Mathematical Exploration of the Consecutive Sums Binomial Sequence or P(n) Sequence in Pascal's Triangle

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Abstract

This paper presents a detailed exploration of what I call the "Consecutive Sums Binomial Sequence" or P(n) sequence derived from Pascal's Triangle. The sequence is characterized by the property that each of its terms, when expressed as a product of specific binomial coefficients forming a hexagonal pattern around elements in Pascal's Triangle, results in a square number. This paper aims to define the sequence formally, elucidate its properties, and provide a mathematical proof of its characteristics.

1 Introduction

Pascal's Triangle is a well-known mathematical construct that finds applications in combinatorics, probability, and algebra. Within this triangle, interesting patterns and sequences emerge, one of which is the Consecutive Sums Binomial Sequence. This sequence is formed by calculating products of binomial coefficients arranged in a hexagonal pattern around certain terms in Pascal's Triangle.

2 Sequence Definition and Binomial Coefficients

2.1 Binomial Coefficients

In combinatorial mathematics, a binomial coefficient $\binom{n}{k}$ counts the number of ways to choose a subset of k elements, disregarding their order, from a larger set of n elements. It is also the entry in the nth row and kth column of Pascal's Triangle and is calculated using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where $n!$ denotes the factorial of n.

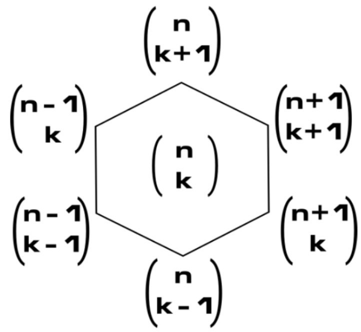

Figure 1: Hexagon Product constructed using binomial coefficients.

2.2 Hexagon Products

The formation of a hexagon in Pascal's Triangle is central to defining the Consecutive Sums Binomial Sequence. For each term n greater than 1, the hexagon is constructed using the binomial coefficients located at specific positions relative to n.

3 Consecutive Sums Binomial Sequence Formation

3.1 Sequence Construction

The Consecutive Sums Binomial Sequence is constructed by calculating the product $P(n)$ for each term n starting from $n = 2$. For $n = 1$, the sequence term is defined as 0, as it is not possible to form a hexagon around the first element in Pascal's Triangle.

3.2 Special Case for $N(1)$

The first term of the sequence $N(1)$ is defined as 0. This is because at the very beginning of Pascal's Triangle, there aren't enough surrounding elements to form a hexagonal pattern. This sets the foundation for the sequence with the hexagonal pattern becoming applicable from $N(2)$ onward.

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4 Mathematical Proof of Sequence Properties

4.1 Detailed Calculation for Specific Terms

We demonstrate the methodology using $N(3)$ as an example. This requires calculating the product of the binomial coefficients that form the hexagon around the 3rd element in Pascal's Triangle.

4.1.1 Proof for $n = 3$

- The hexagon involves the following binomial coefficients: - $\binom{3}{4} = 1$ (since $\binom{n}{n} = 1$) - $\binom{4}{2} = 6$ - $\binom{2}{1} = 2$ - $\binom{2}{2} = 1$ - $\binom{4}{3} = 4$ - $\binom{3}{2} = 3$ - Multiplying these we get P(3) $= 1 \times 6 \times 2 \times 1 \times 4 \times 3 = 144$, a perfect square.

Simplification of $P(n)$ for General n

Given the product $P(n)$ formed by the hexagon in Pascal's Triangle:

$$P(n) = \binom{n}{k+1} \times \binom{n+1}{k} \times \binom{n-1}{k-1} \times \binom{n-1}{k} \times \binom{n+1}{k+1} \times \binom{n}{k-1}$$

where $k = n - 1$.

Substituting $k = n - 1$, the expression simplifies to:

$$P(n) = \binom{n}{n} \times \binom{n+1}{n-1} \times \binom{n-1}{n-2} \times \binom{n-1}{n-1} \times \binom{n+1}{n} \times \binom{n}{n-2}$$

This simplifies to:

$$P(n) = 1 \times \frac{(n+1)n}{2} \times (n-1) \times 1 \times (n+1) \times \frac{n(n-1)}{2}$$

Expanding and simplifying further, we get:

$$P(n) = \left(\frac{(n+1)n}{2}\right)^2 \times (n-1)^2$$

4.2 Parity and Square Nature of $P(n)$

Given the established formula for $P(n)$:

$$P(n) = \left(\frac{(n+1)n}{2}\right)^2 \times (n-1)^2$$

We examine its parity — whether $P(n)$ is odd or even — and confirm its inherent nature as a square number.

