MODULE-2

CHAPTER 1

SEQUENCE & SERIES

Chapter Contents 01

IMPORTANT NOTES

CHAPTER 1

SEQUENCE & SERIES

1. DEFINITION :

Sequence :

A succession of terms a_1, a_2, a_3, a_4........ formed according to some rule or law.

Examples are : 1, 4, 9, 16, 25

–1, 1, –1, 1,........

$\frac{x}{1!}, \frac{x^2}{2!}, \frac{x^3}{3!}, \frac{x^4}{4!}$, ...

A finite sequence has a finite (i.e. limited) number of terms, as in the first example above. An infinite sequence has an unlimited number of terms, i.e. there is no last term, as in the second and third examples.

Series :

The indicated sum of the terms of a sequence. In the case of a finite sequence a_1, a_2, a_3,................, a_n the corresponding series is $a_1 + a_2 + a_3 + ... + a_n = \sum_{k=1}^{n} a_k$. This series has a finite or limited number of terms and is called a finite series.

2. ARITHMETIC PROGRESSION (A.P.) :

A.P. is a sequence whose terms differ by a fixed number. This fixed number is called the common difference. If a is the first term & d the common difference, then A.P. can be written as

$$a, a + d, a + 2d, ..., a + (n - 1) d, ...$$

(a) n^{th} term of AP $T_n = a + (n-1)d$, where $d = t_n - t_{n-1}$

(b) The sum of the first n terms : $S_n = \frac{n}{2}[a+\ell] = \frac{n}{2}[2a+(n-1)d]$

where ℓ is n^{th} term.

Note :

(i) n^{th} term of an A.P. is of the form $An + B$ i.e. a linear expression in 'n', in such a case the coefficient of n is the common difference of the A.P. i.e. A.

(ii) Sum of first 'n' terms of an A.P. is of the form $An^2 + Bn$ i.e. a quadratic expression in 'n', in such case the common difference is twice the coefficient of n^2. i.e. 2A

(iii) Also n^{th} term $T_n = S_n - S_{n-1}$

Illustrations

Illustration 1 : If (x + 1), 3x and (4x + 2) are first three terms of an A.P. then its 5th term is -

(A) 14 (B) 19 (C) 24 (D) 28

Solution : (x + 1), 3x, (4x + 2) are in AP

$\Rightarrow 3x - (x + 1) = (4x + 2) - 3x \qquad \Rightarrow x = 3$

$\therefore\ a = 4,\ d = 9 - 4 = 5 \qquad \Rightarrow T_5 = 4 + (4)5 = 24$ **Ans. (C)**

Illustration 2 : The sum of first four terms of an A.P. is 56 and the sum of it's last four terms is 112. If its first term is 11 then find the number of terms in the A.P.

Solution : $a + a + d + a + 2d + a + 3d = 56$

$4a + 6d = 56$

$44 + 6d = 56$ (as a = 11)

$6d = 12$ hence $d = 2$

Let total number of terms = n

Now sum of last four terms.

$a + (n - 1)d + a + (n - 2)d + a + (n - 3)d + a + (n - 4)d = 112$

$\Rightarrow 4a + (4n - 10)d = 112 \qquad \Rightarrow 44 + (4n - 10)2 = 112$

$\Rightarrow 4n - 10 = 34$

$\Rightarrow n = 11$ **Ans.**

Illustration 3 : The sum of first n terms of two A.Ps. are in ratio $\frac{7n+1}{4n+27}$. Find the ratio of their 11th terms.

Solution : Let a_1 and a_2 be the first terms and d_1 and d_2 be the common differences of two A.P.s respectively then

$$\frac{\frac{n}{2}[2a_1 + (n-1)d_1]}{\frac{n}{2}[2a_2 + (n-1)d_2]} = \frac{7n+1}{4n+27} \Rightarrow \frac{a_1 + \left(\frac{n-1}{2}\right)d_1}{a_2 + \left(\frac{n-1}{2}\right)d_2} = \frac{7n+1}{4n+27}$$

For ratio of 11th terms

$$\frac{n-1}{2} = 10 \quad \Rightarrow \quad n = 21$$

so ratio of 11th terms is $\frac{7(21)+1}{4(21)+27} = \frac{148}{111} = \frac{4}{3}$ **Ans.**

Do yourself - 1

1. Write down the sequence whose n^{th} terms is :

(a) $\frac{2^n}{n}$ (b) $\frac{3+(-1)^n}{3^n}$

2. For an A.P, show that $t_m + t_{2n+m} = 2t_{m+n}$

3. If the sum of p terms of an A.P. is q and the sum of its q terms is p, then find the sum of its (p + q) term.

Find the

4. sum 49, 44, 39.... to 17 terms.

5. sum $\frac{3}{4},\frac{2}{3},\frac{7}{12},$....... to 19 terms.

6. sum 1.3,–3.1, –7.5,.... to 10 terms

7. sum a – 3b, 2a – 5b, 3a – 7b,.... to 40 terms.

3. PROPERTIES OF A.P. :

(a) If each term of an A.P. is increased, decreased, multiplied or divided by the some nonzero number, then the resulting sequence is also an A.P.

(b) Three numbers in A.P. : $a-d, a, a+d$

Four numbers in A.P. : $a-3d, a-d, a+d, a+3d$

Five numbers in A.P. : $a-2d, a-d, a, a+d, a+2d$

Six numbers in A.P. : $a-5d, a-3d, a-d, a+d, a+3d, a+5d$ etc.

(c) The common difference can be zero, positive or negative.

(d) k^{th} term from the last = $(n-k+1)^{th}$ term from the beginning (If total number of terms = n).

(e) The sum of the two terms of an AP equidistant from the beginning & end is constant and equal to the sum of first & last terms. $\Rightarrow T_k + T_{n-k+1} = \text{constant} = a + \ell$.

(f) Any term of an AP (except the first) is equal to half the sum of terms which are equidistant from it. $a_n = (1/2)(a_{n-k} + a_{n+k}),\ k < n$

For $k = 1,\ a_n = (1/2)(a_{n-1} + a_{n+1})$; For $k = 2,\ a_n = (1/2)(a_{n-2} + a_{n+2})$ and so on.

(g) If a, b, c are in AP, then 2b = a + c.

Illustrations

Illustration 4 : Four numbers are in A.P. If their sum is 20 and the sum of their squares is 120, then the middle terms are -

(A) 2, 4 (B) 4, 6 (C) 6, 8 (D) 8, 10

Solution : Let the numbers are $a - 3d, a - d, a + d, a + 3d$

given, $a - 3d + a - d + a + d + a + 3d = 20 \quad \Rightarrow 4a = 20 \Rightarrow a = 5$

and $(a - 3d)^2 + (a - d)^2 + (a + d)^2 + (a + 3d)^2 = 120 \quad \Rightarrow 4a^2 + 20d^2 = 120$

$\Rightarrow 4 \times 5^2 + 20d^2 = 120 \quad \Rightarrow d^2 = 1 \Rightarrow d = \pm 1$

Hence numbers are 2, 4, 6, 8 or 8, 6, 4, 2 **Ans. (B)**

Illustration 5 : If $a_1, a_2, a_3, \ldots\ldots\ldots, a_n$ are in A.P. where $a_i > 0$ for all i, show that :

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{(n-1)}{\sqrt{a_1}+\sqrt{a_n}}$$

Solution : L.H.S. $= \dfrac{1}{\sqrt{a_1}+\sqrt{a_2}}+\dfrac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots\ldots+\dfrac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$

$$=\frac{1}{\sqrt{a_2}+\sqrt{a_1}}+\frac{1}{\sqrt{a_3}+\sqrt{a_2}}+\ldots\ldots+\frac{1}{\sqrt{a_n}+\sqrt{a_{n-1}}}$$

$$=\frac{\sqrt{a_2}-\sqrt{a_1}}{(a_2-a_1)}+\frac{\sqrt{a_3}-\sqrt{a_2}}{(a_3-a_2)}+\ldots\ldots+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}$$

Let 'd' is the common difference of this A.P.

then $a_2 - a_1 = a_3 - a_2 = \ldots\ldots. = a_n - a_{n-1} = d$

Now L.H.S.

$$=\frac{1}{d}\left\{\sqrt{a_2}-\sqrt{a_1}+\sqrt{a_3}-\sqrt{a_2}+\ldots\ldots+\sqrt{a_{n-1}}-\sqrt{a_{n-2}}+\sqrt{a_n}-\sqrt{a_{n-1}}\right\}=\frac{1}{d}\left\{\sqrt{a_n}-\sqrt{a_1}\right\}$$

$$=\frac{a_n-a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)}=\frac{a_1+(n-1)d-a_1}{d\left(\sqrt{a_n}+\sqrt{a_1}\right)}=\frac{1}{d}\frac{(n-1)d}{(\sqrt{a_n}+\sqrt{a_1})}=\frac{n-1}{\sqrt{a_n}+\sqrt{a_1}}=\text{R.H.S.}$$

Do yourself - 2

1. Find the sum of first 24 terms of the A.P. a_1, a_2, a_3......, if it is know that

$a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.

2. Find the number of terms common to the two A.P.'s 3, 7, 11, ... 407 and 2, 9, 16, ..., 709

3. In an A.P. the first term is 2, the last term is 29, the sum is 155; find the common difference.

4. The sum of 15 terms of an A.P. is 600, and the common difference is 5; find the first term.

5. The third term of an A.P. is 18, and the seventh term is 30; find the sum of 17 terms.

6. The sum of three numbers in A.P. is 27, and their product is 504; find them.

7. The sum of three numbers in A.P. is 12, and the sum of their cubes is 408; find them.

8. Find the sum of 15 terms of the series whose n^{th} term is $4n + 1$.

9. In an AP if the sum of 7 terms is 49, and the sum of 17 terms is 289, find the sum of n terms.

10. The sum of four integers in A.P. is 24, and their product is 945; find them.

11. If the sum of n terms of an A.P. is $2n + 3n^2$, find the r^{th} term.

12. If the sum of m terms of an A.P. is to the sum of n terms as m^2 to n^2, show that the m^{th} term is to the n^{th} term as $2m - 1$ is to $2n - 1$.

4. GEOMETRIC PROGRESSION (G.P.) :

G.P. is a sequence of non zero numbers each of the succeeding term is equal to the preceeding term multiplied by a constant. Thus in a GP the ratio of successive terms is constant. This constant factor is called the COMMON RATIO of the sequence & is obtained by dividing any term by the immediately previous term. Therefore a, ar, ar^2, ar^3, ar^4, ... is a GP with 'a' as the first term & 'r' as common ratio.

(a) n^{th} term ; $T_n = a\, r^{n-1}$

(b) Sum of the first n terms; $S_n = \dfrac{a(r^n - 1)}{r - 1}$, if $r \neq 1$

(c) Sum of infinite G.P. , $S_\infty = \dfrac{a}{1-r}$; $0 < |r| < 1$

5. PROPERTIES OF GP :

(a) If each term of a G.P. be multiplied or divided by the some non-zero quantity, then the resulting sequence is also a G.P.

(b) Three consecutive terms of a GP : a/r, a, ar ;

Four consecutive terms of a GP : a/r^3, a/r, ar, ar^3 & so on.

(c) If a, b, c are in G.P. then $b^2 = ac$.

(d) If in a G.P, the product of two terms which are equidistant from the first and the last term, is constant and is equal to the product of first and last term. $\Rightarrow T_k . T_{n-k+1}$ = constant = $a.\ell$

(e) If each term of a G.P. be raised to the same power, then resulting sequence is also a G.P.

(f) In a G.P., $T_r^2 = T_{r-k} . T_{r+k}$, $k < r$, $r \neq 1$

(g) If the terms of a given G.P. are chosen at regular intervals, then the new sequence is also a G.P.

(h) If $a_1, a_2, a_3.....a_n$ is a G.P. of positive terms, then $\log a_1, \log a_2,.....\log a_n$ is an A.P. and vice-versa.

(i) If $a_1, a_2, a_3.....$ and $b_1, b_2, b_3.....$ are two G.P.'s then $a_1b_1, a_2b_2, a_3b_3.....$ & $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}..........$ is also in G.P.

Illustrations

Illustration 6 : If a, b, c, d and p are distinct real numbers such that

$(a^2+b^2+c^2)p^2 - 2p(ab+bc+cd)+(b^2+c^2+d^2) \leq 0$ then a, b, c, d are in

(A) A.P. (B) G.P. (C) H.P. (D) None of these

Solution : Here, the given condition $(a^2+b^2+c^2)p^2 - 2p(ab+bc+ca)+b^2+c^2+d^2 \leq 0$

$\Rightarrow (ap-b)^2 + (bp-c)^2 + (cp-d)^2 \leq 0$

$\because$ a square can not be negative

$\therefore \quad ap-b=0, bp-c=0, cp-d=0 \Rightarrow p = \frac{b}{a} = \frac{c}{b} = \frac{d}{c} \Rightarrow$ a, b, c, d are in G.P.

Ans. (B)

Illustration 7 : If a, b, c are in G.P., then the equations $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root if $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in -

(A) A.P. (B) G.P. (C) H.P. (D) None of these

Solution : a, b, c are in G.P $\Rightarrow b^2=ac$

Now the equation $ax^2+2bx+c=0$ can be rewritten as $ax^2+2\sqrt{ac}x+c=0$

$$\Rightarrow\left(\sqrt{a}x+\sqrt{c}\right)^2=0\Rightarrow x=-\sqrt{\frac{c}{a}},-\sqrt{\frac{c}{a}}$$

If the two given equations have a common root, then this root must be $-\sqrt{\frac{c}{a}}$.

Thus $d\frac{c}{a}-2e\sqrt{\frac{c}{a}}+f=0 \;\Rightarrow\; \frac{d}{a}+\frac{f}{c}=\frac{2e}{c}\sqrt{\frac{c}{a}}=\frac{2e}{\sqrt{ac}}=\frac{2e}{b} \;\Rightarrow\; \frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in A.P.

Ans. (A)

Illustration 8 : A number consists of three digits which are in G.P. the sum of the right hand and left hand digits exceeds twice the middle digit by 1 and the sum of the left hand and middle digits is two third of the sum of the middle and right hand digits. Find the numbers.

Solution : Let the three digits be a, ar and ar^2 then number is

$100a + 10ar + ar^2$ ....(i)

Given, $a + ar^2 = 2ar + 1$

or $a(r^2 - 2r + 1) = 1$

or $a(r-1)^2 = 1$ ....(ii)

Also given $a + ar = \frac{2}{3}(ar + ar^2)$

$\Rightarrow 3 + 3r = 2r + 2r^2$ $\Rightarrow 2r^2 - r - 3 = 0$ $\Rightarrow (r+1)(2r-3) = 0$

$\therefore$ $r = -1, 3/2$

for $r=-1$, $a=\frac{1}{(r-1)^2}=\frac{1}{4}\notin I$ $\therefore r\neq -1$

for $r = 3/2$, $a=\frac{1}{\left(\frac{3}{2}-1\right)^2}=4$ {from (ii)}

From (i), number is $400 + 10.4.\frac{3}{2}+4.\frac{9}{4}=469$ **Ans.**

Illustration 9 : Find the value of $0.32\overline{58}$

Solution : Let $R = 0.32\overline{58} \Rightarrow R = 0.32585858....$ .... (i)

Here number of figures which are not recurring is 2 and number of figures which are recurring is also 2.

then $100\,R = 32.585858......$ ....(ii)

and $10000\,R = 3258.5858.....$ ....(iii)

Subtracting (ii) from (iii) , we get

$9900\,R = 3226 \Rightarrow R = \dfrac{1613}{4950}$

Aliter Method : $R = .32 + .0058 + .0058 + .000058 + ...$

$$= .32 + \frac{58}{10^4}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + ...\infty\right) = .32 + \frac{58}{10^4}\left(\frac{1}{1 - \frac{1}{100}}\right)$$

$$= \frac{32}{100} + \frac{58}{9900} = \frac{3168 + 58}{9900} = \frac{3226}{9900} = \frac{1613}{4950}$$

Do yourself - 3

1. Find a three digit number whose consecutive digits form a G.P. If we subtract 792 from this number, we get a number consisting of the same digits written in the reverse order. Now, if we increase the second digit of the required number by 2, then the resulting digits will form an A.P.

2. If the third term of G.P. is 4, then find the product of first five terms.

3. If a, b, c are respectively the p^{th}, q^{th} and r^{th} terms of the given G.P., then show that $(q - r)\log a + (r - p)\log b + (p - q)\log c = 0$, where $a, b, c > 0$.

4. Find three numbers in G.P., whose sum is 52 and the sum of whose products in pairs is 624.

5. The rational number which equals the number $2.\overline{357}$ with recurring decimal is -

(A) $\dfrac{2357}{999}$ (B) $\dfrac{2379}{997}$ (C) $\dfrac{785}{333}$ (D) $\dfrac{2355}{1001}$

Find the

6. sum $\frac{3}{4}, 1\frac{1}{2}, 3, ...$ to 8 terms.

7. sum 1,5,25,... to p terms.

8. sum $1, \sqrt{3}, 3, ...$ to 12 terms.

9. sum $\frac{8}{5}, -1, \frac{5}{8}, ...$ upto infinity

10. sum .45,.015,.0005,..upto infinity

11. sum $3, \sqrt{3}, 1, ...$ upto infinity

12. The sum of the first 6 terms of a G.P. is 9 times the sum of the first 3 terms; find the common ratio.

13. The fifth term of a G.P. is 81, and the second term is 24; find the series.

14. The sum of three numbers in G.P. is 38, and their product is 1728; find them.

15. The continued product of three numbers in G.P. is 216, and the sum of the product of them in pairs is 156; find the numbers.

16. The sum of an infinite number of terms of a G.P. is 4, and the sum of their cubes is 192; find the series.

17. The first two terms of an infinite G.P. are together equal to 5, and every term is 3 times the sum of all the terms that follow it; find the seires.

18. Sum of following series : $x + a$, $x^2 + 2a$, $x^3 + 3a$... to n terms.

6. HARMONIC PROGRESSION (H.P.) :

A sequence is said to be in H.P. if the reciprocal of its terms are in AP.

If the sequence a_1, a_2, a_3, ..., a_n is an HP then $1/a_1$, $1/a_2$,........., $1/a_n$ is an AP . Here we do not have the formula for the sum of the n terms of an HP. The general form of a harmonic progression is $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, ... \frac{1}{a+(n-1)d}$

Note : No term of any H.P. can be zero.

(i) If a, b, c are in HP, then $b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$

Illustrations

Illustration 10 : The sum of three numbers are in H.P. is 37 and the sum of their reciprocals is 1/4. Find the numbers.

Solution : Three numbers are in H.P. can be taken as

$$\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$$

then $\frac{1}{a-d} + \frac{1}{a} + \frac{1}{a+d} = 37$ ....(i)

and $a - d + a + a + d = \frac{1}{4} \quad \Rightarrow \quad a = \frac{1}{12}$

from (i), $\frac{12}{1-12d} + 12 + \frac{12}{1+12d} = 37 \quad \Rightarrow \quad \frac{12}{1-12d} + \frac{12}{1+12d} = 25$

$\Rightarrow \quad \frac{24}{1-144d^2} = 25 \quad \Rightarrow \quad 1 - 144d^2 = \frac{24}{25} \quad \Rightarrow \quad d^2 = \frac{1}{25 \times 144}$

$\therefore \quad d = \pm\frac{1}{60}$

$\therefore \quad a - d, a, a + d$ are $\frac{1}{15}, \frac{1}{12}, \frac{1}{10}$ or $\frac{1}{10}, \frac{1}{12}, \frac{1}{15}$

Hence, three numbers in H.P. are 15, 12, 10 or 10, 12, 15 **Ans.**

Illustration 11 : Suppose a is a fixed real number such that $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz}$

If p, q, r are in A.P., then prove that x, y, z are in H.P.

Solution : $\because$ p, q, r are in A.P.

$\therefore \quad q - p = r - q$ .... (i)

$\Rightarrow \quad p - q = q - r = k$ (let)

given $\frac{a-x}{px} = \frac{a-y}{qy} = \frac{a-z}{rz} \quad \Rightarrow \quad \frac{\frac{a}{x}-1}{p} = \frac{\frac{a}{y}-1}{q} = \frac{\frac{a}{z}-1}{r}$

$\Rightarrow \quad \frac{\left(\frac{a}{x}-1\right)-\left(\frac{a}{y}-1\right)}{p-q} = \frac{\left(\frac{a}{y}-1\right)-\left(\frac{a}{z}-1\right)}{q-r}$ (by law of proportion)

$\Rightarrow \quad \frac{\frac{a}{x}-\frac{a}{y}}{k} = \frac{\frac{a}{y}-\frac{a}{z}}{k}$ {from (i)}

$$\Rightarrow \quad a\left(\frac{1}{x}-\frac{1}{y}\right)=a\left(\frac{1}{y}-\frac{1}{z}\right) \Rightarrow \frac{1}{x}-\frac{1}{y}=\frac{1}{y}-\frac{1}{z}$$

$$\therefore \quad \frac{2}{y}=\frac{1}{x}+\frac{1}{z}$$

$$\therefore \quad \frac{1}{x},\frac{1}{y},\frac{1}{z} \text{ are in A.P.}$$

Hence x, y, z are in H.P.

Do yourself - 4

1. If the 7th term of a H.P. is 8 and the 8th term is 7. Then find the 28th term.

2. In a H.P., if 5th term is 6 and 3rd term is 10. Find the 2nd term.

3. If the pth, qth and rth terms of a H.P. are a,b,c respectively, then prove that $\frac{q-r}{a}+\frac{r-p}{b}+\frac{p-q}{c}=0$.

4. Find the fourth term in each of the following series :

(a) $2,2\frac{1}{2},3\frac{1}{3},\ldots$ (b) $1\frac{1}{2},2,3\ldots$ (c) $3\frac{3}{5},6,18,\ldots$

5. If a,b,c be in H.P., show that a : a – b = a + c : a – c.

6. If the mth term of a H.P. be equal to n, and the nth term be equal to m, prove that the (m + n)th term is equal to $\frac{mn}{m+n}$.

7. MEANS

(a) ARITHMETIC MEAN :

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c. So A.M. of a and c = $\frac{a+c}{2}=b$.

n-ARITHMETIC MEANS BETWEEN TWO NUMBERS :

If a,b be any two given numbers & a, A_1, A_2, ..., A_n, b are in AP, then A_1, A_2,........A_n are the 'n' A.M's between a & b then. A_1 = a + d , A_2 = a + 2d ,......, A_n= a + nd or b – d,

where $d=\frac{b-a}{n+1}$

$$\Rightarrow A_1=a+\frac{b-a}{n+1},\quad A_2=a+\frac{2(b-a)}{n+1},\ldots\ldots.$$

Note: Sum of n A.M's inserted between a & b is equal to n times the single A.M. between a & b

i.e. $\sum_{r=1}^{n} A_r = nA$ where A is the single A.M. between a & b.

(b) GEOMETRIC MEAN :

If a, b, c are in G.P., then b is the G.M. between a & c, $b^2 = ac$. So G.M. of a and c $= \sqrt{ac} = b$

n-GEOMETRIC MEANS BETWEEN TWO NUMBERS :

If a, b are two given positive numbers & a, G_1, G_2, ..., G_n, b are in G.P. Then G_1, G_2, G_3 ,.......G_n are 'n' G.Ms between a & b. where $b = ar^{n+1} \Rightarrow r = (b/a)^{1/n+1}$

$$G_1 = a(b/a)^{1/n+1}, \qquad G_2 = a(b/a)^{2/n+1}\ldots\ldots\ldots\ldots, \qquad G_n = a(b/a)^{n/n+1}$$

$$= ar, \qquad = ar^2, \quad \ldots\ldots\ldots\ldots \qquad = ar^n = b/r$$

Note : The product of n G.Ms between a & b is equal to n^{th} power of the single G.M. between a & b i.e. $\prod_{r=1}^{n} G_r = (G)^n$ where G is the single G.M. between a & b

(c) HARMONIC MEAN :

If a, b, c are in H.P., then b is H.M. between a & c. So H.M. of a and c = $\dfrac{2ac}{a+c}$ = b.

Insertion of 'n' HM's between a and b :

a, H_1, H_2, H_3 ,........, H_n, b $\rightarrow$ H.P

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots\ldots\ldots\ldots \frac{1}{H_n}, \frac{1}{b} \rightarrow \text{A.P.}$$

$$\frac{1}{b} = \frac{1}{a} + (n+1)D \quad \Rightarrow \quad D = \frac{\frac{1}{b} - \frac{1}{a}}{n+1}$$

$$\frac{1}{H_n} = \frac{1}{a} + n\left(\frac{\frac{1}{b} - \frac{1}{a}}{n+1}\right)$$

Important note :

(i) If A, G, H, are respectively A.M., G.M., H.M. between two positive number a & b then

(a) $G^2 = AH$ (A, G, H constitute a GP) (b) $A \geq G \geq H$ (c) $A = G = H \Leftrightarrow a = b$

(ii) Let $a_1, a_2, \ldots\ldots, a_n$ be n positive real numbers, then we define their arithmetic mean (A), geometric mean (G) and harmonic mean (H) as $A = \dfrac{a_1 + a_2 + \ldots.. + a_n}{n}$

$G = (a_1 a_2 \ldots\ldots\ldots. a_n)^{1/n}$ and $H = \dfrac{n}{\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \ldots. \dfrac{1}{a_n}\right)}$

It can be shown that $A \geq G \geq H$. Moreover equality holds at either place if and only if $a_1 = a_2 = \ldots\ldots = a_n$

Illustrations

Illustration 12 : If $2x^3 + ax^2 + bx + 4 = 0$ (a and b are positive real numbers) has 3 real roots, then prove that $a + b \geq 6(2^{1/3} + 4^{1/3})$.

Solution : Let α, β, γ be the roots of $2x^3 + ax^2 + bx + 4 = 0$. Given that all the coefficients are positive, so all the roots will be negative.

Let $\alpha_1 = -\alpha,\ \alpha_2 = -\beta,\ \alpha_3 = -\gamma \quad \Rightarrow \quad \alpha_1 + \alpha_2 + \alpha_3 = \dfrac{a}{2}$

$\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = \dfrac{b}{2}$

$\alpha_1\alpha_2\alpha_3 = 2$

Applying AM ≥ GM, we have

$$\frac{\alpha_1 + \alpha_2 + \alpha_3}{3} \geq (\alpha_1\alpha_2\alpha_3)^{1/3} \quad \Rightarrow \quad a \geq 6 \times 2^{1/3}$$

Also $\dfrac{\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_1\alpha_3}{3} > (\alpha_1\alpha_2\alpha_3)^{2/3} \Rightarrow \quad b \geq 6 \times 4^{1/3}$

Therefore $a + b \geq 6(2^{1/3} + 4^{1/3})$.

Illustration 13 : If $a_i > 0\ \forall\ i \in N$ such that $\prod_{i=1}^{n} a_i = 1$,

then prove that $(1 + a_1)(1 + a_2)(1 + a_3).....(1 + a_n) \geq 2^n$

Solution : Using A.M. ≥ G.M.

$1 + a_1 \geq 2\sqrt{a_1}$

$1 + a_2 \geq 2\sqrt{a_2}$

$\vdots$

$1 + a_n \geq 2\sqrt{a_n} \Rightarrow (1 + a_1)(1 + a_2)........(1 + a_n) \geq 2^n(a_1a_2a_3.....a_n)^{1/2}$

As $a_1a_2a_3.....a_n = 1$

Hence $(1 + a_1)(1 + a_2)..........(1 + a_n) \geq 2^n$.

Illustration 14 : If a, b, x, y are positive natural numbers such that $\frac{1}{x} + \frac{1}{y} = 1$ then prove that

$\frac{a^x}{x} + \frac{b^y}{y} \geq ab$.

Solution : Consider the positive numbers a^x, a^x,.......y times and b^y, b^y,......x times

For all these numbers,

$$AM = \frac{\{a^x + a^x + ...y\ \text{time}\} + \{b^y + b^y + ...x\ \text{times}\}}{x + y} = \frac{ya^x + xa^y}{(x + y)}$$

$$GM = \left\{\left(a^x.a^x......y\ \text{times}\right)\left(b^y.b^y......x\ \text{times}\right)\right\}^{\frac{1}{(x+y)}} = \left[\left(a^{xy}\right).\left(b^{xy}\right)\right]^{\frac{1}{(x+y)}} = (ab)^{\frac{xy}{(x+y)}}$$

As $\frac{1}{x} + \frac{1}{y} = 1$, $\frac{x + y}{xy} = 1$, i.e, $x + y = xy$

So using AM ≥ GM $\frac{ya^x + xa^y}{x + y} \geq (ab)^{\frac{xy}{x+y}}$

$\therefore \quad \frac{ya^x + xa^y}{xy} \geq ab$ or $\frac{a^x}{x} + \frac{a^y}{y} \geq ab.$

Do yourself - 5

1. If $\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}}$ is the G.M. between a & b then find the value of 'n'.

2. If b is the harmonic mean between a and c, then prove that $\dfrac{1}{b-a} + \dfrac{1}{b-c} = \dfrac{1}{a} + \dfrac{1}{c}$.

3. Insert 19 arithmetic means between $\dfrac{1}{4}$ and $-9\dfrac{3}{4}$.

4. Insert 17 arithmetic means between $3\dfrac{1}{2}$ and $-41\dfrac{1}{2}$

5. Insert 3 geometric means between $2\dfrac{1}{4}$ and $\dfrac{4}{9}$.

6. Insert 5 geometric means between $3\dfrac{5}{9}$ and $40\dfrac{1}{2}$.

7. Insert two harmonic means between 5 and 11.

8. Insert four harmonic means between $\dfrac{2}{3}$ and $\dfrac{2}{13}$.

9. If between any two quantities there be inserted two arithmetic means A_1, A_2; two geometric means G_1, G_2 ; and two harmonic means H_1, H_2;
show that $G_1G_2 : H_1H_2 = A_1 + A_2 : H_1 + H_2$.

8. ARITHMETICO - GEOMETRIC SERIES :

A series, each term of which is formed by multiplying the corresponding term of an A.P. & G.P. is called the Arithmetico-Geometric Series , e.g. $1 + 3x + 5x^2 + 7x^3 + ...$

Here 1, 3, 5, ... are in A.P. & 1, x, x^2, x^3 .... are in G.P.

(a) SUM OF N TERMS OF AN ARITHMETICO-GEOMETRIC SERIES :

Let $S_n = a + (a+d)r + (a+2d)r^2 + ... + [a+(n-1)d]r^{n-1}$

then $S_n = \dfrac{a}{1-r} + \dfrac{dr(1-r^{n-1})}{(1-r)^2} - \dfrac{[a+(n-1)d]\ r^n}{1-r}$, $r \neq 1$

(b) SUM TO INFINITY :

If $0<|r|<1$ & $n\to\infty$, then $\lim\limits_{n\to\infty} r^n = 0,\ S_\infty = \dfrac{a}{1-r}+\dfrac{dr}{(1-r)^2}$

Illustrations

Illustration 15 : Find the sum of series $4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \ldots\ldots\ldots \infty$.

Solution : Let $S = 4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \ldots\ldots\ldots \infty$

$-Sx = -4x + 9x^2 - 16x^3 + 25x^4 - 36x^5 + \ldots\ldots\ldots \infty$

On subtraction, we get

$S(1 + x) = 4 - 5x + 7x^2 - 9x^3 + 11x^4 - 13x^5 + \ldots\ldots \infty$

$-S(1 + x)x = -4x + 5x^2 - 7x^3 + 9x^4 - 11x^5 + \ldots\ldots \infty$

On subtraction, we get

$S(1 + x)^2 = 4 - x + 2x^2 - 2x^3 + 2x^4 - 2x^5 + \ldots\ldots \infty$

$= 4 - x + 2x^2 (1 - x + x^2 - \ldots\ldots\ldots\infty) = 4 - x + \dfrac{2x^2}{1+x} = \dfrac{4+3x+x^2}{1+x}$

$S = \dfrac{4+3x+x^2}{(1+x)^3}$ **Ans.**

Illustration 16 : Find the sum of series upto n terms $\left(\dfrac{2n+1}{2n-1}\right)+3\left(\dfrac{2n+1}{2n-1}\right)^2+5\left(\dfrac{2n+1}{2n-1}\right)^3+\ldots\ldots\ldots$.

