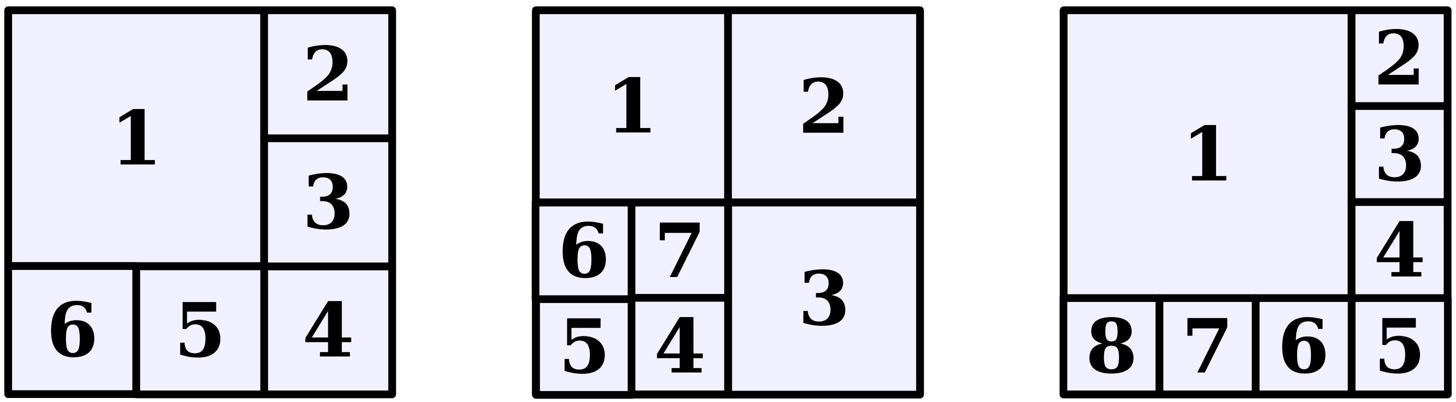

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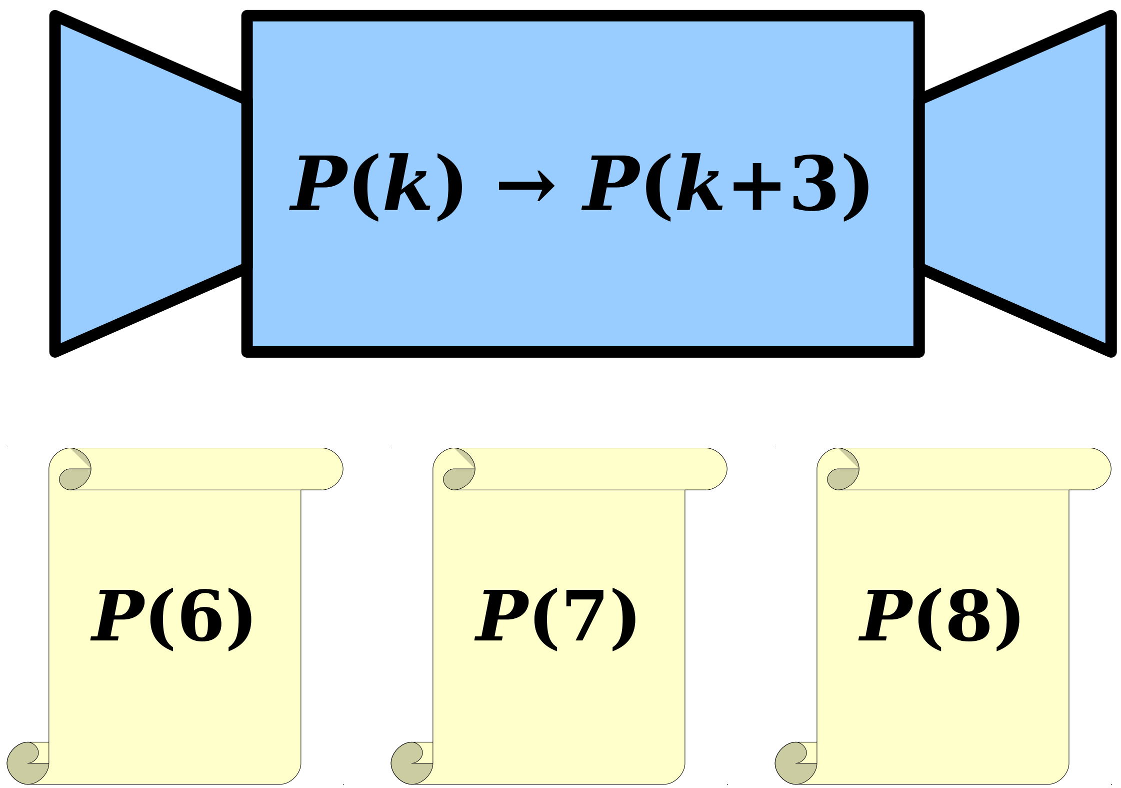
Why This Works

- This induction has three consecutive base cases and takes steps of size three.
- Thinking back to our “induction machine” analogy:



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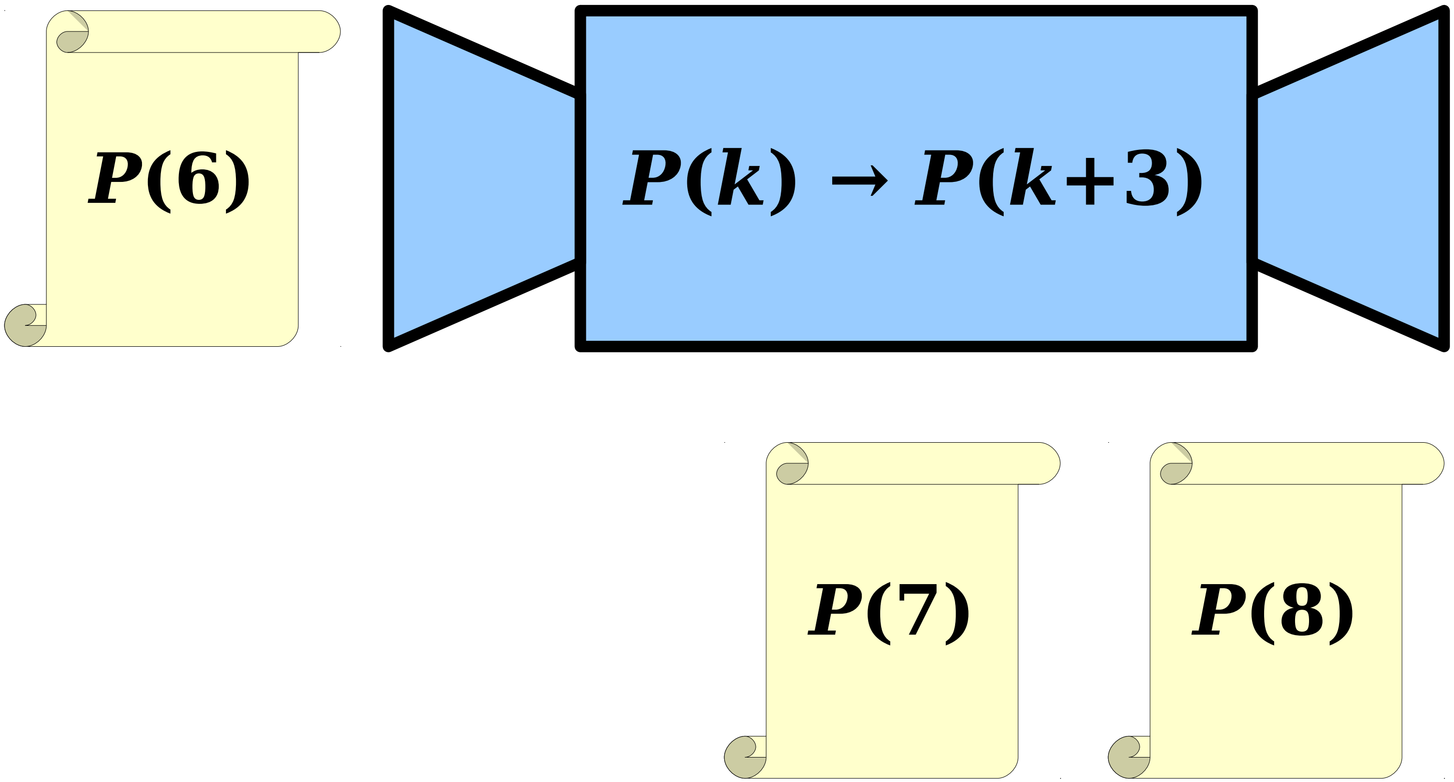

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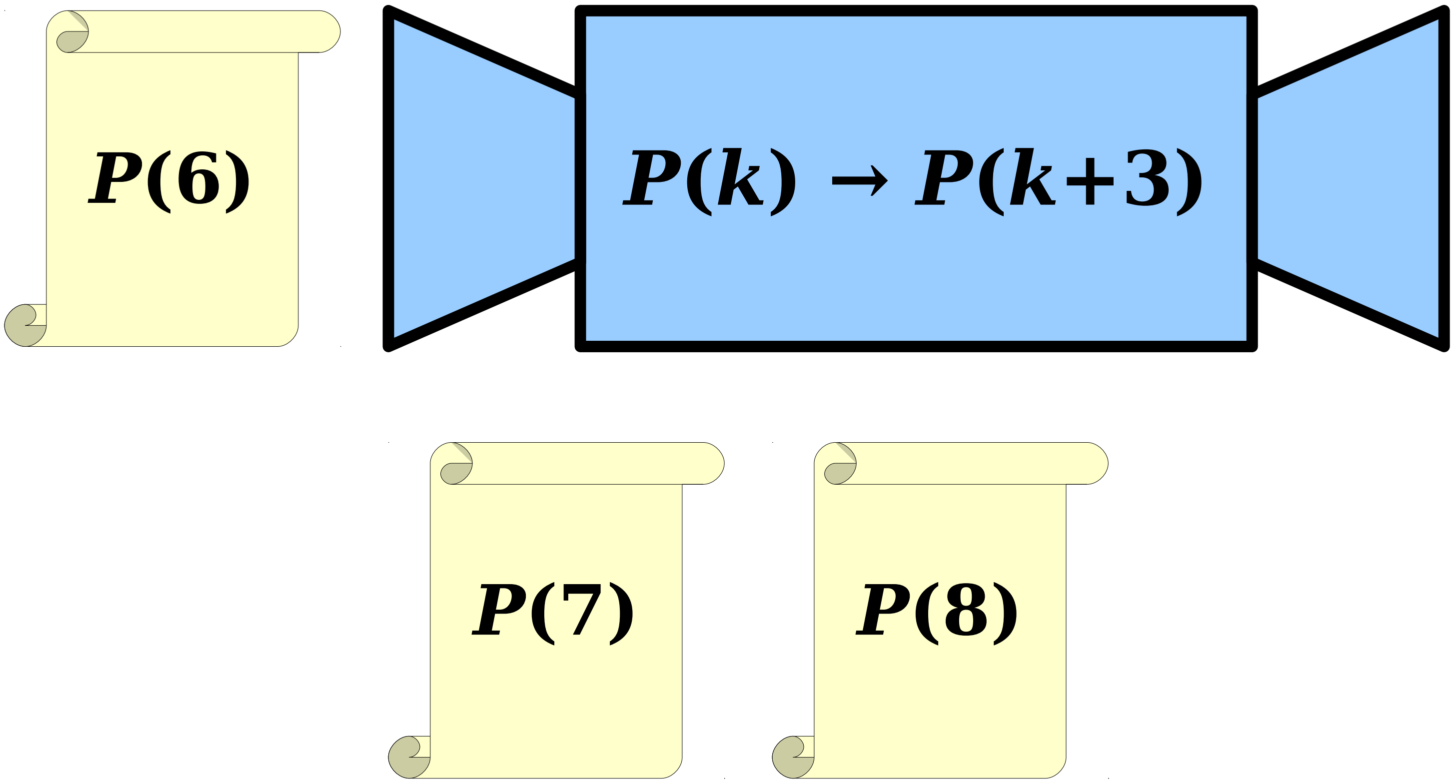

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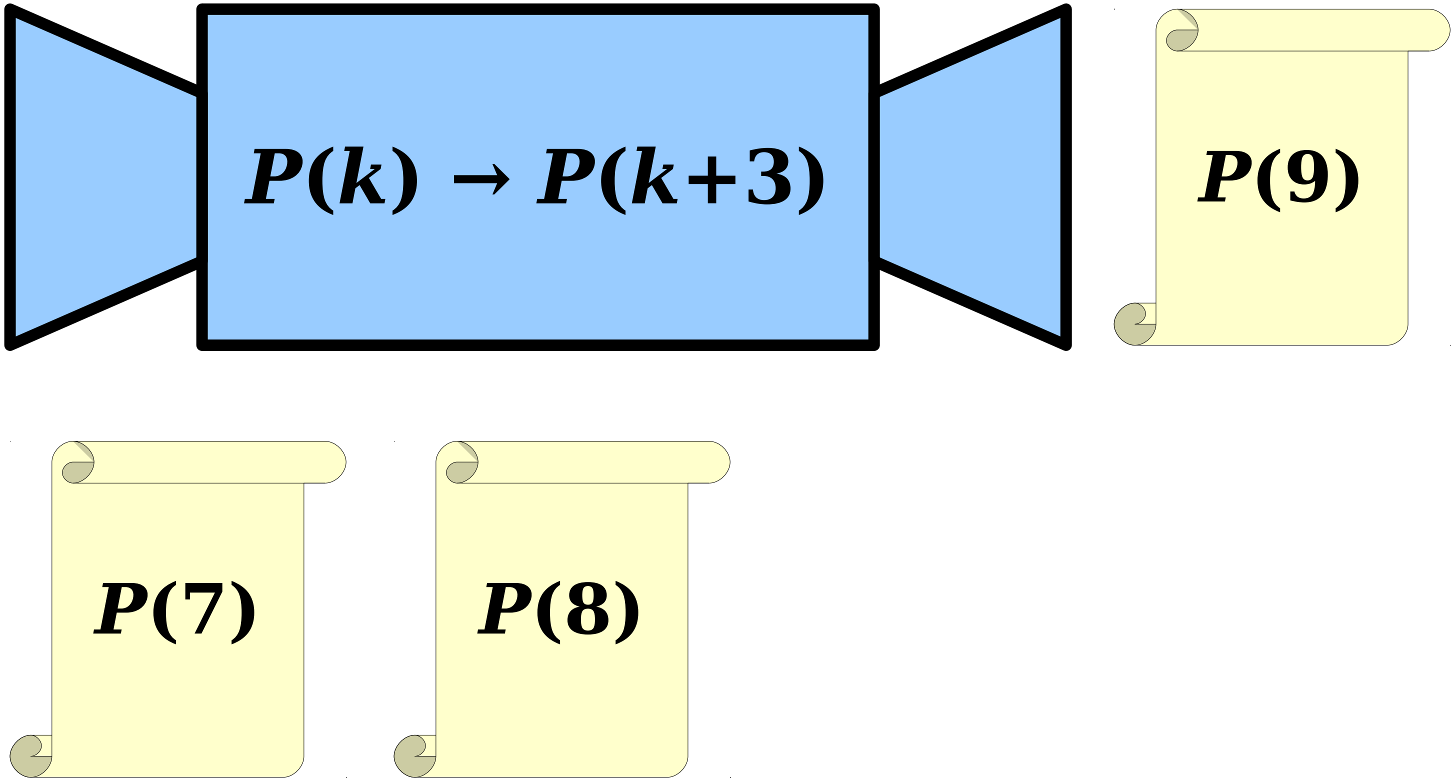

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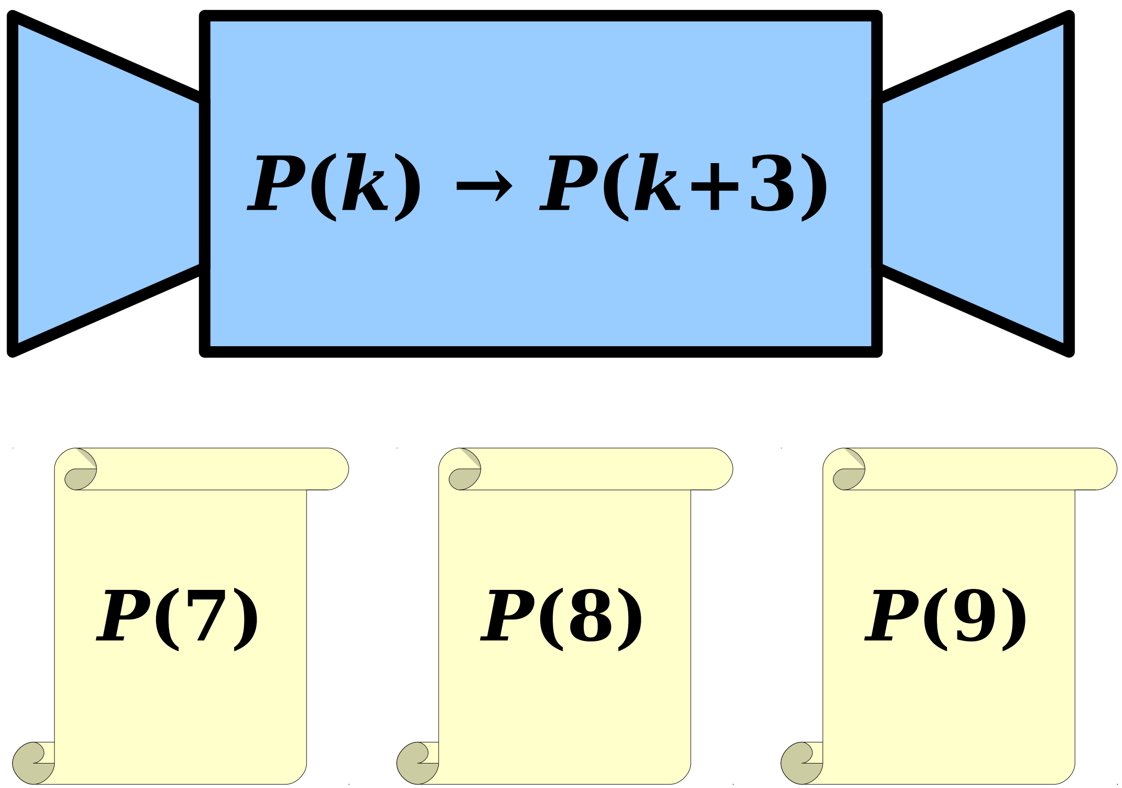

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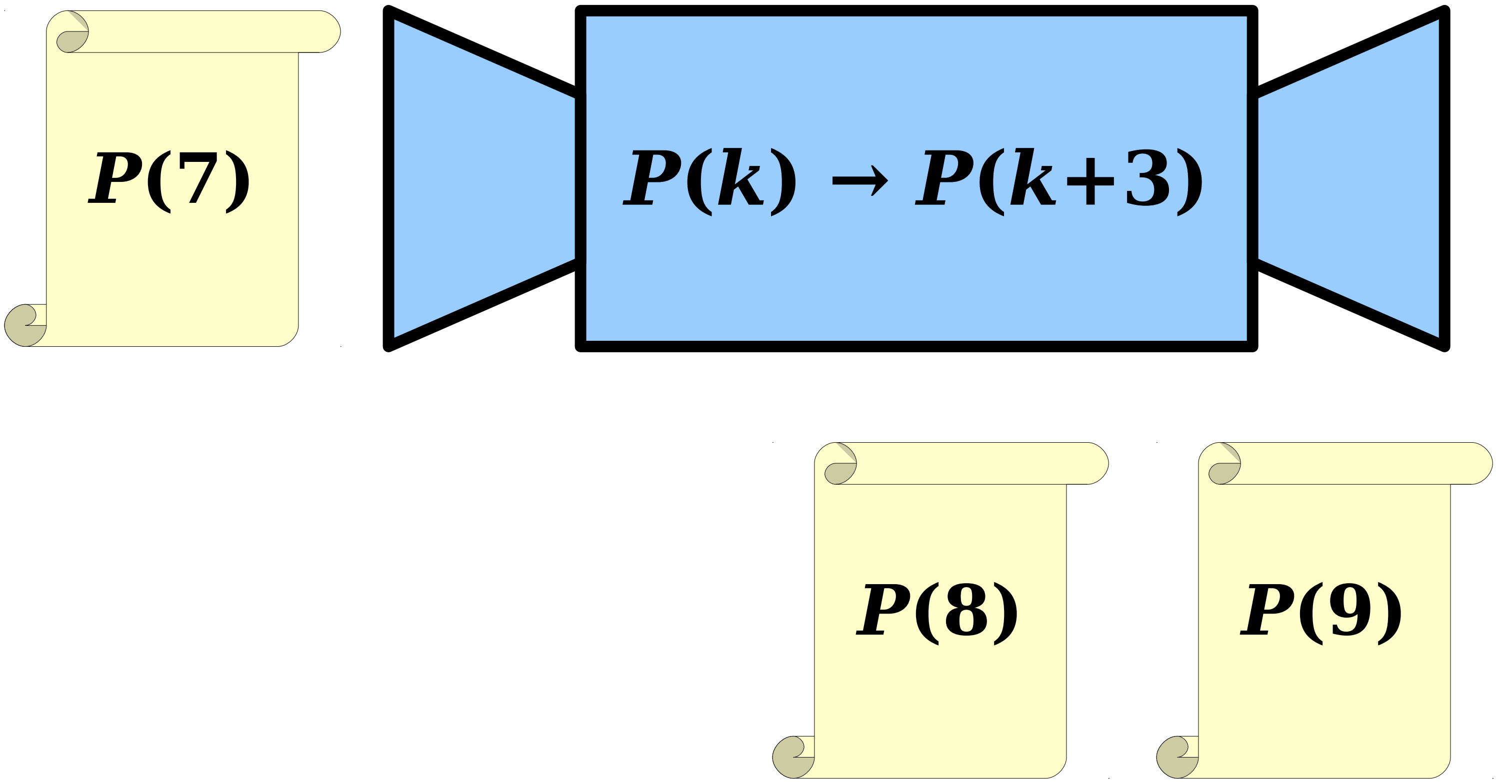

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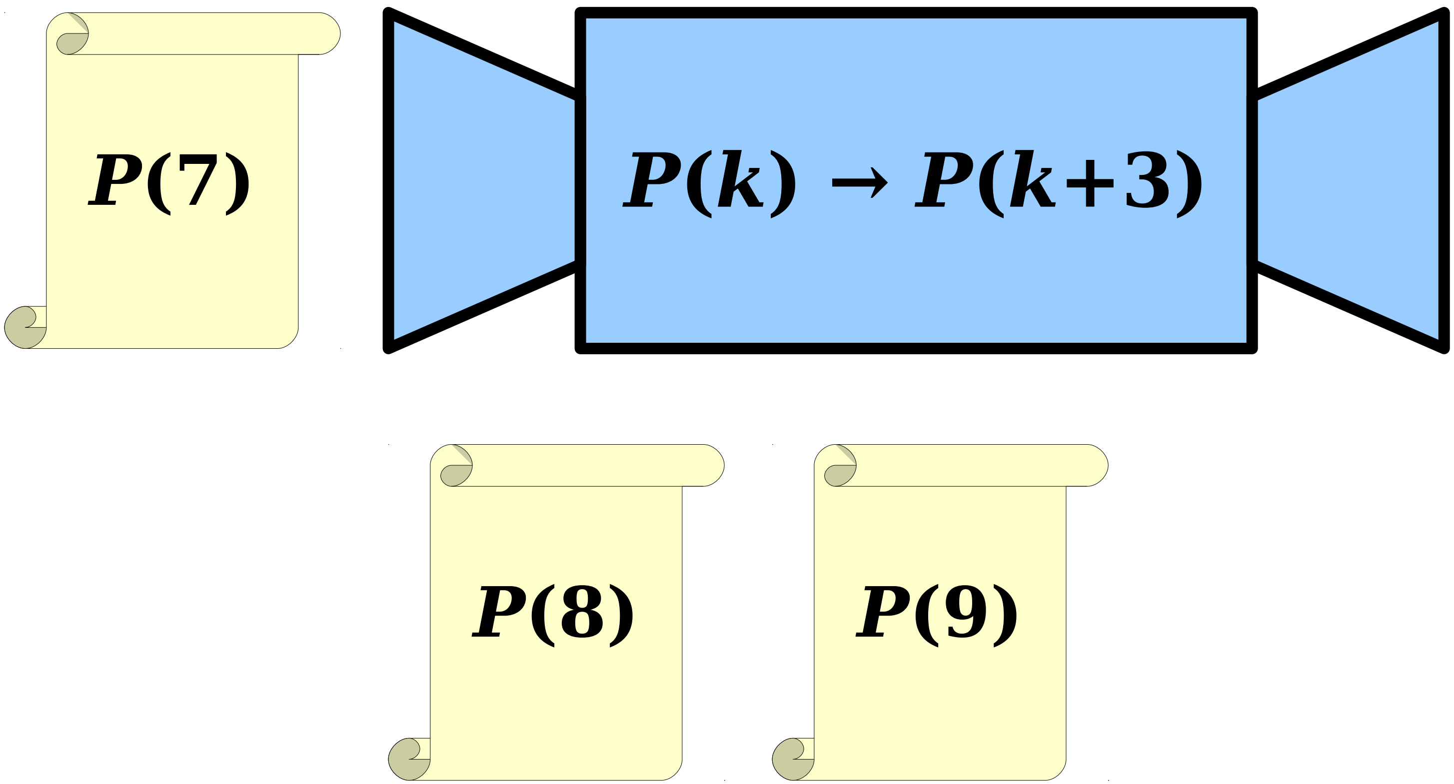

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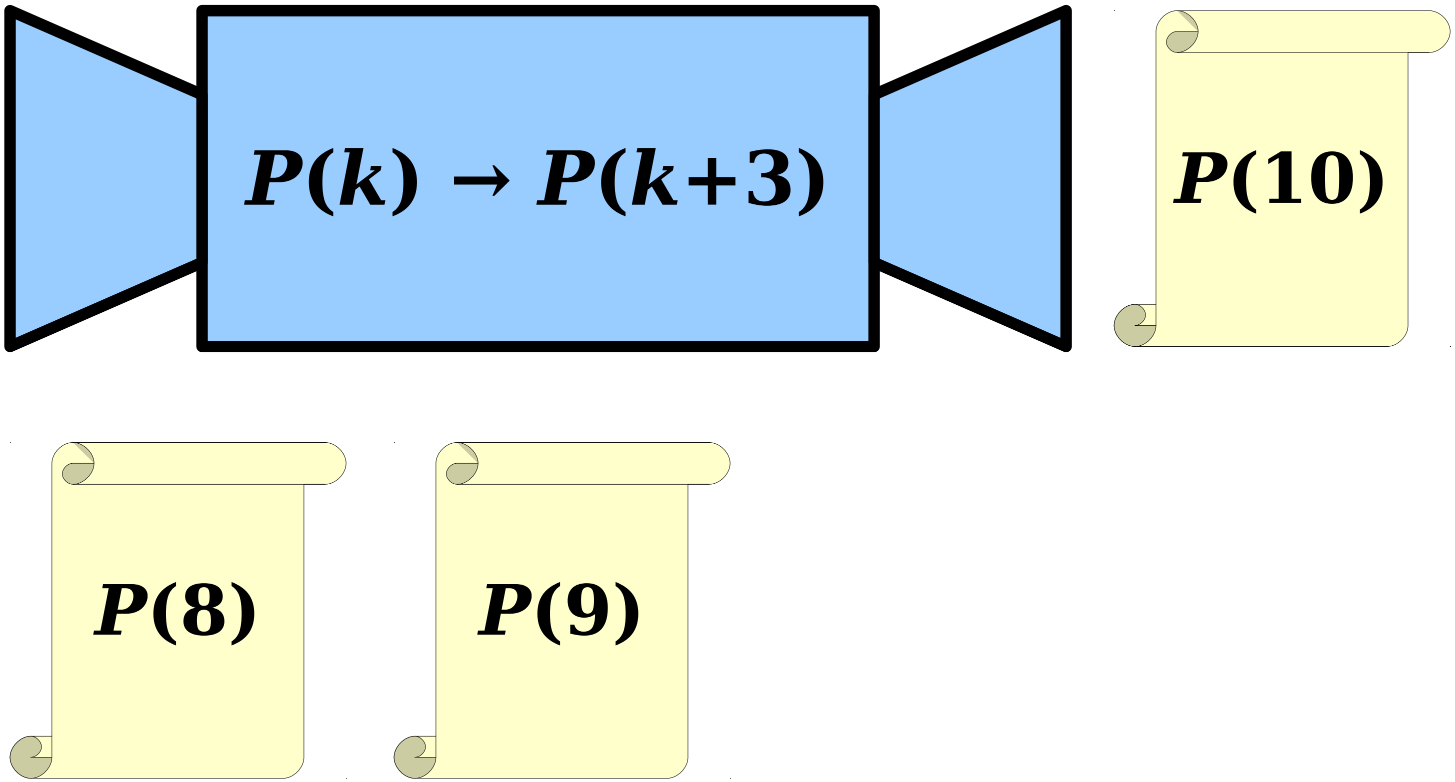

