Comprehensive Exam
(Fall 2022)

Algorithms

SHORT QUESTIONS (Answer all five questions, each carrying 8 points.)

1. Solve the following recurrences to get their runtime upper bounds:

$$T(n) = T(n-2) + T(n/2) + n$$
$$T(n) = T(n^{1/2}) + \lg_2 n$$

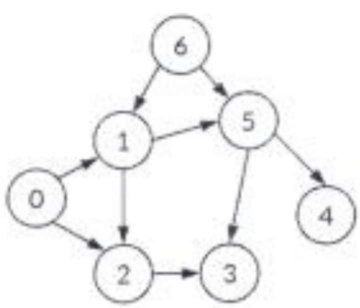
2. The Master Method aims to solve $T(n) = a \cdot T(n/b) + f(n)$ with constants $a \geq 1$ and n being an exact power of b (> 1). Show by the recursion tree technique that $T(n)$ equals

$$\Theta(n^{\log_b a}) + \sum_{j=0}^{\log_b n - 1} a^j \cdot f(n/b^j)$$

3. Prove that for an open-address hash table with load factor α (< 1), the expected number of probes in unsuccessful search under uniform hashing is at most $1/(1-\alpha)$. (Hint: consider the probability of i probes to learn key absence.)

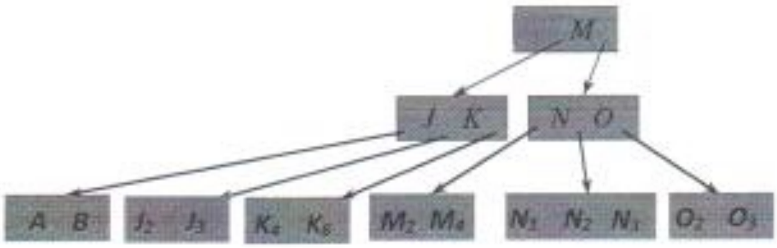
4. For a B-tree of height h with the minimum node degree of $t \geq 2$, derive the maximum number of keys that can be stored in such a B-tree.

5. Follow depth-first search (*DFS*), starting from Node 0, to traverse all nodes of the graph shown below. Mark (a) the type of every edge and (b) the discovery and the finish times of each node.



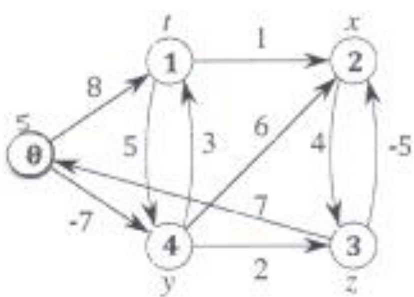
LONG QUESTIONS (Answer all four questions, each carrying 15 points.)

1. Given the initial B-tree with the minimum node degree of $t = 3$ below, show the results (a) after deleting the key of M_2, (b) followed by inserting the key of L, (c) then by deleting the key of J_2, (d) then by inserting the key of O_1, with $O < O_1 < O_2$, (e) then by deleting K, and (f) then by deleting M. (Show the result after each deletion and after each insertion.)

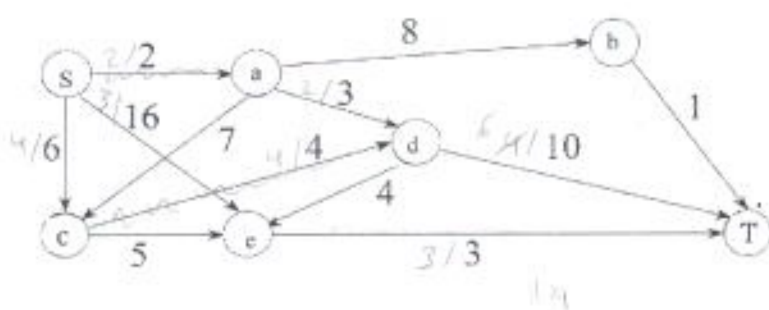


2. The Floyd-Warshall algorithm (FW) obtains all pairs of shortest paths in a weighted directed graph. Consider the graph given below. What is the recursive equation of $d_{i,j}^{(k)}$ for the shortest-path weight of any path between i and j with intermediate vertices $\in \{1, 2, 3, \ldots, k\}$?

   Derive all distance matrices $D^{(k)}$ following FW so that the $d_{i,j}^{(n)}$ element of final matrix $D^{(n)}$ denotes $\delta(i, j)$ for every vertex pair $<i, j>$ for all $i, j \in \{1, 2, 3, 4, 5\}$.



3. Follow the Ford-Fulkerson Algorithm to compute the max flow from Node S to Node T of the flow network given below. Show the corresponding residual network at each step of the Ford-Fulkerson Algorithm.



4. Consider the matrix-chain multiplication problem for four matrices A_1, A_2, A_3, A_4, with their sizes being 30×50, 50×40, 40×60, and 60×20, respectively. Follow the tabular, bottom-up method in the procedure of MATRIX-CHAIN-ORDER below to construct tables that keep respectively entry $m[i, j]$ for all $1 \le i, j \le 4$ and entry $s[i, j]$ for $1 \le i \le 3$ and $2 \le j \le 4$ to get the optimal parenthesized multiplication result.

(a) Construct the two tables, with their entry values shown.

(b) Show the parenthesized multiplication of the matrix-chain.

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MATRIX-CHAIN-ORDER(p)
n = p.length − 1
let m[1..n, 1..n] and s[1..n − 1, 2..n] be new tables
for i = 1 to n
    m[i, i] = 0
for l = 2 to n              // l is the chain length
    for i = 1 to n − l + 1
        j = i + l − 1
        m[i, j] = ∞
        for k = i to j − 1
            q = m[i, k] + m[k + 1, j] + p_{i−1} p_k p_j
            if q < m[i, j]
                m[i, j] = q
                s[i, j] = k
return m and s
```

Good Luck!