Local synchronization in complex networks of coupled oscillators

John Stout, Matthew Whiteway, Edward Ott, Michelle Girvan, and Thomas M. Antonsen

Citation: Chaos **21**, 025109 (2011); doi: 10.1063/1.3581168
View online: http://dx.doi.org/10.1063/1.3581168
View Table of Contents: http://scitation.aip.org/content/aip/journal/chaos/21/2?ver=pdfcov
Published by the AIP Publishing

Local synchronization in complex networks of coupled oscillators

John Stout,[1,a)] Matthew Whiteway,[2,b)] Edward Ott,[3,c)] Michelle Girvan,[3,d)] and Thomas M. Antonsen[3,e)]
[1]*Department of Physics, North Carolina State University, Raleigh, North Carolina 27695, USA*
[2]*Department of Physics and Astronomy, University of Oklahoma, Norman, Oklahoma 73019, USA*
[3]*Institute for Research in Electronics and Applied Physics, University of Maryland, College Park, Maryland 20742, USA*

(Received 4 February 2011; accepted 31 March 2011; published online 28 June 2011)

We investigate the effects that network topology, natural frequency distribution, and system size have on the path to global synchronization as the overall coupling strength between oscillators is increased in a Kuramoto network. In particular, we study the scenario recently found by Gómez-Gardeñes *et al.* [Phys. Rev. E **73**, 056124 (2006)] in which macroscopic global synchronization emerges through a process whereby many small synchronized clusters form, grow, and merge, eventually leading to a macroscopic giant synchronized component. Our main result is that this scenario is robust to an increase in the number of oscillators or a change in the distribution function of the oscillators' natural frequencies, but becomes less prominent as the number of links per oscillator increases. © *2011 American Institute of Physics*. [doi:10.1063/1.3581168]

The Kuramoto model has been used extensively as a tool for understanding systems consisting of a large number of coupled oscillators with heterogeneous frequencies. Although Kuramoto's original formulation was for uniform all-to-all coupling between the oscillators, the case of nontrivial network coupling is often relevant in applications and has begun to receive substantial attention. A central issue is how collective behavior arises as the coupling strengths are increased. In the classical, all-to-all coupled cased, in the limit of a large number of oscillators, there is a phase-transition-type onset of collective behavior at a critical coupling strength. Here we investigate this transition for networks, and, specifically address the case in which the average number of links to an oscillator is not large, considering the possible effects of network topology and size, as well as the distribution of the intrinsic oscillator frequencies.

I. INTRODUCTION

In the Kuramoto model for large systems of N coupled oscillators ($N \gg 1$), as the coupling strength between oscillators increases, these systems exhibit a transition from incoherent behavior to coherent global oscillatory behavior of a suitable system-wide average (called the "order parameter"). This average characterizes the macroscopic state of the system. Recently, Gómez-Gardeñes *et al.* considered network coupling between the oscillators and have investigated the microscopic precursors to global coherence in systems for which the average number of links per oscillator is small compared to N. They discovered that, for low coupling strengths, the oscillators form many small, locally synchronized clusters and, as the coupling strength increases, these clusters grow and merge eventually forming a giant component of synchronized nodes that display a net macroscopic coherence. Their study was done for fixed N, fixed average number of links per node (also called the average "degree" and henceforth denoted as $\langle d \rangle$), and for a particular distribution of the oscillators' natural frequencies. They also compared the behavior of these networks with both scale-free and Erdös–Rényi topologies.

In the present paper, we further investigate this effect and its dependence on such factors as the network size N, the average degree $\langle d \rangle$, and the oscillators' natural frequency distribution. A main result from our study is that the previously discovered effect is largely independent of N and the natural frequency distribution, but is very strongly dependent on $\langle d \rangle$, becoming much less prominent as $\langle d \rangle$ increases.

