The Unit Power Rayleigh Distribution and Applications

A project report
submitted in partial fulfillment
of the requirement of degree
of
Master of Science
in
Mathematics

Submitted by:

Mahesh Patel
(Enrollment Number:22616024)

Under the Supervision of

Dr. A. Gangopadhyay

Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee - 247667, India
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Candidate's Declaration

I hereby declare that the project titled **The Unit Power Rayleigh Distribution and Applications** in partial fulfillment of the requirements of the Degree of **M.Sc. (Mathematics)** and submitted in the **Department of Mathematics, Indian Institute of Technology Roorkee**.

This work has been carried out from January 2023 to April 2023 under the supervision of **Dr. A. Gangopadhyay**, Department of Mathematics, Indian Institute of Technology Roorkee.

This work has not been submitted by me for the award of any other degree of this or any other institute.

Date: April 2024
Place: Roorkee

Mahesh Patel
Enrollment No.: 22616024
M.Sc.(Mathematics)

This is to certify that the above statement made by the candidate is correct to the best of my knowledge.

Aditi Gangopadhyay

A. Gangopadhyay
Associate Professor
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee, India

Abstract

The Rayleigh distribution is commonly used to model the lifespan of objects or service times. This paper introduces a new distribution called the "Unit Power Rayleigh Distribution" with support on the unit interval $(0, 1)$. We explore its statistical properties such as density function, survival function, hazard function, moments, Quantile function, and order statistic. We estimate the unknown parameters using the maximum likelihood method. Simulation schemes are designed to illustrate its behavior. Finally, we apply the model to a real-world dataset and demonstrate its superiority over its special cases by comparing the goodness of fit.

Contents

1 Introduction

The Rayleigh distribution, introduced by Lord Rayleigh, is widely used in fields such as communication theory, physical science, engineering, and medical imaging. It is often employed to model the lifetime of objects whose longevity depends on their age, like mechanical parts, electronic components, wind speed, communication systems, and material strength. Numerous researchers have contributed to defining the characteristics and statistical techniques for this distribution, including Beckmann (University of Colorado)[1] provided foundational insights into its characteristics and generalizations. Dey, Dey, and Kundu [2]examined estimation techniques for the two-parameter Rayleigh distribution. Meanwhile, Al-Noor and Assi (2020) [3] investigated the properties and applications of the Rayleigh-Rayleigh distribution.
The Power Rayleigh distribution was obtained by many researcher including Bhat, Ahmad, Almetwally, Yehia, Alsadat, and Tolba [4], and Kilany, Mahmoud, and El-Refai [5]. Additionally, Almetwally, Afify, and Hamedani (2021) explored its properties and engineering applications in the Marshall-Olkin Alpha Power Rayleigh Distribution[6] by using power transformation $z = y^{\frac{1}{\beta}}$.
While the Rayleigh and Power Rayleigh distributions are versatile, their applications are sometimes limited by their inherent properties, such as the range of possible values they can take. To address scenarios requiring a bounded support, particularly in the interval between zero and one, the Unit Power Rayleigh (UPR) distribution was developed. The UPR distribution is particularly useful in modeling phenomena where the outcomes are naturally restricted to a unit scale, such as probabilities, proportions, or any scenario where standardization of data to a unit scale is crucial.

The UPR distribution not only supports bounded applications but also connects with other well-established distributions like the standard uniform and unit-Weibull distributions. This relationship provides a theoretical foundation that is useful for statistical inference and further studies in applied statistics. The purpose of this project is to introduce and study the properties of the Unit Power Rayleigh (UPR) distribution, which is a special case of the PR distribution with support on the unit interval $(0, 1)$ and arises from a transformation $X = \mathrm{e}^{-Z}$ on the Power Rayleigh distribution. It has connections with well-known distributions such as the standard uniform distribution, the power function distribution, and the unit-Rayleigh distribution,Unit-Weibull distribution.

In the following sections, we discuss the statistical properties and reliability measures of the UPR distribution, including the shapes of the probability density function, hazard function, moments, quantile function, skewness, kurtosis. We also present the maximum likelihood estimation and inference for the UPR distribution, derive the distribution of order statistics, and conduct a simulation study to generate random samples following the UPR distribution. Finally, we apply the UPR distribution to real data sets and compare its fit with some other distributions.

2 Preliminaries

2.1 Introduction to Probability Distributions

This subsection serves as the foundational base for understanding the various types of probability distributions, how they are defined mathematically, and their application in describing real-world phenomena. Here, I'll delve deeper into the concepts of probability density functions (PDFs), cumulative distribution functions (CDFs), and provide practical examples to illustrate these concepts.

What is a Probability Distribution? A probability distribution describes how the values of a random variable are distributed. It defines the probabilities of occurrence of different possible outcomes for an experiment. Distributions can be classified into two types:

- **Discrete Probability Distributions:** These describe the probabilities of discrete outcomes that take specific values and can be counted. A classic example is a dice roll, where the outcomes (1 through 6) are finite and discrete.

- **Continuous Probability Distributions:** These apply to outcomes that can take any value from a continuous range. For example, the exact height of individuals in a population or the time it takes for a chemical reaction to complete can be modeled with continuous distributions.

Probability Density Function (PDF)

The Probability Density Function (PDF) is used to specify the probability of the random variable falling within a particular range of values, as opposed to taking on any one specific value. The PDF is integral in continuous probability distributions. The probability that a continuous random variable X lies in an interval $[a, b]$ is given by the integral of its PDF over that range:

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,dx$$

Where $f_X(x)$ is the PDF of X. For instance, the PDF of a standard normal distribution $\mathcal{N}(0, 1)$ is:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

This function describes the likelihood of different outcomes and is bell-shaped, indicating that values near the mean (zero) are more likely than values far from the mean.

Cumulative Distribution Function (CDF)

The Cumulative Distribution Function (CDF) of a random variable X is defined as the probability that X will take a value less than or equal to x. Mathematically, it is expressed as:

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)\,dt$$

For discrete variables, the CDF is the sum of the probabilities of the outcomes up to and including x. The CDF provides a way to quickly evaluate the probability of observing certain outcomes without performing multiple integrations or summations.

For example, the CDF of the exponential distribution with rate λ is:

$$F(x) = 1 - e^{-\lambda x}, \text{ for } x \geq 0$$

This function is particularly useful for modeling "time until event" scenarios, such as the time until failure of a mechanical component.

2.2 The Rayleigh Distribution

The Rayleigh distribution is a continuous probability distribution named after the British physicist Lord Rayleigh. It is particularly suited for modeling the magnitude of a vector whose components are independent, normally distributed random variables with zero mean and equal variance. This distribution is prevalent in fields like signal processing, reliability engineering, and meteorology.Numerous researchers have dedicated their efforts to developing statistical techniques and defining the characteristics of this distribution, includingKundu, D., and Gupta, R. D. (2009),Ramesh, A. N., and Reza, A. (2004),Gupta, R. D., Kundu, D., and Nagar, D. K. (2005).

Definition and Derivation

The Rayleigh distribution arises when considering the magnitude of a two-dimensional vector formed from two independent standard normal random variables. If N_1 and N_2 are two independent Gaussian random variables with mean 0 and variance α^2, then the variable $Y = \sqrt{N_1^2 + N_2^2}$ follows a Rayleigh distribution. The probability density function (PDF) of the Rayleigh distribution is given by: .

$$f_Y(y; \alpha) = \frac{y}{\alpha^2} e^{\frac{-y^2}{2\alpha^2}}, \quad y > 0, \alpha > 0. \tag{1}$$

Probability Density Function

The PDF of the Rayleigh distribution as stated above is skewed to the right, with its mode at σ and no values less than zero. The function increases from zero, peaks, and then declines exponentially, reflecting typical behavior in scenarios like wind speed where there is a most probable value not at the minimum.

