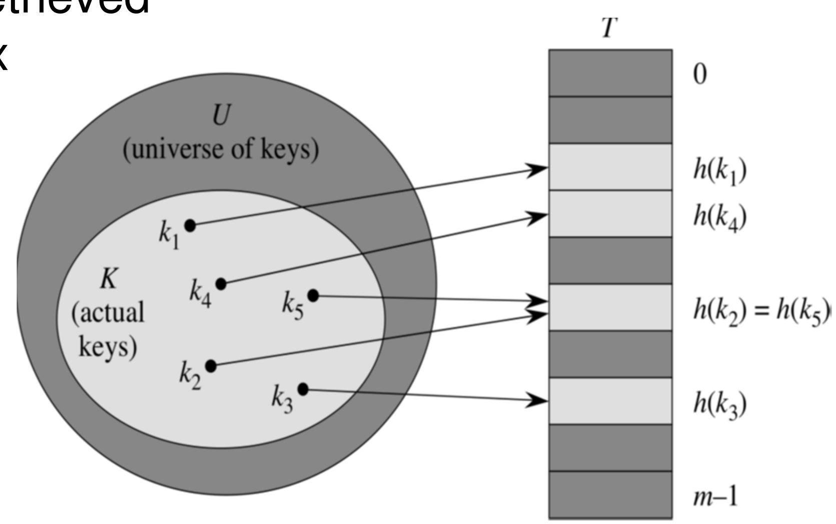
Part II

Data Structures

§ Three Types of Data Structures Considered:

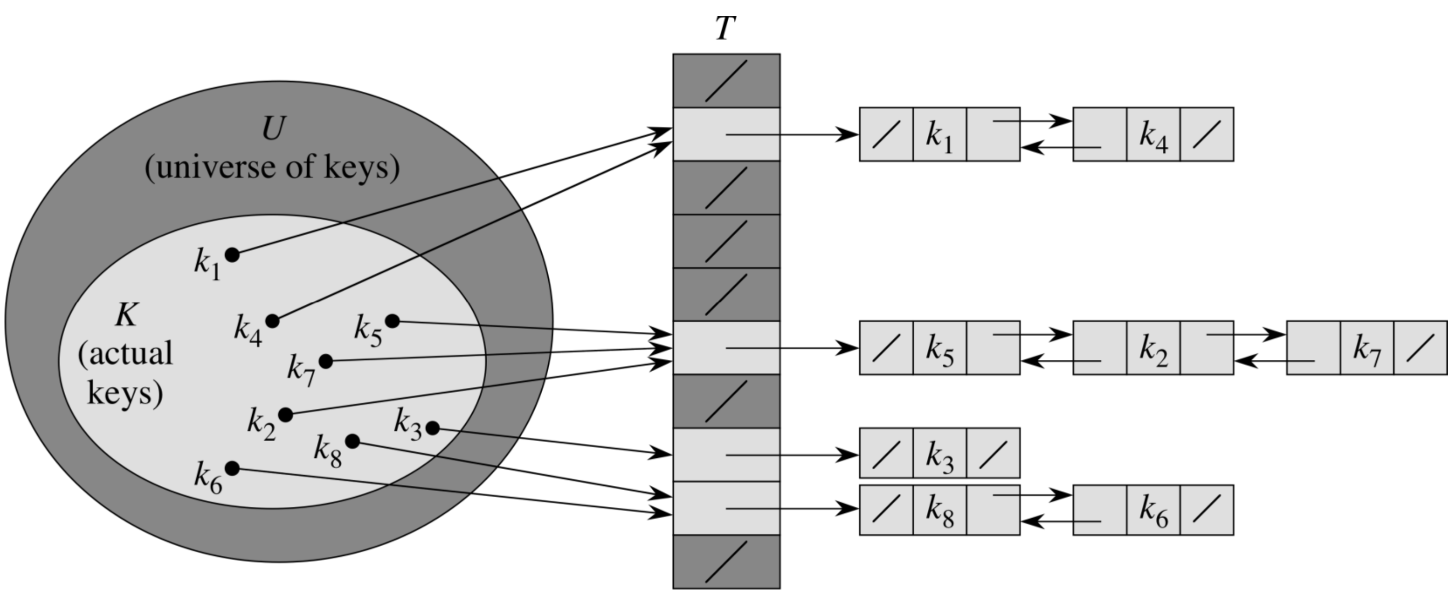
- Hash Tables
- Trees
- Heaps

- **Hash Table**
  - – suitable for large key space with small numbers of actual keys stored
  - – average search time over the hash table equal to $O(1)$
  - – collision-handling strategy required
  - – keys must be stored to ensure right items retrieved
  - – hash function maps a key to one table index



Hash Tables (continued)

- **Hash Table with Collision Resolution by Chaining**
  - multiple elements with different keys mapped to the same table entry (collisions)
  - they can be chained, with a lookup going through the chain
  - for load factor of α = n/m, where n (or m) is the number of elements (or table entries)
  - under uniform distribution, one search (be successful or not) takes Θ(1+α), according to Theorems 11.1 and 11.2
  - if hash table size (m) grows in proportion to n, then the mean search time is Θ(1)
  - there are other more efficient ways for handling collisions



Hash Tables (continued)

- **Hash Functions**
  - division method
  - multiplication method

1. Division method: (more restrictive on the suitable m)

$h(k) = k \bmod m$, where the choice of $m = 2^p - 1$ is poor but a *prime* not close to an exact power of 2 is often good

For example, given $n = 2000$ character strings to be stored in a hash table with $m = 701$ entries, then an unsuccessful search takes some 3 accesses

Hash Tables (continued)

2. Multiplication method:

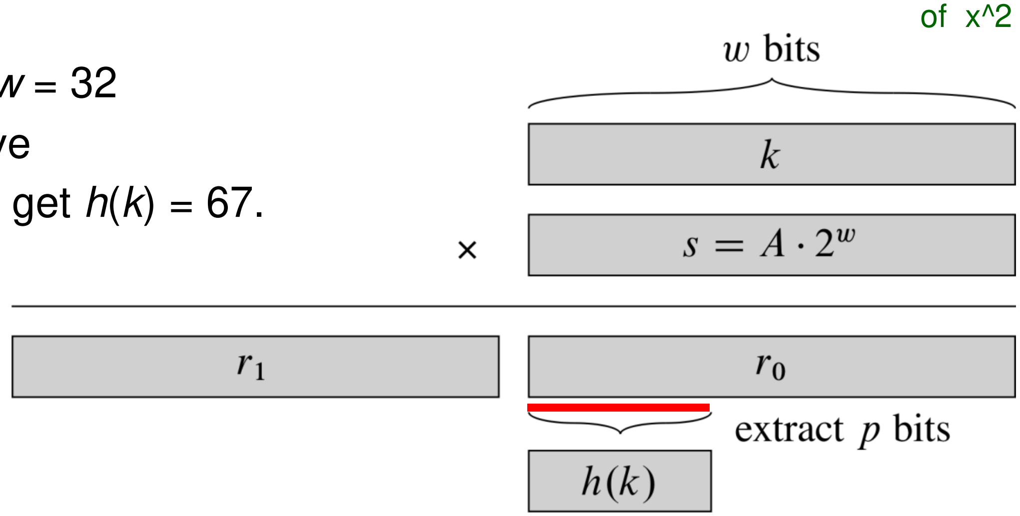
○ Multiply k by a constant A, with $0 < A < 1$ and extract the fraction part of $k \cdot A$, e.g.,
$h(k) = \lfloor m\,(kA \mod 1) \rfloor$, where "$kA \mod 1$" equals $kA - \lfloor kA \rfloor$

– table size (m) can be arbitrary, e.g., $m = 2^p$
– let A be a fraction of the form $s/2^w$ as shown in the figure below
– perform multiplication of w-bit k and w-bit s, to get a $2w$-bit product
– product denoted by $r_1 \cdot 2^w + r_0$, then p-bit hash value is obtained from r_0

– while arbitrary A works, $A \approx (\sqrt{5}-1)/2 = 0.6180339887 \ldots$ recommended

Golden ratio: a root of x^2 + x -1 = 0.

+ Let $k = 123456$, $p = 14$, $m = 2^{14}$, $w = 32$ and $A = 2654435769/2^{32}$, we have $k \cdot s = (76300 \cdot 2^{32}) + 17612864$ to get $h(k) = 67$.



<u>Hash Tables</u> (continued)

- **Open Addressing** (to deal with collision-chaining)
  - – elements stored inside the table
  - – calculating probe sequence of given key, <u>instead of using pointers</u>:
    $\langle h(k, 0), h(k, 1), h(k, 2), \ldots, h(k, m-1) \rangle$
  - – if probe sequence is a permutation of $\langle 0, 1, 2, \ldots, m-1 \rangle$, every table entry is a candidate location for the element
  - – the probe sequence is fixed for a given key
  - – key has to be stored in the table entry

<u>HASH-INSERT(T, k)</u>

```
i = 0
repeat
    j = h(k, i)
    if T[j] == NIL
        T[j] = k
        return j
    else i = i + 1
until i == m
error "hash table overflow"
```

<u>HASH-SEARCH(T, k)</u>

```
i = 0
repeat
    j = h(k, i)
    if T[j] == k
        return j
    i = i + 1
until T[j] == NIL or i = m
return NIL
```

Hash Tables (continued)

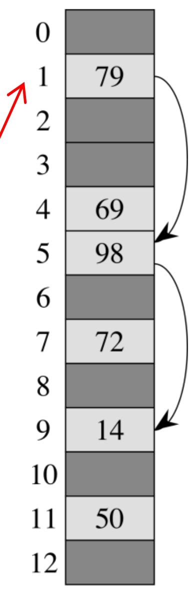
- **Uniform Hashing Analysis on Open Addressing**
  - key probe sequence equal likely in one of $m!$ permutations of $\langle 0, 1, 2, \ldots., m-1\rangle$
  - probe sequences defined for open addressing
    + linear probing: m different sequences
      - dictated by $h(k, i) = (h'(k) + i) \bmod m$
      - subject to *primary clustering*
    + quadratic probing: m different sequences
      - dictated by $h(k, i) = (h'(k) + c_1 \cdot i + c_2 \cdot i^2) \bmod m$
      - subject to *secondary clustering*, i.e., $h(k_1, 0) = h(k_2, 0) \Longrightarrow$ both keys follow the same probe sequence
    + double hashing: m^2 different sequences
      - dictated by $h(k, i) = (h_1(k) + i \cdot h_2(k)) \bmod m$
      - many good choices: $h_2(k)$ relatively prime to m; m itself a prime; m a power of 2 and $h_2(k)$ always an odd number; or below:

        m a prime and m' slightly less than m (e.g., m-1 or m-2) with

        $h_1(k) = k \bmod m$

        $h_2(k) = 1 + (k \bmod m')$

e.g., for k =14 under m = 13, m' = 11; upon inserting k = 14.