II. THE KURAMOTO MODEL

The study of large systems of coupled oscillators is of interest due to the ubiquity of these systems in a wide range of fields including physics, biology, and the social sciences. Insight for this general type of system can be obtained using the Kuramoto model,

$$\dot{\theta}_j = \omega_j + \frac{K}{N}\sum_{i=1}^{N} \sin(\theta_i - \theta_j), \tag{1}$$

where oscillator j has phase angle θ_j and natural frequency ω_j, K is the strength of the coupling between oscillators, and N is the number of oscillators in the system. Typically, the N values of the ω_j's are randomly chosen according to a probability distribution function, $g(\omega)$. Without loss of generality, the frequency at which $g(\omega)$ is maximum can be shifted to $\omega = 0$, and, in what follows, we will additionally assume

[a)]Electronic mail: jestout@ncsu.edu
[b)]Electronic mail: matthew.r.whiteway-1@ou.edu
[c)]Electronic mail: edott@umd.edu
[d)]Electronic mail: girvan@umd.edu
[e)]Electronic mail: antonsen@umd.ed

that $g(\omega)$ is symmetric and nondecreasing with respect to $\omega = 0$.

The coupling constant K governs the strength of the interaction between oscillator nodes. For $K < K_c$ (where K_c is a critical value of K) the system exhibits incoherent behavior and the oscillators lack any synchronization. As K passes through K_c, the system displays an emergent collective behavior characterized by a clumping of phases around a central average phase angle. An order parameter,

$$r = \left| \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j} \right|, \tag{2}$$

quantifies the degree of coherence in the system. When $K < K_c$ and as $N \to \infty$, $r = 0$ corresponds to global incoherence. For $N \to \infty$ and $K > K_c$ we have that $r > 0$, and as $K \to \infty$, $r \to 1$, corresponding to complete synchronization. For $N \to \infty$, Kuramoto was able to obtain analytical results for K_c and for r when $K > K_c$.

Recently, the Kuramoto model has been used to investigate the dynamics of connected networks. In these models, the connections between oscillators are governed by some sort of network scheme, and the Kuramoto model becomes

$$\dot{\theta}_j = \omega_j + \frac{K}{N} \sum_{i=1}^{N} A_{ij} \sin(\theta_i - \theta_j), \tag{3}$$

where A_{ij} describes the structure of the oscillator network and is called the adjacency matrix— $A_{ij} = 1$ if a link exists between oscillators i and j, and is 0 otherwise. For undirected networks, A_{ij} is a symmetric matrix. When the network has large average degree $\langle d \rangle \gg 1$, Restrepo *et al.*[7] showed that the system described by Eq. (3) experienced a transition similar to that for the $N \gg 1$ globally connected model at a critical value $K = K_c$, given approximately by

$$K_c \cong \bar{K}_c = \frac{2N}{\pi g(0) \lambda_1}, \tag{4}$$

where λ_1 denotes the maximum eigenvalue of the adjacency matrix A_{ij}.

The order parameter described by Eq. (2) can still provide useful information about the global synchronization of the network, but we also desire a parameter that describes the degree of synchronization between all *connected* oscillators. For this purpose, we use another order parameter, r_{link}, introduced in Ref. 3, defined as

$$r_{\text{link}} = \frac{1}{2N_l} \sum_{i,j} A_{ij} \left| \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_{\tau_r}^{\tau_r + \Delta t} e^{i(\theta_i - \theta_j)} \, dt \right|. \tag{5}$$

This parameter measures the amount of local synchronization in a given network by comparing the relative phases of each pair of connected oscillators. In this equation, N_l is the total number of links, and A_{ij} is the adjacency matrix of the network. In practice, we choose τ_r to be sufficiently large so that the system is able to attain an equilibrium state and Δt to be large enough to attain a stable time average. r_{link} can be thought of as the fraction of links that connect synchronized oscillators.

III. THE EFFECTS OF NETWORK STRUCTURE

In this paper, we study how varying certain network parameters affect global and local synchronization patterns. We will look at undirected Erdös–Rènyi random networks[4] (denoted henceforth as ER) and undirected Barabási–Albert scale-free networks[5] (denoted as BA). We will first look at the effect the structure of these networks has on the evolution of the saturated system state as K increases from low values to $K > K_c$.

