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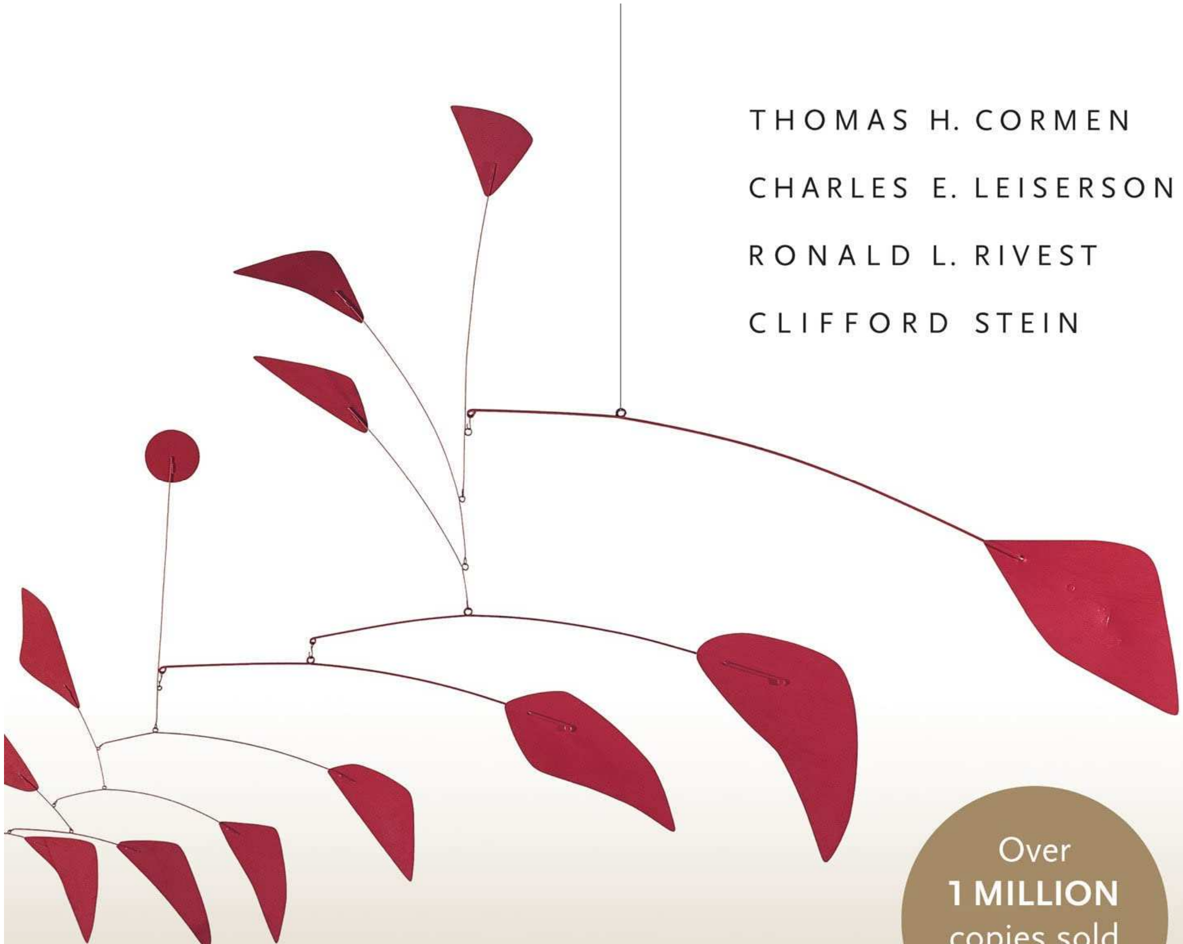

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INTRODUCTION TO ALGORITHMS

FOURTH EDITION

CSCE 500

Design and Analysis of Algorithms

Fall 2022

August 22, 2022

Instructor: Nian-Feng Tzeng
Office: Rm. OLVR 354 (× 2-6304)
Class meeting: MW 10:00 – 11:15, OLVR 113

Textbook and Supplemental Materials:

1. Introduction to Algorithms, Fourth Edition, by Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein, The MIT Press, 2022, ISBN: 978–026204630–5.
2. Published articles supplementary to covered topics.

Course Description:

This course provides a comprehensive coverage of modern computer algorithms, aiming at in-depth treatment of algorithmic design and analysis with elementary explanation while keeping mathematical rigor. Based on the textbook of "Introduction to Algorithms", this class covers the topics listed below in sequence.

(1) Foundations.
(2) Data Structures – hash tables, trees, data structures for disjoint sets.
(3) Design and Analysis Techniques – dynamic programming, greedy algorithms, amortized analysis.
(4) Graph Algorithms – spanning trees, shortest paths, maximum flow, matchings in bipartite graphs.
(5) Selected Topics – NP-completeness, approximation algorithms, parallel algorithms, on-line algorithms.

Each covered topic starts with the description of pertinent algorithms in English and/or in the pseudocode(s), followed by their careful complexity analyses.

Course Requirements:

1. Homework (10%)
2. Midterm exams (2) (50%)
3. Final exam (comprehensive) (40%)

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Analyzing Algorithms

§ Run Time Analysis

– Order of growth
– Worst case analysis
– Average case analysis

§ Insertion Sort for Array A[*i*]

– idea: insert A[*i*] into sorted subarrays: A[1 : *i*-1]
– repeat insertion until A[*i*] is fully sorted

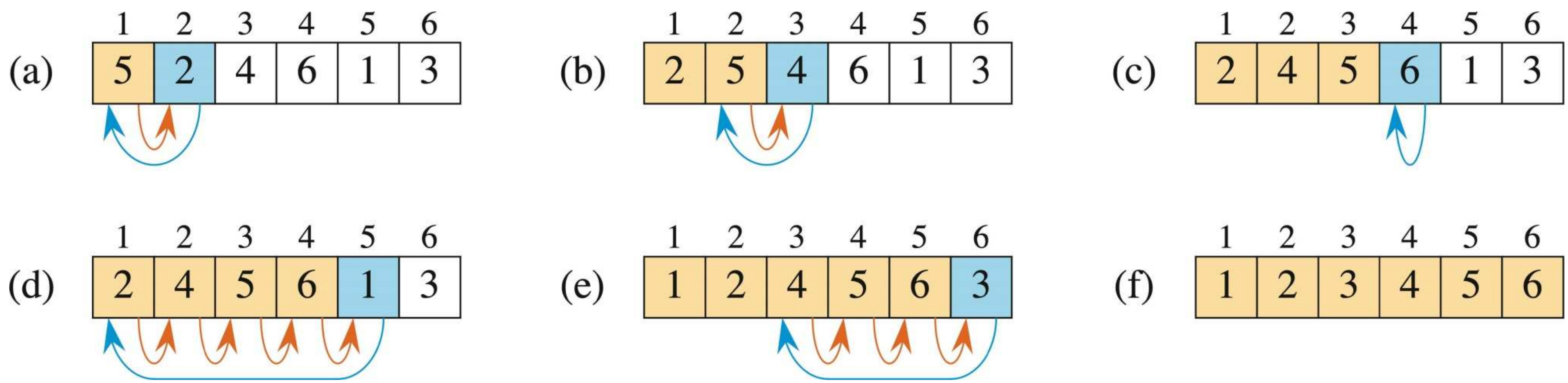
INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$key = A[i]$	c_2	$n-1$
// Insert $A[i]$ into the sorted subarray $A[1:i-1]$.	0	$n-1$
$j = i - 1$	c_4	$n-1$
while $j > 0$ and $A[j] > key$	c_5	$\sum_{i=2}^{n} t_i$
$A[j+1] = A[j]$	c_6	$\sum_{i=2}^{n}(t_i - 1)$
$j = j - 1$	c_7	$\sum_{i=2}^{n}(t_i - 1)$
$A[j+1] = key$	c_8	$n-1$

