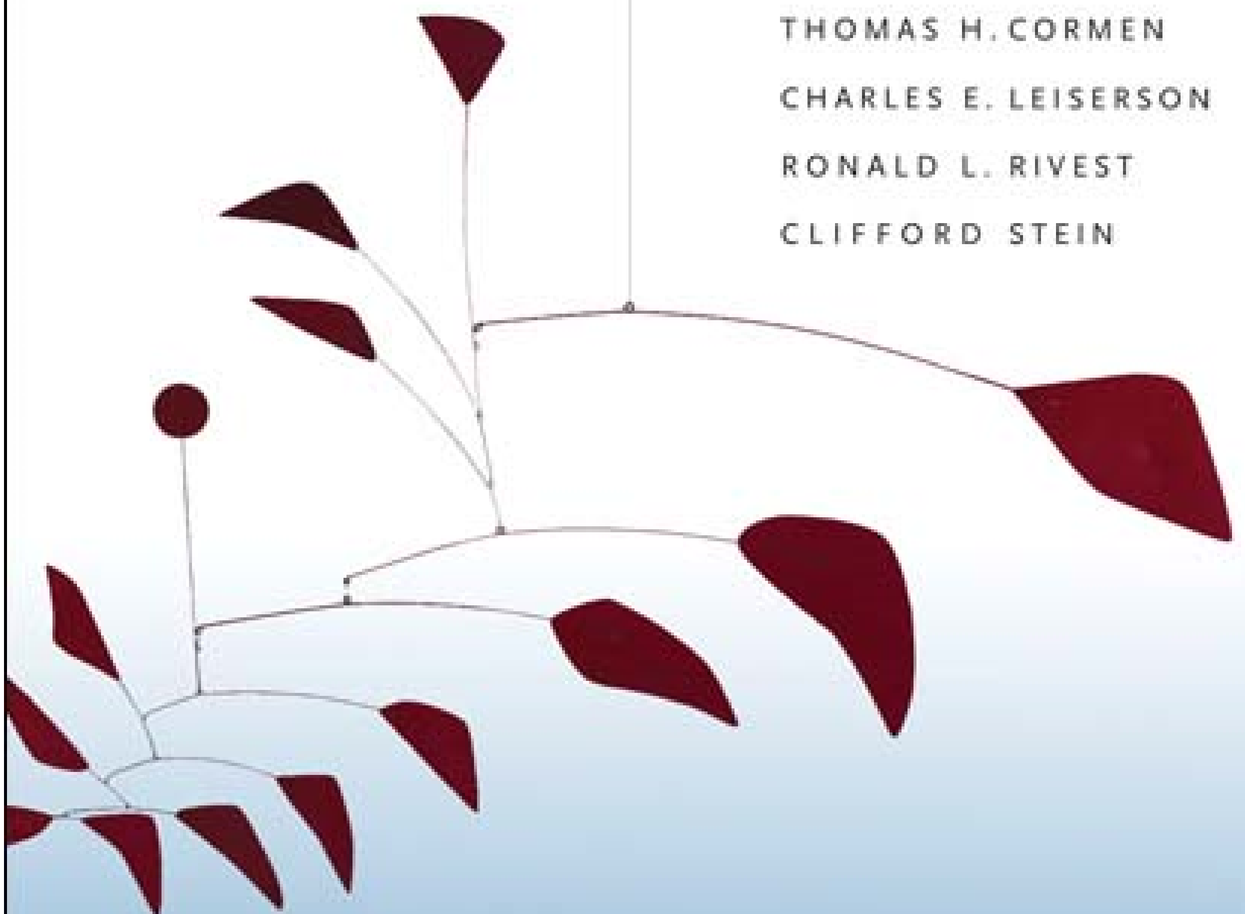
THOMAS H. CORMEN
CHARLES E. LEISERSON
RONALD L. RIVEST
CLIFFORD STEIN



INTRODUCTION TO

ALGORITHMS

THIRD EDITION

CSCE 500

Design and Analysis of Algorithms

Fall 2021

August 20, 2021

Instructor: Nian-Feng Tzeng
Office: Rm. OLVR 354 (× 2-6304)
Class meeting: MW 10:00 – 11:15, OLVR 113

Textbook and Supplemental Materials:

1. Introduction to Algorithms, Third Edition, by Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein, The MIT Press, 2009, ISBN: 978–0–262–03384–8.
2. Published articles supplementary to covered topics.

Course Description:

This course provides a comprehensive coverage of modern computer algorithms, aiming at in-depth treatment of algorithmic design and analysis with elementary explanation while keeping mathematical rigor. Based on the textbook of "Introduction to Algorithms", this class covers the topics listed below in sequence.

(1) Foundations.
(2) Data Structures – hash tables, trees, heaps.
(3) Design and Analysis Techniques – dynamic programming, greedy algorithms, amortized analysis.
(4) Graph Algorithms – spanning trees, shortest paths, maximum flow.
(5) Selected Topics – NP-completeness, approximation algorithms, multithreaded algorithms, string matching.

Each covered topic starts with the description of pertinent algorithms in English and/or in the pseudocode(s), followed by their careful complexity analyses.

Course Requirements:

1. Homework (10%)
2. Midterm exams (2) (50%)
3. Final exam (comprehensive) (40%)

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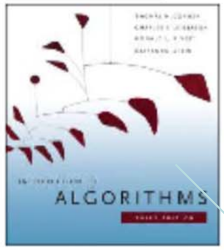

Introduction to Algorithms, third edition

By Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest and Clifford Stein

return

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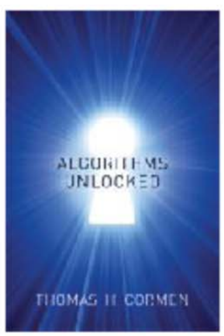
Hardcover | $92.00 Text | £63.95 | ISBN: 9780262033848 | 1312 pp. | 8 x 9 in | 235 b&w illus. | July 2009

Paperback | $70.00 Text | £43.95 | ISBN: 9780262533058 | 1312 pp. | 8 x 9 in | 235 b&w illus. | July 2009

Paperback edition is not for sale in the U.S. or Canada.

Of Related Interest



Algorithms Unlocked

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About the Authors

Thomas H. Cormen is Professor of Computer Science and former Director of the Institute for Writing and Rhetoric at Dartmouth College. He is the coauthor (with Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein) of the leading textbook on computer algorithms, *Introduction to Algorithms* (third edition, MIT Press, 2009).