We now discuss the results of numerical experiments in which we used a fourth order Runge–Kutta method to integrate Eq. (3) forward in time.

Erdös–Rényi networks are formed by first specifying the total number of nodes. Links are randomly established between each oscillator pair, according to a constant probability of connection. This results in a Poisson degree distribution if the number of nodes is large.

Barabási–Albert networks are created as follows. An initial, fully connected core of nodes is first formed. Then, new nodes are randomly attached, one at a time, with links to higher degree nodes having a higher probability of connection. This strategy leads to a very heterogenous degree distribution that is characterized by a few very high degree nodes and many low degree nodes. The degree distributions of BA networks follow a power law when the number of nodes is large. Such networks are referred to as *scale free* and represent a wide variety of real world networks, from protein interaction networks[9] to the Internet.[10]

Figure 1 shows r and r_{link} plotted against a scaled value of the coupling constant K for both ER and BA networks.

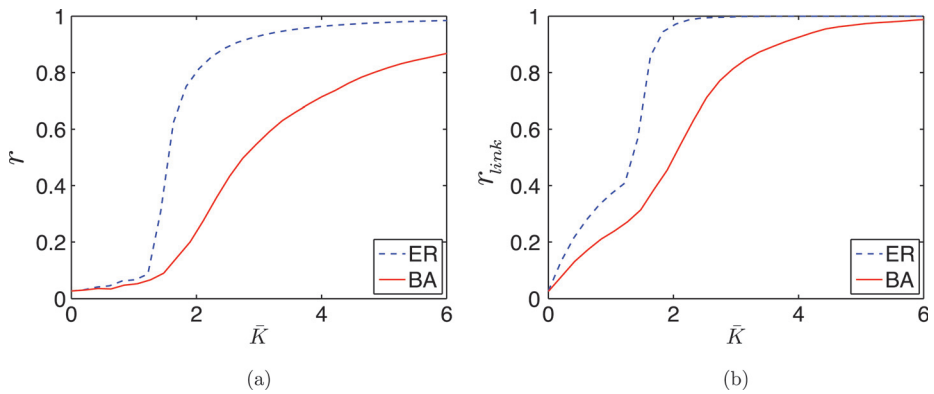

FIG. 1. (Color online) Order parameters r and r_{link} given by Eqs. (2) and (5), respectively, vs. increasing normalized coupling strength $\bar{K}$. Both networks consist of 1000 oscillators, $\langle d \rangle = 6$. Oscillator natural frequencies from a Gaussian distribution. The curves are for Erdös–Rényi (ER) and Barabási–Albert scale-free (BA) networks.

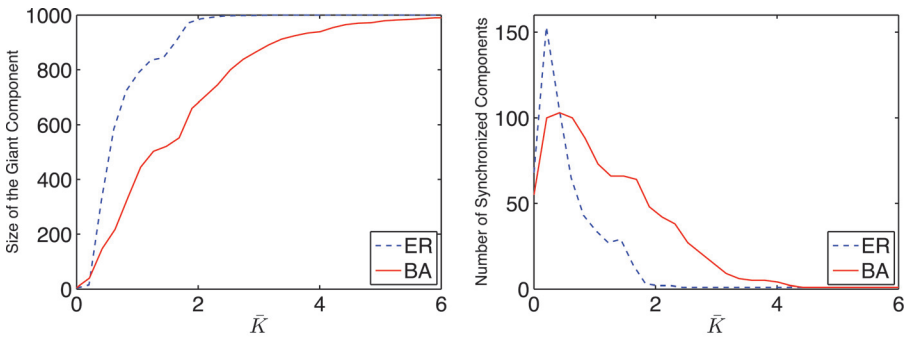

FIG. 2. (Color online) Giant component size and the number of synchronized components vs. increasing normalized coupling strength $\bar{K}$. Both networks consist of 1000 oscillators, $\langle d \rangle = 6$. Oscillator natural frequencies pulled from a Gaussian distribution.