Insert A(*i*) properly

Complexity: $T(n) = \sum_{i=1}^{8} c_i$

Analyzing Algorithms (continued)

- **Insertion Sort for Array A[*i*]**
  - idea: insert A[*i*] into sorted subarrays: A[1 : *i*-1]
  - repeat insertion until A[*i*] is fully sorted



Worst-Case Complexity:

$$T(n) = \sum_{i=1}^{8} c_i = O(n^2)$$

INSERTION-SORT(A, n)	*cost*	*times*
for $i = 2$ **to** n	c_1	n
$key = A[i]$	c_2	$n-1$
// Insert $A[i]$ into the sorted subarray $A[1:i-1]$.	0	$n-1$
$j = i - 1$	c_4	$n-1$
while $j > 0$ and $A[j] > key$	c_5	$\sum_{i=2}^{n} t_i$
$A[j+1] = A[j]$	c_6	$\sum_{i=2}^{n} (t_i - 1)$
$j = j - 1$	c_7	$\sum_{i=2}^{n} (t_i - 1)$
$A[j+1] = key$	c_8	$n-1$

Designing Algorithms

- **Divide-and-Conquer Approaches with Recursive Nature**
  - Divide the problem
  - Conquer subproblems recursively
  - Combine solutions to subproblems

- **Example: Merge Sort**
  - two sorted subarrays: A[*p* .. *q*] & A[*q*+1 .. *r*]
  - merge the two sorted subarrays
  - merging takes Θ(*n*) time

conquer separately

```
MERGE-SORT(A, p, r)
  if p ≥ r                          // zero or one element?
      return
  q = ⌊(p + r)/2⌋                   // midpoint of A[p : r]
  MERGE-SORT(A, p, q)               // recursively sort A[p : q]
  MERGE-SORT(A, q + 1, r)           // recursively sort A[q + 1 : r]
  // Merge A[p : q] and A[q + 1 : r] into A[p : r].
  MERGE(A, p, q, r)
```

```
MERGE(A, p, q, r)
  n_L = q − p + 1          // length of A[p : q]
  n_R = r − q              // length of A[q + 1 : r]
  let L[0 : n_L − 1] and R[0 : n_R − 1] be new arrays
  for i = 0 to n_L − 1     // copy A[p : q] into L[0 : n_L − 1]
      L[i] = A[p + i]
  for j = 0 to n_R − 1     // copy A[q + 1 : r] into R[0 : n_R − 1]
      R[j] = A[q + j + 1]
  i = 0                    // i indexes the smallest remaining element in L
  j = 0                    // j indexes the smallest remaining element in R
  k = p                    // k indexes the location in A to fill
  // As long as each of the arrays L and R contains an unmerged element,
  //     copy the smallest unmerged element back into A[p : r].
  while i < n_L and j < n_R
      if L[i] ≤ R[j]
          A[k] = L[i]
          i = i + 1
      else A[k] = R[j]
          j = j + 1
      k = k + 1
  // Having gone through one of L and R entirely, copy the
  //     remainder of the other to the end of A[p : r].
  while i < n_L
      A[k] = L[i]
      i = i + 1
      k = k + 1
  while j < n_R
      A[k] = R[j]
      j = j + 1
      k = k + 1
```

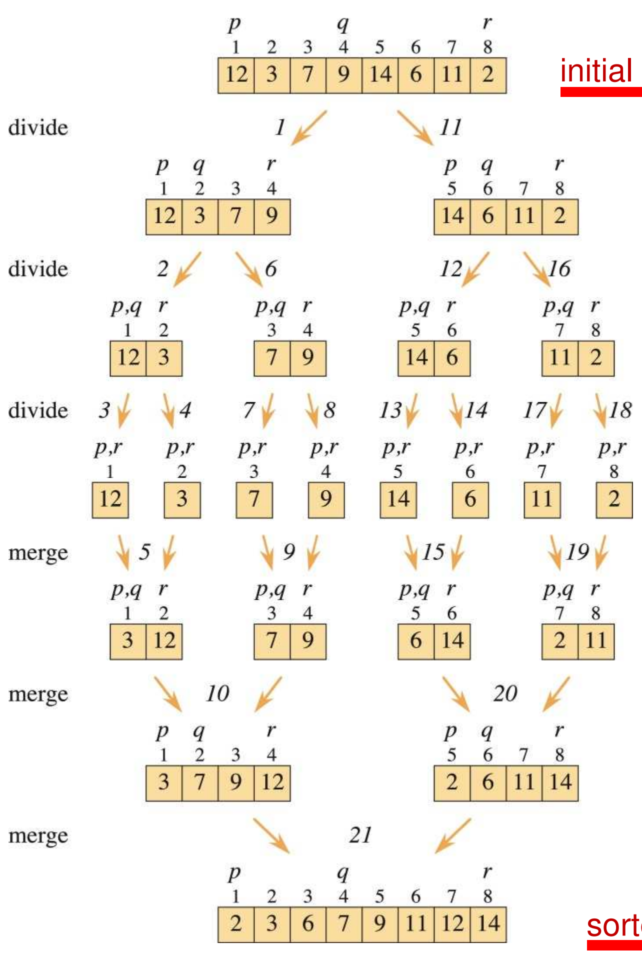
Analyzing Algorithms

§ Analysis of Divide-and-Conquer Algorithms

- Merge Sort
- Time complexity:

$$T(n) = \begin{cases} c_1 & \text{if } n = 1 \\ \underline{2T(n/2) + c_2 \cdot (n)} & \text{if } n > 1 \end{cases}$$

```
MERGE-SORT(A, p, r)
  if p < r                        // check for base case
      q = ⌊(p + r)/2⌋             // divide
      MERGE-SORT(A, p, q)         // conquer
      MERGE-SORT(A, q + 1, r)     // conquer
      MERGE(A, p, q, r)           // combine
```