Solution : For $x \neq 1$, let

$S = x + 3x^2 + 5x^3 + \ldots\ldots + (2n - 3)x^{n-1} + (2n - 1)x^n$ .... (i)

$\Rightarrow$ $xS = x^2 + 3x^3 + \ldots\ldots + (2n - 5)x^{n-1} + (2n - 3)x^n + (2n - 1)x^{n+1}$ .... (ii)

Subtracting (ii) from (i), we get

$(1 - x)S = x + 2x^2 + 2x^3 + \ldots\ldots\ldots + 2x^{n-1} + 2x^n - (2n - 1)x^{n+1}$

$$= x + \frac{2x^2(1-x^{n-1})}{1-x} - (2n-1)x^{n+1}$$

$$= \frac{x}{1-x}[1 - x + 2x - 2x^n - (2n-1)x^n + (2n-1)x^{n+1}]$$

$$\Rightarrow \quad S = \frac{x}{(1-x)^2}[(2n-1)x^{n+1} - (2n+1)x^n + 1 + x]$$

Thus $\left(\frac{2n+1}{2n-1}\right) + 3\left(\frac{2n+1}{2n-1}\right)^2 + \ldots\ldots\ldots + (2n-1)\left(\frac{2n+1}{2n-1}\right)^n$

$$= \left(\frac{2n+1}{2n-1}\right)\left(\frac{2n-1}{2}\right)^2\left[(2n-1)\left(\frac{2n+1}{2n-1}\right)^{n+1} - (2n+1)\left(\frac{2n+1}{2n-1}\right)^n + 1 + \frac{2n+1}{2n-1}\right]$$

$$= \frac{4n^2-1}{4} \cdot \frac{4n}{2n-1} = n(2n+1)$$ **Ans.**

Do yourself - 6

1. Find sum to n terms of the series $3 + 5 \times \frac{1}{4} + 7 \times \frac{1}{4^2} + \ldots\ldots\ldots$

Find

2. sum $1 + 2a + 3a^2 + 4a^3 + \ldots$ to n terms.

3. sum $1 + \frac{3}{4} + \frac{7}{16} + \frac{15}{64} + \frac{31}{256} + \ldots$ to infinity.

4. sum $\frac{2}{3} + \frac{3}{3^2} + \frac{2}{3^3} + \frac{3}{3^4} + \frac{2}{3^5} + \frac{3}{3^6} + \ldots$ to infinity.

5. Find the sum of n terms of the series the r^{th} term of which is $(2r + 1)\,2^r$.

9. SIGMA NOTATIONS (Σ)

THEOREMS :

(a) $\sum_{r=1}^{n}(a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$ **(b)** $\sum_{r=1}^{n} k\,a_r = k\sum_{r=1}^{n} a_r$ **(c)** $\sum_{r=1}^{n} k = nk$; where k is a constant.

10. RESULTS

(a) $\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$ (sum of the first n natural numbers)

(b) $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ (sum of the squares of the first n natural numbers)

(c) $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4} = \left[\sum_{r=1}^{n} r\right]^2$ (sum of the cubes of the first n natural numbers)

(d) $\sum_{r=1}^{n} r^4 = \frac{n}{30}(n+1)(2n+1)(3n^2+3n-1)$

(e) $\sum_{r=1}^{n}(2r-1) = n^2$ (sum of first n odd natural numbers)

(f) $\sum_{r=1}^{n} 2r = n(n+1)$ (sum of first n even natural numbers)

Note :

If n^{th} term of a sequence is given by $T_n = an^3 + bn^2 + cn + d$ where a, b, c, d are constants,

then sum of n terms $S_n = \Sigma T_n = a\Sigma n^3 + b\Sigma n^2 + c\Sigma n + \Sigma d$

This can be evaluated using the above results.

Illustrations

Illustration 17 : Find the sum of the series to n terms whose n^{th} terms is 3n + 2.

Solution : $S_n = \Sigma T_n = \Sigma(3n + 2) = 3\Sigma n + \Sigma 2 = \frac{3(n+1)n}{2} + 2n = \frac{n}{2}(3n+7)$

Illustration 18 : If $T_k = k^3 + 3^k$, then find $\sum_{k=1}^{n} T_k$.

Solution : $$\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 3^k = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3(3^n-1)}{3-1} = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3}{2}(3^n-1)$$

Illustration 19 : Find the value of the expression $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$

Solution : $$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1 = \sum_{i=1}^{n}\sum_{j=1}^{i} j$$

$$= \sum_{i=1}^{n} \frac{i(i+1)}{2} = \frac{1}{2}\left[\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i\right] = \frac{1}{2}\left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{12}[2n+1+3] = \frac{n(n+1)(n+2)}{6}.$$

Illustration 20 : Sum up to 16 terms of the series $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} + \ldots\ldots$ is

(A) 450 (B) 456 (C) 446 (D) none of these

Solution : $$t_n = \frac{1^3+2^3+3^3+\ldots.+n^3}{1+3+5+\ldots.(2n-1)} = \frac{\left\{\frac{n(n+1)}{2}\right\}^2}{\frac{n}{2}\{2+2(n-1)\}} = \frac{\frac{n^2(n+1)^2}{4}}{n^2} = \frac{(n+1)^2}{4} == \frac{n^2}{4} + \frac{n}{2} + \frac{1}{4}$$

$$\therefore \quad S_n = \Sigma t_n = \frac{1}{4}\Sigma n^2 + \frac{1}{2}\Sigma n + \frac{1}{4}\Sigma 1 \quad = \frac{1}{4}.\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}.\frac{n(n+1)}{2} + \frac{1}{4}.n$$

$$\therefore \quad S_{16} = \frac{16.17.33}{24} + \frac{16.17}{4} + \frac{16}{4} = 446$$

Ans. (C)

11. METHOD OF DIFFERENCE :

Some times the n^{th} term of a sequence or a series can not be determined by the method, we have discussed earlier. So we compute the difference between the successive terms of given sequence for obtained the n^{th} terms.

If $T_1, T_2, T_3,........,T_n$ are the terms of a sequence then some times the terms $T_2 - T_1, T_3 - T_2,.........$ constitute an AP/GP. n^{th} term of the series is determined & the sum to n terms of the sequence can easily be obtained.

Case 1 :

(a) If difference series are in A.P., then

Let $T_n = an^2 + bn + c$, where a, b, c are constant

(b) If difference of difference series are in A.P.

Let $T_n = an^3 + bn^2 + cn + d$, where a, b, c, d are constant

Case 2 :

(a) If difference are in G.P., then

Let $T_n = ar^n + b$, where r is common ratio & a, b are constant

(b) If difference of difference are in G.P., then

Let $T_n = ar^n + bn + c$, where r is common ratio & a, b, c are constant

Determine constant by putting n = 1, 2, 3 .... n and putting the value of T_1, T_2, T_3 ......

and sum of series $(S_n) = \sum T_n$

Illustrations

Illustration 21 : Find the sum of n terms of the series 3 + 7 + 14 + 24 + 37 + ...

Solution : Clearly here the differences between the successive terms are

7 – 3, 14 – 7, 24 – 14, ... i.e. 4, 7, 10, 13,........., which are in A.P.

Let $S = 3 + 7 + 14 + 24 + ... + T_n$

$S = \quad 3 + 7 + 14 + ... + T_{n-1} + T_n$

Subtracting, we get

$0 = 3 + [4 + 7 + 10 + 13 + \ldots\ldots (n-1) \text{ terms}] - T_n$

$\therefore \quad T_n = 3 + S_{n-1}$ of an A.P. whose a = 4 and d = 3.

$$\therefore \quad T_n = 3 + \left(\frac{n-1}{2}\right)(2.4+(n-2)3) = \frac{6+(n-1)(3n+2)}{4} \quad \text{or,} \quad T_n = \frac{1}{2}(3n^2-n+4)$$

Now putting n = 1, 2, 3,........, n and adding

$$\therefore \quad S_n = \frac{1}{2}[3\Sigma n^2 - \Sigma n + 4n] = \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right] = \frac{n}{2}(n^2+n+4)$$

Ans.

Aliter Method :

Let $T_n = an^2 + bn + c$

Now, $T_1 = 3 = a + b + c$ .....(i)

$T_2 = 7 = 4a + 2b + c$ .....(ii)

$T_3 = 14 = 8a + 3b + c$ .....(iii)

Solving (i), (ii) & (iii) we get

$$a = \frac{3}{2}, b = -\frac{1}{2} \ \& \ c = 2 \qquad \therefore \qquad T_n = \frac{1}{2}(3n^2-n+4)$$

$$\Rightarrow s_n = \Sigma T_n = \frac{1}{2}[3\Sigma n^2 - \Sigma n + 4n] = \frac{1}{2}\left[3\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right] = \frac{n}{2}(n^2+n+4)$$

Ans.

Illustration 22 : Find the sum of n-terms of the series 1 + 4 + 10 + 22 + ...

Solution : Let $S = 1 + 4 + 10 + 22 + \ldots\ldots + T_n$ .... (i)

$S = \quad 1 + 4 + 10 + \ldots\ldots + T_{n-1} + T_n$ .... (ii)

(i) – (ii) $\Rightarrow T_n = 1 + (3 + 6 + 12 + \ldots\ldots + T_n - T_{n-1})$

$$T_n = 1 + 3\left(\frac{2^{n-1}-1}{2-1}\right)$$

$T_n = 3 \cdot 2^{n-1} - 2$

So $\quad S_n = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2$

$$= 3\left(\frac{2^n-1}{2-1}\right) - 2n = 3.2^n - 2n - 3$$ **Ans.**

Aliter Method :

Let $T_n = ar^n + b$, where r = 2

Now $T_1 = 1 = ar + b$ ....(i)

$T_2 = 4 = ar^2 + b$ ....(ii)

Solving (i) & (ii), we get

$$a = \frac{3}{2}, b = -2$$

$\therefore$ $T_n = 3.2^{n-1} - 2$

$\Rightarrow$ $S_n = \Sigma T_n = 3\Sigma 2^{n-1} - \Sigma 2$

$$= 3\left(\frac{2^n - 1}{2-1}\right) - 2n = 3.2^n - 2n - 3$$ **Ans.**

Illustration 23 : Find the general term and sum of n terms of the series

1 + 5 + 19 + 49 + 101 + 181 + 295 + ...

Solution : The sequence of difference between successive term 4, 14, 30, 52, 80 ......

The sequence of the second order difference is 10, 16, 22, 28, ... clearly it is an A.P.

So, let n^{th} term

$T_n = an^3 + bn^2 + cn + d$

$a + b + c + d = 1$ .... (i)

$8a + 4b + 2c + d = 5$ .... (ii)

$27a + 9b + 3c + d = 19$ .... (iii)

$64a + 16b + 4c + d = 49$ .... (iv)

from (i), (ii), (iii) & (iv)

$a = 1, b = -1, c = 0, d = 1 \Rightarrow T_n = n^3 - n^2 + 1$

$$\therefore S_n = \sum(n^3 - n^2 + 1) = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6} + n = \frac{n(n^2-1)(3n+2)}{12} + n$$

Do yourself - 7

1. Find the sum of the series upto n terms $1+\frac{1+2}{2}+\frac{1+2+3}{3}+\frac{1+2+3+4}{4}+$ ....

2. Find the sum of 'n' terms of the series whose n^{th} term is

(a) $3n^2 + 2n$. (b) $3n^2 - n$. (c) $n^3+\frac{3}{2}n$ (d) $n(n + 2)$

(e) $n^2(2n + 3)$ (f) $3^n - 2^n$ (g) $3(4^n + 2n^2) - 4n^3$

3. Sum the following series to n terms :

(a) 1 . 4 . 7 + 2 . 5 . 8 + 3 . 6 . 9 + ...

(b) 1 . 5 . 9 + 2 . 6 . 10 + 3 . 7 . 11 + ...

4. Find the n^{th} term and the sum of n terms of the series :

(a) 4, 14, 30, 52, 80, 114, ...

(b) 9, 16, 29, 54, 103, ...

Illustrations

Miscellaneous Illustration :

Illustration 24 : If $\sum_{r=1}^{n} T_r = \frac{n}{8}(n+1)(n+2)(n+3)$, then find $\sum_{r=1}^{n}\frac{1}{T_r}$.

Solution : $\because \quad T_n = S_n - S_{n-1}$

$$=\sum_{r=1}^{n}T_r-\sum_{r=1}^{n-1}T_r=\frac{n(n+1)(n+2)(n+3)}{8}-\frac{(n-1)n(n+1)(n+2)}{8}$$

$$=\frac{n(n+1)(n+2)}{8}[(n+3)-(n-1)]$$

$$T_n=\frac{n(n+1)(n+2)}{8}(4)=\frac{n(n+1)(n+2)}{2}$$

$$\Rightarrow \quad \frac{1}{T_n}=\frac{2}{n(n+1)(n+2)}=\frac{(n+2)-n}{n(n+1)(n+2)}=\frac{1}{n(n+1)}-\frac{1}{(n+1)(n+2)} \quad \text{........ (i)}$$

Let $V_n=\frac{1}{n(n+1)}$

$$\therefore \quad \frac{1}{T_n}=V_n-V_{n+1}$$

Putting $n = 1, 2, 3, ... n$

$$\Rightarrow \quad \frac{1}{T_1}+\frac{1}{T_2}+\frac{1}{T_3}+.......+\frac{1}{T_n}=(V_1-V_{n+1}) \Rightarrow \sum_{r=1}^{n}\frac{1}{T_r}=\frac{n^2+3n}{2(n+1)(n+2)}$$

Illustration 25 : Find the sum of n terms of the series $1 \cdot 3 \cdot 5 + 3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + \ldots\ldots.$

Solution : The n^{th} term is $(2n - 1)(2n + 1)(2n + 3)$

$T_n = (2n - 1)(2n + 1)(2n + 3)$

$T_n = \frac{1}{8}(2n-1)(2n + 1)(2n + 3)\{(2n + 5) - (2n - 3)\}$

$= \frac{1}{8}(V_n - V_{n-1})$ [Let $V_n = (2n - 1)(2n + 1)(2n + 3)(2n + 5)$]

$S_n = \sum T_n = \frac{1}{8}[V_n - V_0]$

$\therefore \quad S_n = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{8} + \frac{15}{8} = n(2n^3 + 8n^2 + 7n - 2)$ **Ans.**

Illustration 26 : The series of natural numbers is divided into groups (1), (2, 3, 4), (5, 6, 7, 8, 9) .... and so on. Show that the sum of the numbers in n^{th} group is $n^3 + (n - 1)^3$

Solution : The groups are (1), (2, 3, 4), (5, 6, 7, 8, 9) ....

The number of terms in the groups are 1, 3, 5......

$\therefore$ The number of terms in the n^{th} group $= (2n - 1)$

the last term of the n^{th} group is n^2

If we count from last term common difference should be -1

So the sum of numbers in the n^{th} group $= \left(\frac{2n-1}{2}\right)\left\{2n^2 + (2n-2)(-1)\right\}$

$= (2n - 1)(n^2 - n + 1) = 2n^3 - 3n^2 + 3n - 1 = n^3 + (n - 1)^3$

Illustration 27 : Find the natural number 'a' for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfied $f(x+y) = f(x).\, f(y)$ for all natural number x,y and further $f(1) = 2$.

Solution : It is given that

$f(x+y) = f(x)\, f(y)$ and $f(1) = 2$

$f(1+1) = f(1)\, f(1) \Rightarrow f(2) = 2^2$, $f(1+2) = f(1)\, f(2) \Rightarrow f(3) = 2^3$, $f(2 + 2) = f(2)\, f(2) \Rightarrow f(4) = 2^4$

Similarly $f(k) = 2^k$ and $f(a) = 2^a$

Hence , $\sum_{k=1}^{n} f(a+k) = \sum_{k=1}^{n} f(a)f(k) = f(a)\sum_{k=1}^{n} f(k) = 2^a \sum_{k=1}^{n} 2^k = 2^a\{2^1 + 2^2 + \ldots\ldots\ldots + 2^n\}$

$= 2^a\left\{\frac{2(2^n - 1)}{2-1}\right\} = 2^{a+1}(2^n - 1)$

But $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$

$2^{a+1}(2^n - 1) = 16(2^n - 1)$

$\therefore \quad 2^{a+1} = 2^4$

$\therefore \quad a + 1 = 4 \quad \Rightarrow \quad a = 3$ **Ans.**

ANSWER KEY

Do yourself-1

1. **(a)** $\frac{2}{1},\frac{4}{2},\frac{8}{3},\frac{16}{4},.......,$ **(b)** $\frac{2}{3},\frac{4}{9},\frac{2}{27},\frac{4}{81},.......;$ **3.** $-(p+q)$

4. 153 **5.** 0 **6.** –185 **7.** 820a – 1680b

Do yourself-2

1. 900 **2.** 14 **3.** 3 **4.** 5 **5.** 612 **6.** 4,9,14

7. 1,4,7 **8.** 495 **9.** n^2 **10.** 3,5,7,9 **11.** 6r – 1

Do yourself-3

1. 931 **2.** 4^5 **3.** 4, 12, 36 **4.** C **5.** 612 **6.** $191\frac{1}{4}$

7. $\frac{1}{4}\left(5^p-1\right)$ **8.** $364\left(\sqrt{3}+1\right)$ **9.** $\frac{64}{65}$ **10.** $\frac{27}{58}$ **11.** $\frac{3\left(3+\sqrt{3}\right)}{2}$

12. 2 **13.** 16,24,36,... **14.** 8,12,18 **15.** 2,6,18 **16.** $6,-3,1\frac{1}{2},...$

17. $4,1,\frac{1}{4}...$ **18.** $\frac{x\left(x^n-1\right)}{x-1}+\frac{\left(n\left(n+1\right)\right)a}{2}$

Do yourself-4

1. 2 **2.** 15 **4.** **(a)** 5 **(b)** 6 **(c)** –18

Do yourself-5

1. $\frac{1}{2}$ **3.** $-\frac{1}{4},-\frac{3}{4},...,-9\frac{1}{4}$ **4.** $1,-1\frac{1}{2},...,-39.$ **5.** $\frac{3}{2},1,\frac{2}{3}$

6. $\frac{16}{3},8,...,27$ **7.** $6\frac{1}{9}.7\frac{6}{7}$ **8.** $\frac{2}{5},\frac{2}{7},\frac{2}{9},\frac{2}{11}$

Do yourself-6

1. $4+\frac{8}{9}\left(1-\frac{1}{4^{n-1}}\right)-\left(\frac{2n+1}{3\times 4^{n-1}}\right)$

2. $\frac{1-a^n}{(1-a)^2}-\frac{na^n}{1-a}$

3. $\frac{8}{3}$

4. $\frac{9}{8}$

5. $n.2^{n+2}-2^{n+1}+2$

Do yourself-7

1. $\frac{n(n+3)}{4}$

2. **(a)** $\frac{n(n+1)(2n+3)}{2}$

(b) $n^2(n+1)$

(c) $\frac{1}{4}n(n+1)(n^2+n+3)$

(d) $\frac{1}{6}n(n+1)(2n+7)$

(e) $\frac{1}{2}n(n+1)(n^2+3n+1)$

(f) $\frac{1}{2}(3^{n+1}+1)-2^{n+1}$

(g) $4^{n+1}-4-n(n+1)(n^2-n-1)$

3. **(a)** $\frac{n}{4}(n+1)(n+6)(n+7)$

(b) $\frac{n}{4}(n+1)(n+8)(n+9)$

4. **(a)** $3n^2+n;\ n(n+1)^2$

(b) $3 \, . \, 2^n+n+2;\ 6(2^n-1)+\frac{n(n+5)}{2}$

EXERCISE (O-1)

SINGLE CORRECT ONLY

1. If $a_1, a_2, a_3, \ldots, a_n, \ldots$ are in A.P. such that $a_4 - a_7 + a_{10} = m$, then the sum of first 13 terms of this A.P., is :

(A) 15 m (B) 10 m (C) 12 m (D) 13 m

2. If a, b, c are in AP, then $(a - c)^2$ equals

(A) $4(b^2 - ac)$ (B) $4(b^2 + ac)$ (C) $4b^2 - ac$ (D) $b^2 - 4ac$

3. Let $a_1, a_2, a_3, \ldots$ be an A.P. such that $\frac{a_1 + a_2 + \ldots + a_p}{a_1 + a_2 + a_3 + \ldots + a_q} = \frac{p^2}{q^2}$; $p \neq q$. Then $\frac{a_6}{a_{21}}$ is equal to :

(A) $\frac{121}{1861}$ (B) $\frac{11}{41}$ (C) $\frac{121}{1681}$ (D) $\frac{41}{11}$

4. Given sum of the first n terms of an A. P. is $2n + 3n^2$. Another A. P. is formed with the same first term and double of the common difference, the sum of n terms of the new A. P. is :-

(A) $n + 4n^2$ (B) $n^2 + 4n$ (C) $3n + 2n^2$ (D) $6n^2 - n$

5. If the sum of n terms of an AP is $Pn + Qn^2$, where P, Q are constants, then its common difference is

(A) 2Q (B) P + Q (C) 2P (D) P – Q

6. The first term of an infinite G.P. is 1 and every term is equals to the sum of the successive terms, then its fourth term will be-

(A) $\frac{1}{2}$ (B) $\frac{1}{8}$ (C) $\frac{1}{4}$ (D) $\frac{1}{16}$

7. If $a = \sum_{n=0}^{\infty} x^n, b = \sum_{n=0}^{\infty} y^n, c = \sum_{n=0}^{\infty} (xy)^n$ where $|x|, |y| < 1$; then-

(A) abc = a + b + c (B) ab + bc = ac + b (C) ac + bc = ab + c (D) ab + ac = bc + a

8. If r > 1 and $x = a + \frac{a}{r} + \frac{a}{r^2} + ...\text{to } \infty$, $y = b - \frac{b}{r} + \frac{b}{r^2} - ...\text{to } \infty$ and $z = c + \frac{c}{r^2} + \frac{c}{r^4} + ...\text{to } \infty$, then $\frac{xy}{z} =$

(A) $\frac{ab}{c}$ (B) $\frac{ac}{b}$ (C) $\frac{bc}{a}$ (D) None of these

9. In a GP, first term is 1. If $4T_2 + 5T_3$ is minimum, then its common ratio is

(A) $\frac{2}{5}$ (B) $-\frac{2}{5}$ (C) $\frac{3}{5}$ (D) $-\frac{3}{5}$

10. Given a sequence of 4 numbers, first three of which are in G.P. and the last three are in A.P. with common difference six. If first and last terms of this sequence are equal, then the last term is :

(A) 8 (B) 16 (C) 2 (D) 4

11. If G be the GM between x and y, then the value of $\frac{1}{G^2 - x^2} + \frac{1}{G^2 - y^2}$ is equal to

(A) G^2 (B) $\frac{2}{G^2}$ (C) $\frac{1}{G^2}$ (D) $3G^2$

12. If a, b, c are in HP, then $\frac{a-b}{b-c}$ is equal to

(A) $\frac{a}{b}$ (B) $\frac{b}{a}$ (C) $\frac{a}{c}$ (D) $\frac{c}{b}$

13. If a, b and c are positive real numbers then $\frac{a}{b} + \frac{b}{c} + \frac{c}{a}$ is greater than or equal to

(A) 3 (B) 6 (C) 27 (D) 5

14. If $a_1, a_2, a_3 \ldots a_n \in R^+$ and $a_1.a_2.a_3....a_n = 1$, then minimum value of $(1+a_1+a_1^2)(1+a_2+a_2^2)$ $(1+a_3+a_3^2) \ldots (1+a_n+a_n^2)$ is equal to :-

(A) 3^{n+1} (B) 3^n (C) 3^{n-1} (D) none of these

15. If a, b, c are positive real numbers such that $ab^2c^3 = 64$ then minimum value of $\left(\frac{1}{a}+\frac{2}{b}+\frac{3}{c}\right)$ is equal to:-

(A) 6 (B) 2 (C) 3 (D) None of these

16. The value of $1^2 + 3^2 + 5^2 + \ldots\ldots\ldots + 25^2$ is :

(A) 1728 (B) 1456 (C) 2925 (D) 1469

17. The sum of the series : $(2)^2 + 2(4)^2 + 3(6)^2 + \ldots$ upto 10 terms is :

(A) 11300 (B) 12100 (C) 12300 (D) 11200

18. $2 + 4 + 7 + 11 + 16 + \ldots\ldots\ldots$ to n terms =

(A) $\frac{1}{6}(n^2 + 3n + 8)$ (B) $\frac{n}{6}(n^2 + 3n + 8)$ (C) $\frac{1}{6}(n^2 - 3n + 8)$ (D) $\frac{n}{6}(n^2 - 3n + 8)$

19. The sum $\frac{3}{1^2}+\frac{5}{1^2+2^2}+\frac{7}{1^2+2^2+3^2}+\ldots$ upto 11-terms is :-

(A) $\frac{11}{4}$ (B) $\frac{60}{11}$ (C) $\frac{7}{2}$ (D) $\frac{11}{2}$

20. The sum of the series : $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \ldots$ up to 10 terms, is:

(A) $\frac{22}{13}$ (B) $\frac{18}{11}$ (C) $\frac{20}{11}$ (D) $\frac{16}{9}$

EXERCISE (O-2)

SINGLE CORRECT ONLY

1. If for an A.P. $a_1, a_2, a_3, \ldots, a_n, \ldots$

$a_1 + a_3 + a_5 = -12$ and $a_1 a_2 a_3 = 8$, then the value of $a_2 + a_4 + a_6$ equals

(A) – 12 (B) – 16 (C) – 18 (D) – 21

2. If the sum of the first 11 terms of an arithmetical progression equals that of the first 19 terms, then the sum of its first 30 terms, is

(A) equal to 0 (B) equal to – 1 (C) equal to 1 (D) non unique

3. Let $s_1, s_2, s_3 \ldots\ldots$ and $t_1, t_2, t_3 \ldots\ldots$ are two arithmetic sequences such that $s_1 = t_1 \neq 0$; $s_2 = 2t_2$

and $\sum_{i=1}^{10} s_i = \sum_{i=1}^{15} t_i$. Then the value of $\frac{s_2 - s_1}{t_2 - t_1}$ is

(A) 8/3 (B) 3/2 (C) 19/8 (D) 2

4. If $x \in R$, the numbers $(5^{1+x} + 5^{1-x})$, a/2, $(25^x + 25^{-x})$ form an A.P. then 'a' must lie in the interval

(A) [1, 5] (B) [2, 5] (C) [5, 12] (D) $[12, \infty)$

5. Along a road lies an odd number of stones placed at intervals of 10 m. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried out the job starting with the stone in the middle, carrying stones in succession, thereby covering a distance of 4.8 km. Then the number of stones is

(A) 15 (B) 29 (C) 31 (D) 35

6. In an A.P. with first term 'a' and the common difference d $(a, d \neq 0)$, the ratio 'ρ' of the sum of the first n terms to sum of n terms succeeding them does not depend on n. Then the ratio $\frac{a}{d}$ and the ratio 'ρ', respectively are

(A) $\frac{1}{2}, \frac{1}{4}$ (B) $2, \frac{1}{3}$ (C) $\frac{1}{2}, \frac{1}{3}$ (D) $\frac{1}{2}, 2$

7. Let a_n, $n \in N$ is an A.P. with common difference 'd' and all whose terms are non-zero. If n approaches infinity, then the sum $\frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \ldots\ldots + \frac{1}{a_n a_{n+1}}$ will approach

(A) $\frac{1}{a_1 d}$ (B) $\frac{2}{a_1 d}$ (C) $\frac{1}{2a_1 d}$ (D) $a_1 d$

8. The arithmetic mean of the nine numbers in the given set {9, 99, 999, ... 999999999} is a 9 digit number N, all whose digits are distinct. The number N does not contain the digit

(A) 0 (B) 2 (C) 5 (D) 9

9. If $\dfrac{1+3+5+.....\text{upto n terms}}{4+7+10+....\text{upto n terms}}=\dfrac{20}{7\log_{10}x}$ and

$n=\log_{10}x+\log_{10}x^{\frac{1}{2}}+\log_{10}x^{\frac{1}{4}}+\log_{10}x^{\frac{1}{8}}+.......+\infty$, then x is equal to

(A) 10^3 (B) 10^5 (C) 10^6 (D) 10^7

10. If $a \neq 1$ and $ln\, a^2 + (ln\, a^2)^2 + (ln\, a^2)^3 + ... = 3\left(ln\, a+(ln\, a)^2+(ln\, a)^3+(ln\, a)^4+.......\right)$ then 'a' is equal to

(A) $e^{1/5}$ (B) $\sqrt{e}$ (C) $\sqrt[3]{e}$ (D) $\sqrt[4]{e}$

11. If a, b, c are distinct positive real in H.P., then the value of the expression, $\dfrac{b+a}{b-a}+\dfrac{b+c}{b-c}$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

12. An H.M. is inserted between the number 1/3 and an unknown number. If we diminish the reciprocal of the inserted number by 6, it is the G.M. of the reciprocal of 1/3 and that of the unknown number. If all the terms of the respective H.P. are distinct then

(A) the unknown number is 27 (B) the unknown number is 1/27

(C) the H.M. is 15 (D) the G.M. is 21

13. If abcd = 1 where a, b, c, d are positive reals then the minimum value of

$a^2 + b^2 + c^2 + d^2 + ab + ac + ad + bc + bd + cd$ is

(A) 6 (B) 10 (C) 12 (D) 20

14. **Statement-1:** If $27\,abc \geq (a + b + c)^3$ and $3a + 4b + 5c = 12$ then $\dfrac{1}{a^2}+\dfrac{1}{b^3}+\dfrac{1}{c^5}=10$; where a, b, c are positive real numbers.

Statement-2: For positive real numbers A.M. ≥ G.M.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

15. If $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$ where a, b, c are in A.P. and $|a| < 1$, $|b| < 1$, $|c| < 1$, then x, y, z are in-

(A) HP

(B) Arithmetic - Geometric Progression

(C) AP

(D) GP

16. If $S = 1^2 + 3^2 + 5^2 + ... + (99)^2$ then the value of the sum $2^2 + 4^2 + 6^2 + ... + (100)^2$ is

(A) S + 2550

(B) 2S

(C) 4S

(D) S + 5050

17. For which positive integers *n* is the ratio, $\dfrac{\sum_{k=1}^{n} k^2}{\sum_{k=1}^{n} k}$ an integer?

(A) odd *n* only

(B) even *n* only

(C) $n = 1 + 6k$ only, where $k \geq 0$ and $k \in I$

(D) $n = 1 + 3k$, integer $k \geq 0$

MORE THAN ONE CORRECT :

18. Let a_1, a_2, a_3 .... and b_1, b_2, b_3 ...... be arithmetic progressions such that $a_1 = 25$, $b_1 = 75$ and $a_{100} + b_{100} = 100$. Then

(A) the difference between successive terms in progression 'a' is opposite of the difference in progression 'b'.

(B) $a_n + b_n = 100$ for any *n*.

(C) $(a_1 + b_1), (a_2 + b_2), (a_3 + b_3)$, ... are in A.P.