4.2.1 Parity of $P(n)$

The parity of a number indicates its divisibility by 2. An even number is divisible by 2 without a remainder, while an odd number when divided by 2 yields a remainder of 1. For $P(n)$, the parity is contingent on the value of n:

- If n is even, $\frac{(n+1)n}{2}$ results from multiplying an odd with an even number and dividing by 2, which could be odd or even. Consequently, $P(n)$'s parity hinges on whether $\frac{n}{2}$ is odd or even. If $\frac{n}{2}$ is odd, then $P(n)$ is odd; if it is even, then $P(n)$ is even.
- If n is odd, $n-1$ is even, making the term $\frac{(n+1)n}{2}$ invariably even, which in turn renders $P(n)$ even.

4.2.2 Square Nature of $P(n)$

A square number is one that can be expressed as the square of an integer, and its square root is also an integer. Given that $P(n)$ is the product of two squared integers, it is inherently a square number. Thus, for any integer n, $P(n)$ will always have an integer square root, signifying that $P(n)$ is invariably a perfect square.

4.3 Consecutive Sums Binomial Sequence $N(n)$

The sequence illustrates the properties of the Consecutive Sums Binomial Sequence pattern in Pascal's Triangle, where each term $N(n)$ is the product of specific binomial coefficients forming a hexagon. The square roots of these terms follow a sequence that aligns with the sum of consecutive numbers, starting from the nth triangular number.

The Consecutive Sums Binomial Sequence $N(n)$ is defined as follows:

1. $N(1) = 0$
2. $N(2) = 9$
3. $N(3) = 144$
4. $N(4) = 900$
5. $N(5) = 3600$
6. $N(6) = 11025$
7. $N(7) = 28224$
8. $N(8) = 63504$
9. $N(9) = 129600$
10. $N(10) = 245025$

11. $N(11) = 435600$
12. $N(12) = 736164$
13. $N(13) = 1192464$
14. $N(14) = 1863225$
15. $N(15) = 2822400$
16. $N(16) = 4161600$
17. $N(17) = 5992704$
18. $N(18) = 8450649$
19. $N(19) = 11696400$
20. $N(20) = 15920100$
21. ...

This sequence continues indefinitely (look at appendix up to N(99)), extending to infinity.

5 The Triangle of Consecutive Sums

5.1 Formation of the Triangle

This formulation leads to a sequence where each term represents the cumulative sum of a growing series of consecutive numbers, forming a structure analogous to a triangle when visualized.

An important characteristic of the Consecutive Sums Binomial Sequence is its alignment with a triangular pattern of sums. Each term $N(n)$ in the sequence correlates with the sum of a series of consecutive numbers. This series starts just after the nth triangular number (T_n) and includes exactly n terms, extending up to and including the next triangular number (T_{n+1}).

The formula for the nth term of the Consecutive Sums Binomial Sequence, $N(n)$, can be expressed using triangular numbers. The nth triangular number T_n is given by $T_n = \frac{n(n+1)}{2}$, and the $(n+1)$th triangular number T_{n+1} is $T_{n+1} = \frac{(n+1)(n+2)}{2}$.

Therefore, the formula for $N(n)$ using these triangular numbers is:

$$N(n) = \left(\sum_{i=T_n+1}^{T_{n+1}} i \right)^2$$

This equation expresses that $N(n)$ is the square of the sum of consecutive numbers starting from the number immediately after T_n (which is $T_n + 1$) and ending at T_{n+1}.

n	$N(n)$	Triangle of Consecutive Sums
1	0	0^2
2	9	$(1+2)^2$
3	144	$(3+4+5)^2$
4	900	$(6+7+8+9)^2$
5	3600	$(10+11+12+13+14)^2$
6	11025	$(15+16+17+18+19+20)^2$
7	28224	$(21+22+23+24+25+26+27)^2$
8	63504	$(28+29+30+31+32+33+34+35)^2$
9	129600	$(36+37+38+39+40+41+42+43+44)^2$
$\vdots$	$\vdots$	$\vdots$

