Part III

Advanced Design and Analysis Techniques

§ Two Types of Techniques:

- Dynamic Programming
- Greedy Algorithms

- **Dynamic Programming**
  - subproblems *overlap* (→ i.e., subproblems are repeated)
  - problem solved by *combining solutions* to its subproblems
  - *saving solutions* to common subproblems *in table(s)* for efficiency
  - commonly applicable to optimization problems

- **Greedy Algorithms**
  - following locally optimal choice (instead of exhaustive search)
  - simpler and more efficient approach
  - work for wide ranges of problems
  - yield optimal solutions to certain problems, e.g., combinatorial structures - *matroids*

(where a matroid possesses linear independence, i.e., part solutions are all independent, accelerating combinatorial optimization)

Dynamic Programming (continued)

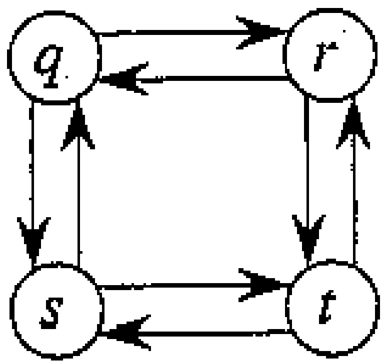
- **Four Steps Involved**
  – characterize structure of optimal solution **(check applicability,** i.e., if optimal substructures exist**)**
  – recursively define expression of optimal solution (**establish recursive formulation**)
  – compute and keep value of optimal solution (**keep subproblem solutions in bottom-up way**)
  – construct optimal solution from kept information

- **Example for Optimal Rod Cutting**
  – solve each subproblem *once*
  – *time-memory tradeoff*: additional memory involved for time complexity reduction
  – full solution in polynomial time if the number of *distinct subproblems* is polynomial
  – two approaches for dynamic programming implementation:
    - top-down with memoization †
    - bottom-up method, with *smallest* subproblems solved first and kept in table(s)

† Here, memoization is from “memo” and *not* from “memory.”

<u>Dynamic Programming</u> (continued)

- **First of four Steps**
  – characterize the structure of optimal solution (**whether applicable**)



Longest simple path from *q* to *t*:
q → r → t
whereas longest simple path from *q* to *r*:
q → s → t → r
and longest simple path from *r* to *t*:
r → q → s → t

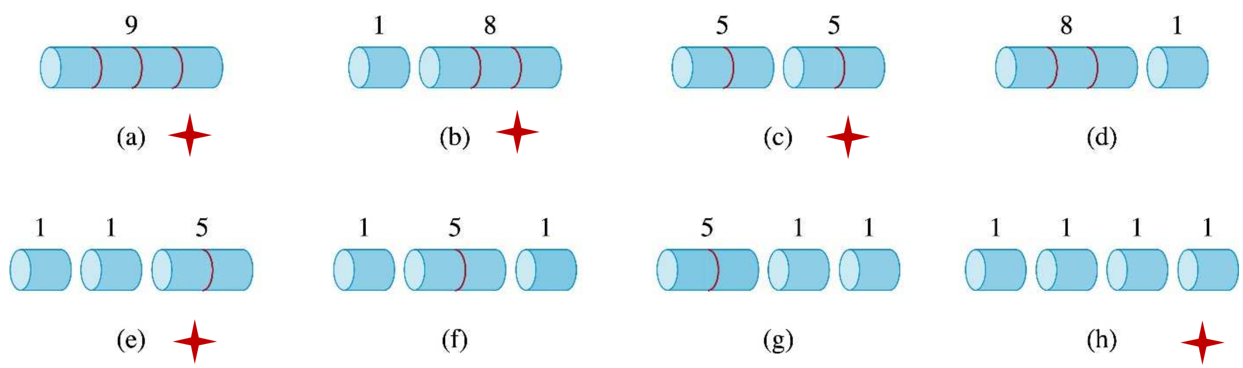
Figure 15.6 A directed graph showing that the problem of finding a longest simple path in an unweighted directed graph does not have optimal substructure. The path $q \to r \to t$ is a longest simple path from q to t, but the subpath $q \to r$ is not a longest simple path from q to r, nor is the subpath $r \to t$ a longest simple path from r to t.

Dynamic programming is applicable *if and only if* <u>optimal solution</u> to it <u>can be composed</u> from optimal solutions to its subproblems. Note while problems solvable by divide-&-conquer are applicable, other problems may also be applicable, as long as their solutions can be obtained from the solutions to their subproblems.

Dyanmic Programming (continued)

- **Rod cutting problem**
  + rod with length n has $2^{n\text{-}1}$ ways (which equal leaf nodes of recursion tree) to cut
  + number of *distinct subproblems* far smaller, due to many identical subproblems
  + this problem exhibits *optimal substructure*, i.e., optimal solution consists of optimal solutions to related subproblems, which can be solved independently

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30



only five (5) distinct subproblems exist

Recursive top-down implementation

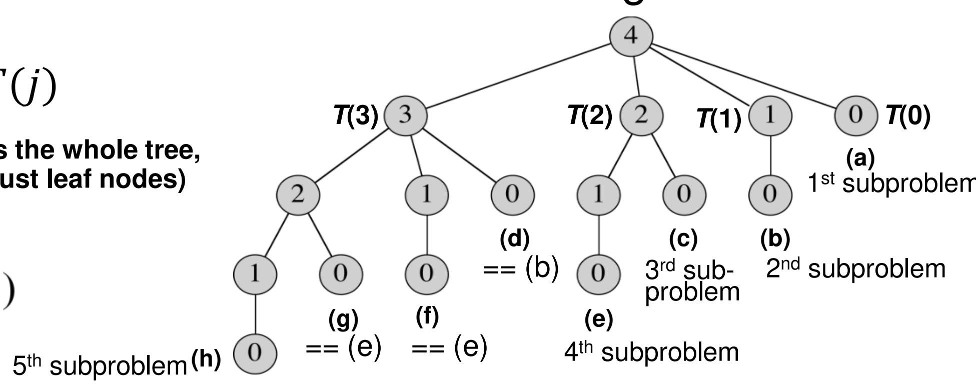
```
CUT-ROD(p, n)
  if n == 0
      return 0
  q = −∞
  for i = 1 to n
      q = max(q, p[i] + CUT-ROD(p, n − i))
  return q
```

root

$$T(n) = 1 + \sum_{j=0}^{n-1} T(j)$$

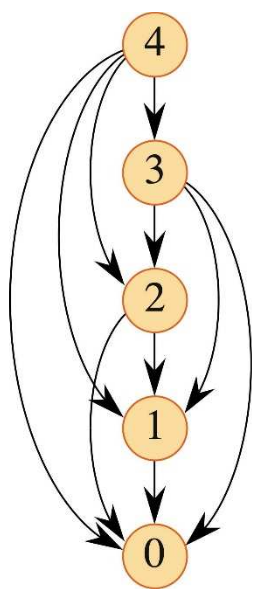
$= 2^n$ **(covers the whole tree, not just leaf nodes)**