(D) $\sum_{r=1}^{100} (a_r + b_r) = 10000$

EXERCISE (S-1)

1. In an AP of which 'a' is the Ist term, if the sum of the Ist p terms is equal to zero, show that the sum of the next q terms is $-\left(\frac{aq(p+q)}{p-1}\right)$

2. The interior angles of a convex polygon form an arithmetic progression with a common difference of 4°. Determine the number of sides of the polygon if its largest interior angle is 172°.

3. If a > 0, then minimum value of $a + 2a^2 + a^3 + 15 + a^{-1} + a^{-3} + a^{-4}$ is

4. The sequence $a_1, a_2, a_3, ... a_{98}$ satisfies the relation $a_{n+1} = a_n + 1$ for n = 1, 2, 3, ...97 and has the sum equal to 4949. Evaluate $\sum_{k=1}^{49} a_{2k}$.

5. There are nAM's between 1 & 31 such that 7th mean : $(n-1)^{th}$ mean = 5 : 9, then find the value of n.

6. The first term of an arithmetic progression is 1 and the sum of the first nine terms equal to 369. The first and the ninth term of a geometric progression with real common ratio coincide with the first and the ninth term of the arithmetic progression. Find the seventh term of the geometric progression.

7. For an increasing G.P. $a_1,a_2,a_3........., a_n$, if $a_6 = 4a_4$, $a_9 - a_7 = 192$, then the value of $\sum_{i=1}^{\infty}\frac{1}{a_i}$ is

8. Find three numbers a,b,c between 2 & 18 such that ;

(i) their sum is 25

(ii) the numbers 2,a,b are consecutive terms of an AP &

(iii) the numbers b,c,18 are consecutive terms of a G.P.

9. If the 10^{th} term of a HP is 21 and 21^{st} term of the same HP is 10, then find the 210^{th} term.

10. The p^{th} term T_p of H.P. is q(p + q) and q^{th} term T_q is p(p + q) when p > 2, q > 2 (p ≠ q). Prove that

(a) $T_{p+q} = pq$; (b) $T_{pq} = p + q$; (c) $T_{p+q} > T_{pq}$

11. **(a)** The harmonic mean of two numbers is 4. The arithmetic mean A & the geometric mean G satisfy the relation $2A + G^2 = 27$. Find the two numbers.

(b) The AM of two numbers exceeds their GM by 15 & HM by 27. Find the numbers.

12. If a, b, c, d > 0 such that a + 2b + 3c + 4d = 50, then find the maximum value of $\left(\frac{a^2b^4c^3d}{16}\right)^{1/10}$

13. If number of coins earned in n^{th} game is $n2^{n+2} - 2^n$ and total number of coins earned in first 10 games is $10(B.2^{10} + 1)$, where $B \in N$, then the value of B is

14. Find the n^{th} term and the sum to n terms of the sequence :

(i) 1 + 5 + 13 + 29 + 61 +...... (ii) 6 + 13 + 22 + 33 +........

15. Sum the following series to n terms and to infinity (where it is finite and defined) :

(i) $\frac{1}{1.4.7}+\frac{1}{4.7.10}+\frac{1}{7.10.13}+.....$ (ii) $\sum_{r=1}^{n} r(r+1)(r+2)(r+3)$ (iii) $\sum_{r=1}^{n}\frac{1}{4r^2-1}$

EXERCISE (JM)

1. If m is the A.M. of two distinct real numbers l and n (l, n > 1) and G_1, G_2 and G_3 are three geometric means between l and n, then $G_1^4 + 2G_2^4 + G_3^4$ equals - **[JEE(Main)-2015]**

(1) $4\ lmn^2$ (2) $4\ l^2m^2n^2$ (3) $4\ l^2mn$ (4) $4\ lm^2n$

2. If the 2^{nd}, 5^{th} and 9^{th} terms of a non-constant A.P. are in G.P., then the common ratio of this G.P. is:- **[JEE(Main)-2016]**

(1) $\frac{7}{4}$ (2) $\frac{8}{5}$ (3) $\frac{4}{3}$ (4) 1

3. If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + ...,$ is $\frac{16}{5}$m, then m is equal to :- **[JEE(Main)-2016]**

(1) 99 (2) 102 (3) 101 (4) 100

4. If, for a positive integer n, the quadratic equation,

$x(x + 1) + (x + 1)(x + 2) + ... + (x + n - 1)(x + n) = 10n$

has two consecutive integral solutions, then n is equal to : **[JEE(Main)-2017]**

(1) 11 (2) 12 (3) 9 (4) 10

5. For any three positive real numbers a, b and c, $9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$. Then : **[JEE(Main)-2017]**

(1) a, b and c are in G.P. (2) b, c and a are in G.P.

(3) b, c and a are in A.P. (4) a, b and c are in A.P.

6. Let $a_1, a_2, a_3, ... , a_{49}$ be in A.P. such that $\sum_{k=0}^{12} a_{4k+1} = 416$ and $a_9 + a_{43} = 66$.

If $a_1^2 + a_2^2 + ... + a_{17}^2 = 140m$, then m is equal to- **[JEE(Main)-2018]**

(1) 68 (2) 34 (3) 33 (4) 66

7. Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series $1^2 + 2\cdot2^2 + 3^2 + 2\cdot4^2 + 5^2 + 2\cdot6^2 + ...$. If $B - 2A = 100\lambda$, then λ is equal to : **[JEE(Main)-2018]**

(1) 248 (2) 464 (3) 496 (4) 232

8. If a, b and c be three distinct real numbers in G. P. and a + b + c = xb, then x cannot be : **[JEE(Main)-2019]**

(1) 4 (2) –3 (3) –2 (4) 2

9. Let $a_1, a_2, ..., a_{30}$ be an A. P., $S = \sum_{i=1}^{30} a_i$ and $T = \sum_{i=1}^{15} a_{(2i-1)}$. If $a_5 = 27$ and $S - 2T = 75$, then a_{10} is equal to :

[JEE(Main)-2019]

(1) 57 (2) 47 (3) 42 (4) 52

10. The sum of the follwing series

$$1+6+\frac{9\left(1^2+2^2+3^2\right)}{7}+\frac{12\left(1^2+2^2+3^2+4^2\right)}{9}+\frac{15\left(1^2+2^2+....+5^2\right)}{11}+.... \text{ up to 15 terms, is :}$$

[JEE(Main)-2019]

(1) 7820 (2) 7830 (3) 7520 (4) 7510

11. Let x, y be positive real numbers and m, n positive integers. The maximum value of the expression $\frac{x^m y^n}{(1+x^{2m})(1+y^{2n})}$ is :- **[JEE(Main)-2019]**

(1) $\frac{1}{2}$ (2) $\frac{1}{4}$ (3) $\frac{m+n}{6mn}$ (4) 1

12. If α, β and γ are three consecutive terms of a non-constant G.P. such that the equations $\alpha x^2 + 2\beta x + \gamma = 0$ and $x^2 + x - 1 = 0$ have a common root, then $\alpha(\beta + \gamma)$ is equal to :

[JEE(Main)-2019]

(1) $\beta\gamma$ (2) 0 (3) $\alpha\gamma$ (4) $\alpha\beta$

13. The sum of all natural numbers 'n' such that $100 < n < 200$ and H.C.F. $(91, n) > 1$ is :

[JEE(Main)- 2019]

(1) 3221 (2) 3121 (3) 3203 (4) 3303

14. If three distinct numbers a,b,c are in G.P. and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, then which one of the following statements is correct?

[JEE(Main)-2019]

(1) d, e, f are in A.P. (2) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in G.P.

(3) $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P. (4) d, e, f are in G.P.

15. Let the sum of the first n terms of a non-constant A.P., $a_1, a_2, a_3, ...$ be $50n + \frac{n(n-7)}{2}A$, where A is a constant. If d is the common difference of this A.P., then the ordered pair (d, a_{50}) is equal to **[JEE(Main)- 2019]**

(1) (A, 50+46A) (2) (A, 50+45A) (3) (50, 50+46A) (4) (50, 50+45A)

16. The sum $\frac{3\times1^3}{1^2}+\frac{5\times\left(1^3+2^3\right)}{1^2+2^2}+\frac{7\times\left(1^3+2^3+3^3\right)}{1^2+2^2+3^2}+.......$ upto 10^{th} term, is : **[JEE(Main)- 2019]**

(1) 660 (2) 620 (3) 680 (4) 600

17. If $a_1, a_2, a_3, ..., a_n$ are in A.P. and $a_1 + a_4 + a_7 + ... + a_{16} = 114$, then $a_1 + a_6 + a_{11} + a_{16}$ is equal to : **[JEE(Main)- 2019]**

(1) 38 (2) 98 (3) 76 (4) 64

18. Let a, b and c be in G. P. with common ratio r, where $a \neq 0$ and $0 < r \leq \frac{1}{2}$. If 3a, 7b and 15c are the first three terms of an A. P., then the 4th term of this A. P. is : **[JEE(Main)- 2019]**

(1) $\frac{7}{3}a$ (2) a (3) $\frac{2}{3}a$ (4) 5a

19. If α and β are the roots of the equation $375x^2 - 25x - 2 = 0$, then $\lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\alpha^r + \lim\limits_{n\to\infty}\sum\limits_{r=1}^{n}\beta^r$ is equal to : **[JEE(Main)- 2019]**

(1) $\frac{21}{346}$ (2) $\frac{29}{358}$ (3) $\frac{1}{12}$ (4) $\frac{7}{116}$

20. If the sum of the first 40 terms of the series, 3 + 4 + 8 + 9 + 13 + 14 + 18 +19 +… is (102)m, then m is equal to : **[JEE(Main)- 2020]**

(1) 20 (2) 5 (3) 10 (4) 25

21. Let $a_1, a_2, a_3,...$ be a G.P. such that $a_1 < 0$, $a_1 + a_2 = 4$ and $a_3 + a_4 = 16$. If $\sum\limits_{i=1}^{9} a_i = 4\lambda$, then λ is equal to : **[JEE(Main)- 2020]**

(1) –171 (2) 171 (3) $\frac{511}{3}$ (4) –513

22. Five numbers are in A.P., whose sum is 25 and product is 2520. If one of these five numbers is $-\frac{1}{2}$, then the greatest number amongst them is : **[JEE(Main)- 2020]**

(1) $\frac{21}{2}$ (2) 27 (3) 16 (4) 7

23. The greatest positive integer k, fr which $49^k + 1$ is a factor of the sum $49^{125} + 49^{124} + ... 49^2 + 49 + 1$, is : **[JEE(Main)- 2020]**

(1) 32 (2) 60 (3) 63 (4) 65

24. The sum, $\sum\limits_{n=1}^{7}\frac{n(n+1)(2n+1)}{4}$ is equal to ________. **[JEE(Main)- 2020]**

25. The sum $\sum\limits_{k=1}^{20}(1+2+3+...+k)$ is .... **[JEE(Main)- 2020]**

26. The number of terms common to the two A.P.'s 3, 7, 11, ..., 407 and 2, 9, 16, ..., 709 is ______. **[JEE(Main)- 2020]**

27. The product $2^{\frac{1}{4}} \cdot 4^{\frac{1}{16}} \cdot 8^{\frac{1}{48}} \cdot 16^{\frac{1}{128}} \cdot ....$ to ∞ is equal to : **[JEE(Main)- 2020]**

(1) $2^{\frac{1}{2}}$ (2) $2^{\frac{1}{4}}$ (3) 2 (4) 1

EXERCISE (JA)

1. If the sum of first *n* terms of an A.P. is cn^2, then the sum of squares of these *n* terms is

(A) $\frac{n(4n^2-1)c^2}{6}$ (B) $\frac{n(4n^2+1)c^2}{3}$ (C) $\frac{n(4n^2-1)c^2}{3}$ (D) $\frac{n(4n^2+1)c^2}{6}$

[JEE 2009, 3 (–1)]

2. Let $a_1, a_2, a_3 .... a_{11}$ be real numbers satisfying

$a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3,4.........11$.

If $\frac{a_1^2 + a_2^2 + ... + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + ... + a_{11}}{11}$ is equal to **[JEE 2010, 3+3]**

3. The minimum value of the sum of real numbers a^{-5}, a^{-4}, $3a^{-3}$, 1, a^8 and a^{10} with $a > 0$ is

[JEE 2011, 4]

4. Let $a_1, a_2, a_3, ..., a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^{p} a_i, 1 \le p \le 100$. For any integer n with $1 \le n \le 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n, then a_2 is **[JEE 2011, 4]**

5. Let a_1, a_2, a_3, ... be in harmonic progression with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is **[JEE 2012, 3 (–1)]**

(A) 22 (B) 23 (C) 24 (D) 25

6. Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} k^2$. Then S_n can take value(s) **[JEE-Advanced 2013, 4, (–1)]**

(A) 1056 (B) 1088 (C) 1120 (D) 1332

7. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller to the numbers on the removed cards is k, then k – 20 = **[JEE-Advanced 2013, 4, (–1)]**

8. Let a,b,c be positive integers such that $\frac{b}{a}$ is an integer. If a,b,c are in geometric progression and the arithmetic mean of a,b,c is b + 2, then the value of $\frac{a^2 + a - 14}{a+1}$ is

[JEE(Advanced)-2014, 3]

9. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is **[JEE 2015, 4M, –0M]**

10. Let $b_i > 1$ for i = 1, 2, ..., 101. Suppose $\log_e b_1, \log_e b_2, ...,\log_e b_{101}$ are in Arithmetic Progression (A.P.) with the common difference $\log_e 2$. Suppose $a_1, a_2, ..., a_{101}$ are in A.P. such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + ... + b_{51}$ and $s = a_1 + a_2 + ... + a_{51}$ then

[JEE(Advanced)-2016, 3(–1)]

(A) $s > t$ and $a_{101} > b_{101}$ (B) $s > t$ and $a_{101} < b_{101}$

(C) $s < t$ and $a_{101} > b_{101}$ (D) $s < t$ and $a_{101} < b_{101}$

11. The sides of the right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side ? **[JEE(Advanced)-2017, 3]**

12. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ... , and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ... Then, the number of elements in the set $X \cup Y$ is ______ **[JEE(Advanced)-2018, 3]**

ANSWER KEY

EXERCISE (O-1)

1. D	**2.** A	**3.** B	**4.** D	**5.** A	**6.** B	**7.** C
8. A	**9.** B	**10.** A	**11.** C	**12.** C	**13.** A	**14.** B
15. C	**16.** C	**17.** B	**18.** B	**19.** D	**20.** C	

EXERCISE (O-2)

1. D	**2.** A	**3.** C	**4.** D	**5.** C	**6.** C	**7.** A
8. A	**9.** B	**10.** D	**11.** B	**12.** B	**13.** B	**14.** D
15. A	**16.** D	**17.** D	**18.** A,B,C,D			

EXERCISE (S-1)

2. 12 **3.** 22 **4.** 2499 **5.** n = 14 **6.** 27 **7.** 2

8. a = 5 , b = 8 , c = 12 **9.** 1 **11.** (a) 6, 3 (b) 120, 30 **12.** 5 **13.** 7

14. (i) $2^{n+1} - 3$; $2^{n+2} - 4 - 3n$ (ii) $n^2 + 4n + 1$; $(1/6)\,n\,(n+1)(2n+13) + n$

15. **(i)** $s_n = (1/24) - [1/\{6(3n+1)(3n+4)\}]$; $s_\infty = 1/24$

(ii) $(1/5)\,n\,(n+1)(n+2)(n+3)(n+4)$

(iii) $n/(2n+1)$

EXERCISE (JM)

1. 4	**2.** 3	**3.** 3	**4.** 1	**5.** 3	**6.** 2	**7.** 1
8. 4	**9.** 4	**10.** 1	**11.** 2	**12.** 1	**13.** 2	**14.** 3
15. 1	**16.** 1	**17.** 3	**18.** 2	**19.** 3	**20.** 1	**21.** 1
22. 3	**23.** 3	**24.** 504	**25.** 1540.00	**26.** 14	**27.** 1	

EXERCISE (JA)

1. C	**2.** 0	**3.** 8	**4.** 9 or 3	**5.** D	**6.** A,D	**7.** 5
8. 4	**9.** 9	**10.** B	**11.** 6	**12.** 3748		

CHAPTER 2

COMPOUND ANGLES

Chapter Contents 02

IMPORTANT NOTES

CHAPTER 2

COMPOUND ANGLES

TRIGONOMETRIC RATIOS & IDENTITIES

1. INTRODUCTION TO TRIGONOMETRY :

The word 'trigonometry' is derived from the Greek words 'trigon' and 'metron' and it means 'measuring the sides of a triangle'. The subject was originally developed to solve geometric problems involving triangles. It was studied by sea captains for navigation, surveyor to map out the new lands, by engineers and others. Currently, trigonometry is used in many areas such as the science of seismology, designing electric circuits, describing the state of an atom, predicting the heights of tides in the ocean, analysing a musical tone and in many other areas.

(a) **Measurement of angles :** Commonly two systems of measurement of angles are used.

(i) **Sexagesimal or English System :** Here 1 right angle = 90° (degrees)

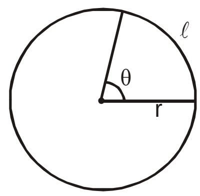
1° = 60' (minutes)

1' = 60" (seconds)

(ii) **Circular system :** Here an angle is measured in radians. One radian corresponds to the angle subtended by an arc of length 'r' at the centre of the circle of radius r. It is a constant quantity and does not depend upon the radius of the circle.

(b) Relation between the these systems : $\dfrac{D}{90}=\dfrac{R}{\pi/2}$

(c) If θ is the angle subtended at the centre of a circle of radius 'r',



by an arc of length 'ℓ' then $\dfrac{\ell}{r}=\theta$.

Note that here ℓ, r are in the same units and θ is always in radians.

Illustrations

Illustration 1 : If the arcs of same length in two circles subtend angles of 60° and 75° at their centres. Find the ratio of their radii.

Solution : Let r_1 and r_2 be the radii of the given circles and let their arcs of same length 's' subtend angles of 60° and 75° at their centres.

$$\text{Now, } 60^\circ=\left(60\times\frac{\pi}{180}\right)^c=\left(\frac{\pi}{3}\right)^c \text{ and } 75^\circ=\left(75\times\frac{\pi}{180}\right)^c=\left(\frac{5\pi}{12}\right)^c$$

$$\therefore\quad \frac{\pi}{3}=\frac{s}{r_1} \text{ and } \frac{5\pi}{12}=\frac{s}{r_2}$$

$$\Rightarrow\quad \frac{\pi}{3}r_1=s \text{ and } \frac{5\pi}{12}r_2=s \Rightarrow \frac{\pi}{3}r_1=\frac{5\pi}{12}r_2 \Rightarrow 4r_1=5r_2 \Rightarrow r_1:r_2=5:4\ \textbf{Ans.}$$

Do yourself - 1

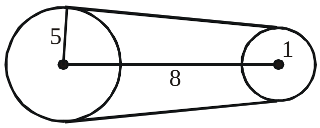
1. The radius of a circle is 30 cm. Find the length of an arc of this circle if the length of the chord of the arc is 30 cm.

2. A man is running around a regular hexagonal field of side length 6m, so that he always at a distance 3 metre from the nearest boundary. Find the length of path travelled by him in one round.

3. Convert the following measurement into radians :

(a) 25° 30' 30"   (b) 10° 42' 30"   (c) 9° 18' 42"

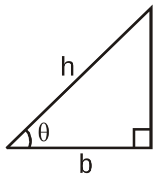
4. A belt is tied up across two circular pulleys of radii 5m and 1m respectively whose centres are seperated at a distance 8m. (as shown). Find the length of the belt required.



5. Find the number of degrees, minutes and seconds in the angle at the centre of a circle, whose radius is 5m, which is subtended by an arc of length 6 m. (Consider π = 22/7)

2. T-RATIOS (or Trigonometric functions) :

In a right angle triangle



$$\sin\theta = \frac{p}{h};\ \cos\theta = \frac{b}{h};\ \tan\theta = \frac{p}{b}; \operatorname{cosec}\theta = \frac{h}{p};\ \sec\theta = \frac{h}{b} \text{ and } \cot\theta = \frac{b}{p}$$

'p' is perpendicular ; 'b' is base and 'h' is hypotenuse.

Note : The quantity by which the cosine falls short of unity i.e. $1 - \cos\theta$, is called the versed sine θ of θ and also by which the sine falls short of unity i.e. $1 - \sin\theta$ is called the coversed sine of θ.

3. BASIC TRIGONOMETRIC IDENTITIES :

(1) $\sin\theta . \operatorname{cosec}\theta = 1$

(2) $\cos\theta . \sec\theta = 1$

(3) $\tan\theta . \cot\theta = 1$

(4) $\tan\theta = \frac{\sin\theta}{\cos\theta}$ & $\cot\theta = \frac{\cos\theta}{\sin\theta}$

(5) $\sin^2\theta + \cos^2\theta = 1$ or $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$

(6) $\sec^2\theta - \tan^2\theta = 1$ or $\sec^2\theta = 1 + \tan^2\theta$ or $\tan^2\theta = \sec^2\theta - 1$

(7) $\sec\theta + \tan\theta = \dfrac{1}{\sec\theta - \tan\theta}$

(8) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$ or $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$ or $\cot^2\theta = \operatorname{cosec}^2\theta - 1$

(9) $\operatorname{cosec}\theta + \cot\theta = \dfrac{1}{\operatorname{cosec}\theta - \cot\theta}$

(10) Expressing trigonometrical ratio in terms of each other :

	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
$\sin\theta$	$\sin\theta$	$\sqrt{1-\cos^2\theta}$	$\dfrac{\tan\theta}{\sqrt{1+\tan^2\theta}}$	$\dfrac{1}{\sqrt{1+\cot^2\theta}}$	$\dfrac{\sqrt{\sec^2\theta-1}}{\sec\theta}$	$\dfrac{1}{\operatorname{cosec}\theta}$
$\cos\theta$	$\sqrt{1-\sin^2\theta}$	$\cos\theta$	$\dfrac{1}{\sqrt{1+\tan^2\theta}}$	$\dfrac{\cot\theta}{\sqrt{1+\cot^2\theta}}$	$\dfrac{1}{\sec\theta}$	$\dfrac{\sqrt{\operatorname{cosec}^2\theta-1}}{\operatorname{cosec}\theta}$
$\tan\theta$	$\dfrac{\sin\theta}{\sqrt{1-\sin^2\theta}}$	$\dfrac{\sqrt{1-\cos^2\theta}}{\cos\theta}$	$\tan\theta$	$\dfrac{1}{\cot\theta}$	$\sqrt{\sec^2\theta-1}$	$\dfrac{1}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\cot\theta$	$\dfrac{\sqrt{1-\sin^2\theta}}{\sin\theta}$	$\dfrac{\cos\theta}{\sqrt{1-\cos^2\theta}}$	$\dfrac{1}{\tan\theta}$	$\cot\theta$	$\dfrac{1}{\sqrt{\sec^2\theta-1}}$	$\sqrt{\operatorname{cosec}^2\theta-1}$
$\sec\theta$	$\dfrac{1}{\sqrt{1-\sin^2\theta}}$	$\dfrac{1}{\cos\theta}$	$\sqrt{1+\tan^2\theta}$	$\dfrac{\sqrt{1+\cot^2\theta}}{\cot\theta}$	$\sec\theta$	$\dfrac{\operatorname{cosec}\theta}{\sqrt{\operatorname{cosec}^2\theta-1}}$
$\operatorname{cosec}\theta$	$\dfrac{1}{\sin\theta}$	$\dfrac{1}{\sqrt{1-\cos^2\theta}}$	$\dfrac{\sqrt{1+\tan^2\theta}}{\tan\theta}$	$\sqrt{1+\cot^2\theta}$	$\dfrac{\sec\theta}{\sqrt{\sec^2\theta-1}}$	$\operatorname{cosec}\theta$

Illustrations

Illustration 2 : If $\sin\theta + \sin^2\theta = 1$, then prove that

$$\cos^{12}\theta + 3\cos^{10}\theta + 3\cos^8\theta + \cos^6\theta - 1 = 0$$

Solution : Given that $\sin\theta = 1 - \sin^2\theta = \cos^2\theta$

L.H.S. $= \cos^6\theta(\cos^2\theta + 1)^3 - 1 = \sin^3\theta(1 + \sin\theta)^3 - 1$

$= (\sin\theta + \sin^2\theta)^3 - 1 = 1 - 1 = 0$

Illustration 3 : $4(\sin^6\theta + \cos^6\theta) - 6(\sin^4\theta + \cos^4\theta)$ is equal to

(A) 0 (B) 1 (C) –2 (D) none of these

Solution : $4[(\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)] - 6[(\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta]$

$= 4[1 - 3\sin^2\theta\cos^2\theta] - 6[1 - 2\sin^2\theta\cos^2\theta]$

$= 4 - 12\sin^2\theta\cos^2\theta - 6 + 12\sin^2\theta\cos^2\theta = -2$ **Ans.(C)**

ALLEN CAREER INSTITUTE OVERSEAS

Do yourself - 2

1. If $\cot\theta = \frac{4}{3}$, then find the value of $\sin\theta$, $\cos\theta$ and $\text{cosec}\theta$ in first quadrant.

2. If $\sin\theta + \text{cosec}\theta = 2$, then find the value of $\sin^8\theta + \text{cosec}^8\theta$

Prove the following statements in their respective valid domains :

3. $\cos^4 A - \sin^4 A + 1 = 2\cos^2 A$

4. $(\sin A + \cos A)(1 - \sin A \cos A) = \sin^3 A + \cos^3 A$.

5. $\dfrac{\sin A}{1+\cos A} + \dfrac{1+\cos A}{\sin A} = 2\,\text{cosec}\,A$

6. $\cos^6 A + \sin^6 A = 1 - 3\sin^2 A\cos^2 A$.

7. $\sqrt{\dfrac{1-\sin A}{1+\sin A}} = \sec A - \tan A \quad (-90^\circ < A < 90^\circ)$

8. $\dfrac{1}{\cot A + \tan A} = \sin A \cos A$

9. $\dfrac{1-\tan A}{1+\tan A} = \dfrac{\cot A - 1}{\cot A + 1}$

10. $\dfrac{\tan A}{1-\cot A} + \dfrac{\cot A}{1-\tan A} = \sec A\,\text{cosec}\,A + 1$

11. $\dfrac{\cos A}{1-\tan A} + \dfrac{\sin A}{1-\cot A} = \sin A + \cos A$.

12. $\sec^4 A - \sec^2 A = \tan^4 A + \tan^2 A$.

13. $\cot^4 A + \cot^2 A = \text{cosec}^4 A - \text{cosec}^2 A$.

14. $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A = \tan^2 A.\sin^2 A$

15. $(1 + \cot A - \text{cosec}A)(1 + \tan A + \sec A) = 2$

16. $\dfrac{1}{\text{cosec}\,A - \cot A} - \dfrac{1}{\sin A} = \dfrac{1}{\sin A} - \dfrac{1}{\text{cosec}\,A + \cot A}$

17. $\dfrac{\cot A.\cos A}{\cot A + \cos A} = \dfrac{\cot A - \cos A}{\cot A.\cos A}$

18. $\left(\dfrac{1}{\sec^2\alpha - \cos^2\alpha} + \dfrac{1}{\text{cosec}^2\alpha - \sin^2\alpha}\right)\cos^2\alpha\sin^2\alpha = \dfrac{1-\cos^2\alpha\sin^2\alpha}{2+\cos^2\alpha\sin^2\alpha}$

19. $\sin^8 A - \cos^8 A = (\sin^2 A - \cos^2 A)(1 - 2\sin^2 A\cos^2 A)$

20. $\dfrac{\tan A+\sec A-1}{\tan A-\sec A+1}=\dfrac{1+\sin A}{\cos A}$

21. $(\tan\alpha + \text{cosec}\ \beta)^2 - (\cot\beta - \sec\alpha)^2 = 2\tan\alpha\cot\beta(\text{cosec}\ \alpha + \sec\beta)$.

22. $2\sec^2\alpha - \sec^4\alpha - 2\,\text{cosec}^2\alpha + \text{cosec}^4\alpha = \cot^4\alpha - \tan^4\alpha$.

23. If $\sin\theta$ equals to $\dfrac{x^2-y^2}{x^2+y^2}$, find the value of $\cos\theta$ and $\cot\theta$, where $\theta \in (0,90^\circ)$.

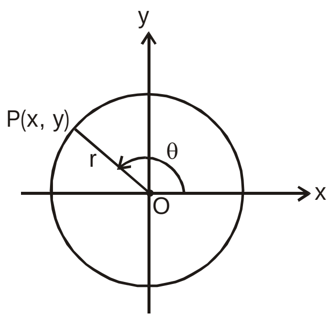
24. If $\sin\theta=\dfrac{m^2+2mn}{m^2+2mn+2n^2}$, prove that $\tan\theta=\dfrac{m^2+2mn}{2mn+2n^2}$, where $\theta \in (0,90^\circ)$.

25. If $\cos\theta-\sin\theta=\sqrt{2}\sin\theta$, prove that $\cos\theta+\sin\theta=\sqrt{2}\cos\theta$.

26. Prove that $\text{cosec}^6\alpha - \cot^6\alpha = 3\,\text{cosec}^2\alpha\cot^2\alpha + 1$.

27. Express $2\sec^2 A - \sec^4 A - 2\text{cosec}^2 A + \text{cosec}^4 A$ in terms of $\tan A$.

4. NEW DEFINITION OF T-RATIOS :



By using rectangular coordinates the definitions of trigonometric functions can be extended to angles of any size in the following way (see diagram). A point P is taken with coordinates (x, y). The radius vector OP has length r and the angle θ is taken as the directed angle measured anticlockwise from the x-axis. The three main trigonometric functions are then defined in terms of r and the coordinates x and y.

$\sin\theta = y/r$,

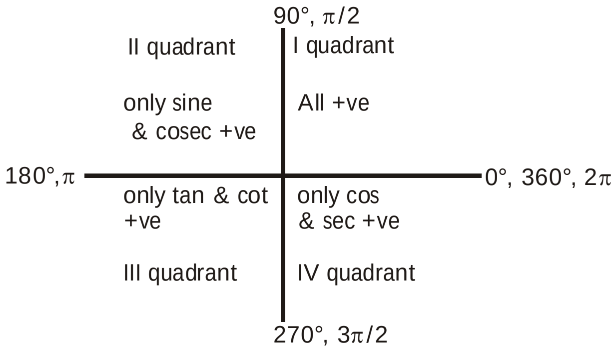
$\cos\theta = x/r$

$\tan\theta = y/x$,

(The other function are reciprocals of these)

This can give negative values of the trigonometric functions.