Note that the horizontal axis in Fig. 1 is $\bar{K} = K/\bar{K}_c$, where $\bar{K}_c$ is given by Eq. (4). This allows the direct comparison of different networks, because the onset of synchronization is predicted[7] to occur at $\bar{K} = 1$ for all networks (assuming $\langle d \rangle \gg 1$). Figure 1 shows that r_{link} for both networks increases noticeably before any substantial change in their respective r values, which suggests that a certain level of local coherence is achieved even in the absence of global synchronization.

Following the methodology of Ref. 3, this assertion is further supported by Fig. 2 which is a plot of both the size of the largest, or giant, component and the number of synchronized components versus $\bar{K}$ for both ER and BA networks. In order to obtain these measures, we create a "distance matrix,"

$$D_{ij} = A_{ij} \left| \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_{\tau_r}^{\tau_r + \Delta t} e^{i(\theta_i - \theta_j)} \, dt \right|. \tag{6}$$

r_{link} is the sum of Eq. (6) over all connected oscillator pairs normalized by the number of links N_l in the network. D_{ij} is a matrix whose entries vary from 0 to 1. An entry of 0 indicates that the two oscillators are unconnected or completely unsynchronized, while an entry of 1 indicates complete synchronization between i and j. Since r_{link} represents the fraction of synchronized links, we filter D_{ij} for the largest $N_l \, r_{\text{link}}$ entries. A new matrix is then made, with the remaining filtered entries set to 1 and the rest to 0, thus defining a putative effective adjacency matrix of synchronized links. The number of unconnected subnetworks (excluding singletons) defines the number of synchronized components, and the largest of such component is the giant component.

Figure 2 demonstrates that, for both our ER and BA networks, the number of synchronized components initially increases as the coupling is increased. This corresponds to unsynchronized connected oscillators becoming synchronized as the coupling is increased. As a result many small groups form, and, as the coupling strength is further increased, these small groups begin to coalesce, thereby decreasing the overall number of synchronized components while increasing the size of each component.

The onset of global synchronization for ER networks appears to occur from the coalescence of many small groups into larger groups, which eventually merge to achieve global synchronization. From Table I we see that there are many synchronized groups of 3, 4, 5, and 6 or more oscillators for small values of $\bar{K}$. As $\bar{K}$ is increased these numbers drop sharply, along with the number of single unsynchronized oscillators. By the time $\bar{K} = 1.22$, there are 8 groups of 3 or more oscillators that account for 875 oscillators, or 87.5% of the total. However, at this coupling strength the size of the giant component is 835. The giant component does not develop from many large components spontaneously forming into one, but rather from the addition of many small components to an already present giant component.

The BA network, on the other hand, has a smaller maximum number of synchronized components and a larger number of single unsynchronized oscillators than the ER network. According to Table I the number of synchronized groups of 3, 4, 5, or 6, or more oscillators, is much smaller than that of the ER network for small $\bar{K}$. Even when the maximum number of synchronized components is reached at $\bar{K} = 0.41$, single oscillators still account for 55.3% of the BA network. When $\bar{K} = 1.22$ there are 20 groups of 3 or more oscillators that account for 646 oscillators (giant component = 503 oscillators), much less than the 8 groups of the ER network. The giant component of the BA network grows mainly by the addition of single oscillators or groups of two oscillators as opposed to groups of more oscillators.

TABLE I. The number of components of size 1 (not counted as a synchronized component), 2, 3, 4, 5, and 6+, as well as the total number of synchronized components (vertical axis of Fig. 2).