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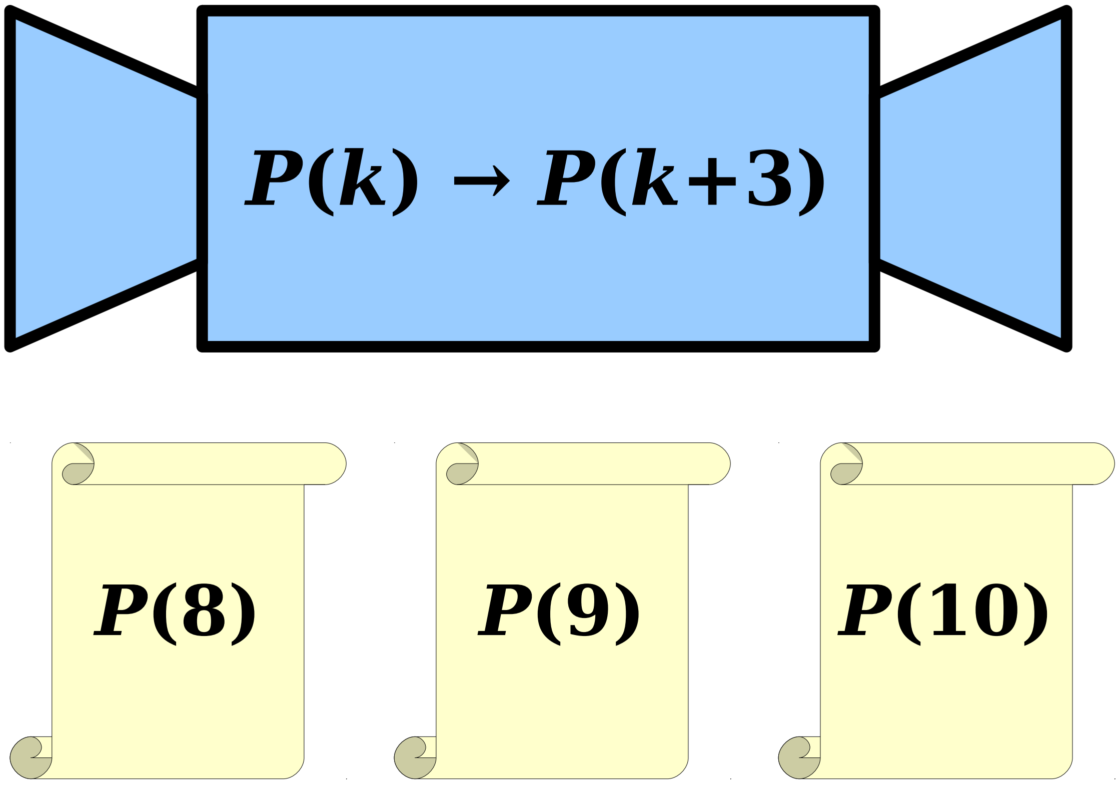

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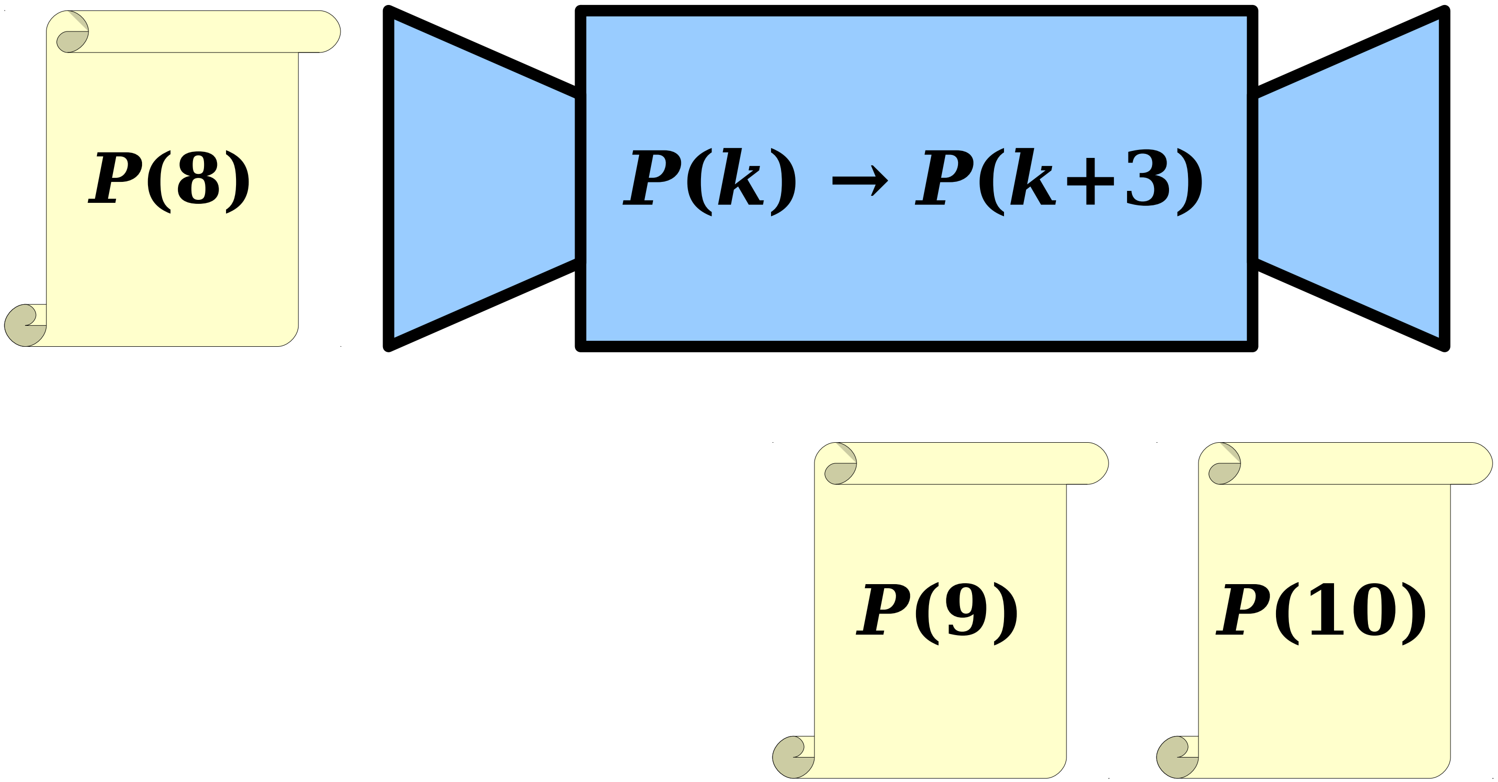

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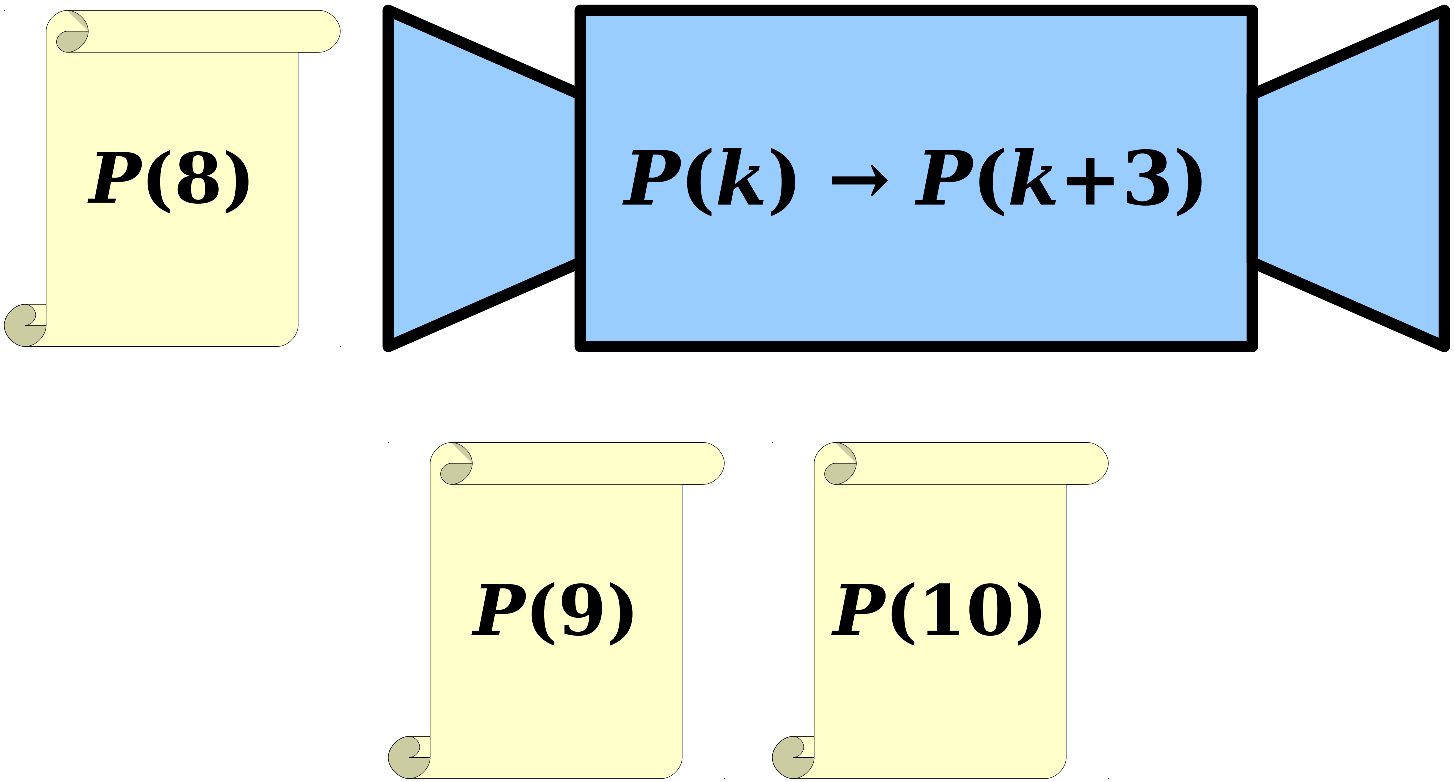

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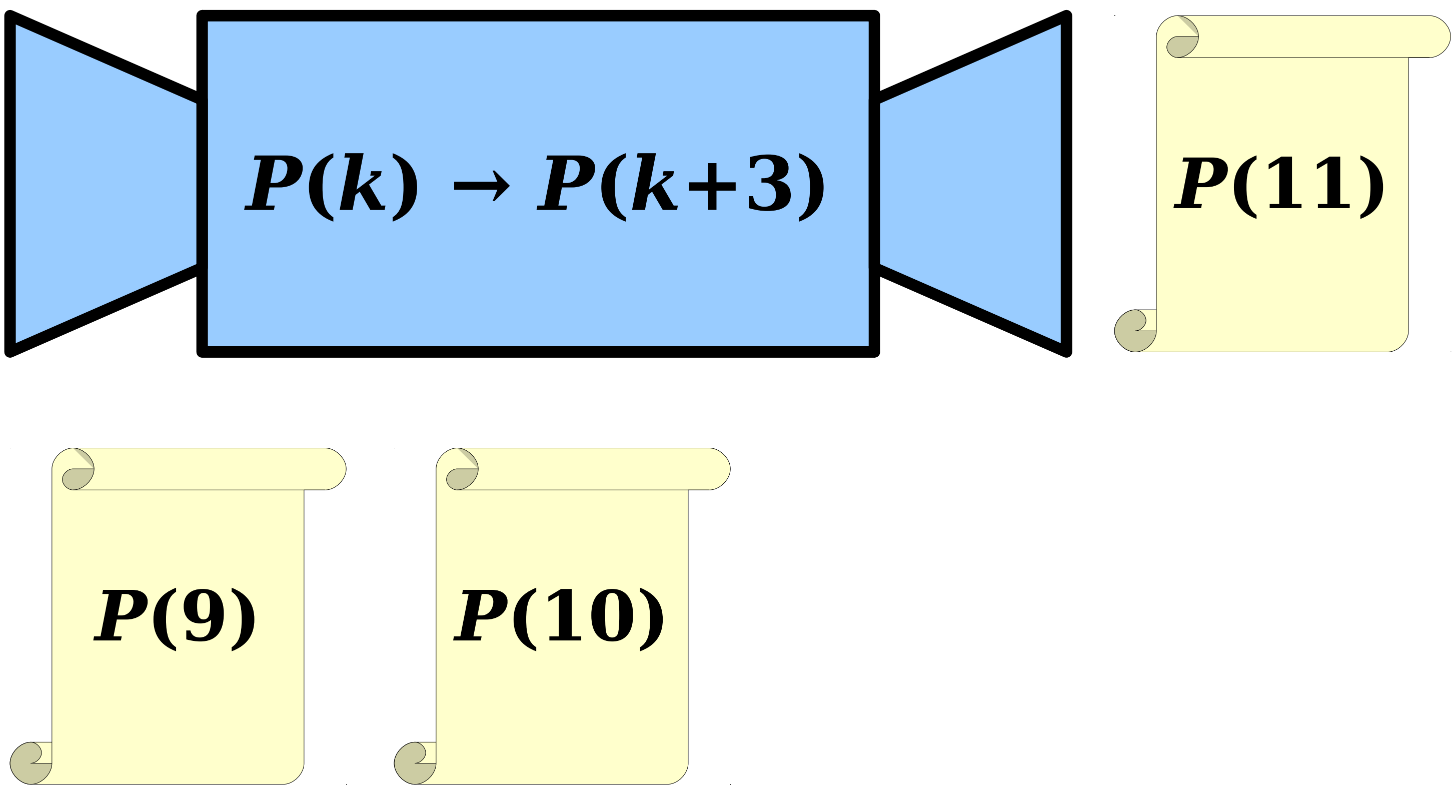

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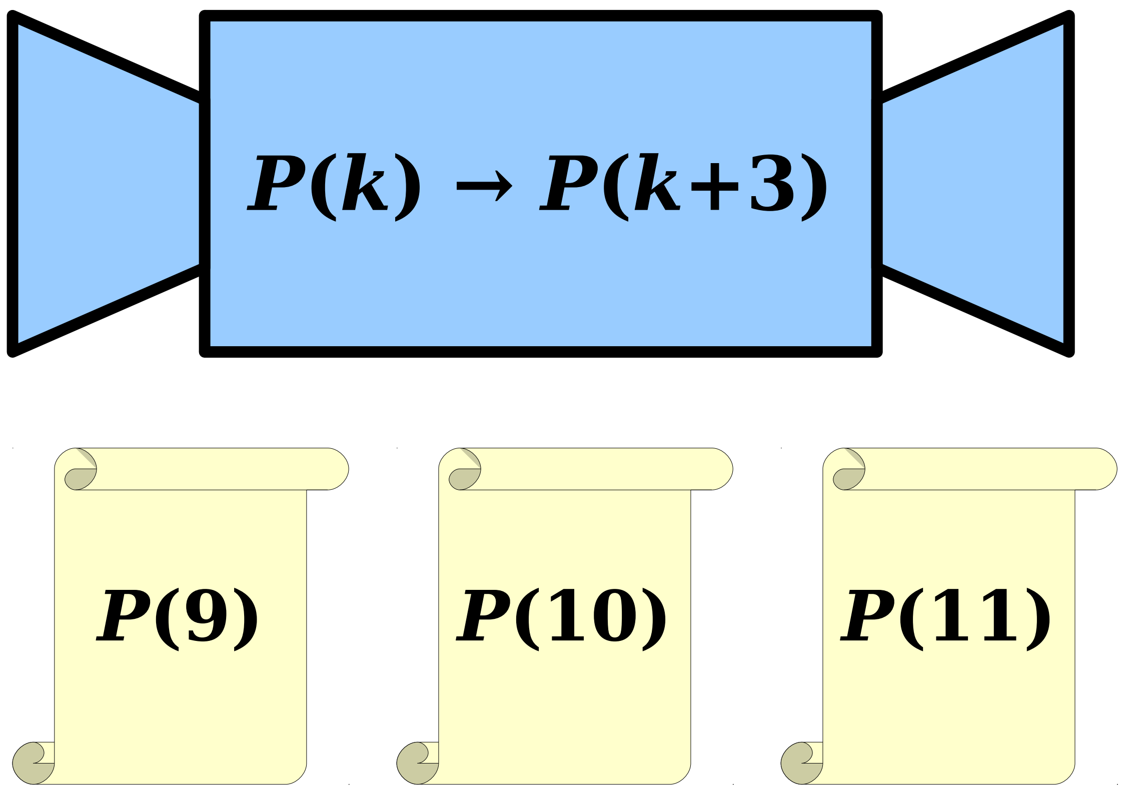


Generalizing Induction

- When doing a proof by induction:
  - Feel free to use multiple base cases.
  - Feel free to take steps of sizes other than one.
- Just be careful to make sure you cover all the numbers you think that you're covering!

Time-Out for Announcements!

Problem Set One

- Problem Set 1 due Friday, October 3 at the start of class.
  - Stop by office hours with questions!
  - Ask questions on Piazza!
  - Email the staff list (cs103-aut1415-staff@lists.stanford.edu) with questions!

Checkpoint Problems

- Problem Set 1 Checkpoints graded; feedback available online at Scoryst.
  - Please review this feedback before submitting the rest of the problems – the point of the checkpoint is to get useful feedback!
- In the future, please only submit one copy of the checkpoint per group – it *dramatically* simplifies grading.
- Submitted in the wrong category! No worries! We've got it handled.

A Quick Apology

Solution Sets

- Solution sets for the discussion problems and for the checkpoint problem are available in hardcopy at lecture or in the Gates B wing open space near Keith's office.
- SCPD students – we'll send you an email with information about solutions soon.

Piazza Questions

- We've gotten a lot of private questions on Piazza that are really interesting and probably super useful to other students.
- If you're asking a Piazza question that doesn't give away hints or answers to problem set questions, consider making it public so that everyone can learn!

GTGTC

- Girls Teaching Girls to Code (GTGTC) has an opening on their admin team.
- They run day-long tutoring sessions where high-school girls get mentorship from Stanford students. At one of their events last year, they had 40 Stanford students mentor 200 high-schoolers!
- Interested? Apply at **http://bit.ly/gtgtc-app**, or contact Jessie Duan at **jduan1@stanford.edu**.

Your Questions

“What are some of the most interesting/unusual proof techniques?”

“Most of the proofs in the homework can be solved algebraically without the use of words – the algebra suffices as an explanation. Especially when you use proof by contradiction. How do you know you're being superfluous with the explanations?”

“Is a program a proof? If so, suppose we have a program that is guaranteed to prove or disprove some fact. Assume that it is guaranteed to terminate, but will only do so thousands of years from now. Is it still a proof?”

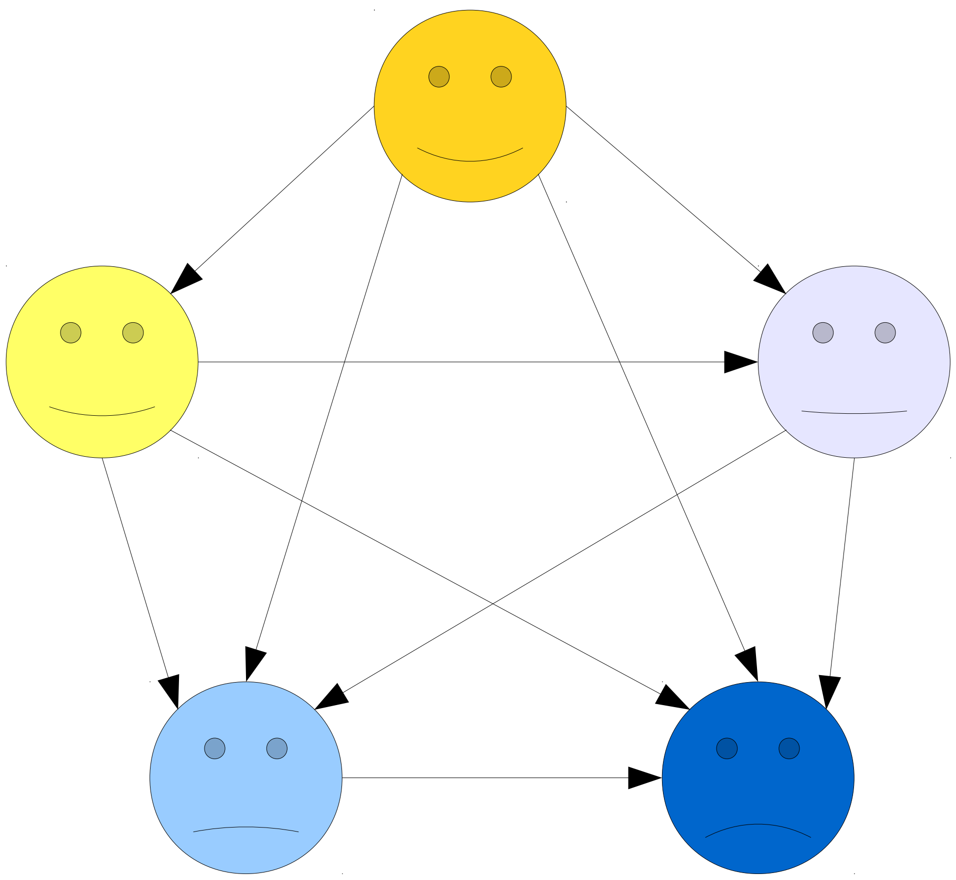
“Keith, why did you choose academia over industry?”

Back to CS103!

Example: Tournaments

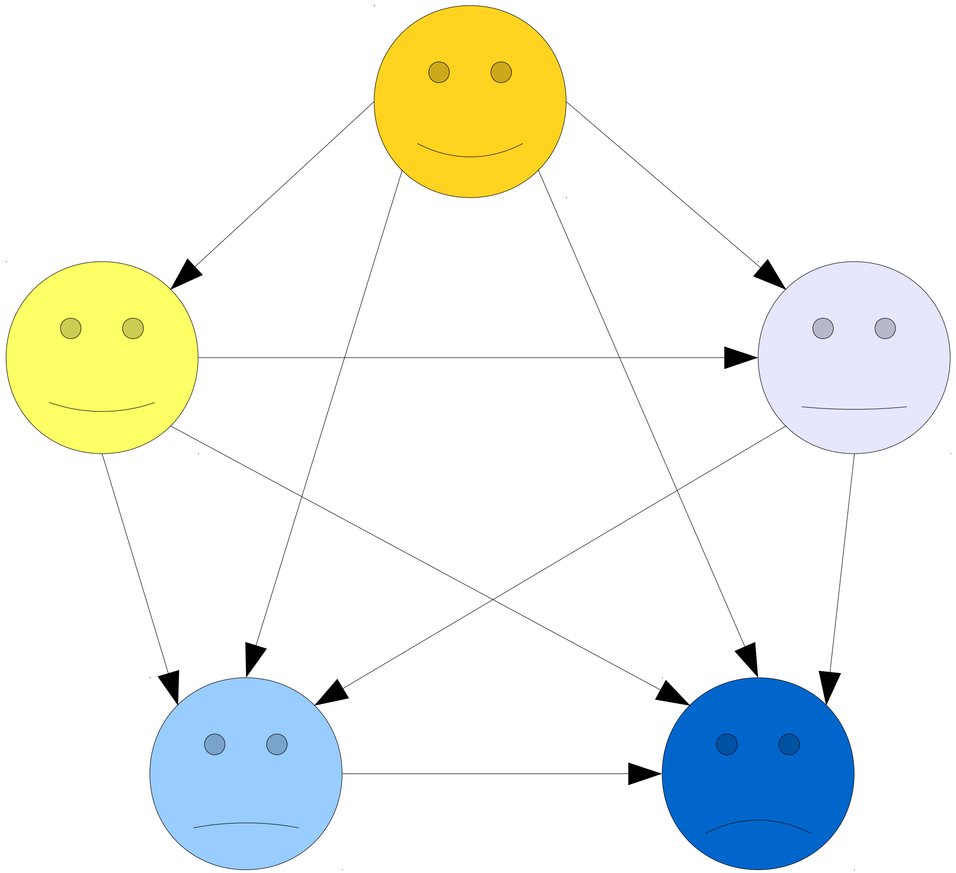
Tournaments

- A ***tournament*** is a contest for $n \geq 1$ people.
- Each person plays exactly one game against each other person, and there are no ties.
- The result can be visualized in a picture like this one, which is called a ***tournament graph***.



Victory Chains

- A ***victory chain*** in a tournament is a way of lining up the players so that every player beat the player that comes after them.