Cumulative Distribution Function

The cumulative distribution function (CDF) for the Rayleigh distribution is derived by integrating the PDF and is expressed as:

$$F_Y(y; \alpha) = 1 - e^{\frac{-y^2}{2\alpha^2}}, \alpha > 0 \tag{2}$$

The CDF indicates the probability that the variable Z takes a value less than or equal to z.

Properties

The mean and variance of the Rayleigh distribution are:

$$\mu = \sigma \sqrt{\frac{\pi}{2}}$$

$$\sigma^2 = \frac{4 - \pi}{2} \alpha^2$$

These properties describe its central tendency and dispersion.

Applications

The Rayleigh distribution is used to model:

- The magnitude of wind speed and vibrations in engineering.
- The resultant amplitude of a signal that has passed through multiple paths in communication theory.

2.3 The Power Rayleigh Distribution

The Power Rayleigh Distribution is an extension of the Rayleigh Distribution that incorporates a shape parameter through a power transformation.This was obtained by many researcher like O. Alzatmi and S. Nadarajah (2016),S. Nadarajah and S. Kotz (2006), R. Chiroma and M. M. Mohd (2016), Mohamed A.W. Mahmoud , N.M.Kilany L.H.El-Refai (2020) This adaptation allows for more flexible modeling capabilities to accommodate a wider range of data behaviors by adjusting the skewness and tail behaviors of the distribution.

Power Transformation

A power transformation is applied to a standard Rayleigh Distribution to derive the Power Rayleigh Distribution. The transformation modifies the original variable Y with a Rayleigh distribution into a new variable Z through the relationship $Z = Y^{1/\beta}$, where β is a positive real number known as the shape parameter. This transformation adjusts the skewness and heaviness of the tails of the distribution.

The idea behind using a power transformation is to finely control the spread and shape of the distribution more than what is possible with the original Rayleigh distribution. The parameter β allows the distribution to model data that exhibit different levels of right-skewness, providing a useful tool for fitting data that do not conform neatly to the assumptions of more common distributions.

Cumulative Distribution Function (CDF)

So we can find the cumulative distribution function of the z as

$$F(z) = P(Z \leq z) = P\left(Y \leq z^{\beta}\right) = F_y\left(z^{\beta}\right), \beta > 0$$

Hence we got CDF as:

$$F_Z(z;\alpha,\beta) = 1 - e^{\frac{-z^{2\beta}}{2\alpha^2}}, \quad \alpha,\beta > 0 \tag{3}$$

Probability Density Function (PDF) Now we can find the Probability density function of Power Rayleigh distribution as

$$\begin{aligned} f_Z(z;\alpha,\beta) &= \frac{d}{dz}F_y\left(z^{\beta}\right) \\ &= \frac{\beta}{\alpha^2}z^{2\beta-1}e^{\frac{-z^2\beta}{2\alpha^2}}, \quad z > 0, \alpha, \beta > 0 \end{aligned} \tag{4}$$

where α is a scale parameter and β is a shape parameter.

The introduction of the Power Rayleigh Distribution with these transformations allows for greater flexibility in statistical modeling, particularly when dealing with data exhibiting varying levels of skewness and tail weight, making it a valuable tool in fields requiring precise modeling of diverse data types.

2.4 Survival and Hazard Functions

In reliability engineering and survival analysis, understanding the time until an event occurs is crucial. Survival and hazard functions are instrumental in providing insights into the probability of an event (such as failure or death) occurring at a specific time or over a period. This section explains both the survival and hazard functions, along with their mathematical definitions and implications.

Survival Function

The survival function, also known as the survivor function, $S(t)$, is defined as the probability that a subject will survive beyond time t. Mathematically, the survival function is the complement of the cumulative distribution function (CDF) of the lifetime T of the subject, given by:

$$S(t) = P(T > t) = 1 - F(t)$$

where $F(t)$ is the CDF of the time until the event occurs.

For example, if the lifetime T follows a Rayleigh distribution with parameter α, then the survival function $S(t)$ can be expressed as:

$$S(t) = e^{-\frac{t^2}{2\alpha^2}}, \quad t \geq 0$$

This expression shows that the probability of survival decreases exponentially with the square of time, which is characteristic of systems where the risk accumulates over time.

Hazard Function

The hazard function, $h(t)$, also known as the failure rate or the force of mortality, measures the instant risk of failure at time t, conditional on survival until time t or later. It is defined as the rate at which failures occur at time t divided by the probability of surviving until that time. The hazard function can be mathematically defined as:

$$h(t) = \frac{f(t)}{S(t)}$$

where $f(t)$ is the probability density function (PDF) of the lifetime T. The hazard function can also be derived from the cumulative hazard function $H(t)$, which is the integral of the hazard function:

$$H(t) = \int_0^t h(u)\, du = -\ln(S(t))$$

Thus, the hazard function itself is:

$$h(t) = \frac{d}{dt}H(t) = -\frac{d}{dt}\ln(S(t))$$

For a Rayleigh distributed lifetime, using the previously provided survival function, the hazard function $h(t)$ is:

$$h(t) = \frac{t}{\alpha^2}, \quad t \geq 0$$

This indicates a linearly increasing failure rate, which is typical for components where "wear and tear" increase the likelihood of failure over time.

Applications

Both survival and hazard functions are extensively used in various fields. In medical studies, the survival function is crucial for understanding patient survival times, while the hazard function is vital for identifying periods of increased risk of relapse or death. In engineering, these functions assist in maintenance planning and lifespan prediction of components.

Understanding these functions and their relationships with the underlying probability distributions provides a comprehensive toolset for analyzing lifetime data, crucial for effective decision-making in healthcare, engineering, and many other fields.

3 Development of the Unit Power Rayleigh Distribution

In this section, we explore the development of the Unit Power Rayleigh (UPR) Distribution by applying a transformation to the Power Rayleigh Distribution. This transformation constrains the distribution within the unit interval $(0, 1)$, making it particularly useful for modeling proportions and probabilities, where variables are naturally constrained to these limits.

3.1 Conceptual Basis for the Transformation

The UPR Distribution is derived by applying a transformation that modifies the Power Rayleigh Distribution to fit within the unit interval. This is achieved by using a transformation function that effectively maps the potentially unbounded support of the Power Rayleigh Distribution onto the bounded interval $(0, 1)$.

3.2 Mathematical Formulation of the Transformation

The transformation used to derive the UPR Distribution from the Power Rayleigh Distribution is expressed as:

$$X = \exp(-Z)$$

where Z is a random variable following the Power Rayleigh Distribution. This transformation maps the range of Z from $[0, \infty)$ to the interval $(0, 1)$ on X.