Table 1: The Consecutive Sums Binomial Sequence and its corresponding triangular sums

n	$\sqrt{N(n)}$	Triangle of Consecutive Sums
1	0	0
2	3	1 + 2
3	12	3 + 4 + 5
4	30	6 + 7 + 8 + 9
5	60	10 + 11 + 12 + 13 + 14
6	105	15 + 16 + 17 + 18 + 19 + 20
7	168	21 + 22 + 23 + 24 + 25 + 26 + 27
8	252	28 + 29 + 30 + 31 + 32 + 33 + 34 + 35
9	360	36 + 37 + 38 + 39 + 40 + 41 + 42 + 43 + 44
$\vdots$	$\vdots$	$\vdots$

Table 2: Square Roots in the Consecutive Sums Binomial Sequence and Corresponding Sums

6 Digital Root Phenomenon in the Consecutive Sums Binomial Sequence

A fascinating observation about the Consecutive Sums Binomial Sequence $N(n)$ is that the digital roots of its terms consistently add up to 9. This pattern emerges across all values of $N(n)$, providing an intriguing aspect of the sequence that transcends its combinatorial origin. The digital root of a number is the single-digit value obtained by iteratively summing the digits of the number until only a single digit remains.

6.1 Analysis of Digital Roots in $N(n)$

Let's consider the first few terms of the Consecutive Sums Binomial Sequence and their digital roots, computationally this is true up to N(99) found in the appendix:

- $N(1) = 0$ — Digital Root = 0
- $N(2) = 9$ — Digital Root = 9
- $N(3) = 144$ — Digital Root = 1 + 4 + 4 = 9
- $N(4) = 900$ — Digital Root = 9 + 0 + 0 = 9
- $N(5) = 3600$ — Digital Root = 3 + 6 + 0 + 0 = 9
- $N(6) = 11025$ —Digital Root = 1 + 1 + 0 + 2 + 5 = 9
- $N(7) = 28224$ —Digital Root = 2 + 8 + 2 + 2 + 4 = 18, then 1 + 8 = 9
- $N(8) = 63504$ —Digital Root = 6 + 3 + 5 + 0 + 4 = 18, then 1 + 8 = 9
- $N(9) = 129600$ —Digital Root = 1 + 2 + 9 + 6 + 0 + 0 = 18, then 1 + 8 = 9
- ...

6.2 Proposed Explanation for the Digital Root Phenomenon

In exploring the underlying reasons for this consistent digital root behavior in the Consecutive Sums Binomial Sequence, a key observation is made regarding the structure of the sequence and its connection to digital roots. For any integer $n \geq 2$, the digital root of the product of the binomial coefficients forming a hexagon in Pascal's Triangle, which defines $N(n)$, is consistently 9.

Theorem and Proof

Theorem: For each $n \geq 2$, the digital root of the Consecutive Sums Binomial Sequence $P(n)$ is consistently 9.

Proof:

Let the Consecutive Sums Binomial Sequence be defined by the product of binomial coefficients:

$$P(n) = \binom{n}{n} \times \binom{n+1}{n-1} \times \binom{n-1}{n-2} \times \binom{n-1}{n-1} \times \binom{n+1}{n} \times \binom{n}{n-2}$$

To analyze the divisibility by 3 of each binomial coefficient in $P(n)$, we apply Kummer's theorem. Kummer's theorem states that the highest power of a prime p (here $p = 3$) dividing a binomial coefficient $\binom{n}{k}$ is equal to the number of carries when adding k and $n - k$ in base p.

For each binomial coefficient in $P(n)$, perform the following base 3 analysis:

- $\binom{n+1}{n-1}$: Convert $n - 1$ and 2 to base 3, add them, and count the carries.

- $\binom{n-1}{n-2}$: Convert $n-2$ and 1 to base 3, add them, and count the carries.
- $\binom{n+1}{n}$: Convert n and 1 to base 3, add them, and count the carries.
- $\binom{n}{n-2}$: Convert $n-2$ and 2 to base 3, add them, and count the carries.

Example Analysis for $P(5)$:
In the case of $P(5)$, we have a product of several binomial coefficients:

$$P(5) = \binom{5}{5} \times \binom{6}{4} \times \binom{4}{3} \times \binom{4}{4} \times \binom{6}{5} \times \binom{5}{3}$$

- $\binom{5}{5} = 1$ and $\binom{4}{4} = 1$: These coefficients are 1, so no carries are involved.
- $\binom{6}{4}$: Convert 4 (11 in base 3) and 2 (02 in base 3) and add: $11_{(3)} + 02_{(3)} = 13_{(3)}$ (1 carry).
- $\binom{4}{3}$: Convert 3 (10 in base 3) and 1 (01 in base 3) and add: $10_{(3)} + 01_{(3)} = 11_{(3)}$ (no carry).
- $\binom{6}{5}$: Convert 5 (12 in base 3) and 1 (01 in base 3) and add: $12_{(3)} + 01_{(3)} = 13_{(3)}$ (1 carry).
- $\binom{5}{3}$: Convert 3 (10 in base 3) and 2 (02 in base 3) and add: $10_{(3)} + 02_{(3)} = 12_{(3)}$ (no carry).

