Mathematical Exploration of the Honeycomb Sequence in Pascal's Triangle

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Abstract

This paper presents a detailed exploration of a sequence termed the "Honeycomb Sequence," derived from Pascal's Triangle. The sequence is characterized by the property that each of its terms, when expressed as a product of specific binomial coefficients forming a hexagonal pattern around elements in Pascal's Triangle, results in a square number. This paper aims to define the sequence formally, elucidate its properties, and provide a mathematical proof of its characteristics.

1 Introduction

Pascal's Triangle is a well-known mathematical construct that finds applications in combinatorics, probability, and algebra. Within this triangle, interesting patterns and sequences emerge, one of which is the Honeycomb Sequence. This sequence is formed by calculating products of binomial coefficients arranged in a hexagonal pattern around certain terms in Pascal's Triangle.

Pascal's Triangle, named after the French mathematician Blaise Pascal, is a triangular array of numbers with applications spanning various fields of mathematics, including combinatorics, algebra, and probability theory. Each entry in Pascal's Triangle is a binomial coefficient, representing the number of ways to choose a subset of elements from a larger set. The triangle is constructed such that each number is the sum of the two numbers directly above it.

The fascinating aspect of Pascal's Triangle lies in the numerous patterns and sequences it contains, many of which have significant mathematical implications. The Honeycomb Sequence, derived from this triangle, is one such sequence that showcases the intricate patterns hidden within Pascal's Triangle. It is formed by identifying hexagonal patterns of binomial coefficients around certain elements in the triangle.

2 Sequence Definition and Binomial Coefficients

2.1 Binomial Coefficients

In combinatorial mathematics, a binomial coefficient $\binom{n}{k}$ counts the number of ways to choose a subset of k elements, disregarding their order, from a larger set of n elements. It is also the entry in the nth row and kth column of Pascal's Triangle and is calculated using the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where $n!$ denotes the factorial of n.

2.2 Hexagon Products

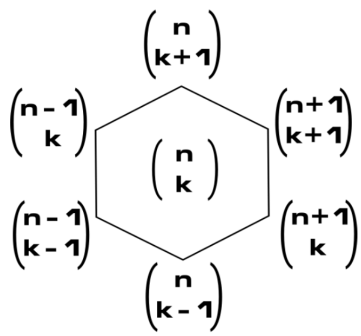

Figure 1: Hexagon Product constructed using binomial coefficients.

The formation of a hexagon in Pascal's Triangle is central to defining the Honeycomb Sequence. For each term n greater than 1, the hexagon is constructed using the binomial coefficients located at specific positions relative to n. These positions are:

- $\binom{n}{k+1}$: The coefficient in the nth row and $(k+1)$th column.
- $\binom{n+1}{k}$: The coefficient directly below $\binom{n}{k+1}$.
- $\binom{n-1}{k-1}$: The coefficient directly above $\binom{n}{k+1}$ and to the left.
- $\binom{n-1}{k}$: The coefficient to the left of $\binom{n}{k+1}$.
- $\binom{n+1}{k+1}$: The coefficient to the right of $\binom{n}{k+1}$.
- $\binom{n}{k-1}$: The coefficient directly above $\binom{n}{k+1}$ and to the right.

These coefficients are multiplied to give the product $P(n)$, defining the sequence term $N(n)$.

3 Honeycomb Sequence Formation

3.1 Sequence Construction

The Honeycomb Sequence is constructed by calculating the product $P(n)$ for each term n starting from $n = 2$. For $n = 1$, the sequence term is defined as

0, as it is not possible to form a hexagon around the first element in Pascal's Triangle.

3.2 Product Calculation

For each term n, the product of the binomial coefficients forming the hexagon is calculated. This involves determining the values of these coefficients from Pascal's Triangle and multiplying them together.

3.3 Special Case for $N(1)$

The first term of the sequence $N(1)$ is defined as 0. This is because at the very beginning of Pascal's Triangle, there aren't enough surrounding elements to form a hexagonal pattern. This sets the foundation for the sequence with the hexagonal pattern becoming applicable from $N(2)$ onward.

4 Mathematical Proof of Sequence Properties

4.1 Detailed Calculation for Specific Terms

We demonstrate the methodology using $N(3)$ as an example. This requires calculating the product of the binomial coefficients that form the hexagon around the 3rd element in Pascal's Triangle.