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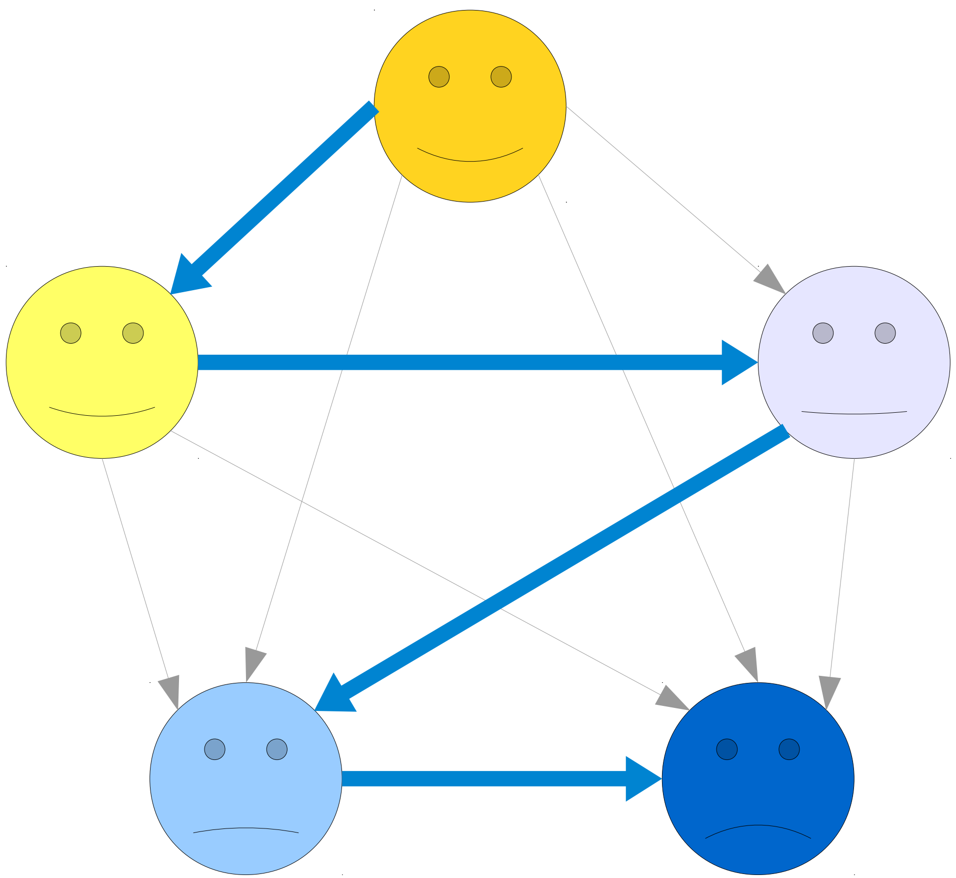

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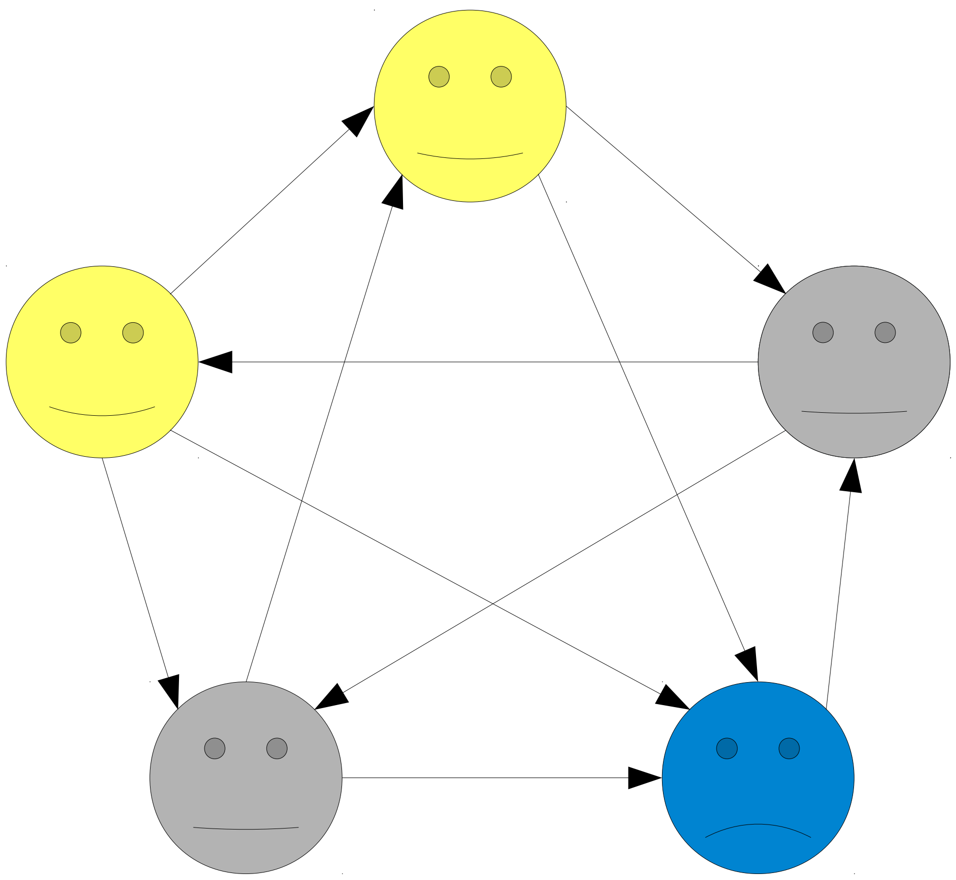

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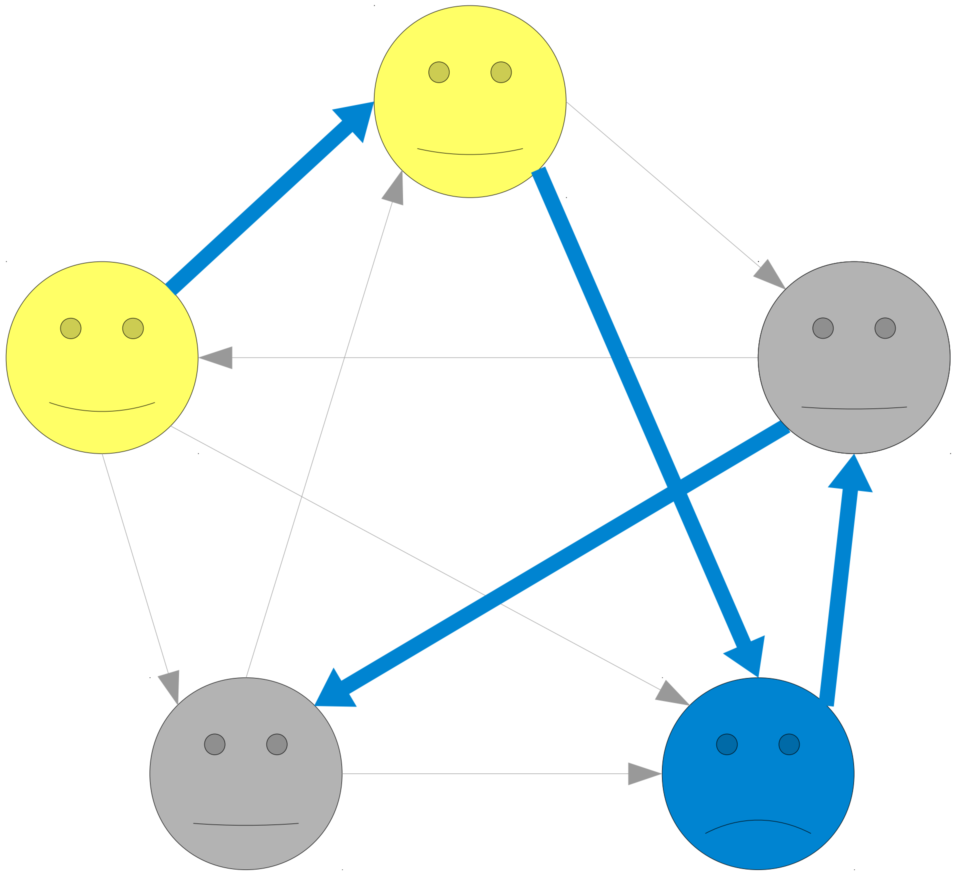


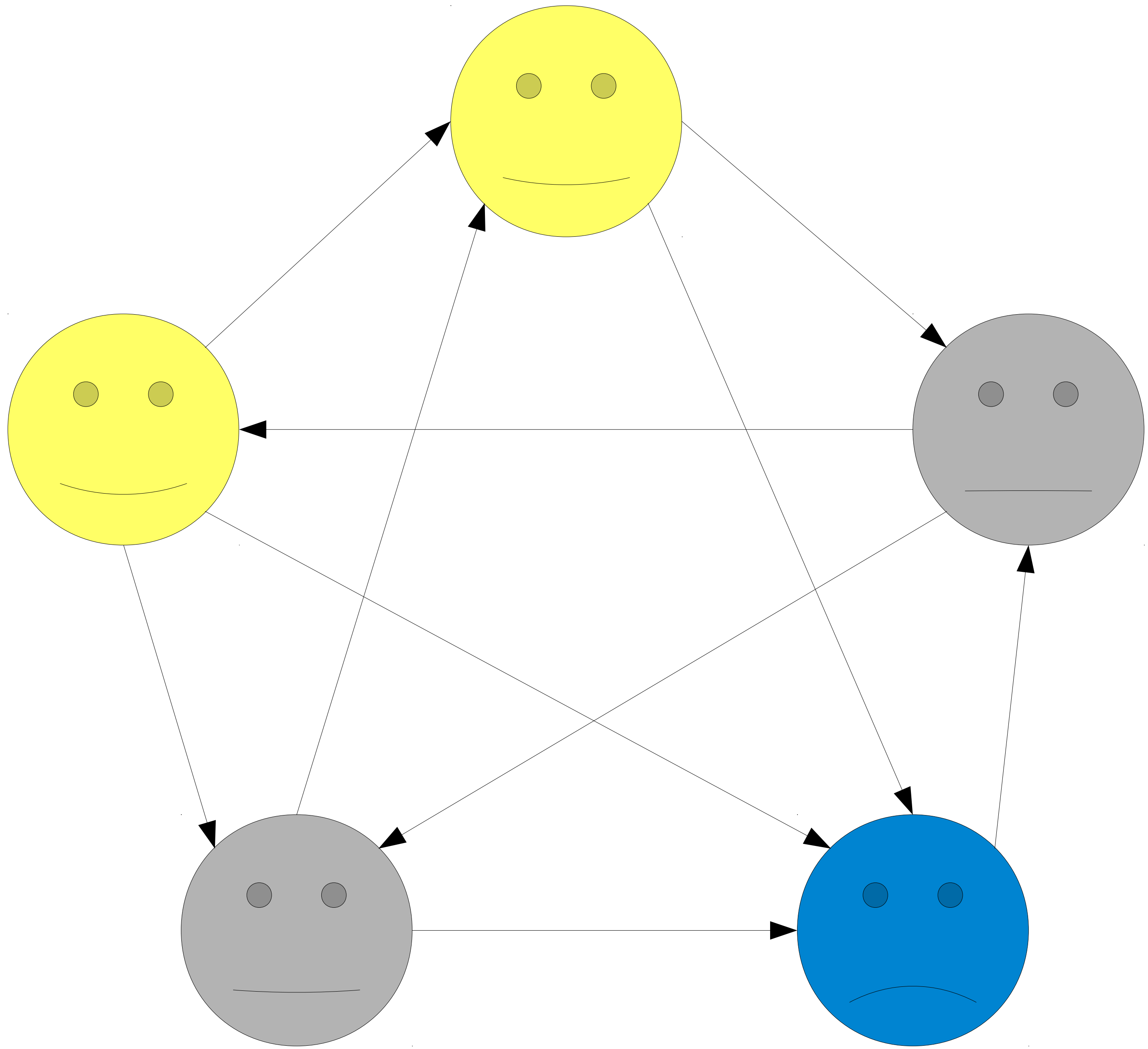
Theorem: Every tournament, regardless of the outcome has a victory chain.

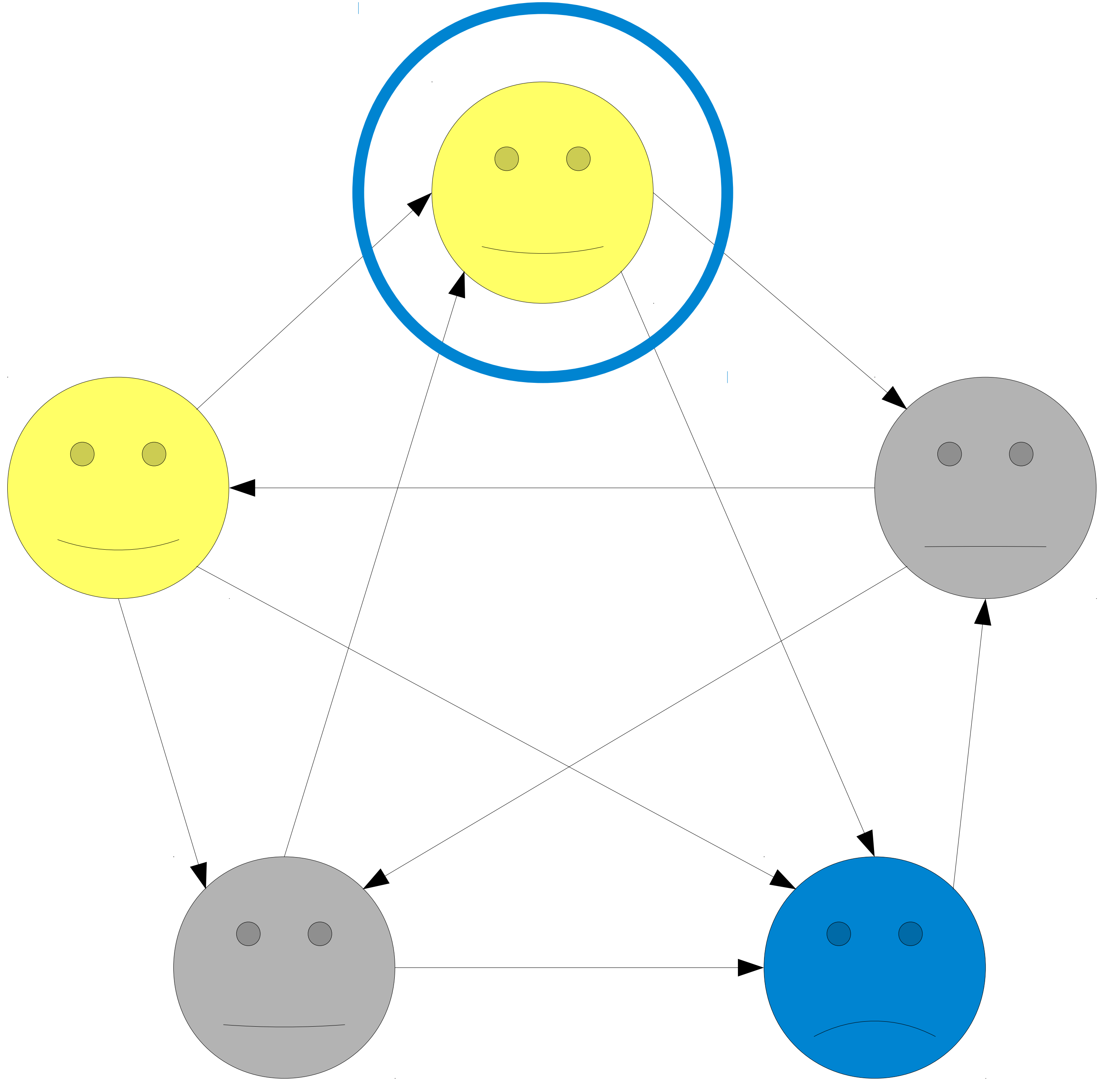
Thinking Inductively

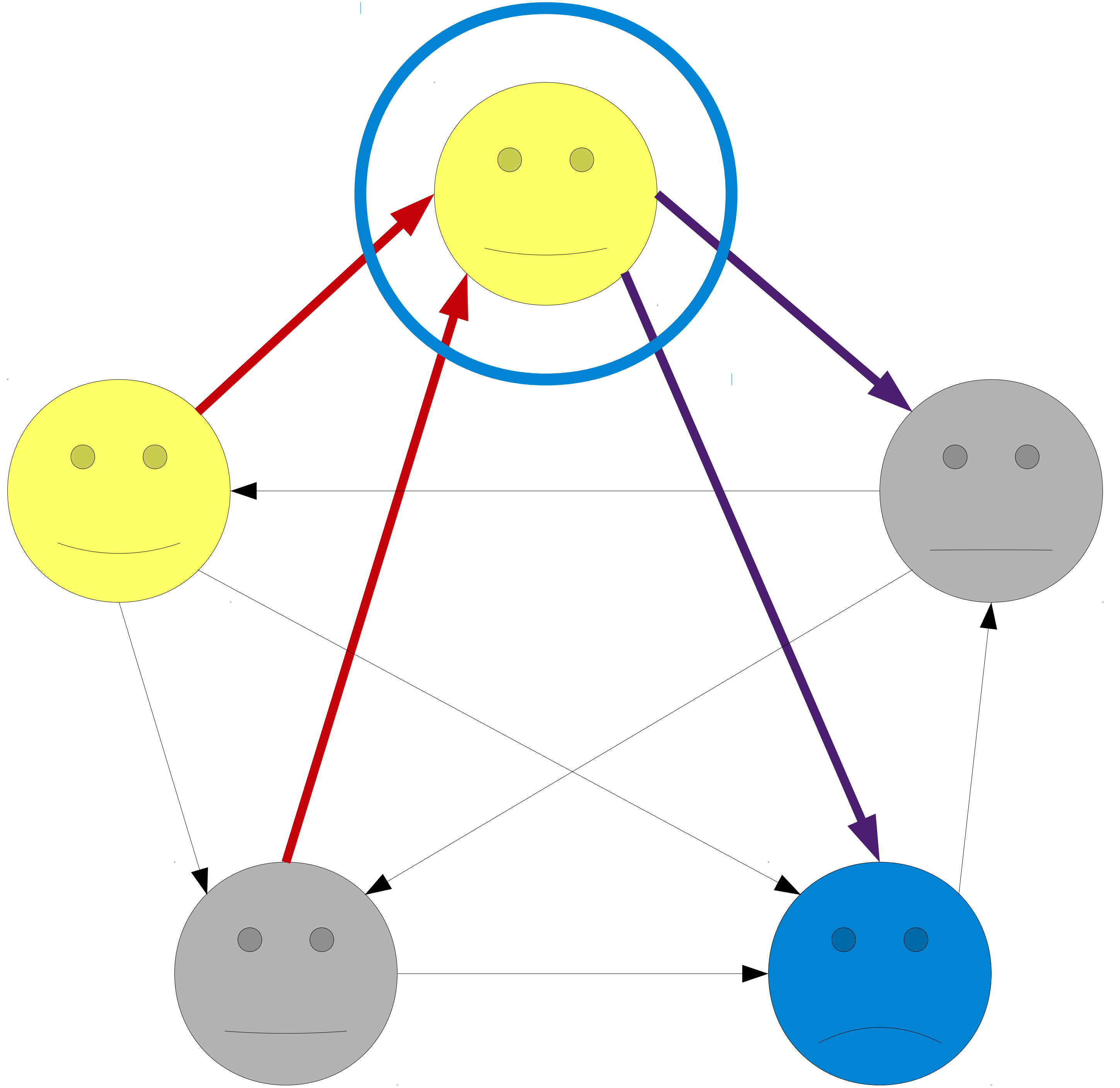
- The inductive step in an inductive proof uses the fact that the result is true for a smaller number (k) to prove that the result is true for a larger number ($k+1$).
- In most inductive proofs, the proof that the result is true for $k+1$ explicitly tries to simplify the $k+1$ case into the k case.
  - Counterfeit coins: Turn $k+1$ weighings into k weighings.
  - `MU` puzzle: Turn a sequence of $k+1$ events into a sequence of k events.
  - Triangulation: Turn a polygon of $k+1$ vertices into one with k vertices.
  - Square subdivision: Use a subdivision into k to get one for $k+3$.

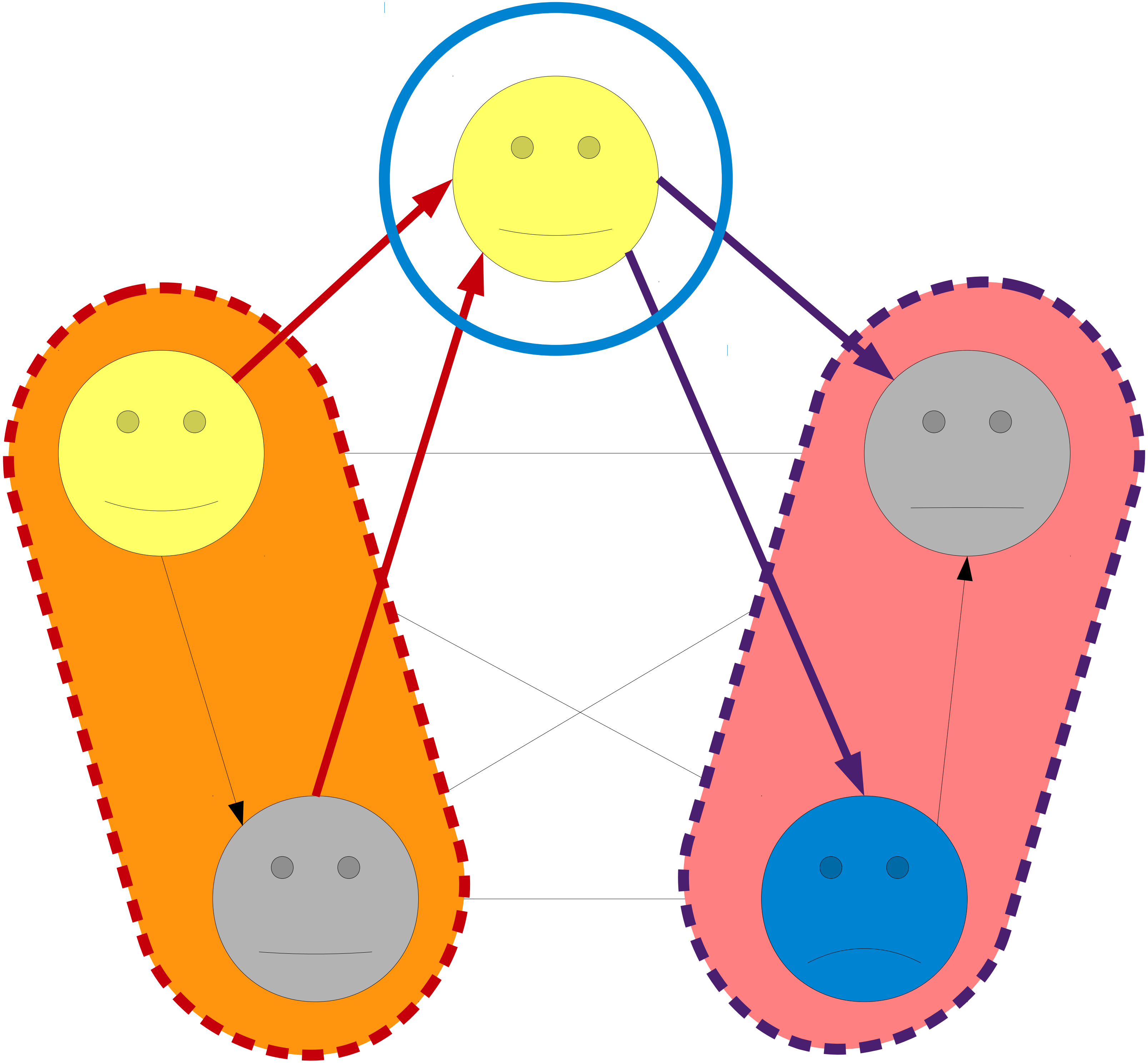
Thinking Inductively

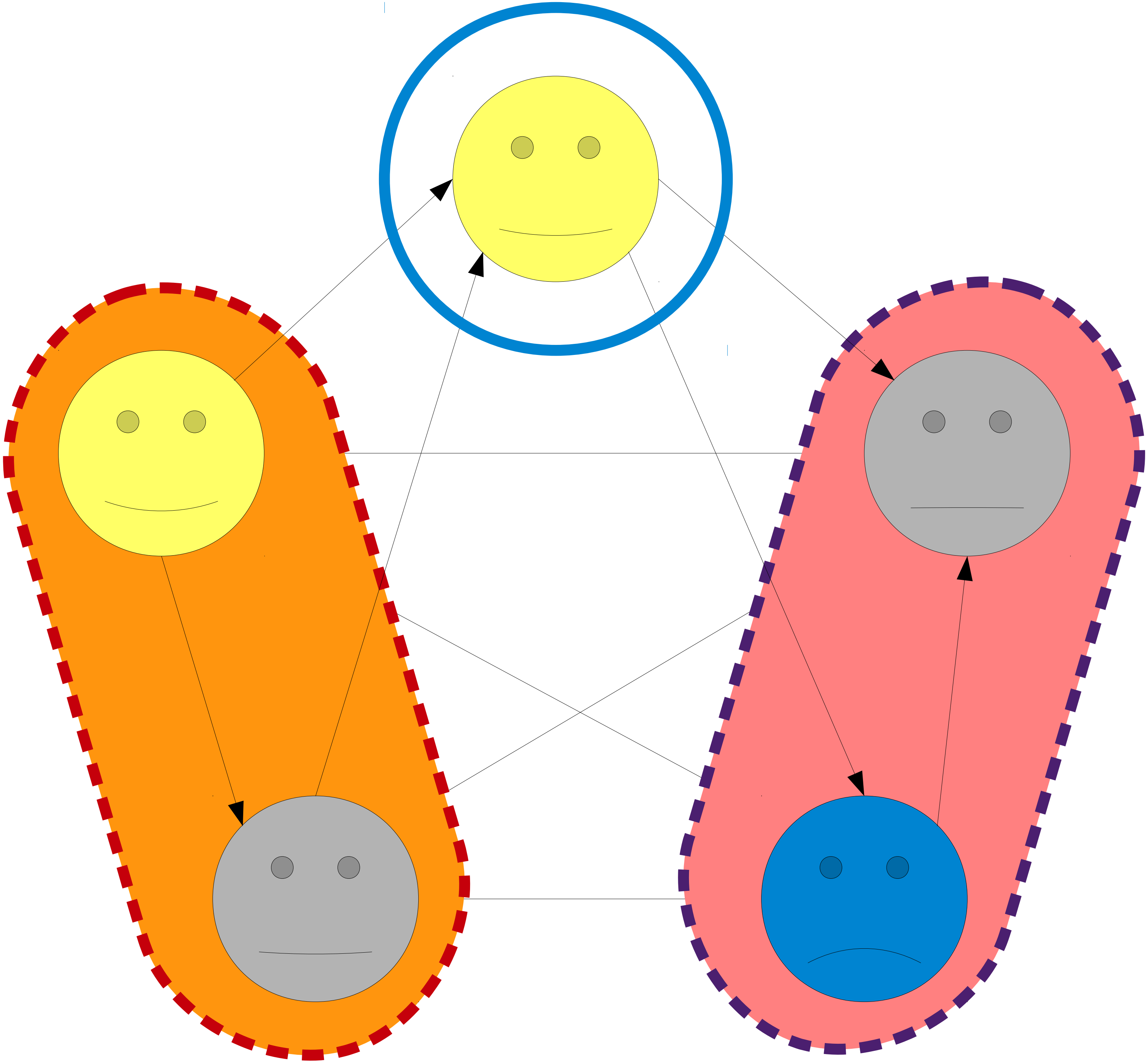
- For our victory chain proof, we will simplify the problem by turning the larger tournament into two smaller tournaments.
- We'll inductively argue that, since those smaller tournaments each have victory chains, the larger tournament must have a victory chain.