Hash Tables (continued)

- **Open-Address Hashing Analysis**

Theorem 1:

For an open-address hash table with load factor $\alpha = n/m < 1$, the expected number of probes in unsuccessful search under uniform hashing is at most $1/(1 - \alpha)$.

Note: unsuccessful search ends at seeing an empty entry.

.

+ if α is a constant, an unsuccessful search runs in $O(1)$ time
+ inserting an element into an open-address hash table with load factor α takes at most $1/(1 - \alpha)$ probes on an average under uniform hashing, because it has $(1-\alpha)$ for 1 probe, plus prob. $\alpha(1-\alpha)$ to take 2 probes, plus prob. $\alpha^2(1-\alpha)$ to take 3 probes, etc., yielding $1+\alpha+\alpha^2+\alpha^3+ \ldots = 1/(1 - \alpha)$

e.g., for α = 50/100, we have 2 probes.

Theorem 2:

For an open-address hash table with load factor $\alpha < 1$, the expected number of probes in a successful search under uniform hashing is at most $\frac{1}{\alpha} \cdot \ln\frac{1}{1-\alpha}$.

► successful search for a key equals the sequence of inserting the key

2^{nd} key insertion takes ≤ $1/(1-(1/m))$ probes, on an average, when α is $1/m$
3^{rd} key takes ≤ $1/(1-(2/m))$ probes, on an average, when α is $2/m$
:
$i+1^{th}$ key takes ≤ $1/(1-(i/m))$ probes

Mean no. of probes equals results over all n keys inserted

This is because **insertion and probes** follow the same hash function.
We have: $1/n\,(\sum_{i=0}^{n-1} 1/(1-i/m))$

e.g., for α = 50/100, we have ≈ 1.39 probes, as ln(2) ~ 0.693. (base e ~ 2.718)

<u>Hash Tables</u> (continued)

- **Universal Hashing**

  For the hash function of $h_{ab}(k) = ((a \cdot k + b) \bmod p) \bmod m$, where p is a <u>large prime</u> number, <u>with $p > m$</u>, $a \in \{1, 2, \ldots p\text{-}1\}$ and $b \in \{0, 1, 2, p\text{-}1\}$, the collection of such hash functions is *universal*.

In this case, m can be <u>any number</u> and does not have to be a prime.
For example, for $p = 17$ and $m = 8$, we have $h_{34}(15) = 7$.

((3*15+4) mod 17) mod 8 = 7

Hash Tables (continued)

- **Perfect Hashing**
  - set of keys is *static* (never changed), then excellent worst case performance exists
  - search takes $O(1)$ in memory accesses in the worst case
  - two levels of hashing, with secondary hash table S_j to hold those hashed to slot j
  - unlikely to have collisions in S_j (with n_j keys) if its size m_j equals n_j^2 $\rightarrow$ easy to determine a collision-free secondary hash function
  - total storage required for both primary and secondary hash tables is $O(n)$

Theorem 3:

Given n_j keys for a hash table sized $m_j = n_j^2$, the probability of having collisions $< \mathbf{1/2}$.

This theorem implies that one can easily (with a few trials) identify **collision-free secondary** $h_j(k)$ with proper a_j and b_j

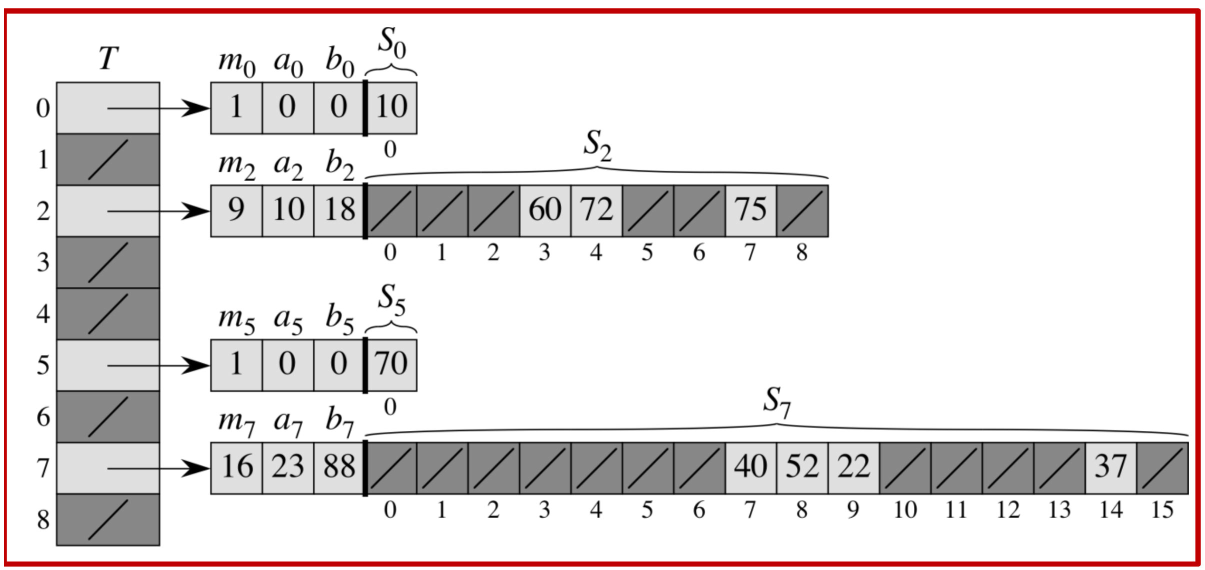


Figure 11.6 Using perfect hashing to store the set $K = \{10, 22, 37, 40, 60, 70, 75\}$. The primary hash function is $h(k) = ((a \cdot k + b) \bmod p) \bmod m$, where $a = 3$, $b = 42$, $p = 101$, and $m = 9$.

For example, $h(75) = 2$, so Key 75 hashes to slot 2 of table T. A secondary hash table S_j stores all keys hashing to Slot j. The size of hash table S_j is m_j, and the associated hash function is $h_j(k) = ((a_j k + b_j) \bmod p) \bmod m_j$. If $h_2(75) = 7$, Key 75 is stored in Slot 7 of secondary hash table $\boldsymbol{S_2}$. There are no collisions in any of the secondary hash tables, and so searching takes a constant time in the worst case.

Hash Tables (continued)

- **Perfect Hashing**
  - total expected storage for both primary and secondary hash tables is $O(n)$.

Theorem 4:

Given n keys for an outer hash table sized $m = n$ and n_j keys present in Slot j under outer hash, we have $E\left[\sum_{j=0}^{m-1} n_j^2\right] < \mathbf{2n}$. (Note: this deals with the "expected" storage size.)

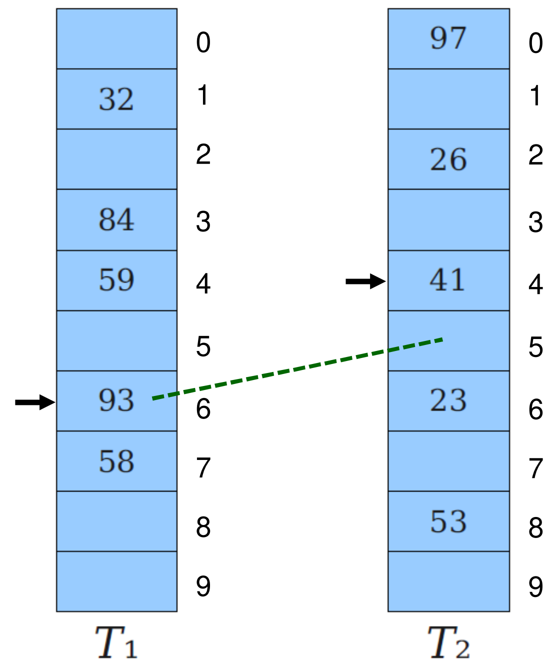
Corollary 1:

With n keys for a hash table sized $m = n$ and n_j keys present in Slot j under the outer hash and the sizes of secondary hash tables equal to $m_j = n_j^2$, we have the probability of total storage used for secondary hash tables $\geq 4n$ less than ½.

This corollary implies that one can easily (with a few trials) identify **good primary hash** $h(k)$ with proper a and b that lead to a reasonable amount of total storage required.

Hash Tables (continued)

- **Cuckoo Hashing**§
  - <u>move keys around</u> upon insertion for worst <u>probe</u> case of $O(1)$
    - + quick search but possibly lengthy insertion
    - + multiple hash functions required for separate hash tables
  - delete keys in the worst-case of $O(1)$
  - example with two hash functions below:
    - + two hash tables, each with m elements
    - + two hash functions, h_1 and h_2, one for a table
    - + each key x at either $h_1(x)$ or $h_2(x)$
    - + new key '10' with $h_1(10) = 6$ and $h_2(10) = 4$
      - * move key '93' around, if $h_2(93)=5$, for insertion



  - Algorithm
    1. insert "new key" to T_1 located at h_1(new key), when available
    2. otherwise, "new key" displaces "existing key" in T_1; place "existing key" in T_2 located at h_2(existing key); repeat the displacement process, if needed.

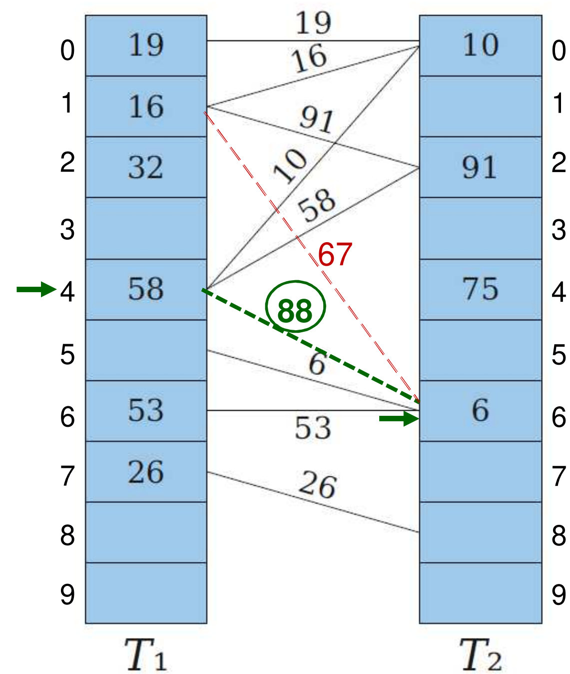
§ R. Pagh and F. Rodler, "Cuckoo Hashing," *Journal of Algorithms*, Aug. 2001, pp. 121-133.

Hash Tables (continued)

- **Cuckoo Graph**
  – derived from cuckoo hash tables
    + each table entry is a node
    + each key is an edge, which links the entries that can hold the key
    + an insertion adds a new edge to the graph
      * let $h_1(88) = 4$ and $h_2(88) = 6$
      * displace key '58' in T_1 to make room for key '88'
      * displace key '91' in T_2 to make room for key '58'
      * displace key '16' in T_1 to make room for key '91'
      * displace key '10' in T_2 to make room for key '16'
      * repeat ….

