Algorithms
Comprehensive Exam
(Spring 2020)

SHORT QUESTIONS (Answer all six questions, each carrying 7 points.)

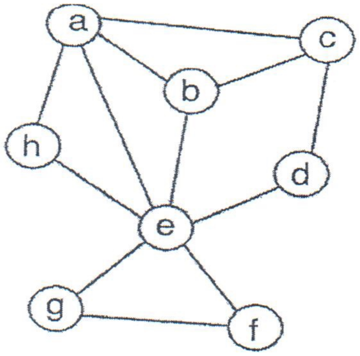
1. Solve these recurrences to get the upper bounds on the runtime:

$$\square(\square) = 2\square(\frac{\square}{2}) + \square^2$$

$$\square(\square) = \square(\frac{1}{4}\square) + \square(\frac{3}{4}\square) + \square$$

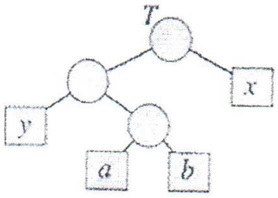
2. True or False: Under the simple uniform hashing assumption, the probability that three specific data elements (say 1, 2 and 3) are hashed to the same slot (i.e., $h(1) = h(2) = h(3)$) is $\frac{1}{\square^2}$, where m is the number of slots.

3. BFS/DFS:
   - Give the visited node order for each type of search of the given graph, starting with node a.
   - Detection of Bipartite Graph: Bipartite means the vertices can be colored red or black such that no edge links vertices of the same color. Please modify either the BFS or DFS algorithm to detect whether the given graph is Bipartite or not. Please explain your modifications.

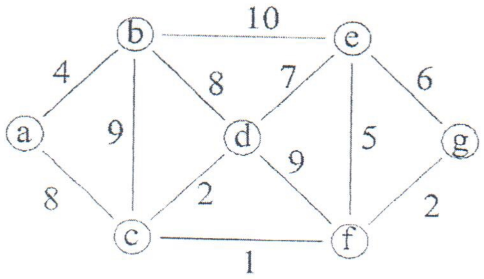


4. Sketch a proof of the Lemma below, using the tree provided.

Let C be an alphabet in which each character c in C has frequency $c.freq$. Let x and y be two characters in C having the lowest frequencies. Then there exists an optimal prefix code for C where the codewords for x and y have the same length and differ only in the least bit.



5. Please use Prim's Algorithm and Kruskal's Algorithm to compute the given graph's Minimum Spanning Tree, respectively. For each algorithm, please write down the edge picked in each step. For example, Step 1: (v1, v2).



6. A pseudo code for computing FIB(4) via $F_i = F_{i\text{-}1} + F_{i\text{-}2}$ without multithreading support and the DAG (directed acyclic graph) denoting its computation are shown below.

(a) Write a code version with multithreading support.

(b) Label (from 1 to 17) on the DAG under P_1 (with one thread); also label (from 1 to 8) on the DAG under P_∞ (with an infinite thread count).