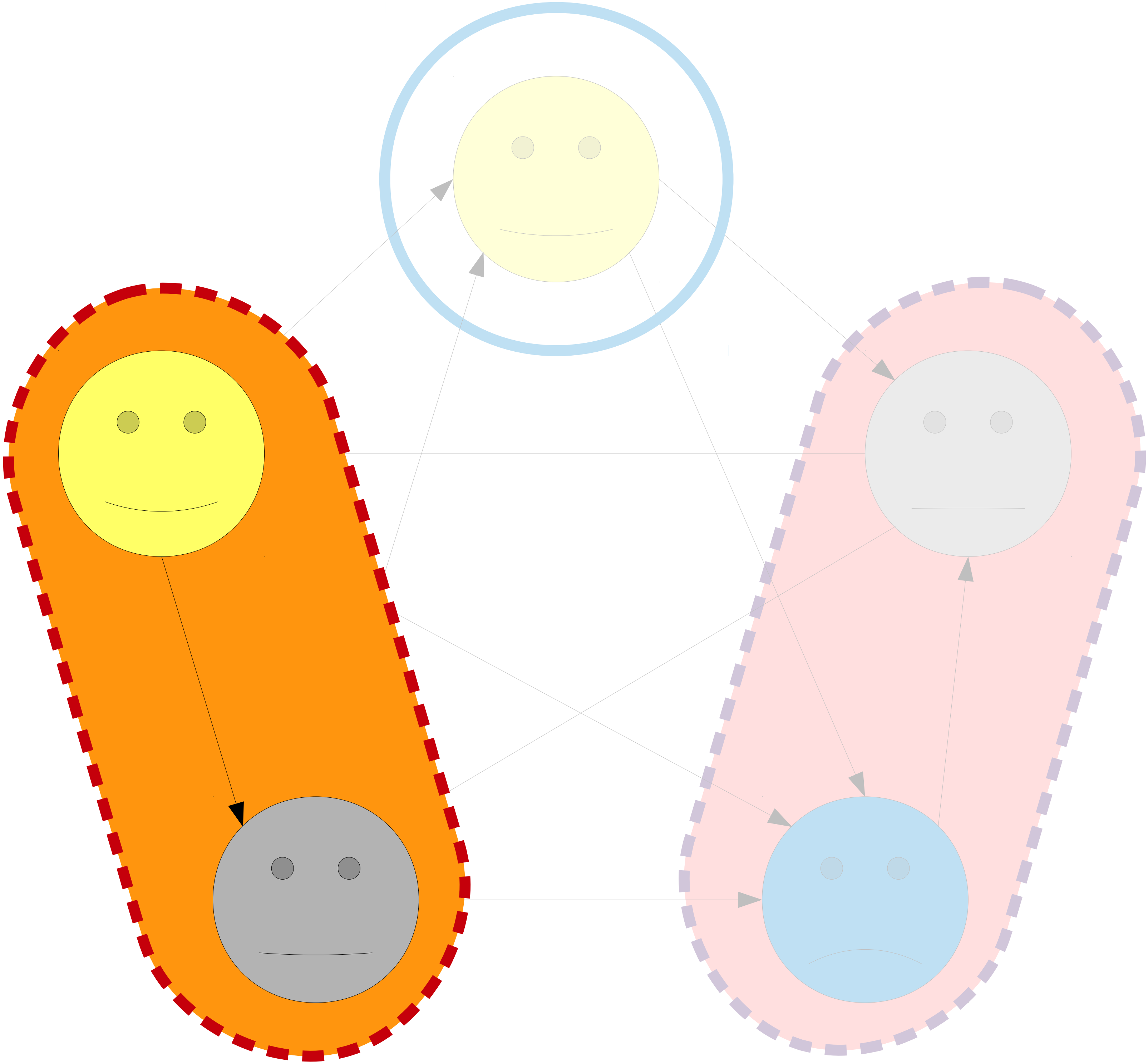


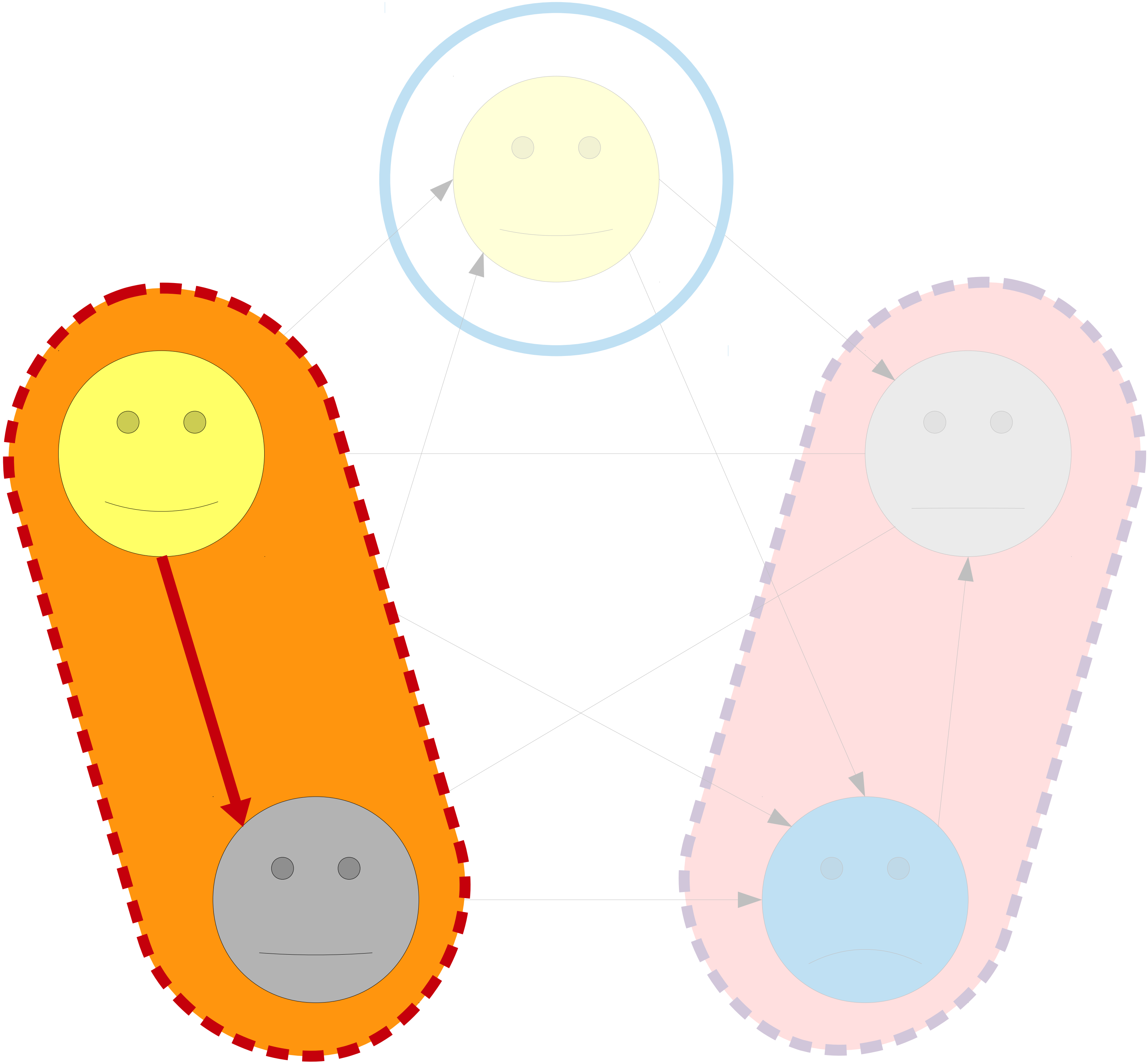


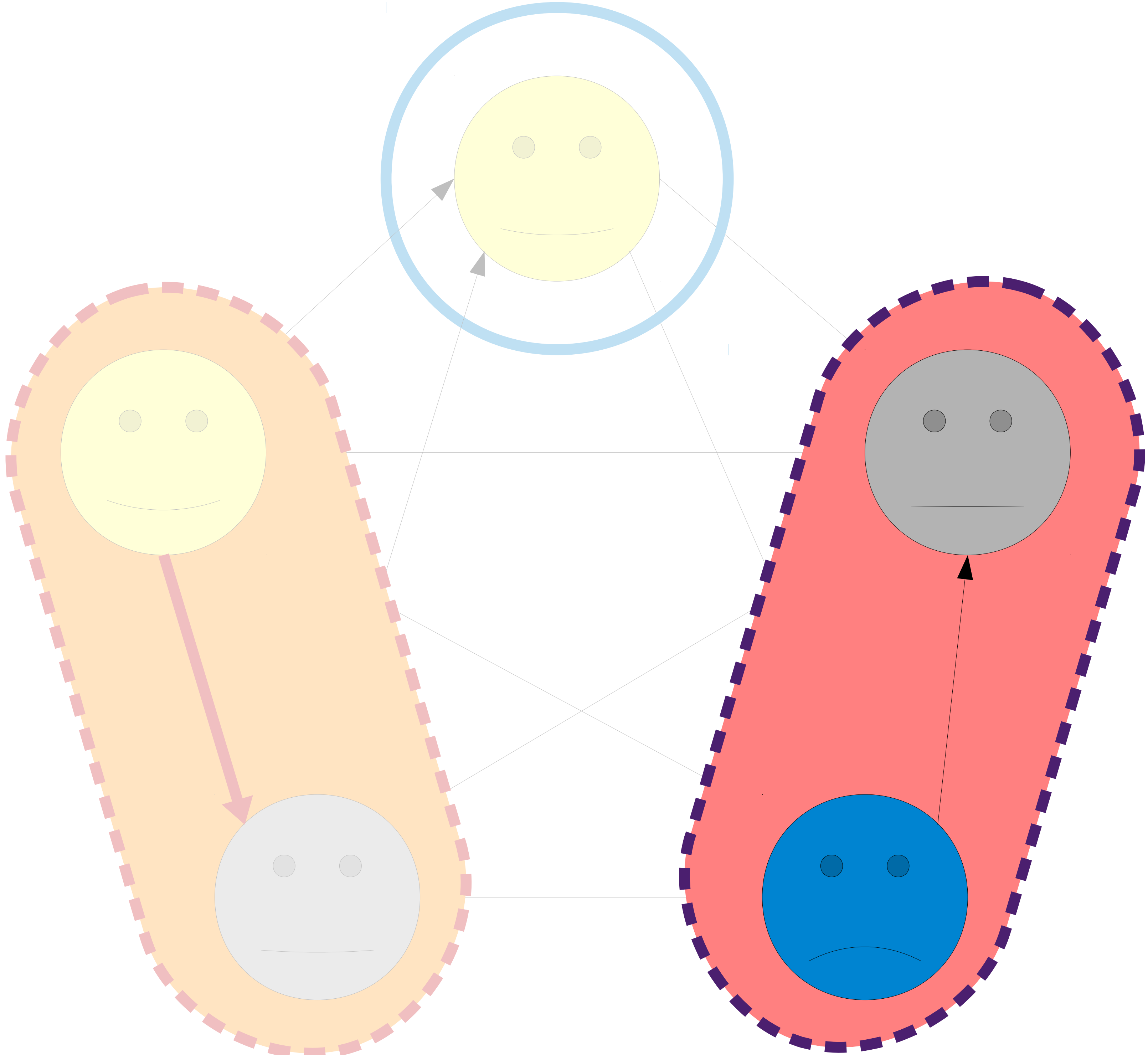


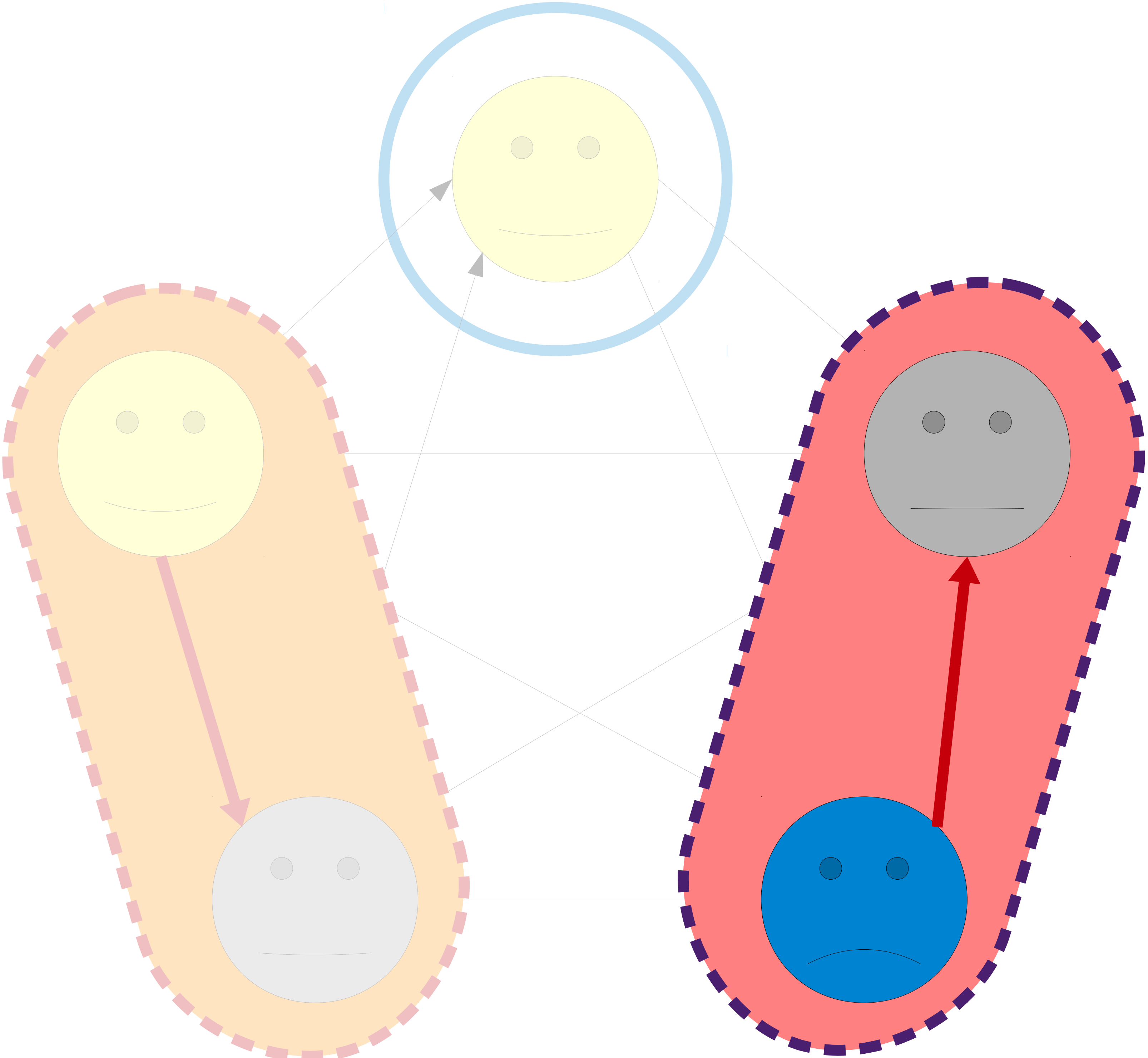


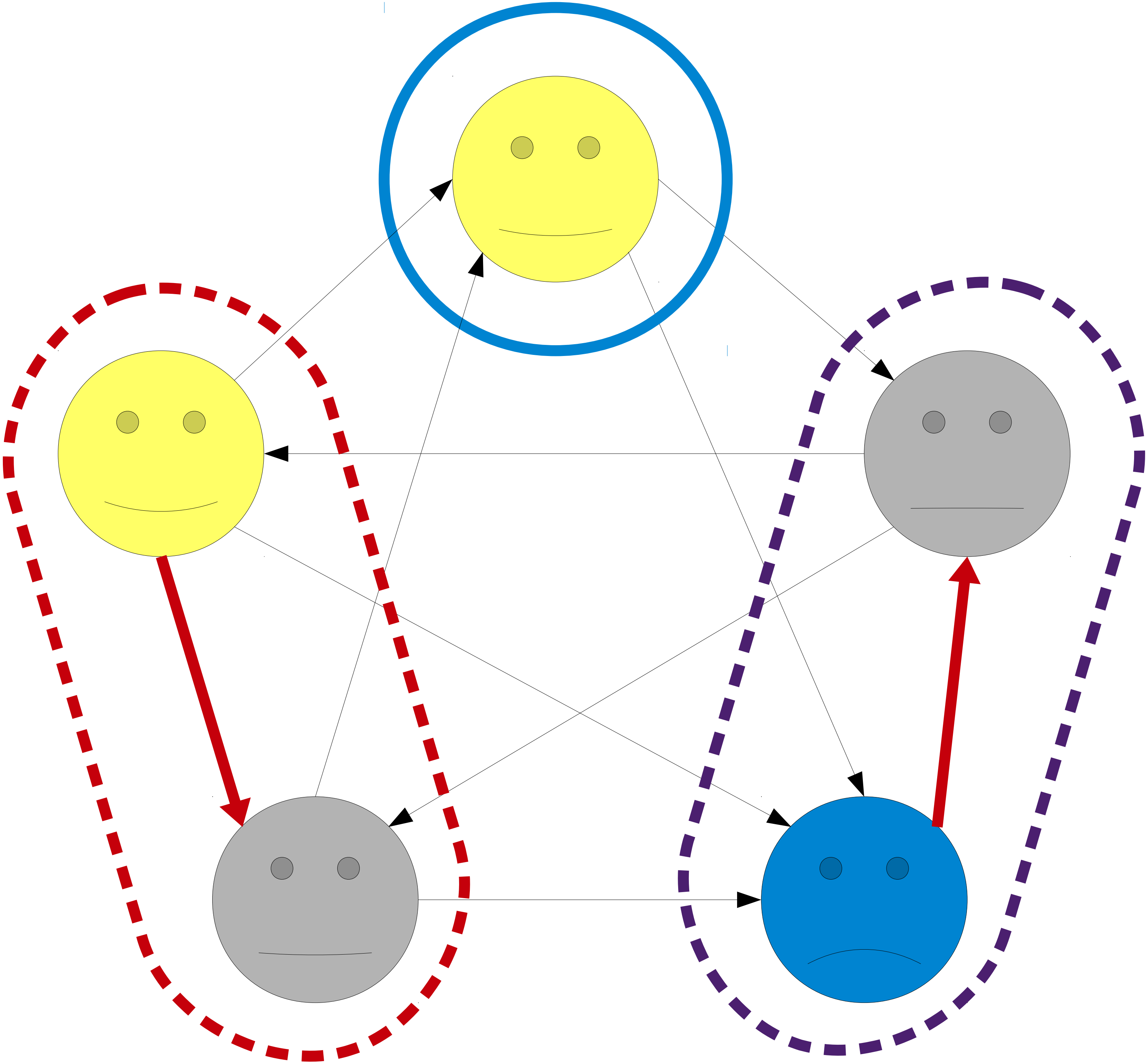


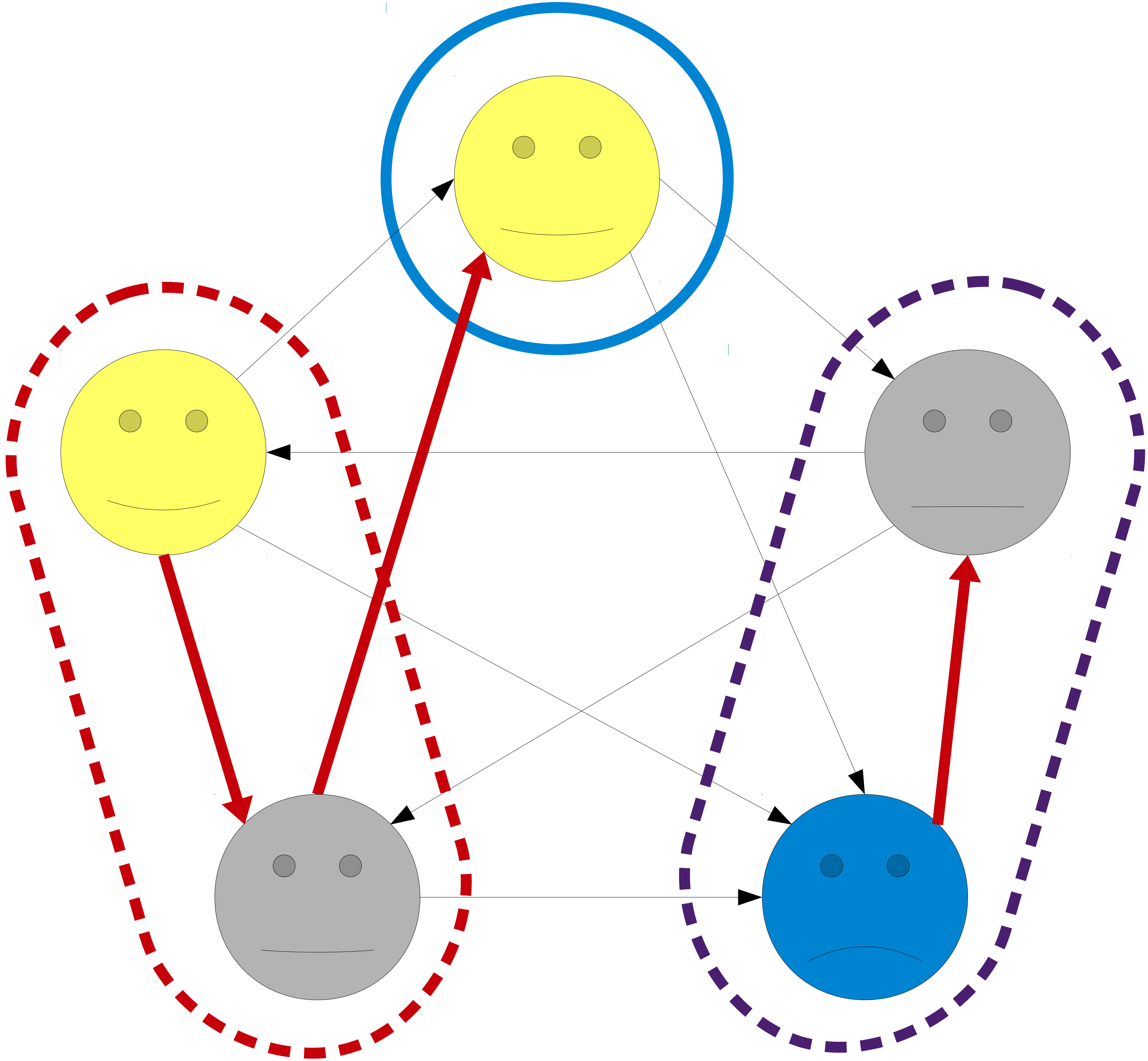


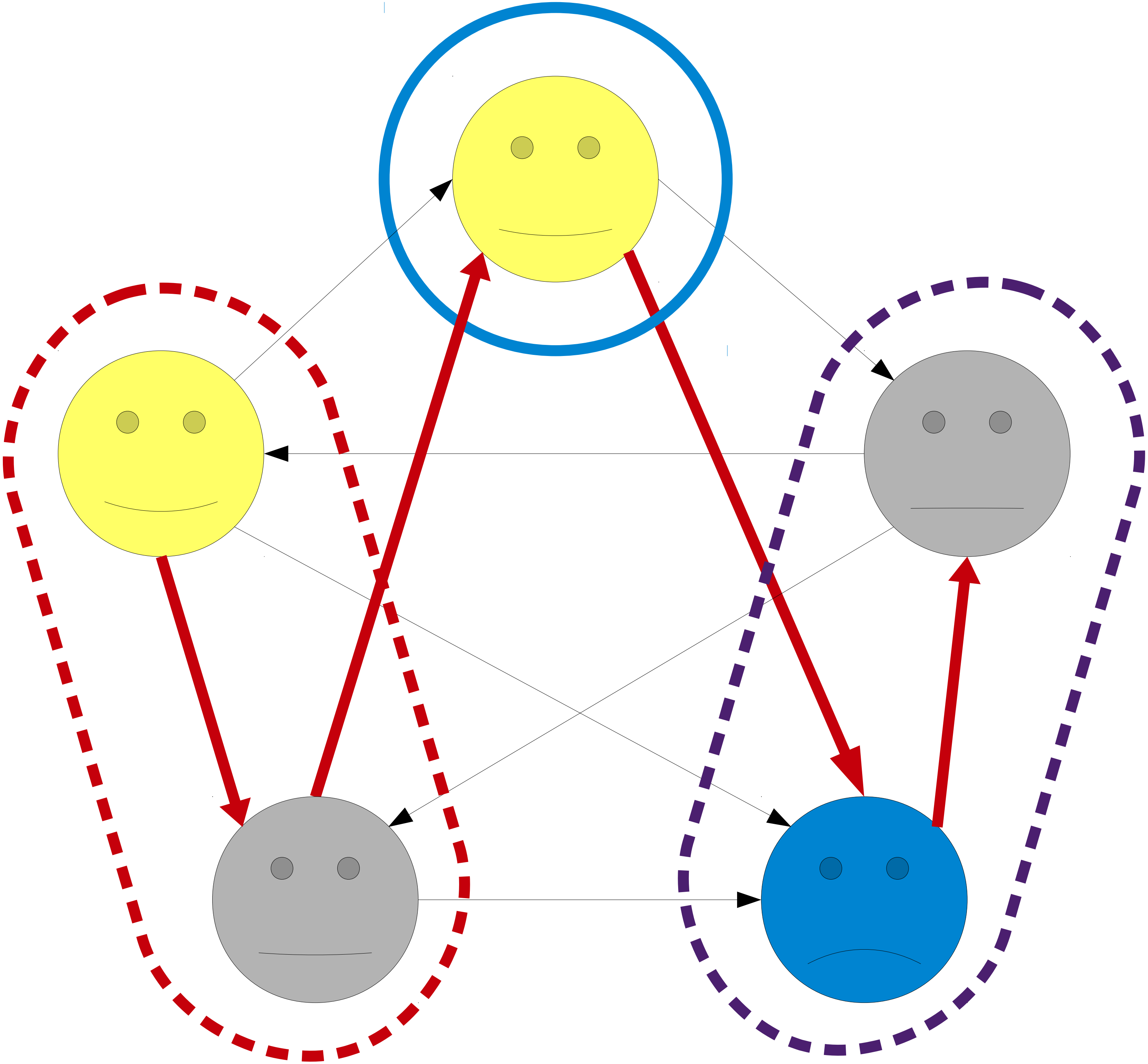


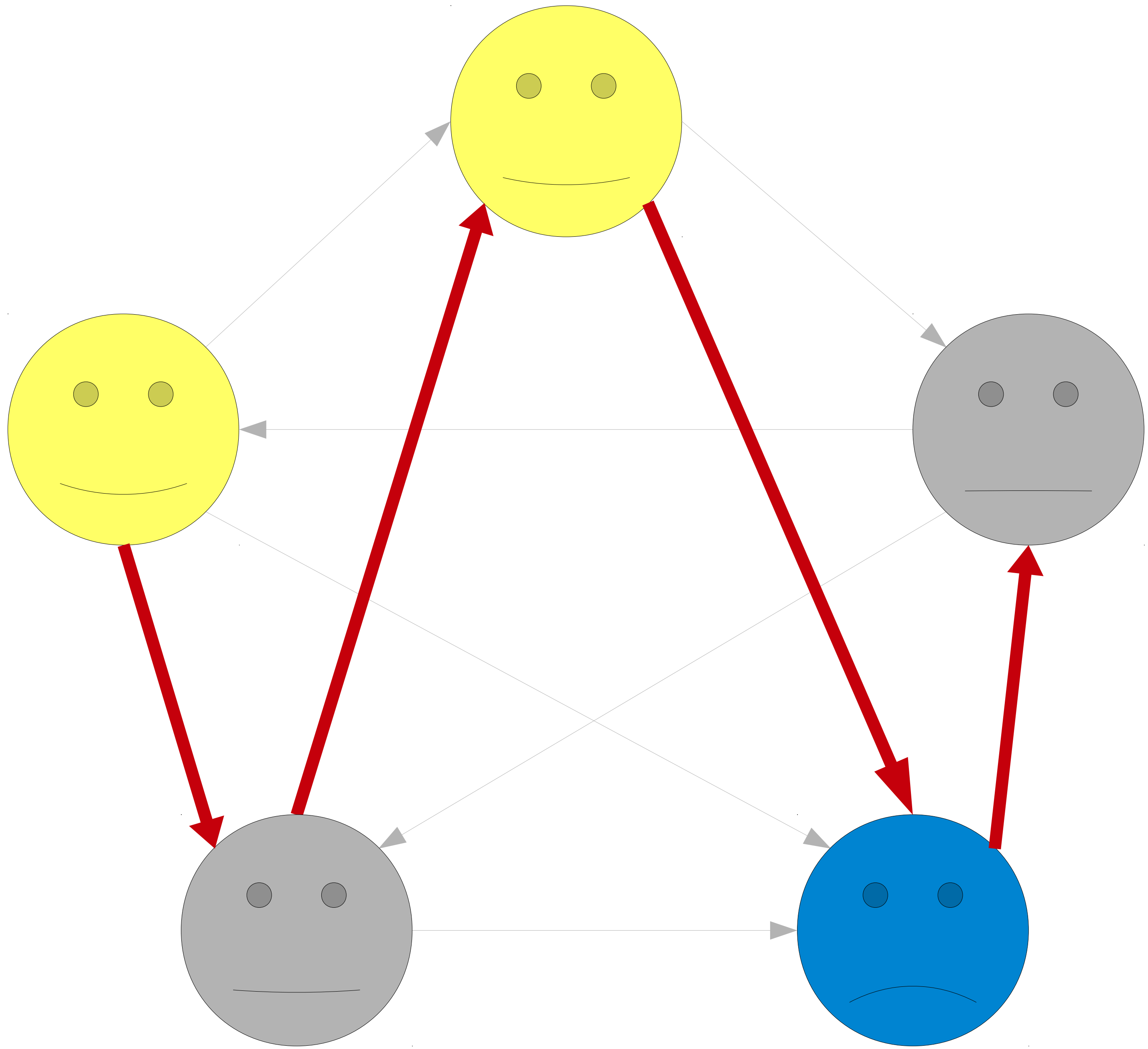












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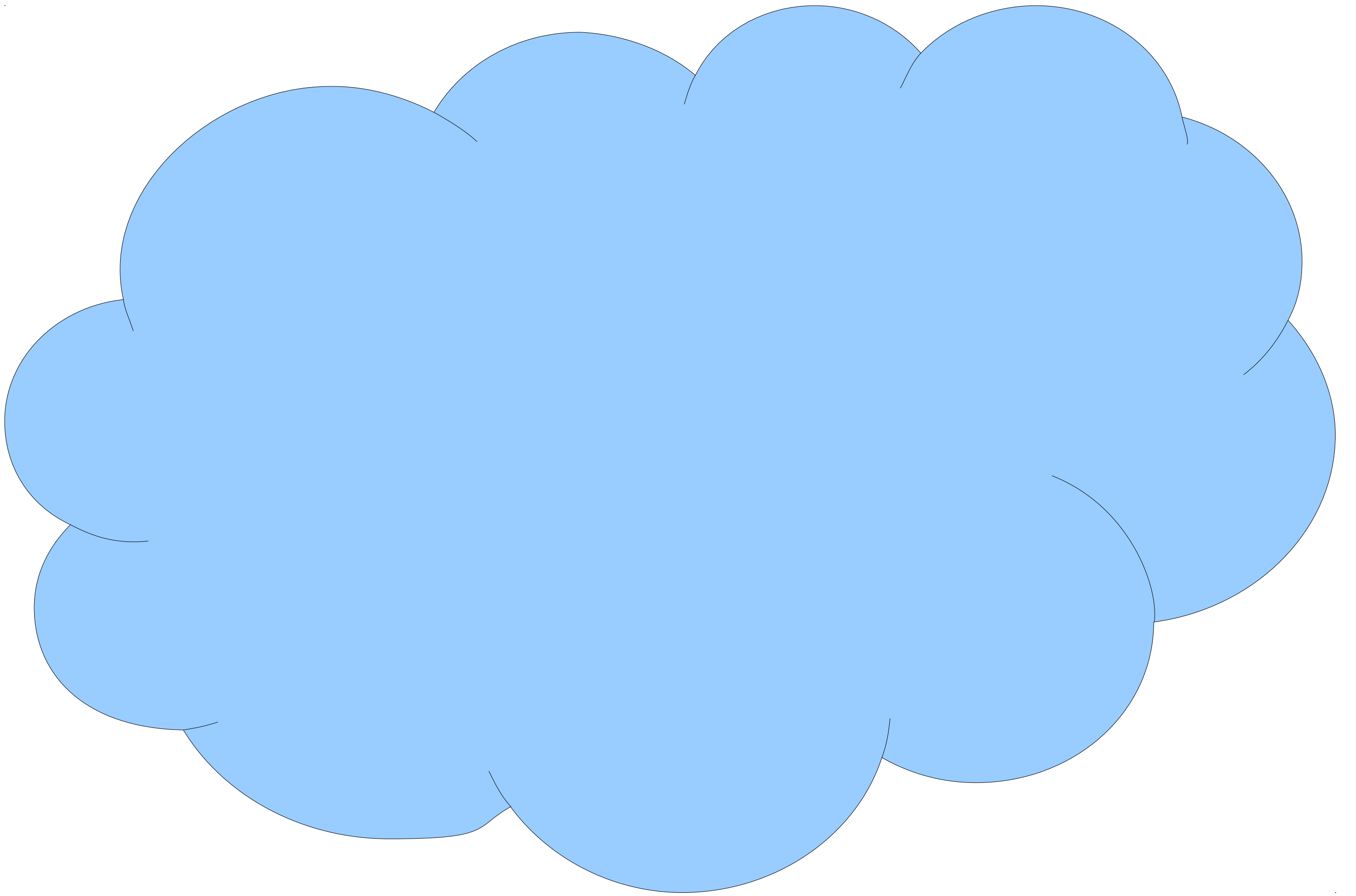