5. SIGNS OF TRIGONOMETRIC FUNCTIONS IN DIFFERENT QUADRANTS :



6. TRIGONOMETRIC FUNCTIONS OF ALLIED ANGLES :

(a) $\sin(2n\pi + \theta) = \sin\theta$, $\cos(2n\pi + \theta) = \cos\theta$, where $n \in I$

(b)

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$
$\sin(90° - \theta) = \cos\theta$	$\cos(90° - \theta) = \sin\theta$
$\sin(90° + \theta) = \cos\theta$	$\cos(90° + \theta) = -\sin\theta$
$\sin(180° - \theta) = \sin\theta$	$\cos(180° - \theta) = -\cos\theta$
$\sin(180° + \theta) = -\sin\theta$	$\cos(180° + \theta) = -\cos\theta$
$\sin(270° - \theta) = -\cos\theta$	$\cos(270° - \theta) = -\sin\theta$
$\sin(270° + \theta) = -\cos\theta$	$\cos(270° + \theta) = \sin\theta$
$\sin(360° - \theta) = -\sin\theta$	$\cos(360° - \theta) = \cos\theta$
$\sin(360° + \theta) = \sin\theta$	$\cos(360° + \theta) = \cos\theta$

Do yourself - 3

1. If $\sin A = \frac{11}{61}$, find tanA, cosA, and sec A.

2. If $\cos\theta = \frac{4}{5}$, find sin θ and cot θ.

3. If $\tan\theta = \frac{1}{\sqrt{7}}$, find the value of $\frac{\operatorname{cosec}^2\theta - \sec^2\theta}{\operatorname{cosec}^2\theta + \sec^2\theta}$

4. If $\cot\theta = \frac{15}{8}$, find sin θ and cosec θ.

5. If $2 \sin \theta = 2 - \cos \theta$, find $\sin \theta$.

6. If $8 \sin \theta = 4 + \cos \theta$, find $\sin \theta$.

7. If $\tan \theta + \sec \theta = 1.5$, find $\sin \theta$.

8. If $\cot \theta + \operatorname{cosec} \theta = 5$, find $\cos \theta$.

9. If $3 \sec^4 \theta + 8 = 10 \sec^2 \theta$, find the values of $\tan \theta$.

10. If $\tan^2 \theta + \sec \theta = 5$, find $\cos \theta$.

11. If $\tan \theta + \cot \theta = 2$, find $\sin \theta$.

12. If $\sec^2 \theta = 2 + 2 \tan \theta$, find $\tan \theta$.

13. If $\tan\theta = \dfrac{2x(x+1)}{2x+1}$, find $\sin \theta$ and $\cos \theta$.

7. VALUES OF T-RATIOS OF SOME STANDARD ANGLES :

Angles	**0°**	**30°**	**45°**	**60°**	**90°**	**180°**	**270°**
T-ratio	**0**	**$\pi/6$**	**$\pi/4$**	**$\pi/3$**	**$\pi/2$**	**π**	**$3\pi/2$**
$\sin\theta$	0	1/2	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0	–1
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	1/2	0	–1	0
$\tan\theta$	0	$1/\sqrt{3}$	1	$\sqrt{3}$	N.D.	0	N.D.
$\cot\theta$	N.D.	$\sqrt{3}$	1	$1/\sqrt{3}$	0	N.D.	0
$\sec\theta$	1	$2/\sqrt{3}$	$\sqrt{2}$	2	N.D.	–1	N.D.
$\operatorname{cosec}\theta$	N.D.	2	$\sqrt{2}$	$2/\sqrt{3}$	1	N.D.	–1

N.D. $\rightarrow$ Not Defined

(a) $\sin n\pi = 0$; $\cos n\pi = (-1)^n$; $\tan n\pi = 0$ where $n \in I$

(b) $\sin(2n+1)\dfrac{\pi}{2} = (-1)^n$; $\cos(2n+1)\dfrac{\pi}{2} = 0$ where $n \in I$

Do yourself - 4

1. **Verify the following identities for A = 30º as well as for A = 45º**

(a) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

(b) $\sin 2A = 2\sin A \cos A$ (c) $\cos 3A = 4\cos^3 A - 3\cos A$

(d) $\sin 3A = 3\sin A - 4\sin^3 A$ (e) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

2. **Find the value of**

(a) $\sin^2 30^\circ + \sin^2 45^\circ + \sin^2 60^\circ$ (b) $\tan^2 30^\circ + \tan^2 45^\circ + \tan^2 60^\circ$

(c) $\sin 30^\circ \cos 60^\circ + \sin 30^\circ \sin 60^\circ$ (d) $\cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ$

Illustrations

Illustration 4 : If $\sin\theta = -\dfrac{1}{2}$ and $\tan\theta = \dfrac{1}{\sqrt{3}}$ then θ is equal to -

(A) 30° (B) 150° (C) 210° (D) none of these

Solution : Let us first find out θ lying between 0 and 360°.

Since $\sin\theta = -\dfrac{1}{2} \Rightarrow \theta = 210^\circ$ or 330° and

$\tan\theta = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = 30^\circ$ or 210°

Hence , $\theta = 210^\circ$ or $\dfrac{7\pi}{6}$ is the value satisfying both. **Ans. (C)**

Do yourself - 5

1. **(i)** If $\cos\theta = -\dfrac{1}{2}$ and $\pi < \theta < \dfrac{3\pi}{2}$, then find the value of $4\tan^2\theta - 3\mathrm{cosec}^2\theta$.

(ii) Prove that : (a) $\cos 570^\circ \sin 510^\circ + \sin(-330^\circ)\cos(-390^\circ) = 0$

(b) $\tan\dfrac{11\pi}{3} - 2\sin\dfrac{9\pi}{3} - \dfrac{3}{4}\mathrm{cosec}^2\dfrac{\pi}{4} + 4\cos^2\dfrac{17\pi}{6} = \dfrac{3-2\sqrt{3}}{2}$

2. **Evaluate :**

(a) $\sin 420^\circ \cos 390^\circ + \cos(-300^\circ)\sin(-330^\circ)$

(b) $\tan 225^\circ \cot 405^\circ + \tan 765^\circ \cot 675^\circ$

3. What are the values of $\cos A - \sin A$ and $\tan A + \cot A$ when A has the values

(a) $\dfrac{2\pi}{3}$ (b) $\dfrac{7\pi}{4}$ (c) $\dfrac{11\pi}{3}$ (d) $\dfrac{5\pi}{4}$

4. Express the following quantities in terms of the ratios of a positive angle, which is less than 45°

(a) sin(–65°) (b) cos(–928°) (c) tan1145° (d) cot(–1054°)

5. What is the sign of sinA + cosA for the following values of A ?

(a) 140° (b) –356° (c) –1125°.

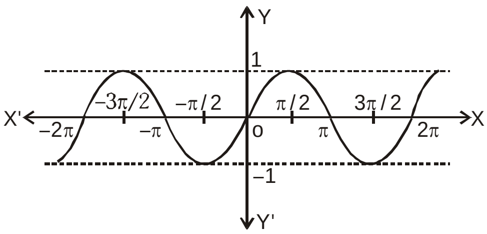
6. What is the sign of sinA – cosA for the following values of A ?

(a) 215° (b) –634° (c) – 457°

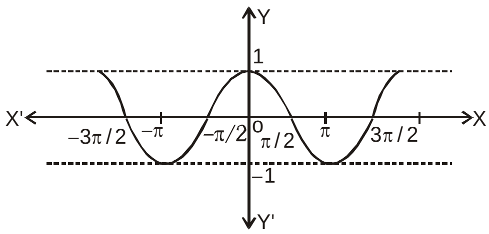
7. Find the sines and cosines of all angles in the first four quadrant whose tangents are equal to cos135°.

8. GRAPH OF TRIGONOMETRIC FUNCTIONS :

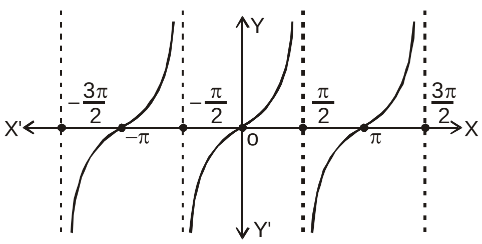
(i) y = sinx



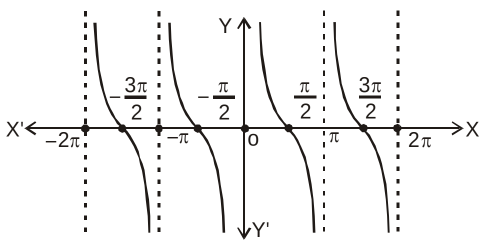
(ii) y = cosx



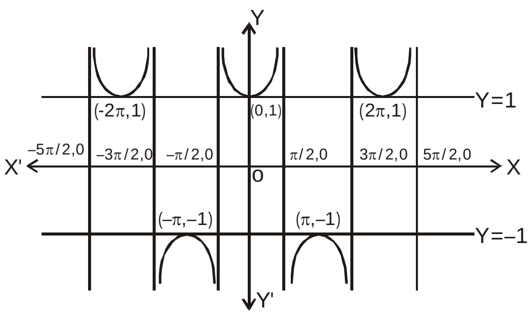
(iii) y = tanx



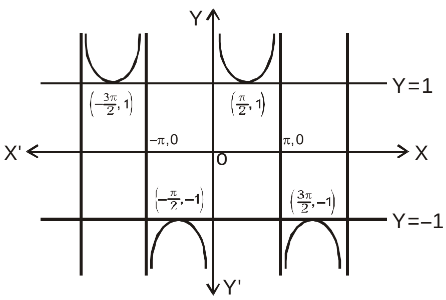
(iv) y = cotx



(v) y = secx



(vi) y = cosecx



9. DOMAINS, RANGES AND PERIODICITY OF TRIGONOMETRIC FUNCTIONS:

T-Ratio	Domain	Range	Period
sin x	R	[–1,1]	2π
cos x	R	[–1,1]	2π
tan x	$R-\{(2n+1)\pi/2 \,;\, n\in I\}$	R	π
cot x	$R-\{n\pi : n \in I\}$	R	π
sec x	$R-\{(2n+1)\pi/2 : n \in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π
cosec x	$R-\{n\pi : n \in I\}$	$(-\infty,-1]\cup[1,\infty)$	2π

Do yourself - 6

1. Prove that the equation $\sin\theta = x + \frac{1}{x}$ is impossible if x be real.

2. Show that the equation $\sec^2\theta = \frac{4xy}{(x+y)^2}$ is only possible when x = y.

3. The number of real solutions of the equation $\sin(e^x) = 2^x + 2^{-x}$ is -

(A) 1 (B) 0 (C) 2 (D) Infinite

4. Draw graphs of

(a) y = sin2x (b) y = 2cos3x (c) y = 4 tanx

5. Find number of solutions of the equation $\sin\pi x + x^2 + 1 = 2x$

10. TRIGONOMETRIC RATIOS OF THE SUM & DIFFERENCE OF TWO ANGLES :

(i) sin (A + B) = sin A cos B + cos A sin B.

(ii) sin (A – B) = sin A cos B – cos A sin B.

(iii) cos (A + B) = cos A cos B – sin A sin B

(iv) cos (A – B) = cos A cos B + sin A sin B

(v) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$ **(vi)** $\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$

(vii) $\cot(A+B) = \frac{\cot B \cot A - 1}{\cot B + \cot A}$ **(viii)** $\cot(A-B) = \frac{\cot B \cot A + 1}{\cot B - \cot A}$

Some more results :

(i) $\sin^2 A - \sin^2 B = \sin(A+B).\sin(A-B) = \cos^2 B - \cos^2 A$.

(ii) $\cos^2 A - \sin^2 B = \cos(A+B).\cos(A-B)$.

Illustrations

Illustration 5 : Prove that $\sqrt{3}$ cosec20° – sec20° = 4.

Solution : L.H.S. $= \dfrac{\sqrt{3}}{\sin 20°} - \dfrac{1}{\cos 20°} = \dfrac{\sqrt{3}\cos 20° - \sin 20°}{\sin 20°.\cos 20°}$

$$= \frac{4\left(\frac{\sqrt{3}}{2}\cos 20° - \frac{1}{2}\sin 20°\right)}{2\sin 20° \cos 20°} = \frac{4(\sin 60.\cos 20° - \cos 60°.\sin 20°)}{\sin 40°}$$

$$= 4.\frac{\sin(60° - 20°)}{\sin 40°} = 4.\frac{\sin 40°}{\sin 40°} = 4 = \text{R.H.S.}$$

Illustration 6 : Prove that tan70° = cot70° + 2cot40° **.**

Solution : L.H.S. $= \tan 70° = \tan(20° + 50°) = \dfrac{\tan 20° + \tan 50°}{1 - \tan 20° \tan 50°}$

or tan70° – tan20° tan50° tan70° = tan20° + tan50°

or tan70° = tan70° tan50° tan20° + tan20° + tan50° = 2 tan 50° +tan20°

= cot70° + 2cot40° = R.H.S.

Do yourself - 7

1. If $\sin A = \dfrac{3}{5}$ and $\cos B = \dfrac{9}{41}$, $0 < A, B < \dfrac{\pi}{2}$, then find the value of the following :

(a) sin(A + B) (b) sin(A – B)

(c) cos(A + B) (d) cos(A – B)

2. If x + y = 45°, then prove that :

(a) (1 + tanx)(1 + tany) = 2 (b) (cotx – 1)(coty – 1) = 2

(Remember these results)

3. If $\sin\alpha = \dfrac{3}{5}$ and $\cos\beta = \dfrac{9}{41}$, find the value of sin(α – β) and cos(α + β).

4. If $\sin\alpha = \dfrac{15}{17}$ and $\cos\beta = \dfrac{12}{13}$, find the values of sin(α + β), cos(α – β), and tan(α + β)

Prove that

5. cos(45° – A) cos(45° – B) – sin(45° – A) sin(45° – B) = sin(A + B).

6. $\sin(45^\circ + A)\cos(45^\circ - B) + \cos(45^\circ + A)\sin(45^\circ - B) = \cos(A - B)$.

7. $\frac{\sin(A-B)}{\cos A\cos B}+\frac{\sin(B-C)}{\cos B\cos C}+\frac{\sin(C-A)}{\cos C\cos A}=0$

8. $\sin 105^\circ + \cos 105^\circ = \cos 45^\circ$

9. $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$

10. $\cos(\alpha + \beta)\cos\gamma - \cos(\beta + \gamma)\cos\alpha = \sin\beta\sin(\gamma - \alpha)$

11. $\sin(n + 1)A \sin(n - 1)A + \cos(n + 1)A \cos(n - 1)A = \cos 2A$.

12. $\sin(n + 1)A \sin(n + 2)A + \cos(n + 1)A \cos(n + 2)A = \cos A$.

13. If $\tan A=\frac{1}{2}$ and $\tan B=\frac{1}{3}$, find the values of $\tan(2A + B)$ and $\tan(2A - B)$

14. If $\tan\alpha=\frac{5}{6}$ and $\tan\beta=\frac{1}{11}$, prove that $\alpha+\beta=\frac{\pi}{4}$

Prove that

15. $\tan\left(\frac{\pi}{4}+\theta\right)\times\tan\left(\frac{3\pi}{4}+\theta\right)=-1$

16. $\cot\left(\frac{\pi}{4}+\theta\right)\cot\left(\frac{\pi}{4}-\theta\right)=1$

17. $1+\tan A\tan\frac{A}{2}=\tan A\cot\frac{A}{2}-1=\sec A$

18. $\tan 3A \tan 2A \tan A = \tan 3A - \tan 2A - \tan A$.

11. FORMULAE TO TRANSFORM THE PRODUCT INTO SUM OR DIFFERENCE:

(i) $2\sin A\cos B = \sin(A + B) + \sin(A - B)$.

(ii) $2\cos A\sin B = \sin(A + B) - \sin(A - B)$.

(iii) $2\cos A\cos B = \cos(A + B) + \cos(A - B)$

(iv) $2\sin A\sin B = \cos(A - B) - \cos(A + B)$

Illustrations

Illustration 7: If sin2A = λ sin2B, then prove that $\frac{\tan(A+B)}{\tan(A-B)}=\frac{\lambda+1}{\lambda-1}$.

Solution : Given sin2A = λ sin2B

$$\Rightarrow \quad \frac{\sin 2A}{\sin 2B}=\frac{\lambda}{1}$$

Applying componendo & dividendo,

$$\frac{\sin 2A+\sin 2B}{\sin 2B-\sin 2A}=\frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \quad \frac{2\sin\left(\frac{2A+2B}{2}\right)\cos\left(\frac{2A-2B}{2}\right)}{2\cos\left(\frac{2B+2A}{2}\right)\sin\left(\frac{2B-2A}{2}\right)}=\frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \quad \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin\{-(A-B)\}}=\frac{\lambda+1}{1-\lambda}$$

$$\Rightarrow \quad \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\times-\sin(A-B)}=\frac{\lambda+1}{-(\lambda-1)}$$

$$\Rightarrow \quad \frac{\sin(A+B)\cos(A-B)}{\cos(A+B)\sin(A-B)}=\frac{\lambda+1}{\lambda-1} \Rightarrow \tan(A+B)\cot(A-B)=\frac{\lambda+1}{\lambda-1}$$

$$\Rightarrow \quad \frac{\tan(A+B)}{\tan(A-B)}=\frac{\lambda+1}{\lambda-1}$$

Do yourself - 8

1. Simplify $\frac{\sin 75°-\sin 15°}{\cos 75°+\cos 15°}$

Express follwing as a sum or difference of angles used in arguments :

2. 2sin5θ sin7θ.

3. 2 sin54º sin66º

4. cos(36º – A) cos(36º + A) + cos(54º + A) cos(54º – A) = cos2A.

5. cos A sin (B – C) + cos B sin (C – A) + cos C sin (A – B) = 0.

6. sin(45º + A) sin (45º – A) = $\frac{1}{2}$ cos2A

7. sin (β – γ) cos (α – δ) + sin (γ – α) cos (β – δ) + sin (α – β) cos(γ – δ) = 0.

8. $2\cos\frac{\pi}{13}\cos\frac{9\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}=0$

12. FORMULAE TO TRANSFORM SUM OR DIFFERENCE INTO PRODUCT :

(i) $\sin C + \sin D = 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(ii) $\sin C - \sin D = 2\cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$

(iii) $\cos C + \cos D = 2\cos\left(\frac{C+D}{2}\right)\cos\left(\frac{C-D}{2}\right)$

(iv) $\cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

Illustrations

Illustration 8 : $\dfrac{\sin 5\theta + \sin 2\theta - \sin\theta}{\cos 5\theta + 2\cos 3\theta + 2\cos^2\theta + \cos\theta}$ is equal to -

(A) $\tan\theta$ (B) $\cos\theta$ (C) $\cot\theta$ (D) none of these

Solution **:** L.H.S.$= \dfrac{2\sin 2\theta\cos 3\theta + \sin 2\theta}{2\cos 3\theta.\cos 2\theta + 2\cos 3\theta + 2\cos^2\theta} = \dfrac{\sin 2\theta[2\cos 3\theta + 1]}{2[\cos 3\theta(\cos 2\theta + 1) + (\cos^2\theta)]}$

$$= \frac{\sin 2\theta[2\cos 3\theta + 1]}{2[\cos 3\theta(2\cos^2\theta) + \cos^2\theta]} = \frac{\sin 2\theta(2\cos 3\theta + 1)}{2\cos^2\theta(2\cos 3\theta + 1)} = \tan\theta$$ **Ans. (A)**

Do yourself - 9

1. Prove that

(a) $(\sin 3A + \sin A)\sin A + (\cos 3A - \cos A)\cos A = 0$

(b) $\dfrac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta} = \tan 2\theta$

Prove that

2. $\dfrac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = \tan\theta$

3. $\dfrac{\cos 2B + \cos 2A}{\cos 2B - \cos 2A} = \cot(A+B)\cot(A-B)$

4. $\dfrac{\sin 2A+\sin 2B}{\sin 2A-\sin 2B}=\dfrac{\tan(A+B)}{\tan(A-B)}$

5. $\cos(A + B) + \sin(A – B) = 2\sin(45^\circ + A)\cos(45^\circ + B)$

6. $\dfrac{\cos 3A-\cos A}{\sin 3A-\sin A}+\dfrac{\cos 2A-\cos 4A}{\sin 4A-\sin 2A}=\dfrac{\sin A}{\cos 2A\cos 3A}$

7. $\dfrac{\tan 5\theta+\tan 3\theta}{\tan 5\theta-\tan 3\theta}=4\cos 2\theta\cos 4\theta$

8. $\dfrac{\cos 3\theta+2\cos 5\theta+\cos 7\theta}{\cos\theta+2\cos 3\theta+\cos 5\theta}=\cos 2\theta-\sin 2\theta\tan 3\theta$

9. $\dfrac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}=\tan 4A$

10. $\dfrac{\sin(A-C)+2\sin A+\sin(A+C)}{\sin(B-C)+2\sin B+\sin(B+C)}=\dfrac{\sin A}{\sin B}$

11. $\dfrac{\sin A-\sin 5A+\sin 9A-\sin 13A}{\cos A-\cos 5A-\cos 9A+\cos 13A}=\cot 4A$

12. $\dfrac{\sin A+\sin B}{\sin A-\sin B}=\tan\dfrac{A+B}{2}\cot\dfrac{A-B}{2}$

13. $\dfrac{\sin A+\sin B}{\cos A+\cos B}=\tan\dfrac{A+B}{2}$

14. $\dfrac{\sin A-\sin B}{\cos B-\cos A}=\cot\dfrac{A+B}{2}$

15. $\dfrac{\cos(A+B+C)+\cos(-A+B+C)+\cos(A-B+C)+\cos(A+B-C)}{\sin(A+B+C)+\sin(-A+B+C)-\sin(A-B+C)+\sin(A+B-C)}=\cot B$

16. $\cos\left\{\theta+\left(n-\dfrac{3}{2}\right)\phi\right\}-\cos\left\{\theta+\left(n+\dfrac{3}{2}\right)\phi\right\}=2\sin\dfrac{3\phi}{2}.\sin(\theta+n\phi)$

17. $\sin\dfrac{\theta}{2}\sin\dfrac{7\theta}{2}+\sin\dfrac{3\theta}{2}\sin\dfrac{11\theta}{2}=\sin 2\theta\sin 5\theta$

18. $\cos 2\theta \cos \frac{\theta}{2} - \cos 3\theta \cos \frac{9\theta}{2} = \sin 5\theta \sin \frac{5\theta}{2}$

19. $\frac{2\sin(A-C)\cos C - \sin(A-2C)}{2\sin(B-C)\cos C - \sin(B-2C)} = \frac{\sin A}{\sin B}$

20. $\frac{\sin A \sin 2A + \sin 3A \sin 6A + \sin 4A \sin 13A}{\sin A \cos 2A + \sin 3A \cos 6A + \sin 4A \cos 13A} = \tan 9A$

21. $\frac{\cos 2A \cos 3A - \cos 2A \cos 7A + \cos A \cos 10A}{\sin 4A \sin 3A - \sin 2A \sin 5A + \sin 4A \sin 7A} = \cot 6A \cot 5A$

13. TRIGONOMETRIC RATIOS OF SUM OF MORE THAN TWO ANGLES :

(i) $\sin(A+B+C) = \sin A\cos B\cos C + \sin B\cos A\cos C + \sin C\cos A\cos B - \sin A\sin B\sin C$

$= \Sigma \sin A \cos B \cos C - \Pi \sin A$

$= \cos A \cos B \cos C\,[\tan A + \tan B + \tan C - \tan A \tan B \tan C]$

(ii) $\cos(A+B+C) = \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

$= \Pi \cos A - \Sigma \sin A \sin B \cos C$

$= \cos A \cos B \cos C\,[1 - \tan A \tan B - \tan B \tan C - \tan C \tan A]$

(iii) $\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A} = \frac{S_1 - S_3}{1 - S_2}$

14. TRIGONOMETRIC RATIOS OF MULTIPLE ANGLES :

(a) Trigonometrical ratios of an angle 2θ in terms of the angle θ :

(i) $\sin 2\theta = 2\sin\theta\cos\theta = \frac{2\tan\theta}{1+\tan^2\theta}$

(ii) $\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$

(iii) $1 + \cos 2\theta = 2\cos^2\theta$ **(iv)** $1 - \cos 2\theta = 2\sin^2\theta$

(v) $\tan\theta = \frac{1-\cos 2\theta}{\sin 2\theta} = \frac{\sin 2\theta}{1+\cos 2\theta}$ **(vi)** $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

Illustrations

Illustration 9: Prove that : $\dfrac{2\cos 2A+1}{2\cos 2A-1}=\tan(60^\circ+A)\tan(60^\circ-A)$.

Solution : R.H.S. = tan(60° + A) tan(60° – A)

$$=\left(\frac{\tan 60^\circ+\tan A}{1-\tan 60^\circ\tan A}\right)\left(\frac{\tan 60^\circ-\tan A}{1+\tan 60^\circ\tan A}\right)=\left(\frac{\sqrt{3}+\tan A}{1-\sqrt{3}\tan A}\right)\left(\frac{\sqrt{3}-\tan A}{1+\sqrt{3}\tan A}\right)$$

$$=\frac{3-\tan^2 A}{1-3\tan^2 A}=\frac{3-\dfrac{\sin^2 A}{\cos^2 A}}{1-3\dfrac{\sin^2 A}{\cos^2 A}}=\frac{3\cos^2 A-\sin^2 A}{\cos^2 A-3\sin^2 A}==\frac{2\cos^2 A+\cos^2 A-2\sin^2 A+\sin^2 A}{2\cos^2 A-2\sin^2 A-\sin^2 A-\cos^2 A}$$

$$=\frac{2(\cos^2 A-\sin^2 A)+\cos^2 A+\sin^2 A}{2(\cos^2 A-\sin^2 A)-(\sin^2 A+\cos^2 A)}=\frac{2\cos 2A+1}{2\cos 2A-1}=\text{L.H.S.}$$

Do yourself - 10

Prove that :

1. $\dfrac{\sin 2A}{1+\cos 2A}=\tan A$ (Remember)

2. $\dfrac{\sin 2A}{1-\cos 2A}=\cot A$ (Remember)

3. tanA – cotA = –2 cot2A. (Remember)

4. $\dfrac{1+\sin 2\theta+\cos 2\theta}{1+\sin 2\theta-\cos 2\theta}=\cot\theta$

5. tanA + cotA = 2 cosec2A

6. $\dfrac{1-\cos A+\cos B-\cos(A+B)}{1+\cos A-\cos B-\cos(A+B)}=\tan\dfrac{A}{2}\cot\dfrac{B}{2}$

7. $\dfrac{\sec 8A-1}{\sec 4A-1}=\dfrac{\tan 8A}{\tan 2A}$

8. $\dfrac{1+\tan^2(45°-A)}{1-\tan^2(45°-A)}=\text{cosec } 2A$

9. $\dfrac{\sin^2 A-\sin^2 B}{\sin A\cos A-\sin B\cos B}=\tan(A+B)$

10. $\tan\left(\dfrac{\pi}{4}+\theta\right)-\tan\left(\dfrac{\pi}{4}-\theta\right)=2\tan 2\theta$

11. $\cot(A+15°)-\tan(A-15°)=\dfrac{4\cos 2A}{1+2\sin 2A}$

12. $\dfrac{\sin(n+1)A+2\sin nA+\sin(n-1)A}{\cos(n-1)A-\cos(n+1)A}=\cot\dfrac{A}{2}$

13. $\sin 3A+\sin 2A-\sin A=4\sin A\cos\dfrac{A}{2}\cos\dfrac{3A}{2}$

14. $\tan 2A=(\sec 2A+1)\sqrt{\sec^2 A-1}$

15. $\cos^3 2\theta+3\cos 2\theta=4(\cos^6\theta-\sin^6\theta)$

16. $1+\cos^2 2\theta=2(\cos^4\theta+\sin^4\theta)$

17. $\sec^2 A(1+\sec 2A)=2\sec 2A$

18. $\text{cosec}A-2\cot 2A\cos A=2\sin A$

19. $\cot A=\dfrac{1}{2}\left(\cot\dfrac{A}{2}-\tan\dfrac{A}{2}\right)$

20. $\dfrac{2\cos 2^n\theta+1}{2\cos\theta+1}=(2\cos\theta-1)(2\cos 2\theta-1)(2\cos 2^2\theta-1)....(2\cos 2^{n-1}\theta-1)$

(b) Trigonometrical ratios of an angle 3θ in terms of the angle θ :

(i) $\sin 3\theta=3\sin\theta-4\sin^3\theta$.

(ii) $\cos 3\theta=4\cos^3\theta-3\cos\theta$.

(iii) $\tan 3\theta=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$

Illustrations

Illustration 10: Show that sin12°.sin48°.sin54° = 1/8

Solution : L.H.S. $= \frac{1}{2}[\cos 36° - \cos 60°]\sin 54° = \frac{1}{2}\left[\cos 36° \sin 54° - \frac{1}{2}\sin 54°\right]$

$$= \frac{1}{4}[2\cos 36° \sin 54° - \sin 54°] = \frac{1}{4}[\sin 90° + \sin 18° - \sin 54°]$$

$$= \frac{1}{4}[1 - (\sin 54° - \sin 18°)] = \frac{1}{4}[1 - 2\sin 18° \cos 36°]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 18°}{\cos 18°}\cos 18° \cos 36°\right] = \frac{1}{4}\left[1 - \frac{\sin 36° \cos 36°}{\cos 18°}\right]$$

$$= \frac{1}{4}\left[1 - \frac{2\sin 36° \cos 36°}{2\cos 18°}\right] = \frac{1}{4}\left[1 - \frac{\sin 72°}{2\sin 72°}\right] = \frac{1}{4}\left[1 - \frac{1}{2}\right] = \frac{1}{8} = \text{R.H.S.}$$

Illustration 11: Prove that : tanA + tan(60° + A) + tan(120° + A) = 3tan3A

Solution : L.H.S. = tanA + tan(60° + A) + tan(120° + A)

= tanA + tan(60° + A) + tan{180° –(60° – A)}

= tanA + tan(60° + A) – tan(60° – A)

[$\because$ tan(180° – θ) = –tanθ]

$$= \tan A + \frac{\tan 60° + \tan A}{1 - \tan 60° \tan A} - \frac{\tan 60° - \tan A}{1 + \tan 60° \tan A} = \tan A + \frac{\sqrt{3} + \tan A}{1 - \sqrt{3}\tan A} - \frac{\sqrt{3} - \tan A}{1 + \sqrt{3}\tan A}$$

$$= \tan A + \frac{\sqrt{3} + \tan A + 3\tan A + \sqrt{3}\tan^2 A - \sqrt{3} + \tan A + 3\tan A - \sqrt{3}\tan^2 A}{(1 - \sqrt{3}\tan A)(1 + \sqrt{3}\tan A)}$$

$$= \tan A + \frac{8\tan A}{1 - 3\tan^2 A} = \frac{\tan A - 3\tan^3 A + 8\tan A}{1 - 3\tan^2 A}$$

$$= \frac{9\tan A - 3\tan^3 A}{1 - 3\tan^2 A} = 3\left(\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}\right) = 3\tan 3A = \text{R.H.S.}$$

Do yourself - 11

1 Prove that :

(a) $\cot\theta \cot(60° - \theta) \cot(60° + \theta) = \cot 3\theta$

(b) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(c) $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

Prove that

2. $\sin\alpha \sin(60^\circ - \alpha) \sin(60^\circ + \alpha) = \frac{1}{4}\sin 3\alpha$

3. $\cos\alpha \cos(60^\circ - \alpha) \cos(60^\circ + \alpha) = \frac{1}{4}\cos 3\alpha$

4. $\cot\alpha + \cot(60^\circ + \alpha) - \cot(60^\circ - \alpha) = 3\cot 3\alpha$.

5. $\cos 4\alpha = 1 - 8\cos^2\alpha + 8\cos^4\alpha$

6. $\sin 4A = 4\sin A\cos^3 A - 4\cos A\sin^3 A$.

7. $\cos 6\alpha = 32\cos^6\alpha - 48\cos^4\alpha + 18\cos^2\alpha - 1$.