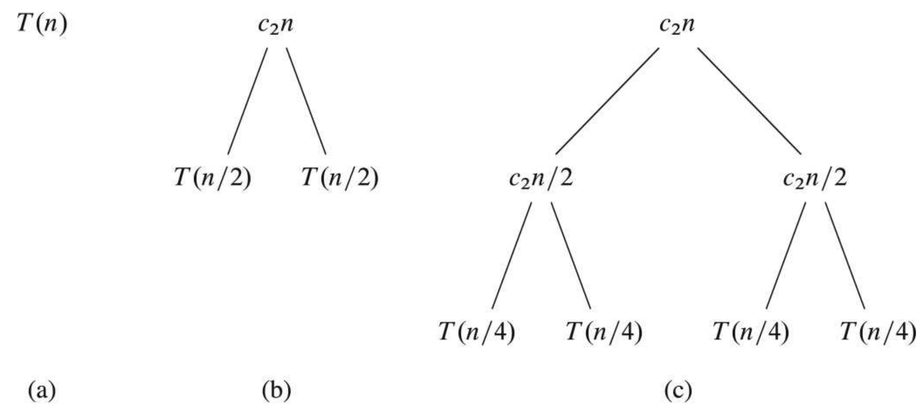


initial sequence

sorted sequence

Analyzing Algorithms (continued)

- **Evaluating: $2T(n/2) + c_2\cdot(n)$**
  – Recursion tree, shown right, equal to $c_2 n\cdot \lg(n) + c_1 n = \Theta(n\cdot \lg n)$



Another approach for solution:

$T(n) = 2T(n/2) + c_2\cdot(n)$

$= 2(2T(n/4)+c_2\cdot(n/2)) + c_2\cdot(n)$

$= 2^2 T(n/4) + 2\cdot c_2\cdot(n)$

$= 2^2(2T(n/8) + c_2\cdot(n/4)) + 2\cdot c_2\cdot(n)$

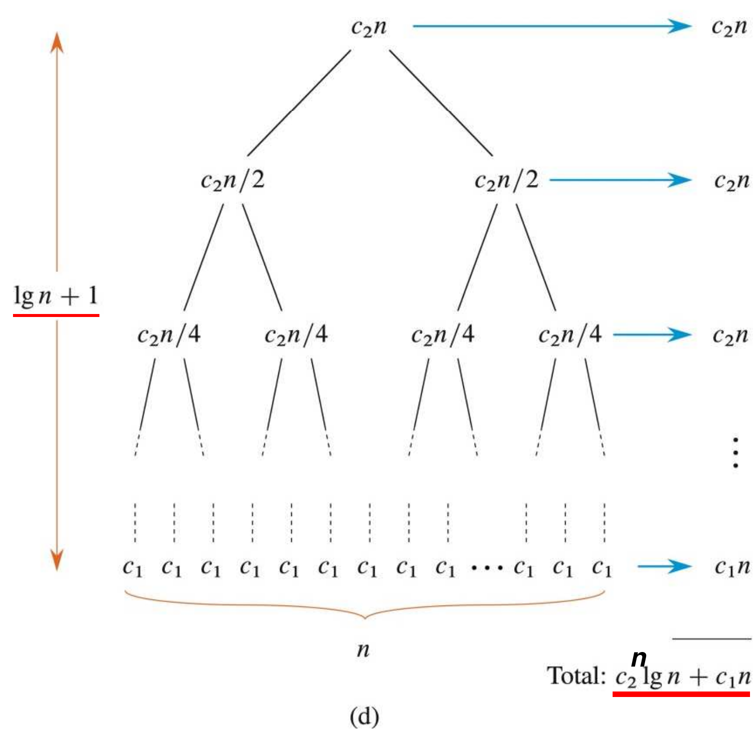
$= 2^3 T(n/8) + 3\cdot c_2\cdot(n)$

....

....

$= n\cdot T(1) + \lg(n)\cdot c_2\cdot(n)$

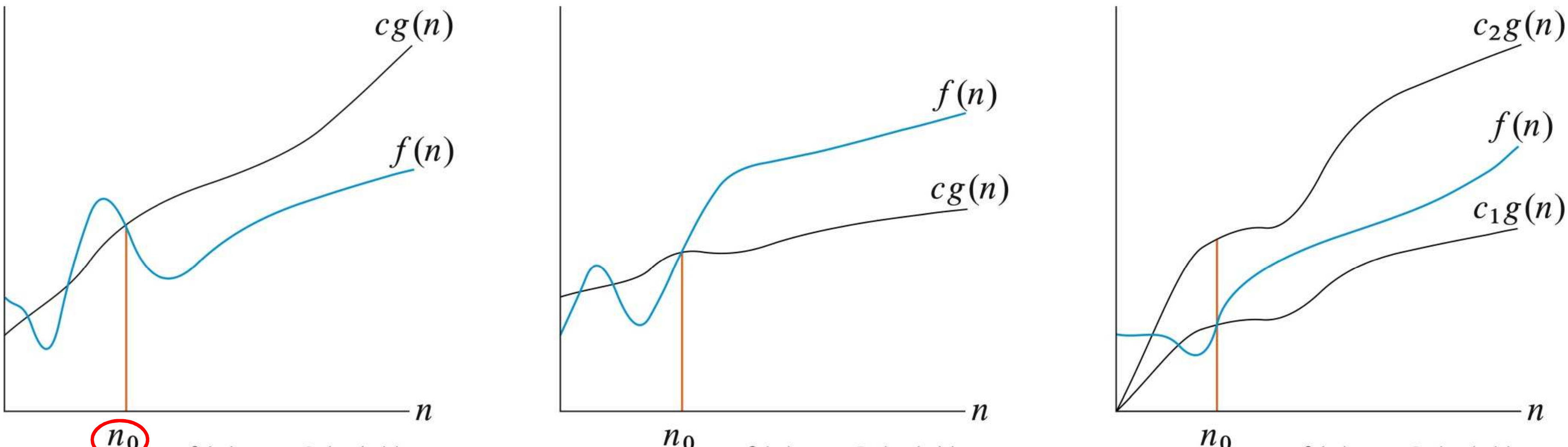
$= n\cdot c_1 + c_2\cdot(n)\lg(n)$



Growth of Functions

§ Asymptotic Notations of Running Times

- *O*-notation: upper-bounding a function to within a constant factor
- Ω-notation: lower-bounding a function to within a constant factor
- Θ-notation: bounding a function to within constant factors



a constant $f(n) = O(g(n))$ (a)