$\bar{K}$	1	2	3	4	5	6+	Synchronized components
(a) ER network							
0.21	457	74	35	14	12	18	153
0.41	290	55	19	11	5	17	107
0.61	199	35	9	7	4	10	65
0.81	151	23	9	3	2	6	43
1.01	117	20	6	3	1	4	34
1.22	93	16	4	3	2	2	27
(b) BA network							
0.21	691	74	13	5	2	6	100
0.41	553	67	11	11	5	9	103
0.61	450	63	15	6	3	13	100
0.81	389	59	12	8	2	7	88
1.01	338	49	11	6	1	6	73
1.22	308	46	11	2	2	5	66

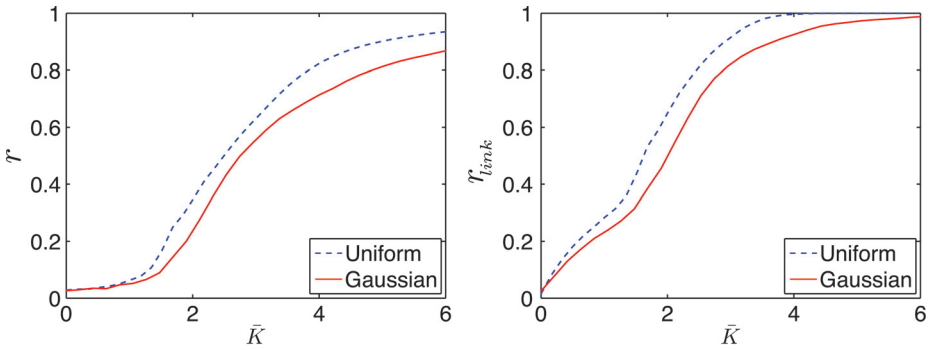

FIG. 3. (Color online) r and r_{link} vs. increasing normalized coupling strength $\bar{K}$ for Gaussian and uniform natural frequency distributions. These results are for BA networks consisting of 1000 oscillators, $\langle d \rangle = 6$, though ER networks produce similar results.

This contrasts with what is observed in the ER case where the giant component typically grows by a merger of synchronized groups of 3 or more oscillators. This behavior of the BA system also explains its slower route to local and global synchronization, as evidenced in Fig. 1, compared to the ER network.

Our conclusions above are consistent with those of Ref. 3, which were obtained at fixed N and $\langle d \rangle$ values similar to those used above.

IV. THE EFFECTS OF FREQUENCY DISTRIBUTION

In this section we explore how the natural frequency distribution of the oscillators affects local and global synchronization properties. We consider only natural frequency distributions that are symmetric about a central point, which we take to be zero without any loss of generality.[8] The two frequency distributions that we use are a uniform distribution between $-\pi$ and π and a Gaussian distribution, both centered at zero.

In Ref. 3, only the uniform $g(\omega)$ was investigated. However, the character of the transition at $K = K_c$ for $N \to \infty$ can be shown[8] to be very different for uniform $g(\omega)$ as opposed to a continuous, unimodal $g(\omega)$ (e.g., a Gaussian). In particular, in the former case r jumps discontinuously from $r = 0$ to $r > 0$ as K increases through K_c (as in a first order phase transition), while, in the case of a continuous $g(\omega)$ the transition at $K = K_c$ is continuous (as in a second order phase transition). Thus, the question naturally arises as to whether the transitional phenomena observed in Ref. 3 might be different for different types of $g(\omega)$.

Figure 3 shows the same information for the order parameters presented in Sec III, but now only considers a single network structure (BA) and for both Gaussian and uniform natural frequency distributions. Even at coupling strengths below the critical value, the network with the uniform distribution shows a faster increase in r and r_{link}. This behavior arises from the fact that the uniform distribution has no tail, and therefore the farthest apart any two frequencies can be 2π. The Gaussian distribution, on the other hand, has tails that extend out to infinity, and the outliers contained in these tails take either more connections or a greater coupling strength in order to synchronize them with the rest of the nodes. With respect to the general route to local and global synchronization, however, the differences between these two natural frequency distributions are qualitatively similar.