8. If $\cos x + \sin x = a, \left(-\frac{\pi}{2} < x < -\frac{\pi}{4}\right)$, then $\cos 2x$ is equal to

(A) a^2 (B) $a\sqrt{(2-a)}$ (C) $a\sqrt{(2+a)}$ (D) $a\sqrt{(2-a^2)}$

9. If $\cos A = \frac{3}{4}$, then the value of expression $32\sin\frac{A}{2}\sin\frac{5A}{2}$ is equal to

(A) 11 (B) –11 (C) 12 (D) 4

15. TRIGONOMETRIC RATIOS OF SUB MULTIPLE ANGLES :

Since the trigonometric relations are true for all values of angle θ, they will be true if instead of θ be substitute $\frac{\theta}{2}$

(i) $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \dfrac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(ii) $\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2} = \dfrac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$

(iii) $1 + \cos\theta = 2\cos^2\frac{\theta}{2}$ **(iv)** $1 - \cos\theta = 2\sin^2\frac{\theta}{2}$

(v) $\tan\frac{\theta}{2}=\frac{1-\cos\theta}{\sin\theta}=\frac{\sin\theta}{1+\cos\theta}$

(vi) $\tan\theta=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$

(vii) $\sin\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{2}}$

(viii) $\cos\frac{\theta}{2}=\pm\sqrt{\frac{1+\cos\theta}{2}}$

(ix) $\tan\frac{\theta}{2}=\pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

(x) $2\sin\frac{\theta}{2}=\pm\sqrt{1+\sin\theta}\pm\sqrt{1-\sin\theta}$

(xi) $2\cos\frac{\theta}{2}=\pm\sqrt{1+\sin\theta}\mp\sqrt{1-\sin\theta}$

(xii) $\tan\frac{\theta}{2}=\frac{\pm\sqrt{1+\tan^2\theta}-1}{\tan\theta}$

for **(vii)** to **(xii)** , we decide the sign of ratio according to value of θ.

Illustrations

Illustration 12 : $\sin 67\frac{1}{2}^\circ+\cos 67\frac{1}{2}^\circ$ is equal to

(A) $\frac{1}{2}\sqrt{4+2\sqrt{2}}$

(B) $\frac{1}{2}\sqrt{4-2\sqrt{2}}$

(C) $\frac{1}{4}\left(\sqrt{4+2\sqrt{2}}\right)$

(D) $\frac{1}{4}\left(\sqrt{4-2\sqrt{2}}\right)$

Solution : $\sin 67\frac{1}{2}^\circ+\cos 67\frac{1}{2}^\circ=\sqrt{1+\sin 135^\circ}=\sqrt{1+\frac{1}{\sqrt{2}}}$ (using cosA + sinA

$=\sqrt{1+\sin 2A}$)

$=\frac{1}{2}\sqrt{4+2\sqrt{2}}$ **Ans.(A)**

Do yourself - 12

1 Find the value of

(a) $\sin\frac{\pi}{8}$ (b) $\cos\frac{\pi}{8}$ (c) $\tan\frac{\pi}{8}$

2. $\frac{\cos A}{1\mp\sin A}=\tan\left(45^\circ\pm\frac{A}{2}\right)$

3. $\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}=\tan\dfrac{\theta}{2}$

4. If $\sin\theta=\dfrac{1}{2}$ and $\sin\phi=\dfrac{1}{3}$, find the values of sin $(\theta+\phi)$ and $\sin(2\theta+2\phi)$.

5. If $\cos\alpha=\dfrac{11}{61}$ and $\sin\beta=\dfrac{4}{5}$, find the values of $\sin^2\dfrac{\alpha-\beta}{2}$ and $\cos^2\dfrac{\alpha+\beta}{2}$, the angles α and β being positive acute angles.

6. If $\cos\alpha=\dfrac{3}{5}$ and $\cos\beta=\dfrac{4}{5}$, find the value of $\cos\dfrac{\alpha-\beta}{2}$, the angles α and β being positive acute angles.

7. Given $\sec\theta=1\dfrac{1}{4}$, find $\tan\dfrac{\theta}{2}$ and $\tan\theta$.

8. Find the values of (a) $\sin 7\dfrac{1}{2}^\circ$ (b) $\cos 7\dfrac{1}{2}^\circ$ (c) $\tan 11\dfrac{1}{4}^\circ$

9. If $\sin\theta+\sin\phi=a$ and $\cos\theta+\cos\phi=b$, find the value of $\tan\dfrac{\theta-\phi}{2}$.

Prove that

10. $(\cos\alpha+\cos\beta)^2+(\sin\alpha-\sin\beta)^2=4\cos^2\dfrac{\alpha+\beta}{2}$.

11. $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2=4\cos^2\dfrac{\alpha-\beta}{2}$.

12. $(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=4\sin^2\dfrac{\alpha-\beta}{2}$

13. $\sec\left(\dfrac{\pi}{4}+\theta\right)\sec\left(\dfrac{\pi}{4}-\theta\right)=2\sec 2\theta$

14. $\tan\left(45^\circ+\dfrac{A}{2}\right)=\sqrt{\dfrac{1+\sin A}{1-\sin A}}=\sec A+\tan A$

15. $\sin^2\left(\dfrac{\pi}{8}+\dfrac{A}{2}\right)-\sin^2\left(\dfrac{\pi}{8}-\dfrac{A}{2}\right)=\dfrac{1}{\sqrt{2}}\sin A$

16. $\cos^2\alpha + \cos^2(\alpha + 120^o) + \cos^2(\alpha - 120^o) = \dfrac{3}{2}$

17. $\cos^4\dfrac{\pi}{8} + \cos^4\dfrac{3\pi}{8} + \cos^4\dfrac{5\pi}{8} + \cos^4\dfrac{7\pi}{8} = \dfrac{3}{2}$

18. $\sin^4\dfrac{\pi}{8} + \sin^4\dfrac{3\pi}{8} + \sin^4\dfrac{5\pi}{8} + \sin^4\dfrac{7\pi}{8} = \dfrac{3}{2}$

16. TRIGONOMETRIC RATIOS OF SOME STANDARD ANGLES :

(i) $\sin 18° = \sin\dfrac{\pi}{10} = \dfrac{\sqrt{5}-1}{4} = \cos 72° = \cos\dfrac{2\pi}{5}$

(ii) $\cos 36° = \cos\dfrac{\pi}{5} = \dfrac{\sqrt{5}+1}{4} = \sin 54° = \sin\dfrac{3\pi}{10}$

(iii) $\sin 72° = \sin\dfrac{2\pi}{5} = \dfrac{\sqrt{10+2\sqrt{5}}}{4} = \cos 18° = \cos\dfrac{\pi}{10}$

(iv) $\sin 36° = \sin\dfrac{\pi}{5} = \dfrac{\sqrt{10-2\sqrt{5}}}{4} = \cos 54° = \cos\dfrac{3\pi}{10}$

(v) $\sin 15° = \sin\dfrac{\pi}{12} = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \cos 75° = \cos\dfrac{5\pi}{12}$

(vi) $\cos 15° = \cos\dfrac{\pi}{12} = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \sin 75° = \sin\dfrac{5\pi}{12}$

(vii) $\tan 15° = \tan\dfrac{\pi}{12} = 2-\sqrt{3} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = \cot 75° = \cot\dfrac{5\pi}{12}$

(viii) $\tan 75° = \tan\dfrac{5\pi}{12} = 2+\sqrt{3} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = \cot 15° = \cot\dfrac{\pi}{12}$

(ix) $\tan(22.5°) = \tan\dfrac{\pi}{8} = \sqrt{2}-1 = \cot(67.5°) = \cot\dfrac{3\pi}{8}$

(x) $\tan(67.5°) = \tan\dfrac{3\pi}{8} = \sqrt{2}+1 = \cot(22.5°) = \cot\dfrac{\pi}{8}$

Illustrations

Illustration 13: Evaluate sin78° – sin66° – sin42° + sin6°.

Solution : The expression = (sin78° – sin42°) – (sin66° – sin6°)

= 2cos(60°) sin(18°) – 2cos36°.sin30°

$$= \sin 18° - \cos 36° = \left(\frac{\sqrt{5}-1}{4}\right) - \left(\frac{\sqrt{5}+1}{4}\right) = -\frac{1}{2}$$

Do yourself - 13

1 Find the value of

(a) $\sin\frac{\pi}{10}+\sin\frac{13\pi}{10}$ (b) $\cos^2 48° - \sin^2 12°$

Evaluate :

2. $\sin^2 72° - \sin^2 60°$

3. $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5}$

4. tan6º tan42º tan66º tan78º

5. $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15}$

6. $16\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{14\pi}{15}$

7. Two parallel chords of a circle, which are on the same side of the centre, subtend angles of 72º and 144º respectively at the centre. Prove that the perpendicular distance between the chords is half the radius of the circle.

8. In any circle prove that the chord which subtends 108º at the centre is equal to the sum of the two chords which subtend angles of 36º and 60º.

9. If $A=\cos\frac{\pi}{9}\cos\frac{4\pi}{9}\cos\frac{16\pi}{9}, B=\cos\frac{2\pi}{9}\cos\frac{8\pi}{9}\cos\frac{32\pi}{9}$, then value of $\left|\frac{1}{16A.B}\right|$ is

10. The value of $\cos\frac{\pi}{10}\cos\frac{2\pi}{10}\cos\frac{4\pi}{10}\cos\frac{8\pi}{10}\cos\frac{16\pi}{10}$ is

(A) $\frac{\sqrt{10+2\sqrt{5}}}{64}$ (B) $-\frac{\cos(\pi/10)}{16}$ (C) $\frac{\cos(\pi/10)}{16}$ (D) $-\frac{\sqrt{10+2\sqrt{5}}}{16}$

17. CONDITIONAL TRIGONOMETRIC IDENTITIES :

If A + B + C = 180°, then

(i) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

(ii) $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$

(iii) $\tan\frac{A}{2}\tan\frac{B}{2}+\tan\frac{B}{2}\tan\frac{C}{2}+\tan\frac{C}{2}\tan\frac{A}{2}=1$

(iv) $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

(v) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

(vi) $\cos 2A + \cos 2B + \cos 2C = -1-4 \cos A \cos B \cos C$

(vii) $\sin A + \sin B + \sin C = 4 \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

(viii) $\cos A + \cos B + \cos C = 1 + 4 \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

Illustrations

Illustration 14: In any triangle ABC, $\sin A - \cos B = \cos C$, then angle B is (Where $\angle B > \angle C$)

(A) $\pi/2$ (B) $\pi/3$ (C) $\pi/4$ (D) $\pi/6$

Solution : We have , $\sin A - \cos B = \cos C$

$\sin A = \cos B + \cos C$

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2}=2\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2}=2\cos\left(\frac{\pi-A}{2}\right)\cos\left(\frac{B-C}{2}\right) \quad \because \; A+B+C=\pi$$

$$\Rightarrow \quad 2\sin\frac{A}{2}\cos\frac{A}{2}=2\sin\frac{A}{2}\cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \quad \cos\frac{A}{2}=\cos\frac{B-C}{2} \quad \text{or} \quad A = B - C \; ; \; \text{But} \; A+B+C=\pi$$

Therefore $2B = \pi \Rightarrow B = \pi/2$ **Ans.(A)**

Illustration 15: If $A + B + C = \frac{3\pi}{2}$, then cos 2A + cos 2B + cos2C is equal to-

(A) 1 – 4cosA cosB cosC (B) 4 sinA sin B sinC

(C) 1 + 2cosA cosB cosC (D) 1 – 4 sinA sinB sinC

Solution : $\cos 2A + \cos 2B + \cos 2C = 2\cos(A+B)\cos(A-B) + \cos 2C$

$$= 2\cos\left(\frac{3\pi}{2} - C\right)\cos(A-B) + \cos 2C \quad \because A + B + C = \frac{3\pi}{2}$$

$$= -2\sin C\cos(A-B) + 1 - 2\sin^2 C = 1 - 2\sin C$$

$$[\cos(A-B) + \sin C]$$

$$= 1 - 2\sin C\left[\cos(A-B) + \sin\left(\frac{3\pi}{2} - (A+B)\right)\right]$$

$$= 1 - 2\sin C[\cos(A-B) - \cos(A+B)] = 1 - 4\sin A\sin B\sin C$$

Ans.(D)

Do yourself - 14

1. If ABCD is a cyclic quadrilateral, then find the value of sinA + sinB – sinC – sinD

2. If $A + B + C = \frac{\pi}{2}$, then find the value of tanA tanB + tanBtanC + tanC tanA

If A + B + C = 180º, then prove that

3. $\sin 2A + \sin 2B - \sin 2C = 4\cos A\cos B\sin C$

4. $\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A\sin B\cos C$

5. $\sin A + \sin B - \sin C = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

6. $\sin^2 A + \sin^2 B - \sin^2 C = 2\sin A\sin B\cos C$

7. $\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C$.

8. $\cos^2A + \cos^2B - \cos^2C = 1 - 2\sin A \sin B \cos C$

9. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}=1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

10. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$

11. $\sin(B + 2C) + \sin(C + 2A) + \sin(A + 2B) = 4\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$.

12. $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}-1=4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}$

13. $\frac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

14. $\sin(B + C - A) + \sin(C + A - B) + \sin(A + B - C) = 4\sin A \sin B \sin C$

15. If $x + y + z = xyz$ prove that

(a) $\frac{3x-x^3}{1-3x^2}+\frac{3y-y^3}{1-3y^2}+\frac{3z-z^3}{1-3z^2}=\frac{3x-x^3}{1-3x^2}\cdot\frac{3y-y^3}{1-3y^2}\cdot\frac{3z-z^3}{1-3z^2}$

(b) $x(1 - y^2)(1 - z^2) + y(1 - z^2)(1 - x^2) + z(1 - x^2)(1- y^2) = 4xyz$

18. MAXIMUM & MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS :

(i) $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum values are $\sqrt{a^2+b^2}, -\sqrt{a^2+b^2}$ respectively.

(ii) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta = 2ab$ where $a, b > 0$

(iii) $-\sqrt{a^2+b^2+2ab\cos(\alpha-\beta)} \le a\cos(\alpha+\theta)+b\cos(\beta+\theta) \le \sqrt{a^2+b^2+2ab\cos(\alpha-\beta)}$ where α and β areknown angles.

(iv) In case a quadratic in $\sin\theta$ & $\cos\theta$ is given then the maximum or minimum values can be obtained by making perfect square.

Illustrations

Illustration 16: Prove that : $-4 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 10$, for all values of θ.

Solution : We have, $5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) = 5\cos\theta + 3\cos\theta\cos\frac{\pi}{3} - 3\sin\theta\sin\frac{\pi}{3}$

$= \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta$

Since, $-\sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le \sqrt{\left(\frac{13}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2}$

$\Rightarrow \quad -7 \le \frac{13}{2}\cos\theta - \frac{3\sqrt{3}}{2}\sin\theta \le 7$

$\Rightarrow \quad -7 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) \le 7$ for all θ.

$\Rightarrow \quad -7 + 3 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 7 + 3$ for all θ.

$\Rightarrow \quad -4 \le 5\cos\theta + 3\cos\left(\theta + \frac{\pi}{3}\right) + 3 \le 10$ for all θ.

Illustration 17: Find the maximum value of $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$ -

(A) 1 (B) 2 (C) 3 (D) 4

Solution : We have $1 + \sin\left(\frac{\pi}{4} + \theta\right) + 2\cos\left(\frac{\pi}{4} - \theta\right)$

$= 1 + \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta) + \sqrt{2}(\cos\theta + \sin\theta) = 1 +$

$\left(\frac{1}{\sqrt{2}} + \sqrt{2}\right)(\cos\theta + \sin\theta)$

$= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) . \sqrt{2}\cos\left(\theta - \frac{\pi}{4}\right)$

$\therefore$ maximum value $= 1 + \left(\frac{1}{\sqrt{2}} + \sqrt{2}\right) . \sqrt{2} = 4$ **Ans. (D)**

Do yourself - 15

1. Find maximum and minimum value of $5\cos\theta + 3\sin\left(\theta + \frac{\pi}{6}\right)$ for all real values of θ.

2. Find the minimum value of $\cos\theta + \cos 2\theta$ for all real values of θ.

3. Find maximum and minimum value of $\cos^2\theta - 6\sin\theta\cos\theta + 3\sin^2\theta + 2$.

4. Find the maximum and minimum values of

(i) $\cos 2x + \cos^2 x$ (ii) $\cos^2\left(\frac{\pi}{4} + x\right) + (\sin x - \cos x)^2$

5. If $\alpha + \beta = 90^\circ$, then find the maximum value of $\sin\alpha.\sin\beta$.

6. Find the maximum and minimum value of $1 + 2\sin x + 3\cos^2 x$

7. Find the minimum value of $4\sec^2 x + 9\text{cosec}^2 x$

8. Find the maximum and minimum value of $9\cos^2 x + 48\sin x.\cos x - 5\sin^2 x - 2$

9. Find the maximum and minimum value of $2\sin\left(\theta + \frac{\pi}{6}\right) + \sqrt{3}\cos\left(\theta - \frac{\pi}{6}\right)$

10. Find minimum value of (i) $3\sin^2 x + 27\text{cosec}^2 x$ (ii) $27\sin^2 x + 3\text{cosec}^2 x$

19. IMPORTANT RESULTS :

(i) $\sin\theta\sin(60^\circ - \theta)\sin(60^\circ + \theta) = \frac{1}{4}\sin 3\theta$

(ii) $\cos\theta.\cos(60^\circ - \theta)\cos(60^\circ + \theta) = \frac{1}{4}\cos 3\theta$

(iii) $\tan\theta\tan(60^\circ - \theta)\tan(60^\circ + \theta) = \tan 3\theta$

(iv) $\cot\theta\cot(60^\circ - \theta)\cot(60^\circ + \theta) = \cot 3\theta$

(v) (a) $\sin^2\theta + \sin^2(60^\circ + \theta) + \sin^2(60^\circ - \theta) = \frac{3}{2}$

(b) $\cos^2\theta + \cos^2(60^\circ + \theta) + \cos^2(60^\circ - \theta) = \frac{3}{2}$

(c) $\tan\theta + \tan(60^\circ + \theta) + \tan(120^\circ + \theta) = 3\tan 3\theta$

(vi) (a) If $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, then $A + B + C = n\pi, n \in I$

(b) If $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$, then $A + B + C = (2n + 1)\frac{\pi}{2}, n \in I$

(vii) $\cos\theta \cos 2\theta \cos 4\theta \ldots \cos(2^{n-1}\theta) = \frac{\sin(2^n\theta)}{2^n \sin\theta}$

(viii) (a) $\cot A - \tan A = 2\cot 2A$ (b) $\cot A + \tan A = 2\operatorname{cosec} 2A$

(ix) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \ldots \sin(\alpha + \overline{n-1}\,\beta) = \frac{\sin\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

(x) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + \overline{n-1}\,\beta) = \frac{\cos\left\{\alpha + \left(\frac{n-1}{2}\right)\beta\right\}\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}$

Do yourself - 16

1 Evaluate $\sin\frac{\pi}{n} + \sin\frac{3\pi}{n} + \sin\frac{5\pi}{n} + \ldots\ldots\ldots$ to n terms

2. Prove that : $\sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin(2n-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$

3. Find the average of $\sin 2^\circ, \sin 4^\circ, \sin 6^\circ, \ldots, \sin 180^\circ$

4. Prove that : $\cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11} = \frac{1}{2}$

5. Find sum of the following series :

(a) $\cos\frac{\pi}{2n+1} + \cos\frac{3\pi}{2n+1} + \cos\frac{5\pi}{2n+1} + \ldots$ up to n terms.

(b) $\sin 2\alpha + \sin 3\alpha + \sin 4\alpha + \ldots + \sin n\alpha$, where $(n+2)\alpha = 2\pi$

6. If $S = \cos^2\frac{\pi}{n} + \cos^2\frac{2\pi}{n} + \ldots + \cos^2\frac{(n-1)\pi}{n}$, then S equals (where $n \geq 2$, $n \in N$)

(A) $\frac{n}{2(n+1)}$ (B) $\frac{1}{2(n-1)}$ (C) $\frac{n-2}{2}$ (D) $\frac{n}{2}$

Illustrations

Miscellaneous Illustration :

Illustration 18: Prove that

$\tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha + 2^n\cot 2^n\alpha = \cot\alpha$

Solution : We know $\tan\theta = \cot\theta - 2\cot 2\theta$ .....(i)

Putting $\theta = \alpha, 2\alpha, 2^2\alpha, \ldots\ldots\ldots\ldots$ in (i), we get

$\tan\alpha = (\cot\alpha - 2\cot 2\alpha)$

$2(\tan 2\alpha) = 2(\cot 2\alpha - 2\cot 2^2\alpha)$

$2^2(\tan 2^2\alpha) = 2^2(\cot 2^2\alpha - 2\cot 2^3\alpha)$

....

$2^{n-1}(\tan 2^{n-1}\alpha) = 2^{n-1}(\cot 2^{n-1}\alpha - 2\cot 2^n\alpha)$

Adding,

$\tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha = \cot\alpha - 2^n\cot 2^n\alpha$

$\therefore$ $\tan\alpha + 2\tan 2\alpha + 2^2\tan^2\alpha + \ldots\ldots + 2^{n-1}\tan 2^{n-1}\alpha + 2^n\cot 2^n\alpha = \cot\alpha$

Illustration 19: If A,B,C and D are angles of a quadrilateral and

$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2} = \frac{1}{4}$, prove that

$A = B = C = D = \pi/2.$

Solution : $\left(2\sin\frac{A}{2}\sin\frac{B}{2}\right)\left(2\sin\frac{C}{2}\sin\frac{D}{2}\right) = 1$

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right) - \cos\left(\frac{C+D}{2}\right)\right\} = 1$$

Since, A + B = 2π – (C + D), the above equation becomes,

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right\}\left\{\cos\left(\frac{C-D}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right\}=1$$

$$\Rightarrow \cos^2\left(\frac{A+B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right\}+1-\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{C-D}{2}\right)=0$$

This is a quadratic equation in $\cos\left(\frac{A+B}{2}\right)$ which has real roots.

$$\Rightarrow \left\{\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{C-D}{2}\right)\right\}^2-4\left\{1-\cos\left(\frac{A-B}{2}\right).\cos\left(\frac{C-D}{2}\right)\right\}\geq 0$$

$$\left(\cos\frac{A-B}{2}+\cos\frac{C-D}{2}\right)^2\geq 4$$

$\Rightarrow \quad \cos\frac{A-B}{2}+\cos\frac{C-D}{2}\geq 2$, Now both $\cos\frac{A-B}{2}$ and $\cos\frac{C-D}{2}\leq 1$

$$\Rightarrow \quad \cos\frac{A-B}{2}=1\ \&\ \cos\frac{C-D}{2}=1$$

$$\Rightarrow \quad \frac{A-B}{2}=0=\frac{C-D}{2}$$

⇒ A = B, C = D.

Similarly A = C, B = D ⇒ A = B = C = D = π/2

ANSWER KEY

Do yourself-1

1. 10π cm **2.** $36 + 6\pi$ **3.** (a) $\left(\frac{3061}{21600}\pi\right)$ (b) $\left(\frac{257}{4320}\pi\right)$ (c) $\frac{5587}{108000}\pi$

4. $\frac{22\pi}{3}+8\sqrt{3}$ **5.** 68° 43' 38"

Do yourself-2

1. $\frac{3}{5},\frac{4}{5},\frac{5}{3}$ **2.** 2 **23.** $\frac{2xy}{x^2+y^2},\frac{2xy}{x^2-y^2}$ **27.** $\left(\frac{1}{\tan^4 A}-\tan^4 A\right)$

Do yourself-3

1. $\left\{\frac{11}{60},\frac{60}{61},\frac{61}{60}\right\};\left\{\frac{-11}{60},\frac{-60}{61},\frac{-61}{60}\right\}$ **2.** $\left\{\frac{3}{5},\frac{4}{3}\right\};\left\{\frac{-3}{5},\frac{-4}{3}\right\}$ **3.** $\left\{\frac{3}{4}\right\}$

4. $\left\{\frac{8}{17},\frac{17}{8}\right\};\left\{\frac{-8}{17},\frac{-17}{8}\right\}$ **5.** $1,\frac{3}{5}$ **6.** $\left\{\frac{3}{5},\frac{5}{13}\right\}$ **7.** $\frac{5}{13}$ **8.** $\left\{\frac{12}{13}\right\}$

9. $\left\{\pm1,\pm\frac{1}{\sqrt{3}}\right\}$ **10.** $\left\{\frac{1}{2},-\frac{1}{3}\right\}$ **11.** $\pm\frac{1}{\sqrt{2}}$

12. $\left\{1\pm\sqrt{2}\right\}$ **13.** $\pm\frac{2x(x+1)}{2x^2+2x+1};\pm\frac{2x+1}{2x^2+2x+1}$

Do yourself-4

2. (a) $\frac{3}{2}$ (b) $4\frac{1}{3}$ (c) $\frac{1+\sqrt{3}}{4}$ (d) $-\frac{\sqrt{3}-1}{2\sqrt{2}}$

Do yourself-5

1. **(i)** 8 **2.** (a) 1 (b) 0

3. (a) $-\left(\frac{\sqrt{3}+1}{2}\right),-\frac{4}{\sqrt{3}}$ (b) $\left(\sqrt{2},-2\right)$ (c) $\frac{\sqrt{3}+1}{2},-\frac{4}{\sqrt{3}}$ (d) 0,2

4. (a) –cos(25°) (b) –cos(28°) (c) cot 25° (d) cot(26°)

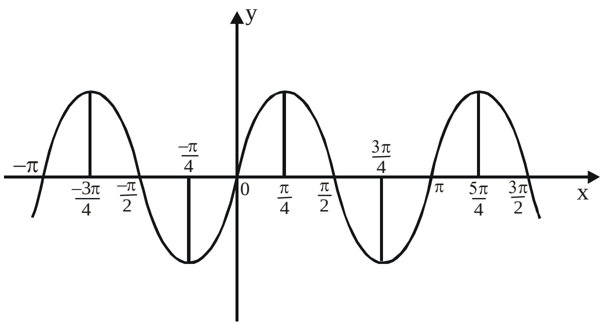
5. (a) negative (b) positive (c) zero

6. (a) positive (b) positive (c) negative **7.** $-\frac{1}{\sqrt{3}}$ and $\frac{\sqrt{2}}{\sqrt{3}}$; $\frac{1}{\sqrt{3}}$ and $-\frac{\sqrt{2}}{\sqrt{3}}$

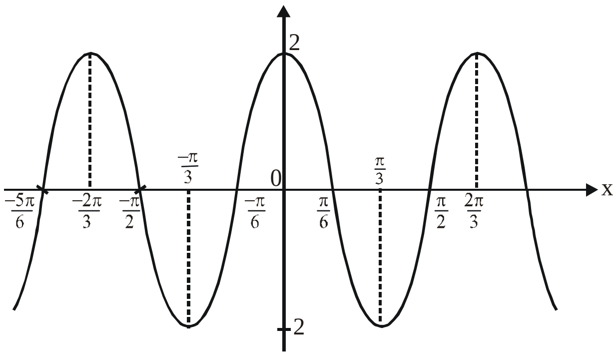
Do yourself-6

3. B

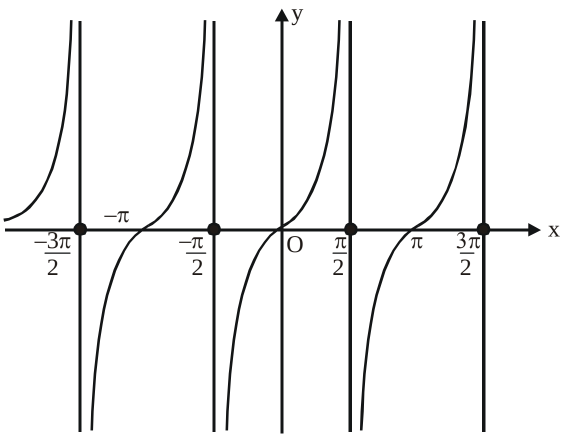
4. (a)



(b)



(c)



5. 2

Do yourself-7

1. (a) $\frac{187}{205}$ (b) $\frac{-133}{205}$ (c) $\frac{-84}{205}$ (d) $\frac{156}{205}$

3. $\sin(\alpha-\beta)=-\frac{133}{205}\text{ or }\frac{187}{205}$; $\cos(\alpha+\beta)=-\frac{84}{205}\text{ or }\frac{156}{205}$

4. $\sin(\alpha+\beta)=\frac{220}{221}\text{ or }\frac{140}{221}$; $\cos(\alpha-\beta)=\pm\frac{171}{221}\text{ or }\pm\frac{21}{221}$; $\sin(\alpha+\beta)=\frac{220}{221}\text{ or }\frac{140}{221}$; $\tan(\alpha+\beta)=\pm\frac{220}{21}\text{ or }\pm\frac{140}{171}$

13. 3 and $\frac{9}{13}$

Do yourself-8

1. $\frac{1}{\sqrt{3}}$ **2.** $\cos2\theta-\cos12\theta$ **3.** $\cos12^o-\cos120^o$

Do yourself-11

8. D **9.** A

Do yourself-12

1 (a) $\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}$ (b) $\sqrt{\frac{\sqrt{2}+1}{2\sqrt{2}}}$ (c) $\sqrt{2}-1$ **4.** $\frac{\pm2\sqrt{2}\pm\sqrt{3}}{6};\frac{\pm7\sqrt{3}\pm4\sqrt{2}}{18}$

5. $\frac{16}{305};\frac{49}{305}$ **6.** $\frac{7}{5\sqrt{2}}$ **7.** $\pm\frac{1}{3};\pm\frac{3}{4}$

8. (a) $\frac{\sqrt{4-\sqrt{2}-\sqrt{6}}}{2\sqrt{2}}$; (b) $\frac{\sqrt{4+\sqrt{2}+\sqrt{6}}}{2\sqrt{2}}$; (c) $\sqrt{4+2\sqrt{2}}-(\sqrt{2}+1)$

9. $\pm\sqrt{\frac{4-a^2-b^2}{a^2+b^2}}$

Do yourself-13

1. (a) $-\frac{1}{2}$ (b) $\frac{\sqrt{5}+1}{8}$ **2.** $\frac{\sqrt{5}-1}{8}$ **3.** $\frac{5}{16}$ **4.** 1 **5.** $\frac{1}{2^7}$ **6.** 1

9. 4 **10.** B

Do yourself-14

1. 0 **2.** 1

Do yourself-15

1. 7 & –7 **2.** $-\frac{9}{8}$ **3.** $4+\sqrt{10}$ & $4-\sqrt{10}$

4. (i) 2, – 1 (ii) 3,0 **5.** $\frac{1}{2}$ **6.** $\left\{\frac{13}{3},-1\right\}$

7. 25 **8.** {25,–25} **9.** $\left\{\sqrt{13},-\sqrt{13}\right\}$

10. (i) 30 (ii) 18

Do yourself-16

1. 0 **3.** $\frac{\cot 1^\circ}{90}$ **5.** (a) $\frac{1}{2}$ (b) 0 **6.** C

EXERCISE (O-1)

1. If $\sin x + \sin^2 x = 1$, then the value of $\cos^2 x + \cos^4 x$ is -

(A) 0 (B) 2 (C) 1 (D) 3

2. $2(\sin^6\theta + \cos^6\theta) - 3(\sin^4\theta + \cos^4\theta) + 1$ is equal to -

(A) 2 (B) 0 (C) 4 (D) 6

3. If $x = y\cos\frac{2\pi}{3} = z\cos\frac{4\pi}{3}]$ then $xy + yz + zx =$

(A) –1 (B) 0 (C) 1 (D) 2

4. The value of $\sin 10^\circ + \sin 20^\circ + \sin 30^\circ + ... + \sin 360^\circ$ is equal to -

(A) 0 (B) 1 (C) $\sqrt{3}$ (D) 2

5. If $\cos(\alpha + \beta) + \sin(\alpha - \beta) = 0$ and $2010\tan\beta + 1 = 0$, then $\tan\alpha$ is equal to

(A) 1 (B) –1 (C) 2010 (D) $\frac{1}{2010}$

6. If $\cos x + \cos y + \cos\alpha = 0$ and $\sin x + \sin y + \sin\alpha = 0$, then $\cot\left(\frac{x+y}{2}\right) =$

(A) $\sin\alpha$ (B) $\cos\alpha$ (C) $\cot\alpha$ (D) $2\sin\alpha$

7. $\frac{\sin(A-C) + 2\sin A + \sin(A+C)}{\sin(B-C) + 2\sin B + \sin(B+C)}$ is equal to -