$g(n)$ is an asymptotical upper bound for $f(n)$
(may or may not be tight)

$f(n) = \Omega(g(n))$ (b)

$g(n)$ is an asymptotical lower bound for $f(n)$
(may or may not be tight)

$f(n) = \Theta(g(n))$ (c)

$g(n)$ is an asymptotically tight bound for $f(n)$

There exist positive constants c_1, c_2, and n_0 s.t. $0 \le c_1 \cdot g(n) \le f(n) \le c_2 \cdot g(n)$ for all $n \ge n_0$

Growth of Functions (continued)

- *o*-notation: not asymptotically-tight upper-bound

o-notation

$$o(g(n)) = \{f(n) : \text{ for any constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le f(n) < cg(n) \text{ for all } n \ge n_0\}\ .$$

Another view, probably easier to use: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0.$

$n^{1.9999} = o(n^2)$
$n^2/\lg n = o(n^2)$
$n^2 \ne o(n^2)$ (just like $2 \not< 2$)
$n^2/1000 \ne o(n^2)$

- ω-notation: not asymptotically-tight lower-bound

ω-notation

$$\omega(g(n)) = \{f(n) : \text{ for any constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le cg(n) < f(n) \text{ for all } n \ge n_0\}$$

Another view, again, probably easier to use: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty.$

$n^{2.0001} = \omega(n^2)$
$n^2 \lg n = \omega(n^2)$
$n^2 \ne \omega(n^2)$

Solutions after Divide-and-Conquer

- **Divide-and-Conquer**
  - leads to recurrences in various forms
  - there are 3 kinds of methods for solving recurrences
    - + **substitution** methods
    - + **recursion-tree** methods
    - + **master methods** to find bounds for recurrences of $T(n) = a \cdot T(n/b) + f(n)$, with $a > 0$ and $b > 1$

- **Example: Multiplying two square matrices sized $n \times n$**
  - $C = A \cdot B$
  - divide each $n \times n$ matrix into four $n/2 \times n/2$ submatrices for multiplying:

$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ to get $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$, with

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \bullet \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$
$$= \begin{pmatrix} A_{11}.B_{11} + A_{12}.B_{21} & A_{11}.B_{12} + A_{12}.B_{22} \\ A_{21}.B_{11} + A_{22}.B_{21} & A_{21}.B_{12} + A_{22}.B_{22} \end{pmatrix}.$$

Hence, we have

$$C_{11} = A_{11}.B_{11} + A_{12}.B_{21}; \qquad C_{12} = A_{11}.B_{12} + A_{12}.B_{22}$$
$$C_{21} = A_{21}.B_{11} + A_{22}.B_{21}; \qquad C_{22} = A_{21}.B_{12} + A_{22}.B_{22}$$

Divide-and-Conquer (continued)

§ **Multiplying two square matrices**

– solution involves eight **MATRIX-MULTIPLY RECURSIVE** calls

```
MATRIX-MULTIPLY-RECURSIVE(A, B, C, n)
if n == 1
// Base case.
    c11 = c11 + a11 · b11
    return
// Divide.
partition A, B, and C into n/2 × n/2 submatrices
    A11, A12, A21, A22; B11, B12, B21, B22;
    and C11, C12, C21, C22; respectively
// Conquer.
MATRIX-MULTIPLY-RECURSIVE(A11, B11, C11, n/2)
MATRIX-MULTIPLY-RECURSIVE(A11, B12, C12, n/2)
MATRIX-MULTIPLY-RECURSIVE(A21, B11, C21, n/2)
MATRIX-MULTIPLY-RECURSIVE(A21, B12, C22, n/2)
MATRIX-MULTIPLY-RECURSIVE(A12, B21, C11, n/2)
MATRIX-MULTIPLY-RECURSIVE(A12, B22, C12, n/2)
MATRIX-MULTIPLY-RECURSIVE(A22, B21, C21, n/2)
MATRIX-MULTIPLY-RECURSIVE(A22, B22, C22, n/2)
```

The time complexity of this procedure:

$$T(n) = \mathrm{D}(n) + 8T(n/2) + \Theta(1)$$

$$= \Theta(n^3).$$

Divide-and-Conquer (continued)

§ **Substitution Methods**: two steps involved
– guess the solution form
– mathematic induction to validate the constants

Substitution method for proving an upper bound on the recurrence of
$T(n) = 2T(\lfloor n/2 \rfloor) + \Theta(n)$ being $T(n) \le c \cdot n \cdot \lg n$ for a constant $c > 0$.

This is due to composing the full solution.

This is obtained by guessing its solution to be $T(n) = O(n \cdot \lg n)$ and then substituting $T(\lfloor n/2 \rfloor) \le c \cdot \lfloor n/2 \rfloor \cdot \lg(\lfloor n/2 \rfloor)$ into the recurrence:

$$\begin{aligned} T(n) &\le 2(c \cdot \lfloor n/2 \rfloor \cdot \lg(\lfloor n/2 \rfloor)) + \Theta(n) \\ &\le c \cdot n \cdot \lg(\lfloor n/2 \rfloor) + \Theta(n) \\ &\le c \cdot n \cdot \lg(n) - c \cdot n \cdot \lg(2) + \Theta(n) \\ &\le c \cdot n \cdot \lg n, \quad \text{for } c \ge 1 \end{aligned}$$

Similar substitution method for proving an upper bound on recurrence
$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(1)$,
which equals $O(n)$.

This is due to composing the full solution.