4.1.1 Proof for $n = 3$

- The hexagon involves the following binomial coefficients: - $\binom{3}{4} = 1$ (since $\binom{n}{n} = 1$) - $\binom{4}{2} = 6$ - $\binom{2}{1} = 2$ - $\binom{2}{2} = 1$ - $\binom{4}{3} = 4$ - $\binom{3}{2} = 3$ - Multiplying these we get P(3) $= 1 \times 6 \times 2 \times 1 \times 4 \times 3 = 144$, a perfect square.

4.2 Simplification of $P(n)$ for General n

Given the product $P(n)$ formed by the hexagon in Pascal's Triangle:

$$P(n) = \binom{n}{k+1} \times \binom{n+1}{k} \times \binom{n-1}{k-1} \times \binom{n-1}{k} \times \binom{n+1}{k+1} \times \binom{n}{k-1}$$

where $k = n - 1$. We simplify each binomial coefficient:

1. $\binom{n}{n} = 1$ (since $\binom{n}{n} = \frac{n!}{n!0!} = 1$)
2. $\binom{n+1}{n-1} = \frac{(n+1)!}{(n-1)!2!}$ (simplify this expression)
3. $\binom{n-1}{n-2} = \frac{(n-1)!}{(n-2)!1!}$
4. $\binom{n-1}{n-1} = 1$

5. $\binom{n+1}{n} = \frac{(n+1)!}{n!1!}$

6. $\binom{n}{n-1} = \frac{n!}{(n-1)!1!}$

The product $P(n)$ simplifies to:

$$P(n) = 1 \times \frac{(n+1)n}{2} \times (n-1) \times 1 \times (n+1) \times n$$

$$P(n) = \frac{n^2(n+1)^2(n-1)}{2}$$

This is the general expression for $N(n)$.

4.3 Conditions for $N(n)$ Being a Square Number

The expression $\frac{n^2(n+1)^2(n-1)}{2}$ suggests that $N(n)$ might be a square number under certain conditions:

1. For $N(n)$ to be a square number, the term $\frac{n^2(n+1)^2(n-1)}{2}$ must simplify to a perfect square.

2. The factor $\frac{(n-1)}{2}$ is critical here. If $n-1$ is even, then $\frac{(n-1)}{2}$ is an integer, and the expression can potentially be a square number.

3. If n is odd, then $n-1$ is even, making $\frac{(n-1)}{2}$ an integer. Hence, for odd values of n, $N(n)$ has a higher likelihood of being a square number.

4.4 Honeycomb Sequence $N(n)$

The sequence illustrates the properties of the honeycomb pattern in Pascal's Triangle, where each term $N(n)$ is the product of specific binomial coefficients forming a hexagon. The square roots of these terms follow a sequence that aligns with the sum of consecutive numbers, starting from the nth triangular number.

The Honeycomb Sequence $N(n)$ is defined as follows:

1. $N(1) = 0$

2. $N(2) = 9$

3. $N(3) = 144$

4. $N(4) = 900$

5. $N(5) = 3600$

6. $N(6) = 11025$

7. $N(7) = 28224$

8. $N(8) = 63504$
9. $N(9) = 129600$
10. $N(10) = 245025$
11. $N(11) = 435600$
12. $N(12) = 736164$
13. $N(13) = 1192464$
14. $N(14) = 1863225$
15. $N(15) = 2822400$
16. $N(16) = 4161600$
17. $N(17) = 5992704$
18. $N(18) = 8450649$
19. $N(19) = 11696400$
20. $N(20) = 15920100$
21. ...

This sequence continues indefinitely (look at appendix up to N(99)), extending to infinity.

5 The Triangle of Consecutive Sums

5.1 Formation of the Triangle

This formulation leads to a sequence where each term represents the cumulative sum of a growing series of consecutive numbers, forming a structure analogous to a triangle when visualized.

An important characteristic of the Honeycomb Sequence is its alignment with a triangular pattern of sums. Each term $N(n)$ in the sequence correlates with the sum of a series of consecutive numbers. This series starts just after the nth triangular number (T_n) and includes exactly n terms, extending up to and including the next triangular number (T_{n+1}).