(A) $\tan A$ (B) $\frac{\sin A}{\sin B}$ (C) $\frac{\cos A}{\cos B}$ (D) $\frac{\sin C}{\cos B}$

8. The expression $\frac{\sin 8\theta\cos\theta - \sin 6\theta\cos 3\theta}{\cos 2\theta\cos\theta - \sin 3\theta\sin 4\theta}$ is equals -

(A) $\tan\theta$ (B) $\tan 2\theta$ (C) $\sin 2\theta$ (D) $\cos 2\theta$

9. $\frac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} =$

(A) $\frac{1}{2}\tan\theta$ (B) $\frac{1}{2}\cot\theta$ (C) $\tan\theta$ (D) $\cot\theta$

10. If $\tan x + \tan y = 25$ and $\cot x + \cot y = 30$, then the value of $\tan(x + y)$ is

(A) 150 (B) 200 (C) 250 (D) 100

11. If $A = \tan 6^\circ \tan 42^\circ$ and $B = \cot 66^\circ \cot 78^\circ$, then -

(A) $A = 2B$ (B) $A = 1/3B$ (C) $A = B$ (D) $3A = 2B$

12. In a right angled triangle the hypotenuse is $2\sqrt{2}$ times the perpendicular drawn from the opposite vertex. Then the other acute angles of the triangle are

(A) $\frac{\pi}{3}$ and $\frac{\pi}{6}$ (B) $\frac{\pi}{8}$ and $\frac{3\pi}{8}$ (C) $\frac{\pi}{4}$ and $\frac{\pi}{4}$ (D) $\frac{\pi}{5}$ and $\frac{3\pi}{10}$

13. If $\tan\alpha = (1+2^{-x})^{-1}$, $\tan\beta = (1+2^{x+1})^{-1}$, then $\alpha + \beta =$

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

14. If 3 sin α = 5 sin β, then $\dfrac{\tan\dfrac{\alpha+\beta}{2}}{\tan\dfrac{\alpha-\beta}{2}} =$

(A) 1 (B) 2 (C) 3 (D) 4

15. If $\cos\alpha = \dfrac{2\cos\beta - 1}{2 - \cos\beta}$ then $\tan\dfrac{\alpha}{2} \cdot \cot\dfrac{\beta}{2}$ has the value equal to {where $\alpha, \beta \in (0, \pi)$}

(A) 2 (B) $\sqrt{2}$ (C) 3 (D) $\sqrt{3}$

16. If $\tan B = \dfrac{n\sin A\cos A}{1 - n\cos^2 A}$ then $\tan(A + B)$ equals

(A) $\dfrac{\sin A}{(1-n)\cos A}$ (B) $\dfrac{(n-1)\cos A}{\sin A}$ (C) $\dfrac{\sin A}{(n-1)\cos A}$ (D) $\dfrac{\sin A}{(n+1)\cos A}$

17. The value of $\operatorname{cosec}\dfrac{\pi}{18} - \sqrt{3}\sec\dfrac{\pi}{18}$ is a

(A) surd (B) rational which is not integral

(C) negative integer (D) natural number

18. The value of $\cot x + \cot(60^\circ + x) + \cot(120^\circ + x)$ is equal to :

(A) $\cot 3x$ (B) $\tan 3x$

(C) $3\tan 3x$ (D) $\dfrac{3 - 9\tan^2 x}{3\tan x - \tan^3 x}$

19. If $\dfrac{5\pi}{2} < x < 3\pi$, then the value of the expression $\dfrac{\sqrt{1-\sin x} + \sqrt{1+\sin x}}{\sqrt{1-\sin x} - \sqrt{1+\sin x}}$ is

(A) $-\cot\dfrac{x}{2}$ (B) $\cot\dfrac{x}{2}$ (C) $\tan\dfrac{x}{2}$ (D) $-\tan\dfrac{x}{2}$

20. The value of $\cot 7\dfrac{1}{2}^\circ + \tan 67\dfrac{1}{2}^\circ - \cot 67\dfrac{1}{2}^\circ - \tan 7\dfrac{1}{2}^\circ$ is :

(A) a rational number (B) irrational number (C) $2(3 + 2\sqrt{3})$ (D) $2(3 - \sqrt{3})$

21. If $x + y = 3 - \cos 4\theta$ and $x - y = 4\sin 2\theta$ then

(A) $x^4 + y^4 = 9$ (B) $\sqrt{x} + \sqrt{y} = 16$

(C) $x^3 + y^3 = 2(x^2 + y^2)$ (D) $\sqrt{x} + \sqrt{y} = 2$

22. If $A = \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}$ and $B = \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7}$ then $\sqrt{A^2+B^2}$ is equal to

(A) 1 (B) $\sqrt{2}$ (C) 2 (D) $\sqrt{3}$

23. If tan A + tan B + tan C = tan A. tan B. tan C, then -

(A) A,B,C must be angles of a triangle

(B) the sum of any two of A,B,C is equal to the third

(C) A+B+C must be n integral multiple of π

(D) None of these

24. If $f(x) = \frac{\sin 3x}{\sin x}$, $x \neq n\pi$, then the range of values of f(x) for real values of x is -

(A) [–1,3] (B) $(-\infty,-1]$ (C) $(3, +\infty)$ (D) [–1,3)

25. Maximum and minimum value of $2\sin^2\theta - 3\sin\theta + 2$ is -

(A) $\frac{1}{4}, -\frac{7}{4}$ (B) $\frac{1}{4}, \frac{21}{4}$ (C) $\frac{21}{4}, -\frac{3}{4}$ (D) $7, \frac{7}{8}$

26. For $\theta \in (0, \pi/2)$, the maximum value of $\sin\left(\theta+\frac{\pi}{6}\right)+\cos\left(\theta+\frac{\pi}{6}\right)$ is attained at θ =

(A) $\frac{\pi}{12}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

27. Minimum value of the expression $\cos^2\theta - (6\sin\theta\cos\theta) + 3\sin^2\theta + 2$, is -

(A) $4+\sqrt{10}$ (B) $4-\sqrt{10}$ (C) 0 (D) 4

28. The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is :-

(A) $\frac{1}{16}$ (B) $\frac{1}{8}$ (C) $\frac{1}{2}$ (D) 1

29. The exact value of $\frac{96\sin 80°\sin 65°\sin 35°}{\sin 20° + \sin 50° + \sin 110°}$ is equal to

(A) 12 (B) 24 (C) –12 (D) 48

30. If *m* and *n* are positive integers satisfying

$1 + \cos 2\theta + \cos 4\theta + \cos 6\theta + \cos 8\theta + \cos 10\theta = \frac{\cos m\theta \cdot \sin n\theta}{\sin\theta}$ then $(m+n)$ is equal to

(A) 9 (B) 10 (C) 11 (D) 12

EXERCISE (O-2)

MULTIPLE OBJECTIVE TYPE :

1. If cosx = tanx, then which of the following is/are true ?

(A) $\frac{1}{\sin x}+\cos^4 x=1$

(B) $\frac{1}{\sin x}+\cos^4 x=2$

(C) $\cos^4x + \cos^2x = 1$

(D) $\cos^4x + \cos^2x = 2$

2. If $\tan A=-\frac{1}{2}$ and $\tan B=-\frac{1}{3}$, then A + B =

(A) $\frac{\pi}{4}$

(B) $\frac{3\pi}{4}$

(C) $\frac{5\pi}{4}$

(D) $\frac{7\pi}{4}$

3. If $\cos(A-B)=\frac{3}{5}$ and tanA tanB = 2, then which of the following is/are correct ?

(A) $\cos A \cos B = -\frac{1}{5}$

(B) $\sin A\sin B=\frac{2}{5}$

(C) $\cos(A+B)=-\frac{1}{5}$

(D) $\sin A\cos B=\frac{4}{5}$

4. If $\sin t+\cos t=\frac{1}{5}$, then $\tan\frac{t}{2}$ is equal to

(A) –1

(B) $-\frac{1}{3}$

(C) 2

(D) $-\frac{1}{6}$

5. If $\tan^2\alpha + 2\tan\alpha.\tan2\beta = \tan^2\beta + 2\tan\beta.\tan2\alpha$, then

(A) $\tan^2\alpha + 2\tan\alpha.\tan2\beta = 0$

(B) $\tan\alpha + \tan\beta = 0$

(C) $\tan^2\beta + 2\tan\beta.\tan2\alpha = 1$

(D) $\tan\alpha = \tan\beta$

6. If $3\sin\beta = \sin(2\alpha + \beta)$, then $\tan(\alpha + \beta) - 2\tan\alpha$ is

(A) independent of α

(B) independent of β

(C) dependent of both α and β

(D) independent of α but dependent of β

7. If $L = \cos^2 84^\circ + \cos^2 36^\circ + \cos 36^\circ \cos 84^\circ$

$M = \cot 73^\circ \cot 47^\circ \cot 13^\circ$

$N = 4 \sin 156^\circ \sin 84^\circ \sin 36^\circ$, then which of the following option(s) is(are) correct?

(A) L < 1

(B) M > tan 2

(C) $N>\sin\frac{\pi}{4}$

(D) 0 < LMN

8. If $\frac{\cos 3x}{\cos x}=\frac{1}{3}$ for some angle x, $0\le x\le\frac{\pi}{2}$, which of the following is/are true ?

(A) $\frac{\sin 3x}{\sin x}=\frac{7}{3}$

(B) $\cos 2x=\frac{2}{3}$

(C) $\tan x=\frac{1}{\sqrt{5}}$

(D) $\sin 2x=\frac{2\sqrt{5}}{6}$

9. If $\alpha+\beta+\gamma=2\pi$, then

(A) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(B) $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1$

(C) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$

(D) $\tan\frac{\alpha}{4}\tan\frac{\beta}{4}+\tan\frac{\beta}{4}\tan\frac{\gamma}{4}+\tan\frac{\gamma}{4}\tan\frac{\alpha}{4}=1$

10. If x + y = z, then $\cos^2x+\cos^2y+\cos^2z-2\cos x\cos y\cos z$ is equal to

(A) $\cos^2z$ (B) $\sin^2z$ (C) $\cos(x+y-z)$ (D) 1

11. If A,B,C are angles of a triangle ABC and tanA tanC = 3; tan B tanC = 6 then, which is (are) correct?

(A) $A=\frac{\pi}{4}$

(B) tanA tanB = 2

(C) $\frac{\tan A}{\tan C}=3$

(D) tanB = 2 tanA

12. Which of the following is/are true ?

(A) $\sin\left(\frac{\pi}{4}+\theta\right)\sin\left(\frac{\pi}{4}-\theta\right)=\frac{\cos 2\theta}{2}$

(B) In a ΔABC, if tanA = 2, tanB = 3, then tanC = 1

(C) Minimum value of $4\tan^2\theta+\cot^2\theta$ is 4 (wherever defined)

(D) Range of $3\sin^2\theta+4\sin\theta\cos\theta+5\cos^2\theta$ is $\left[4-\sqrt{5},\,4+\sqrt{5}\right]$

13. Let $f_n(\theta)=\sum_{n=0}^{n}\frac{1}{4^n}\sin^4(2^n\theta)$. Then which of the following alternative(s) is/are correct?

(A) $f_2\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$ (B) $f_3\left(\frac{\pi}{8}\right)=\frac{2+\sqrt{2}}{4}$ (C) $f_4\left(\frac{3\pi}{2}\right)=1$ (D) $f_5(\pi)=0$

14. Let $y=\frac{\cos x+\cos 2x+\cos 3x+\cos 4x+\cos 5x+\cos 6x+\cos 7x}{\sin x+\sin 2x+\sin 3x+\sin 4x+\sin 5x+\sin 6x+\sin 7x}$, then which of the following hold good?

(A) The value of y when $x = \pi/8$ is not defined.

(B) The value of y when $x = \pi/16$ is 1.

(C) The value of y when $x = \pi/32$ is $\sqrt{2}-1$.

(D) The value of y when $x = \pi/48$ is $2+\sqrt{3}$.

Paragraph for Question Nos. 15 to 17

Consider the polynomial $P(x) = (x - \cos 36°)(x - \cos 84°)(x - \cos 156°)$

15. The coefficient of x^2 is

(A) 0 (B) 1 (C) $-\frac{1}{2}$ (D) $\frac{\sqrt{5}-1}{2}$

16. The coefficient of x is

(A) $\frac{3}{2}$ (B) $-\frac{3}{2}$ (C) $-\frac{3}{4}$ (D) zero

17. The absolute term in P(x) has the value equal to

(A) $\frac{\sqrt{5}-1}{4}$ (B) $\frac{\sqrt{5}-1}{16}$ (C) $\frac{\sqrt{5}+1}{16}$ (D) $\frac{1}{16}$

Paragraph for Question 18 to 19

Let a,b,c are respectively the sines and p,q,r are respectively the cosines of $\alpha, \alpha+\frac{2\pi}{3}$ and $\alpha+\frac{4\pi}{3}$, then

18. The value of (a + b + c) + (ab + bc + ca) is

(A) 0 (B) $\frac{3}{4}$ (C) 1 (D) $-\frac{3}{4}$

19. The value of (qc – rb) is-

(A) 0 (B) $-\frac{\sqrt{3}}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) depends on α

INTEGER TYPE

20. If acute angle A = 3B and $\sin A = \frac{4}{5}$ then $\left|\frac{3\sec B - 4\operatorname{cosec} B}{2}\right|$ is

21. If the value of $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} + \cos\frac{7\pi}{7} = -\frac{\ell}{2}$. Find the value of ℓ.

22. If $\cot(\theta - \alpha)$, $3\cot\theta$, $\cot(\theta + \alpha)$ are in AP (where, $\theta \neq \frac{n\pi}{2}, \alpha \neq k\pi, n, k \in I$), then $\frac{2\sin^2\theta}{\sin^2\alpha}$ is equal to

23. If $k_1 = \tan 27\theta - \tan\theta$ and $k_2 = \frac{\sin\theta}{\cos 3\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin 9\theta}{\cos 27\theta}$ then $\left(\frac{k_1}{k_2}\right)$ is equal to

24. The value of the expression $\frac{1 - 4\sin 10° \sin 70°}{2\sin 10°}$ is

MATRIX MATCH TYPE

25. In the following matrix match **Column-I** has some quantities and **Column-II** has some comments or other quantities

Match the each element in **Column-I** with corresponding element(s) in **Column-II**

Column-I	**Column-II**
(A) The value of $\left|4\left(2\cos^3\frac{\pi}{7} - \cos^2\frac{\pi}{7} - \cos\frac{\pi}{7}\right)\right|$ is	(P) 4
(B) If A + B + C = π and cosA = cosB. cosC then tanB. tanC has the value equal to	(Q) 8
(C) $4\left(\frac{\cos 20° + 8\sin 70° \sin 50° \sin 10°}{\sin^2 80°}\right)$ is equal to	(R) 2
(D) The maximum value of $12\sin\theta - 9\sin^2\theta$ is	(S) 1
	(T) 6

EXERCISE (S-1)

1. If $X=\sin\left(\theta+\frac{7\pi}{12}\right)+\sin\left(\theta-\frac{\pi}{12}\right)+\sin\left(\theta+\frac{3\pi}{12}\right), Y=\cos\left(\theta+\frac{7\pi}{12}\right)+\cos\left(\theta-\frac{\pi}{12}\right)+\cos\left(\theta+\frac{3\pi}{12}\right)$,

then prove that $\frac{X}{Y}-\frac{Y}{X}=2\tan 2\theta$.

2. Prove that :

(a) $\tan 20° . \tan 40° . \tan 60° . \tan 80° = 3$

(b) $\sin^4\frac{\pi}{16}+\sin^4\frac{3\pi}{16}+\sin^4\frac{5\pi}{16}+\sin^4\frac{7\pi}{16}=\frac{3}{2}$

(c) $\cos^2\alpha + \cos^2(\alpha+\beta) - 2\cos\alpha\ \cos\beta\ \cos(\alpha+\beta) = \sin^2\beta$

(d) $(4\cos^2 9° - 3)(4\cos^2 27° - 3) = \tan 9°$.

3. If $m\tan(\theta - 30°) = n\tan(\theta + 120°)$, show that $\cos 2\theta = \frac{m+n}{2(m-n)}$.

4. If $\cos(\alpha+\beta)=\frac{4}{5}$; $\sin(\alpha-\beta)=\frac{5}{13}$ & α, β lie between 0 & $\frac{\pi}{4}$, then find the value of $\tan 2\alpha$.

5. If the value of the expression $\sin 25°.\sin 35°.\sin 85°$ can be expressed as $\frac{\sqrt{a}+\sqrt{b}}{c}$, where $a,b,c \in N$ and are in their lowest form, find the value of $(a + b + c)$.

6. If $\alpha + \beta = \gamma$, prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 + 2\cos\alpha\cos\beta\cos\gamma$.

7. Let $P(k)=\left(1+\cos\frac{\pi}{4k}\right)\left(1+\cos\frac{(2k-1)\pi}{4k}\right)\left(1+\cos\frac{(2k+1)\pi}{4k}\right)\left(1+\cos\frac{(4k-1)\pi}{4k}\right)$, then find the value of **(a)** P(5) and **(b)** P(6).

8. Calculate without using trigonometric tables :

(a) $4\cos 20° - \sqrt{3}\cot 20°$

(b) $\dfrac{2\cos 40° - \cos 20°}{\sin 20°}$

(c) $\cos^6\dfrac{\pi}{16} + \cos^6\dfrac{3\pi}{16} + \cos^6\dfrac{5\pi}{16} + \cos^6\dfrac{7\pi}{16}$

(d) $\tan10° - \tan50° + \tan70°$

9. Given that $(1 + \tan 1°)(1 + \tan2°)....(1 + \tan45°) = 2^n$, find n.

10. (a) If $y = 10\cos^2x - 6\sin x\cos x + 2\sin^2x$, then find the greatest & least value of y.

(b) If $y = 1 + 2\sin x + 3\cos^2x$, find the maximum & minimum values of y $\forall\, x \in R$.

(c) If $y = 9\sec^2x + 16\operatorname{cosec}^2x$, find the minimum value of y for all permissible value of x.

(d) If $a \le 3\cos\left(\theta + \dfrac{\pi}{3}\right) + 5\cos\theta + 3 \le b$, find a and b, where a is the minimum value & b is the maximum value.

EXERCISE (JM)

1. In a ΔPQR, if $3 \sin P + 4 \cos Q = 6$ and $4 \sin Q + 3 \cos P = 1$, then the angle R is equal to : **[AIEEE-2012]**

(1) $\frac{3\pi}{4}$ (2) $\frac{5\pi}{6}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{4}$

2. The expression $\frac{\tan A}{1-\cot A}+\frac{\cot A}{1-\tan A}$ can be written as **[JEE-MAIN 2013]**

(1) $\sin A \cos A + 1$ (2) $\sec A \operatorname{cosec} A + 1$

(3) $\tan A + \cot A$ (4) $\sec A + \operatorname{cosec} A$

3. Let $f_K(x) = \frac{1}{k}\left(\sin^k x + \cos^k x\right)$ where $x \in R$ and $k \geq 1$. Then $f_4(x) - f_6(x)$ equals: **[JEE-MAIN 2014]**

(1) $\frac{1}{6}$ (2) $\frac{1}{3}$ (3) $\frac{1}{4}$ (4) $\frac{1}{12}$

4. If $5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$, then the value of $\cos 4x$ is :- **[JEE-MAIN 2017]**

(1) $-\frac{7}{9}$ (2) $-\frac{3}{5}$ (3) $\frac{1}{3}$ (4) $\frac{2}{9}$

5. For any $\theta \in \left(\frac{\pi}{4},\frac{\pi}{2}\right)$, the expression $3(\sin\theta - \cos\theta)^4 + 6(\sin\theta + \cos\theta)^2 + 4\sin^6\theta$ equals: **[JEE-MAIN 2019]**

(1) $13 - 4\cos^6\theta$ (2) $13 - 4\cos^4\theta + 2\sin^2\theta\cos^2\theta$

(3) $13 - 4\cos^2\theta + 6\cos^4\theta$ (4) $13 - 4\cos^2\theta + 6\sin^2\theta\cos^2\theta$

6. The value of $\cos\frac{\pi}{2^2}\cdot\cos\frac{\pi}{2^3}\cdot\ldots\ldots\cdot\cos\frac{\pi}{2^{10}}\cdot\sin\frac{\pi}{2^{10}}$ is : **[JEE-MAIN 2019]**

(1) $\frac{1}{256}$ (2) $\frac{1}{2}$ (3) $\frac{1}{512}$ (4) $\frac{1}{1024}$

7. Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ for k = 1, 2, 3, ... Then for all x $\in$ R, the value of $f_4(x) - f_6(x)$ is equal to :- **[JEE-MAIN 2019]**

(1) $\frac{5}{12}$ (2) $\frac{-1}{12}$ (3) $\frac{1}{4}$ (4) $\frac{1}{12}$

8. The maximum value of $3\cos\theta + 5\sin\left(\theta - \frac{\pi}{6}\right)$ for any real value of θ is : **[JEE-MAIN 2019]**

(1) $\sqrt{19}$ (2) $\frac{\sqrt{79}}{2}$ (3) $\sqrt{31}$ (4) $\sqrt{34}$

9. Let α and β be two real roots of the equation $(k + 1)\tan^2 x - \sqrt{2} \cdot \lambda \tan x = (1 - k)$, where $k(\neq -1)$ and λ are real numbers. If $\tan^2(\alpha + \beta) = 50$, then a value of λ is ; **[JEE-MAIN 2020]**

(1) 5 (2) 10 (3) $5\sqrt{2}$ (4) $10\sqrt{2}$

10. The value of $\cos^3\left(\frac{\pi}{8}\right)\cdot\cos\left(\frac{3\pi}{8}\right) + \sin^3\left(\frac{\pi}{8}\right)\cdot\sin\left(\frac{3\pi}{8}\right)$ is : **[JEE-MAIN 2020]**

(1) $\frac{1}{4}$ (2) $\frac{1}{\sqrt{2}}$ (3) $\frac{1}{2\sqrt{2}}$ (4) $\frac{1}{2}$

EXERCISE (JA)

1. Let $\theta \in \left(0, \frac{\pi}{4}\right)$ and $t_1 = (\tan\theta)^{\tan\theta}, t_2 = (\tan\theta)^{\cot\theta}, t_3 = (\cot\theta)^{\tan\theta}, t_4 = (\cot\theta)^{\cot\theta}$, then -

[JEE 06, 3M,–1M]

(A) $t_1 > t_2 > t_3 > t_4$ (B) $t_4 > t_3 > t_1 > t_2$ (C) $t_3 > t_1 > t_2 > t_4$ (D) $t_2 > t_3 > t_1 > t_4$

2. **(a)** If $\frac{\sin^4 x}{2} + \frac{\cos^4 x}{3} = \frac{1}{5}$, then **[JEE 2009, 4 + 4]**

(A) $\tan^2 x = \frac{2}{3}$

(B) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{1}{125}$

(C) $\tan^2 x = \frac{1}{3}$

(D) $\frac{\sin^8 x}{8} + \frac{\cos^8 x}{27} = \frac{2}{125}$

(b) For $0 < \theta < \frac{\pi}{2}$, the solution(s) of $\sum_{m=1}^{6} \operatorname{cosec}\left(\theta + \frac{(m-1)\pi}{4}\right)\operatorname{cosec}\left(\theta + \frac{m\pi}{4}\right) = 4\sqrt{2}$ is (are)-

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{12}$ (D) $\frac{5\pi}{12}$

3. **(a)** The maximum value of the expression $\frac{1}{\sin^2\theta + 3\sin\theta\cos\theta + 5\cos^2\theta}$ is

(b) Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3}+1$ apart. If the chords subtend at the center, angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$ where $k > 0$, then the value of $[k]$ is -

[**Note :** $[k]$ denotes the largest integer less than or equal to k] **[JEE 2010, 3+3]**

4. Let $P = \{\theta : \sin\theta - \cos\theta = \sqrt{2}\cos\theta\}$ and $Q = \{\theta : \sin\theta + \cos\theta = \sqrt{2}\sin\theta\}$ be two sets. Then

(A) $P \subset Q$ and $Q - P \neq \varnothing$ (B) $Q \not\subset P$ **[JEE 2011,3]**

(C) $P \not\subset Q$ (D) $P = Q$

5. The value of $\displaystyle\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4}+\frac{(k-1)\pi}{6}\right)\sin\left(\frac{\pi}{4}+\frac{k\pi}{6}\right)}$ is equal to

[JEE(Advanced)-2016, 3(–1)]

(A) $3-\sqrt{3}$ (B) $2(3-\sqrt{3})$ (C) $2(\sqrt{3}-1)$ (D) $2(2+\sqrt{3})$

6. Let α and β be nonzero real numbers such that $2(\cos\beta - \cos\alpha) + \cos\alpha \cos\beta = 1$. Then which of the following is/are true? **[JEE(Advanced)-2017, 4]**

(A) $\tan\left(\frac{\alpha}{2}\right)-\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

(B) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)-\tan\left(\frac{\beta}{2}\right)=0$

(C) $\tan\left(\frac{\alpha}{2}\right)+\sqrt{3}\tan\left(\frac{\beta}{2}\right)=0$

(D) $\sqrt{3}\tan\left(\frac{\alpha}{2}\right)+\tan\left(\frac{\beta}{2}\right)=0$

ANSWER KEY

EXERCISE (O-1)

1.	C	**2.**	B	**3.**	B	**4.**	A	**5.**	B	**6.**	C	**7.**	B
8.	B	**9.**	D	**10.**	A	**11.**	C	**12.**	B	**13.**	B	**14.**	D
15.	D	**16.**	A	**17.**	D	**18.**	D	**19.**	D	**20.**	B	**21.**	D
22.	B	**23.**	C	**24.**	D	**25.**	D	**26.**	A	**27.**	B	**28.**	B
29.	B	**30.**	C										

EXERCISE (O-2)

1.	B,C	**2.**	B,D	**3.**	B,C	**4.**	B,C	**5.**	B,C,D	**6.**	A,B
7.	A,B,C,D	**8.**	A,B,C,D	**9.**	A,D	**10.**	C,D	**11.**	A,B,D	**12.**	A,B,C,D
13.	C,D	**14.**	B,D	**15.**	A	**16.**	C	**17.**	B	**18.**	D
19.	C	**20.**	**5**	**21.**	**3**	**22.**	**3**	**23.**	**2**	**24.**	**1**

25. (A)→(S); (B)→(R); (C)→(Q); (D)→(P)

EXERCISE (S-1)

4. $\frac{56}{33}$ **5.** 24 **7.** (a) $\frac{3-\sqrt{5}}{32}$; (b) $\frac{2-\sqrt{3}}{16}$

8. (a) –1, (b) $\sqrt{3}$, (c) $\frac{5}{4}$, (d) $\sqrt{3}$ **9.** n = 23

10. (a) $y_{max}=11$, $y_{min}=1$; (b) $y_{max}=\frac{13}{3}$, $y_{min}=-1$; (c) 49; (d) a = –4 & b = 10

EXERCISE (JM)

1.	3	**2.**	2	**3.**	4	**4.**	1	**5.**	1	**6.**	3	**7.**	4	**8.**	1
9.	2	**10.**	3												

EXERCISE (JA)

1. B **2.** **(a)** A, B; **(b)** C,D **3.** (a) 2; (b) k = 3 **4.** D **5.** C

6. **Bonus**

CHAPTER 3

TRIGONOMETRIC EQUATION

Chapter Contents 03

IMPORTANT NOTES

CHAPTER 3

TRIGONOMETRIC EQUATION

1. TRIGONOMETRIC EQUATION :

An equation involving one or more trigonometrical ratios of unknown angles is called a trigonometrical equation.

2. SOLUTION OF TRIGONOMETRIC EQUATION :

A value of the unknown angle which satisfies the given equation is called a solution of the trigonometric equation.

(a) **Principal solution :-** The solution of the trigonometric equation lying in the interval $[0, 2\pi)$.

(b) **General solution :-** Since all the trigonometric functions are many one & periodic, hence there are infinite values of θ for which trigonometric functions have the same value. All such possible values of θ for which the given trigonometric function is satisfied is given by a general formula. Such a general formula is called general solution of trigonometric equation.

(c) **Particular solution :-** The solution of the trigonometric equation lying in the given interval.

3. GENERAL SOLUTIONS OF SOME TRIGONOMETRIC EQUATIONS (TO BE REMEMBERED) :

(a) If $\sin\theta = 0$, then $\theta = n\pi$, $n \in I$ (set of integers)

(b) If $\cos\theta = 0$, then $\theta = (2n+1)\frac{\pi}{2}$, $n \in I$

(c) If $\tan\theta = 0$, then $\theta = n\pi$, $n \in I$

(d) If $\sin\theta = \sin\alpha$, then $\theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$, $n \in I$

(e) If $\cos\theta = \cos\alpha$, then $\theta = 2n\pi \pm \alpha$, $n \in I$, $\alpha \in [0,\pi]$

(f) If $\tan\theta = \tan\alpha$, then $\theta = n\pi + \alpha$, $n \in I$, $\alpha \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

(g) If $\sin\theta = 1$, then $\theta = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2}$, $n \in I$

(h) If $\cos\theta = 1$ then $\theta = 2n\pi$, $n \in I$

(i) If $\sin^2\theta = \sin^2\alpha$ or $\cos^2\theta = \cos^2\alpha$ or $\tan^2\theta = \tan^2\alpha$, then $\theta = n\pi \pm \alpha$, $n \in I$

(j) For $n \in I$, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$, $n \in I$

$\sin(n\pi + \theta) = (-1)^n \sin\theta$ $\cos(n\pi + \theta) = (-1)^n \cos\theta$

(k) $\cos n\pi = (-1)^n,\ n \in I$

If n is an odd integer, then $\sin\frac{n\pi}{2} = (-1)^{\frac{n-1}{2}}, \cos\frac{n\pi}{2} = 0,$

$$\sin\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n-1}{2}}\cos\theta$$

$$\cos\left(\frac{n\pi}{2}+\theta\right) = (-1)^{\frac{n+1}{2}}\sin\theta$$

Illustrations

Illustration 1 : Find the set of values of x for which $\frac{\tan 3x - \tan 2x}{1+\tan 3x.\tan 2x} = 1.$

Solution : We have, $\frac{\tan 3x - \tan 2x}{1+\tan 3x.\tan 2x} = 1 \Rightarrow \tan(3x - 2x) = 1 \Rightarrow \tan x = 1$

$\Rightarrow \tan x = \tan\frac{\pi}{4} \Rightarrow x = n\pi + \frac{\pi}{4},\ n \in I$ {using $\tan\theta = \tan\alpha \Leftrightarrow \theta = n\pi + \alpha$)

But for this value of x, tan 2x is not defined.

Hence the solution set for x is ϕ. **Ans.**

Do yourself - 1

1. Find general solutions of the following equations :

(a) $\sin\theta = \frac{1}{2}$ (b) $\cos\left(\frac{3\theta}{2}\right) = 0$ (c) $\tan\left(\frac{3\theta}{4}\right) = 0$

(d) $\cos^2 2\theta = 1$ (e) $\sqrt{3}\sec 2\theta = 2$ (f) $\text{cosec}\left(\frac{\theta}{2}\right) = -1$

2. cos 15 x = sin 5x if

(A) $x = -\frac{\pi}{20} + \frac{n\pi}{5}, n \in I$ (B) $x = \frac{\pi}{40} + \frac{n\pi}{10}, n \in I$

(C) $x = \frac{3\pi}{20} + \frac{n\pi}{5}, n \in I$ (D) $x = -\frac{3\pi}{40} + \frac{n\pi}{10}, n \in I$

3. tan (pπ/4) = cot (qπ/4) if :

(A) p + q = 0 (B) p + q = 2n + 1

(C) p + q = 2n (D) p + q = 2 (2n + 1)

4.. Solve $\tan 2\theta = \tan\frac{2}{\theta}$.