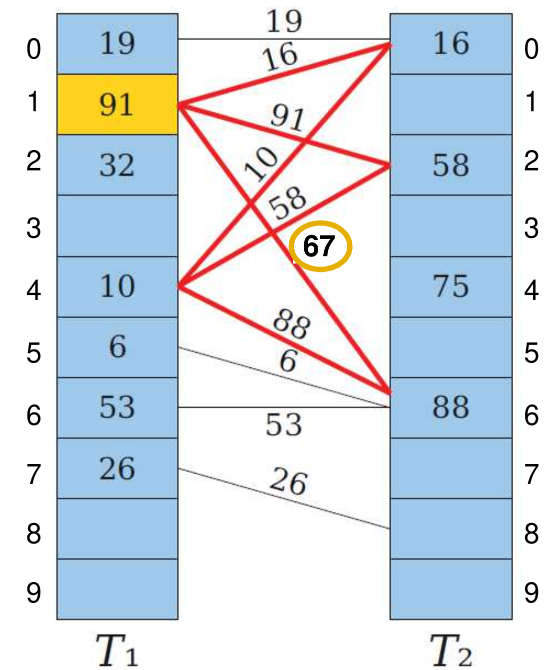
    + insertion done by tracing a path over the graph

    + what about inserting next key "67" with
      $h_1(67) = 1$ and $h_2(67) = 6$?



Hash Tables (continued)

- **Cuckoo Graph**
  – insertion succeeds if and only if the new edge (defined by the new key) contains at most one cycle



Proof sketch:

1. each edge denotes a key and requires a table entry (i.e., node) to hold it
2. for a cycle exists, the number of its edges equals the number of its nodes involved, so that all nodes (table entries) are taken
3. if a new edge (after added) is on two cycles the edge must link two taken nodes AND all nodes on the two cycles have been used
4. the new key (edge) *cannot* be accommodated.

Side note: two cycles (one with 4 distinct nodes and the other with at least one distinct node) that share one edge include (at least) 5 distinct nodes but contain (at least) 6 distinct edges; impossible.

Binary Search Trees

§ Binary Search Tree Property

– Given tree node x, if node y is in the left (or right) subtree of x, we have
$y.key \leq x.key$ (or $y.key \geq x.key$).

Theorem 1

INORDER-TREE-WALK(x) across an n-node subtree rooted at x takes $\Theta(n)$.

(because the tree height may equal n)

Proof.

The tree walk has to visit every node, and hence, the time complexity $T(n)$ is lower bounded by $\Omega(n)$.

Next, considering that INORDER-TREE-WALK(x) is called on x whose left and right subtrees have k and $(n\text{-}k\text{-}1)$ nodes, respectively, we have

$T(n) \leq T(k) + T(n\text{-}k\text{-}1) + d$ for some $d > 0$.

By the substitution method, we prove that $T(n) \leq (c+d)n + c$, for a constant c, below:

$$\begin{aligned}
T(n) &\leq T(k) + T(n\text{-}k\text{-}1) + d \\
&\leq ((c+d)k + c) + ((c+d)(n\text{-}k\text{-}1) + c) + d \\
&= (c+d)n + c - (c+d) + c + d \\
&= \underline{(c+d)n + c} \quad \text{(upper bounded)}
\end{aligned}$$

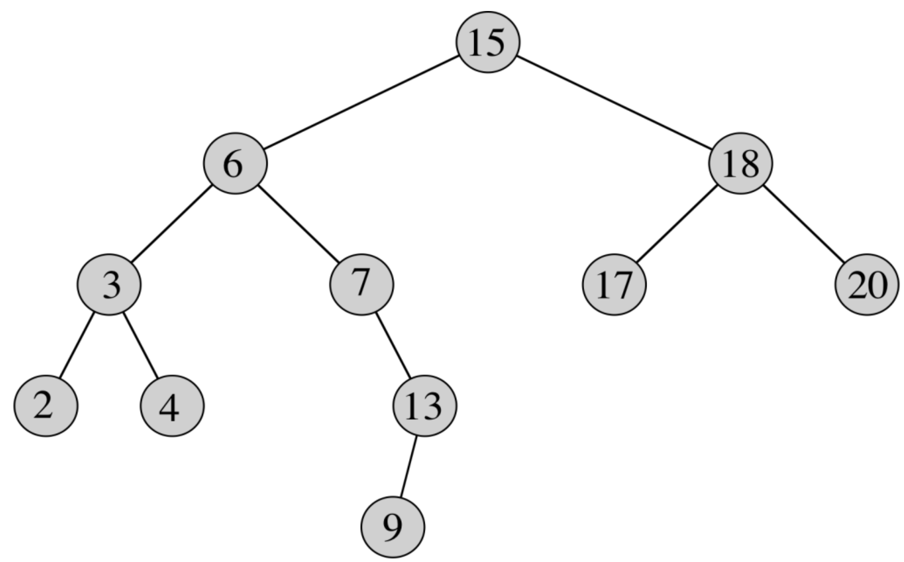
<u>Binary Search Trees</u> (continued)

- **Querying binary search trees**
  - searching
  - successor and predecessor
  - insertion and deletion

  - Searching via TREE-SEARCH in time complexity $O(h)$

$\text{T}{\scriptsize \text{REE}}\text{-S}{\scriptsize \text{EARCH}}(x, k)$

if x == NIL or k == $key[x]$
    return x
if $k < x.key$
    return $\text{T}{\scriptsize \text{REE}}\text{-S}{\scriptsize \text{EARCH}}(x.left, k)$
else return $\text{T}{\scriptsize \text{REE}}\text{-S}{\scriptsize \text{EARCH}}(x.right, k)$



<u>search path</u> from root to *key* = 13 is
$15 \rightarrow 6 \rightarrow 7 \rightarrow 13$

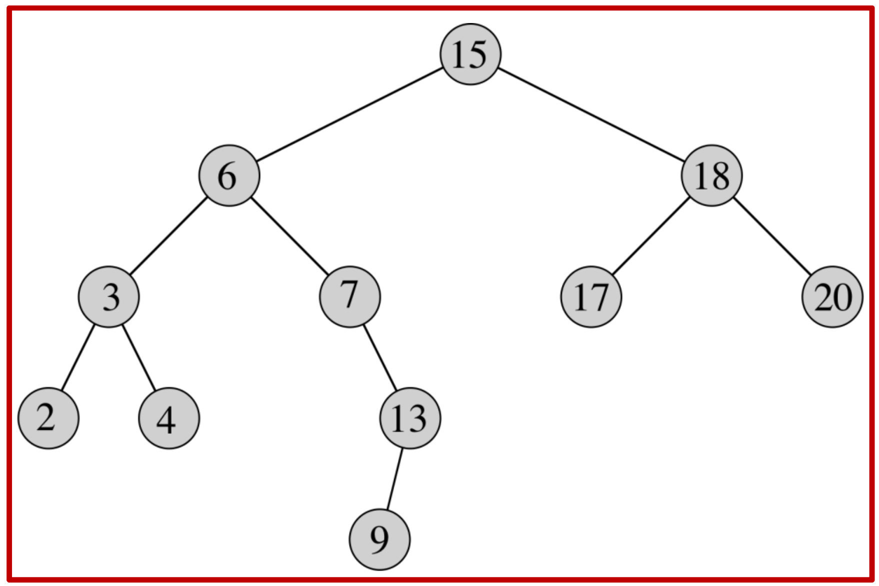
Binary Search Trees (continued)

– Successor and predecessor
  + by inorder tree walk
  + separate results for non-empty right subtree and for empty right subtree
  + time complexity for a tree of height *h* is $O(h)$

TREE-SUCCESSOR(x)

```
if x.right ≠ NIL
    return TREE-MINIMUM(x.right)
y = x.p                          y is the parent node of x.
while y ≠ NIL and x == y.right
    x = y                        climb up the tree.
    y = y.p
return y
```

If node *x* has no right subtree, its successor is the lowest ancestor of *x* whose left subtree contains *x*.



successor of *key* = 7 is 9
successor of *key* = 13 is 15

<u>Binary Search Trees</u> (continued)

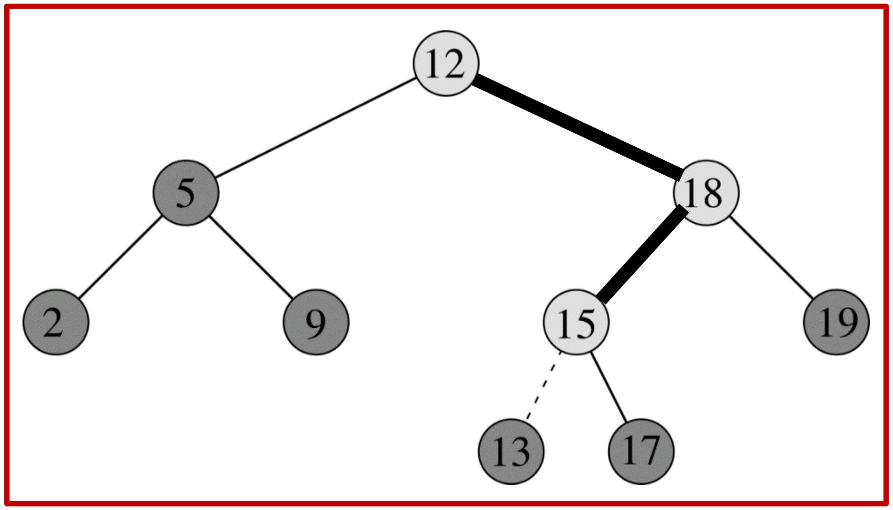
– Insertion
  + tree node (say, *z*) to be added, with *z.key* = *v*, *z.left* = nil, *z.right* = nil
  + *z* added to one appropriate node with an ***absent*** left or right child (never a full node)
  + tree in-order walk, with the <u>trailing pointer *y*</u> maintained
  + time complexity for a tree of height *h* is <u>*O*(*h*)</u>

<u>TREE-INSERT(T, z)</u>

```
y = NIL
x = T.root
while x ≠ NIL
    y = x
    if z.key < x.key
        x = x.left
    else x = x.right
z.p = y          // y is the parent node of z.
if y == NIL
    T.root = z          // tree T was empty
elseif z.key < y.key
    y.left = z
else y.right = z
```

y is the parent node of *x* after *x* is reassigned to move down

tree traversal downward until the insertion point, with the added node as a **leaf node**



Inserting *key* = 13 into binary search tree, with affected path marked.