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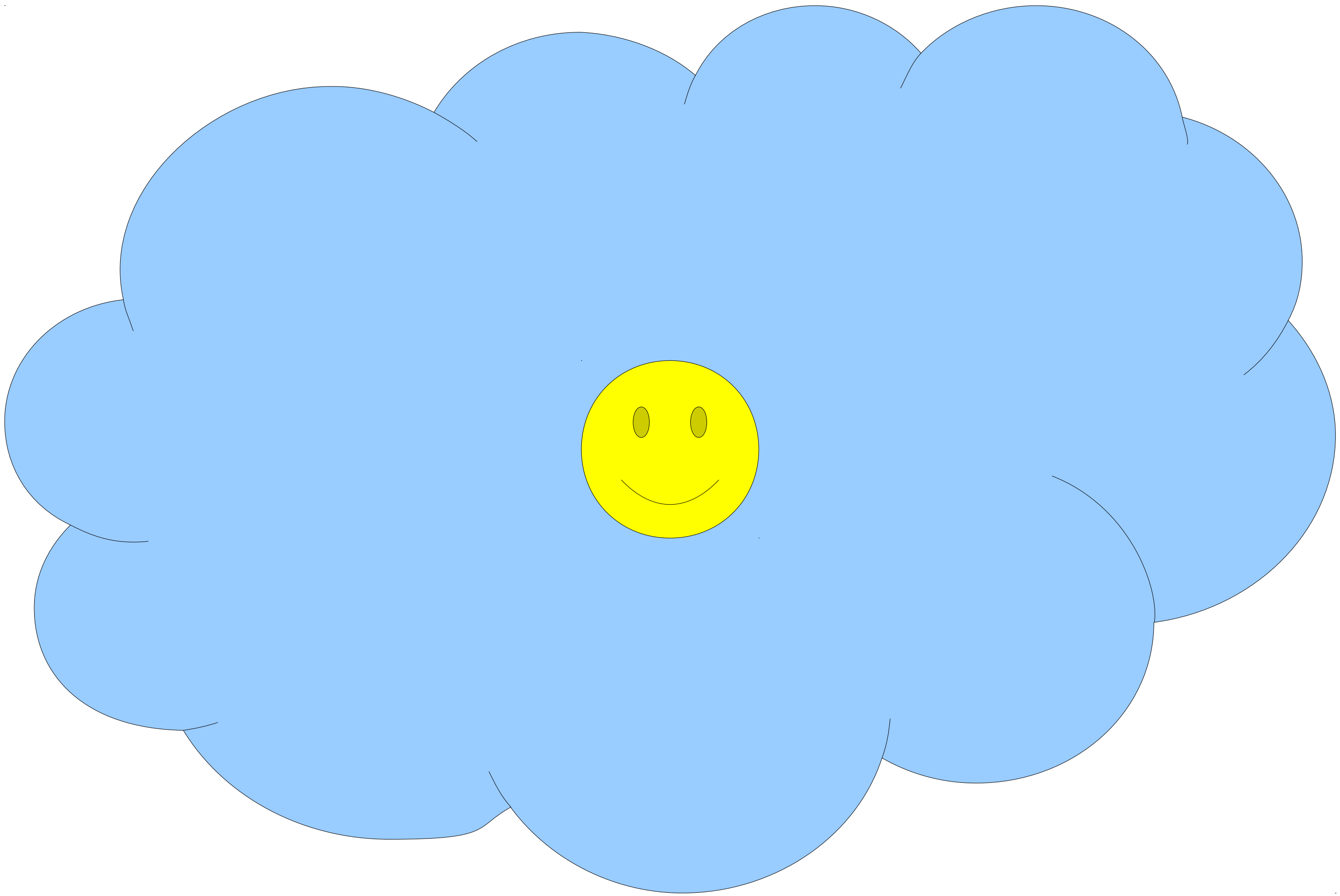

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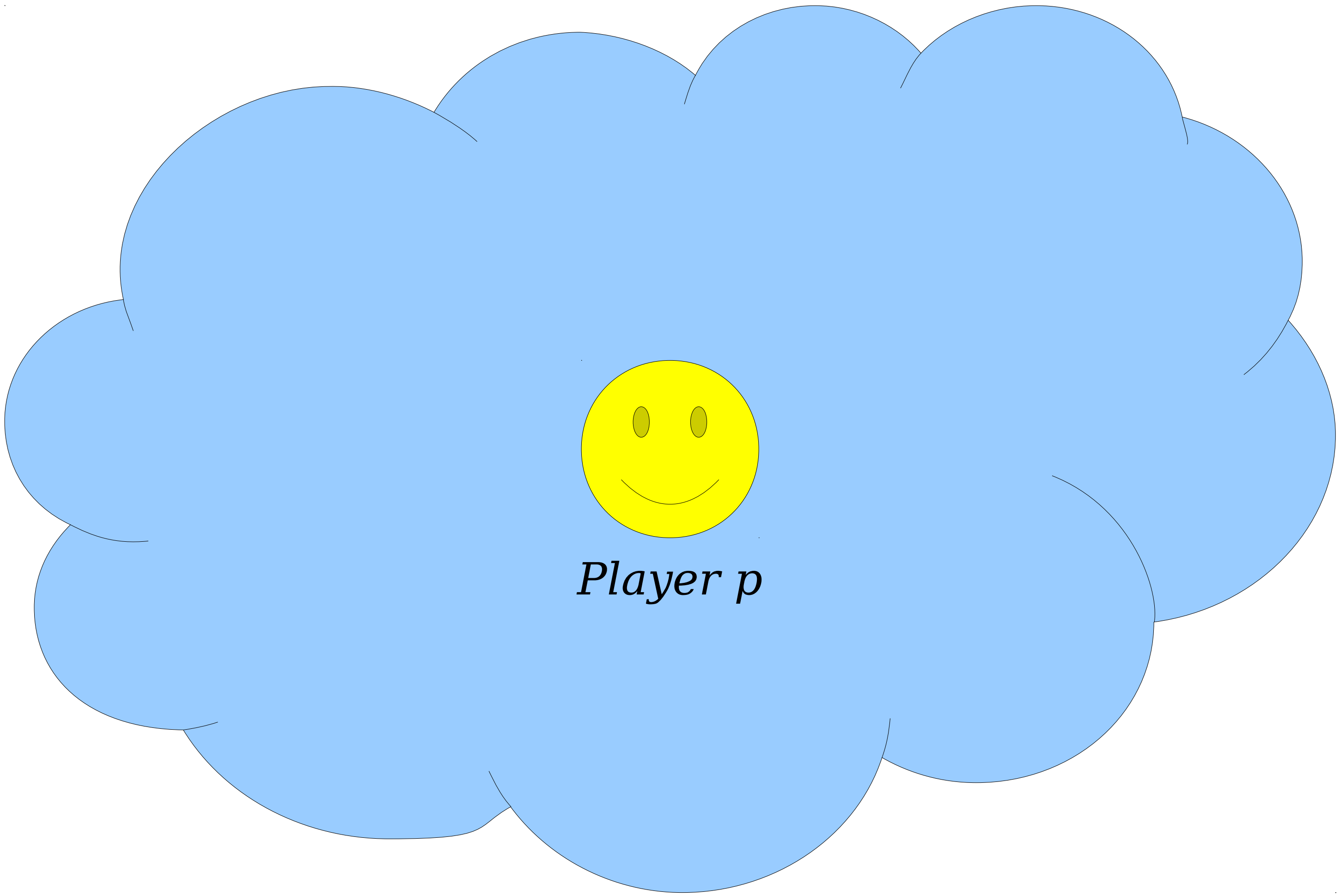

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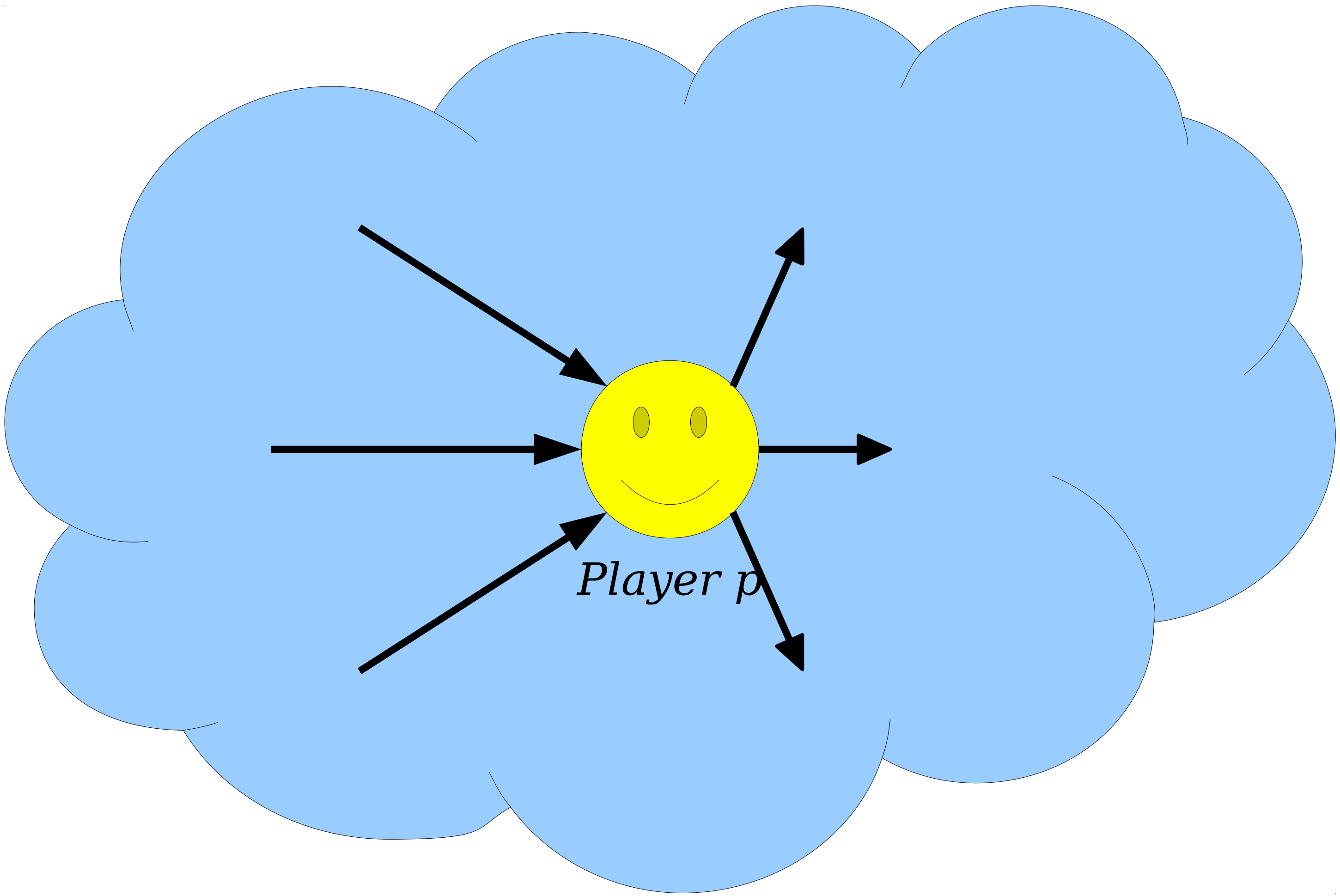

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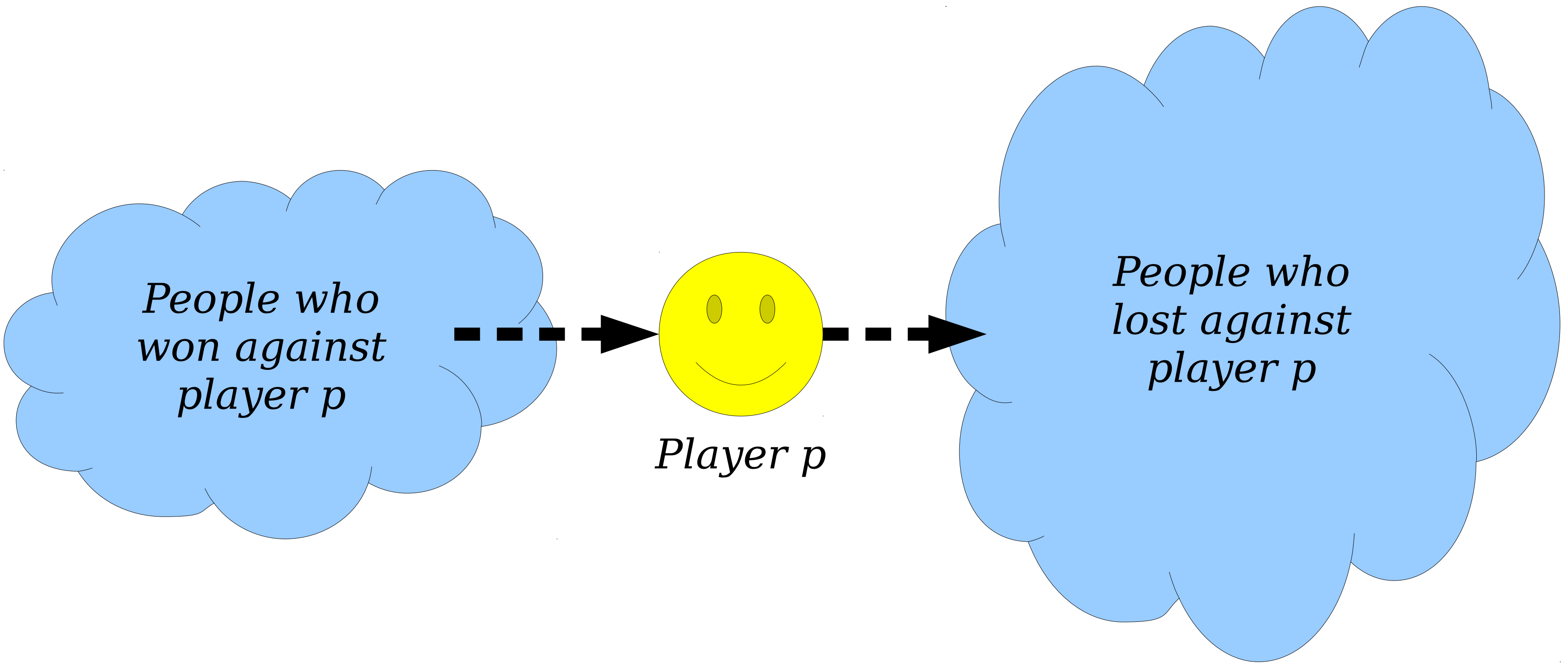

The Idea

- Suppose that every tournament with at most k players has a victory chain.
- Take a tournament T with $k+1$ players.
- Choose any one player p.
- Form the subtournaments T_0 and T_1 of all players who beat p and lost to p, respectively.
- Get victory chains from T_0 and T_1.
- Splice those chains together through p.

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This is the key idea behind an inductive proof – we're reducing the problem to smaller copies of itself.

Writing the Proof: A First Attempt

We're going to run into trouble in the middle of this proof. Don't worry – we'll see how to fix it.

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As a base ca… no players has a victory … t of all 0 of the players … er beat the next player i…

At this point, we're stuck. We know that tournaments with ***exactly*** *k* players must have a victory chain, but we're not assuming anything about tournaments with 0, 1, 2, …, $k-1$ players. Therefore, we can't necessarily say anything about these subtournaments.

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What if we made some additional assumptions so that we *can* say something about these smaller tournaments?

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We're now assuming that the result is true for 0, 1, 2, 3, …, k. Now, we can continue to make progress!

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Consider any tournament T with $k+1$ players. Choose any one player p and form two subtournaments, a subtournament T_0 of all the players who beat p and a subtournament T_1 of all the players who lost to p. Both of these tournaments have between 0 and k players. Therefore, by our inductive hypothesis, these two subtournaments have victory chains, call them V_0 and V_1.

Theorem: Every tournament has a victory chain.

Proof: Let $P(n)$ be the statement "every tournament with n players has a victory chain." We will prove by induction that $P(n)$ is true for all $n \in \mathbb{N}$, from which the theorem follows.

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What We Just Did

- In a normal inductive step, we assume that $P(k)$ is true and prove $P(k+1)$.
- In this type of inductive step, we assume $P(0)$, $P(1)$, …, and $P(k)$ are true before we prove $P(k+1)$.
- That way, when we found *any* kind of smaller tournament, we knew something about its structure.
- This type of proof has a name!

Complete Induction

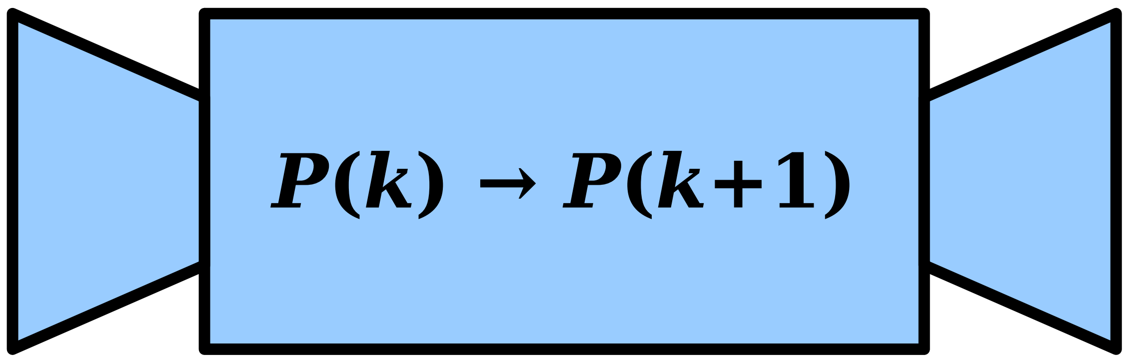
- If the following are true:
  - $P(0)$ is true, and
  - If $P(0)$, $P(1)$, $P(2)$, …, $P(k)$ are true, then $P(k+1)$ is true as well.

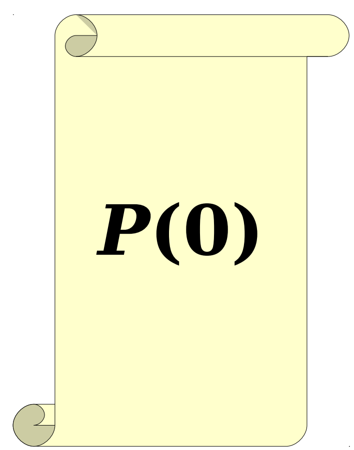
  then $P(n)$ is true for all $n \in \mathbb{N}$.
- This is called the ***principle of complete induction*** or the ***principle of strong induction***.
  - (A note: this also works starting from a number other than 0; just modify what you're assuming appropriately.)

That's a *lot* of assumptions to make!

Why is this legal?

Review: Induction as a Machine





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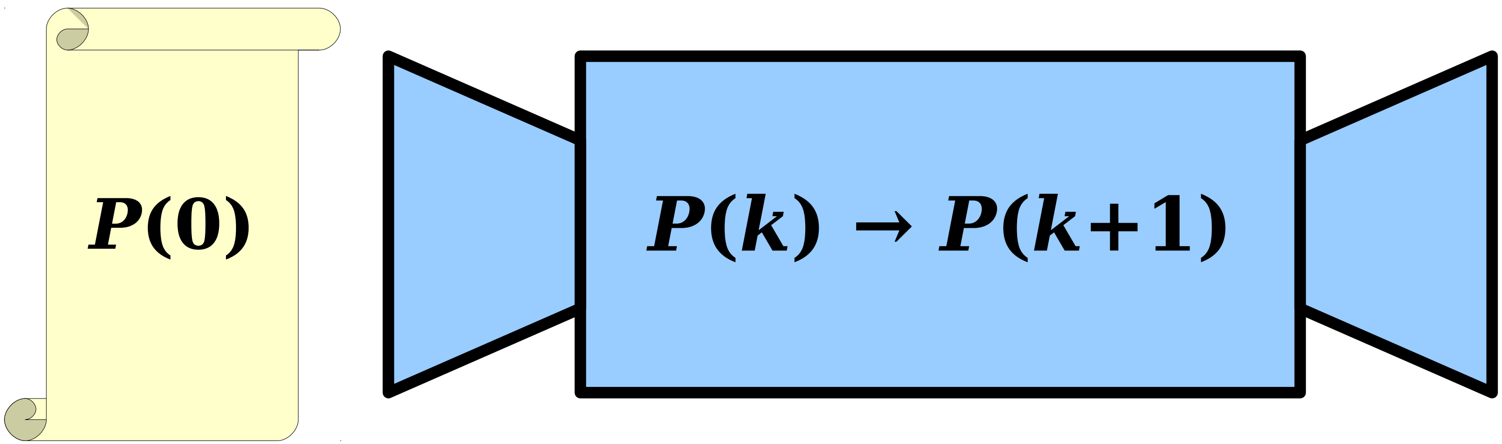

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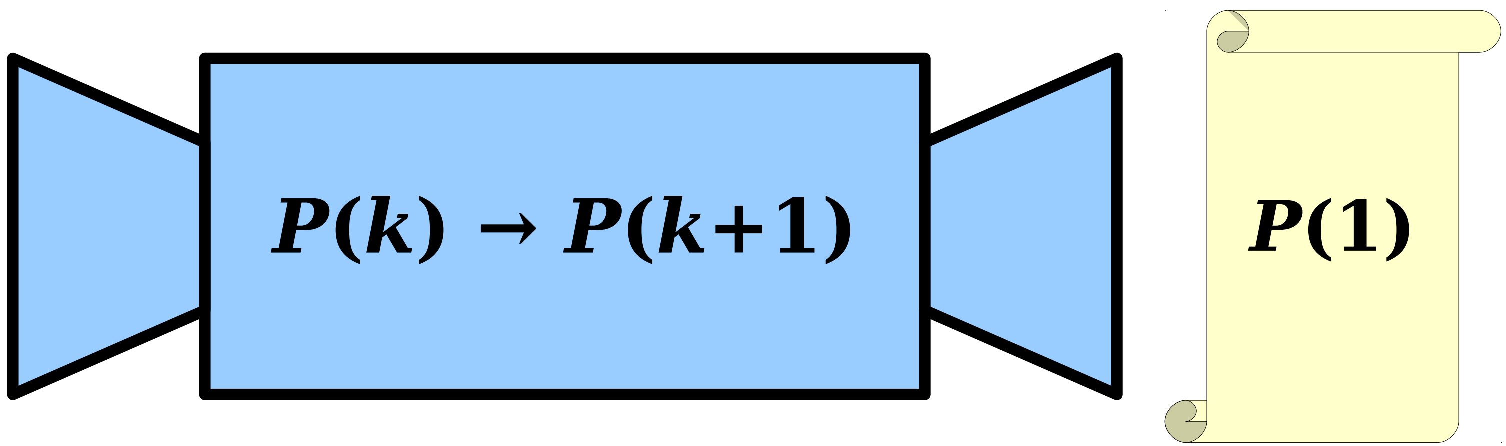

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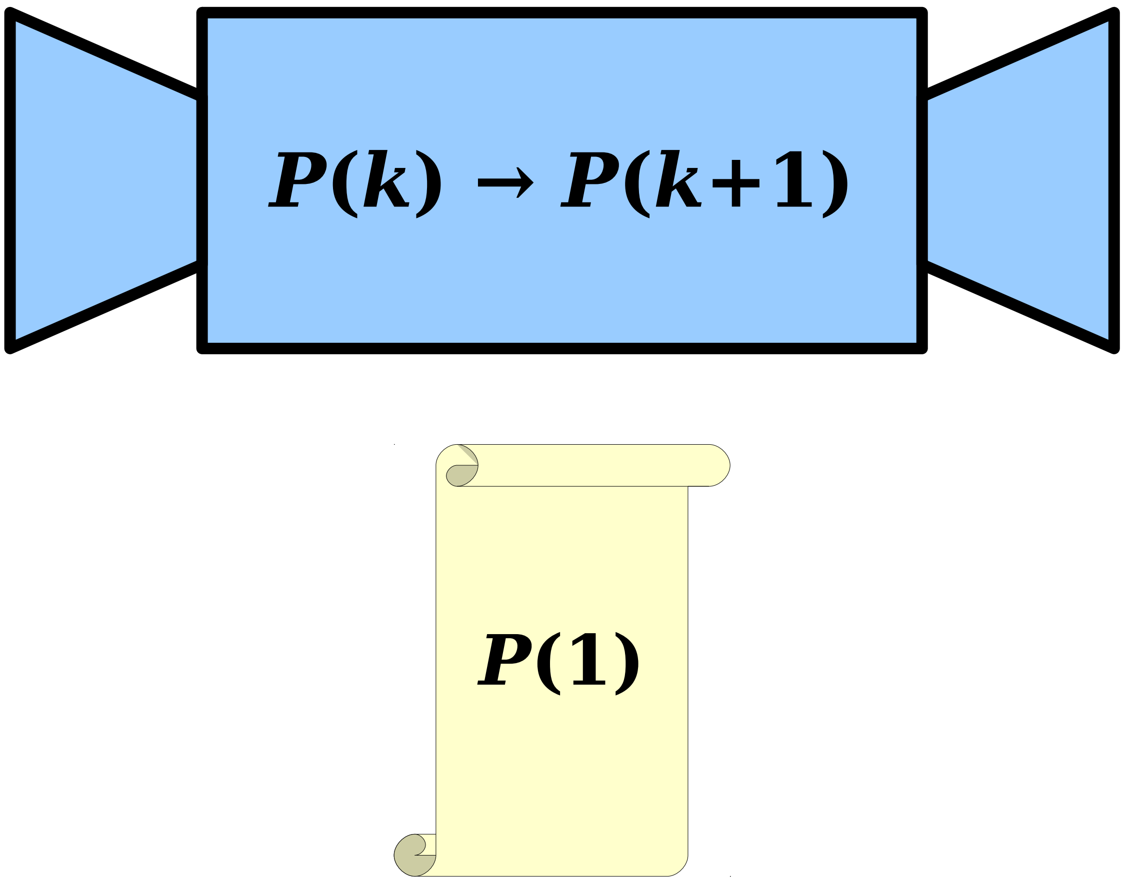

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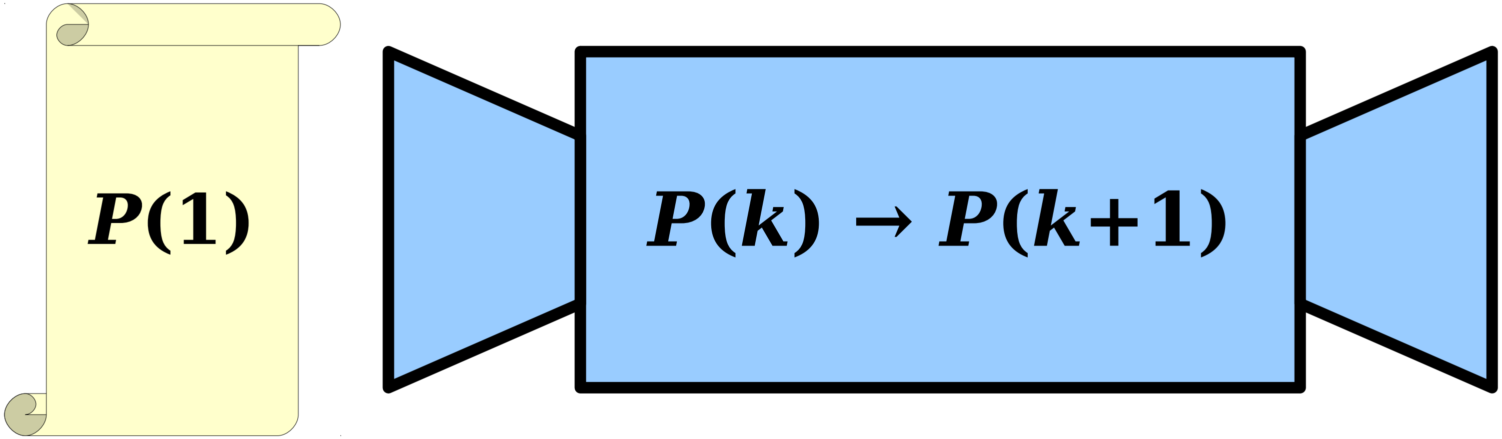