Recursion tree showing recursive calls



Dynamic Programming (continued)

- **First three steps of typical dynamic programming**
  - – characterize structure of optimal solution (**applicability**)
  - – recursively define value of optimal solutions to subproblems (**recursion formulation**)
  - – compute & keep value of optimal solution to every subproblem (**keep subsolutions**)



Subproblem graph, being collapsed recursion tree

MEMOIZED-CUT-ROD(p, n)

let $r[0..n]$ be a new array // rely on this added array to keep earlier solutions
for $i = 0$ **to** n
  $r[i] = -\infty$
return MEMOIZED-CUT-ROD-AUX(p, n, r)

Running time complexity: $\Theta(n^2)$
(as it equals $n-1$, n-2, …,1 for i =1, 2 …, obtainable from recursion tree in preceding page, to visit leftmost path for calculation + the level just below the root for references)

MEMOIZED-CUT-ROD-AUX(p, n, r)

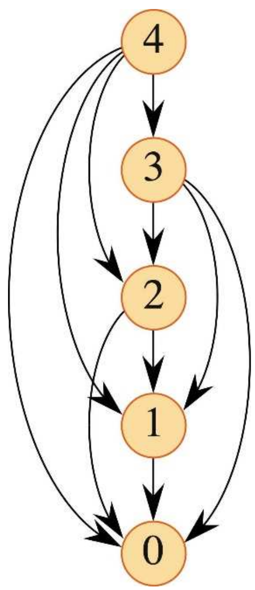
if $r[n] \geq 0$
  return $r[n]$] reference to subproblem solved before
if $n == 0$
  $q = 0$
else $q = -\infty$
  for $i = 1$ **to** n
    $q = \max(q, p[i] + \text{MEMOIZED-CUT-ROD-AUX}(p, n-i, r))$
$r[n] = q$
return q

top-down, same as divide-&-conquer method, but with memoization

Dynamic Programming (continued)

- Bottom-up method
  - same asymptotic complexity as the top-down method, with running time complexity of $\Theta(n^2)$
  - often with smaller constants, due to avoiding recursive calls (employed by top-down counterpart)

length i	1	2	3	4	5	6	7	8	9	10
price p_i	1	5	8	9	10	17	17	20	24	30



Subproblem graph, being collapsed recursion tree

```
BOTTOM-UP-CUT-ROD(p, n)
let r[0..n] be a new array
r[0] = 0
for j = 1 to n
    q = -∞
    for i = 1 to j
        q = max(q, p[i] + r[j - i])
    r[j] = q
return r[n]
```

bottom-up, permitting table lookups

Dynamic Programming (continued)

- **Fourth step of typical dynamic programming**
  - reconstruct a solution to the problem
  - the solution of rod-cutting problem includes the choice of every cut (the cut listed in array ***s*** below), besides the optimal value (given in array ***r*** below)

```
EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
let r[0..n] and s[0..n] be new arrays
r[0] = 0
for j = 1 to n
    q = -∞
    for i = 1 to j
        if q < p[i] + r[j - i]
            q = p[i] + r[j - i]
            s[j] = i        // keep the cut location
    r[j] = q
return r and s
```

Solution of the problem printed by:

```
PRINT-CUT-ROD-SOLUTION(p, n)
(r, s) = EXTENDED-BOTTOM-UP-CUT-ROD(p, n)
while n > 0
    print s[n]
    n = n - s[n]
```

‡ PRINT-CUT-ROD-SOLUTION(*p*, 3) prints "3" while PRINT-CUT-ROD-SOLUTION(*p*, 8) prints "2" & "6"

EXTENDED-BOTTOM-UP-CUT-ROD(*p*, 8) returns ***r*** and ***s***, as follows:

i	0	1	2	3	4	5	6	7	8
$r[i]$	0	1	5	8	10	13	17	18	22
$s[i]$	0	1	2	3	2	2	6	1	2

<u>Dynamic Programming</u> (continued)

- **Elements of dynamic programming**
  - two key elements for optimization problems to apply dynamic programming:
    - + optimal substructure
    - + repeated (overlapped) subproblems
  - e.g., optimal parenthesization of matrix chain: $A_i \bullet A_{i+1} \bullet \ldots \bullet A_j$ consists of optimal solutions to split parenthesizing subproblems $A_i \bullet A_{i+1} \bullet \ldots \bullet A_k$ & $A_{k+1} \bullet A_{k+2} \bullet \ldots \bullet A_j$

<u>Matrix-chain multiplication problem</u>

Given a chain of n matrices $\langle A_1, A_2, \ldots, A_n \rangle$, find full parenthesization of the product $A_1 \bullet A_2 \bullet \ldots \bullet A_n$ that minimizes the number of scalar multiplications.

Let $\langle A_1, A_2, A_3 \rangle$ are of dimensions 10x100, 100x5, and 5x50.
The number of multiplications equals: $10 \bullet 100 \bullet 5 + 10 \bullet 5 \bullet 50 = 7500$ for $((A_1 \bullet A_2) \bullet A_3)$
and equals: $100 \bullet 5 \bullet 50 + 10 \bullet 100 \bullet 50 = 75000$ for $(A_1 \bullet (A_2 \bullet A_3))$

Dynamic Programming (continued)

- **Matrix-chain multiplication problem**
  - given <A_1, A_2, A_3, A_4>, we have ***five distinct ways*** for full parenthesization:

    $(A_1 \bullet (A_2 \bullet (A_3 \bullet A_4)))$
    $(A_1 \bullet ((A_2 \bullet A_3) \bullet A_4))$
    $((A_1 \bullet A_2) \bullet (A_3 \bullet A_4))$
    $((A_1 \bullet (A_2 \bullet A_3)) \bullet A_4)$
    $(((A_1 \bullet A_2) \bullet A_3) \bullet A_4)$

  - let $P(n)$ denote number of *alternative parenthesizations* of a sequence of n matrices and the two split subproducts be: $A_1 \bullet A_2 \bullet \ldots \bullet A_K$ and $A_{k+1} \bullet A_{k+2} \bullet \ldots \bullet A_n$, we have the recurrence below:

$$P(n) = \begin{cases} 1 & \text{if } n = 1\,, \\ \sum_{k=1}^{n-1} P(k)P(n-k) & \text{if } n \geq 2\,. \end{cases}$$

whose time complexity is $\Omega(2^n)$, exponential time complexity

(see pp. 386 for a proof by substitution).