Binary Search Trees (continued)

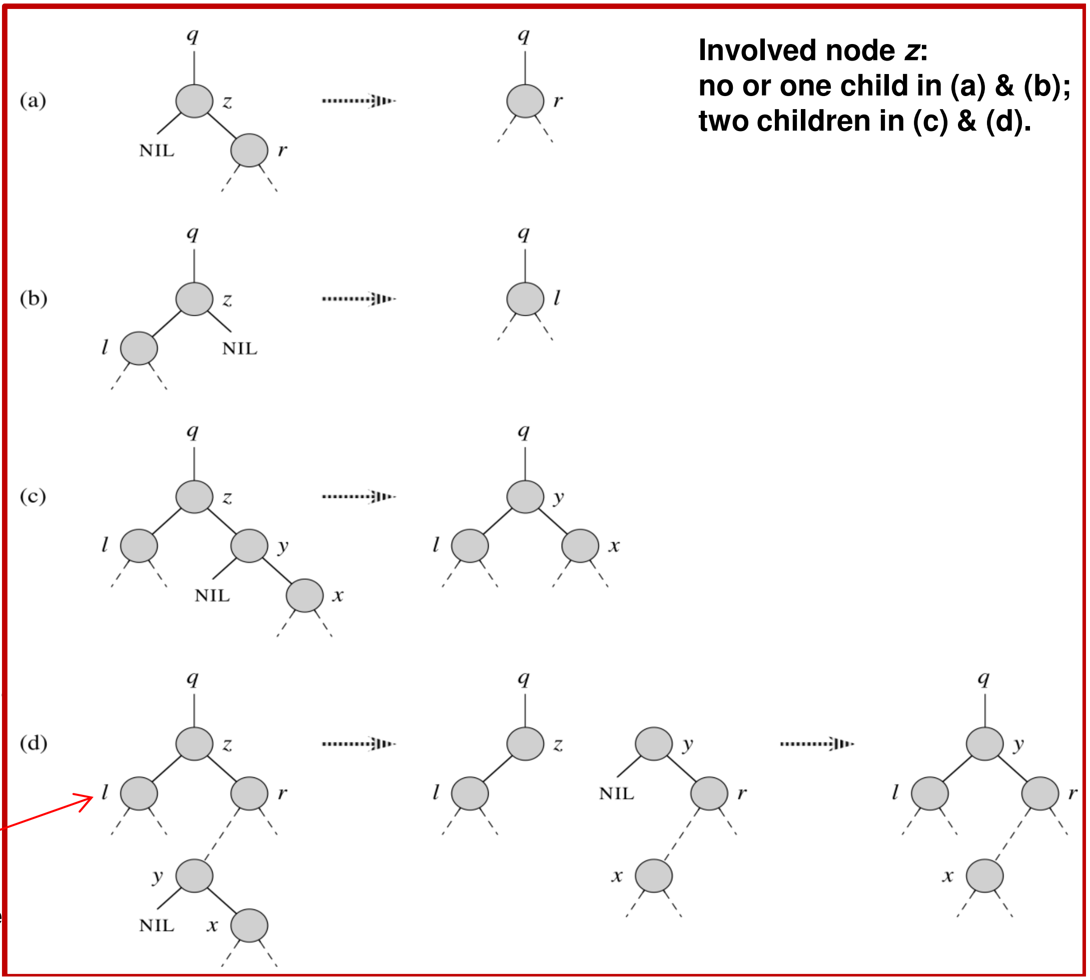
– Deletion (on *z*)

+ three basic cases involved: *z* having no child, exactly one child, two children
+ based on TRANSPLANT that replaces sibling subtree with another subtree
+ time complexity for a tree of height *h* is <u>*O*(*h*)</u>

```
TRANSPLANT(T, u, v)
  if u.p == NIL
      T.root = v
  elseif u == u.p.left
      u.p.left = v
  else u.p.right = v
  if v ≠ NIL
      v.p = u.p
```

```
TREE-DELETE(T, z)
 if z.left == NIL
     TRANSPLANT(T, z, z.right)          // z has no left child
 elseif z.right == NIL
     TRANSPLANT(T, z, z.left)           // z has just a left child
 else // z has two children.
     y = TREE-MINIMUM(z.right)          // y is z's successor
     if y.p ≠ z   <-- This is case (d) shown in the figure.
         // y lies within z's right subtree but not the root of this subtree
         TRANSPLANT(T, y, y.right)  ]
         y.right = z.right          ]  Prepare a subtree rooted at y
         y.right.p = y              ]  whose left subtree is empty.
     // Replace z by y.
     TRANSPLANT(T, z, y)
     y.left = z.left
     y.left.p = y
```

Alternatively, one may cut the left-hand subtree of *z* and move it to become the left-hand subtree of *y*.



Binary Search Trees (continued)

– Randomly build binary search trees
  + worst-case tree height (*h*) being n-1
  + can be shown that $h \geq \lfloor \lg n \rfloor$, which is the best case
  + like quicksort, average case proven to be much closer to best case than worst case
  + special case of creating binary search trees (via insertion alone) with random keys, we have following theorem:

Theorem 2. ← This theorem refers to the special case that the tree is created by insertion alone (and no deletion).

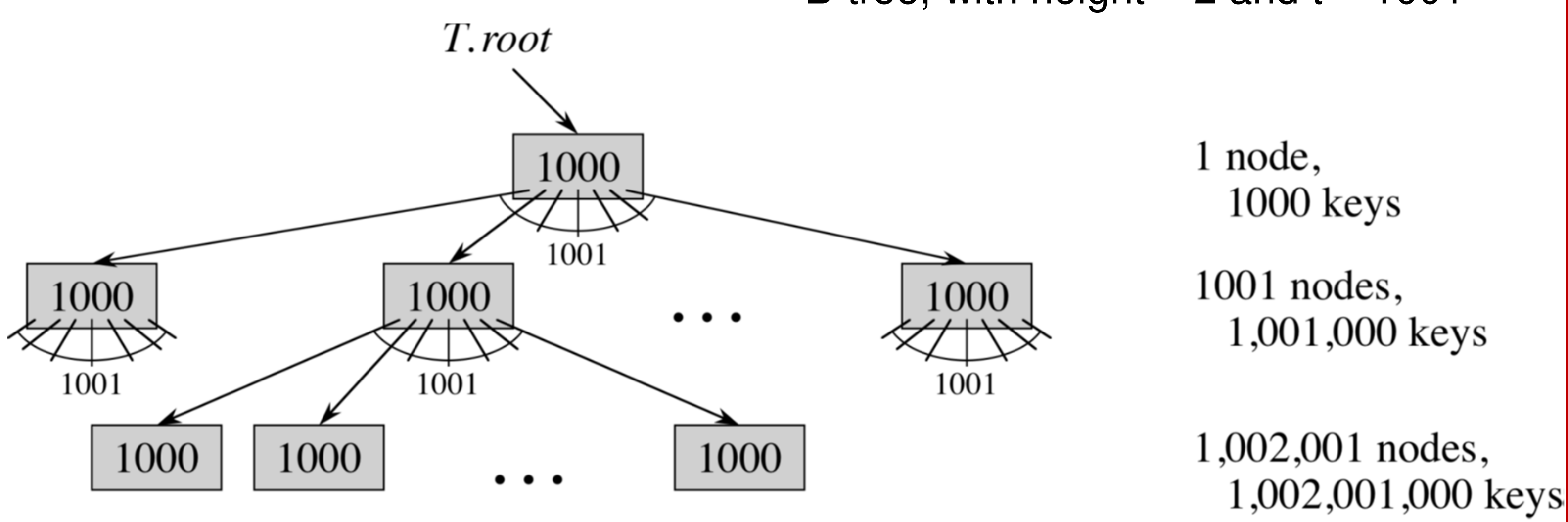
The expected height of a randomly built binary search tree on n distinct keys is $\boldsymbol{O(\lg n)}$.

B-Trees

§ **Balanced Search Trees (B-Trees)** with t — non-root node has ≥ t & ≤ $2t$ children

+ balance achieved by keeping multiple (i.e., ≥ t-1) keys in each non-root node
+ node has at most $2t$-1 keys, called **full node** (with degree = $2t$)
+ keys stored in a node in non-decreasing order
+ node x (with $x.n$ keys) has $x.n$+1 children, pointed by $x.c_1$, $x.c_2$, …., $x.c_{x.n+1}$, then $k_1 \le x.key_1 \le k_2 \le x.key_2 \le \ldots \le x.key_{x.n} \le k_{x.n+1}$, where k_i is a key stored in subtree rooted at $x.c_i$ and $x.key_i$ is a key stored in node x
+ all leaves have the same height
+ simplest B-tree is for t = 2, called 2-3-4 tree (each *internal node* has 2, 3, or 4 children)

B-tree, with height = 2 and t = 1001

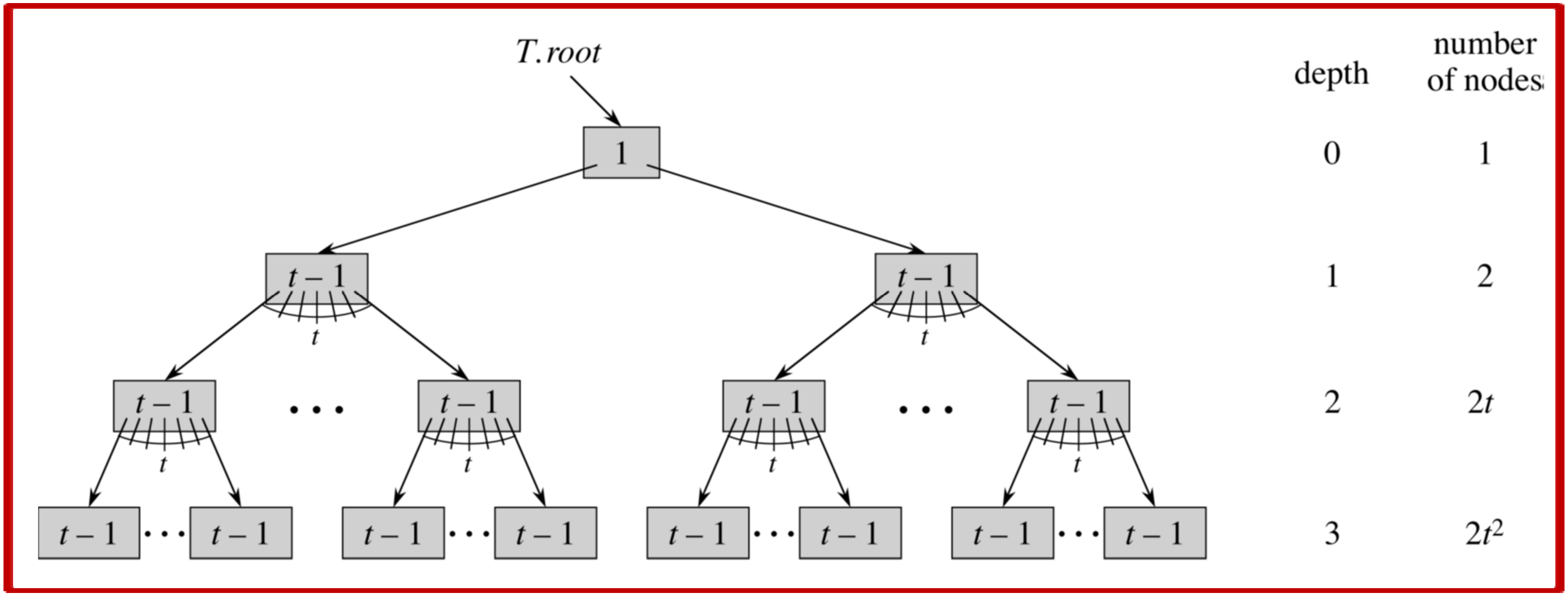


B-Trees (continued)

- **Theorem**

  For any n-key B-tree T of height h and minimum node degree $t \geq 2$ (i.e., number of keys in each non-root node $\geq t$-1), we have $h \leq \log_t \frac{n+1}{2}$.