Charles E. Leiserson is Professor of Computer Science and Engineering at the Massachusetts Institute of Technology.

Ronald L. Rivest is Andrew and Erna Viterbi Professor of Electrical Engineering and Computer Science at the Massachusetts Institute of Technology.

Clifford Stein is Professor of Industrial Engineering and Operations Research at Columbia University.

Endorsements

"As an educator and researcher in the field of algorithms for over two decades, I can unequivocally say that the Cormen et al book is the best textbook that I have ever seen on this subject. It offers an incisive, encyclopedic, and modern treatment of algorithms, and our department will continue to use it for teaching at both the

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Analyzing Algorithms

- **Run Time Analysis**
  - Order of growth
  - Worst case analysis
  - Average case analysis

- **Insertion Sort for Array A[*i*]**
  - idea: insert A[*j*] into sorted subarrays: A[1 .. *j*-1]
  - repeat insertion until A[*i*] is fully sorted

INSERTION-SORT(A, n)	*cost*	*times*
for $j = 2$ **to** n	c_1	n
$\quad key = A[j]$	c_2	$n-1$
$\quad$ // Insert $A[j]$ into the sorted sequence $A[1 \,..\, j-1]$.	0	$n-1$
$\quad i = j-1$	c_4	$n-1$
$\quad$ **while** $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$\qquad A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$\qquad i = i-1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$\quad A[i+1] = key$	c_8	$n-1$

Insert A(*j*) properly

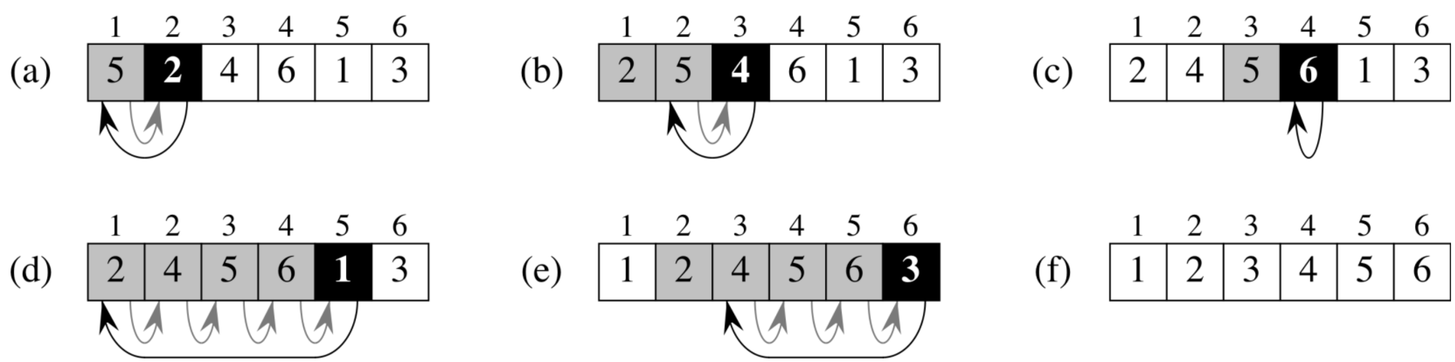
Complexity: $T(n) = \sum_{i=1}^{8} c_i$

Analyzing Algorithms (continued)

§ **Insertion Sort for Array A[*i*]**

– idea: insert A[*j*] into sorted subarrays: A[1 .. *j*-1]

– repeat insertion until A[*i*] is fully sorted



Worst-Case Complexity:

$$T(n) = \sum_{i=1}^{8} c_i = O(n^2)$$

INSERTION-SORT(A, n)	*cost*	*times*
for $j = 2$ **to** n	c_1	n
$key = A[j]$	c_2	$n-1$
// Insert $A[j]$ into the sorted sequence $A[1\,..\,j-1]$.	0	$n-1$
$i = j - 1$	c_4	$n-1$
while $i > 0$ and $A[i] > key$	c_5	$\sum_{j=2}^{n} t_j$
$A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$i = i - 1$	c_7	$\sum_{j=2}^{n} (t_j - 1)$
$A[i+1] = key$	c_8	$n-1$

Designing Algorithms

- **Divide-and-Conquer Approaches with Recursive Nature**
  - Divide the problem
  - Conquer subproblems recursively
  - Combine solutions to subproblems

- **Example: Merge Sort**
  - two sorted subarrays: A[*p* .. *q*] & A[*q*+1 .. *r*]
  - merge the two sorted subarrays
  - merging takes Θ(*n*) time

```
MERGE-SORT(A, p, r)

if p < r                          // check for base case
    q = ⌊(p + r)/2⌋               // divide
    MERGE-SORT(A, p, q)           // conquer
    MERGE-SORT(A, q + 1, r)       // conquer
    MERGE(A, p, q, r)             // combine
```

merge

```
MERGE(A, p, q, r)
  n1 = q − p + 1
  n2 = r − q
  let L[1 .. n1 + 1] and R[1 .. n2 + 1] be new arrays
  for i = 1 to n1
      L[i] = A[p + i − 1]
  for j = 1 to n2
      R[j] = A[q + j]
  L[n1 + 1] = ∞
  R[n2 + 1] = ∞
  i = 1
  j = 1
  for k = p to r
      if L[i] ≤ R[j]
          A[k] = L[i]
          i = i + 1
      else A[k] = R[j]
          j = j + 1
```