Divide-and-Conquer (continued)

§ Substitution Methods

– changing variables and/or function renaming

Substitution method for proving: $T(n) = 2T(\lfloor\sqrt{n}\rfloor) + \lg n$

Rename $m = \lg n$ (and ignore rounding) to get $T(n)$ (after parameter renaming):

$$T(n) = T(2^m) = 2T(2^{m/2}) + m$$

Further renaming $T(2^m)$ as $S(m)$, we have (function renaming)

$S(m) = 2S(m/2) + m$, which has the solution of

$S(m) = O(m \cdot \lg m)$ ← via the prior result

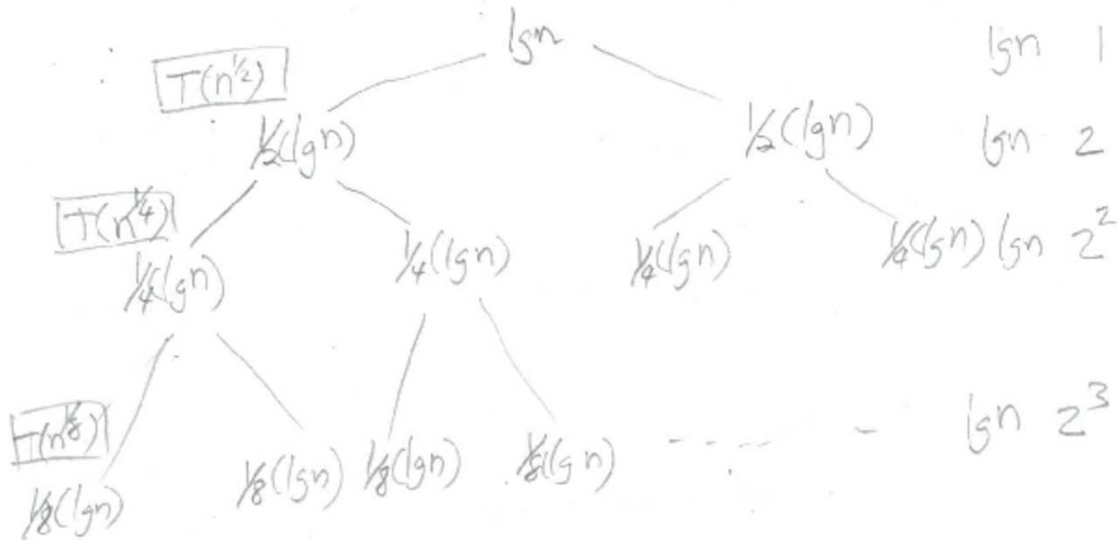
We thus have $T(n) = T(2^m) = S(m) = O(m \cdot \lg m)$

$= O(\lg n \cdot \lg \lg n)$.

Note: this problem can also be solved by the recursion-tree method, described next.

Recursion tree approach for solving

$T(n) = 2T(\sqrt{n}) + \lg n$



total leaf count $= 2^{\lg\lg} = \lg n$

Find out:

$n^{1/2^k}$ = a constant, say, 2

$\frac{1}{2^k}(\lg n) = 1$,

$2^k = \lg n \Rightarrow k = \lg\lg n$, so the tree height $= 1 + \lg\lg n$

So, the solution to $T(n)$

$\leq c(\lg n * \lg\lg n)$

You try to solve $T(n) = T(n/3) + T(2n/3) + cn$

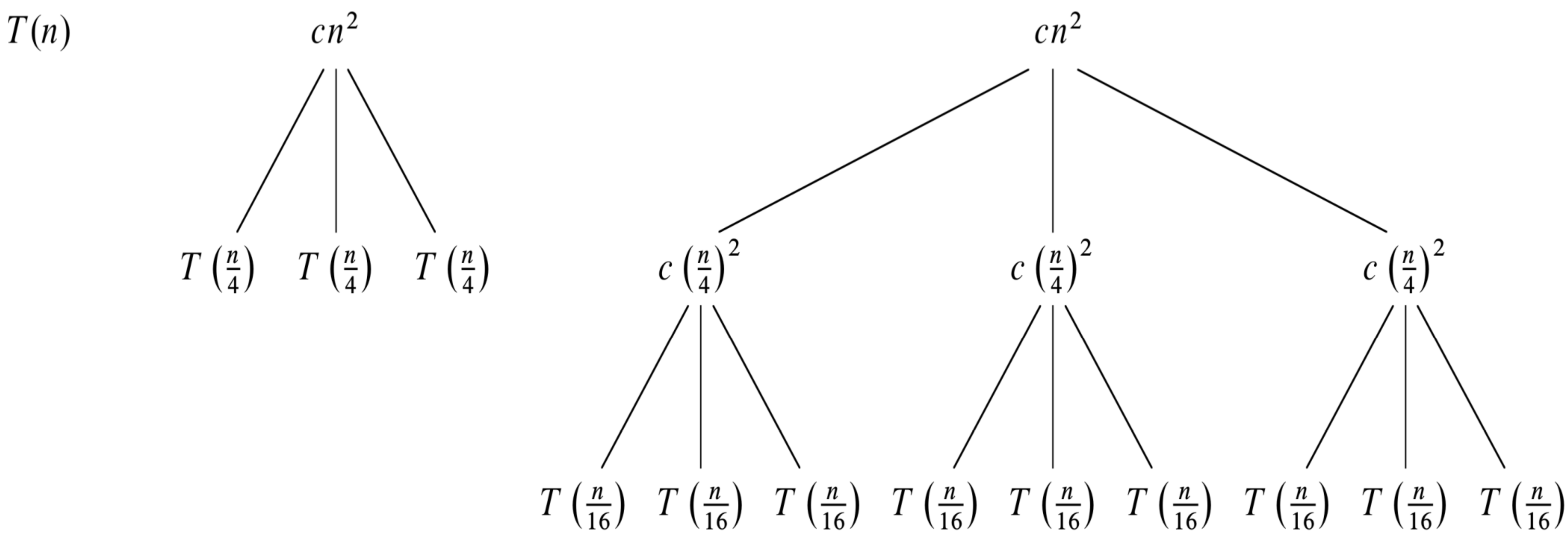
by the substitution method

Divide-and-Conquer (continued)

§ **Recursion-Tree Methods**

– best for generating good complexity bounds in general
– two examples given below