3.3 Derivation of the PDF and CDF

Given the transformation $X = \exp(-Z)$, where Z follows the Power Rayleigh Distribution with the probability density function (PDF):

$$\begin{aligned} f_Z(z; \alpha, \beta) &= \frac{d}{dz} F_y\left(z^\beta\right) \\ &= \frac{\beta}{\alpha^2} z^{2\beta-1} e^{\frac{-z^2\beta}{2\alpha^2}}, \quad z > 0, \alpha, \beta > 0 \end{aligned} \tag{5}$$

and the cumulative density function (CDF) as:

$$F_Z(z; \alpha, \beta) = 1 - e^{\frac{-z^{2\beta}}{2\alpha^2}}, \quad \alpha, \beta > 0 \tag{6}$$

Now we can derive the Cumulative distribution function of Unit Power Rayleigh (UPR) distribution as following :

$$\begin{aligned} F_X(x) &= P(e^{-Z} \leq x) \\ &= P(-Z \leq \ln(x)) \\ &= P(Z \geq -\ln(x)) \\ &= 1 - F_Z(-\ln(x)) \end{aligned}$$

$$F_X(x; \alpha, \beta) = 1 - (1 - \exp\left[\frac{-(-\ln x)^{2\beta}}{2\alpha^2}\right]), \quad 0 < x < 1, \quad \alpha, \beta > 0$$

$$F(x;\alpha,\beta) = \exp\left[\frac{-(-\ln x)^{2\beta}}{2\alpha^2}\right], \quad 0 < x < 1, \quad \alpha,\beta > 0 \tag{7}$$

Now we can find the Probability density function of Power Rayleigh distribution as

$$\begin{aligned} f_X(x;\alpha,\beta) &= \frac{d}{dx}(F_X\,(x)) \\ &= \frac{d}{dx}(\exp\left[\frac{-(-\ln x)^{2\beta}}{2\alpha^2}\right]) \end{aligned}$$

$$f(x;\alpha,\beta) = \frac{\beta}{\alpha^2 x}(-\ln x)^{2\beta-1}\exp\left[\frac{-(-\ln x)^{2\beta}}{2\alpha^2}\right], \quad 0 < x < 1, \quad \alpha,\beta > 0. \tag{8}$$

3.4 Shape

After deriving the probability density function (PDF) for the Unit Power Rayleigh (UPR) distribution, it is instructive to visualize how the distribution behaves across its range. The following graph illustrates the PDF of the UPR distribution for different values of the parameters α and β. This visualization helps in understanding the influence of these parameters on the shape and spread of the distribution.

The shapes of the probability density functions (PDFs) are influenced by the parameters α and β. An increase in α results in a flatter and wider distribution, indicative of a decrease in the peak's height and an increase in the spread. This suggests a lower probability concentration near the mode and a higher variance. Essentially, α controls the scale of the distribution.

Conversely, β affects the distribution's tail and the location of its peak. Higher values of β cause the peak to shift rightward and the tail to become heavier, suggesting a distribution with a tendency for larger values. The peak also broadens with higher β, which points to a less steep decline in the PDF away from the mode.

All observed PDFs maintain a unimodal shape across different α and β values, each exhibiting one distinct peak. The parameter α modulates the peak's spread, while β influences its sharpness and position. Therefore, α is associated with the dispersion of the distribution, and β with its skewness.

In essence, α and β work in tandem to define the overall characteristics of the PDF, with α affecting the general spread and β determining the shape and heaviness of the tail. These parameters are crucial for modeling data with specific distributional properties.

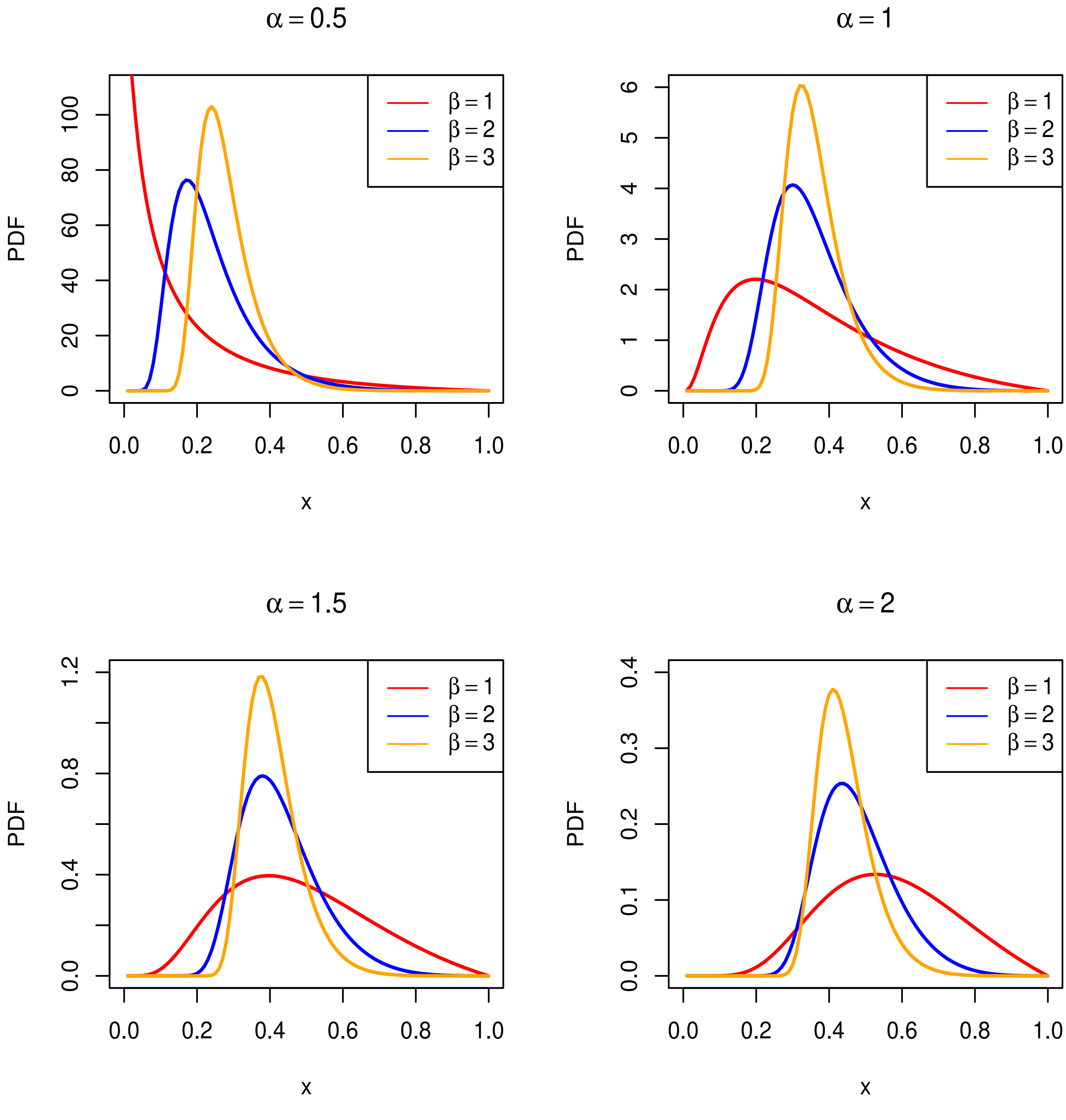

Figure 1: The behavior of PDF of UPR distribution for some selected choices of α and β.