The formula for the nth term of the Honeycomb Sequence, $N(n)$, can be expressed using triangular numbers. The nth triangular number T_n is given by $T_n = \frac{n(n+1)}{2}$, and the $(n+1)$th triangular number T_{n+1} is $T_{n+1} = \frac{(n+1)(n+2)}{2}$.

Therefore, the formula for $N(n)$ using these triangular numbers is:

$$N(n) = \left(\sum_{i=T_n+1}^{T_{n+1}} i \right)^2$$

This equation expresses that $N(n)$ is the square of the sum of consecutive numbers starting from the number immediately after T_n (which is $T_n + 1$) and ending at T_{n+1}.

n	$N(n)$	Triangle of Consecutive Sums
1	0	0^2
2	9	$(1+2)^2$
3	144	$(3+4+5)^2$
4	900	$(6+7+8+9)^2$
5	3600	$(10+11+12+13+14)^2$
6	11025	$(15+16+17+18+19+20)^2$
7	28224	$(21+22+23+24+25+26+27)^2$
8	63504	$(28+29+30+31+32+33+34+35)^2$
9	129600	$(36+37+38+39+40+41+42+43+44)^2$
$\vdots$	$\vdots$	$\vdots$

Table 1: The Honeycomb Sequence and its corresponding triangular sums

n	$\sqrt{N(n)}$	Triangle of Consecutive Numbers
1	0	0
2	3	1 + 2
3	12	3 + 4 + 5
4	30	6 + 7 + 8 + 9
5	60	10 + 11 + 12 + 13 + 14
6	105	15 + 16 + 17 + 18 + 19 + 20
7	168	21 + 22 + 23 + 24 + 25 + 26 + 27
8	252	28 + 29 + 30 + 31 + 32 + 33 + 34 + 35
9	360	36 + 37 + 38 + 39 + 40 + 41 + 42 + 43 + 44
$\vdots$	$\vdots$	$\vdots$

Table 2: Square Roots in the Honeycomb Sequence and Corresponding Sums

6 Digital Root Phenomenon in the Honeycomb Sequence

A fascinating observation about the Honeycomb Sequence $N(n)$ is that the digital roots of its terms consistently add up to 9. This pattern emerges across all values of $N(n)$, providing an intriguing aspect of the sequence that transcends its combinatorial origin. The digital root of a number is the single-digit value obtained by iteratively summing the digits of the number until only a single digit remains.

6.1 Analysis of Digital Roots in $N(n)$

Let's consider the first few terms of the Honeycomb Sequence and their digital roots, computationally this is true up to N(99) found in the appendix:

- $N(1) = 0$ — Digital Root = 0
- $N(2) = 9$ — Digital Root = 9
- $N(3) = 144$ — Digital Root = 1 + 4 + 4 = 9
- $N(4) = 900$ — Digital Root = 9 + 0 + 0 = 9
- $N(5) = 3600$ — Digital Root = 3 + 6 + 0 + 0 = 9
- $N(6) = 11025$ —Digital Root = 1 + 1 + 0 + 2 + 5 = 9
- $N(7) = 28224$ —Digital Root = 2 + 8 + 2 + 2 + 4 = 18, then 1 + 8 = 9
- $N(8) = 63504$ —Digital Root = 6 + 3 + 5 + 0 + 4 = 18, then 1 + 8 = 9
- $N(9) = 129600$ —Digital Root = 1 + 2 + 9 + 6 + 0 + 0 = 18, then 1 + 8 = 9
- ...

6.2 Proposed Explanation for the Digital Root Phenomenon

In exploring the underlying reasons for this consistent digital root behavior in the Honeycomb Sequence, a key observation is made regarding the structure of the sequence and its connection to digital roots. For any integer $n \geq 2$, the digital root of the product of the binomial coefficients forming a hexagon in Pascal's Triangle, which defines $N(n)$, is consistently 9. These coefficients are:

$$P(n) = \binom{n}{k+1} \times \binom{n+1}{k} \times \binom{n-1}{k-1} \times \binom{n-1}{k} \times \binom{n+1}{k+1} \times \binom{n}{k-1}, \quad (1)$$

with $k = n - 1$.