For recurrence: $T(n) = 3T(n/4) + \Theta(n^2)$

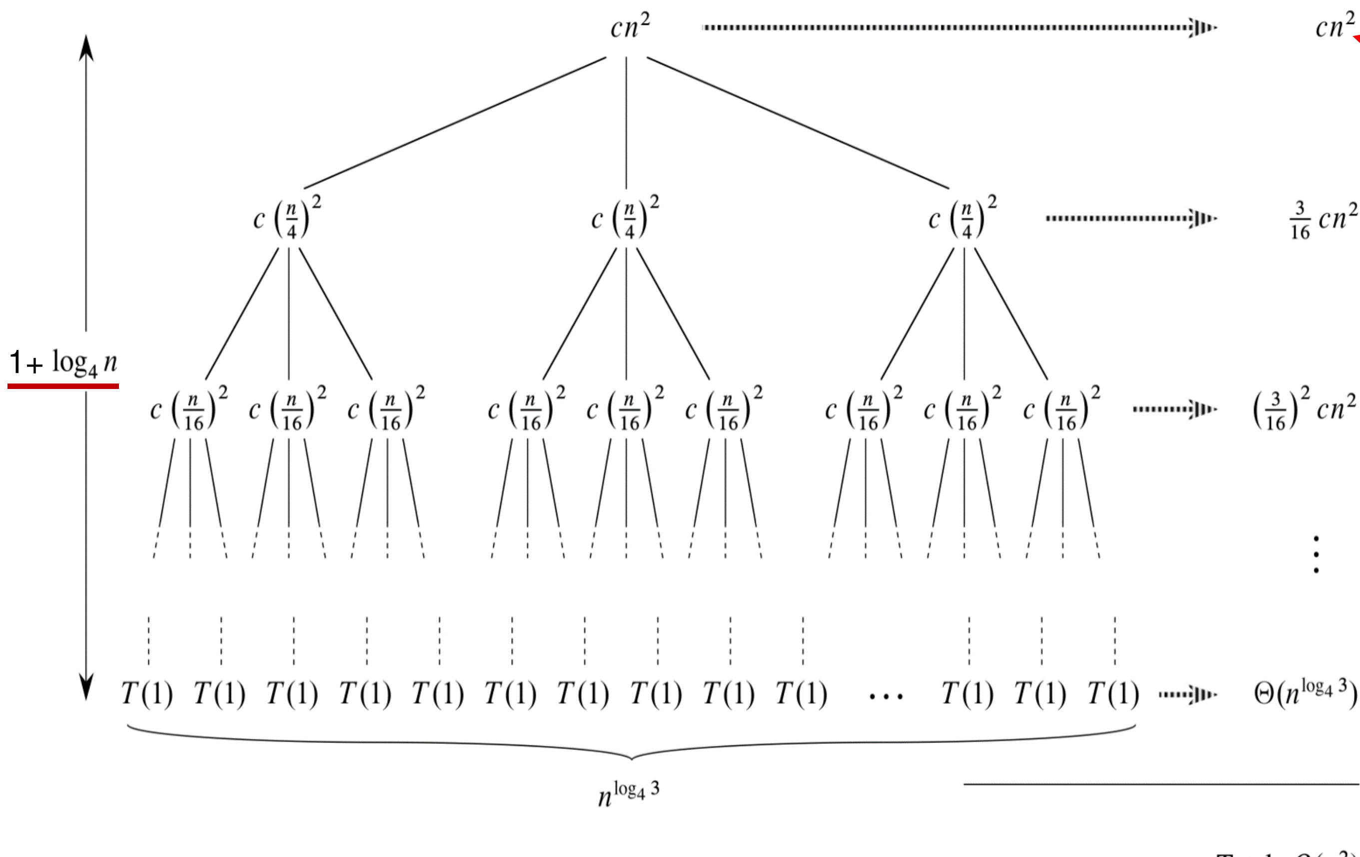


(a) (b) (c)

Divide-and-Conquer (continued)

§ Recursion-Tree Methods

For recurrence: $T(n) = 3T(n/4) + \Theta(n^2)$



(d)

$T(n)$ satisfies $\Omega(n^2)$.

3 elements

3^2 elements

$3^{\log_4 n}$ elements

$$T(n) = \sum_{i=0}^{\log_4 n - 1} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3})$$

$$< \sum_{i=0}^{\infty} \left(\frac{3}{16}\right)^i cn^2 + \Theta(n^{\log_4 3}) = \frac{1}{1-\left(\frac{3}{16}\right)} cn^2 + \Theta(n^{\log_4 3}) = O(n^2).$$

Use the property of: $v^{a \cdot b} = (v^a)^b = (v^b)^a$.

Let $3^{\log_4 n} = x$.

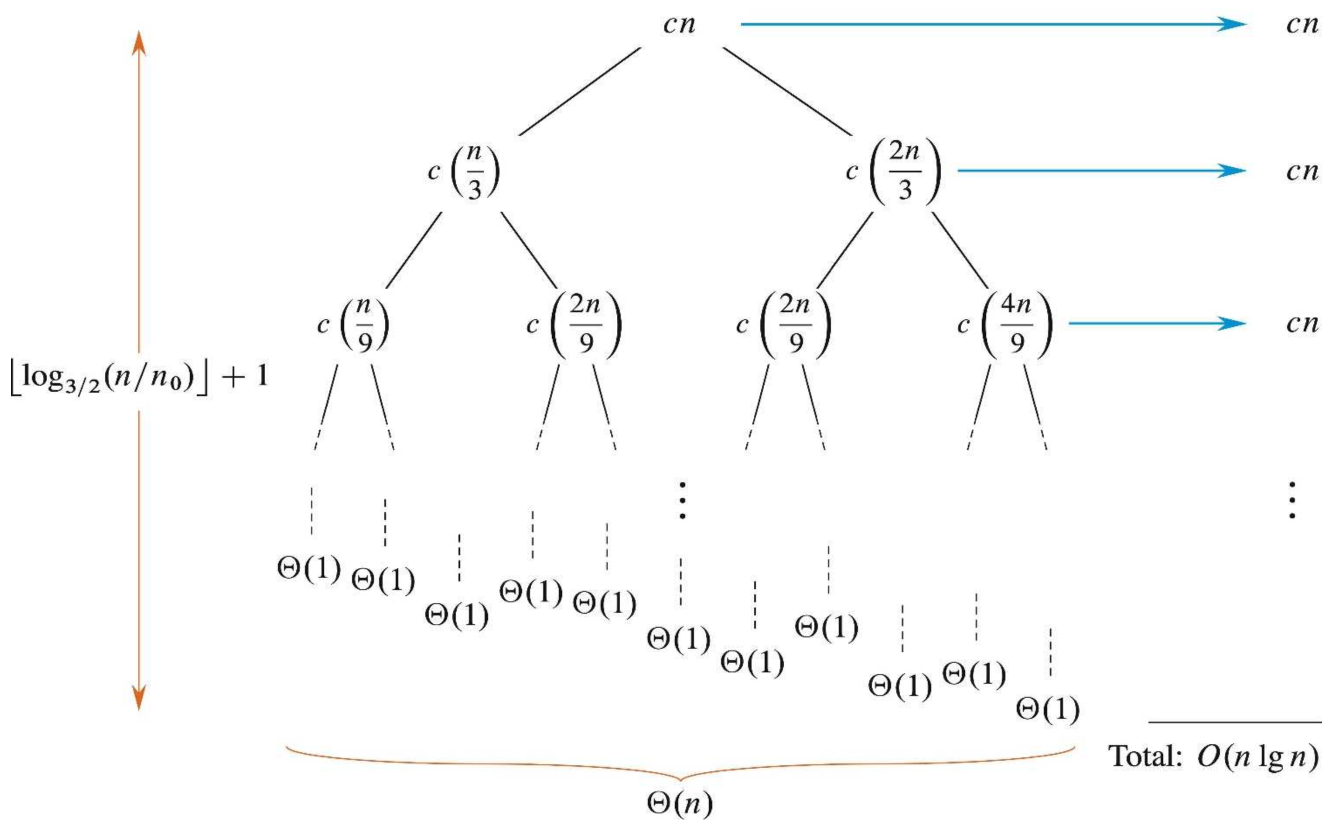
After taking $\log_4$ on both sides, we have: $(\log_4 n) \cdot (\log_4 3) = \log_4 x$.