4 Statistical Properties

4.1 Quantile Function

The Quantile function can be define as

$$Q(p) = F^{-1}(p) \text{ for } 0 < p < 1$$

Hence for Unit Power Rayleigh Distribution Quantile Function is given by

$$Q(p) = \exp[-\left(-2\alpha^2 \ln(p)\right)^{\frac{1}{2\beta}}] \tag{9}$$

- The First quartile of the UPR distribution is given by

$$Q_1 = Q(\frac{1}{4}) = \exp[-\left(2\alpha^2 \ln(4)\right)^{\frac{1}{2\beta}}]$$

- The Second quartile (Median) of the UPR distribution is given by

$$Q_2 = Q(\frac{1}{2}) = \exp[-\left(2\alpha^2 \ln(2)\right)^{\frac{1}{2\beta}}]$$

- The of Third quartile the UPR distribution is given by

$$Q_3 = Q(\frac{3}{4}) = \exp[-\left(2\alpha^2 \ln(4/3)\right)^{\frac{1}{2\beta}}]$$

4.2 Survival and Hazard Rate Functions

survival function $S(x)$ of UPR is distribution is given by

$$S(x) = 1 - F(x)$$

$$S(x;\alpha,\beta) = 1 - \exp\left[\frac{-(-\ln x)^{2\beta}}{2\alpha^2}\right] \tag{10}$$

And the Hazard rate function of UPR distribution is

$$h(x) = \frac{f(x)}{S(x)}$$

$$h(x;\alpha,\beta) = \frac{\beta(-\ln x)^{2\beta-1}\exp\left(\frac{-(-\ln x)^{2\beta}}{2\alpha^2}\right)}{\alpha^2 x[1 - \exp\left(\frac{-(-\ln x)^{2\beta}}{2\alpha^2}\right)]} \tag{11}$$

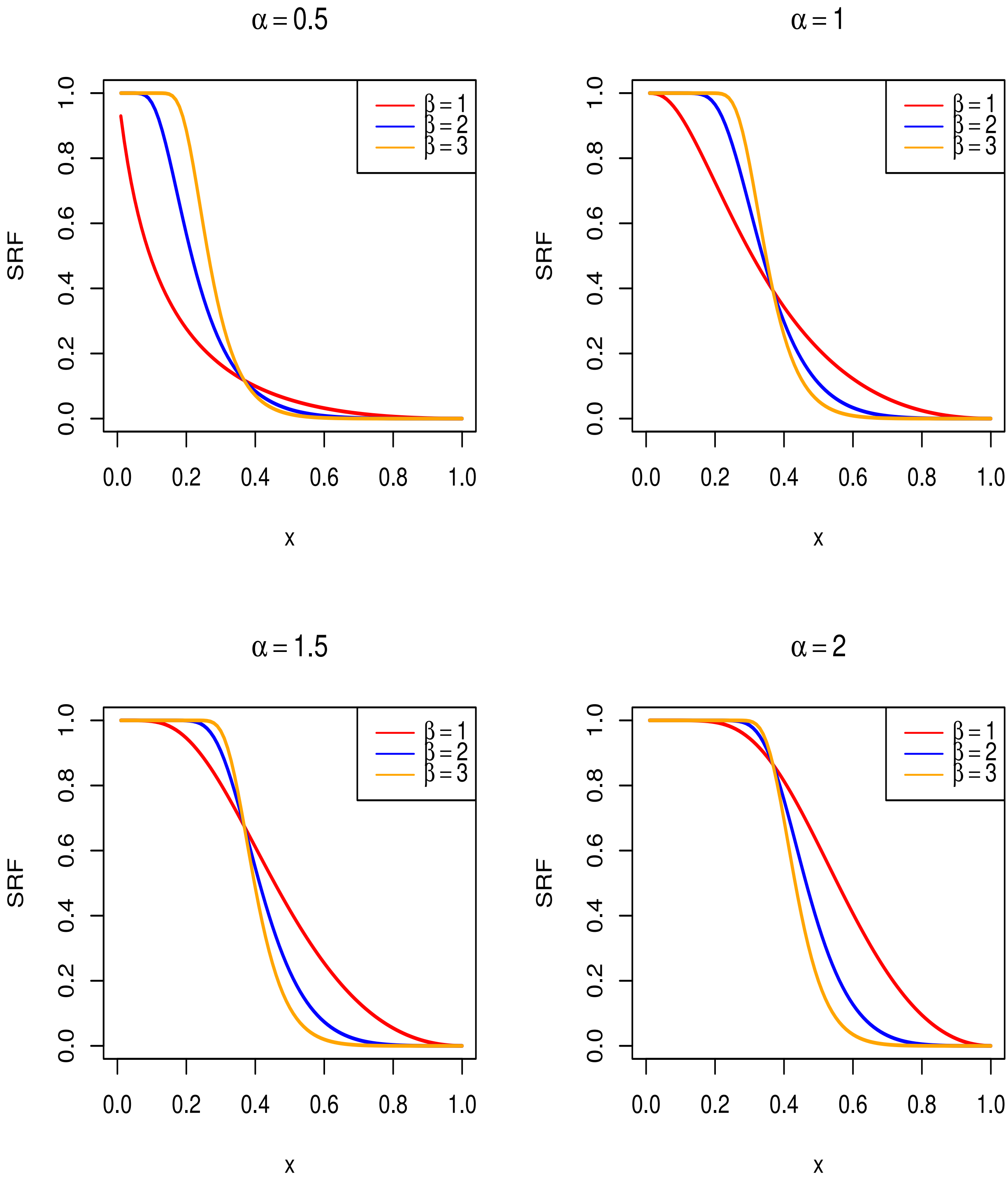

Figure 2: The behavior of Survival function of UPR distribution for some selected choices of α and β