  + Proof follows the fact of $n \geq 1 + (t-1) \cdot \sum_{i=1}^{h} 2 \cdot t^{i-1}$ illustrated in the figure below (to show the <u>minimum</u> possible number of keys in a B-tree with height = 3).



B-Trees (continued)

- **Basic operations**
  - + Searching for record with key = k: each internal node ***x*** makes an ($x.n$ +1)-way branching decision; returning (y, i) → node ***y*** and its index i with $y.key_i$ = k

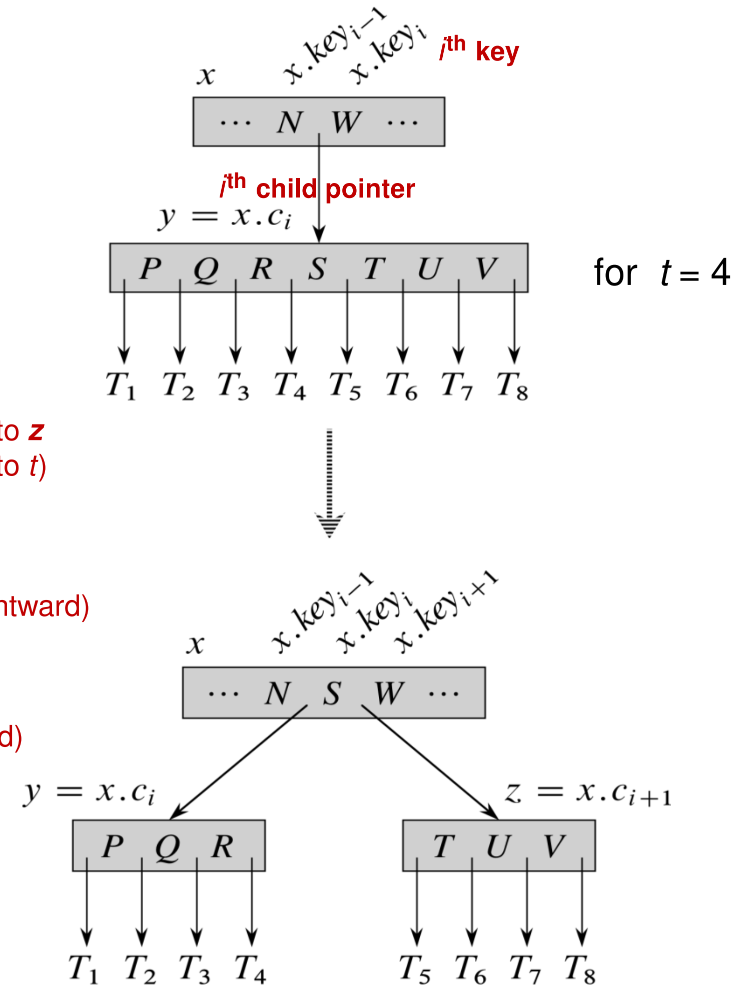
```
B-TREE-SEARCH(x, k)
i = 1
while i ≤ x.n and k > x.key_i
    i = i + 1
if i ≤ x.n and k == x.key_i
    return (x, i)
elseif x.leaf
    return NIL
else DISK-READ(x.c_i)
    return B-TREE-SEARCH(x.c_i, k)
```

x.leaf: Boolean variable to indicate if it is a leaf node

B-Trees (continued)

- **Basic operations**
  + Splitting a full node: for *non-full internal node* ***x*** and a *full child* of ***x*** (say, ***y*** = $x.c_i$), ***y*** is split at median key *S*, which moves up into ***x***, with those > *S* forming a new subtree

```
B-TREE-SPLIT-CHILD(x, i)                      (split i-th child of x)
z = ALLOCATE-NODE()
y = x.c_i                                     (Node x's i-th child is full)
z.leaf = y.leaf                               (status – leaf?)
z.n = t - 1
for j = 1 to t - 1
    z.key_j = y.key_{j+t}
if not y.leaf
    for j = 1 to t                            (move child pointers from y to z
        z.c_j = y.c_{j+t}                      with pointer # from 1 to t)
y.n = t - 1
for j = x.n + 1 downto i + 1
    x.c_{j+1} = x.c_j                         (shift child pointers in x rightward)
x.c_{i+1} = z
for j = x.n downto i
    x.key_{j+1} = x.key_j                     (shift keys in x rightward)
x.key_i = y.key_t                             (move middle key in y up)
x.n = x.n + 1
DISK-WRITE(y)
DISK-WRITE(z)
DISK-WRITE(x)
```



B-Trees (continued)

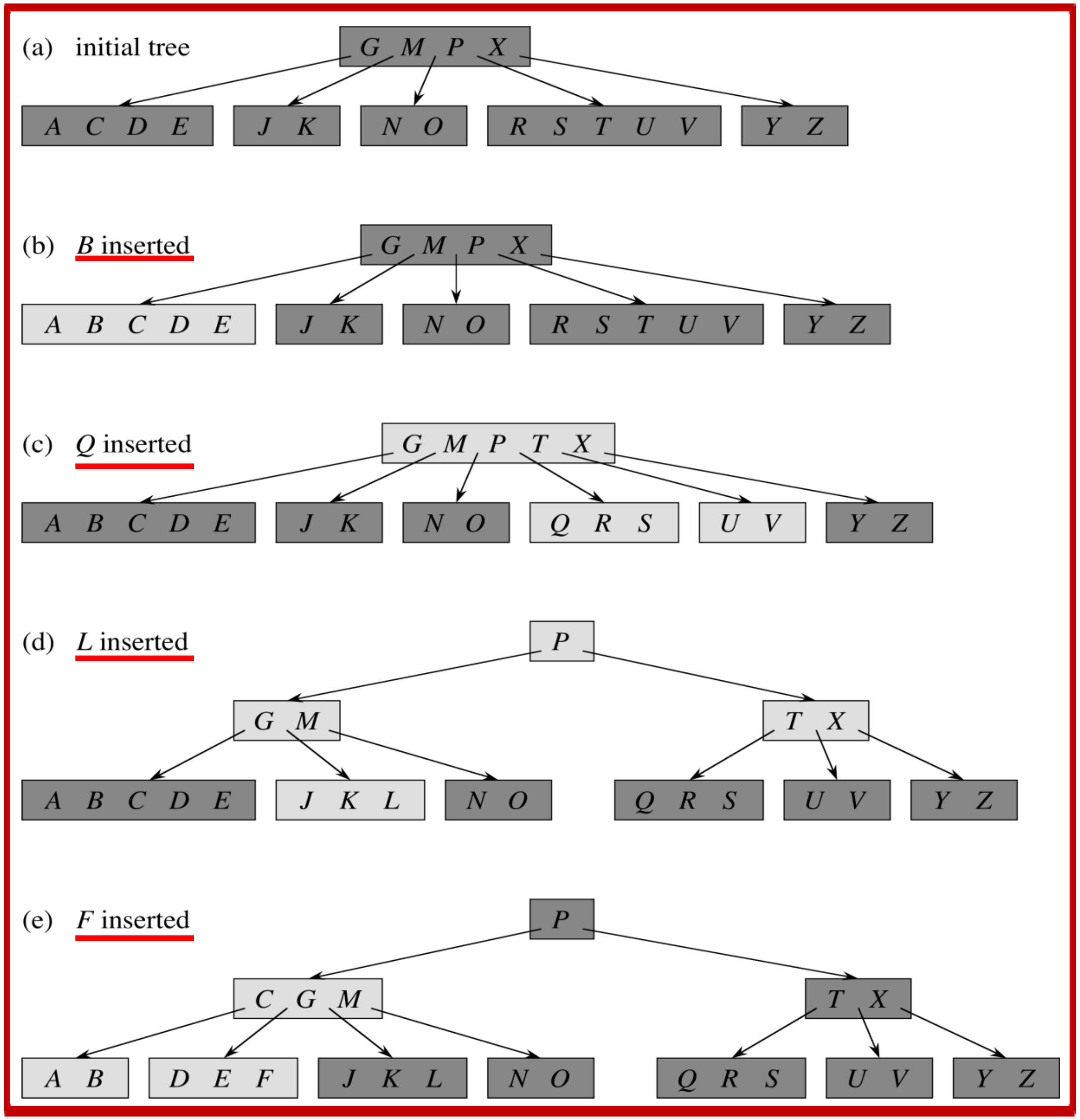
- **Basic operations**
  - + Inserting a key: B-tree ***T*** of height *h*, takes *O*(*h*) disk accesses (nodes kept as disk pages)
    - – *can't* insert the key into a newly created node *nor* into an internal node directly
      $\longrightarrow$ so can insert it only to a **leaf node**
    - – **never** descend through a **full node**, achieved by B-TREE-SPLIT-CHILD, to avoid back-tracking altogether
    - – root split to increase height by 1

```
B-TREE-INSERT(T, k)
r = T.root
if r.n == 2t − 1            // splitting the root?
    s = ALLOCATE-NODE()
    T.root = s
    s.leaf = FALSE
    s.n = 0
    s.c_1 = r
    B-TREE-SPLIT-CHILD(s, 1)   (split 1st child of S)
    B-TREE-INSERT-NONFULL(s, k)
else B-TREE-INSERT-NONFULL(r, k)
```

```
B-TREE-INSERT-NONFULL(x, k)
i = x.n
if x.leaf                   (upon a leaf, key is inserted therein)
    while i ≥ 1 and k < x.key_i
        x.key_{i+1} = x.key_i   (shift key in x rightwar
        i = i − 1
    x.key_{i+1} = k
    x.n = x.n + 1
    DISK-WRITE(x)
else while i ≥ 1 and k < x.key_i
        i = i − 1
    i = i + 1
    DISK-READ(x.c_i)        (descend one level)
    if x.c_i.n == 2t − 1    // full child?
        B-TREE-SPLIT-CHILD(x, i)
        if k > x.key_i
            i = i + 1
    B-TREE-INSERT-NONFULL(x.c_i, k)
```