Dynamic Programming (continued)

- **Applying dynamic programming to matrix-chain multiplication**

  + four-step sequence:

    1. characterize structure of optimal parenthesization (**applicability**)
    2. establish recursive solution approach (**recursive formulation**)
    3. compute optimal costs (**memoization or table storage**) for subproblems
    4. construct an optimal solution for the problem

Recursive solution approach

– let $m[i, j]$ be the minimum number of scalar multiplications for matrix sequence $A_{i..j}$ with two split subsequences $A_i \cdot A_{i+1} \cdot \ldots \cdot A_K$ & $A_{k+1} \cdot A_{k+2} \cdot \ldots \cdot A_j$. We have:

$$m[i, j] = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j \quad \text{for a given } k$$

– consider every possible k, $i \le k \le j$-1, we have

$$m[i, j] = \begin{cases} 0 & \text{if } i = j \,, \\ \min\limits_{i \le k < j} \{m[i, k] + m[k+1, j] + p_{i-1} p_k p_j\} & \text{if } i < j \,. \end{cases}$$

– finally, we define $s[i, j]$ to be a value of k at which we split the product $A_i A_{i+1} \cdots A_j$ in an optimal parenthesization.

Dynamic Programming (continued)

- Recursive solution approach (without avoiding repeated subproblem computation)
  - from recurrence below directly leads to ***exponential complexity***:

$$m[i,j] = \begin{cases} 0 & \text{if } i = j\ , \\ \min\limits_{i \le k < j} \{m[i,k] + m[k+1,j] + p_{i-1}p_k p_j\} & \text{if } i < j\ . \end{cases}$$

Time complexity of Line 5 (for *i* = 1 and *j* = n):

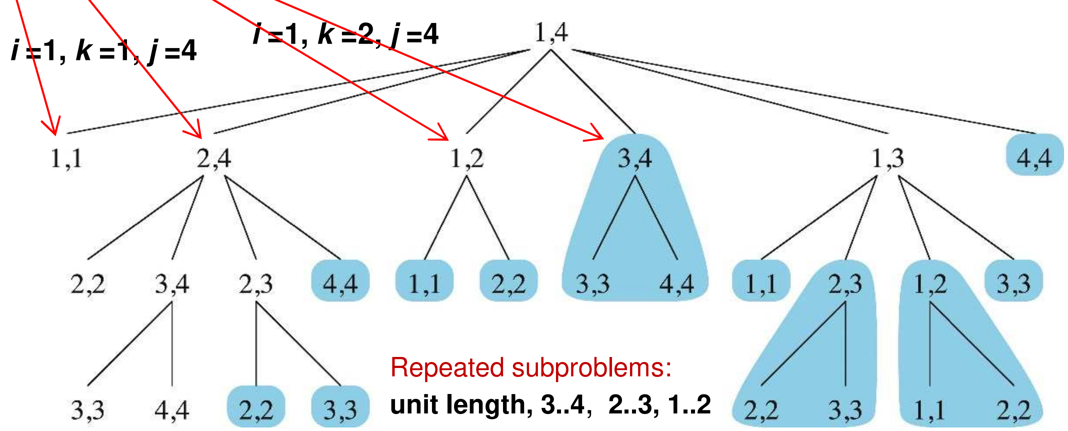
$$T(n) \geq 1 + \sum_{k=1}^{n-1}(T(k) + T(n-k) + 1)$$

$$\geq 2^{n-1}$$ (proof details given in pp. 386)

RECURSIVE-MATRIX-CHAIN(p, i, j)

```
1  if i == j
2      return 0
3  m[i, j] = ∞
4  for k = i to j − 1
5      q = RECURSIVE-MATRIX-CHAIN(p, i, k)
            + RECURSIVE-MATRIX-CHAIN(p, k + 1, j)
            + p_{i−1} p_k p_j
6      if q < m[i, j]
7          m[i, j] = q
8  return m[i, j]
```

// recursive calls, following divide-&-conquer alone
// without properly memoizing results in *m*[*i*, *j*] from shortest chains upward



Dynamic Programming (continued)

- **Applying dynamic programming to matrix-chain multiplication**

Computing optimal costs

– relatively ***few distinct*** subproblems (i.e., subproblems are overlapping)
– two auxiliary tables: *m*[1.. n, 1.. n] to store the *m*[*i*, *j*] cost and *s*[1.. n-1, 2 .. n] to store *k*, *i* ≤ *k* ≤ *j*-1, at which a split yields the lowest cost in computing *m*[*i*, *j*]

MATRIX-CHAIN-ORDER(*p*)

There are Θ(n^2) *distinct* subproblems, to yield time complexity of Θ(n^3).

```
n = p.length − 1
let m[1..n, 1..n] and s[1..n − 1, 2..n] be new tables
for i = 1 to n
    m[i, i] = 0                  // nil initial cost for unit length stored in m[i, i]
for l = 2 to n                   // l is the chain length, for memoizing results in m[i, j] from shortest chains upward
    for i = 1 to n − l + 1
        j = i + l − 1            // range from element i to element j for the chain length of l
        m[i, j] = ∞
        for k = i to j − 1
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q      // memoization
                s[i, j] = k
return m and s
```

<u>Dynamic Programming</u> (continued)

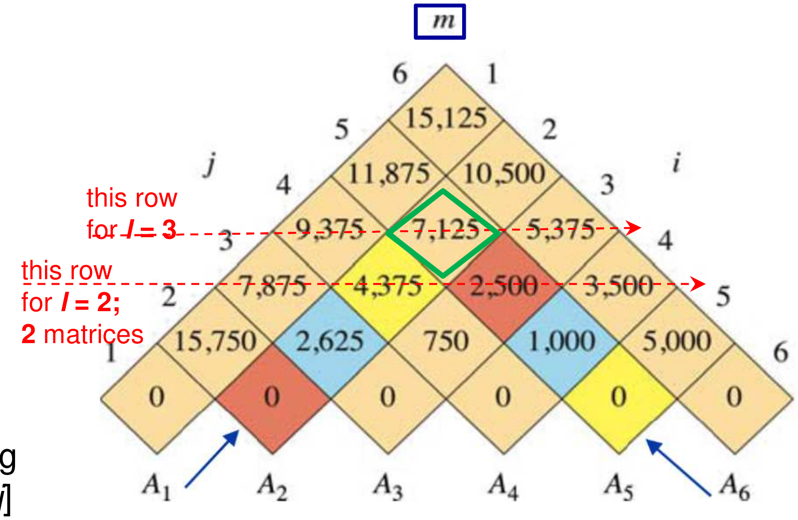
Computing optimal costs

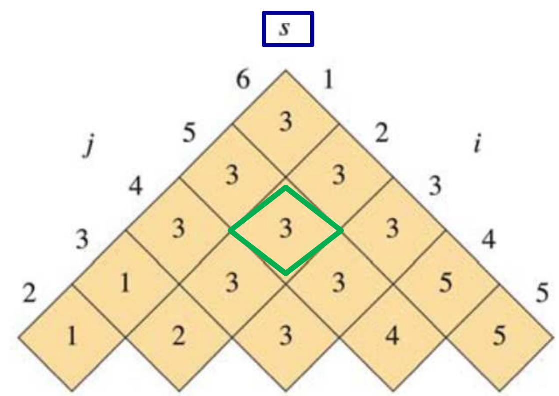
MATRIX-CHAIN-ORDER(p)