(c) What is the inherent parallelism expression in general?

```
FIB($n$)
if $n \le 1$
    return $n$
else $x$ = FIB($n - 1$)
    $y$ = FIB($n - 2$)
    return $x + y$
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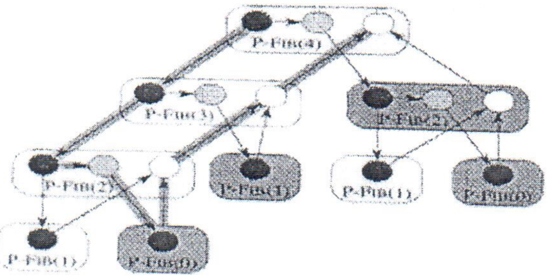
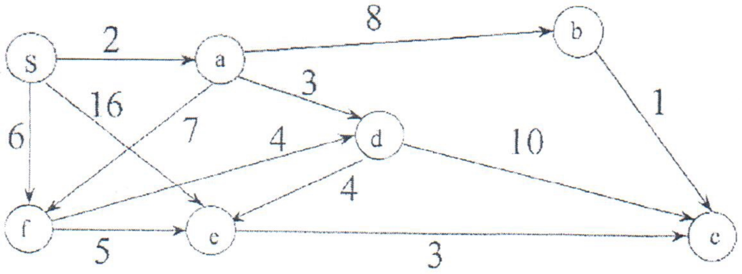
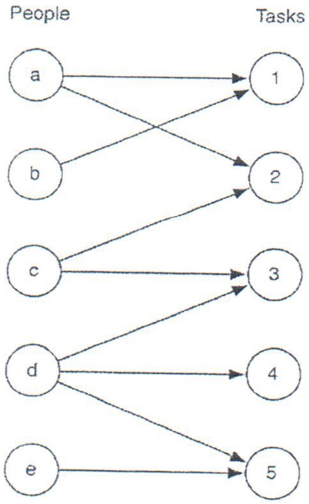


LONG QUESTIONS (Answer all four questions, each carrying 15 points.)

1. Please apply Dijkstra's algorithm and Bellman-Ford algorithm to compute the shortest paths between S and all other vertices on the given graph shown below, respectively. Please show each step.



2. The given Bipartite Graph below shows all the possible allocation of people a, b, c, d, and e to tasks 1, 2, 3, 4, and 5. A task can be handled by one person only and a person can be assigned to one job only. Apply Ford-Fulkerson algorithm to find a maximal matching such that as many tasks as possible are handled. Please show each step. (Hint: modify the existing graph to a max-flow network.)



3. Consider the matrix-chain multiplication problem for four matrices A_1, A_2, A_3, A_4, with their sizes being 30×10, 10×50, 50×40, and 40×20, respectively. Follow the tabular, bottom-up method in the procedure of MATRIX-CHAIN-ORDER below to construct tables that keep respectively entry $m[i, j]$ for all $1\leq i, j \leq 4$ and entry $s[i, j]$ for $1\leq i \leq 3$ and $2\leq j \leq 4$ to get the optimal parenthesized multiplication result.

(a) Construct the two tables, with their entry values shown.

(b) Show the parenthesized multiplication of the matrix-chain.

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MATRIX-CHAIN-ORDER($p$)
$n = p.length - 1$
let $m[1..n, 1..n]$ and $s[1..n-1, 2..n]$ be new tables
for $i = 1$ to $n$
    $m[i, i] = 0$
for $l = 2$ to $n$                // $l$ is the chain length
    for $i = 1$ to $n - l + 1$
        $j = i + l - 1$
        $m[i, j] = \infty$
        for $k = i$ to $j - 1$
            $q = m[i, k] + m[k+1, j] + p_{i-1} p_k p_j$
            if $q < m[i, j]$
                $m[i, j] = q$
                $s[i, j] = k$
return $m$ and $s$
```
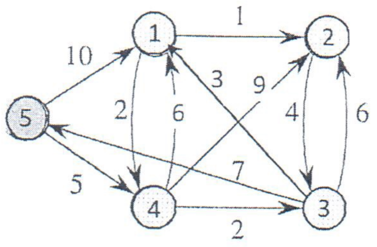

4. The Floyd-Warshall algorithm (FW) obtains all pairs of shortest paths in a weighted directed graph. Consider the graph given below. What is the recursive equation of $d_{i,j}^{(k)}$ for the shortest-path weight of any path between i and j with intermediate vertices ϵ $\{1, 2, 3, \ldots, k\}$?

Derive all distance matrices $D^{(k)}$ following FW so that the $d_{i,j}^{(n)}$ element of final matrix $D^{(n)}$ denotes $\delta(i, j)$ for every vertex pair $<i, j>$ for all $i, j \in \{1, 2, 3, 4, 5\}$.



Good Luck!