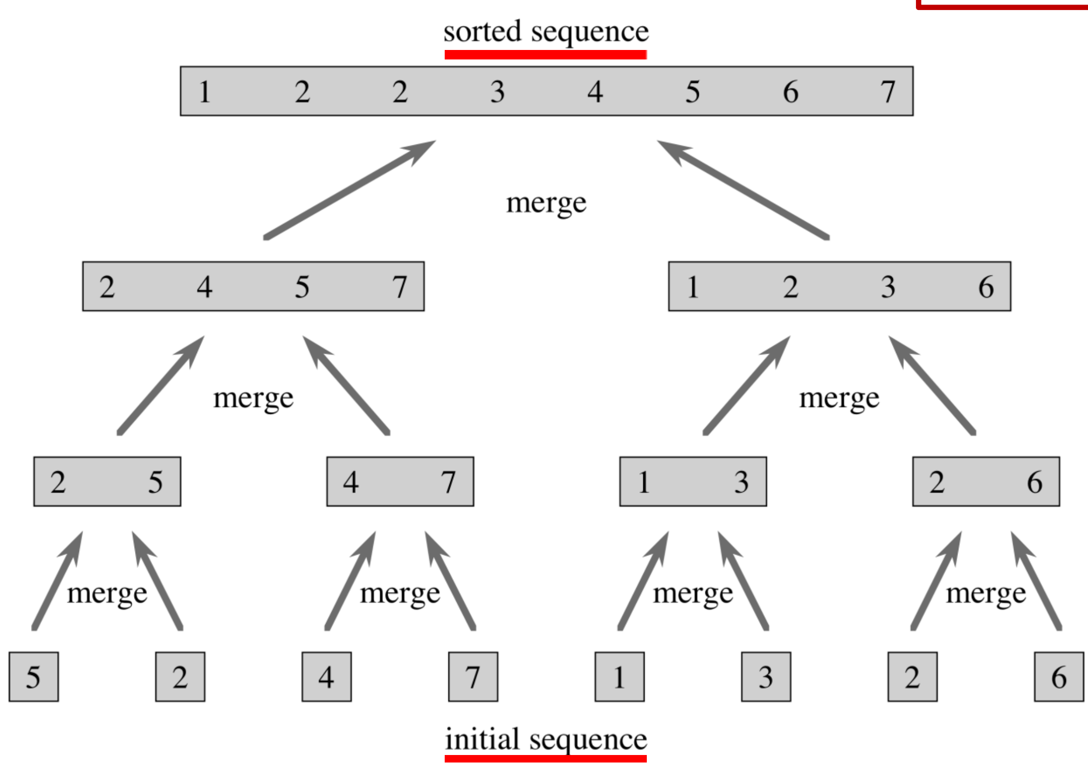
stoppers for half arrays

Analyzing Algorithms

- **Analysis of Divide-and-Conquer Algorithms**
  - Merge Sort
  - Time complexity:

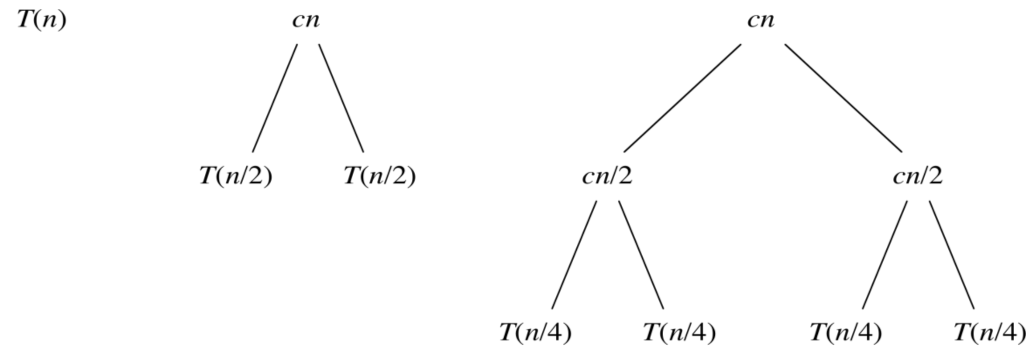
$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(n/2) + c\cdot(n) & \text{if } n > 1 \end{cases}$$

```
MERGE-SORT(A, p, r)
  if p < r                          // check for base case
      q = ⌊(p + r)/2⌋               // divide
      MERGE-SORT(A, p, q)           // conquer
      MERGE-SORT(A, q + 1, r)       // conquer
      MERGE(A, p, q, r)             // combine
```



Analyzing Algorithms (continued)

§ **Evaluating: $2T(n/2) + c\cdot(n)$**
- Recursion tree, shown right
- $cn \cdot \lg(n) + cn = \Theta(n \cdot \lg n)$



(a) (b) (c)

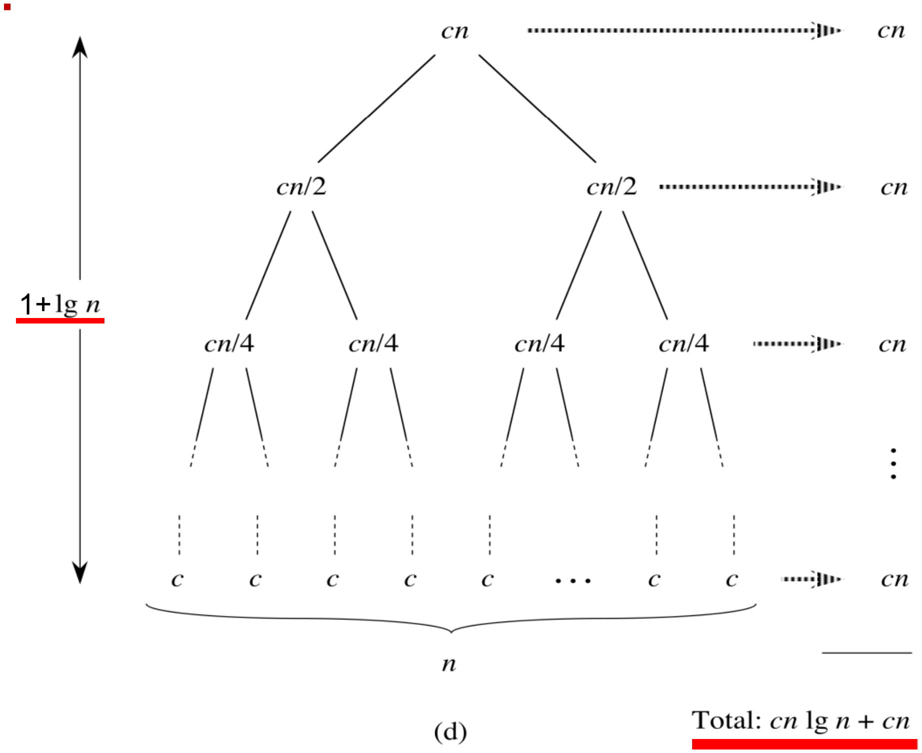
Another approach for solution:

$$T(n) = 2T(n/2) + c\cdot(n)$$
$$= 2(2T(n/4)+c\cdot(n/2)) + c\cdot(n)$$
$$= 2^2T(n/4) + 2\cdot c\cdot(n)$$
$$= 2^2(2T(n/8) + c\cdot(n/4)) + 2\cdot c\cdot(n)$$
$$= 2^3T(n/8) + 3\cdot c\cdot(n)$$

....