4. IMPORTANT POINTS TO BE REMEMBERED WHILE SOLVING TRIGONOMETRIC EQUATIONS :

(a) For equations of the type $\sin\theta = k$ or $\cos\theta = k$, one must check that $|k| \le 1$.

(b) Avoid squaring the equations, if possible, because it may lead to extraneous solutions. Reject extra solutions if they do not satisfy the given equation.

(c) Do not cancel the common variable factor from the two sides of the equations which are in a product because we may loose some solutions.

(d) The answer should not contain such values of θ, which make any of the terms undefined or infinite.

(i) Check that denominator is not zero at any stage while solving equations.

(ii) If $\tan\theta$ or $\sec\theta$ is involved in the equations, θ should not be odd multiple of $\frac{\pi}{2}$.

(iii) If $\cot\theta$ or $\text{cosec}\,\theta$ is involved in the equation, θ should not be multiple of π or 0.

5. DIFFERENT STRATEGIES FOR SOLVING TRIGONOMETRIC EQUATIONS :

(a) Solving trigonometric equations by factorisation.

e.g. $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\therefore\quad (2\sin x - \cos x)(1 + \cos x) - (1 - \cos^2 x) = 0$

$\therefore\quad (1 + \cos x)(2\sin x - \cos x - 1 + \cos x) = 0$

$\therefore\quad (1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow\quad \cos x = -1 \text{ or } \sin x = \frac{1}{2}$

$\Rightarrow\quad \cos x = -1 = \cos\pi \quad \Rightarrow \quad x = 2n\pi + \pi = (2n+1)\pi,\ n \in I$

or $\quad \sin x = \frac{1}{2} = \sin\frac{\pi}{6} \quad \Rightarrow \quad x = k\pi + (-1)^k\frac{\pi}{6},\ k \in I$

Illustrations

Illustration 2 : If $\frac{1}{6}\sin\theta$, $\cos\theta$ and $\tan\theta$ are in G.P. then the general solution for θ is -

(A) $2n\pi \pm \frac{\pi}{3}$ (B) $2n\pi \pm \frac{\pi}{6}$ (C) $n\pi \pm \frac{\pi}{3}$ (D) none of these

Solution : Since, $\frac{1}{6}\sin\theta$, $\cos\theta$, $\tan\theta$ are in G.P.

$\Rightarrow \quad \cos^2\theta = \frac{1}{6}\sin\theta \,.\, \tan\theta \quad \Rightarrow \quad 6\cos^3\theta + \cos^2\theta - 1 = 0$

$\therefore \quad (2\cos\theta - 1)(3\cos^2\theta + 2\cos\theta + 1) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2}$ (other values of $\cos\theta$ are imaginary)

$\Rightarrow \quad \cos\theta = \cos\frac{\pi}{3} \quad \Rightarrow \quad \theta = 2n\pi \pm \frac{\pi}{3}, n \in I.$ **Ans. (A)**

(b) Solving of trigonometric equation by reducing it to a quadratic equation.

e.g. $6 - 10\cos x = 3\sin^2 x$

$\therefore \quad 6 - 10\cos x = 3 - 3\cos^2 x \quad \Rightarrow \quad 3\cos^2 x - 10\cos x + 3 = 0$

$\Rightarrow \quad (3\cos x - 1)(\cos x - 3) = 0 \quad \Rightarrow \quad \cos x = \frac{1}{3}$ or $\cos x = 3$

Since $\cos x = 3$ is not possible as $-1 \le \cos x \le 1$

$\therefore \quad \cos x = \frac{1}{3} = \cos\left(\cos^{-1}\frac{1}{3}\right) \quad \Rightarrow \quad x = 2n\pi \pm \cos^{-1}\left(\frac{1}{3}\right), n \in I$

Illustrations

Illustration 3 : Solve $\sin^2\theta - \cos\theta = \frac{1}{4}$ for θ and write the values of θ in the interval $0 \le \theta \le 2\pi$.

Solution : The given equation can be written as

$1 - \cos^2\theta - \cos\theta = \frac{1}{4} \quad \Rightarrow \quad \cos^2\theta + \cos\theta - 3/4 = 0$

$\Rightarrow \quad 4\cos^2\theta + 4\cos\theta - 3 = 0 \quad \Rightarrow \quad (2\cos\theta - 1)(2\cos\theta + 3) = 0$

$\Rightarrow \quad \cos\theta = \frac{1}{2}, -\frac{3}{2}$

Since, $\cos\theta = -3/2$ is not possible as $-1 \le \cos\theta \le 1$

$\therefore \quad \cos\theta = \dfrac{1}{2} \quad \Rightarrow \quad \cos\theta = \cos\dfrac{\pi}{3} \quad \Rightarrow \quad \theta = 2n\pi \pm \dfrac{\pi}{3}, n \in I$

For the given interval, n = 0 and n = 1.

$\Rightarrow \quad \theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$ **Ans.**

Illustration 4 : Find the number of solutions of $\tan x + \sec x = 2\cos x$ in $[0, 2\pi]$.

Solution : Here, $\tan x + \sec x = 2\cos x \quad \Rightarrow \quad \sin x + 1 = 2\cos^2 x$

$\Rightarrow \quad 2\sin^2 x + \sin x - 1 = 0 \quad \Rightarrow \quad \sin x = \dfrac{1}{2}, -1$

But $\sin x = -1 \Rightarrow x = \dfrac{3\pi}{2}$ for which $\tan x + \sec x = 2\cos x$ is not defined.

Thus $\sin x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

$\Rightarrow$ number of solutions of $\tan x + \sec x = 2\cos x$ is 2. **Ans.**

Illustration 5 : Solve the equation $5\sin^2 x - 7\sin x\cos x + 16\cos^2 x = 4$

Solution : To solve this equation we use the fundamental formula of trigonometric identities,

$\sin^2 x + \cos^2 x = 1$

writing the equation in the form,

$5\sin^2 x - 7\sin x \cdot \cos x + 16\cos^2 x = 4(\sin^2 x + \cos^2 x)$

$\Rightarrow \quad \sin^2 x - 7\sin x\cos x + 12\cos^2 x = 0$

dividing by $\cos^2 x$ on both side we get,

$\tan^2 x - 7\tan x + 12 = 0$

Now it can be factorized as :

$(\tan x - 3)(\tan x - 4) = 0$

$\Rightarrow \quad \tan x = 3, 4$

i.e., $\tan x = \tan(\tan^{-1}3)$ or $\tan x = \tan(\tan^{-1}4)$

$\Rightarrow \quad x = n\pi + \tan^{-1}3$ or $x = n\pi + \tan^{-1}4, n \in I$. **Ans.**

Illustration 6 : If $x \neq \frac{n\pi}{2}$, $n \in I$ and $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1$, then find the general solutions of x.

Solution : As $x \neq \frac{n\pi}{2} \Rightarrow \cos x \neq 0, 1, -1$

So, $(\cos x)^{\sin^2 x - 3\sin x + 2} = 1 \quad \Rightarrow \quad \sin^2 x - 3\sin x + 2 = 0$

$\therefore \quad (\sin x - 2)(\sin x - 1) = 0 \quad \Rightarrow \quad \sin x = 1, 2$

where sinx = 2 is not possible and sinx = 1 which is also not possible as $x \neq \frac{n\pi}{2}$

$\therefore$ no general solution is possible. **Ans.**

Illustration 7 : Solve the equation $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \cdot \cos x$.

Solution : $\sin^4 x + \cos^4 x = \frac{7}{2}\sin x \cdot \cos x \quad \Rightarrow (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x = \frac{7}{2}\sin x \cdot \cos x$

$\Rightarrow \quad 1 - \frac{1}{2}(\sin 2x)^2 = \frac{7}{4}(\sin 2x) \Rightarrow 2\sin^2 2x + 7\sin 2x - 4 = 0$

$\Rightarrow \quad (2\sin 2x - 1)(\sin 2x + 4) = 0 \Rightarrow \sin 2x = \frac{1}{2}$ or $\sin 2x = -4$ (which is not possible)

$\Rightarrow \quad 2x = n\pi + (-1)^n \frac{\pi}{6}, n \in I$

i.e., $x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}, n \in I$ **Ans.**

Do yourself - 2

1. Solve the following equations :

(a) $3\sin x + 2\cos^2 x = 0$ (b) $\sec^2 2\alpha = 1 - \tan 2\alpha$

(c) $7\cos^2\theta + 3\sin^2\theta = 4$ (d) $4\cos\theta - 3\sec\theta = \tan\theta$

2. Solve the equation : $2\sin^2\theta + \sin^2 2\theta = 2$ for $\theta \in (-\pi, \pi)$.

3. Solve $\cos^3 x + \cos^2 x - 4\cos^2\frac{x}{2} = 0$

4. Solve $\cot^2\theta + 3\text{cosec}\theta + 3 = 0$

5. The general solution of the equation, $\frac{1 - \sin x + ... + (-1)^n \sin^n x + ...\infty}{1 + \sin x + ... + \sin^n x + ...\infty} = \frac{1 - \cos 2x}{1 + \cos 2x}$ is

(A) $(-1)^n \pi/3 + n\pi$ (B) $(-1)^n \pi/6 + n\pi$ (C) $(-1)^{n+1} \pi/6 + n\pi$ (D) $(-1)^{n-1} \pi/3 + n\pi$

6. If sum of all solutions of the equation $6\sin\frac{x}{2} = \sec\frac{x}{2}$

7. If sum of all possible values of $x \in (0, 2\pi)$ satisfying the equation

$2\cos x \cdot \operatorname{cosec} x - 4\cos x - \operatorname{cosec} x = -2$, is $\frac{k\pi}{4}$, then k is equal to

(A) 9 (B) 12 (C) 16 (D) 32

(c) Solving trigonometric equations by introducing an auxilliary argument.

Consider, $a\sin\theta + b\cos\theta = c$ .... (i)

$$\therefore \quad \frac{a}{\sqrt{a^2+b^2}}\sin\theta + \frac{b}{\sqrt{a^2+b^2}}\cos\theta = \frac{c}{\sqrt{a^2+b^2}}$$

equation (i) has a solution only if $|c| \le \sqrt{a^2+b^2}$

let $\frac{a}{\sqrt{a^2+b^2}} = \cos\phi$, $\frac{b}{\sqrt{a^2+b^2}} = \sin\phi$ & $\phi = \tan^{-1}\frac{b}{a}$

by introducing this auxillary argument ϕ, equation (i) reduces to

$\sin(\theta+\phi) = \frac{c}{\sqrt{a^2+b^2}}$ Now this equation can be solved easily.

Illustrations

Illustration 8 : Find the number of distinct solutions of $\sec x + \tan x = \sqrt{3}$, where $0 \le x \le 3\pi$.

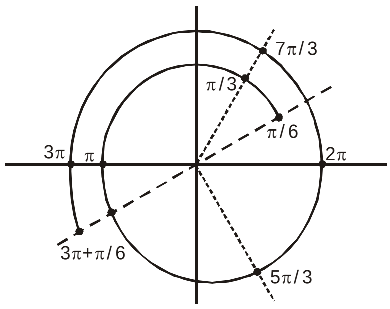
Solution : Here, $\sec x + \tan x = \sqrt{3} \Rightarrow 1 + \sin x = \sqrt{3}\cos x$

or $\sqrt{3}\cos x - \sin x = 1$

dividing both sides by $\sqrt{a^2+b^2}$ i.e. $\sqrt{4} = 2$, we get

$$\Rightarrow \frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \frac{1}{2}$$

$$\Rightarrow \cos\frac{\pi}{6}\cos x - \sin\frac{\pi}{6}\sin x = \frac{1}{2} \Rightarrow \cos\left(x+\frac{\pi}{6}\right) = \frac{1}{2}$$



As $0 \le x \le 3\pi$

$$\frac{\pi}{6} \le x + \frac{\pi}{6} \le 3\pi + \frac{\pi}{6}$$

$$\Rightarrow x + \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \Rightarrow x = \frac{\pi}{6}, \frac{3\pi}{2}, \frac{13\pi}{6}$$

But at $x = \frac{3\pi}{2}$, tanx and secx is not defined.

$\therefore$ Total number of solutions are 2. **Ans.**

Illustration 9 : Prove that the equation kcosx – 3sinx = k + 1 possess a solution iff $k \in (-\infty, 4]$.

Solution : Here, k cosx – 3sinx = k + 1, could be re-written as :

$$\frac{k}{\sqrt{k^2+9}}\cos x - \frac{3}{\sqrt{k^2+9}}\sin x = \frac{k+1}{\sqrt{k^2+9}}$$

or $\cos(x+\phi) = \frac{k+1}{\sqrt{k^2+9}}$, where $\tan\phi = \frac{3}{k}$

which possess a solution only if $-1 \le \frac{k+1}{\sqrt{k^2+9}} \le 1$

i.e., $\left|\frac{k+1}{\sqrt{k^2+9}}\right| \le 1$

i.e., $(k+1)^2 \le k^2+9$

i.e., $k^2 + 2k + 1 \le k^2 + 9$

or $k \le 4$

⇒ The interval of k for which the equation (kcosx – 3sinx = k + 1) has a solution is $(-\infty, 4]$. **Ans.**

Do yourself - 3

1 Solve the following equations :

(a) $\sin x + \sqrt{2} = \cos x$.

(b) $\mathrm{cosec}\theta = 1 + \cot\theta$.

2. Solve: $\cos\theta + \sin\theta = \cos 2\theta + \sin 2\theta$.

3. Solve $8\sin x = \frac{\sqrt{3}}{\cos x} + \frac{1}{\sin x}$

(d) Solving trigonometric equations by transforming sum of trigonometric functions into product.

e.g. $\cos 3x + \sin 2x - \sin 4x = 0$

$\cos 3x - 2 \sin x \cos 3x = 0$

$\Rightarrow$ $(\cos 3x)(1 - 2\sin x) = 0$

$\Rightarrow$ $\cos 3x = 0$ or $\sin x = \frac{1}{2}$

$\Rightarrow$ $\cos 3x = 0 = \cos\frac{\pi}{2}$ or $\sin x = \frac{1}{2} = \sin\frac{\pi}{6}$

$\Rightarrow$ $3x = 2n\pi \pm \frac{\pi}{2}$ or $x = m\pi + (-1)^m \frac{\pi}{6}$

$\Rightarrow$ $x = \frac{2n\pi}{3} \pm \frac{\pi}{6}$ or $x = m\pi + (-1)^m \frac{\pi}{6}$; $(n, m \in I)$

Illustration 10 : Solve : $\cos\theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$

Solution : We have $\cos\theta + \cos 7\theta + \cos 3\theta + \cos 5\theta = 0$

$\Rightarrow$ $2\cos 4\theta \cos 3\theta + 2\cos 4\theta \cos\theta = 0 \Rightarrow \cos 4\theta(\cos 3\theta + \cos\theta) = 0$

$\Rightarrow$ $\cos 4\theta(2\cos 2\theta \cos\theta) = 0$

$\Rightarrow$ Either $\cos\theta = 0 \Rightarrow \theta = (2n_1 + 1)\pi/2, \; n_1 \in I$

or $\cos 2\theta = 0 \Rightarrow \theta = (2n_2 + 1)\frac{\pi}{4}, n_2 \in I$

or $\cos 4\theta = 0 \Rightarrow \theta = (2n_3 + 1)\frac{\pi}{8}, \; n_3 \in I$ **Ans.**

(e) Solving trigonometric equations by transforming a product into sum.

e.g. $\sin 5x . \cos 3x = \sin 6x . \cos 2x$

$\sin 8x + \sin 2x = \sin 8x + \sin 4x$

$\therefore$ $2\sin 2x . \cos 2x - \sin 2x = 0$

$\Rightarrow$ $\sin 2x(2 \cos 2x - 1) = 0$

$\Rightarrow$ $\sin 2x = 0$ or $\cos 2x = \frac{1}{2}$

$\Rightarrow$ $\sin 2x = 0 = \sin 0$ or $\cos 2x = \frac{1}{2} = \cos\frac{\pi}{3}$

$\Rightarrow$ $2x = n\pi + (-1)^n \times 0, \; n \in I$ or $2x = 2m\pi \pm \frac{\pi}{3}, \; m \in I$

$\Rightarrow$ $x = \frac{n\pi}{2}, n \in I$ or $x = m\pi \pm \frac{\pi}{6}, m \in I$

Illustration 11 : Solve : $\cos\theta \cos2\theta \cos3\theta = \frac{1}{4}$; where $0 \le \theta \le \pi$.

Solution : $\frac{1}{2}(2\cos\theta \cos3\theta)\cos2\theta = \frac{1}{4} \Rightarrow (\cos2\theta + \cos4\theta)\cos2\theta = \frac{1}{2}$

$\Rightarrow \frac{1}{2}[2\cos^2 2\theta + 2\cos4\theta \cos2\theta] = \frac{1}{2} \Rightarrow 1 + \cos4\theta + 2\cos4\theta\cos2\theta = 1$

$\therefore \cos4\theta(1 + 2\cos2\theta) = 0$

$\cos4\theta = 0$ or $(1 + 2\cos2\theta) = 0$

Now from the first equation : $2\cos4\theta = 0 = \cos(\pi/2)$

$\therefore 4\theta = \left(n + \frac{1}{2}\right)\pi \Rightarrow \theta = (2n+1)\frac{\pi}{8}, n \in I$

for $n = 0, \theta = \frac{\pi}{8}$; $n = 1, \theta = \frac{3\pi}{8}$; $n = 2, \theta = \frac{5\pi}{8}$; $n = 3, \theta = \frac{7\pi}{8}$ $(\because 0 \le \theta \le \pi)$

and from the second equation :

$\cos2\theta = -\frac{1}{2} = -\cos(\pi/3) = \cos(\pi - \pi/3) = \cos(2\pi/3)$

$\therefore 2\theta = 2k\pi \pm 2\pi/3 \therefore \theta = k\pi \pm \pi/3, k \in I$

again for $k = 0, \theta = \frac{\pi}{3}$; $k = 1, \theta = \frac{2\pi}{3}$ $(\because 0 \le \theta \le \pi)$

$\therefore \theta = \frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$ **Ans.**

Illustration 12 : The general solution of the trigonometric equation $\sin x \cos 2x + \sin 2x \cos 5x = \sin 3x \cos 5x$, is

(A) $\frac{n\pi}{3}$ (B) $\frac{2n\pi}{9}$ (C) $2n\pi$ (D) $\frac{n\pi}{3} \cup \frac{2n\pi}{9}$

(where $n \in I$)

Solution : $2\sin x \cos 2x + 2\sin 2x \cos 5x = 2\sin 3x \cos 5x$

$\sin 3x - \sin x + \sin 7x - \sin 3x = \sin 8x - \sin 2x$

$\sin 7x - \sin x = \sin 8x - \sin 2x$

$2 \cos 4x \sin 3x = 2 \cos 5x \sin 3x$

$2 \sin 3x [\cos 5x - \cos 4x] = 0$

$\sin 3x = 0 \quad \Rightarrow \quad x = \frac{n\pi}{3}$

if $\cos 5x - \cos 4x = 0$

$2 \sin\frac{9x}{2} \sin\frac{x}{2} = 0$

$\therefore \quad x = \frac{2n\pi}{9}$ or $2n\pi$

Hence general solution is $\frac{n\pi}{3} \cup \frac{2n\pi}{9}, n \in I$]

Do yourself - 4

1. Solve $4\sin\theta \sin2\theta \sin4\theta = \sin3\theta$.

2. Solve for x : $\sin x + \sin3x + \sin5x = 0$.

3. Solve$\sin2x + 5\sin x + 1 + 5\cos x = 0$

4. Solve$3\cos x + 3\sin x + \sin3x - \cos3x = 0$

5. Solve$(1 - \sin2x) (\cos x - \sin x) = 1 - 2\sin^2x$.

6. The number of integral values of a for which the equation $\cos 2x + a \sin x = 2a - 7$ possesses a solution is

(A) 2 (B) 3 (C) 4 (D) 5

(f) Solving equations by a change of variable :

(i) Equations of the form P (sin x ± cos x, sin x. cos x) = 0, where P (y,z) is a polynomial, can be solved by the substitution :

$\cos x \pm \sin x = t \quad \Rightarrow \quad 1 \pm 2 \sin x. \cos x = t^2$.

Illustrations

Illustration 13 : Solve : $\sin x + \cos x = 1 + \sin x . \cos x$.

Solution : put $\sin x + \cos x = t$

$\Rightarrow \sin^2x + \cos^2x + 2\sin x . \cos x = t^2$

$\Rightarrow 2\sin x \cos x = t^2 - 1 \quad (\because \sin^2x + \cos^2x = 1)$

$$\Rightarrow \sin x.\cos x = \left(\frac{t^2-1}{2}\right)$$

Substituting above result in given equation, we get :

$$t = 1 + \frac{t^2-1}{2}$$

$\Rightarrow 2t = t^2 + 1 \Rightarrow t^2 - 2t + 1 = 0$

$\Rightarrow (t - 1)^2 = 0 \quad \Rightarrow \quad t = 1$

$\Rightarrow \sin x + \cos x = 1$

Dividing both sides by $\sqrt{1^2+1^2}$ i.e. $\sqrt{2}$, we get

$$\Rightarrow \frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x = \frac{1}{\sqrt{2}} \Rightarrow \quad \cos x \cos\frac{\pi}{4} + \sin x.\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos\left(x-\frac{\pi}{4}\right) = \cos\frac{\pi}{4} \quad \Rightarrow \quad x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$$

$$\Rightarrow x = 2n\pi \text{ or } x = 2n\pi + \frac{\pi}{2} = (4n+1)\frac{\pi}{2},\ n \in I$$

(ii) Equations of the form of $a\sin x + b\cos x + d = 0$, where a, b & d are real numbers can be solved by changing sin x & cos x into their corresponding tangent of half the angle.

Illustration 14 : Solve : $3 \cos x + 4 \sin x = 5$

Solution :

$$\Rightarrow 3\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)+4\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right)=5$$

$$\Rightarrow \frac{3-3\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{8\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}=5$$

$$\Rightarrow\ 3 - 3\tan^2\frac{x}{2} + 8\tan\frac{x}{2} = 5 + 5\tan^2\frac{x}{2} \ \Rightarrow\ 8\tan^2\frac{x}{2} - 8\tan\frac{x}{2} + 2 = 0$$

$$\Rightarrow\ 4\tan^2\frac{x}{2} - 4\tan\frac{x}{2} + 1 = 0 \ \Rightarrow\ \left(2\tan\frac{x}{2} - 1\right)^2 = 0$$

$$\Rightarrow\ 2\tan\frac{x}{2} - 1 = 0 \Rightarrow \tan\frac{x}{2} = \frac{1}{2} = \tan\left(\tan^{-1}\frac{1}{2}\right)$$

$$\Rightarrow\ \frac{x}{2} = n\pi + \tan^{-1}\left(\frac{1}{2}\right),\ n \in I \ \Rightarrow\ x = 2n\pi + 2\tan^{-1}\frac{1}{2},\ n \in I$$

(g) Solving trigonometric equations with the use of the boundness of the functions involved.

Illustrations

Illustration 15 : Solve the equation $(\sin x + \cos x)^{1+\sin 2x} = 2$, when $0 \le x \le \pi$.

Solution : We know, $-\sqrt{a^2+b^2} \le a\sin\theta + b\cos\theta \le \sqrt{a^2+b^2}$ and $-1 \le \sin\theta \le 1$.

$\therefore$ $(\sin x + \cos x)$ admits the maximum value as $\sqrt{2}$

and $(1 + \sin 2x)$ admits the maximum value as 2.

Also $\left(\sqrt{2}\right)^2 = 2$.

$\therefore$ the equation could hold only when, $\sin x + \cos x = \sqrt{2}$ and $1 + \sin 2x = 2$

Now, $\sin x + \cos x = \sqrt{2}$ $\Rightarrow$ $\cos\left(x - \frac{\pi}{4}\right) = 1$

$\Rightarrow$ $x = 2n\pi + \pi/4,\ n \in I$ ...... (i)

and $1 + \sin 2x = 2$ $\Rightarrow$ $\sin 2x = 1 = \sin\frac{\pi}{2}$

$\Rightarrow$ $2x = m\pi + (-1)^m\frac{\pi}{2},\ m \in I$ $\Rightarrow$ $x = \frac{m\pi}{2} + (-1)^m\frac{\pi}{4}$ ...... (ii)

The value of x in $[0, \pi]$ satisfying equations (i) and (ii) is $x = \frac{\pi}{4}$ (when n = 0 & m =0)

Ans.

Note : $\sin x + \cos x = -\sqrt{2}$ and $1 + \sin 2x = 2$ also satisfies but as $x \ge 0$, this solution is not in domain.

Illustration 16 : Solve for x and y : $2^{\frac{1}{\cos^2 x}}\sqrt{y^2-y+1/2}\le 1$

Solution :

$$2^{\frac{1}{\cos^2 x}}\sqrt{y^2-y+1/2}\le 1 \qquad \text{....... (i)}$$

$$2^{\frac{1}{\cos^2 x}}\sqrt{\left(y-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\le 1$$

Minimum value of $2^{\frac{1}{\cos^2 x}} = 2$

Minimum value of $\sqrt{\left(y-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=\frac{1}{2}$

$\Rightarrow$ Minimum value of $2^{\frac{1}{\cos^2 x}}\sqrt{y^2-y+\frac{1}{2}}$ is 1

$\Rightarrow$ (i) is possible when $2^{\frac{1}{\cos^2 x}}\sqrt{\left(y-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}=1$

$\Rightarrow$ $\cos^2 x = 1$ and $y = 1/2$ $\Rightarrow \cos x = \pm 1 \Rightarrow$ $x = n\pi$, where $n \in I$.

Hence $x = n\pi$, $n \in I$ and $y = 1/2$. **Ans.**

Illustration 17 : The number of solution(s) of $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2+\frac{1}{x^2}$, $0 \le x \le \pi/2$, is/are -

(A) 0 (B) 1 (C) infinite (D) none of these

Solution : Let $y = 2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2+\frac{1}{x^2}$ $\Rightarrow$ $y = (1 + \cos x)\sin^2 x$ and $y = x^2 + \frac{1}{x^2}$

when $y = (1 + \cos x)\sin^2 x$ = (a number < 2)(a number $\le$ 1) $\Rightarrow$ $y < 2$ .... (i)

and when $y = x^2 + \frac{1}{x^2} = \left(x-\frac{1}{x}\right)^2 + 2 \ge 2$ $\Rightarrow y \ge 2$ .... (ii)

No value of y can be obtained satisfying (i) and (ii), simultaneously

$\Rightarrow$ No real solution of the equation exists. **Ans. (A)**

Note: If L.H.S. of the given trigonometric equation is always less than or equal to k and RHS is always greater than k, then no solution exists. If both the sides are equal to k for same value of θ, then solution exists and if they are equal for different values of θ, then solution does not exist.

Illustrations

Illustration 18 : The number of ordered pairs (x, y) satisfying $|x|+|y|=3$ and $\sin\left(\frac{\pi x^2}{3}\right)=1$ is less than or equal to

(A) 7 (B) 8 (C) 9 (D) 10

Solution : $\sin\left(\frac{\pi}{3}x^2\right) = 1$

$$\frac{\pi}{3}x^2 = 2n\pi + \frac{\pi}{2};\ n \in I$$

$$\frac{\pi}{3}x^2 = (4n+1)\frac{\pi}{2}$$

$$x^2 = \frac{3}{2}(4n+1);\ n \in I$$

only n = 0 and n = 1 is possible.

$$x^2 = \frac{3}{2} \text{ or } x^2 = \frac{15}{2}$$

$$\therefore\ x = \pm\sqrt{\frac{3}{2}} \text{ or } x = \pm\sqrt{\frac{15}{2}}.$$

for each value of x will get 2 values of y, Hence 8 ordered pairs.]

Do yourself - 5

1. If $x^2 - 4x + 5 - \sin y = 0$, $y \in [0, 2\pi)$, then -
(A) x = 1, y = 0 (B) x = 1, y = π/2 (C) x = 2, y = 0 (D) x = 2, y = π/2

2. If $\sin x + \cos x = \sqrt{y + \frac{1}{y}}$, $y > 0$, $x \in [0, \pi]$, then find the least positive value of x satisfying the given condition.

3. Solve $\sin 3x + \cos 2x = -2$

4. Solve $\sqrt{3\sin 5x - \cos^2 x - 3} = 1 - \sin x$

5. The number of real solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is-

(A) 0 (B) 1 (C) 2 (D) infinitely many

6. If $\sin^3\alpha + \cos^3\beta + 6\sin\alpha \cdot \cos\beta = 8$, where $\alpha, \beta \in [0, 2\pi]$ then number of ordered pairs (α, β) is equal to

(A) 0 (B) 1 (C) 2 (D) 3

6. TRIGONOMETRIC INEQUALITIES :

There is no general rule to solve trigonometric inequations and the same rules of algebra are valid provided the domain and range of trigonometric functions should be kept in mind.

Illustrations

Illustration 19 : Find the solution set of inequality $\sin x > 1/2$.

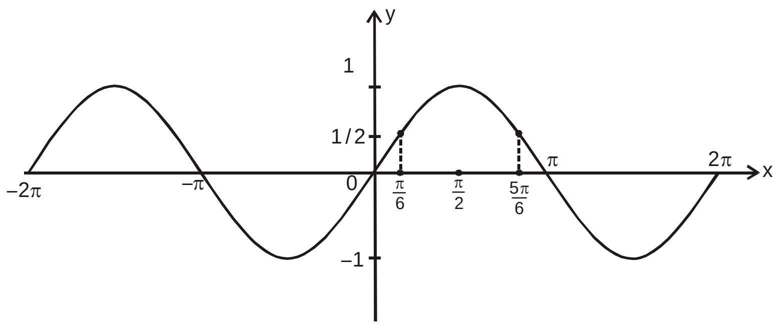
Solution : When $\sin x = \frac{1}{2}$, the two values of x between 0 and 2π are $\pi/6$ and $5\pi/6$.

From the graph of $y = \sin x$, it is obvious that between 0 and 2π,

$\sin x > \frac{1}{2}$ for $\pi/6 < x < 5\pi/6$

Hence, $\sin x > 1/2$

$\Rightarrow \quad 2n\pi + \pi/6 < x < 2n\pi + 5\pi/6,\ n \in I$

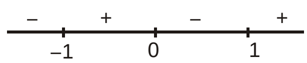


Thus, the required solution set is $\bigcup_{n\in I}\left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$ **Ans.**

Illustration 20 : Find the values of α lying between 0 and π for which the inequality : $\tan\alpha > \tan^3\alpha$ is valid.

Solution : We have : $\tan\alpha - \tan^3\alpha > 0 \Rightarrow \tan\alpha\,(1 - \tan^2\alpha) > 0$

$\Rightarrow \quad (\tan\alpha)(\tan\alpha + 1)(\tan\alpha - 1) < 0$



So $\quad \tan\alpha < -1,\ 0 < \tan\alpha < 1$

$\therefore \quad$ Given inequality holds for $\alpha \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$ **Ans.**

Do yourself - 6

(i) Find the solution set of the inequality : $\cos x \geq -1/2$.

(ii) Find the values of x in the interval $[0, 2\pi]$ for which $4\sin^2 x - 8\sin x + 3 \leq 0$.