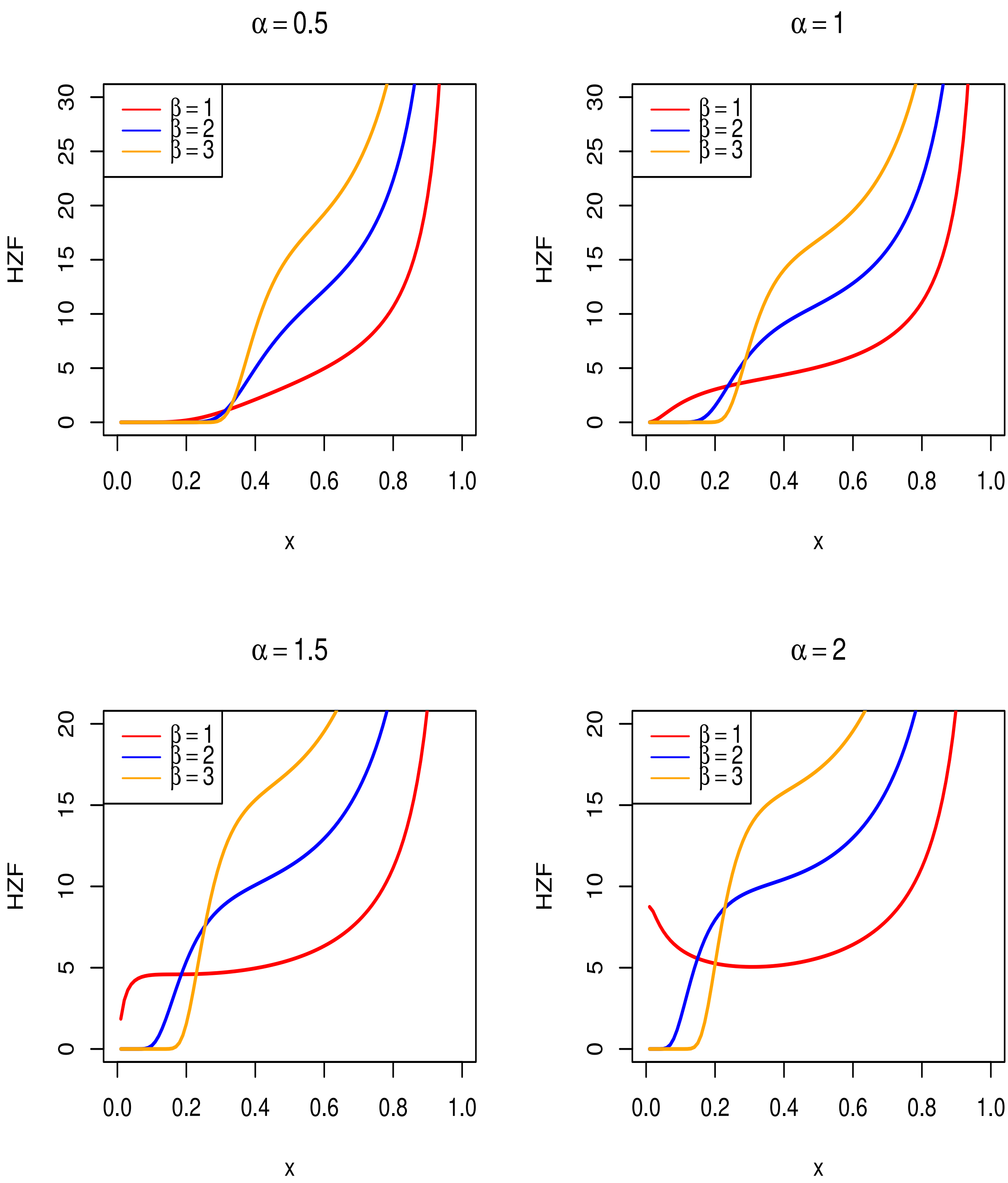

Figure 3: The behavior of Hazard rate function of UPR distribution for some selected choices of α and β

4.3 Moments

We can find the r th raw moment of UPR distribution as

$$\mu_r' = E\left(X^r\right) = E\left(e^{-rZ}\right) = M_Z(-r)[7]$$

Where $M_Z(-r)$is Moment generation function of Z, which follows Power Rayleigh Distribution

$$\begin{aligned} M_Z(t) &= \int_0^\infty e^{tx} f_Z(z)dz \\ &= 1 + \sum_{i=1}^{\infty} \frac{t^i}{i!} 2^{\frac{i}{2\beta}} \left(\frac{1}{\alpha^2}\right)^{\frac{-i}{2\beta}} \Gamma\left(1 + \frac{i}{2\beta}\right)^{[5]} \end{aligned}$$

Hence the r th raw moment of the UPR distribution is given as:

$$\mu_r' = 1 + \sum_{i=1}^{\infty} \frac{(-r)^i}{i!} 2^{\frac{i}{2\beta}} \left(\frac{1}{\alpha^2}\right)^{\frac{-i}{2\beta}} \Gamma\left(1 + \frac{i}{2\beta}\right) \tag{12}$$

Mean and Variance: We can find mean by putting r=1 in equation (12),and then by putting $r = 2$ we can find second moment and now By using the mean and second moment we can caculate varinace as follows

$$\sigma^2 = E[X^2] - (E[X])^2 = \mu_2' - (\mu_1')^2 = \mu_2' - (\mu)^2 \tag{13}$$

Skewness and kurtosis : Now we can find skewness and kurtosis measures from the below expressions by substituting raw moments

$$\text{skewness} = \frac{\mu_3' - 3\mu_2'\mu + 2\mu^3}{\sigma^3} \tag{14}$$

$$\text{kurtosis} = \frac{\mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4}{\sigma^4} \tag{15}$$

Now we can see calculated Mean, Variance,Skewness and kurtosis at different different values of alpha and betain Table 1 , Table 2 , Table 3 , and Table 4 respectively.

Table 1: Mean of UPR for various values of α and β

α/β	1	2	3	5
0.5	0.5618177	0.4774909	0.4433379	0.4104342
1	0.34432046	0.3563400	0.3604293	0.3635252
1.5	0.22627654	0.2862055	0.3120536	0.3339861
2	0.15726154	0.2387865	0.2784121	0.3131681
3	0.08622910	0.1775317	0.2327069	0.2841881
5	0.03595948	0.1130789	0.1793232	0.2485880

Table 2: Variance of UPR for various values of α and β

α/β	1	2	3	5
0.5	0.02868125	0.010789002	0.005416085	0.002108607
1	0.03870496	0.012299070	0.005812890	0.002172091
1.5	0.03502803	0.012155832	0.005807151	0.002174511
2	0.02865928	0.011015063	0.005679370	0.002159059
3	0.10815600	0.009913664	0.005322628	0.002112077
5	18.34587	0.007194775	0.004610563	0.002010780

Table 3: Skewness Matrix for various values of α and β

α/β	1	2	3	5
0.5	1.243318×10^{-1}	6.531961×10^{-1}	8.577634×10^{-1}	1.012871×10^{0}
1	7.126940×10^{-1}	8.984603×10^{-1}	9.959107×10^{-1}	1.074470×10^{0}
1.5	1.230782×10^{0}	1.093908×10^{0}	1.096685×10^{0}	1.115634×10^{0}
2	2.3627×10^{1}	1.264608×10^{0}	1.179425×10^{0}	1.147468×10^{0}
3	1.297782×10^{18}	1.565801×10^{0}	1.315355×10^{0}	1.196478×10^{0}
5	1.139267×10^{7}	2.088942×10^{0}	1.527335×10^{0}	1.266065×10^{0}

Table 4: Kurtosis Matrix for various values of α and β

α/β	1	2	3	5
0.5	2.381068×10^{0}	3.346023×10^{0}	4.032743×10^{0}	4.710738×10^{0}
1	2.889072×10^{0}	3.913803×10^{0}	4.462948×10^{0}	4.949900×10^{0}
1.5	7.735039×10^{1}	4.499879×10^{0}	4.818530×10^{0}	5.118164×10^{0}
2	4.037712×10^{14}	5.109342×10^{0}	5.137094×10^{0}	5.252976×10^{0}
3	1.417083×10^{31}	6.406425×10^{0}	5.713100×10^{0}	5.468581×10^{0}
5	9.215566×10^{8}	9.340069×10^{0}	6.743809×10^{0}	5.791667×10^{0}

A Special Case : Now if we take a particular case $\beta = 1/2$, i.e. then r^{th} raw moment will be :

$$\mu_r' = E\left(X^r\right) = \frac{1}{1+2r\alpha^2}, \quad r = 1, 2, \ldots \tag{16}$$

Now for this parcticulat case the mean and variance are given by

$$\mu = E\left(X\right) = \frac{1}{1+2\alpha^2}$$

$$\sigma^2 = E\left(X^2\right) - \left(E\left(X\right)\right)^2 = \frac{4\alpha^4}{(1+2\alpha^2)^2(4\alpha^2+1)}$$

Now by putiing $\alpha = 1/\sqrt{2}$ in the last expressions, i.e., the standard uniform distribution, we obtain

$$\mu = \frac{1}{2}, \quad \sigma^2 = \frac{1}{12}, \quad \text{skewness } = 0 \text{ and kurtosis } = \frac{9}{5}$$