The total number of carries for $P(5)$ is 2, suggesting that $P(5)$ is divisible by 3^2 or 9.

Generalization for $n \geq 2$:
For any $n \geq 2$, a similar base 3 analysis can be performed. Given that in the sequence of $n, n-1$, and $n+1$, at least one number is divisible by 3, their squares (which appear in the binomial coefficients) will be divisible by 9.

This, combined with the fact that we are multiplying several such terms, increases the chances that the total product will have multiple carries in base 3 addition, thus being divisible by 9. Hence, the digital root of $P(n)$ is consistently 9 for each $n \geq 2$.

7 Conclusion

This study has looked at the Consecutive Sums Binomial Sequence within Pascal's Triangle, a sequence characterized by square numbers formed from the product of binomial coefficients in a hexagonal pattern. This discovery highlights not only the combinatorial richness of Pascal's Triangle but also its deep connections to number theory. The sequence's alignment with triangular sums and the consistent digital root phenomenon further accentuate its mathematical significance. These findings offer a novel perspective on Pascal's Triangle, revealing intricate patterns and suggesting potential areas for further research in mathematical theory and its applications.

A Digital Roots of the Consecutive Sums Binomial Sequence from $N(1)$ to $N(99)$

Below is the list of terms $N(n)$ in the Consecutive Sums Binomial Sequence from $N(1)$ to $N(99)$, along with their corresponding digital roots. These calculations demonstrate the consistent pattern where the digital roots add up to 9.

n	$N(n)$	Digital Root of $N(n)$
1	0	0
2	9	9
3	144	9
4	900	9
5	3600	9
6	11025	9
7	28224	9
8	63504	9
9	129600	9
10	245025	9
11	435600	9
12	736164	9
13	1192464	9
14	1863225	9
15	2822400	9
16	4161600	9
17	5992704	9
18	8450649	9
19	11696400	9
20	15920100	9
21	21344400	9
22	28227969	9
23	36869184	9
24	47610000	9
25	60840000	9
26	77000625	9
27	96589584	9
28	120165444	9
29	148352400	9
30	181845225	9
31	221414400	9
32	267911424	9
33	322274304	9
34	385533225	9
35	458816400	9
36	543356100	9
37	640494864	9
38	751691889	9
39	878529600	9
40	1022720400	9
41	1186113600	9
42	1370702529	9
43	1578631824	9
44	1812204900	9
45	2073891600	9
46	2366336025	9
47	2692364544	9
48	3054993984	9
49	3457440000	9
50	3903125625	9

Table 3: Consecutive Sums Binomial Sequence and it's Digital Root $N(1)$ to $N(50)$

n	$N(n)$	Digital Root of $N(n)$
51	4395690000	9
52	4938997284	9
53	5537145744	9
54	6194477025	9
55	6915585600	9
56	7705328400	9
57	8568834624	9
58	9511515729	9
59	10539075600	9
60	11657520900	9
61	12873171600	9
62	14192671689	9
63	15623000064	9
64	17171481600	9
65	18845798400	9
66	20654001225	9
67	22604521104	9
68	24706181124	9
69	26968208400	9
70	29400246225	9
71	32012366400	9
72	34815081744	9
73	37819358784	9
74	41036630625	9
75	44478810000	9
76	48158302500	9
77	52088019984	9
78	56281394169	9
79	60752390400	9
80	65515521600	9
81	70585862400	9
82	75979063449	9
83	81711365904	9
84	87799616100	9
85	94261280400	9
86	101114460225	9
87	108377907264	9
88	116071038864	9
89	124213953600	9
90	132827447025	9
91	141933027600	9
92	151552932804	9
93	161710145424	9
94	172428410025	9
95	183732249600	9
96	195646982400	9
97	208198738944	9
98	221414479209	9
99	235322010000	9

Table 4: Consecutive Sums Binomial Sequence and it's Digital Root $N(51)$ to $N(99)$