....

$$= nT(1) + \lg(n) \cdot c\cdot(n)$$

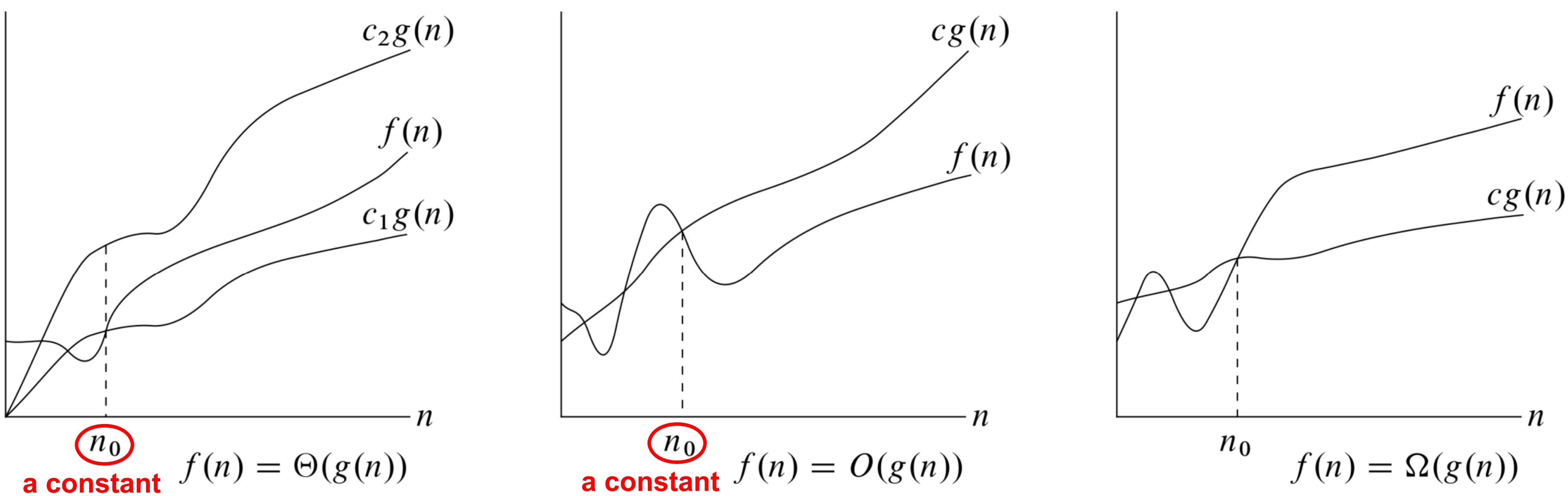


(d)

Growth of Functions

§ **Asymptotic Notations of Running Times**

- Θ-notation: bounding a function to within constant factors
- *O*-notation: upper-bounding a function to within a constant factor
- Ω-notation: lower-bounding a function to within a constant factor



(a) $g(n)$ is an asymptotically tight bound for $f(n)$

(b) $g(n)$ is an asymptotical upper bound for $f(n)$ **(may or may not be tight)**

(c) $g(n)$ is an asymptotical lower bound for $f(n)$ **(may or may not be tight)**

: there exist positive constants c_1, c_2, and n_0 s.t. $0 \le c_1 \cdot g(n) \le f(n) \le c_2 \cdot g(n)$ for all $n \ge n_0$

Growth of Functions (continued)

- *o*-notation: not asymptotically-tight upper-bound

o-notation

$$o(g(n)) = \{ f(n) : \text{for any constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le f(n) < cg(n) \text{ for all } n \ge n_0 \} .$$

Another view, probably easier to use: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0.$

$n^{1.9999} = o(n^2)$
$n^2/\lg n = o(n^2)$
$n^2 \ne o(n^2)$ (just like $2 \nless 2$)
$n^2/1000 \ne o(n^2)$

- ω-notation: not asymptotically-tight lower-bound

ω-notation

$$\omega(g(n)) = \{ f(n) : \text{for any constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le cg(n) < f(n) \text{ for all } n \ge n_0 \}$$

Another view, again, probably easier to use: $\lim_{n \to \infty} \frac{f(n)}{g(n)} = \infty.$

$n^{2.0001} = \omega(n^2)$
$n^2 \lg n = \omega(n^2)$
$n^2 \ne \omega(n^2)$

Solutions after Divide-and-Conquer

§ Divide-and-Conquer

– leads to recurrences in various forms

– there are 3 kinds of methods for solving recurrences

+ **substitution** methods
+ **recursion-tree** methods
+ **master methods** to find bounds for recurrences of $T(n) = a \cdot T(n/b) + f(n)$, with $a \geq 1$ and $b > 1$

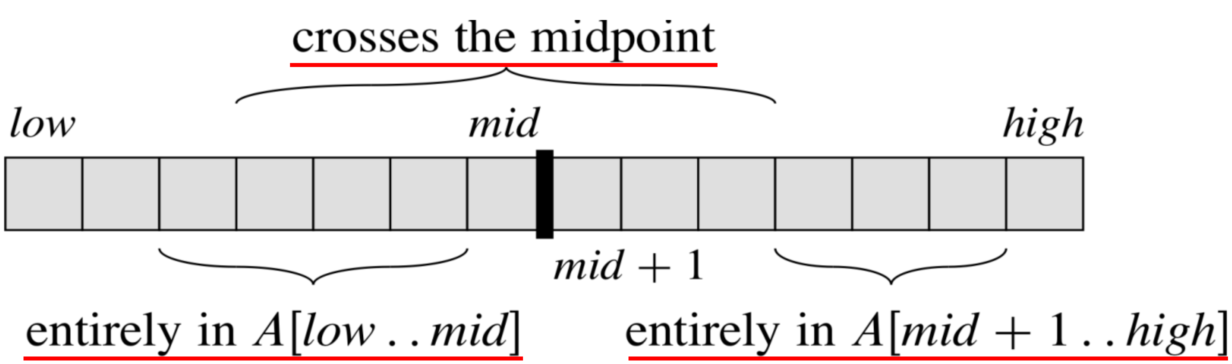
§ Example: Maximum-Subarray Problem (finding such a subarray and its sum)

– divide the problem into two subarrays of (roughly) the same size, finding *mid*

– maximum continuous subarray lies in exactly one of three places, shown below:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
A	13	–3	–25	20	–3	–16	–23	18	20	–7	12	–5	–22	15	–4	7

maximum subarray



(a)

Divide-and-Conquer (continued)