4.4 Order Statistic

Let $x_1, x_2, \ldots\ldots., x_n$ be a n sized random sample from UPR distribution.
and $x_{(1:n)} \leq x_{(2:n)} \leq \ldots\ldots\ldots x_{(n:n)}$ be ordered Sample value
Let $U = X_{j:n}$, Then the probability density function of order statistics is given by

$$f_U(u) = \frac{n!}{(j-1)!(n-j)!} \times F^{j-1}(u) \times \{1 - F(u)\}^{n-j} \times f(u)$$

$$= \frac{[e^{-\frac{(-\ln u)^{2\beta}}{2\alpha^2}}]^j \left(1 - e^{-\frac{(-\ln u)^{2\beta}}{2\alpha^2}}\right)^{n-j} (-\ln u)^{2\beta}(n!).}{u\alpha^2(j-1)!(n-j)!}$$

the cumulative distribution function of the order statistic is given by

$$F_U(u) = \sum_{m=j}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) F^m(u)\{1 - F(u)\}^{n-m}$$

$$F_U(u) = \sum_{m=j}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) \exp\left[\frac{-m(-\ln x)^{2\beta}}{2\alpha^2}\right] \{1 - \exp\left[\frac{-(-\ln x)^{2\beta}}{2\alpha^2}\right]\}^{n-m}$$

5 Methods of Estimations

5.1 Maximum likelihood estimation

Let $x_1, x_2, \ldots\ldots., x_n$ be a n sized random sample from UPR distribution, the sample likelihood function can be defined as:

$$
\begin{aligned}
L(\alpha, \beta, x) &= \prod_{i=1}^{n} f\left(x_{i,\alpha,\beta}\right) \\
&= \left(\frac{\beta}{\alpha^2}\right)^n \prod_{i=1}^{n} \frac{(-\ln x_i)^{2\beta-1} e^{-\sum_{i=1}^{n} \frac{(-\ln x_i)^{2\beta}}{2\alpha^2}}}{(x_i)}
\end{aligned}
$$

with respective sample log-likelihood function

$$
l(\alpha, \beta, x) = n\ln(\beta) - n\ln\left(\alpha^2\right) - \sum_{i=1}^{n} \ln x_i + 2\beta \sum_{i=1}^{n} \ln\left(-\ln x_i\right) - \sum_{i=1}^{n} \ln x_i - \frac{\sum_{i=1}^{n}(-\ln x_i)^{2\beta}}{2\alpha^2}
$$

The maximum likelihood estimators $\hat{\beta}$ and $\hat{\alpha}$ are obtained by solving the following equations

$$
\frac{\partial \ell}{\partial \alpha} = \frac{-2n}{\alpha} + \frac{1}{\alpha^3} \sum_{i=1}^{n} \left(-\ln x_i\right)^{2\beta} = 0 \tag{17}
$$

and

$$
\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} + 2\sum_{i=1}^{n} \ln\left(-\ln x_i\right) - \frac{1}{\alpha^2} \sum_{i=1}^{n} \left(-\ln x_i\right)^{2\beta} \ln\left(-\ln x_i\right) = 0 \tag{18}
$$

From Eqs (17) we will get

$$
\hat{\alpha} = \sqrt{\frac{\sum_{i=1}^{n}(-\ln x_i)^{2\beta}}{2n}} \tag{19}
$$

By substituding $\hat{\alpha}$ in Eq(18), we will get

$$
g(\beta) = \frac{n}{\beta} + 2\sum_{i=1}^{n} \ln\left(-\ln x_i\right) - \frac{2n \sum_{i=1}^{n} \left(-\ln x_i\right)^{2\beta} \ln\left(-\log x_i\right)}{\sum_{i=1}^{n} \left(-\ln x_i\right)^{2\beta}} = 0 \tag{20}
$$

To find the solution to Equation (20), numerical methods like Brent's method [9]can be used. This method is available in the software R through the uniroot function. A benefit of this method is that it doesn't need the derivative $g'(\beta)$ to be calculated, and you can just input a range as an initial estimate for β.

6 Simulation Study

In this section, we conduct Monte Carlo simulations to analyze the finite-sample performance of the maximum likelihood estimators (MLEs) for the parameters of the unit power Rayleigh distribution. The evaluation focuses on estimated bias and root mean-squared error (RMSE). We select sample sizes of $n = 10, 20, 50, 100, 200$, and 500, with parameter values of $\alpha = 0.5, 1, 1.5, 2, 3$, and 5, and $\beta = 1, 2, 3$, and 5. Each unique combination of n, α, and β is assessed.

Table 5: Monte Carlo Estimates with Bias and RMSE for beta=1

n	α	$\hat{\alpha}$	$\hat{\beta}$	Abs Bias α	Abs Bias β	RMSE α	RMSE β
10	0.5	0.4763769	1.165198	0.0236231092	0.165198231	0.09634409	0.38324818
20	0.5	0.4893732	1.073777	0.0106267993	0.073777158	0.06165280	0.21720533
50	0.5	0.4954950	1.028205	0.0045050450	0.028205259	0.03743591	0.12183537
100	0.5	0.4980383	1.014272	0.0019617261	0.014271911	0.02617530	0.08245738
200	0.5	0.4987550	1.007398	0.0012449949	0.007398230	0.01799041	0.05611113
500	0.5	0.4995639	1.002350	0.0004361017	0.002350291	0.01157338	0.03518625
10	1.0	1.0868813	1.166726	0.0868813453	0.166725587	0.33657066	0.38878415
20	1.0	1.0414541	1.081367	0.0414540894	0.081366545	0.18324345	0.22181563
50	1.0	1.0142337	1.030000	0.0142337485	0.029999793	0.10233878	0.12242837
100	1.0	1.0067796	1.014257	0.0067796250	0.014257388	0.07026466	0.08224894
200	1.0	1.0021467	1.006269	0.0021467445	0.006269355	0.04716611	0.05679667
500	1.0	1.0010870	1.002462	0.0010870377	0.002461962	0.03004702	0.03518812
10	1.5	1.8218972	1.170019	0.3218972398	0.170018685	0.99488641	0.39264551
20	1.5	1.6190338	1.073532	0.1190337644	0.073532300	0.40923733	0.21528730
50	1.5	1.5442121	1.029034	0.0442120891	0.029034377	0.21886591	0.12160406
100	1.5	1.5187280	1.013014	0.0187280265	0.013013714	0.14250628	0.08126383
200	1.5	1.5089062	1.006233	0.0089062033	0.006233390	0.09869942	0.05649143
500	1.5	1.5035602	1.002634	0.0035602149	0.002633750	0.06143140	0.03511613
10	2.0	2.7330681	1.171403	0.7330680718	0.171403474	2.80480304	0.39880040
20	2.0	2.2530427	1.077139	0.2530427015	0.077139091	0.77343548	0.21989426
50	2.0	2.0878458	1.029397	0.0878458142	0.029397092	0.36793019	0.12204709
100	2.0	2.0417033	1.013963	0.0417032723	0.013962909	0.23822919	0.08161315
200	2.0	2.0206202	1.006802	0.0206201537	0.006801829	0.16009933	0.05628261
500	2.0	2.0092022	1.003219	0.0092021970	0.003218969	0.10132049	0.03541943
10	3.0	4.8751115	1.173935	1.8751114731	0.173935141	7.79436629	0.39399986
20	3.0	3.5496066	1.072286	0.5496066039	0.072286099	1.65708491	0.21739539
50	3.0	3.2027283	1.030670	0.2027282562	0.030670093	0.72800909	0.12147796
100	3.0	3.0934063	1.014195	0.0934063225	0.014194696	0.46237386	0.08125563
200	3.0	3.0421618	1.006398	0.0421618161	0.006398325	0.30978638	0.05705657
500	3.0	3.0184225	1.002906	0.0184224859	0.002905928	0.18945882	0.03507048