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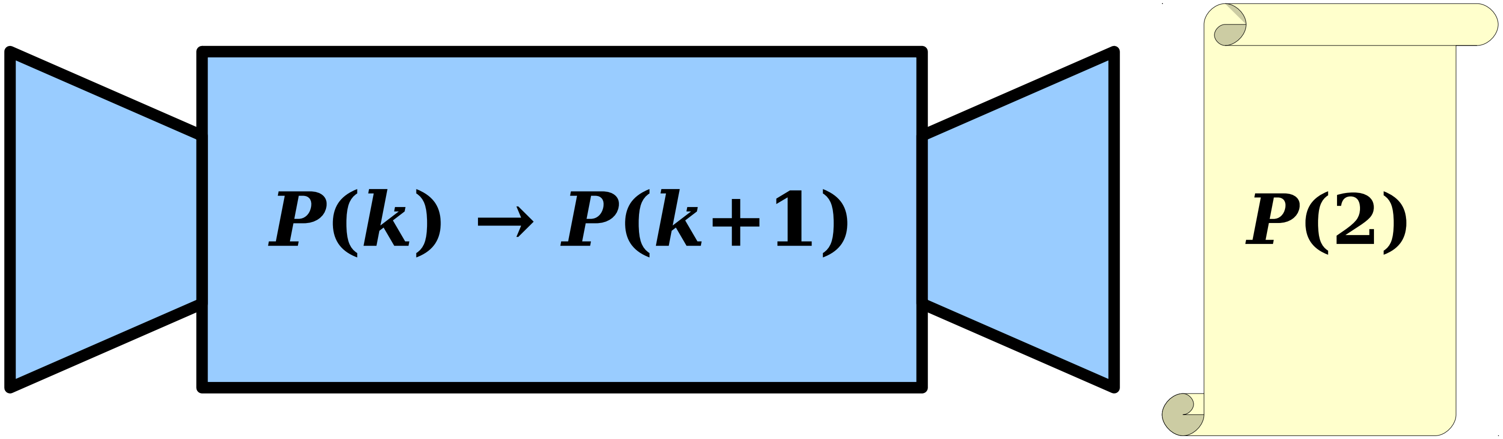

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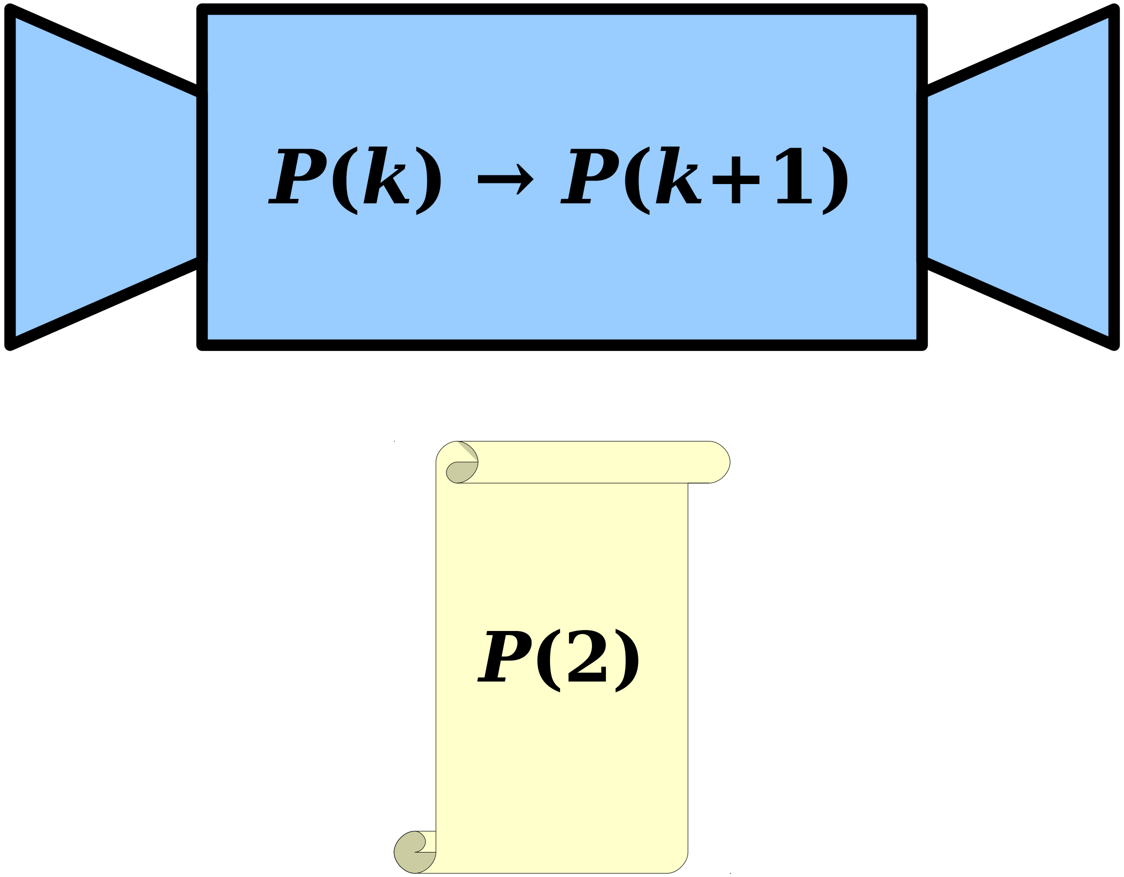

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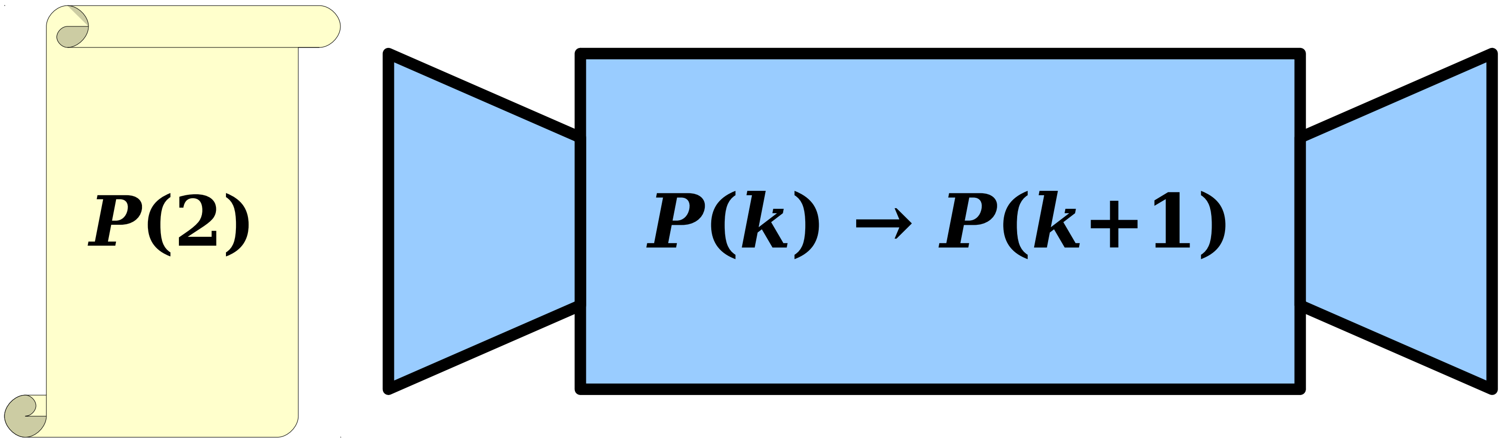

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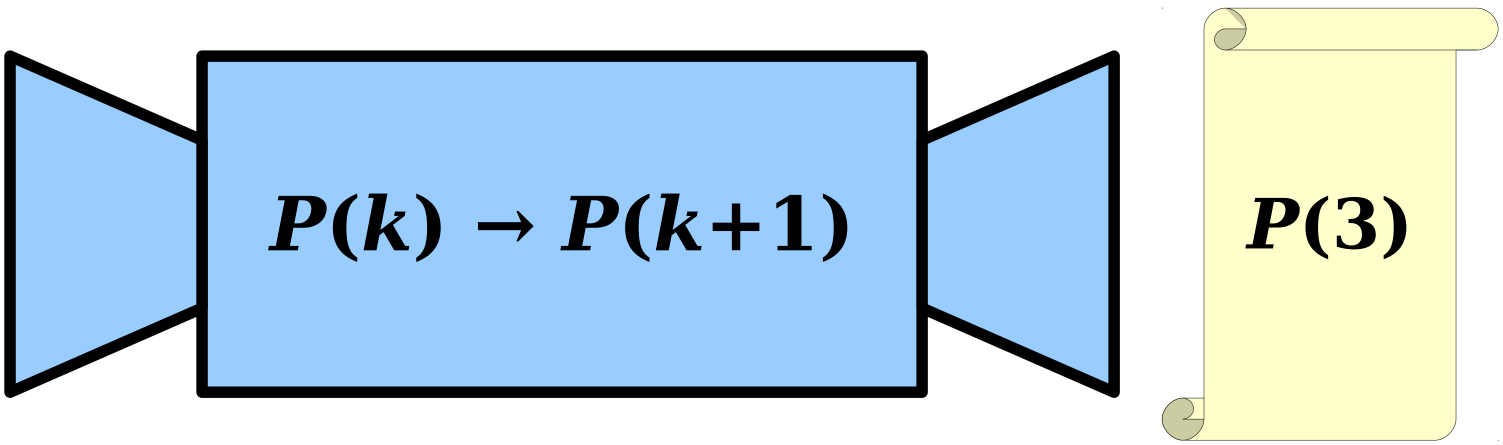

An Observation

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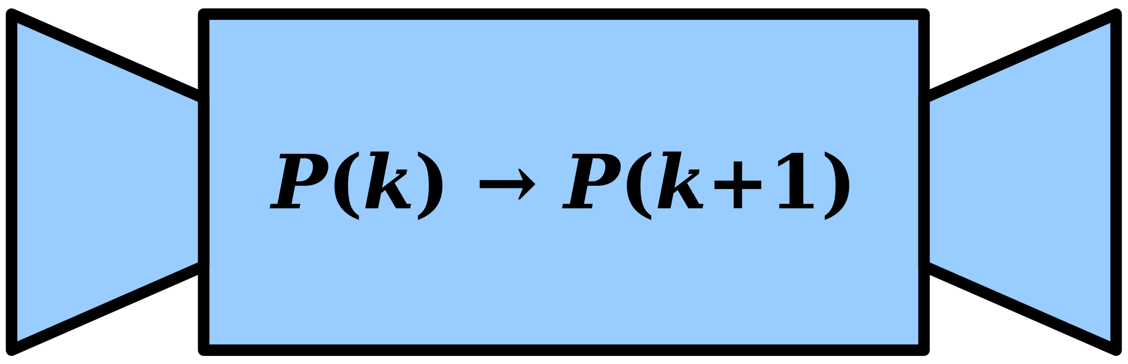

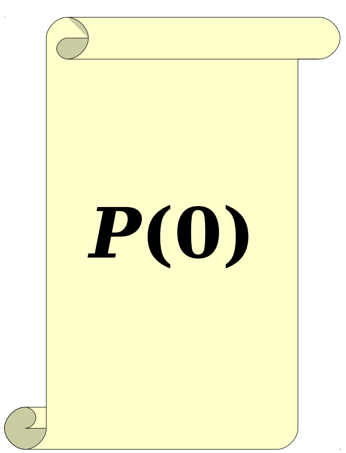

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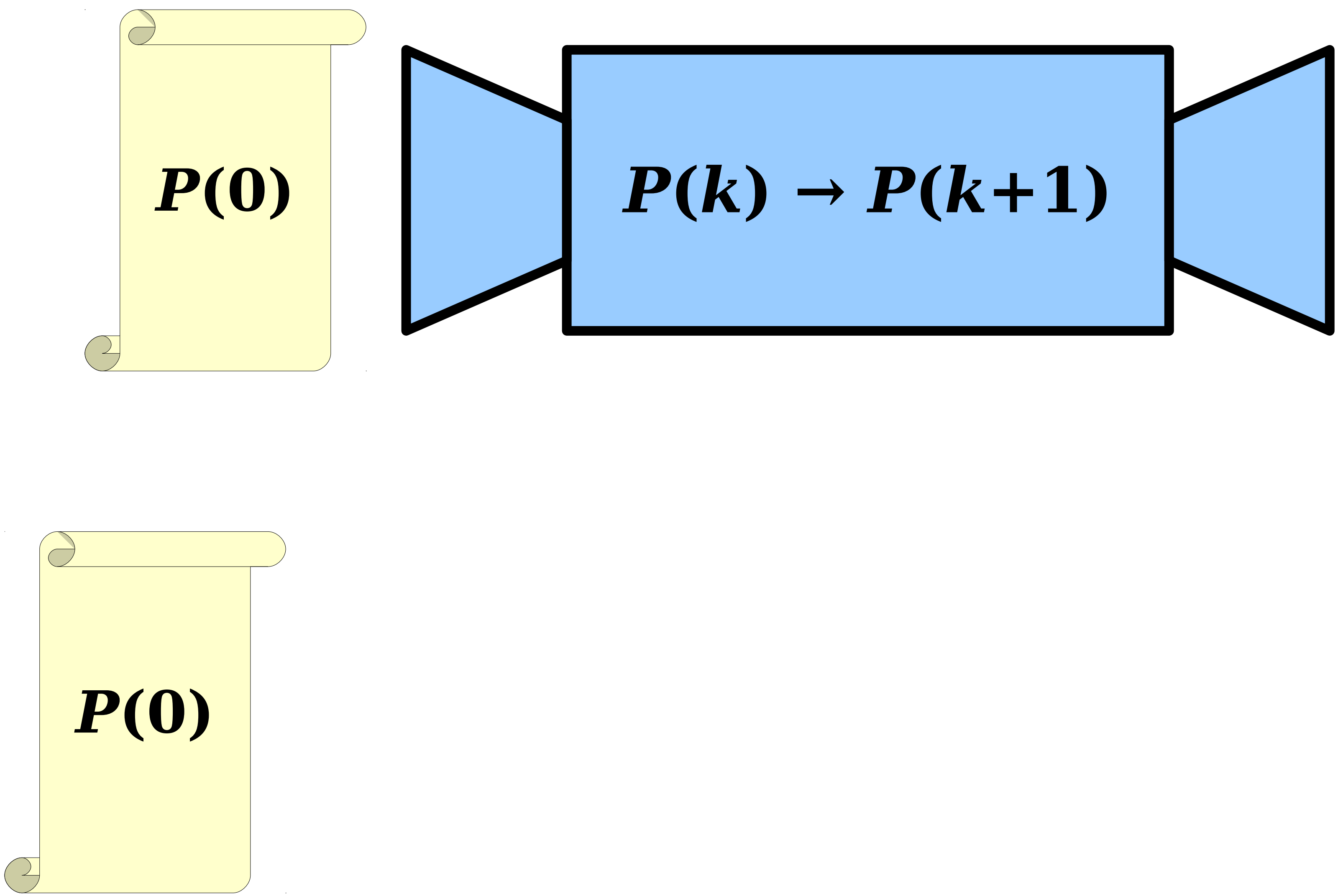

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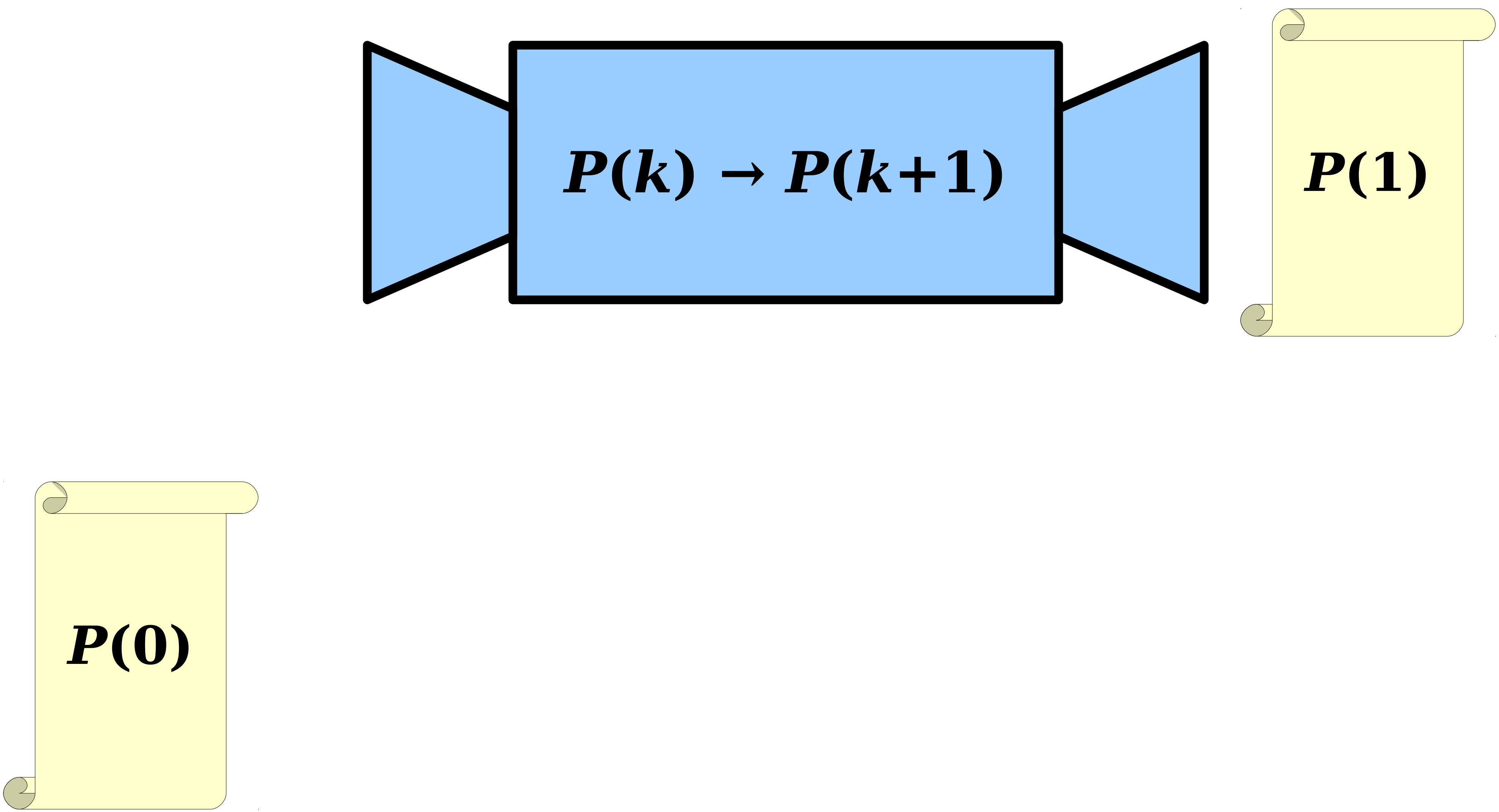

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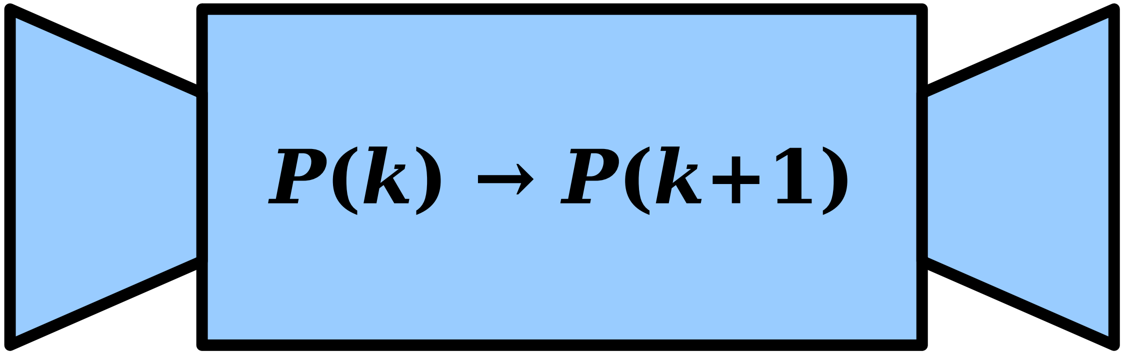

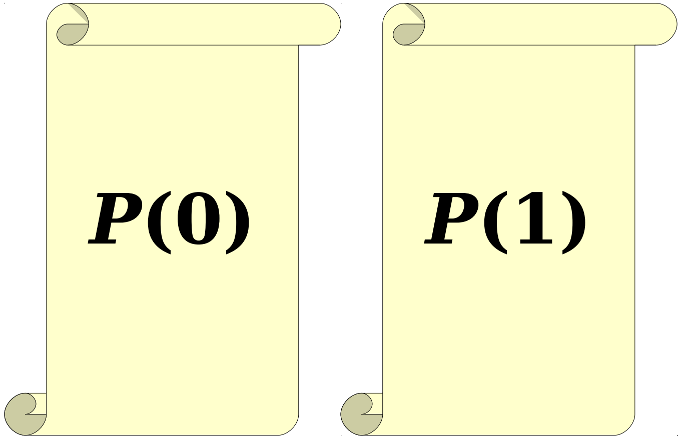

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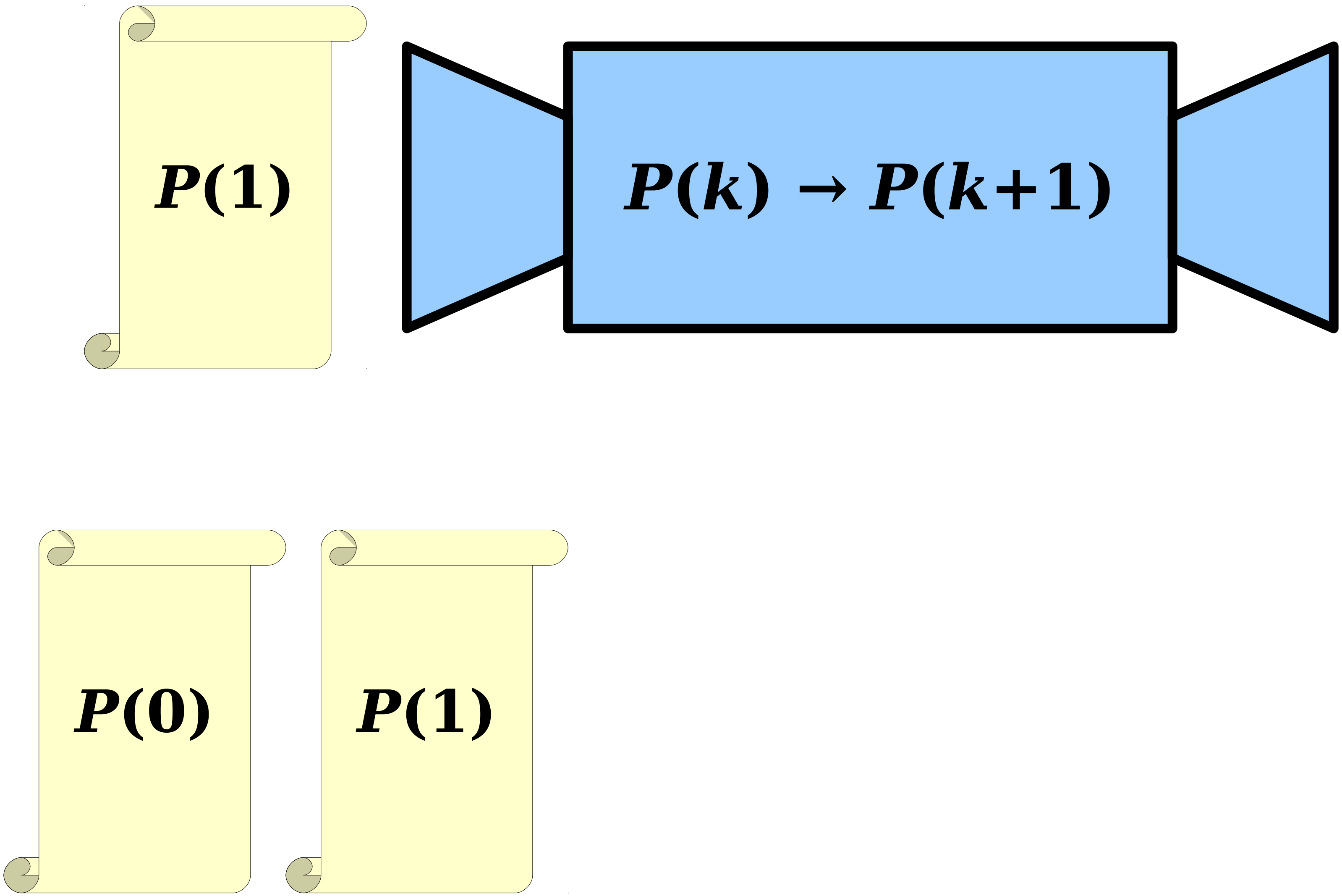

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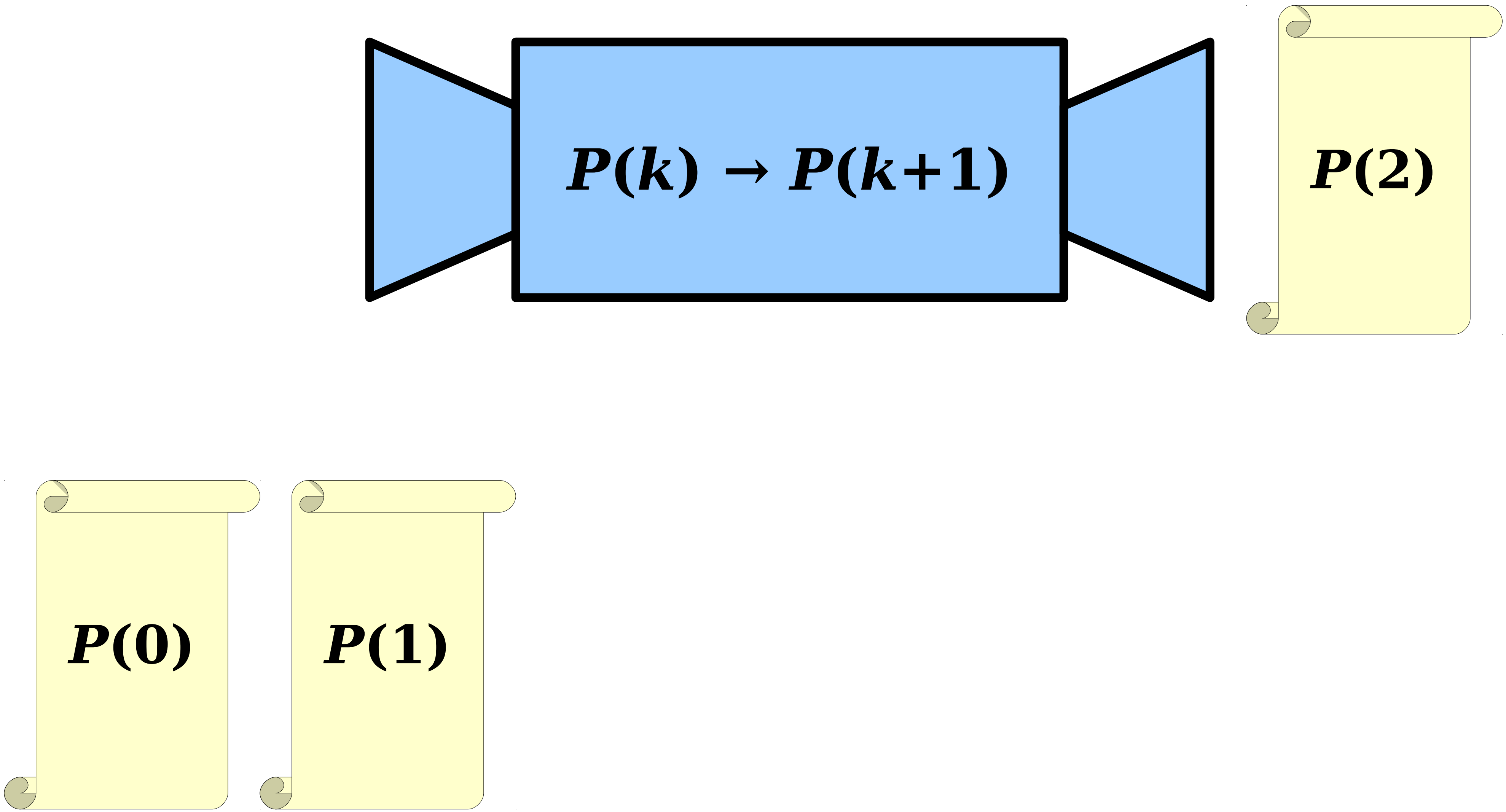