§ **Maximum-Subarray**

– solution lies in exactly one of three places

FIND-MAXIMUM-SUBARRAY$(A, low, high)$

```
if high == low
      return (low, high, A[low])     // resulting sum
else mid = ⌊(low + high)/2⌋
      (left-low, left-high, left-sum) =
            FIND-MAXIMUM-SUBARRAY(A, low, mid)
      (right-low, right-high, right-sum) =
            FIND-MAXIMUM-SUBARRAY(A, mid + 1, high)
      (cross-low, cross-high, cross-sum) =
            FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
      if left-sum ≥ right-sum and left-sum ≥ cross-sum
            return (left-low, left-high, left-sum)
      elseif right-sum ≥ left-sum and right-sum ≥ cross-sum
            return (right-low, right-high, right-sum)
      else return (cross-low, cross-high, cross-sum)
```

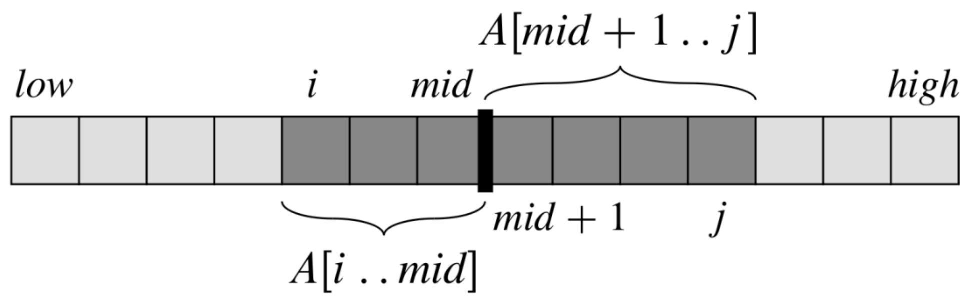
The time complexity of this procedure:

$T(n) = \Theta(1)+2T(n/2)+ \Theta(n)+\Theta(1)$

$= 2T(n/2) + \Theta(n).$

Solutions after Divide-and-Conquer (continued)

§ **Maximum-Subarray**
– crossing the midpoint, comprising two parts



(b)

```
FIND-MAX-CROSSING-SUBARRAY(A, low, mid, high)
  // Find a maximum subarray of the form A[i..mid].
  left-sum = −∞
  sum = 0
  for i = mid downto low          // This searches all the way down to low.
      sum = sum + A[i]
      if sum > left-sum
          left-sum = sum
          max-left = i
  // Find a maximum subarray of the form A[mid+1..j].
  right-sum = −∞
  sum = 0
  for j = mid + 1 to high         // This searches all the way up to high.
      sum = sum + A[j]
      if sum > right-sum
          right-sum = sum
          max-right = j
  // Return the indices and the sum of the two subarrays.
  return (max-left, max-right, left-sum + right-sum)
```

This whole procedure takes $\Theta(n)$ time.

Divide-and-Conquer (continued)

§ **Substitution Methods**: two steps involved
– guess the solution form
– mathematic induction to validate the constants

Substitution method for proving an upper bound on the recurrence of
$T(n) = 2T(\lfloor n/2 \rfloor) + \underline{n}$ being $\underline{T(n) \le c \cdot n \cdot \lg n}$ for a constant $c > 0$.

This is due to composing the full solution.

This is obtained by guessing its solution to be $T(n) = O(n \cdot \lg n)$ and then substituting $T(\lfloor n/2 \rfloor) \le c \cdot \lfloor n/2 \rfloor \cdot \lg(\lfloor n/2 \rfloor)$ into the recurrence:

$$
\begin{aligned}
T(n) &\le 2(c \cdot \lfloor n/2 \rfloor \cdot \lg(\lfloor n/2 \rfloor)) + n \\
&\le c \cdot n \cdot \lg(\lfloor n/2 \rfloor) + n \\
&\le c \cdot n \cdot \lg(n) - c \cdot n \cdot \lg(2) + n \\
&\le c \cdot n \cdot \lg n \quad \text{for } c \ge 1
\end{aligned}
$$

Similar substitution method for proving an upper bound on recurrence
$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \underline{1}$
equal to $O(n)$.

This is due to composing the full solution.

Divide-and-Conquer (continued)

§ **Substitution Methods**

– changing variables and/or function renaming

Substitution method for proving: $T(n) = 2T(\lfloor\sqrt{n}\rfloor) + \lg n$

Rename $m = \lg n$ (and ignore rounding) to get $T(n)$ (after parameter renaming):

$$T(n) = T(2^m) = 2T(2^{m/2}) + m$$

Further renaming $T(2^m)$ as $S(m)$, we have (function renaming)

$S(m) = 2S(m/2) + m$, which has the solution of

$S(m) = O(m \cdot \lg m)$ ← via the prior result

We thus have $T(n) = T(2^m) = S(m) = O(m \cdot \lg m)$

$= O(\lg n \cdot \lg \lg n)$.

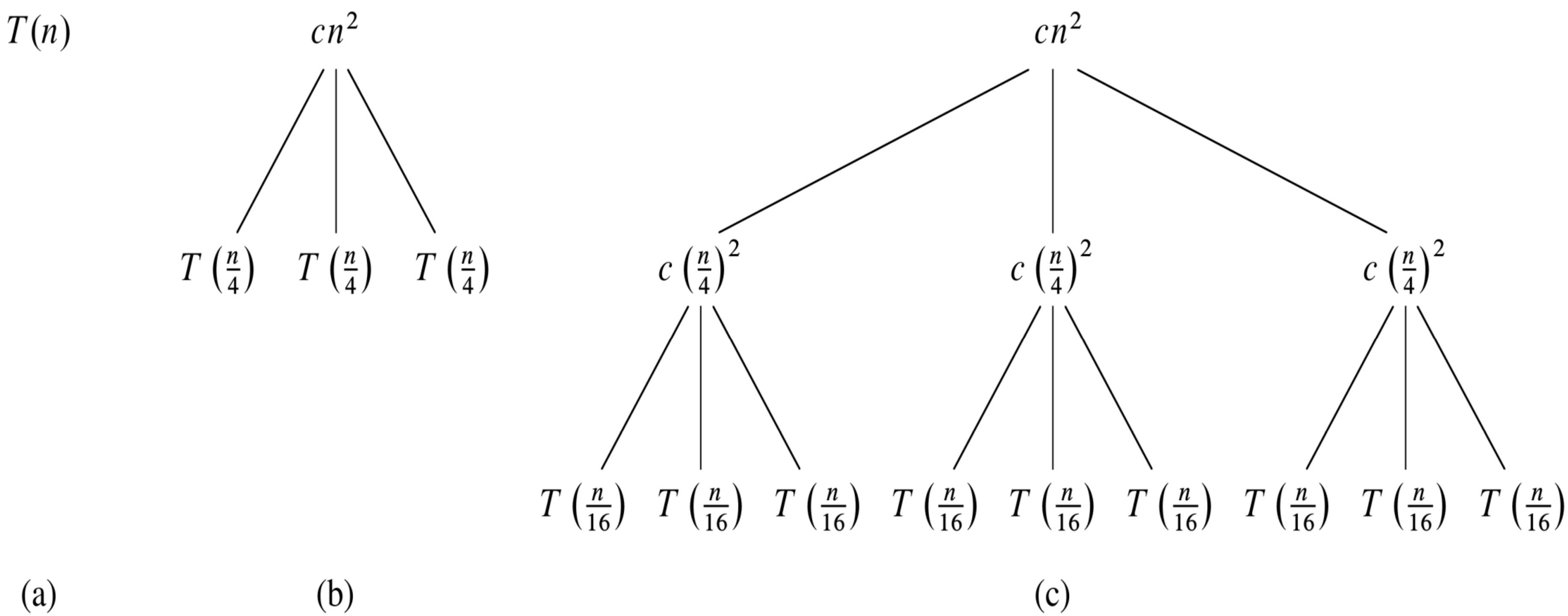
Note: this problem can also be solved by the recursion-tree method, described next.