Time complexity = $\Theta(n^3)$

```
n = p.length − 1
let m[1..n, 1..n] and s[1..n − 1, 2..n] be new tables
for i = 1 to n
    m[i, i] = 0
for l = 2 to n                    // l is the chain length
    for i = 1 to n − l + 1
        j = i + l − 1             // range from element i to element j
        m[i, j] = ∞
        for k = i to j − 1
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q
                s[i, j] = k
return m and s
```

for memoizing results in $m[i, j]$





matrix	A_1	A_2	A_3	A_4	A_5	A_6
dimension	30×35	35×15	15×5	5×10	10×20	20×25

$m[1, 6]$ = 15,125

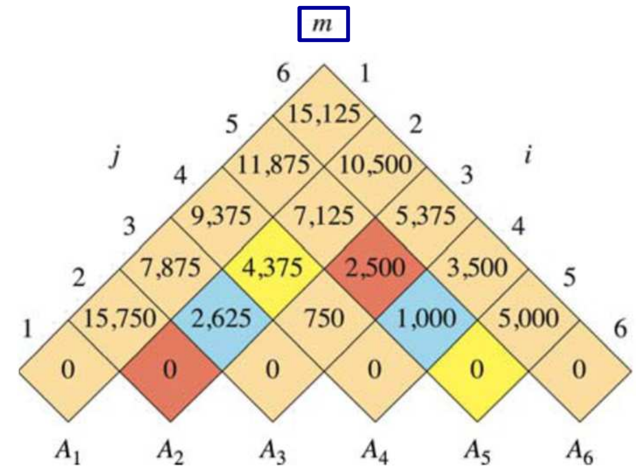
its computation relies on
$m[1, 1]$, $m[2, 6]$; $m[1, 2]$, $m[3, 6]$;
$m[1, 3]$, $m[4, 6]$; $m[1, 4]$, $m[5, 6]$;
$m[1, 5]$, $m[6, 6]$.

$$m[2,5] = \min \begin{cases} m[2,2] + m[3,5] + p_1 p_2 p_5 = 0 + 2500 + 35 \cdot 15 \cdot 20 = 13{,}000\,, \\ m[2,3] + m[4,5] + p_1 p_3 p_5 = 2625 + 1000 + 35 \cdot 5 \cdot 20 = 7125\,, \\ m[2,4] + m[5,5] + p_1 p_4 p_5 = 4375 + 0 + 35 \cdot 10 \cdot 20 = 11{,}375 \end{cases}$$
$$= 7125\,.$$

Dynamic Programming (continued)

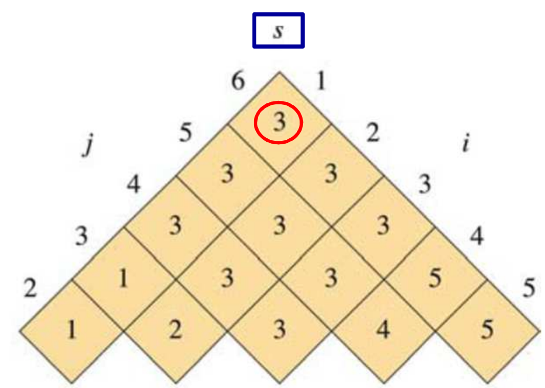
Constructing an optimal solution

– Entry $s(i, j)$ keeps k where optimal parenthesization splits between A_k and A_{k+1}



PRINT-OPTIMAL-PARENS(s, i, j)

```
if i == j
    print "A"_i
else print "("
    PRINT-OPTIMAL-PARENS(s, i, s[i, j])
    PRINT-OPTIMAL-PARENS(s, s[i, j] + 1, j)
    print ")"
```



PRINT-OPTIMAL-PARENS$(s, 1, 6)$ prints the parenthesization $((A_1(A_2A_3))((A_4A_5)A_6))$.

PRINT-OPTIMAL-PARENS(s, 1, s[1, 6] = 3)

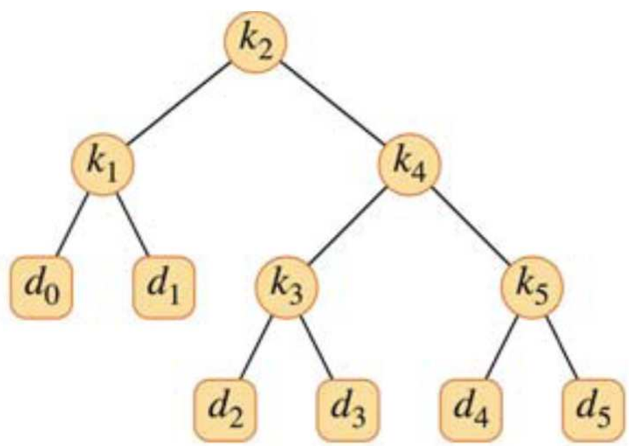
PRINT-OPTIMAL-PARENS(s, s[1, 6]+1=4, 6)

Dynamic Programming (continued)