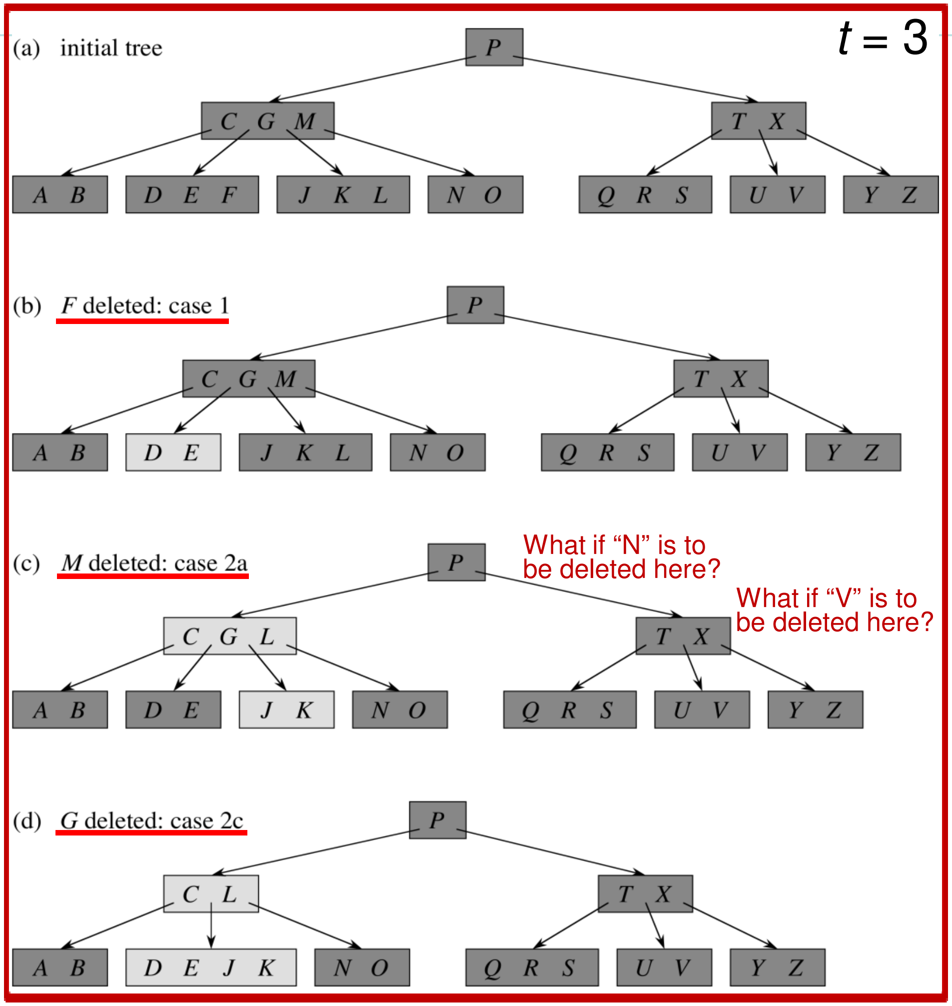
B-Trees (continued)

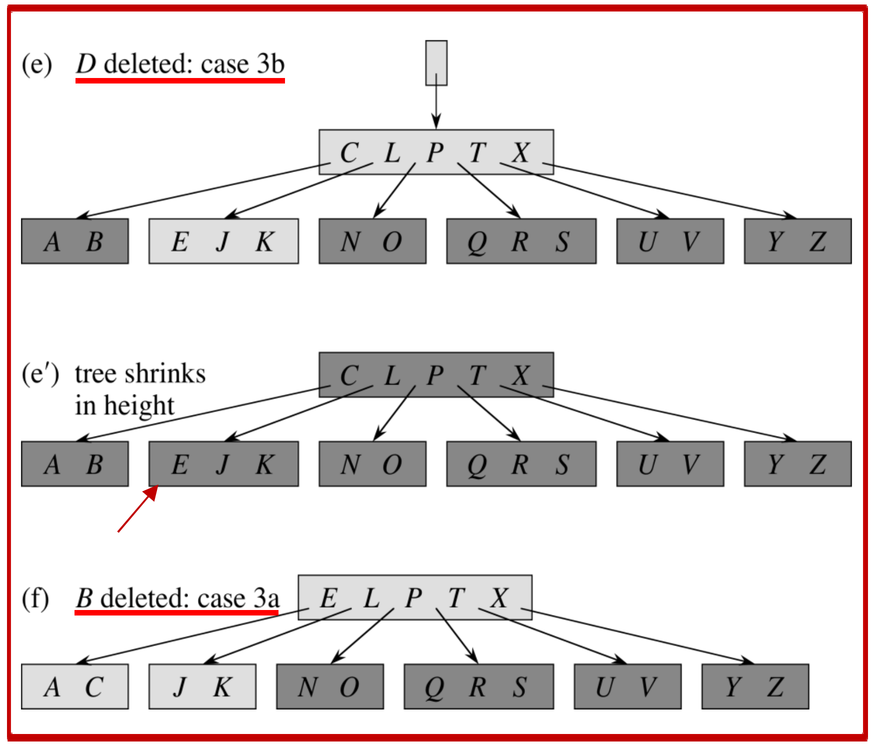
+ Inserting a key: B-tree ***T*** with *t* = 3 and modified nodes lightly shaded



B-Trees (continued)

- **Basic operations**
  - \+ Deleting a key: delete key *k* from the subtree rooted at ***x*** in B-tree ***T*** with the minimum degree of *t* (where the key may be in any node, not just at a leaf node)
    - – number of keys kept in any **internal node** upon descending through would be **at least *t***





B-Trees (continued)

+ Deleting key k from the subtree rooted at $\boldsymbol{x}$ in B-tree $\boldsymbol{T}$ with the minimum degree of t
  – **Deletion Procedure** sketched below:

1. If the key k is in node x and x is a leaf, delete the key k from x.
2. If the key k is in node x and x is an internal node, do the following:
   a. If the child y that precedes k in node x has at least t keys, then find the predecessor k' of k in the subtree rooted at y. Recursively delete k', and replace k by k' in x. (We can find k' and delete it in a single downward pass.)
   b. If y has fewer than t keys, then, symmetrically, examine the child z that follows k in node x. If z has at least t keys, then find the successor k' of k in the subtree rooted at z. Recursively delete k', and replace k by k' in x. (We can find k' and delete it in a single downward pass.)
   c. Otherwise, if both y and z have only $t-1$ keys, merge k and all of z into y, so that x loses both k and the pointer to z, and y now contains $2t-1$ keys. Then free z and recursively delete k from y.
3. If the key k is not present in ^visited internal node x, determine the root $x.c_i$ of the appropriate subtree that must contain k, if k is in the tree at all. If $x.c_i$ has only $t-1$ keys, execute step 3a or 3b as necessary to guarantee that we descend to a node containing at least t keys. Then finish by recursing on the appropriate child of x.
   a. If $x.c_i$ has only $t-1$ keys but has an immediate sibling with at least t keys, give $x.c_i$ an extra key by moving a key from x down into $x.c_i$, moving a key from $x.c_i$'s immediate left or right sibling up into x, and moving the appropriate child pointer from the sibling into $x.c_i$.
   b. If $x.c_i$ and both of $x.c_i$'s immediate siblings have $t-1$ keys, merge $x.c_i$ with one sibling, which involves moving a key from x down into the new merged node to become the median key for that node.

Targeted node with the key

This is Rule 1 - Key k present:
At internal node x which contains the key (say, k) to be deleted, it checks if predecessor or successor of k can be borrowed to replace k (and in this case, only the borrowed key is moved; and its associated pointer stays).

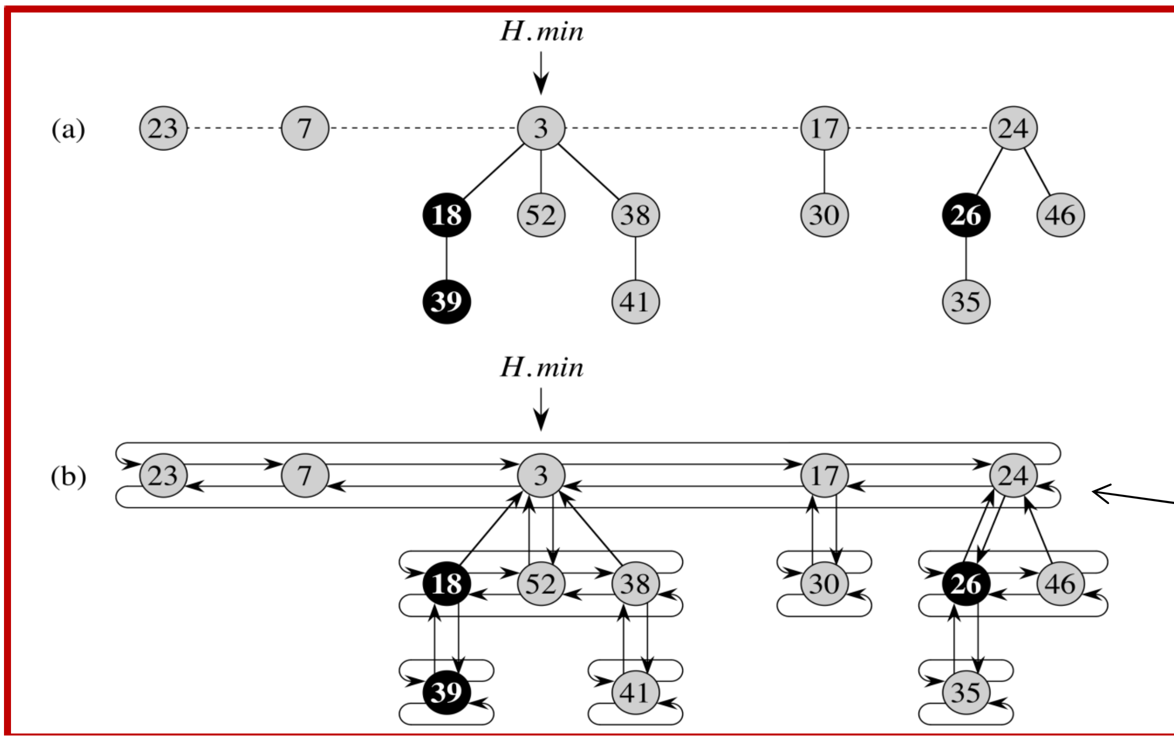
Prepare to walk down the tree

This is Rule 2 - Key k not present:
While at internal node x, it checks if the root of the target child has at least t key
If not, it prepares that root node by borrowing a key (plus associated pointer, i.e., a subtree) from its left or right sibling, if possible.