We then take a power of 4 on both sides to yield: $4^{\log_4 n \cdot \log_4 3} = x$, which becomes:

$[4^{\log_4 n}]^{\log_4 3} = [n]^{\log_4 3} = x$. Hence, $3^{\log_4 n} = n^{\log_4 3}$.

Divide-and-Conquer (continued)

Another recurrence: $T(n) = T(n/3) + T(2n/3) + \Theta(n)$



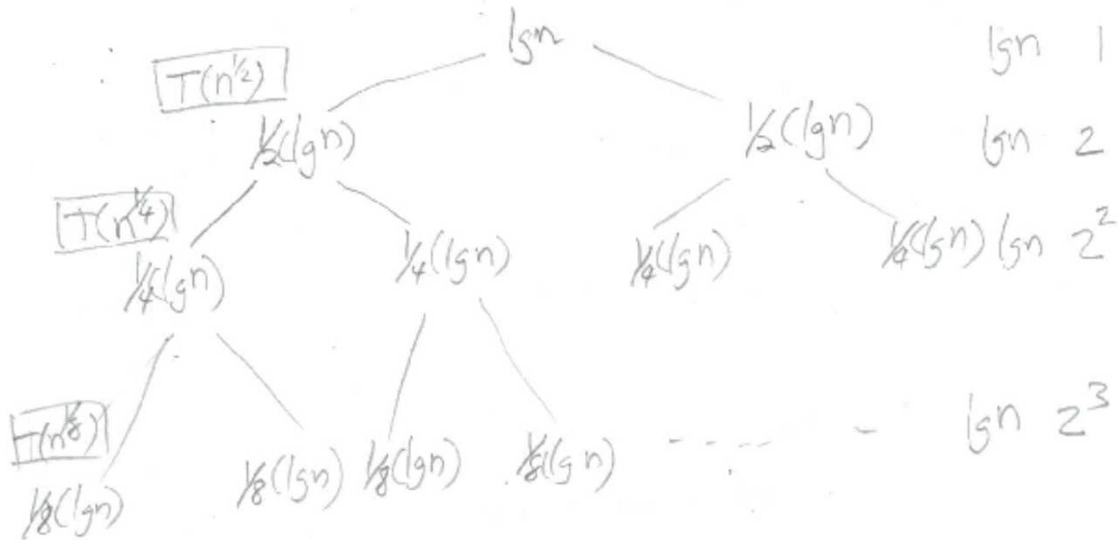
Longest path: $cn \to c(\frac{2}{3})n \to c(\frac{2}{3})^2 n \to c(\frac{2}{3})^3 n \to \cdots \to 1$, we have: $k = \log_{3/2} n$, as $(\frac{2}{3})^k n = 1$

Shortest path: $cn \to c(\frac{1}{3})n \to c(\frac{1}{3})^2 n \to c(\frac{1}{3})^3 n \to \cdots \to 1$ to get $k = \log_3 n$, ~ $(\log_{3/2} n)/2.7$

Similarly, one may show $T(n)$ upper bounded by $O(n \cdot \lg n)$ via substitution method.

Recursion tree approach for solving

$T(n) = 2T(\sqrt{n}) + \lg n$



total leaf count $= 2^{\lg\lg n}$
$= \underline{\lg n}$

Find out :

$n^{1/2^k}$ = a constant, say, $\underline{2}$

$\frac{1}{2^k}(\lg n) = 1$,

$2^k = \lg n \Rightarrow \underline{k = \lg\lg n}$, so the tree height $= 1 + \lg\lg n$

So, the solution to $T(n)$
$\leq c(\lg n * \lg\lg n)$

You try to solve $T(n) = T(n/3) + T(2n/3) + cn$
by the substitution method

Master Method for Solving Recurrences

Recurrences of $T(n) = a \cdot T(n/b) + f(n)$ with constants $a > 0$ and $b > 1$
and $f(n)$ nonnegative function that covers work on dividing the problem and on combining subproblems' results

Theorem 4.1

$T(n) = a \cdot T(n/b) + f(n)$ has following asymptotical bounds:

1. for $f(n) = O(n^{\log_b a - \epsilon})$ with constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
2. for $f(n) = \Theta(n^{\log_b a} \cdot \lg^k n)$ with constant $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \cdot \lg^{k+1} n)$
3. for $f(n) = \Omega(n^{\log_b a + \epsilon})$ with constant $\epsilon > 0$ and $a \cdot f(n/b) \leq c \cdot f(n)$, then $T(n) = \Theta(f(n))$

In the recursion-tree, 2nd level sums to **no more than** 1st level & $f(n)$ is *polynomially larger*

Note: bound is the **larger** of the two: $f(n)$ and $n^{\log_b a}$

In Case 1, $n^{\log_b a}$ is *polynomially larger* than $f(n)$

In Case 2 with $k = 0$ most commonly, $n^{\log_b a}$ and $f(n)$ are of the same size