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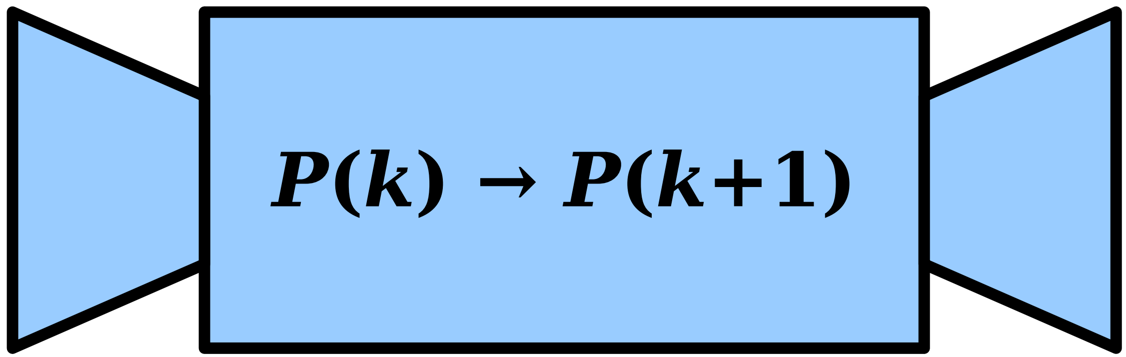

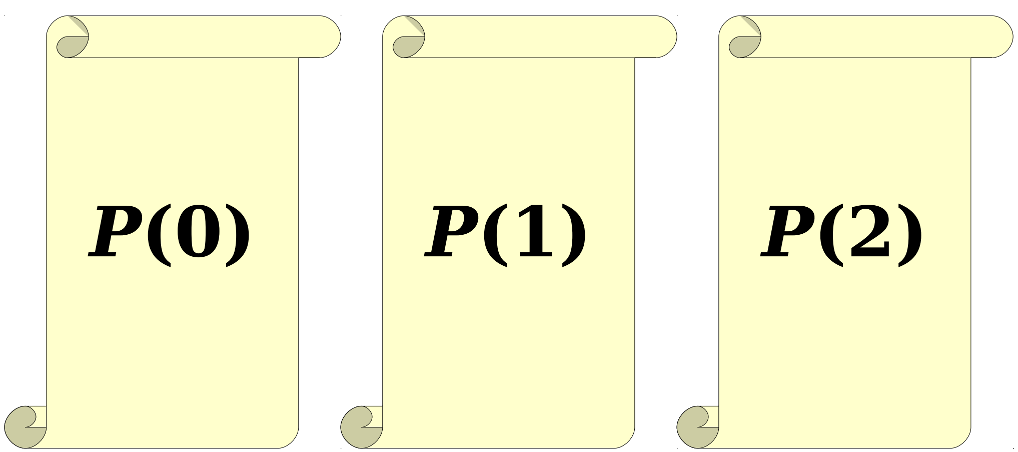

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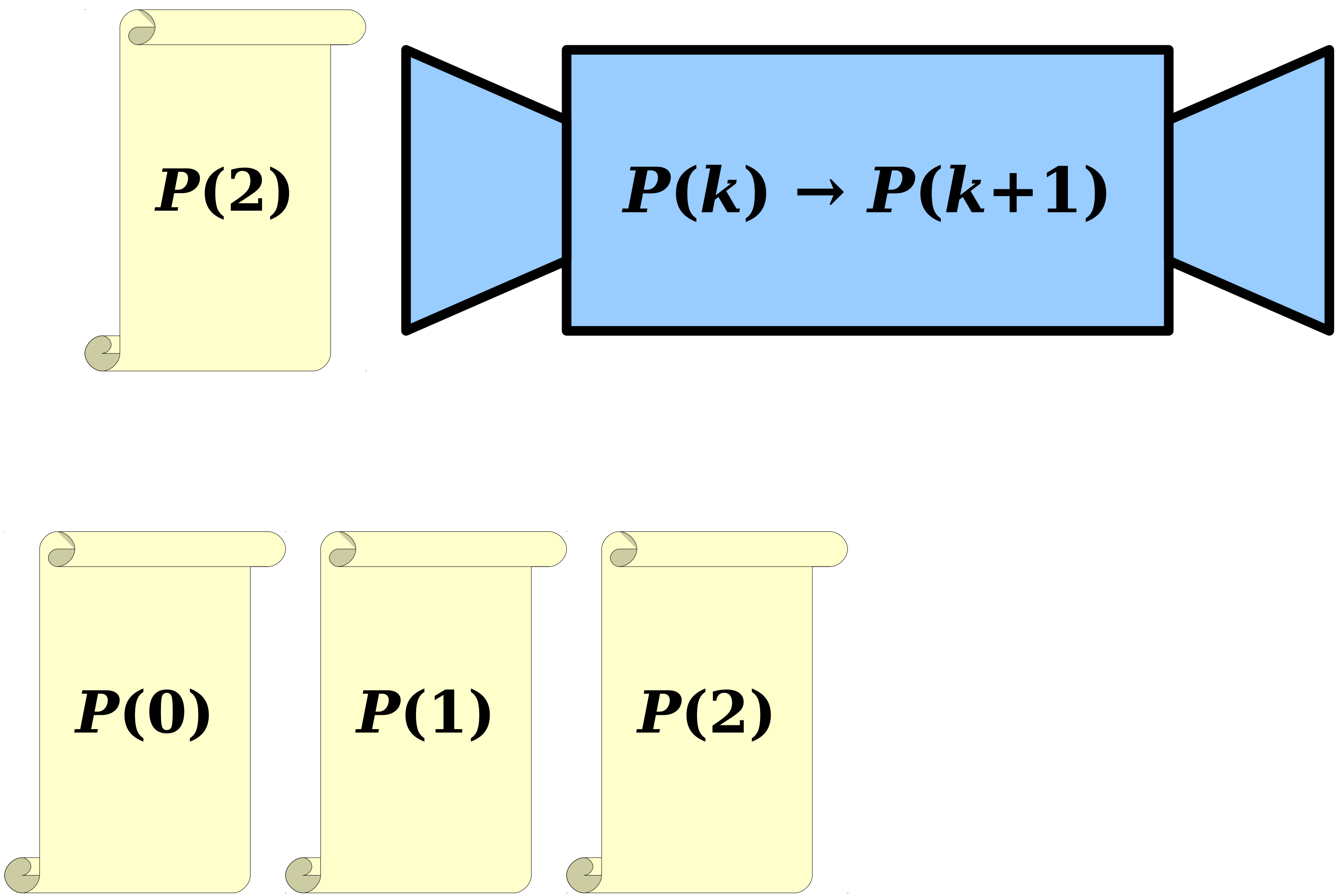

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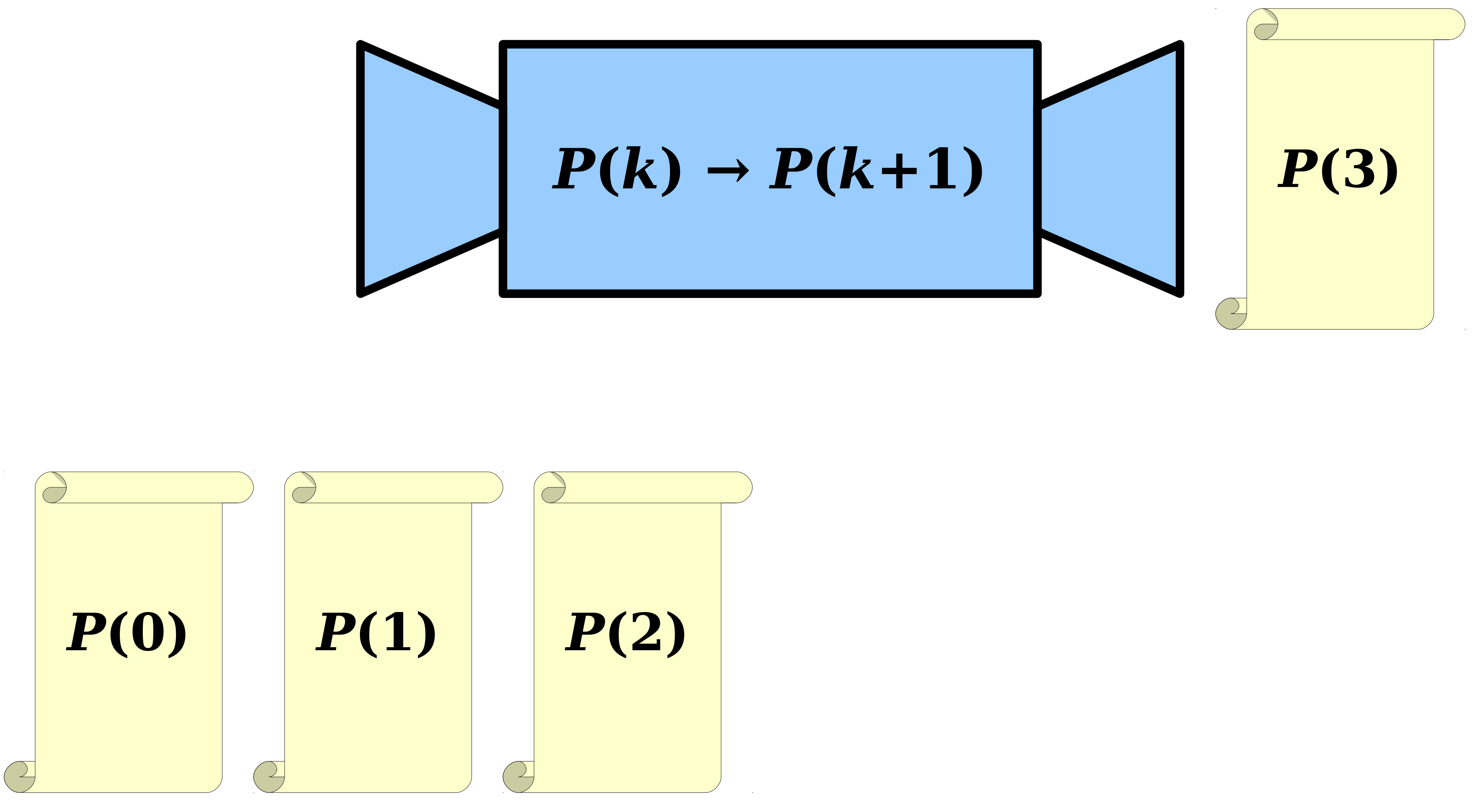
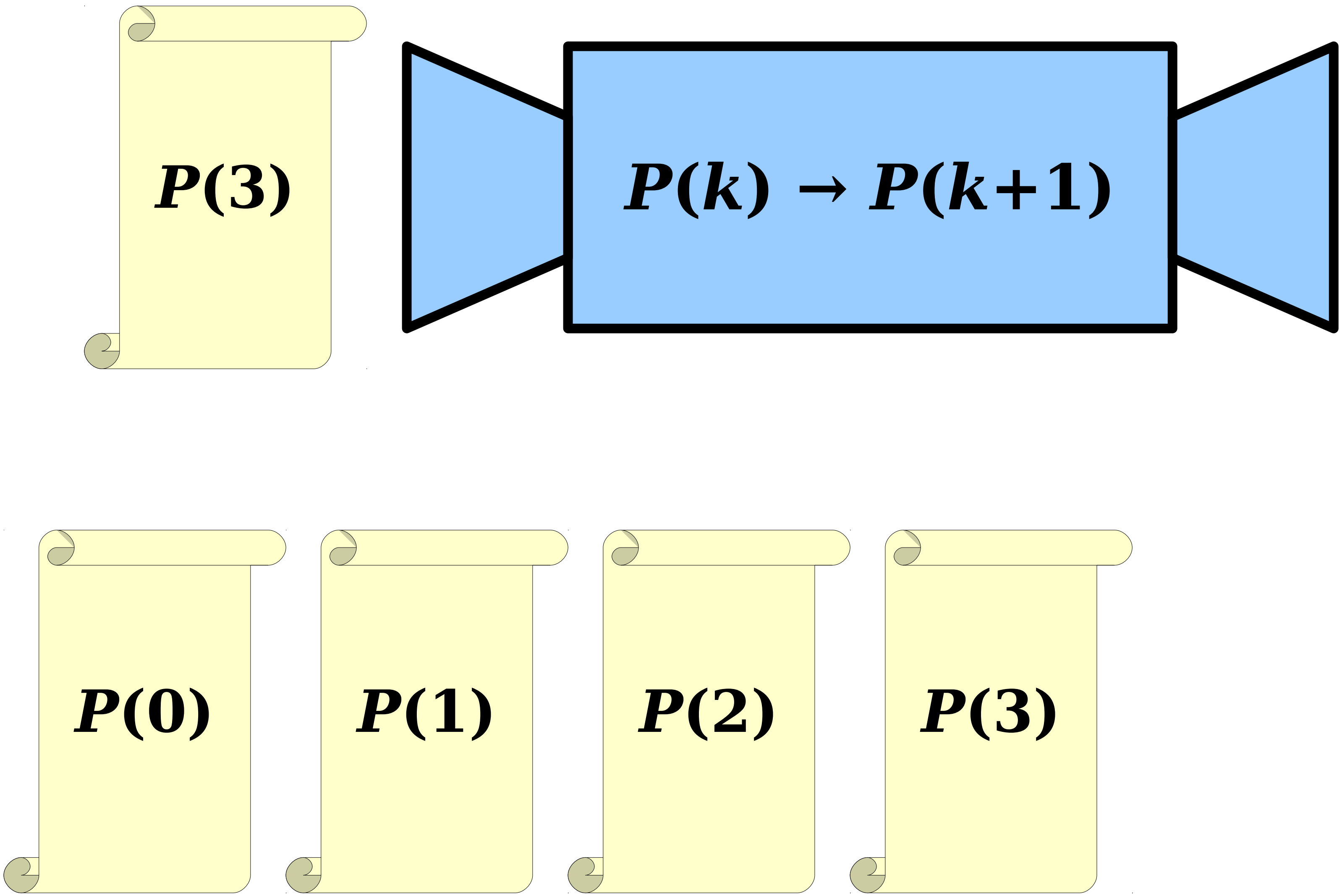


An Observation

$$P(k) \rightarrow P(k+1)$$

$P(0)$ $P(1)$ $P(2)$ $P(3)$

An Observation



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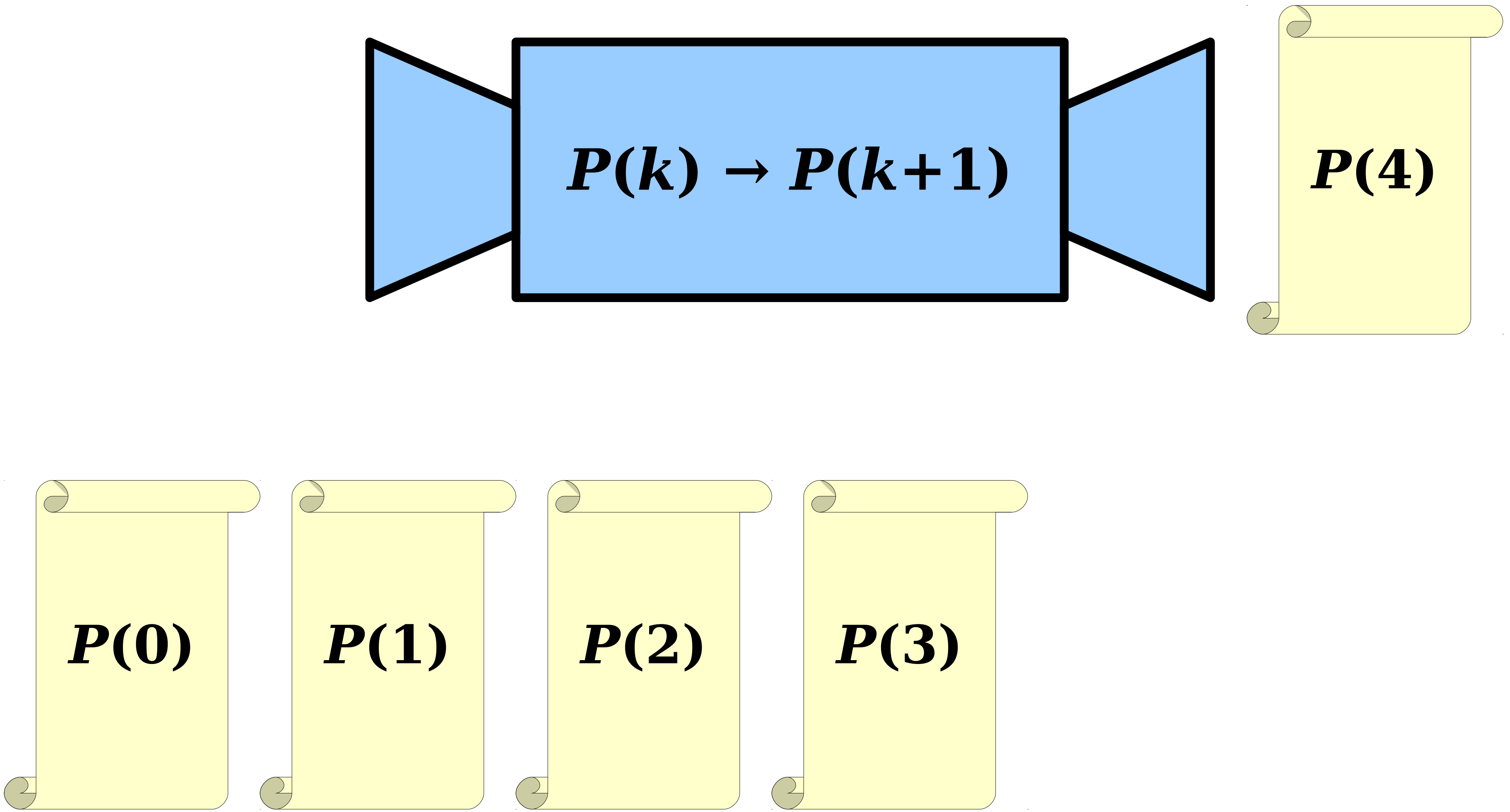

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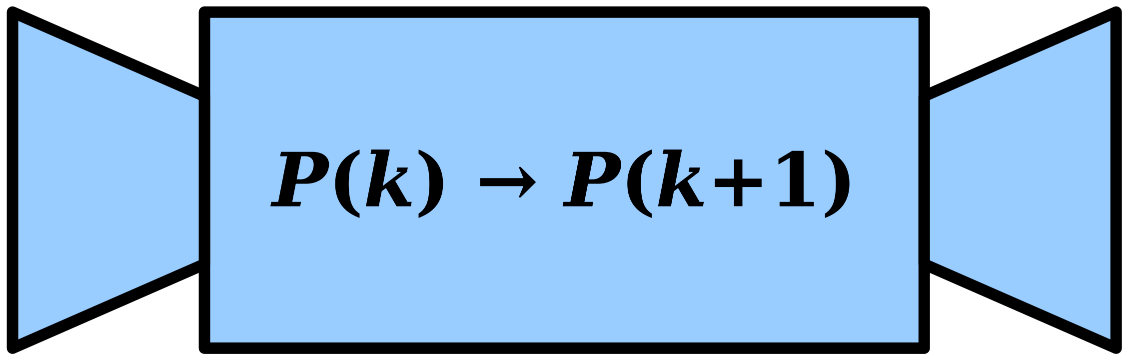

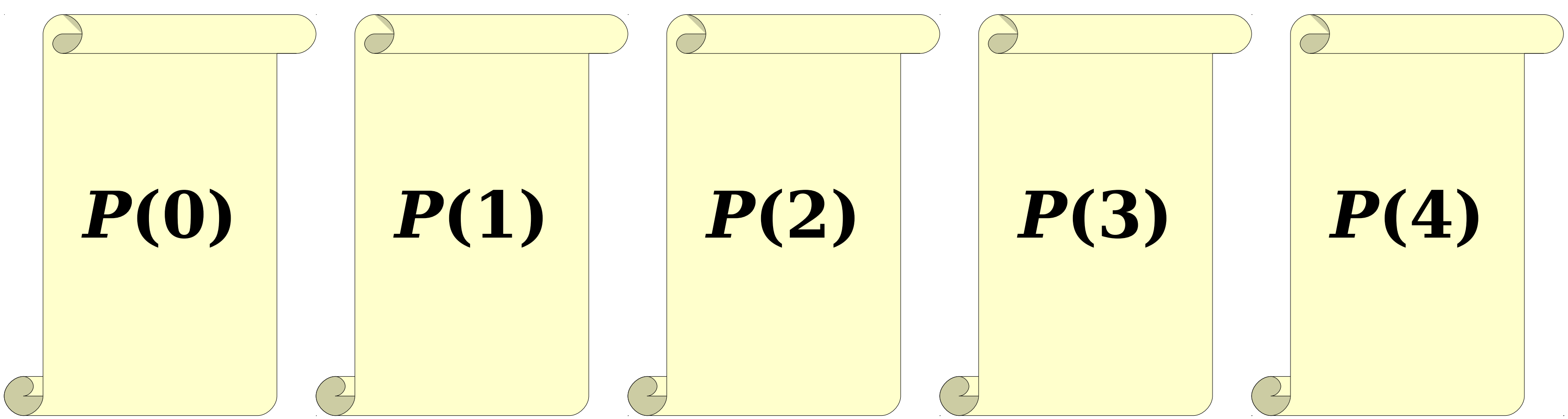

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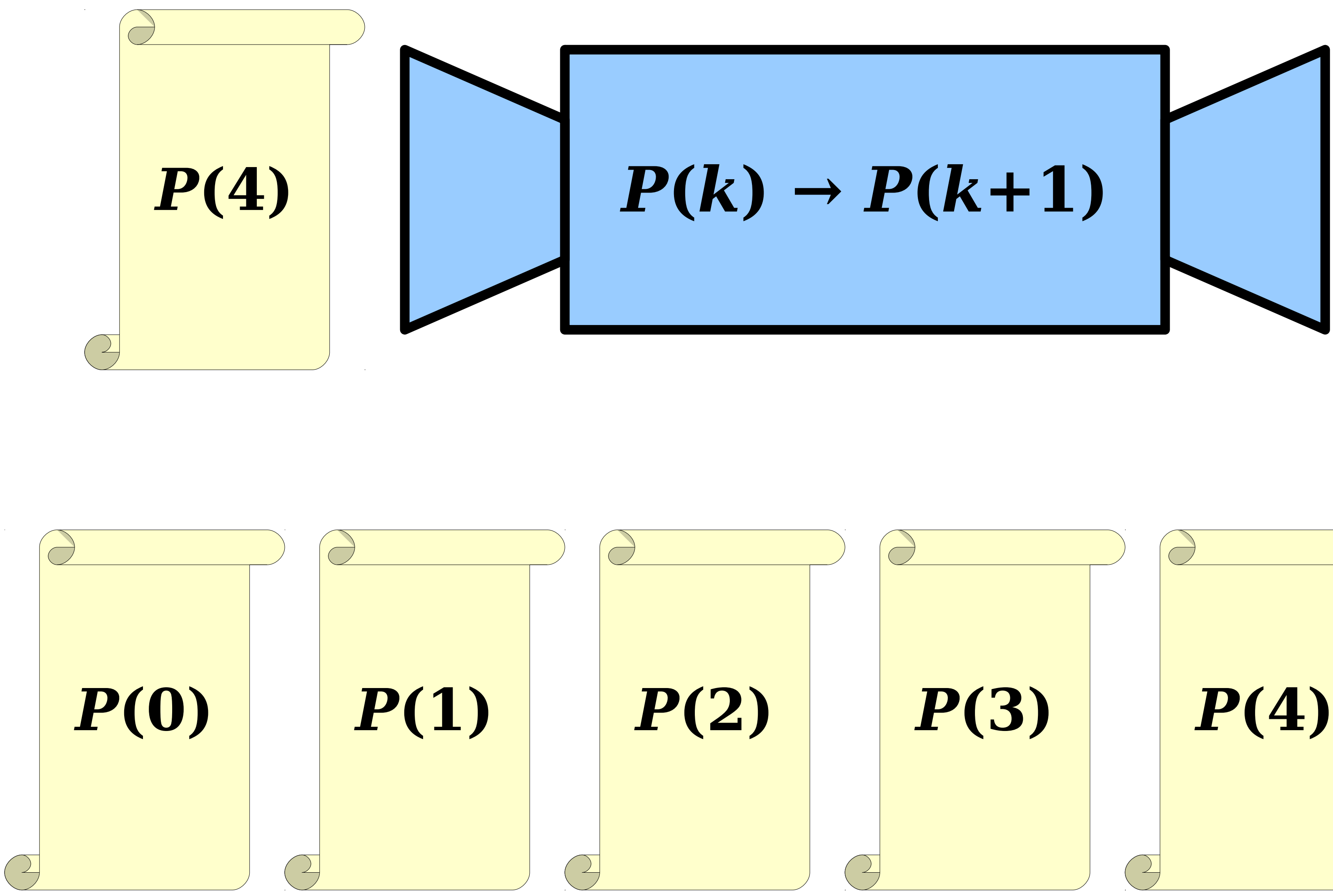

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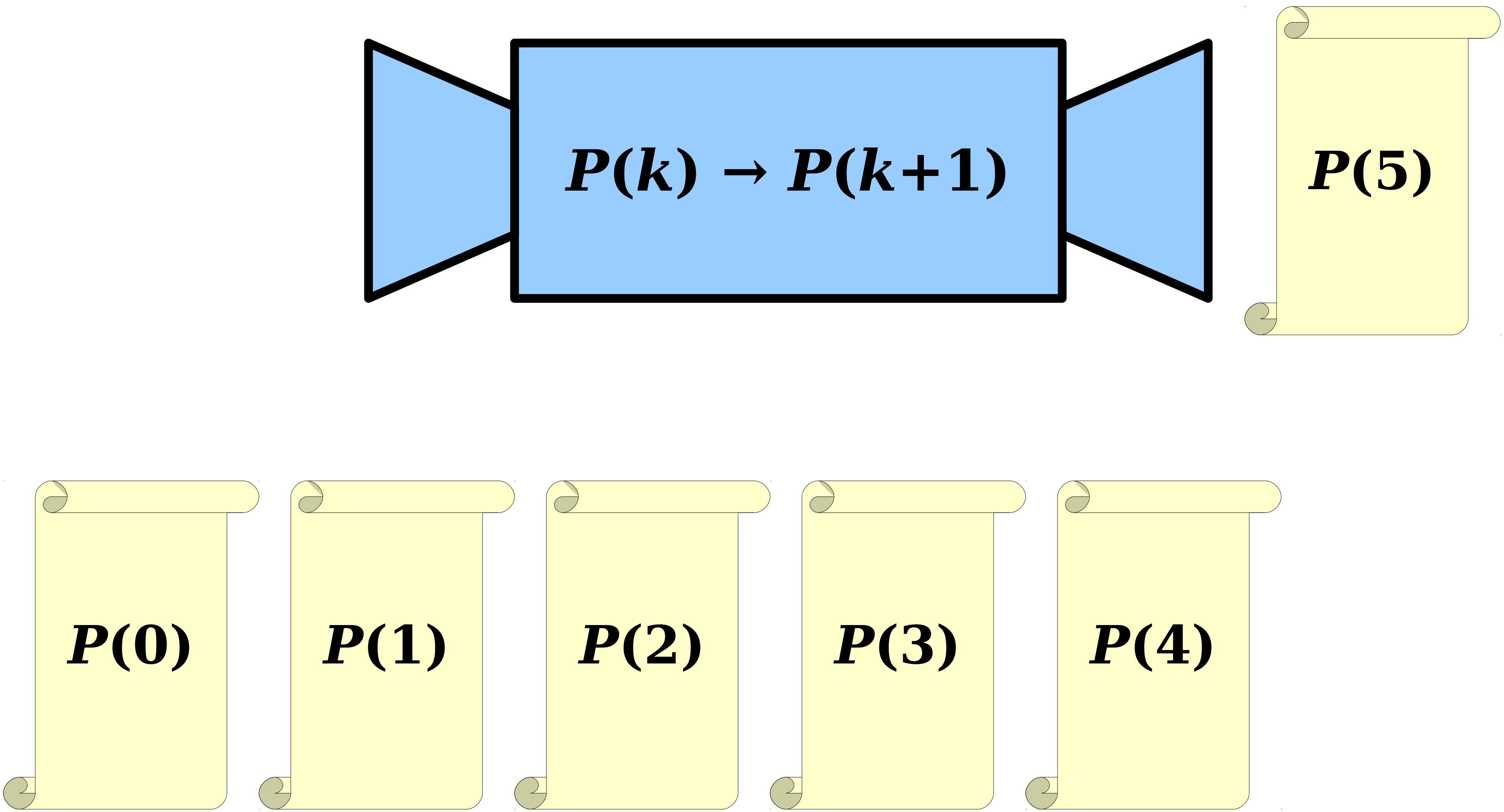