V. DEPENDENCE ON THE NUMBER OF NODES *N* AND THE AVERAGE DEGREE

We now consider what happens to the synchronization properties of a network when we vary N and $\langle d \rangle$. There are three different ways that we do this, which will be covered in the following three subsections: (1) increasing the number of nodes, N, while keeping the average degree, $\langle d \rangle$, fixed; (2) increasing $\langle d \rangle$ while keeping N fixed; and (3) increasing both N and $\langle d \rangle$ while keeping the density of links fixed by keeping $\langle d \rangle / N$ constant.

A. Varying *N* with ⟨*d*⟩ fixed

We specifically desire to know if the behavior discussed earlier (i.e., the evolution of r_∞ and r_{link} with increasing $\bar{K}$) depends on the number of nodes. r_∞ (not shown) shows only very small variations with N. r_{link} also fails to show any significant differences, as seen in Fig. 4(a).

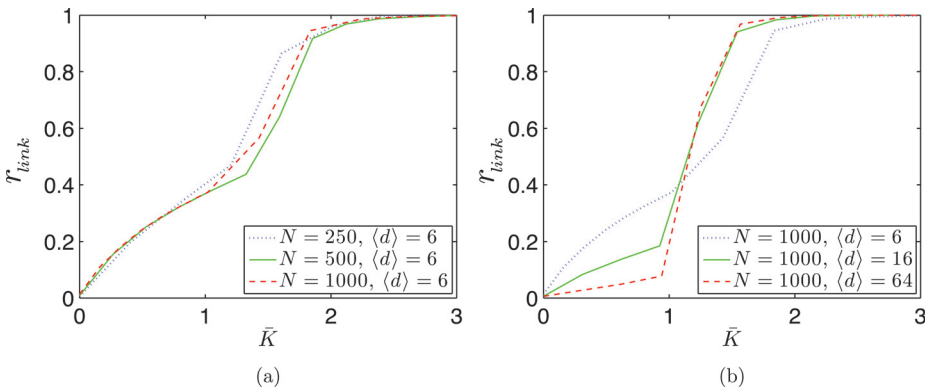

FIG. 4. (Color online) r_{link} vs. increasing normalized coupling strength $\bar{K}$ on an ER network. In (a) the number of oscillators, N, is varied keeping $\langle d \rangle$ constant and in (b) the opposite is done.

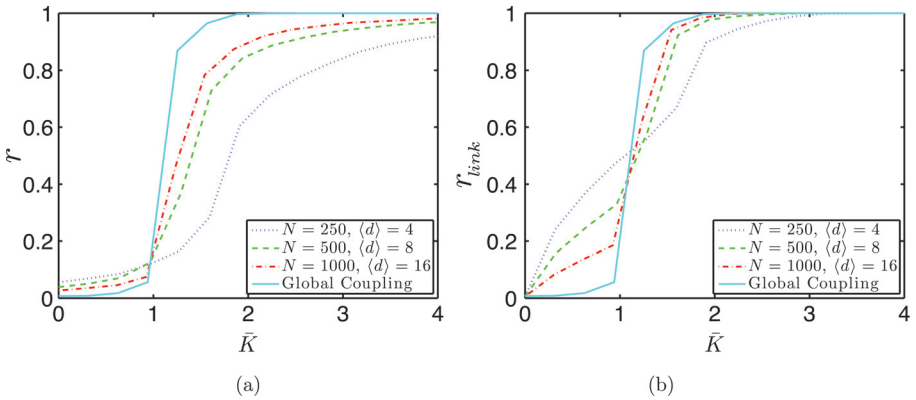

FIG. 5. (Color online) r and r_{link} vs. increasing normalized coupling strength $\bar{K}$ for several different ER network realizations in which $\langle d \rangle / N$ is kept constant, with the globally coupled case included. Similar results were obtained for BA networks.