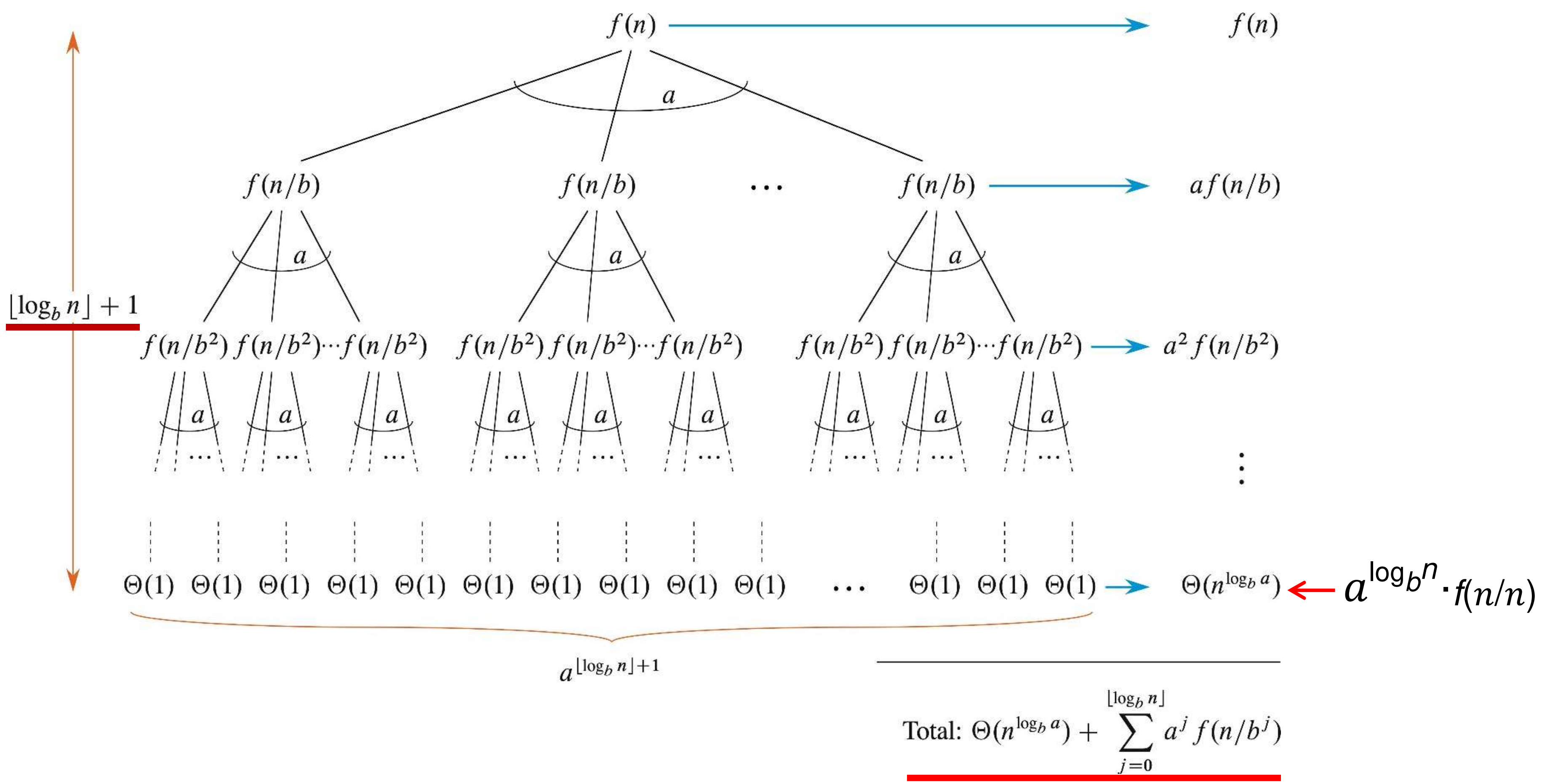
In Case 3, $f(n)$ is *polynomially larger* than $n^{\log_b a}$

Master Method (continued)

Let $T(n) = a \cdot T(n/b) + f(n)$ with constant $a > 0$ and $n \geq 1$ being an exact power of b (> 1). We have:

Lemma 4.2

$$T(n) = \Theta(n^{\log_b a}) + \sum_{j=0}^{\lfloor \log_b n \rfloor} a^j \cdot f(n/b^j)$$



Master Method (continued)

Example Recurrences $T(n) = a \cdot T(n/b) + f(n)$ solved by the master method:

1. for $f(n) = O(n^{\log_b a - \epsilon})$ with constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
2. for $f(n) = \Theta(n^{\log_b a} \cdot \lg^k n)$, then $T(n) = \Theta(n^{\log_b a} \cdot \lg^{k+1} n)$, with $k = 0$ most common
3. for $f(n) = \Omega(n^{\log_b a + \epsilon})$ with constant $\epsilon > 0$ and $a \cdot f(n/b) \leq c \cdot f(n)$, then $T(n) = \Theta(f(n))$

$T(n) = 9T(n/3) + n$

Here, $a = 9$, $b = 3$, and $f(n) = n$

From $n^{\log_3 9} = n^2$, we have $f(n) = n = O(n^{\log_3 9 - 1})$ to get $T(n) = \Theta(n^2)$

$T(n) = T(2n/3) + 1$

Here, $a = 1$, $b = 3/2$, and $f(n) = 1$

From $n^{\log_{3/2} 1} = n^0$, we have $f(n) = 1 = O(n^{\log_{3/2} 1})$ to get $T(n) = \Theta(\lg n)$

$T(n) = 3T(n/4) + n \lg n$

Here, $a = 3$, $b = 4$, and $f(n) = n \lg n$

From $n^{\log_4 3} = O(n^{0.793})$, we have $f(n) = n \lg n = \Omega(n^{\log_4 3 + \epsilon})$ to get $T(n) = \Theta(n \lg n)$

Master Method (continued)

Example Recurrences $T(n) = a \cdot T(n/b) + f(n)$:

1. for $f(n) = O(n^{\log_b a - \epsilon})$ with constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$
2. for $f(n) = \Theta(n^{\log_b a} \cdot \lg^k n)$, then $T(n) = \Theta(n^{\log_b a} \cdot \lg^{k+1} n)$, with $k = 0$ most common
3. for $f(n) = \Omega(n^{\log_b a + \epsilon})$ with constant $\epsilon > 0$ and $a \cdot f(n/b) \leq c \cdot f(n)$, then $T(n) = \Theta(f(n))$

$f(n)$ is polynomially larger

$T(n) = 2T(n/2) + n \lg n$

Here, $a = 2$, $b = 2$, and $f(n) = n \lg n$

From $n^{\log_2 2} = n$, we have $f(n) = n \lg n > n^{\log_2 2}$ but ***not polynomially*** **>** $n^{\log_2 2}$

(use substitution or recursion-tree to solve this)

$T(n) = 7T(n/2) + \Theta(n^2)$

Here, $a = 7$, $b = 2$, and $f(n) = \Theta(n^2)$

From $n^{\log_2 7} = n^{2.8^+}$, we have $f(n) = O(n^{\log_2 7 - \epsilon})$ to get $T(n) = \Theta(n^{\log_2 7})$

Master Method (continued)

Lemma 4.3

Given $g(n) = \sum_{j=0}^{\lfloor \log_b n \rfloor} a^j \cdot f(n/b^j)$ for $a > 0$ and n an exact power of b (> 1), we have:

1. for $f(n) = O(n^{\log_b a - \epsilon})$ with constant $\epsilon > 0$, then $g(n) = \Theta(n^{\log_b a})$
2. for $f(n) = \Theta(n^{\log_b a} \cdot \lg^k n)$, then $g(n) = \Theta(n^{\log_b a} \cdot \lg^{k+1} n)$, with $k = 0$ most common
3. if $a \cdot f(n/b) \leq c \cdot f(n)$ for constant $c < 1$ and for sufficiently large n, $g(n) = \Theta(f(n))$