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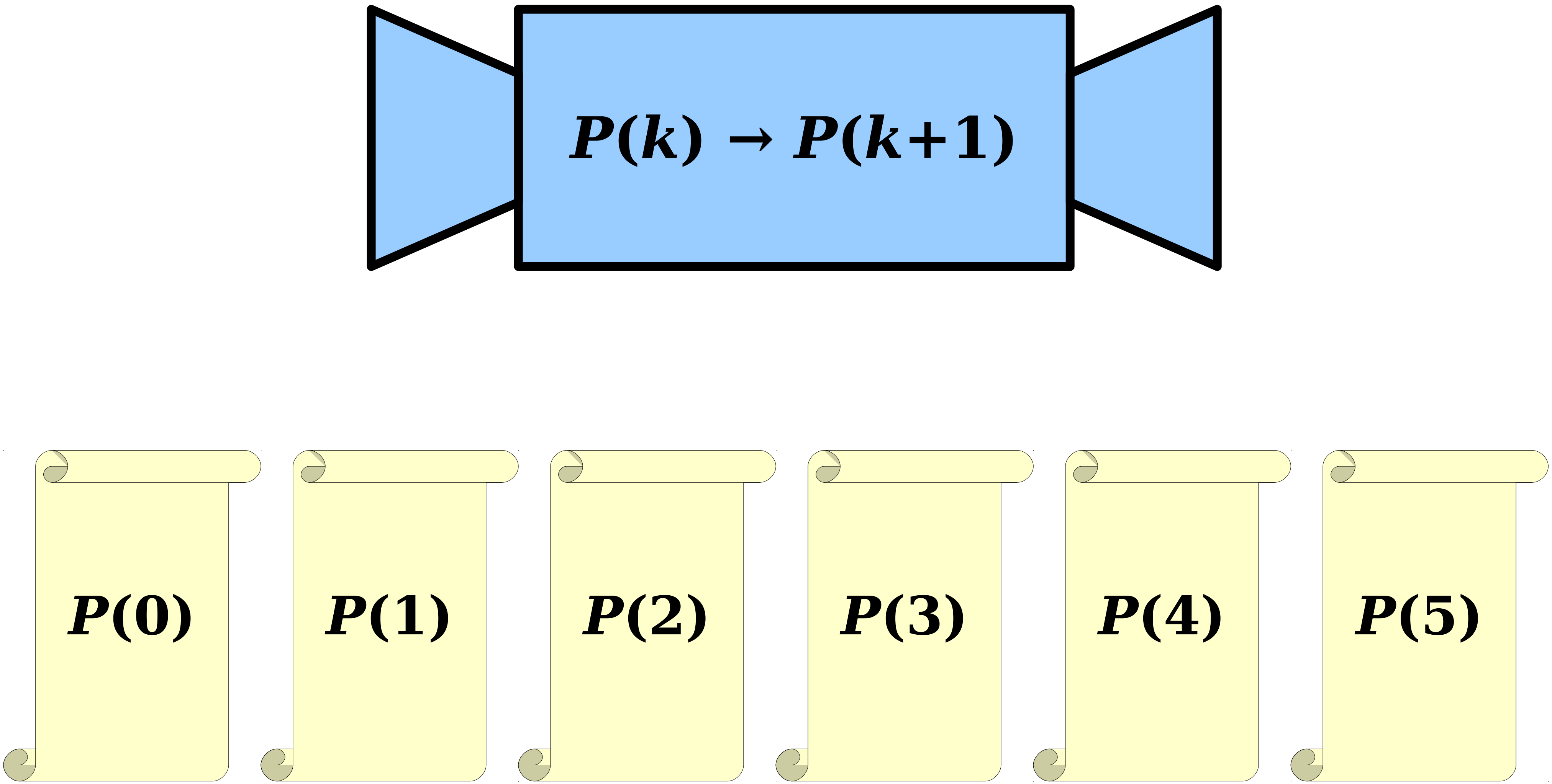
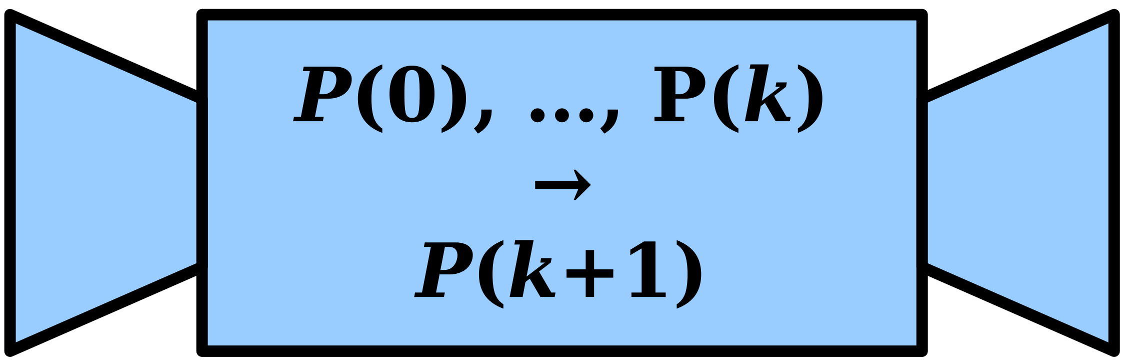
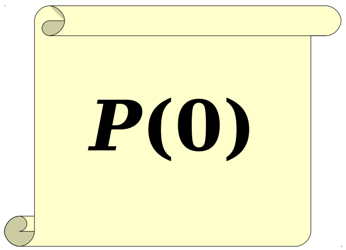


Intuiting Complete Induction





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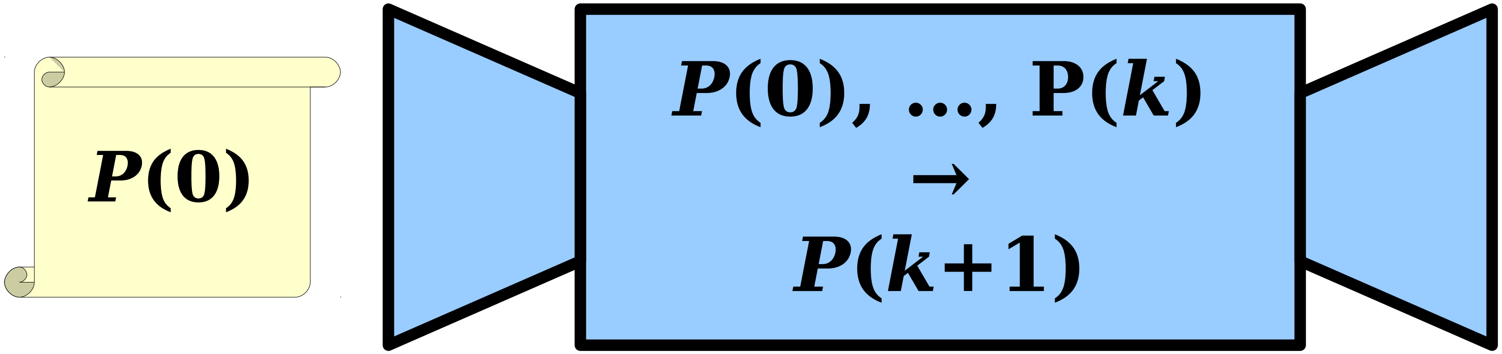

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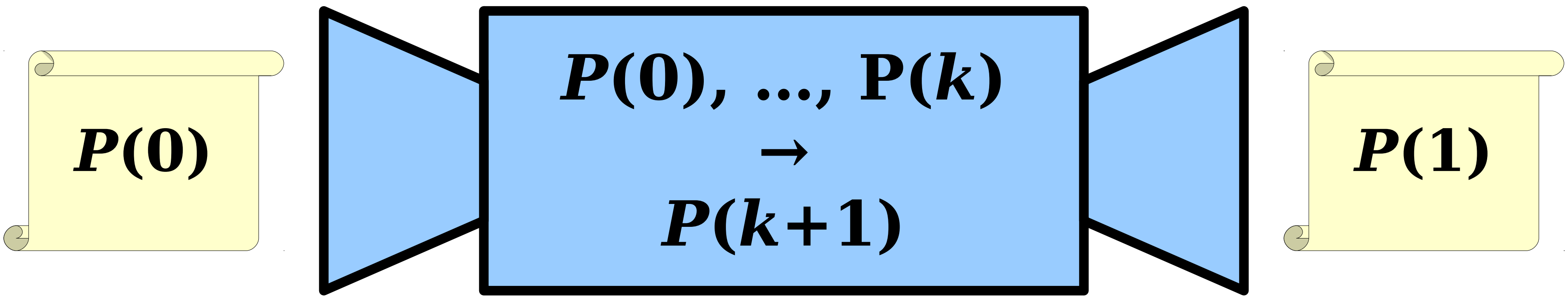

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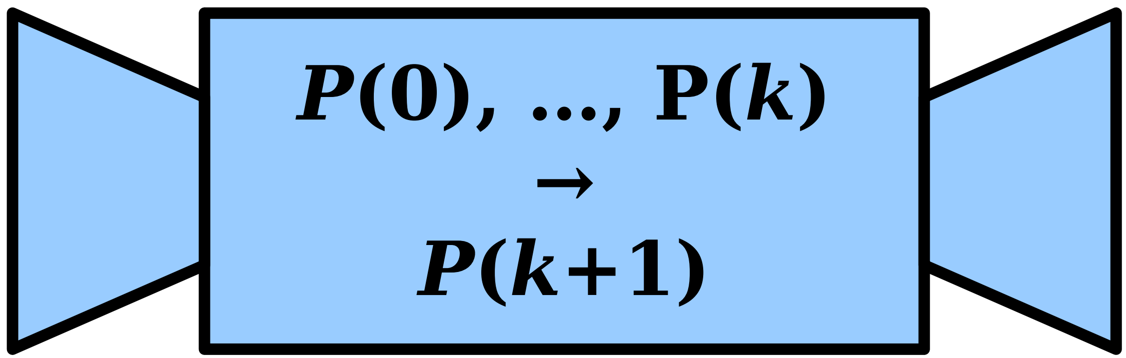

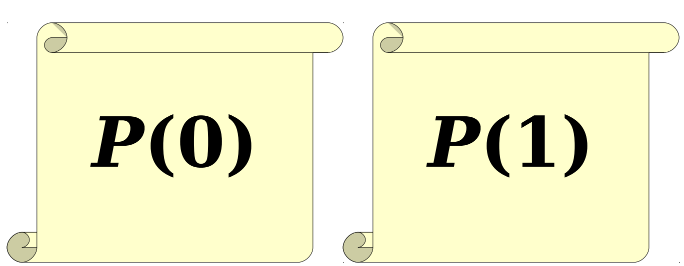

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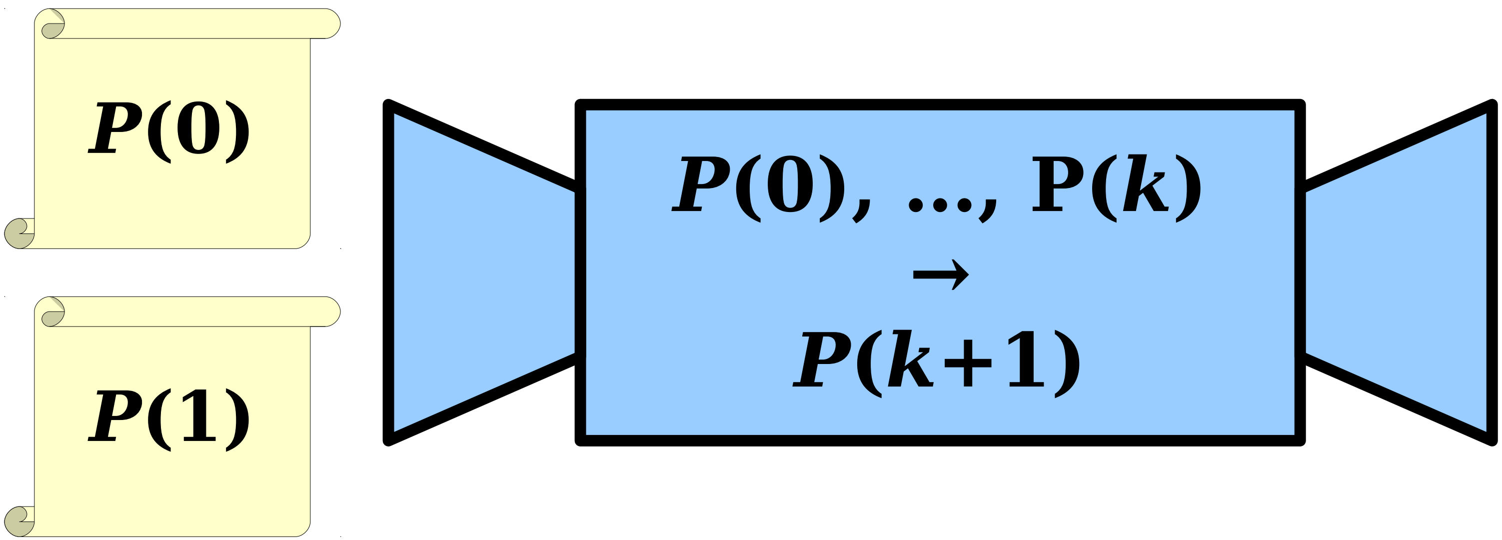

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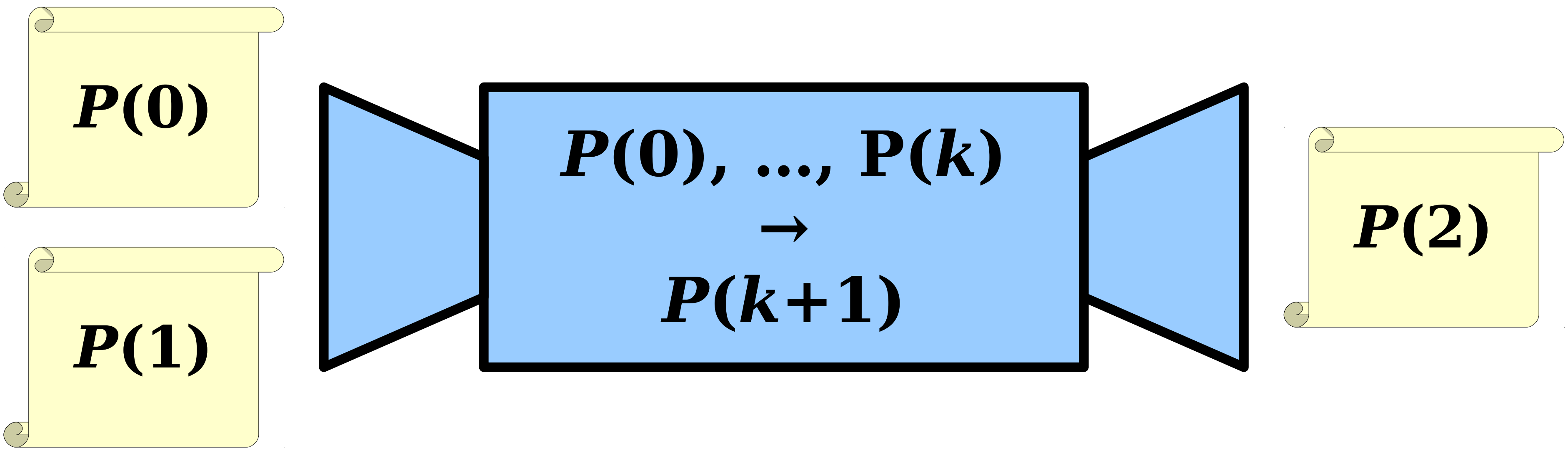

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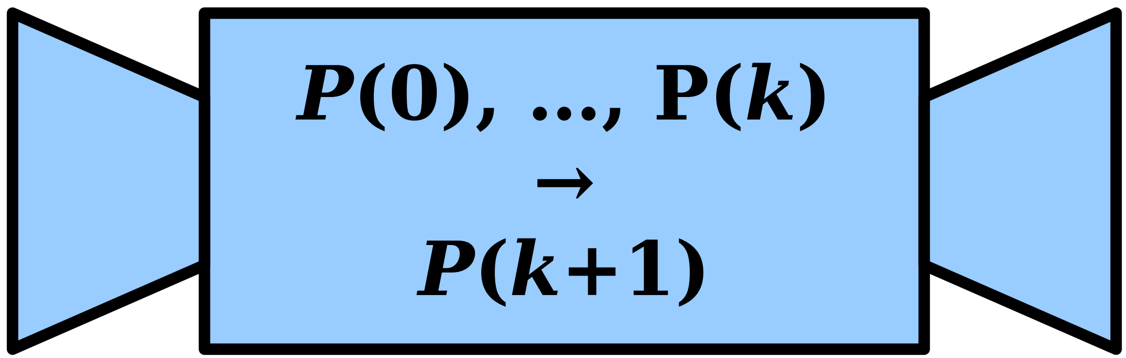

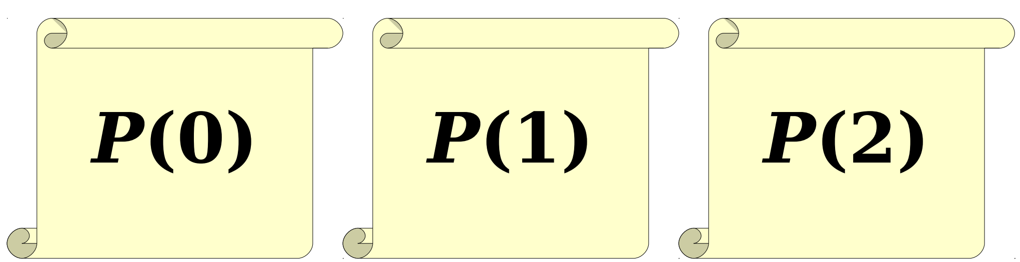

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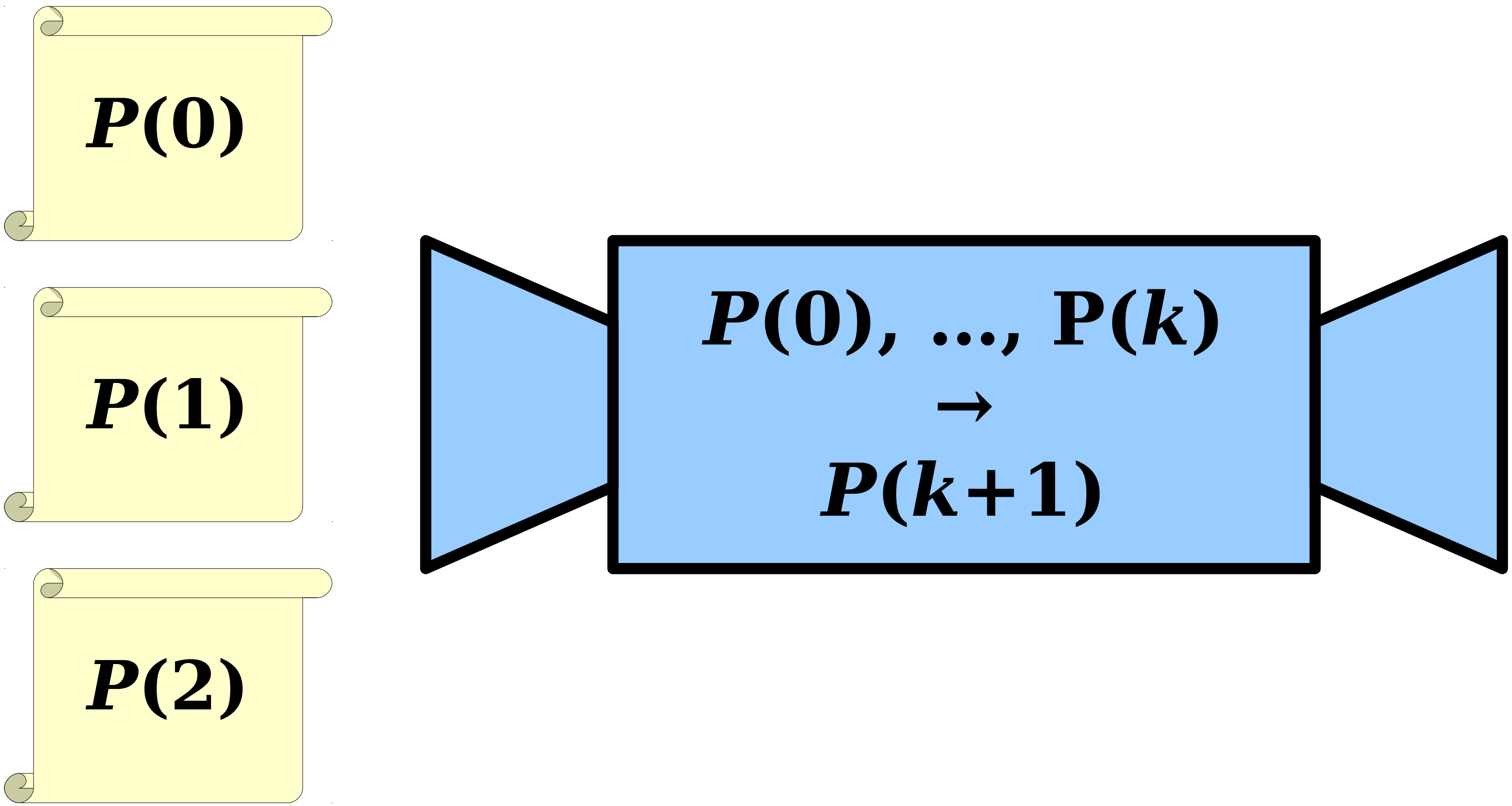

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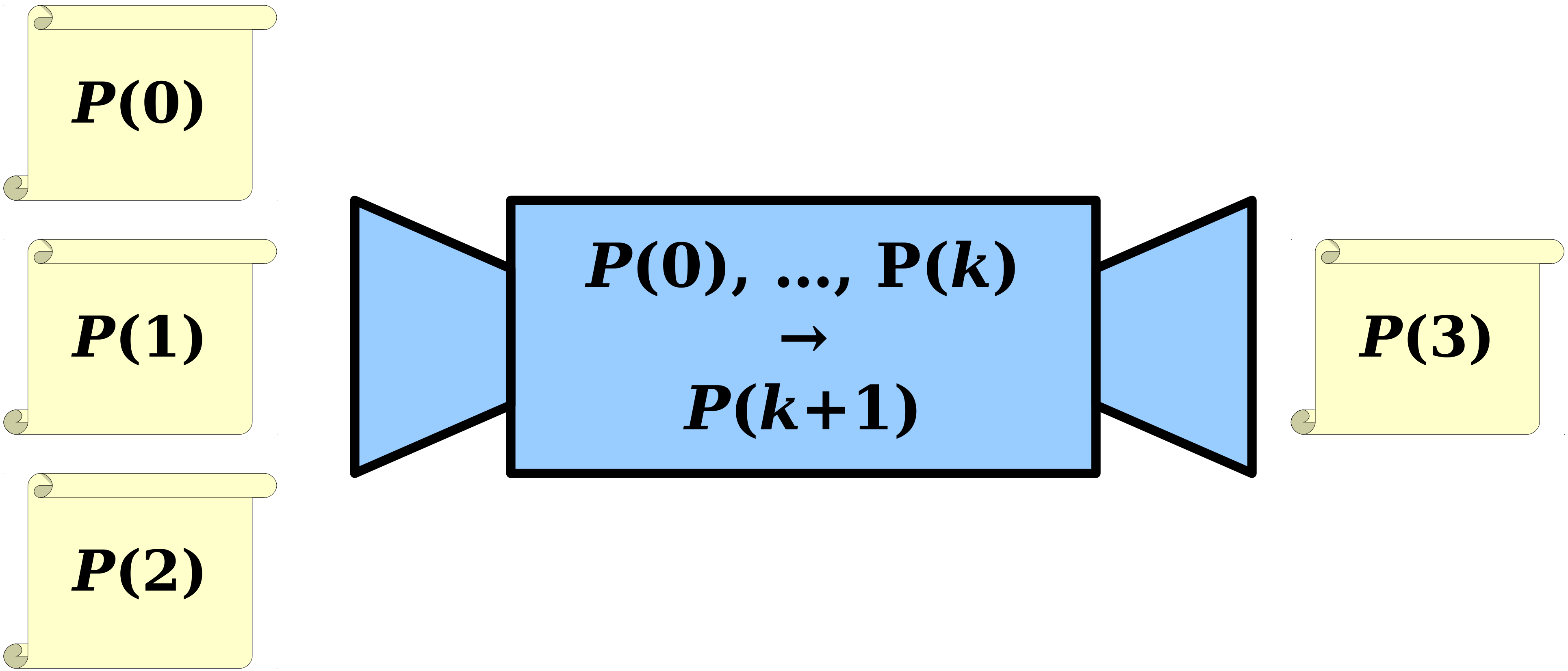

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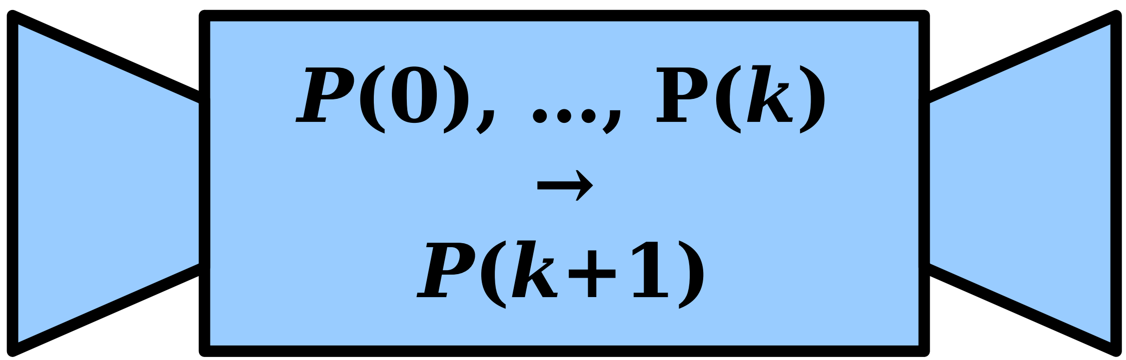

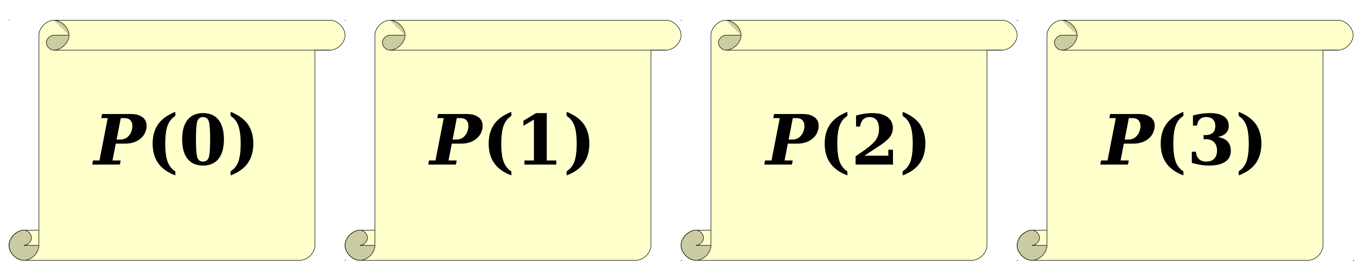

More on Complete Induction

- This type of induction can seem too powerful, almost like it's cheating.
- It's actually perfectly safe!
- We'll do more examples when we come back next time.

Next Time

- **More on Complete Induction**
    - Building an intuition for complete induction.
    - More applications!
- **Graphs**
    - Representing relationships between objects.
    - Graph connectivity.