Table 6: Monte Carlo Estimates with Bias and RMSE for beta=2

n	α	$\hat{\alpha}$	$\hat{\beta}$	Abs Bias α	Abs Bias β	RMSE α	RMSE β
10	0.5	0.4764037	2.344575	0.0235963381	0.344575425	0.09714432	0.79014824
20	0.5	0.4888485	2.153561	0.0111515356	0.153560809	0.06188387	0.44273041
50	0.5	0.4965260	2.058503	0.0034739501	0.058502547	0.03789937	0.24188489
100	0.5	0.4981626	2.026168	0.0018374089	0.026168129	0.02599258	0.16273894
200	0.5	0.4989861	2.013496	0.0010138968	0.013495984	0.01846774	0.11318542
500	0.5	0.4995845	2.005519	0.0004155054	0.005519473	0.01147778	0.06987089
10	1.0	1.0856839	2.341758	0.0856839348	0.341757543	0.33032460	0.77665707
20	1.0	1.0377867	2.149834	0.0377867266	0.149833802	0.18002722	0.43547931
50	1.0	1.0142966	2.057327	0.0142966427	0.057327240	0.10090622	0.24067530
100	1.0	1.0070772	2.030120	0.0070771670	0.030119660	0.06878966	0.16393798
200	1.0	1.0029038	2.013041	0.0029038215	0.013041320	0.04733173	0.11246290
500	1.0	1.0015116	2.005709	0.0015116151	0.005709086	0.02953334	0.07086860
10	1.5	1.8438375	2.344055	0.3438374945	0.344054540	1.30435764	0.78652643
20	1.5	1.6272136	2.154289	0.1272135739	0.154288660	0.43159837	0.44309568
50	1.5	1.5451258	2.059421	0.0451257665	0.059420977	0.21549518	0.24130601
100	1.5	1.5214611	2.029337	0.0214610823	0.029337148	0.14350071	0.16363799
200	1.5	1.5106884	2.015413	0.0106883747	0.015412777	0.10016438	0.11447224
500	1.5	1.5035354	2.005554	0.0035353520	0.005553771	0.06006882	0.07030883
10	2.0	2.8563143	2.348121	0.8563143050	0.348120869	15.29731201	0.78131232
20	2.0	2.2436254	2.152323	0.2436253647	0.152322623	0.76301160	0.43537381
50	2.0	2.0883185	2.057731	0.0883185486	0.057730681	0.36318059	0.24052199
100	2.0	2.0395129	2.028029	0.0395129230	0.028028672	0.23955788	0.16443319
200	2.0	2.0195818	2.013231	0.0195817525	0.013231251	0.16151816	0.11347241
500	2.0	2.0080818	2.005007	0.0080817727	0.005007199	0.09977664	0.07028846
10	3.0	4.7722135	2.339845	1.7722135285	0.339844886	7.90572399	0.77634069
20	3.0	3.5516162	2.148350	0.5516162283	0.148349783	1.63204845	0.43241582
50	3.0	3.1903812	2.056129	0.1903811548	0.056129066	0.72936370	0.24194993
100	3.0	3.0867131	2.026439	0.0867131434	0.026439482	0.45819344	0.16266209
200	3.0	3.0503639	2.015851	0.0503639217	0.015851275	0.31158098	0.11285685
500	3.0	3.0130214	2.003801	0.0130213790	0.003800761	0.18715128	0.07002978

Table 7: Monte Carlo Estimates with Bias and RMSE for beta=3

n	α	$\hat{\alpha}$	$\hat{\beta}$	Abs Bias α	Abs Bias β	RMSE α	RMSE β
10	0.5	0.4745465	3.505656	0.02545349	0.505655777	0.09643349	1.1686396
20	0.5	0.4894497	3.226144	0.01055033	0.226143843	0.06196042	0.6610154
50	0.5	0.4957416	3.085018	0.004258372	0.085017874	0.03809810	0.3644502
100	0.5	0.4978458	3.042514	0.002154154	0.042513984	0.02595963	0.2461202
200	0.5	0.4987311	3.019702	0.001268874	0.019701575	0.01843803	0.1698667
500	0.5	0.4994709	3.009679	0.0005290721	0.009678968	0.01132775	0.1053262
10	1.0	1.0872573	3.503338	0.08725728	0.503338488	0.3376315	1.1465730
20	1.0	1.0360162	3.224513	0.03601617	0.224513086	0.1798812	0.6616435
50	1.0	1.0133995	3.088567	0.01339946	0.088567392	0.1012554	0.3650295
100	1.0	1.0054202	3.039378	0.00542020	0.039378042	0.0682824	0.2427601
200	1.0	1.0029197	3.019217	0.00291970	0.019217354	0.0474908	0.1677287
500	1.0	1.0010743	3.008787	0.00107432	0.008787395	0.0298594	0.1044638
10	1.5	1.8222767	3.496615	0.3222767	0.496615343	1.135672	1.1638813
20	1.5	1.6307178	3.237180	0.1307178	0.237180107	0.4351928	0.6639202
50	1.5	1.5413086	3.083739	0.0413086	0.083739399	0.2159243	0.3667821
100	1.5	1.5212129	3.041233	0.0212129	0.041233409	0.1426009	0.2447855
200	1.5	1.5115247	3.023144	0.0115247	0.023143989	0.1002920	0.1724435
500	1.5	1.5030622	3.005723	0.0030622	0.005723186	0.0605265	0.1040874
10	2.0	2.7257433	3.518569	0.7257433	0.518568500	2.733309	1.1995648
20	2.0	2.2510778	3.227811	0.2510778	0.227811215	0.7719501	0.6625655
50	2.0	2.0804376	3.079171	0.0804376	0.079171098	0.3639228	0.3582555
100	2.0	2.0444247	3.043965	0.0444247	0.043965457	0.2395050	0.2466384
200	2.0	2.0211098	3.020215	0.0211098	0.020215427	0.1626288	0.1693891
500	2.0	2.0059278	3.005727	0.0059278	0.005727201	0.09876269	0.1049561
10	3.0	5.0556178	3.522607	2.0556178	0.522606978	16.12208	1.1802629
20	3.0	3.5522602	3.224347	0.5522602	0.224346685	1.617582	0.6457136
50	3.0	3.1886831	3.084478	0.1886831	0.084477970	0.7260160	0.3644003
100	3.0	3.1010426	3.045761	0.1010426	0.045760793	0.4690498	0.2488823
200	3.0	3.0466407	3.022144	0.0466407	0.022144278	0.3125994	0.1715814
500	3.0	3.0153707	3.006367	0.0153707	0.006366941	0.1911748	0.1059466

The numerical experiments (results in the Table 5,Table 6 and Table 7) yield some clear insights. While the biases of $\widehat{\alpha}$ and $\widehat{\beta}$ tend to zero as the sample size increases, both parameters are positively biased. Additionally, the RMSE of both parameters diminishes with larger sample sizes. Interestingly, as the value of α rises, the corresponding bias increases, and as the value of β increase the bias of β also increase. This indicates that estimating is more accurate for small true values.

7 Application

This section introduces two examples that illustrate the suitability of the proposed UPR distribution. The first example draws from [15], containing 20 measurements of maximum flood levels (in millions of cubic feet per second) recorded at the Susquehanna River in Harrisburg, Pennsylvania. The second example uses data from [?], including 48 observations from 12 core samples taken from petroleum reservoirs, divided into four cross-sections. This data set can be accessed under the name "rock" in [16]. Details of these data sets can be found in Table 8

The proposed two-parameter UPR distribution is evaluated in comparison to other two-parameter distributions that lie within the unit interval (0,1).