B. Varying $\langle d \rangle$ with N fixed

Again r_∞ shows little variation between the networks, but r_{link} displays some interesting behavior. In Fig. 4(b) we see that as the average degree is increased, the amount of local synchronization begins to decrease. Though there is a substantial amount of local synchronization when $\langle d \rangle = 6$, increasing the average degree of the network suppresses this synchronization until at $\langle d \rangle = 64$ it is negligible. This result can be interpreted as follows. Each oscillator is "pulled" a certain amount by its neighbors and when the average degree is small enough, these oscillators are "pulled" into synchronizing with a small cluster of oscillators with similar natural frequencies and phases. As the average degree is increased, each oscillator will be "pulled" in many opposing directions, with the net result that the oscillator fails to synchronize, until the coupling strength reaches the critical point, where the entire system quickly synchronizes.

C. Varying both N and $\langle d \rangle$ with $\langle d \rangle / N$ fixed

Finally we consider the effects of increasing the size of a network while keeping the density of links constant. Figure 5 clearly indicates that the behaviors of r_∞ and r_{link} vary as we increase the network size. The differences in r_∞ are due to the fact that as the network shrinks in size there are less links that can synchronize. With fewer synchronized links, the transition is noisier, and, as expected; this both delays the onset of large r and broadens the transition region. r_{link}, again, displays much more interesting information. As the size of the network (and, therefore, average degree) is increased, the maximum value that r_{link} achieves before the transition to global synchronization is decreased. This implies that local synchronization plays a much more influential role in smaller networks than in larger networks. In fact, as $N \to \infty$, r_{link} appears to approach its behavior for a globally connected network.

VI. CONCLUSION

We have considered Erdös–Rényi and Barabási–Albert networks, as well as two different natural frequency distributions and their effect on the path to global synchronization. The structure of the interaction network has a strong effect on the path to synchronization in systems of coupled oscillators. The homogeneity of the ER network causes it to reach a higher degree of both local and global synchronization faster than the heterogeneous BA network. Although frequency distribution does not have a profound impact on the system's dynamics, a uniform natural frequency distribution tends to synchronize both locally and globally earlier than a Gaussian distribution.

Global synchronization seems to be most affected by the topology of the network, as we can see in Fig. 1. Local synchronization, however, is most directly influenced by the average degree of the network. For low average degree, as originally found in Ref. 3, there is a significant amount of local synchronization, but, for high average degree, the amount of local synchronization is very small when $K < K_c$. In fact, for $N \gg 1$ and global coupling, there is hardly any local synchronization before the transition to synchronization (Fig. 5).

ACKNOWLEDGMENTS

This work was supported by an NSF REU grant to the University of Maryland and by ONR Grant No. N00014-07-1-0734.

[1]M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[2]Y. Kuramoto, *International Conference on Mathematical Problems in Theoretical Physics, Lecture Notes in Physics*, Vol. 39, edited by H. Araki (Springer, Berlin, 1975), p. 420.
[3]J. Gómez-Gardeñes, Y. Moreno, and A. Arenas, Phys. Rev. E **75**, 066106 (2007); Phys. Rev. Lett. **98**, 034101 (2007).
[4]P. Erdös and A. Rényi, Publ. Math. **6**, 290 (1959).
[5]A. L. Barabási and R. Albert, Science **286**, 509 (1999).
[6]J. Gómez-Gardeñes and Y. Moreno, Phys. Rev. E **73**, 056124 (2006).
[7]J. G. Restrepo, E. Ott, and B. R. Hunt, Phys. Rev. E **71**, 036151 (2005).
[8]E. Ott, *Chaos in Dynamical Systems*, 2nd ed. (Cambridge University Press, New York, 2002).
[9]H. Jeong, S. Mason, A. L. Barabasi, and Z. N. Oltvai, Nature **411**, 4142 (2001).
[10]Q. Chen, H. Chang, R. Govindan, S. Jamin, S. J. Shenker, and W. Willinger, *Proceedings of the 21st Annual Joint Conference of the IEEE Computer and Communications Societies*, IEEE Computer Society (Twente, The Netherlands, 2002).