Fibonacci Heaps

§ Fibonacci Heap

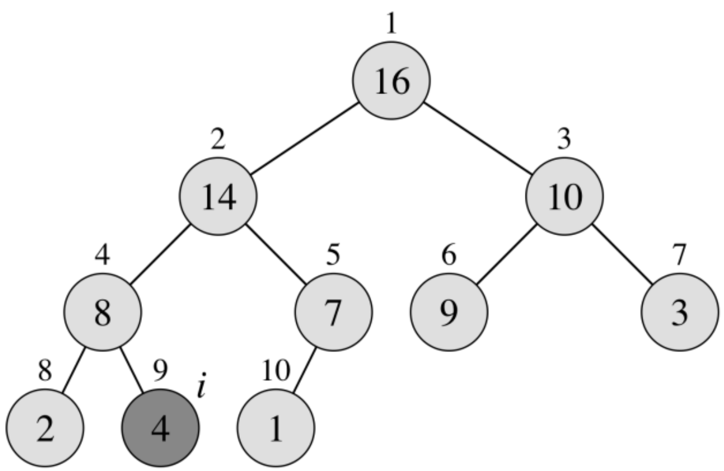
+ a collection of min-heap trees, with their roots doubly linked together
+ belongs to the "mergeable heap" that supports five operations efficiently:
  MAKE-HEAP(); INSERT(H, x); MINIMUM(H); EXTRACT-MIN(H); UNION(H_1, H_2)
+ additionally, the Fibonacci heap supports:
  DECREASE-KEY(H, x, k); DELETE(H, x)
+ several preceding operations run in constant amortized time



Four pointers exist for each node: ***p, child, left, right,*** permitting children of a node to be doubly linked as the child list.

Procedure	Binary heap (worst-case)	Fibonacci heap (amortized)
MAKE-HEAP	$\Theta(1)$	$\Theta(1)$
INSERT (to Root List)	$\Theta(\lg n)$	$\Theta(1)$
MINIMUM	$\Theta(1)$	$\Theta(1)$
EXTRACT-MIN	$\Theta(\lg n)$	$O(\lg n)$
UNION (i.e., Merging)	$\Theta(n)$	$\Theta(1)$
DECREASE-KEY	$\Theta(\lg n)$	$\Theta(1)$
DELETE	$\Theta(\lg n)$	$O(\lg n)$

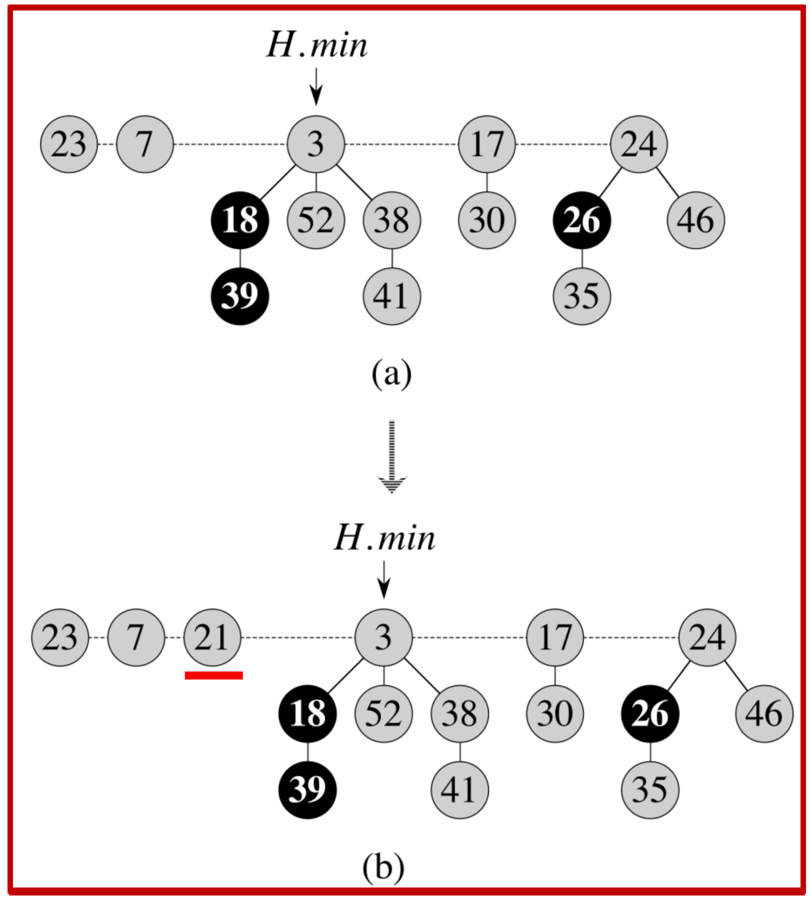
Max Binary Heap



Fibonacci Heaps (continued)

- **Heap operations**
  + inserting a node, with time complexity $O(1)$, by adding the node to the root list

```
FIB-HEAP-INSERT(H, x)
x.degree = 0
x.p = NIL
x.child = NIL
x.mark = FALSE
if H.min == NIL
    create a root list for H containing just x
    H.min = x
else insert x into H's root list
    if x.key < H.min.key
        H.min = x
H.n = H.n + 1
```



Fibonacci Heaps (continued)

- **Uniting two Fibonacci heaps**
  + concatenating the root lists of H_1 and H_2
  + determining the new minimum node

FIB-HEAP-UNION(H_1, H_2)

```
H = MAKE-FIB-HEAP()
H.min = H1.min
concatenate the root list of H2 with the root list of H
if (H1.min == NIL) or (H2.min ≠ NIL and H2.min.key < H1.min.key)
    H.min = H2.min
H.n = H1.n + H2.n
return H
```

<u>Fibonacci Heaps</u> (continued)

- **Extracting minimum node**
  - \+ doing <u>consolidation</u> of trees on the <u>root list</u>, so that one root is left for each degree
  - \+ calling CONSOLIDATE(*H*) to link trees of the same degree

conducting delayed work of consolidating trees in the root list

<u>FIB-HEAP-EXTRACT-MIN(*H*)</u>

```
z = H.min
if z ≠ NIL
    for each child x of z
        add x to the root list of H
        x.p = NIL
    remove z from the root list of H
    if z == z.right     //  z being the root of the only tree and having no child (since any child, if existing, would then have been added to the root list)
        H.min = NIL
    else H.min = z.right
        CONSOLIDATE(H)
    H.n = H.n − 1
return z
```

F_4

Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

Fibonacci Heaps (continued)

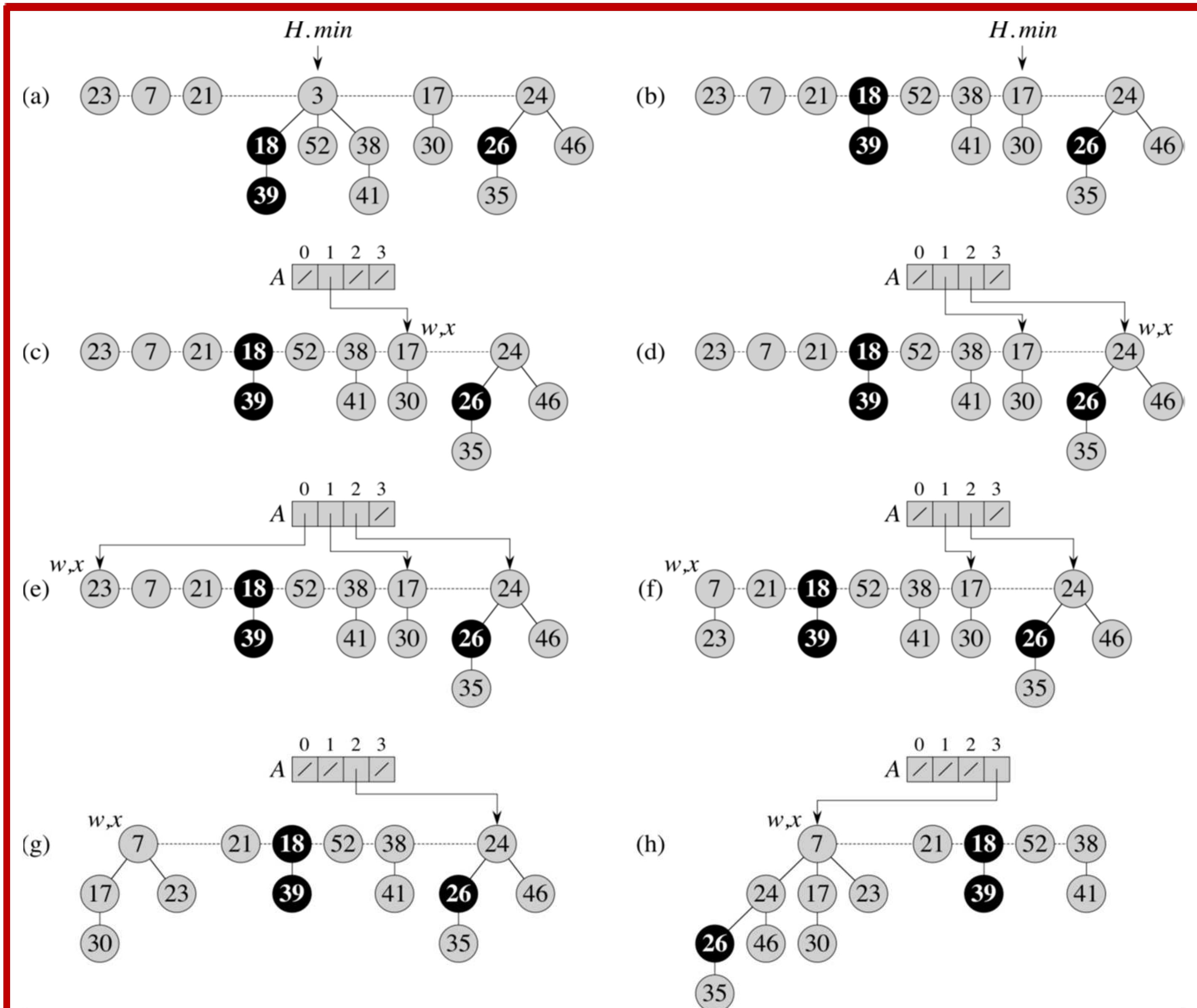
- **CONSOLIDATE(*H*)** for consolidating trees on the root list

```
CONSOLIDATE(H)
let A[0 . . D(H.n)] be a new array    // D(H, n) being the largest degree of a constituent MIN heap
for i = 0 to D(H.n)
    A[i] = NIL                        // auxiliary array initialization
for each node w in the root list of H
    x = w
    d = x.degree
    while A[d] ≠ NIL
        y = A[d]          // another node with the same degree as x existing
        if x.key > y.key
            exchange x with y
        FIB-HEAP-LINK(H, y, x)
        A[d] = NIL
        d = d + 1
    A[d] = x
H.min = NIL
for i = 0 to D(H.n)       // establish the root list and find H.min after consolidation
    if A[i] ≠ NIL
        if H.min == NIL
            create a root list for H containing just A[i]
            H.min = A[i]
        else insert A[i] into H's root list
            if A[i].key < H.min.key
                H.min = A[i]
```

```
FIB-HEAP-LINK(H, y, x)
remove y from the root list of H
make y a child of x, incrementing x.degree
y.mark = FALSE
```