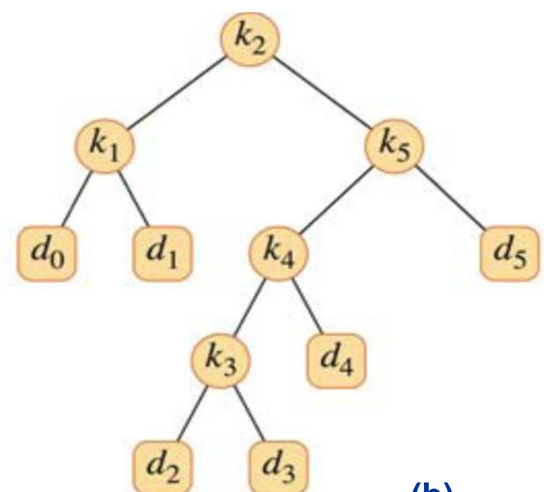
- **Optimal binary search trees (O-BSTs)**
  - given a set of sorted keys $K = \langle k_1, k_2, \ldots k_n \rangle$, with p_i being search visit probability of k_i, build a binary search tree with the *mean search cost* minimized
  - n+1 "dummy keys (ranges)" (i.e., d_j) required to cover those value ranges outside K

contribution = $(\text{depth}_T(k_i)+1)\cdot p_i$

node	depth	probability	contribution
k_1	1	0.15	0.30
k_2	0	0.10	0.10
k_3	2	0.05	0.15
k_4	1	0.10	0.20
k_5	2	0.20	0.60
d_0	2	0.05	0.15
d_1	2	0.10	0.30
d_2	3	0.05	0.20
d_3	3	0.05	0.20
d_4	3	0.05	0.20
d_5	3	0.10	0.40
Total			2.80



(a)



(b)

Figure 15.9 Two binary search trees for a set of $n = 5$ keys with the following probabilities:

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

$$\sum_{i=1}^{n} p_i + \sum_{i=0}^{n} q_i = 1$$

(a) A binary search tree with expected search cost 2.80. **(b)** A binary search tree with expected search cost 2.75. This tree is optimal.

Dynamic Programming (continued)

- **Optimal binary search trees (O-BSTs)**
  - number of binary trees with n nodes: $\Omega(n^4/n^{3/2})$ // see Prob. 12-4, pp. 306
  - optimal solution to the problem obtainable from optimal solutions to its subproblems: root k_r has left subtree, k_i, k_{i+1}, … k_{r-1}, and right subtree, k_{r+1}, k_{r+2}, …, k_j
  - recursive solution: let $e[i, j]$ denote mean cost of searching O-BST with k_i, k_{i+1}, …, k_j then, $e[i, j] = p_r + (e[i, r-1] + w(i, r-1)) + (e[r+1, j] + w(r+1, j))$, with k_r the tree root,

    due to tree height raised by 1

    where $w(i, j)$ is *expected search prob.* over the tree, equal to $w(i,j) = \sum_{l=i}^{j} p_l + \sum_{l=i-1}^{j} q_l$

    We have final $e[i, j]$ as follows, due to $w(i,j) = w(i,r-1) + p_r + w(r+1,j)$:

$$e[i,j] = \begin{cases} q_{i-1} & \text{if } j = i-1 \,, \\ \min\limits_{i \le r \le j} \{e[i,r-1] + e[r+1,j] + w(i,j)\} & \text{if } i \le j \,. \end{cases}$$

due to tree height raised by 1

There are relatively fewer distinct subproblems (i.e., O-BST involving k_i, k_{i+1}, … k_j):

$$C(n, 2) + n = \Theta(n^2)$$

selecting i & j out of n *values*

due to $i = j$

Dynamic Programming (continued)

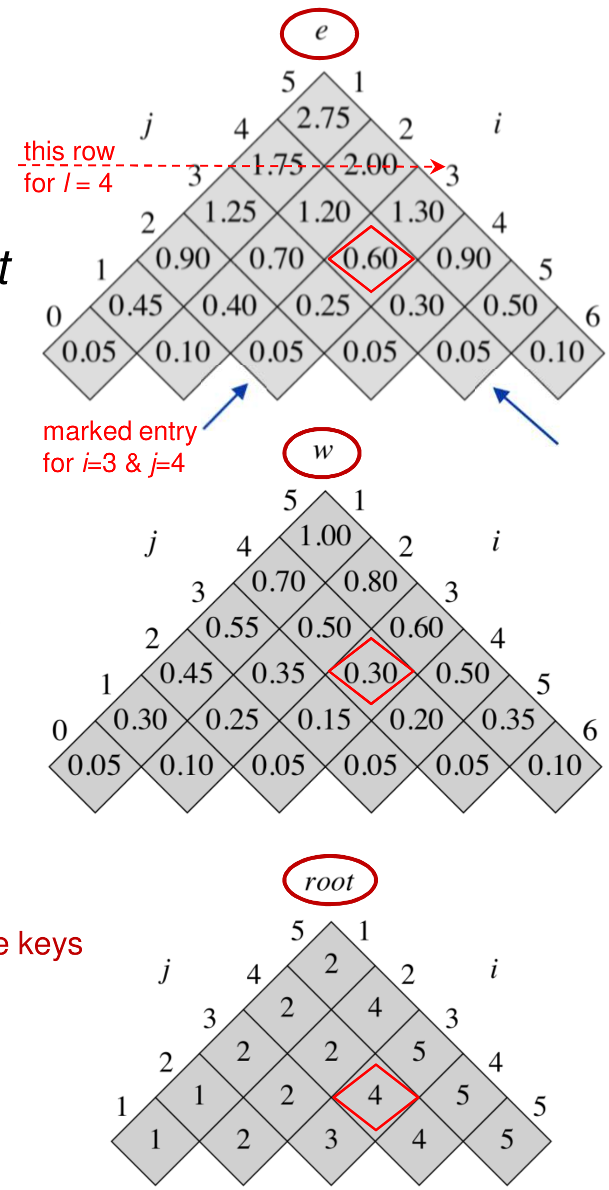
- **Optimal binary search trees (O-BSTs)**
  - compute the expected search cost of an O-BST
  - we store the $e[i, j]$ values in a table $e[1\,..\,n+1, 0\,..\,n]$.
  - keep optimal BST for key subset of k_i, k_{i+1}, …, k_j in *root*

OPTIMAL-BST(p, q, n)

```
let e[1..n+1, 0..n], w[1..n+1, 0..n],
        and root[1..n, 1..n] be new tables
for i = 1 to n+1
    e[i, i-1] = q_{i-1}
    w[i, i-1] = q_{i-1}
for l = 1 to n
    for i = 1 to n-l+1
        j = i+l-1
        e[i, j] = ∞
        w[i, j] = w[i, j-1] + p_j + q_j
        for r = i to j        // search subset of k_i .. k_j with l consecutive keys
            t = e[i, r-1] + e[r+1, j] + w[i, j]
            if t < e[i, j]
                e[i, j] = t
                root[i, j] = r
return e and root
```

Note. $\boldsymbol{p_j}$ & $\boldsymbol{q_j}$

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10



Resulting O-BST obtains from *root* table; see previous slide.

<u>Greedy Algorithm</u>

- **Basics**
  - make locally optimal choice, so only a single subproblem is solved in each step
  - simpler and more efficient
  - powerful and applicable to wide ranges of problems

An Example

<u>Activity selection problem</u>. To find a maximal subset of *compatible activities* from a set of activities, each with a starting time and an ending time.