Divide-and-Conquer (continued)

§ Recursion-Tree Methods

- best for generating good complexity bounds in general
- two examples given below

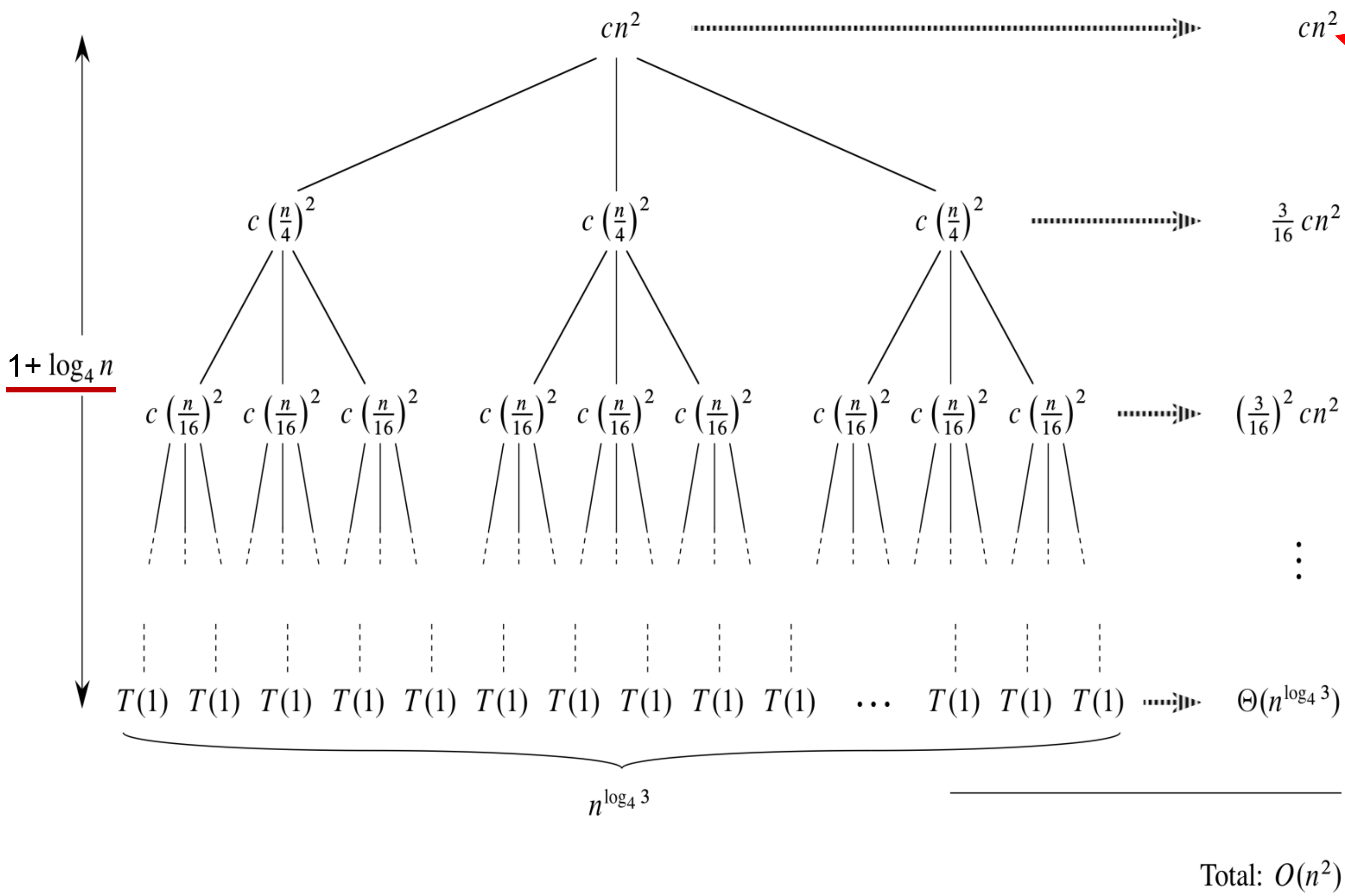
For recurrence: $T(n) = 3T(n/4) + cn^2$



Divide-and-Conquer (continued)

§ Recursion-Tree Methods

For recurrence: $T(n) = 3T(n/4) + cn^2$



$T(n)$ satisfies $\Omega(n^2)$.

3 elements

3^2 elements

$3^{\log_4 n}$ elements

$$T(n) = \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3})$$

$$< \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) = \frac{1}{1-(\frac{3}{16})} cn^2 + \Theta(n^{\log_4 3}) = O(n^2).$$

Use the property of: $v^{a \bullet b} = (v^a)^b = (v^b)^a$.

Let $3^{\log_4 n} = x$.