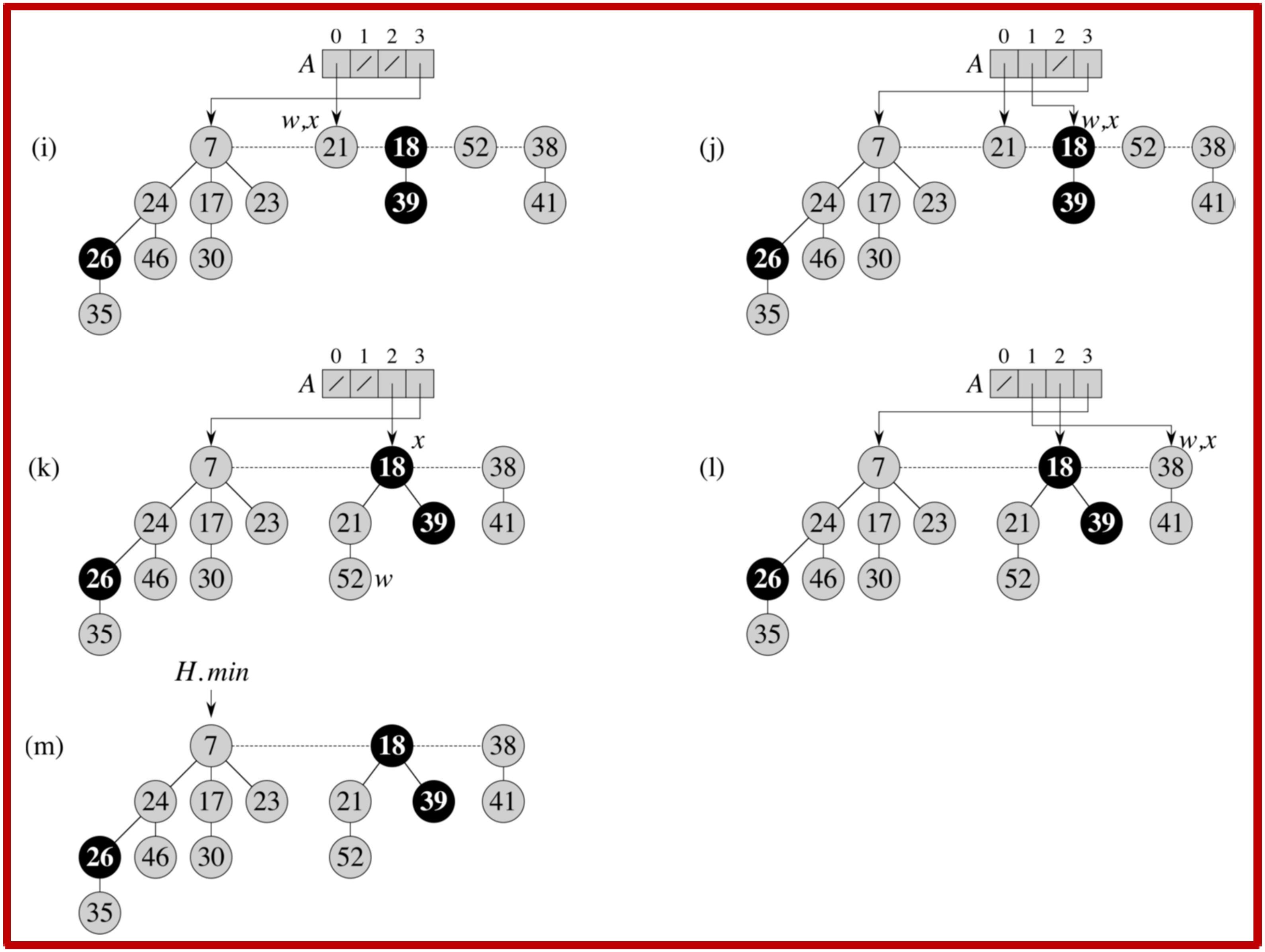
Fibonacci Heaps (continued)

- **Extracting minimum node**



Fibonacci Heaps (continued)

- **Extracting minimum node**



Fibonacci Heaps (continued)

- **Decreasing a key**

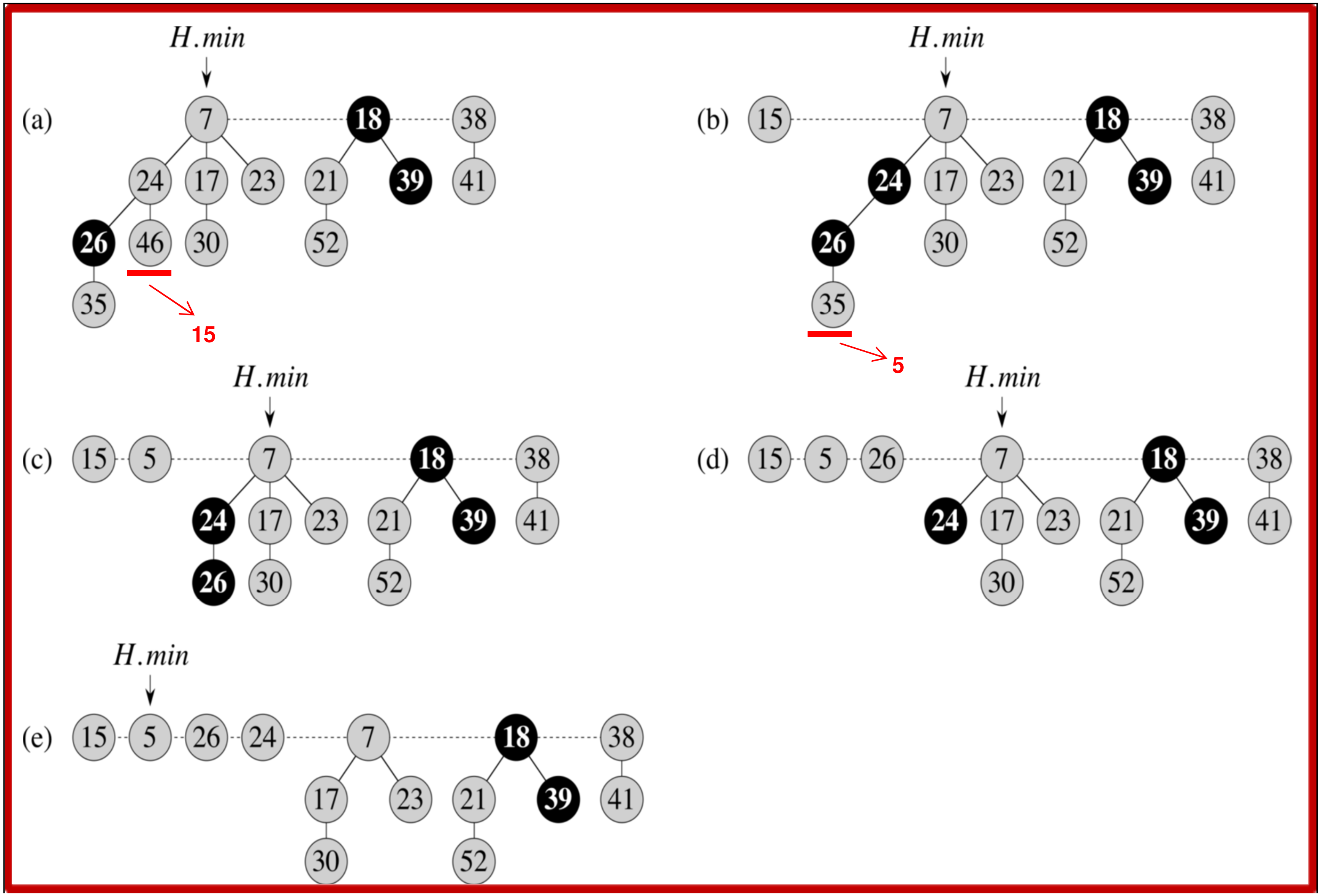
```
FIB-HEAP-DECREASE-KEY(H, x, k)
if k > x.key
    error "new key is greater than current key"
x.key = k
y = x.p
if y ≠ NIL and x.key < y.key
    CUT(H, x, y)
    CASCADING-CUT(H, y)
if x.key < H.min.key
    H.min = x

CUT(H, x, y)
remove x from the child list of y, decrementing y.degree
add x to the root list of H
x.p = NIL
x.mark = FALSE

CASCADING-CUT(H, y)
z = y.p
if z ≠ NIL
    if y.mark == FALSE
        y.mark = TRUE
    else CUT(H, y, z)
        CASCADING-CUT(H, z)
```

Fibonacci Heaps (continued)

- **Decreasing a key**



Fibonacci Heaps (continued)

- **Deleting a key**

FIB-HEAP-DELETE(H, x)

  FIB-HEAP-DECREASE-KEY(H, x, $-\infty$)
  FIB-HEAP-EXTRACT-MIN(H)

Fibonacci Heaps (continued)

- **Heap Bounds**

Lemma 1.

Let x be <u>any node</u> in a Fibonacci heap and $k = x.degree$. We have size(x) $\geq F_{k+2} \geq \phi^k$, where $\phi = (1 + \sqrt{5})/2$.

$= 1.618$

F_4 F_6

Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, …

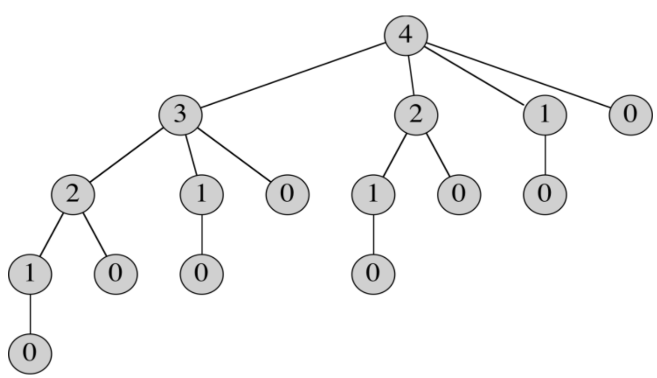
ϕ^k 1.0, 1.6, 2.6, 4.2, 6.9, 11.1, 17.9, 29.0, 47.0

k=1, k=2, k=3, k=4,

In fact, $D(k) \geq$ 2, 4, 8, 16, 32, 64, 128, ….

Lemma 2.

The maximum degree $D(x)$ of any node in an n-node Fibonacci heap is $O(\lg n)$.



Note: the smallest Fibonacci heaps with k = 1, 2, 3, 4, etc. include 2, 4, 8, 16, etc. nodes. Hence, *size*(1) ≥ 2, *size*(2) ≥ 4, *size*(3) ≥ 8, *size*(4) ≥ 16, etc.