Ph.D. Comprehensive Examination
Design and Analysis of Algorithms

Spring 06

Short Questions

Answer 3 of 4 questions.

[S1] Calculate $\sum_{i=1}^{n} \frac{1}{i(i+1)(i+2)(i+3)(i+4)}$.

[S2] Given the recurrence relation

$$T(n) = 2T(\sqrt{n}) + \log_2 n,$$

$$T(2) = 1,$$

obtain a closed-form formula for $T(n)$ and determine its growth rate (Θ). (Hint: let $n = 2^{2^i}$).

[S3] Construct

[a] a finite automaton or a regular expression for the language

$\{ x \in \{0,1\}^* : x$ includes substring "000" but not "111" $\}$.

[b] a context free grammar or pushdown automaton for the language

$\{ a^{3n} b^n : n > 0 \}$

[S4]

[a] Briefly define the following four classes of sets: decidable (recursive), semi-decidable (recursively enumerable), P, and NP.

[b] What is known about the relationships of these four classes? What is not known, but believed to be true? Use a diagram if appropriate.

Long Questions

Answer 3 of 4 questions.

[L1]

[a] Write the definition of binary search tree.

[b] Consider 5 keys and their respective frequences:

keys A, B, C, D, and E, having frequencies (respectively) 7, 10, 5, 8, and 4,

where A < B < C < D < E. Using dynamic programming algorithm, find the optimal binary search tree.

[L2] Consider the use of branch and bound method to solve the traveling salesman problem.

[a] Given a cost matrix $\mathbf{M}$, how to calculate the *value* $V = V(\mathbf{M})$ of the matrix $\mathbf{M}$?

[b] Consider a graph on 4 vertices corresponding to the following cost matrix M:

$$\begin{bmatrix} \infty & 8 & 6 & 7 \\ 8 & \infty & 7 & 4 \\ 6 & 7 & \infty & 6 \\ 7 & 4 & 6 & \infty \end{bmatrix}$$

Using a branch-and-bound method, find the minimum-cost Hamiltonian circuit.

(Hint: Suppose we have a partial solution $\mathbf{X} = (x_1, ..., x_{lev}, -, ..., -)$ $(0 \le lev \le n\text{-}1)$, which represents the path $[1, x_1, \cdots, x_{lev}]$. A completion of $\mathbf{X}$ to a Hamiltonian circuit is a path from x_{lev} to x_1, having as intermediate vertices all elements in the set $\{ 2, ..., n \} - \{ x_1, ..., x_{lev} \}$. Perform the following operations on the cost matrix $\mathbf{M}$: 1) if $lev < n-1$, define $\mathbf{M}[x_{lev}, 1] = \infty$; 2) delete rows $1, x_1, ..., x_{lev-1}$ of $\mathbf{M}$; 3) delete columns $x_1, ..., x_{lev}$ of $\mathbf{M}$. Let this resulting matrix be $\mathbf{M}'(\mathbf{X})$. Then the bounding function is $B(\mathbf{X}) = V(\mathbf{M}'(\mathbf{X})) + \mathbf{M}[1, x_1] + \cdots + \mathbf{M}[x_{lev-1}, x_{lev}]$.)

[L3] Briefly prove each of the following about nondeterministic machines or programs:

[a] Any language accepted by a nondeterministic finite automaton is also accepted by a deterministic finite automaton.

[b] Any language accepted by a nondeterministic Turing machine is also accepted by a deterministic Turing Machine.

[c] The union of two languages in *NP* is also in *NP*.

[L4] Classify each of the following languages as regular, context free but not regular, or decidable but not context free. Prove your answers.

[a] $\{ a^n b^m c^m d^n : n, m \ge 0 \}$

[b] $\{ a^n b^m c^n d^m : n, m \ge 0 \}$

[c] $\{ a^{2n} b^{2m} : n, m \ge 0 \}$

2 Long Questions

Answer 3 of 4 questions

[L1] Assume that the 3-dimensional matching problem (3-DM) has been proved NP-complete. Prove that the sub-set sum problem is NP-complete.

[L2] Consider the use of branch and bound method to solve the traveling salesman problem.

(L2a) Given a cost matrix M, how to calculate the value $V = V(\mathrm{M})$ of the matrix M?
(L2b) Consider a graph on 4 vertices corresponding to the following cost matrix M

$$\mathrm{M} = \begin{bmatrix} \infty & 8 & 6 & 7 \\ 8 & \infty & 7 & 4 \\ 6 & 7 & \infty & 6 \\ 7 & 4 & 6 & \infty \end{bmatrix}$$

Using a branch-and-bound method, find the minimum-cost Hamiltonian circuit.

[L3] Let A and B be in NP, and A be polynomial-time reducible to B. Briefly prove:
(L3a) If B is in P, then A is in P.
(L3b) If A is NP-complete, then B is NP-complete.
(L3c) If A is NP-complete and B is in P, then $P = NP$.

[L4]Classify each of the following languages as regular, context free but not regular, or decidable but not context free. Prove your answers.
(L4a) $\{a^n b^m c^m : n, m \geq 0\}$;
(L4b) $\{a^{n^2} : n \geq 0\}$;
(L4c) $\{a^{2n+1} : n \geq 0\}$.