Given the following set S of activities, **<u>sorted</u> according to their <u>finish times</u>**

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

Compatible activities, a_k and a_l, satisfy $f_k \le s_l$ or $s_k \ge f_l$.

<u>Largest compatible sets:</u> $\{a_1, a_4, a_8, a_{11}\}$ and $\{a_2, a_4, a_9, a_{11}\}$.

This cannot be identified via a greedy algorithm.

Greedy Algorithm (continued)

Given a set of activities, S, sorted according to their finish times, i.e.,

$$f_1 \leq f_2 \leq f_3 \leq \cdots \leq f_{n-1} \leq f_n$$

i	1	2	3	4	5	6	7	8	9	10	11
s_i	1	3	0	5	3	5	6	8	8	2	12
f_i	4	5	6	7	9	9	10	11	12	14	16

Let S_{ij} denote the set of activities that start after a_i finishes and that finish before a_j starts. Find out a maximum set of compatible activities in S_{ij}, with such a set represented by A_{ij}.

If A_{ij} contains a_k, then the solution is left with two subproblems: maximum set of compatible activities in S_{ik} and the maximum set of compatible activities in S_{kj}, namely, $A_{ij} = A_{ik} \cup \{a_k\} \cup A_{kj}$.

Let the size of an optimal solution for S_{ij} be denoted by $c[i, j]$, with a_k in the optimal solution set. We have the recurrence of

$$c[i, j] = c[i, k] + c[k, j] + 1.$$

Consider every possible a_k, we have:

$$c[i, j] = \begin{cases} 0 & \text{if } S_{ij} = \emptyset\ , \\ \max\limits_{a_k \in S_{ij}} \{c[i, k] + c[k, j] + 1\} & \text{if } S_{ij} \neq \emptyset\ . \end{cases}$$

Note: S_i denotes the subset of activities $\{a_{i+1}, a_{i+2}, \ldots, a_n\}$ that start after $\boldsymbol{a_i}$ finishes.

One may adopt dynamic programming to solve the above recurrence (either top-down with memoization or bottom-up with auxiliary table(s)), or preferably to solve *just one* desirable subproblem (instead of *all*) in each step via a greedy choice, called the greedy approach.

Greedy Algorithm (continued)

- **Greedy Choice**
  - activity selection problem chooses the activity which ***finishes the earliest***
  - typically, one follows top-down manner to choose an activity for adding to the optimal solution, then finds maximal compatible activities for left subproblem
  - the greedily chosen activity always part of some optimal solution, as follows:

Theorem 1. This refers to the greedy choice property.

If S_k is nonempty and a_m has the earliest finish time in S_k, then a_m is included in some optimal solution.

Unlike S_{kv}, S_k denotes the subset of activities $\{a_{k+1}, a_{k+2}, \ldots, a_n\}$ that start **after a_k finishes.**

Recursive greedy algorithm

With greedy selection of a_m, only one subproblem (S_{m+1}) is left after selecting a_m.

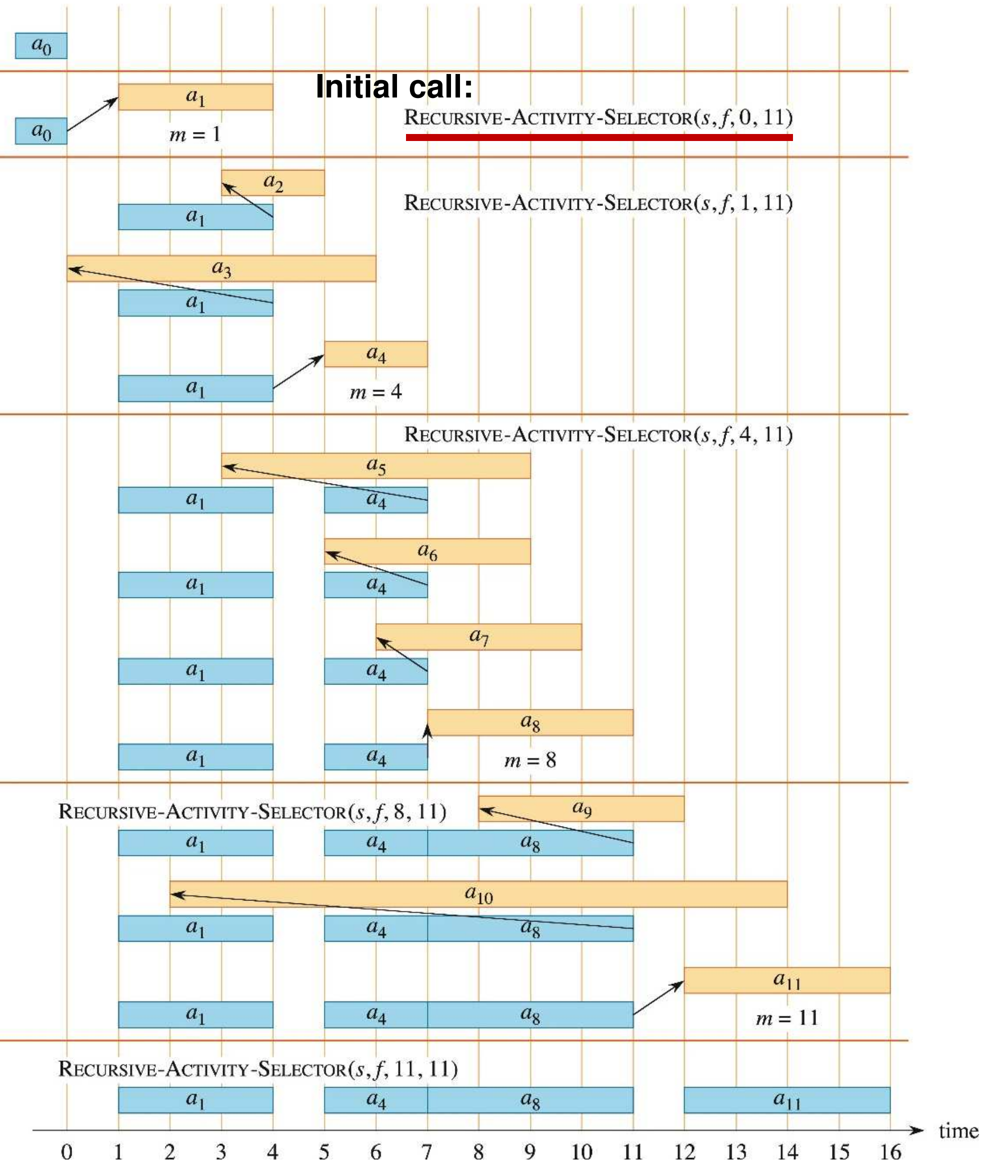
REC-ACTIVITY-SELECTOR(s, f, k, n)

```
m = k + 1
while m ≤ n and s[m] < f[k]      // find the first activity in S_k to finish
    m = m + 1                    after a_k is added to the solution (i.e., compatible with a_k).
if m ≤ n
    return {a_m} ∪ REC-ACTIVITY-SELECTOR(s, f, m, n)
else return Ø
```