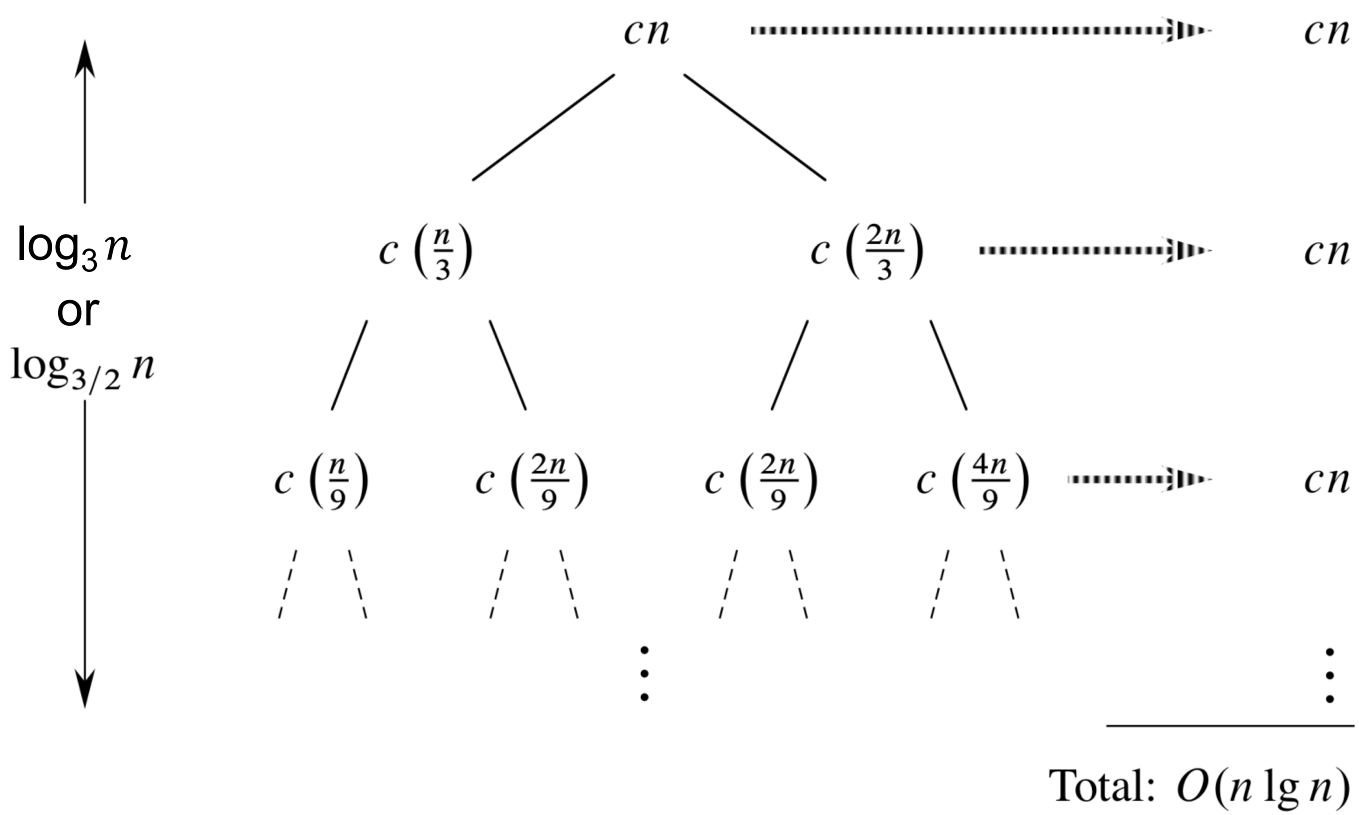
After taking $\log_4$ on both sides, we have: $(\log_4 n) \cdot (\log_4 3) = \log_4 x$.

We then take a power of 4 on both sides to yield: $4^{\log_4 n \cdot \log_4 3} = x$, which becomes:

$[4^{\log_4 n}]^{\log_4 3} = [n]^{\log_4 3} = x$. Hence, $3^{\log_4 n} = n^{\log_4 3}$.

Divide-and-Conquer (continued)

Another recurrence: $T(n) = T(n/3) + T(2n/3) + cn$



Longest path: $cn \rightarrow c(\frac{2}{3})n \rightarrow c(\frac{2}{3})^2 n \rightarrow c(\frac{2}{3})^3 n \rightarrow \cdots \rightarrow 1$, we have: $k = \log_{3/2} n$, as $(\frac{2}{3})^k n = 1$

Shortest path: $cn \rightarrow c(\frac{1}{3})n \rightarrow c(\frac{1}{3})^2 n \rightarrow c(\frac{1}{3})^3 n \rightarrow \cdots \rightarrow 1$ to get $k = \log_3 n$, ~ $(\log_{3/2} n)/2.7$

Similarly, one may show $T(n)$ upper bounded by $O(n \cdot \lg n)$ via substitution method.

Master Method for Solving Recurrences

Recurrences of $T(n) = a \cdot T(n/b) + f(n)$ with constants $a \geq 1$ and $b > 1$
and $f(n)$ asymptotically positive function that covers work on dividing the problem and on combining subproblems' results

Theorem

$T(n) = a \cdot T(n/b) + f(n)$ has following asymptotical bounds:

1. for $f(n) = O(n^{\log_b a - \epsilon})$ with constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
2. for $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \lg n)$
3. for $f(n) = \Omega(n^{\log_b a + \epsilon})$ with constant $\epsilon > 0$ and $a \cdot f(n/b) \leq c \cdot f(n)$, then $T(n) = \Theta(f(n))$

In the recursion-tree, 2[nd] level sums to **no more than** 1[st] level &
$f(n)$ is *polynomially larger*