The digital roots of these binomial coefficients do not exhibit uniformity across all values of n; however, they follow a pattern where certain coefficients consistently have digital roots that contribute to the overall product having a digital root of 9. This phenomenon is influenced by the stabilization of factorial digital roots at 9 beyond a certain point, as binomial coefficients are ratios of factorials. The multiplicative behavior of digital roots, combined with the structured arrangement of binomial coefficients in the Honeycomb Sequence, leads to a product whose overall digital root is consistently 9 for $n \geq 2$.

6.3 Scope and Future Work

While the pattern holds consistently within the analyzed range, a comprehensive mathematical proof or theorem to conclusively explain this digital root phenomenon in the Honeycomb Sequence is beyond the scope of our current exploration. Further in-depth mathematical investigation is needed to understand its universality and underlying principles. Future research may focus on the relationship between digital roots and factorial arithmetic in combinatorial structures, potentially leading to a more profound understanding of this intriguing pattern in the Honeycomb Sequence.

7 Conclusion

The Honeycomb Sequence presents a fascinating exploration of the patterns hidden within Pascal's Triangle. By examining the products of binomial coefficients in a hexagonal pattern, this sequence reveals intricate mathematical properties, notably the formation of square numbers. The alignment of the sequence with a triangular pattern of sums further extends its significance, connecting it to broader mathematical concepts and inviting further exploration. The study of the Honeycomb Sequence highlights the beauty and complexity of Pascal's Triangle and opens doors to new mathematical inquiries and potential applications.

A Digital Roots of the Honeycomb Sequence from $N(1)$ to $N(99)$

Below is the list of terms $N(n)$ in the Honeycomb Sequence from $N(1)$ to $N(99)$, along with their corresponding digital roots. These calculations demonstrate the consistent pattern where the digital roots add up to 9.

n	$N(n)$	Digital Root of $N(n)$
1	0	0
2	9	9
3	144	9
4	900	9
5	3600	9
6	11025	9
7	28224	9
8	63504	9
9	129600	9
10	245025	9
11	435600	9
12	736164	9
13	1192464	9
14	1863225	9
15	2822400	9
16	4161600	9
17	5992704	9
18	8450649	9
19	11696400	9
20	15920100	9
21	21344400	9
22	28227969	9
23	36869184	9
24	47610000	9
25	60840000	9
26	77000625	9
27	96589584	9
28	120165444	9
29	148352400	9
30	181845225	9
31	221414400	9
32	267911424	9
33	322274304	9
34	385533225	9
35	458816400	9
36	543356100	9
37	640494864	9
38	751691889	9
39	878529600	9
40	1022720400	9
41	1186113600	9
42	1370702529	9
43	1578631824	9
44	1812204900	9
45	2073891600	9
46	2366336025	9
47	2692364544	9
48	3054993984	9
49	3457440000	9
50	3903125625	9

Table 3: Honeycomb Sequence and it's Digital Root $N(1)$ to $N(50)$

n	$N(n)$	Digital Root of $N(n)$
51	4395690000	9
52	4938997284	9
53	5537145744	9
54	6194477025	9
55	6915585600	9
56	7705328400	9
57	8568834624	9
58	9511515729	9
59	10539075600	9
60	11657520900	9
61	12873171600	9
62	14192671689	9
63	15623000064	9
64	17171481600	9
65	18845798400	9
66	20654001225	9
67	22604521104	9
68	24706181124	9
69	26968208400	9
70	29400246225	9
71	32012366400	9
72	34815081744	9
73	37819358784	9
74	41036630625	9
75	44478810000	9
76	48158302500	9
77	52088019984	9
78	56281394169	9
79	60752390400	9
80	65515521600	9
81	70585862400	9
82	75979063449	9
83	81711365904	9
84	87799616100	9
85	94261280400	9
86	101114460225	9
87	108377907264	9
88	116071038864	9
89	124213953600	9
90	132827447025	9
91	141933027600	9
92	151552932804	9
93	161710145424	9
94	172428410025	9
95	183732249600	9
96	195646982400	9
97	208198738944	9
98	221414479209	9
99	235322010000	9

Table 4: Honeycomb Sequence and it's Digital Root $N(51)$ to $N(99)$