Greedy Algorithm (continued)

- **Recursive greedy algorithm**

i	s_i	f_i
0	–	0
1	1	4
2	3	5
3	0	6
4	5	7
5	3	9
6	5	9
7	6	10
8	7	11
9	8	12
10	2	14
11	12	16



Idea

The **while** loop checks $a_{k+1}, a_{k+2}, \ldots, a_n$ until it finds an activity a_m that is compatible with a_k (need $s_m \geq f_k$).

- If the loop terminates because a_m is found ($m \leq n$), then recursively solve S_m,
- If the loop never finds a compatible a_m ($m > n$), then just return empty set.

Time

$\Theta(n)$—each activity examined exactly once, assuming that activities are already sorted according to finish times.

Greedy Algorithm (continued)

- **Iterative version of greedy algorithm**

GREEDY-ACTIVITY-SELECTOR(s, f)

$n = s.length$
$A = \{a_1\}$
$k = 1$
for $m = 2$ **to** n
    if $s[m] \geq f[k]$
        $A = A \cup \{a_m\}$ // Activity a_m being most recently added to A.
        $k = m$
return A

<u>Greedy Algorithm</u> (continued)

- **Elements of greedy strategy to yield *global optimum***
  - a sequence of choices, each of which is best at the moment
  - key elements: <u>greedy-choice property</u> and <u>optimal substructure</u>
  - greedy algorithm with three steps listed below, to yield optimal solutions (for matriods, i.e., linear independence)

1. Cast the optimization problem as one in which we make a choice and are left with <u>one subproblem</u> to solve.
2. Prove that there's always an optimal solution that makes the greedy choice, so that the <u>greedy choice is always safe.</u>
3. Demonstrate optimal substructure by showing that, having made the greedy choice, combining an optimal solution to the remaining subproblem with the greedy choice gives an <u>optimal solution to the original problem</u> (with **<u>linear independence</u>**)

<u>*Greedy* ***strategy typically will***</u>

- Make a choice at each step.
- Make the choice *before* solving the subproblems.
- Solve *top-down*.

<u>Greedy Algorithm</u> (continued)

- **Huffman codes**
  - for efficient data compression using <u>variable</u> code length
  - fixed-length code is simple but larger in its encoded footprint
  - <u>prefix codes</u>: no codeword is a prefix of any other codeword

<u>Example</u>

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100

← totally, 100K

Fixed length: 3 bits per symbol type or 300,000 bits for 100K symbols

Variable length: $(45 \cdot 1 + 13 \cdot 3 + 12 \cdot 3 + 16 \cdot 3 + 9 \cdot 4 + 5 \cdot 4) \cdot 1{,}000 = 224{,}000$ bits

savings of ~25%

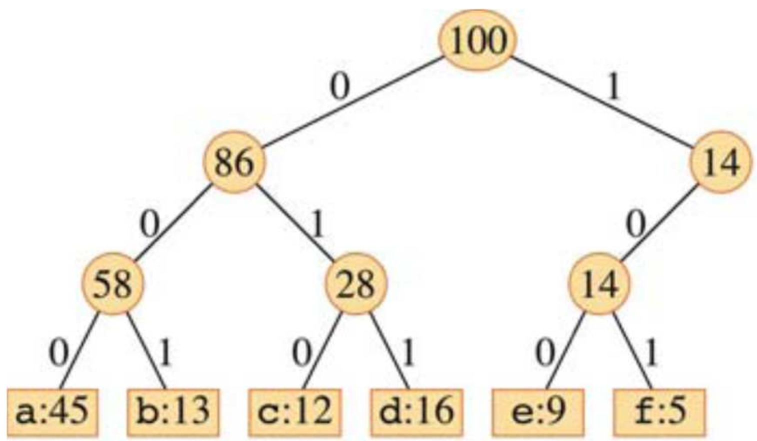
Greedy Algorithm (continued)

- **Prefix codes**
  - no codeword is a prefix of any other codeword
  - simplifying decoding
  - achieving *optimal* data compression for character coding

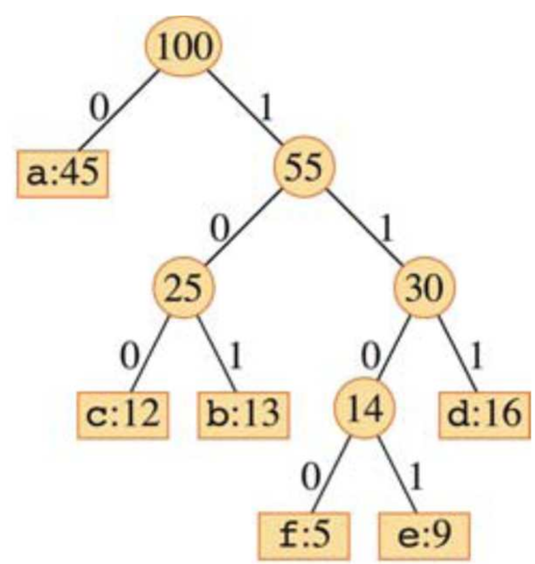
Tree representations of character coding

Number of bits (cost) required of ***T***: $B(T) = \sum_{c \in C} c.freq \cdot d_T(c)$, where $d_T(c)$ is codeword length.

	a	b	c	d	e	f
Frequency (in thousands)	45	13	12	16	9	5
Fixed-length codeword	000	001	010	011	100	101
Variable-length codeword	0	101	100	111	1101	1100



(a) Fixed-length coding, with the same height, but not a full binary tree, for all codewords.



(b) Optimal coding, always denoted by full binary tree with variable heights.