Note: bound is the **larger** of the two: $f(n)$ and $n^{\log_b a}$

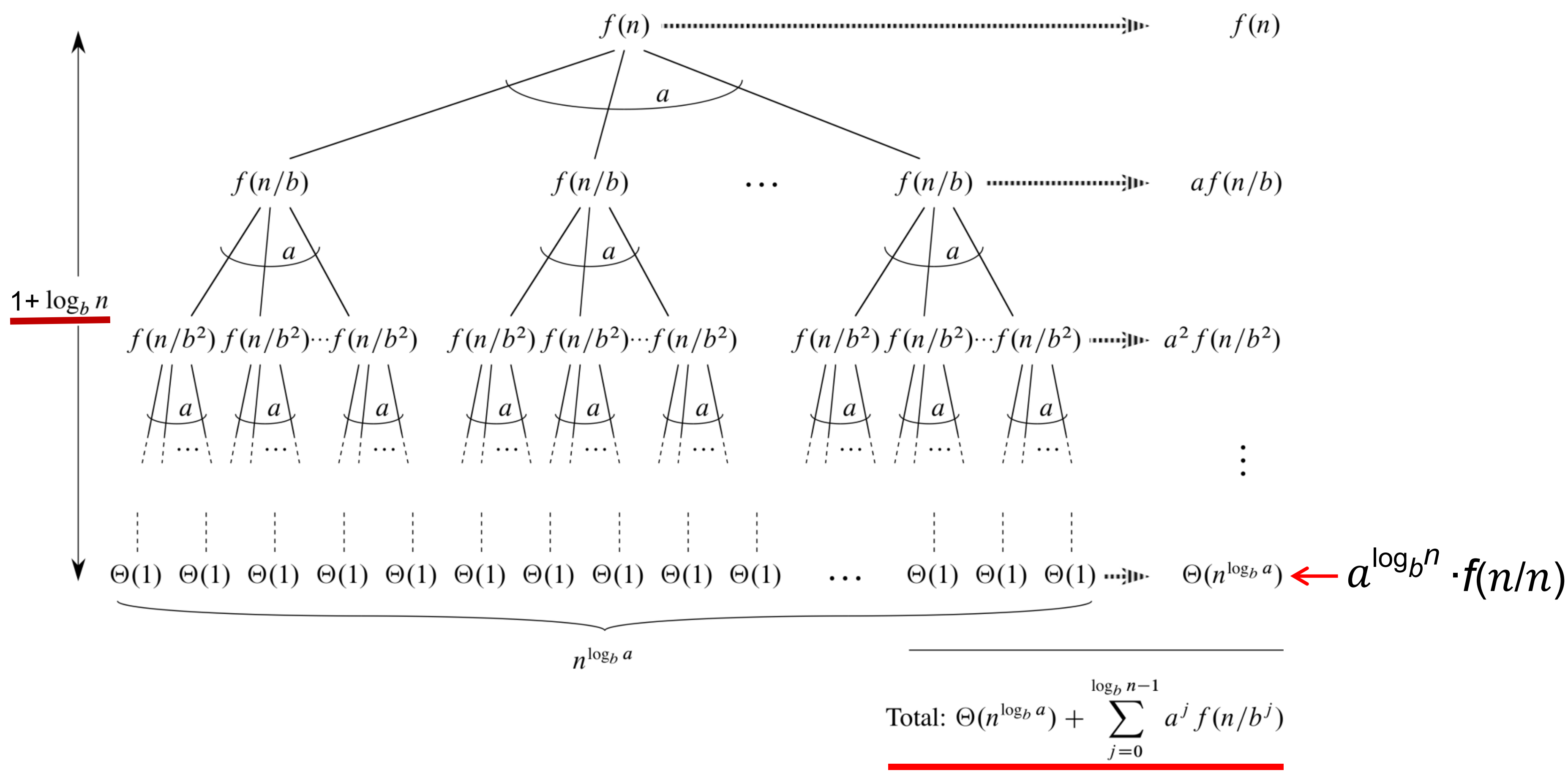
In Case 1, $n^{\log_b a}$ is *polynomially larger* than $f(n)$
In Case 2, $n^{\log_b a}$ and $f(n)$ are of the same size
In Case 3, $f(n)$ is *polynomially larger* than $n^{\log_b a}$

Master Method (continued)

Let $T(n) = a \cdot T(n/b) + f(n)$ with constants $a \geq 1$ and n being an exact power of b (> 1)

Lemma 4.2

$$T(n) = \Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n - 1} a^j \cdot f(n/b^j)$$



Master Method (continued)

Example Recurrences $T(n) = a \cdot T(n/b) + f(n)$ solved by the master method:

1. for $f(n) = O(n^{\log_b a - \epsilon})$ with constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
2. for $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \lg n)$
3. for $f(n) = \Omega(n^{\log_b a + \epsilon})$ with constant $\epsilon > 0$ and $a \cdot f(n/b) \leq c \cdot f(n)$, then $T(n) = \Theta(f(n))$

$T(n) = 9T(n/3) + n$

Here, $a = 9$, $b = 3$, and $f(n) = n$

From $n^{\log_3 9} = n^2$, we have $f(n) = n = O(n^{\log_3 9 - 1})$ to get $T(n) = \Theta(n^2)$

$T(n) = T(2n/3) + 1$

Here, $a = 1$, $b = 3/2$, and $f(n) = 1$

From $n^{\log_{3/2} 1} = n^0$, we have $f(n) = 1 = O(n^{\log_{3/2} 1})$ to get $T(n) = \Theta(\lg n)$

$T(n) = 3T(n/4) + n \lg n$

Here, $a = 3$, $b = 4$, and $f(n) = n \lg n$

From $n^{\log_4 3} = O(n^{0.793})$, we have $f(n) = n \lg n = \Omega(n^{\log_4 3 + \epsilon})$ to get $T(n) = \Theta(n \lg n)$

<u>Master Method</u> (continued)

Example Recurrences $T(n) = a \cdot T(n/b) + f(n)$:

1. for $f(n) = O(n^{\log_b a - \epsilon})$ with constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
2. for $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \cdot \lg n)$
3. for $f(n) = \Omega(n^{\log_b a + \epsilon})$ with constant $\epsilon > 0$ and $a \cdot f(n/b) \le c \cdot f(n)$, then $T(n) = \Theta(f(n))$

$f(n)$ is polynomially larger

$T(n) = 2T(n/2) + n \lg n$

Here, $a = 2$, $b = 2$, and $f(n) = n \lg n$

From $n^{\log_2 2} = n$, we have $f(n) = n \lg n > n^{\log_2 2}$ but ***not polynomially >*** $n^{\log_2 2}$

(use substitution or recursion-tree to solve this)

$T(n) = 7T(n/2) + \Theta(n^2)$

Here, $a = 7$, $b = 2$, and $f(n) = \Theta(n^2)$

From $n^{\log_2 7} = n^{2.8^+}$, we have $f(n) = O(n^{\log_2 7 - \epsilon})$ to get $T(n) = \Theta(n^{\log_2 7})$

Master Method (continued)

Lemma 4.3

Given $g(n) = \sum_{j=0}^{\log_b n - 1} a^j \cdot f(n/b^j)$ for $a \geq 1$ and n an exact power of b (> 1), we have:

1. for $f(n) = O(n^{\log_b a - \epsilon})$ with constant $\epsilon > 0$, then $g(n) = \Theta(n^{\log_b a})$
2. for $f(n) = \Theta(n^{\log_b a})$, then $g(n) = \Theta(n^{\log_b a} \cdot \lg n)$
3. if $a \cdot f(n/b) \leq c \cdot f(n)$ for constant $c < 1$ and for sufficiently large n, $g(n) = \Theta(f(n))$