The proposed two-parameter UPR distribution is compared with the following two-parameter distributions within the unit interval (0,1):

1. Beta distribution:

$$f(y;\alpha,\beta)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}(1-y)^{\beta-1},\quad \alpha,\beta>0.$$

2. Kumaraswamy distribution:

$$f(y;\alpha,\beta)=\alpha\beta y^{\alpha-1}\left(1-y^{\alpha}\right)^{\beta-1},\quad \alpha,\beta>0. \tag{21}$$

3. Johnson S_B distribution:

$$f(y;\alpha,\beta)=\frac{\beta}{\sqrt{2\pi}}\frac{1}{y(1-y)}\exp\left\{-\frac{1}{2}\left[\alpha+\beta\log\left(\frac{y}{1-y}\right)\right]^2\right\},\quad \alpha\in\mathbb{R},\beta>0. \tag{22}$$

4. Unit-Logistic distribution:

$$f(y;\alpha,\beta)=\frac{\beta e^{\alpha}y^{\beta-1}(1-y)^{\beta-1}}{\left[y^{\beta}e^{\alpha}+(1-y)^{\beta}\right]^2},\quad \alpha\in\mathbb{R},\beta>0. \tag{23}$$

5. Simplex distribution:

$$f(y;\alpha,\beta)=\left[2\pi\beta^2\{y(1-y)\}^3\right]^{-\frac{1}{2}}\exp\left\{-\frac{1}{2\beta^2}\left[\frac{(y-\alpha)^2}{y(1-y)\alpha^2(1-\alpha)^2}\right]\right\},\quad \alpha\in(0,1),\beta>0. \tag{24}$$

6.Unit-Gamma distribution:

$$f(y;\alpha,\beta)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}y^{\beta-1}(-\log y)^{\alpha-1},\quad \alpha,\beta>0. \tag{25}$$

7. Extended Arcsine distribution:

$$f(y;\alpha,\beta)=\frac{\alpha\beta}{\pi\sqrt{y-y^2}}\left[1-\frac{2}{\pi}\arcsin(\sqrt{y})\right]^{\alpha-1}\left\{1-\left[1-\frac{2}{\pi}\arcsin(\sqrt{y})\right]^{\alpha}\right\}^{\beta-1},\quad \alpha,\beta>0. \tag{26}$$

8. Exponentiated Topp-Leone distribution:

$$f(y;\alpha,\beta)=2\alpha\beta(1-y)[y(2-y)]^{\alpha-1}\left[1-y^{\alpha}(2-y)^{\alpha}\right]^{\beta-1},\quad \alpha,\beta>0. \tag{27}$$

Table 8: Data of Flood level and Petroleum reservoirs.

Data Set I
$0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40, 0.41, 0.42, 0.42, 0.42, 0.45, 0.48, 0.49, 0.61, 0.65, 0.74$
Data Set I
$0.09, 0.11, 0.12, 0.12, 0.13, 0.14, 0.15, 0.15, 0.15, 0.15, 0.15, 0.16, 0.16, 0.16, 0.16, 0.17, 0.17, 0.18, 0.18, 0.18, 0.18, 0.19, 0.19, 0.20, 0.20, 0.20, 0.20, 0.20, 0.20, 0.23, 0.23, 0.23, 0.23, 0.24, 0.25, 0.26, 0.26, 0.28, 0.28, 0.28, 0.29, 0.31, 0.33, 0.33, 0.34, 0.42, 0.44, 0.46$

Table 9 contains the maximum likelihood estimates (MLEs) for Data Sets 1 and 2. We assess the UPR distribution's performance against eight other distributions using the Kolmogorov-Smirnov (KS) goodness-of-fit test. Furthermore, to compare the UPR distribution with these alternatives, we rely on likelihood-based metrics, including Akaike's Information Criterion (AIC) and the Bayesian Information Criterion (BIC). Table 10 presents a detailed comparison of the results for Data Sets 1 and 2.

After carefully analyzing Table 10, it becomes evident that the UPR distribution surpasses the other competing distributions for both data sets due to its lowest AIC and BIC values.

Table 9: Maximum Likelihood Estimates for Different Distributions

Distribution	Data Set I		Data Set II	
	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\alpha}$	$\widehat{\beta}$
UPR	0.6984781	1.951801	2.596426	2.423518
1	6.6989	9.327	5.9521	21.2169
2	3.3594	11.8001	2.7205	44.6401
3	0.6214	1.9215	2.2156	2.1498
4	1.3602	3.6001	5.2272	3.7998
5	0.4289	1.1012	0.2212	1.1645
6	8.7298	9.7186	17.9498	11.3098
7	9.1629	141.9531	14.1752	101.1459
8	4.6073	4.0439	3.1361	13.6399

Table 10: KS Test Measures (p-values), AIC, and BIC

Distribution	Data Set I			Data Set II		
	KS	AIC	BIC	KS	AIC	BIC
UPR	0.1448(0.7958)	-57.72988	-55.73842	0.12929(0.3987)	-306.7337	-302.9096
1	0.2004(0.4140)	-24.1269	-22.1385	0.1492(0.2845)	-107.2040	-103.4601
2	0.2159(0.3401)	-21.7388	-19.7456	0.1600(0.2129)	-100.9895	-97.2422
3	0.1983(0.4471)	-24.5311	-22.5252	0.1211(0.4295)	-109.9772	-106.2299
4	0.1448(0.8290)	-25.4790	-23.4733	0.0984(0.7520)	-109.9111	-106.1598
5	0.2130(0.3503)	-24.3150	-22.3192	0.1302(0.3914)	-110.1189	-106.3735
6	0.2020(0.4225)	-24.3715	-22.3798	0.1384(0.3281)	-108.2233	-104.4745
7	0.1572(0.7331)	-27.8388	-25.8452	0.1159(0.5702)	-111.9354	-108.1994
8	0.2091(0.3658)	-23.1820	-21.1907	0.1597(0.2175)	-102.7160	-98.9663

8 Conclusion

This Project introduces a new two-parameter distribution, named the UPR distribution, with support on the interval (0, 1). The study provides a detailed analysis of this distribution. The paper derives the maximum likelihood estimators for its parameters and their standard errors. Additionally, a strategy is proposed for initial estimation by linearizing the cumulative distribution function of the UPR. Generating random samples for this distribution is straightforward by transforming samples from the Weibull distribution. A simulation study was conducted to assess the bias and root mean squared error of the maximum likelihood estimators. Applications of the UPR distribution to two real datasets demonstrated a better fit than various well-known two-parameter distributions supported on (0, 1), such as Beta, Kumaraswamy, Johnson SB, unit-Logistic, Simplex, unit-Gamma, extended Arcsine, and exponentiated Topp-Leone distributions. Since the UPR distribution is highly flexible and has a closed-form expression for its quantiles, it is being explored for use in quantile regression models in various applications.

Future Reaserch: I plan to explore additional properties of the UPR distribution, such as interval estimation and coverage probability. I will also conduct further goodness-of-fit tests, including the Anderson-Darling (AD) and Cramér-von Mises (CvM) statistics, and compare the UPR distribution with others by plotting P-P plots. To distinguish the UPR distribution from each competing distribution, I will employ the Vuong likelihood ratio test for non-nested distributions.

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