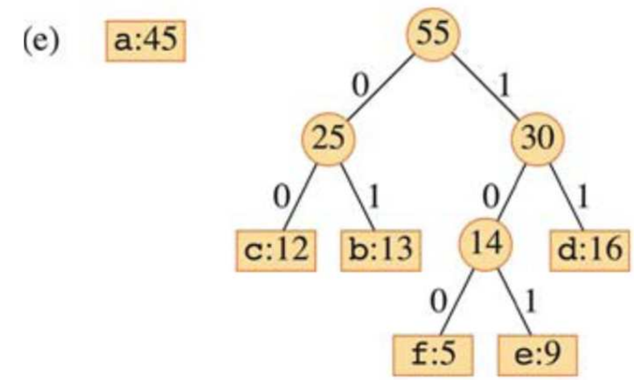
Greedy Algorithm (continued)

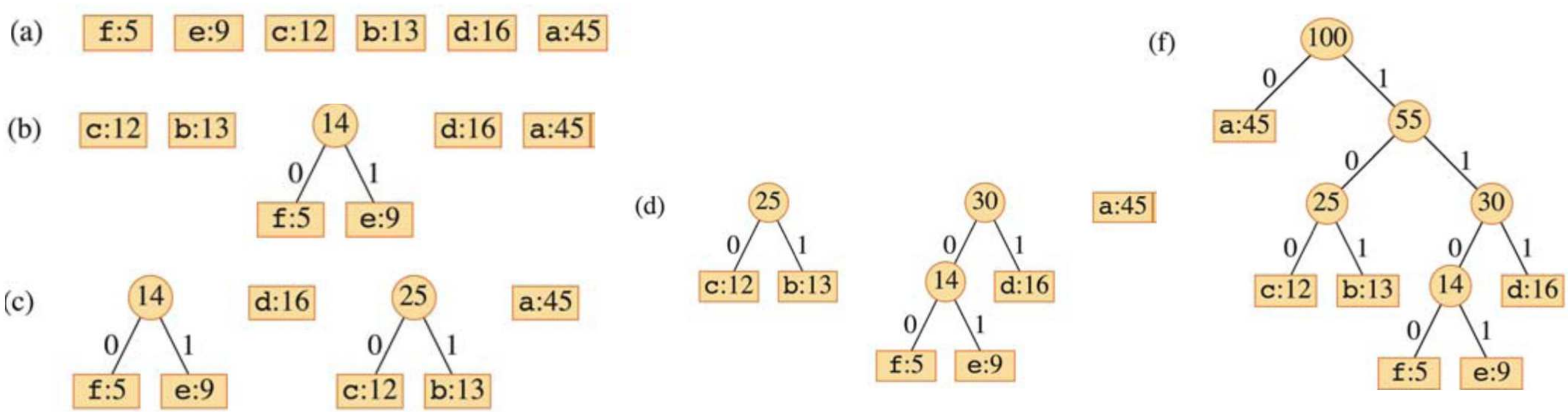
- **Huffman code construction**

HUFFMAN(C) C is a set of n characters and each character $c \in C$ is an object with an attribute $c.freq$

```
n = |C|
Q = C
for i = 1 to n - 1
    allocate a new node z
    z.left = x = EXTRACT-MIN(Q)
    z.right = y = EXTRACT-MIN(Q)
    z.freq = x.freq + y.freq
    INSERT(Q, z)
return EXTRACT-MIN(Q)        // return the root of the tree
```

Example

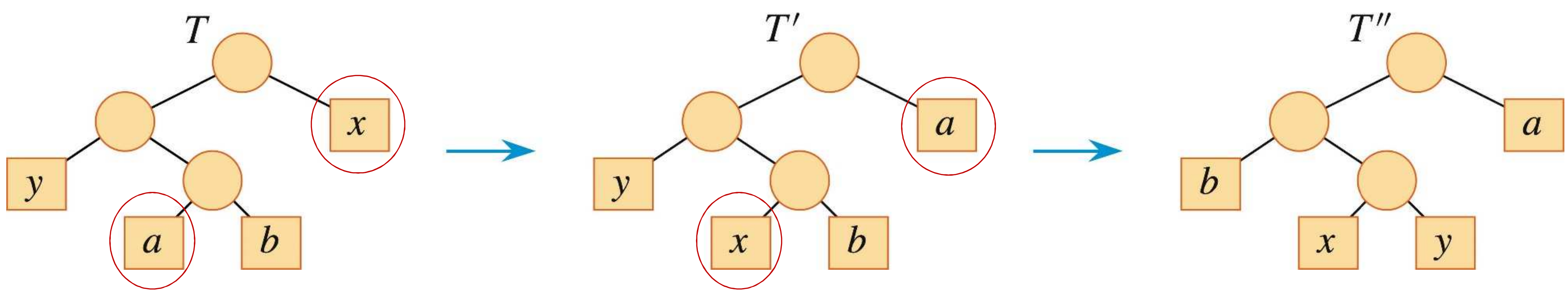




Greedy Algorithm (continued)

Lemma 1

Let C be an alphabet in which each character $c \in C$ has frequency $c.freq$. Let x and y be two characters in C having the lowest frequencies. Then there exists an optimal prefix code for C in which the codewords for x and y have the same length and differ only in the last bit.



T denotes arbitrary optimal prefix code

$$\begin{aligned}
&B(T) - B(T') \\
&\quad = \sum_{c\in C} c.freq \cdot d_T(c) - \sum_{c \in C} c.freq \cdot d_{T'}(c) \\
&\quad = x.freq \cdot d_T(x) + a.freq \cdot d_T(a) - x.freq \cdot d_{T'}(x) - a.freq \cdot d_{T'}(a) \\
&\quad = x.freq \cdot d_T(x) + a.freq \cdot d_T(a) - x.freq \cdot d_T(a) - a.freq \cdot d_T(x) \\
&\quad = (a.freq - x.freq)(d_T(a) - d_T(x)) \\
&\quad \geq 0\,,
\end{aligned}$$

therefore, $B(T) \geq B(T')$

But, $B(T') \geq B(T)$ since T is optimal, so we have $B(T') \equiv B(T)$.

Similarly, $B(T') \geq B(T'')$ to yield $B(T) \geq B(T'')$, since $B(T') \equiv B(T)$.

Given that T is optimal, we have $B(T'') \equiv B(T)$, implying that ***T''*** is another optimal prefix code with x and y differing only in the last bit.

Greedy Algorithm (continued)

Lemma 2

Let C be a given alphabet with frequency $c.freq$ defined for each character $c \in C$. Let x and y be two characters in C with **smallest frequencies.** Let C' be the alphabet C with the characters x and y removed and a new character z added, so that $C' = C - \{x, y\} \cup \{z\}$. Define $freq$ for C' as for C, except that $z.freq = x.freq + y.freq$. Let T' be any tree representing an optimal prefix code for the alphabet C'. Then the tree T, obtained from T' by replacing the leaf node for z with an internal node having x and y as children, represents an optimal prefix code for the alphabet C.

Theorem

Procedure HUFFMAN produces an optimal prefix code.