Illustrations

Miscellaneous Illustration :

Illustration 21 : Solve the following equation : $\tan^2\theta + \sec^2\theta + 3 = 2(\sqrt{2}\sec\theta + \tan\theta)$

Solution : We have $\tan^2\theta + \sec^2\theta + 3 = 2\sqrt{2}\sec\theta + 2\tan\theta$

$\Rightarrow \quad \tan^2\theta - 2\tan\theta + \sec^2\theta - 2\sqrt{2}\sec\theta + 3 = 0$

$\Rightarrow \quad \tan^2\theta + 1 - 2\tan\theta + \sec^2\theta - 2\sqrt{2}\sec\theta + 2 = 0$

$\Rightarrow \quad (\tan\theta - 1)^2 + (\sec\theta - \sqrt{2})^2 = 0 \qquad \Rightarrow \quad \tan\theta = 1$ and $\sec\theta = \sqrt{2}$

As the periodicity of $\tan\theta$ and $\sec\theta$ are not same, we get

$\theta = 2n\pi + \dfrac{\pi}{4},\ n \in I$ **Ans.**

Illustration 22 : Find the solution set of equation $5^{(1 + \log_5 \cos x)} = 5/2$.

Solution : Taking log to base 5 on both sides in given equation :

$(1 + \log_5 \cos x).\ \log_5 5 = \log_5(5/2) \quad \Rightarrow \log_5 5 + \log_5 \cos x = \log_5 5 - \log_5 2$

$\Rightarrow \log_5 \cos x = -\log_5 2 \Rightarrow \cos x = 1/2 \quad \Rightarrow \quad x = 2n\pi \pm \pi/3,\ n \in I$

Ans.

Illustration 23 : If the set of all values of x in $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ satisfying $|4\sin x + \sqrt{2}| < \sqrt{6}$ is $\left(\dfrac{a\pi}{24}, \dfrac{b\pi}{24}\right)$ then find the value of $\left|\dfrac{a-b}{3}\right|$.

Solution : $|4\sin x + \sqrt{2}| < \sqrt{6}$

$\Rightarrow \quad -\sqrt{6} < 4\sin x + \sqrt{2} < \sqrt{6} \qquad \Rightarrow \quad -\sqrt{6} - \sqrt{2} < 4\sin x < \sqrt{6} - \sqrt{2}$

$$\Rightarrow \quad \frac{-(\sqrt{6}+\sqrt{2})}{4} < \sin x < \frac{\sqrt{6}-\sqrt{2}}{4} \quad \Rightarrow \quad -\frac{5\pi}{12} < x < \frac{\pi}{12} \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Comparing with $\frac{a\pi}{24} < x < \frac{b\pi}{24}$, we get, $a = -10$, $b = 2$

$\therefore \left|\frac{a-b}{3}\right| = \left|\frac{-10-2}{3}\right| = 4$ **Ans.**

Illustration 24 : The number of values of x in the interval $[0, 5\pi]$ satisfying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is - **[JEE 98]**

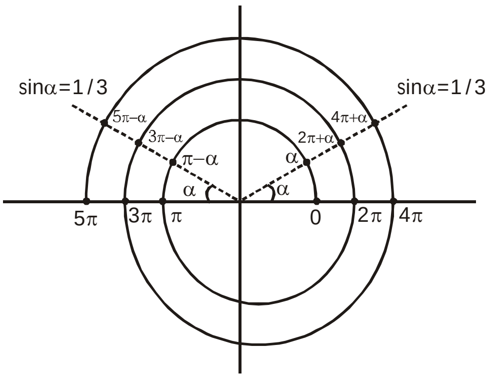
(A) 0 (B) 5 (C) 6 (D) 10

Solution : $3\sin^2 x - 7\sin x + 2 = 0$

$\Rightarrow \quad (3\sin x - 1)(\sin x - 2) = 0$

$\because \quad \sin x \neq 2$

$\Rightarrow \quad \sin x = \frac{1}{3} = \sin\alpha$ (say)



where α is the least positive value of x

such that $\sin\alpha = \frac{1}{3}$.

Clearly $0 < \alpha < \frac{\pi}{2}$. We get the solution,

$x = \alpha, \pi - \alpha, 2\pi + \alpha, 3\pi - \alpha, 4\pi + \alpha$ and $5\pi - \alpha$.

Hence total six values in $[0, 5\pi]$ **Ans. (C)**

ANSWER KEY

Do yourself-1

1. (a) $\theta = n\pi + (-1)^n \frac{\pi}{6}, n \in I$ (b) $\theta = (2n+1)\frac{\pi}{3}, n \in I$ (c) $\theta = \frac{4n\pi}{3}, n \in I$

(d) $\theta = \frac{n\pi}{2}, n \in I$ (e) $\theta = n\pi \pm \frac{\pi}{12}, n \in I$

(f) $\theta = 2n\pi + (-1)^{n+1}\pi, n \in I$

2. A,B,C,D 3. D 4. $\frac{n\pi}{4} \pm \sqrt{1+\frac{n^2\pi^2}{16}}, n \in I$

Do yourself-2

1. (a) $x = n\pi + (-1)^{n+1}\frac{\pi}{6}, n \in I$ (b) $\alpha = \frac{n\pi}{2}$ or $\alpha = \frac{k\pi}{2} + \frac{3\pi}{8}, n, k \in I$

(c) $\theta = n\pi \pm \frac{\pi}{3}, n \in I$

(d) $\theta = n\pi + (-1)^n\alpha$, where $\alpha = \sin^{-1}\left(\frac{\sqrt{17}-1}{8}\right)$ or $\sin^{-1}\left(\frac{-1-\sqrt{17}}{8}\right), n \in I$

2. $\theta = \left\{-\frac{\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{2}\right\}$ 3. $(2n+1)\pi, n \in I$

4. $2n\pi - \frac{\pi}{2}, n \in I$ or $n\pi + (-1)^{n+1}\frac{\pi}{6}, n \in I$ 5. B 6. 3

7. B

Do yourself-3

1. (a) $x = 2n\pi - \frac{\pi}{4}, n \in I$ (b) $2m\pi + \frac{\pi}{2}, m \in I$

2. $2n\pi, n \in I$ or $\frac{2n\pi}{3} + \frac{\pi}{6}, n \in I$ 3. $x = n\pi + \frac{\pi}{6}, n \in I, x = \frac{n\pi}{2} - \frac{\pi}{12}, n \in I$

Do yourself-4

1. $\theta = n\pi$ or $\theta = \frac{m\pi}{3} \pm \frac{\pi}{9}$; $n, m \in I$

2. $x = \frac{n\pi}{3}, n \in I$ and $k\pi \pm \frac{\pi}{3}, k \in I$

3. $n\pi - \frac{\pi}{4}, n \in I$

4. $n\pi - \frac{\pi}{4}, n \in I$

5. $2n\pi + \frac{\pi}{2}, n \in I$ or $2n\pi, n \in I$ or $n\pi + \frac{\pi}{4}, n \in I$

6. D

Do yourself-5

1. D

2. $x = \frac{\pi}{4}$

3. $(4p - 3)\frac{\pi}{2}, p \in I$

4. $2m\pi + \frac{\pi}{2}, m \in I$

5. A

6. C

Do yourself-6

1. $\bigcup\limits_{n \in I}\left[2n\pi - \frac{2\pi}{3}, 2n\pi + \frac{2\pi}{3}\right]$

2. $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

EXERCISE (O-1)

1. If $2\tan^2\theta = \sec^2\theta$, then the general solution of θ -

(A) $n\pi + \frac{\pi}{4}(n\in I)$ (B) $n\pi - \frac{\pi}{4}(n\in I)$ (C) $n\pi \pm \frac{\pi}{4}(n\in I)$ (D) $2n\pi \pm \frac{\pi}{4}(n\in I)$

2. If $\frac{1-\cos 2\theta}{1+\cos 2\theta} = 3$, then the general solution of θ is -

(A) $2n\pi \pm \pi/6$ (B) $n\pi \pm \pi/6$ (C) $2n\pi \pm \pi/3$ (D) $n\pi \pm \pi/3$

where $n \in I$

3. The general value of θ satisfying $\sin^2\theta + \sin\theta = 2$ is-

(A) $n\pi\ (-1)^n\frac{\pi}{6}$ (B) $2n\pi + \frac{\pi}{4}$ (C) $n\pi + (-1)^n\frac{\pi}{2}$ (D) $n\pi + (-1)^n\frac{\pi}{3}$

4. Let $A = \{\theta : \sin(\theta) = \tan(\theta)\}$ and $B = \{\theta : \cos(\theta) = 1\}$ be two sets. Then -

(A) $A = B$ (B) $A \subset B$ and $B - A \neq \phi$

(C) $A \not\subset B$ (D) $B \not\subset A$

5. The solution set of $(5 + 4\cos\theta)(2\cos\theta + 1) = 0$ in the interval $[0,2\pi]$ is :

(A) $\left\{\frac{\pi}{3}, \frac{2\pi}{3}\right\}$ (B) $\left\{\frac{\pi}{3}, \pi\right\}$ (C) $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$ (D) $\left\{\frac{2\pi}{3}, \frac{5\pi}{3}\right\}$

6. The general solution of equation $4\cos^2 x + 6\sin^2 x = 5$ is -

(A) $x = n\pi \pm \frac{\pi}{2}(n\in I)$ (B) $x = n\pi \pm \frac{\pi}{4}(n\in I)$

(C) $x = n\pi \pm \frac{3\pi}{2}(n\in I)$ (D) None of these

7. If $\tan^2\theta - (1 + \sqrt{3})\tan\theta + \sqrt{3} = 0$, then the general value of θ is :

(A) $n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{3}$ (B) $n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{3}$ (C) $n\pi + \frac{\pi}{4}, n\pi - \frac{\pi}{3}$ (D) $n\pi - \frac{\pi}{4}, n\pi - \frac{\pi}{3}$

where $n \in I$

8. The general solution of the equation $\tan^2\alpha + 2\sqrt{3}\tan\alpha = 1$ is given by -

(A) $\alpha = \frac{n\pi}{2} (n \in I)$

(B) $\alpha = (2n+1)\frac{\pi}{2} (n \in I)$

(C) $\alpha = (6n+1)\frac{\pi}{12} (n \in I)$

(D) $\alpha = \frac{n\pi}{12} (n \in I)$

9. The number of solutions of the equation sin 2x – 2cosx + 4 sinx = 4 in the interval [0, 5π] is -

(A) 6
(B) 4
(C) 3
(D) 5

10. The set of angles between 0 and 2π satisfying the equation $4\cos^2\theta - 2\sqrt{2}\cos\theta - 1 = 0$ is

(A) $\left\{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$

(B) $\left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{17\pi}{12}, \frac{23\pi}{12}\right\}$

(C) $\left\{\frac{5\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}\right\}$

(D) $\left\{\frac{\pi}{12}, \frac{7\pi}{12}, \frac{19\pi}{12}, \frac{23\pi}{12}\right\}$

11. If tanθ + tan4θ + tan7θ = tanθ tan4θ tan7θ, then θ =

(A) $\frac{n\pi}{4}$
(B) $\frac{n\pi}{7}$
(C) $\frac{n\pi}{12}$
(D) nπ

where n ∈ I

12. If $\frac{\tan 2\theta + \tan\theta}{1 - \tan\theta\tan 2\theta} = 0$, then the general value of θ is -

(A) nπ ; n ∈ I
(B) $\frac{n\pi}{3}$; n ∈ I
(C) $\frac{n\pi}{4}$
(D) $\frac{n\pi}{6}$; n ∈ I

where n ∈ I

13. The smallest positive angle satisfying the equation 1 + cos3x – 2cos2x = 0, is equal to

(A) 15°
(B) 22.5°
(C) 30°
(D) 45°

14. If $\tan\theta - \sqrt{2}\sec\theta = \sqrt{3}$, then the general solution of θ is -

(A) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{3}$

(B) $n\pi + (-1)^n \frac{\pi}{3} - \frac{\pi}{4}$

(C) $n\pi + (-1)^n \frac{\pi}{3} + \frac{\pi}{4}$

(D) $n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{3}$

where n ∈ I

15. Number of principal solution(s) of the equation $4\cdot 16^{\sin^2 x} = 2^{6\sin x}$ is

(A) 1 (B) 2 (C) 3 (D) 4

16. The most general values of x for which $\sin x + \cos x = \min\limits_{a\in R}\{1, a^2 - 4a + 6\}$ is given by-

(A) $2n\pi$ (B) $2n\pi + \frac{\pi}{2}$ (C) $n\pi + (-1)^n.\frac{\pi}{4} - \frac{\pi}{4}$ (D) None of these

where $n \in I$

17. If the equation $\sin^4 x - (k+2)\sin^2 x - (k+3) = 0$ has a solution then k must lie in the interval :

(A) $(-4, -2)$ (B) $[-3, 2)$ (C) $(-4, -3)$ (D) $[-3, -2]$

18. Number of values of x satisfying the equation $\log_2(\sin x) + \log_{1/2}(-\cos x) = 0$ in the interval $(-\pi, \pi]$ is equal to-

(A) 0 (B) 1 (C) 2 (D) 3

19. The equation sin x cos x = 2 has :

(A) one solution (B) two solutions (C) infinite solutions (D) no solution

20. The number of solutions of the equation $\tan^2 x - \sec^{10} x + 1 = 0$ in (0, 10) is -

(A) 3 (B) 6 (C) 10 (D) 11

21. If $x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$, then the greatest positive solution of $1 + \sin^4 x = \cos^2 3x$ is -

(A) π (B) 2π (C) $\frac{5\pi}{2}$ (D) none of these

22. The number of solutions of the equation $\sin x = x^2 + x + 1$ is-

(A) 0 (B) 1 (C) 2 (D) None

23. **Statement-1:** If $\sin\frac{3x}{2}\cos\frac{5y}{3} = k^8 - 4k^4 + 5$, where $x, y \in R$ then exactly four distinct real values of k are possible.

because

Statement-2: $\sin\frac{3x}{2}$ and $\cos\frac{5y}{3}$ both are less than or equal to one and greater than or equal to – 1.

(A) Statement-1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.

(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.

(C) Statement-1 is true, statement-2 is false.

(D) Statement-1 is false, statement-2 is true.

24. The number of solutions of the equation $2\cos\left(\frac{x}{2}\right) = 3^x + 3^{-x}$ is-

(A) 1 (B) 2 (C) 3 (D) None

25. The equation $2\cos^2\left(\frac{x}{2}\right)\sin^2 x = x^2 + x^{-2},\ 0 < x \le \frac{\pi}{2}$ has

(A) one real solutions (B) more than one real solutions

(C) no real solution (D) none of the above

26. If $0 \le x \le 3\pi$, $0 \le y \le 3\pi$ and $\cos x.\sin y = 1$, then the possible number of values of the ordered pair (x, y) is -

(A) 6 (B) 12 (C) 8 (D) 15

27. Given $a^2 + 2a + \mathrm{cosec}^2\left(\frac{\pi}{2}(a+x)\right) = 0$ then, which of the following holds good?

(A) $a = 1\ ;\ \frac{x}{2} \in I$ (B) $a = -1\ ;\ \frac{x}{2} \in I$

(C) $a \in R\ ;\ x \in \phi$ (D) a , x are finite but not possible to find

28. If the equation $\cot^4 x - 2\,\mathrm{cosec}^2 x + a^2 = 0$ has atleast one solution then, sum of all possible integral values of 'a' is equal to

(A) 4 (B) 3 (C) 2 (D) 0

29. The complete solution set of the inequality $\tan^2 x - 2\sqrt{2}\tan x + 1 \le 0$ is-

(A) $n\pi + \frac{\pi}{8} \le x \le \frac{3\pi}{8} + n\pi,\ n \in I$ (B) $n\pi + \frac{\pi}{4} \le x \le \frac{3\pi}{4} + n\pi,\ n \in I$

(C) $n\pi + \frac{\pi}{16} \le x \le \frac{3\pi}{8} + n\pi,\ n \in I$ (D) $n\pi + \frac{\pi}{3} \le x \le \frac{2\pi}{3} + n\pi,\ n \in I$

30. Number of integral solution(s) of the inequality $2\sin^2 x - 5\sin x + 2 > 0$ in $x \in [0,2\pi]$, is-

(A) 3 (B) 4 (C) 5 (D) 6

EXERCISE (O-2)

[MULTIPLE OBJECTIVE TYPE]

1. The equation $2\sin\frac{x}{2}\cdot\cos^2x + \sin^2x = 2\sin\frac{x}{2}\cdot\sin^2x + \cos^2x$ has a root for which

(A) $\sin 2x = 1$ (B) $\sin 2x = -1$ (C) $\cos x = \frac{1}{2}$ (D) $\cos 2x = -\frac{1}{2}$

2. $\sin^2x - \cos 2x = 2 - \sin 2x$ if

(A) $x = n\pi/2, n \in I$ (B) $\tan x = 3/2$

(C) $x = (2n+1)\pi/2, n \in I$ (D) $x = n\pi + (-1)^n \sin^{-1}(2/3), n \in I$

3. $4\sin^4x + \cos^4x = 1$ if

(A) $x = n\pi$ (B) $x = n\pi \pm \frac{1}{2}\cos^{-1}\left(\frac{1}{5}\right)$

(C) $x = \frac{n\pi}{2}$ (D) none of these, $(n \in I)$

4. Which of the following set of values of x satisfies the equation

$2^{(2\sin^2x - 3\sin x + 1)} + 2^{(2 - 2\sin^2x + 3\sin x)} = 9$ is

(A) $x = x\pi, n \in I$ (B) $x = 2n\pi + \frac{\pi}{2}, n \in I$

(C) $x = n\pi \pm \frac{\pi}{6}, n \in I$ (D) $x = n\pi \pm \frac{\pi}{3}, n \in I$

5. $5\sin^2x + \sqrt{3}\sin x\cos x + 6\cos^2x = 5$ if

(A) $\tan x = -1/\sqrt{3}$ (B) $\sin x = 0$

(C) $x = n\pi + \pi/2, n \in I$ (D) $x = n\pi + \pi/6, n \in I$

6. If sum of all the solution of the equation $\cot x + \csc x + \sec x = \tan x$ in $[0, 2\pi]$ is $\frac{k\pi}{2}$, then the value of k is greater than

(A) 1 (B) 2 (C) 3 (D) 4

7. The equation $\sin x + \cos(k+x) + \cos(k-x) = 2$ has real solution(s), then sin k can be

(A) $\frac{-3}{4}$ (B) $\frac{1}{4}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

8. Which of the following equations have no solution?

(A) $2^{|x|} = \sin x^2$ (B) $3^{\left|\sin\sqrt{x}\right|} = |\cos x|$ (C) $x^2 = -\cos x$ (D) $3x^2 = 1 - 2\cos x$

9. If $m > 0, n < 5, 0 < m + n < 10$ and $\frac{x^2+5}{2} = x - 2\cos(m + nx)$ has atleast one real root, then

(A) the greatest value of $(m + n)$ is 3π. (B) the greatest value of $(m + n)$ is 2π.

(C) the least value of $(m + n)$ is π. (D) the least value of $(m + n)$ is 2π.

10. If $x^4 + 3\cos(ax^2 + bx + c) = 2(x^2 - 2)$ has two solutions with $a, b, c \in (2, 5)$ then

(A) $a + b + c = \pi$ (B) $a - b + c = \pi$ (C) $a + b + c = 3\pi$ (D) $a - b + c = 3\pi$

[COMPREHENSION TYPE]

Paragraph for question no. 11 & 12

Let $f(x) = \cos x + \sin x - 1$ and $g(x) = \sin 2x - 2$.

11. Number of solutions of the equation $f(x) = g(x)$ in $x \in \left[-\pi, \frac{7\pi}{2}\right]$ is equal to

(A) 4 (B) 5 (C) 6 (D) 8

12. If the equation $f(x) = k$ has atleast one real solution then the number of possible integral values of k is equal to

(A) 1 (B) 2 (C) 3 (D) 4

Paragraph for Question 13 to 15

Let $f(x) = \sin^2 x - (a - 1)\sin x + 2(a - 3)$

On the basis of above information, answer the following questions :

13. If $x \in [0,\pi]$ and $f(x) = 0$ has exactly one real root, then 'a' lies in

(A) (3,5) (B) (2,4) (C) (4,5) (D) none of these

14. If $f(x) = 0$ have two real roots in $(0,\pi)$, then $a \in$

(A) (1,2) (B) (3,4) (C) $(3,4) \cup \{5\}$ (D) (3,5)

15. If $f(x) \geq 0\ \forall\ x \in R$ then range of 'a' is

(A) $[2,\infty)$ (B) $[4,\infty)$ (C) $(4,\infty)$ (D) none of these

[MATRIX MATCH TYPE]

16. **Column-I** | **Column-II**

(A) Number of common solutions of the equations $2\cos^2x - 3\cos x + 1 = 0$ and $\tan\left(\frac{3x}{4}\right) + 1 = 0$, where $-\pi < x \le 3\pi$, is equal to — (P) 2

(B) Number of solutions of the equation $\sin^2x + 2\sin x - 4\cos x - \sin 2x = 0$ in $(0, 2\pi)$ is equal to — (Q) 12

(C) If the sum of all possible values of $x \in (0, 2\pi)$ satisfying the equation $2\cos x \operatorname{cosec} x - 4\cos x - \operatorname{cosec} x = -2$ is equal to $\frac{k\pi}{4}$ $(k \in N)$, then the value of k is — (R) 4

(D) Let $x - y = \frac{1}{4}$ and $\cos(\pi x)\cdot\cos(\pi y) = \frac{1}{2\sqrt{2}}$ where $x, y \in (0, 2)$ then the number of ordered pair(s) (x, y) is equal to — (S) 3

(T) 1

17. **Column I** | **Column II**

(A) The number of real roots of the equation $\cos^7x + \sin^4x = 1$ in $(-\pi, \pi)$ is — (P) 1

(B) The number of solutions of the equation $|\cot x| = \cot x + \frac{1}{\sin x}$ $(0 < x < \pi)$ is — (Q) 2

(C) If $\sin\theta + \sin\phi = \frac{1}{2}$ and $\cos\theta + \cos\phi = 2$, then value of $\cot\left(\frac{\theta+\phi}{2}\right)$ is — (R) 0

(D) The number of values of $x \in [-2\pi, 2\pi]$, which satisfy $\operatorname{cosec} x = 1 + \cot x$ — (S) 3

(T) not exists

EXERCISE (S-1)

1. Find all the values of θ satisfying the equation; $\sin\theta + \sin 5\theta = \sin 3\theta$ such that $0 \le \theta \le \pi$.

2. Find the number of principal solution of the equation, $\sin x - \sin 3x + \sin 5x = \cos x - \cos 3x + \cos 5x$.

3. Find all values of θ between $0°$ & $180°$ satisfying the equation ; $\cos 6\theta + \cos 4\theta + \cos 2\theta + 1 = 0$.

4. Find all value of θ, between 0 & π, which satisfy the equation; $\cos\theta \,.\, \cos 2\theta \,.\, \cos 3\theta = 1/4$.

5. Find the general solution of the equation, $\sin \pi x + \cos \pi x = 0$. Also find the sum of all solutions in $[0, 100]$.

6. Find the range of y such that the equation , $y + \cos x = \sin x$ has a real solution. For $y = 1$, find x such that $0 < x < 2\pi$.

7. Find the general values of θ for which the quadratic function $(\sin\theta)\, x^2 + (2\cos\theta)x + \dfrac{\cos\theta + \sin\theta}{2}$ is the square of a linear function.

8. Solve the equation for x, $5^{\frac{1}{2}} + 5^{\frac{1}{2} + \log_5 (\sin x)} = 15^{\frac{1}{2} + \log_{15} \cos x}$

9. Solve the equality: $2 \sin 11x + \cos 3x + \sqrt{3} \sin 3x = 0$

10. Solve for x , the equation $\sqrt{13 - 18\tan x} = 6 \tan x - 3$, where $-2\pi < x < 2\pi$.

EXERCISE (JM)

1. Let A and B denote the statements

$$\mathbf{A:} \cos\alpha + \cos\beta + \cos\gamma = 0$$
$$\mathbf{B:} \sin\alpha + \sin\beta + \sin\gamma = 0$$

If $\cos(\beta-\gamma) + \cos(\gamma-\alpha) + \cos(\alpha-\beta) = -\frac{3}{2}$, then - **[AIEEE 2009]**

(1) Both **A** and **B** are true (2) Both **A** and **B** are false

(3) **A** is true and **B** is false (4) **A** is false and **B** is true

2. The possible values of $\theta \in (0, \pi)$ such that $\sin(\theta) + \sin(4\theta) + \sin(7\theta) = 0$ are: **[AIEEE 2011]**

(1) $\frac{2\pi}{9},\frac{\pi}{4},\frac{4\pi}{9},\frac{\pi}{2},\frac{3\pi}{4},\frac{8\pi}{9}$ (2) $\frac{\pi}{4},\frac{5\pi}{12},\frac{\pi}{2},\frac{2\pi}{3},\frac{3\pi}{4},\frac{8\pi}{9}$

(3) $\frac{2\pi}{9},\frac{\pi}{4},\frac{\pi}{2},\frac{2\pi}{3},\frac{3\pi}{4},\frac{35\pi}{36}$ (4) $\frac{2\pi}{9},\frac{\pi}{4},\frac{\pi}{2},\frac{2\pi}{3},\frac{3\pi}{4},\frac{8\pi}{9}$

3. In a ΔPQR, if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then the angle R is equal to : **[AIEEE-2012]**

(1) $\frac{3\pi}{4}$ (2) $\frac{5\pi}{6}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{4}$

4. If $0 \le x < 2\pi$, then the number of real values of x, which satisfy the equation $\cos x + \cos 2x + \cos 3x + \cos 4x = 0$, is :- **[JEE(Main) 2016]**

(1) 9 (2) 3 (3) 5 (4) 7

5. If sum of all the solutions of the equation $8\cos x\cdot\left(\cos\left(\frac{\pi}{6}+x\right).\cos\left(\frac{\pi}{6}-x\right)-\frac{1}{2}\right)=1$ in $[0, \pi]$ is $k\pi$, then k is equal to : **[JEE(Main) 2018]**

(1) $\frac{13}{9}$ (2) $\frac{8}{9}$ (3) $\frac{20}{9}$ (4) $\frac{2}{3}$

6. If $0 \le x < \frac{\pi}{2}$, then the number of values of x for which $\sin x-\sin 2x+\sin 3x = 0$, is **[JEE(Main) 19]**

(1) 2 (2) 1 (3) 3 (4) 4

7. The sum of all values of $\theta \in \left(0,\frac{\pi}{2}\right)$ satisfying $\sin^2 2\theta+\cos^4 2\theta=\frac{3}{4}$ is : **[JEE(Main) 19]**

(1) $\frac{\pi}{2}$ (2) π (3) $\frac{3\pi}{8}$ (4) $\frac{5\pi}{4}$

8. Let S = {$\theta \in [-2\pi, 2\pi] : 2\cos^2\theta + 3\sin\theta = 0$}. Then the sum of the elements of S is

[JEE(Main) 19]

(1) $\frac{13\pi}{6}$ (2) π (3) 2π (4) $\frac{5\pi}{3}$

9. All the pairs (x, y) that satisfy the inequality $2^{\sqrt{\sin^2 x - 2\sin x + 5}} \cdot \frac{1}{4^{\sin^2 y}} \le 1$ also satisfy the eauation.

[JEE(Main) 19]

(1) $\sin x = |\sin y|$ (2) $\sin x = 2\sin y$ (3) $2|\sin x| = 3\sin y$ (4) $2\sin x = \sin y$

10. The number of solutions of the equation $1 + \sin^4 x = \cos^2 3x,\ x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ is : **[JEE(Main) 19]**

(1) 5 (2) 4 (3) 7 (4) 3

11. Let S be the set of all $\alpha \in R$ such that the equation, $\cos 2x + \alpha\sin x = 2\alpha - 7$ has a solution. Then S is equal to : **[JEE(Main) 19]**

(1) [2, 6] (2) [3,7] (3) R (4) [1,4]

12. If $\sin^4\alpha + 4\cos^4\beta + 2 = 4\sqrt{2}\sin\alpha\cos\beta;\ \alpha,\ \beta \in [0, \pi]$, then $\cos(\alpha + \beta) - \cos(\alpha - \beta)$ is equal to :

[JEE(Main) 2019]

(1) 0 (2) $-\sqrt{2}$ (3) –1 (4) $\sqrt{2}$

13. The number of distinct solutions of the equation $\log_{\frac{1}{2}}|\sin x| = 2 - \log_{\frac{1}{2}}|\cos x|$ in the interval $[0, 2\pi]$, is ______. **[JEE(Main) 20]**

EXERCISE (JA)

1. The number of values of θ in the interval $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ such that $\theta \neq \frac{n\pi}{5}$ for n = 0, ±1,±2 and $\tan\theta = \cot5\theta$ as well as $\sin2\theta = \cos4\theta$, is **[JEE 2010, 3]**

2. The positive integer value of n > 3 satisfying the equation

$$\frac{1}{\sin\left(\frac{\pi}{n}\right)}=\frac{1}{\sin\left(\frac{2\pi}{n}\right)}+\frac{1}{\sin\left(\frac{3\pi}{n}\right)}$$ is **[JEE 2011, 4]**

3. Let $\theta,\ \varphi \in [0,2\pi]$ be such that

$$2\cos\theta(1-\sin\varphi)=\sin^2\theta\left(\tan\frac{\theta}{2}+\cot\frac{\theta}{2}\right)\cos\varphi-1,\ \tan(2\pi-\theta)>0 \text{ and } -1<\sin\theta<-\frac{\sqrt{3}}{2}.$$

Then φ **cannot** satisfy- **[JEE 2012, 4M]**

(A) $0<\varphi<\frac{\pi}{2}$ (B) $\frac{\pi}{2}<\varphi<\frac{4\pi}{3}$ (C) $\frac{4\pi}{3}<\varphi<\frac{3\pi}{2}$ (D) $\frac{3\pi}{2}<\varphi<2\pi$

4. For $x \in (0, \pi)$, the equation $\sin x + 2\sin2x - \sin3x = 3$ has **[JEE(Advanced)-2014, 3(–1)]**

(A) infinitely many solutions (B) three solutions

(C) one solution (D) no solution

5. The number of distinct solutions of equation $\frac{5}{4}\cos^2 2x+\cos^4 x+\sin^4 x+\cos^6 x+\sin^6 x=2$ in the interval $[0, 2\pi]$ is **[JEE 2015, 4M, –0M]**

6. Let $S=\left\{x\in(-\pi,\pi): x\neq 0,\pm\frac{\pi}{2}\right\}$. The sum of all distinct solution of the equation $\sqrt{3}\sec x+\text{cosec}x+2(\tan x-\cot x)=0$ in the set S is equal to -

[JEE(Advanced)-2016, 3(–1)]

(A) $-\frac{7\pi}{9}$ (B) $-\frac{2\pi}{9}$ (C) 0 (D) $\frac{5\pi}{9}$

7. Let a, b, c be three non-zero real numbers such that the equation

$$\sqrt{3}a\cos x+2b\sin x=c,\quad x\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

has two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then the value of $\frac{b}{a}$ is ______

[JEE(Advanced)-2018, 3(0)]

Answer the following by appropriately matching the lists based on the information given in the paragraph

Let $f(x) = \sin(\pi \cos x)$ and $g(x) = \cos(2\pi \sin x)$ be two functions defined for $x > 0$. Define the following sets whose elements are written in the increasing order :

$$X = \{x : f(x) = 0\}, \qquad Y = \{x : f'(x) = 0\}$$

$$Z = \{x : g(x) = 0\}, \qquad W = \{x : g'(x) = 0\}.$$

List-I contains the sets X,Y,Z and W. List -II contains some information regarding these sets.

List-I		**List-II**	
(I)	X	(P)	$\supseteq \left\{\frac{\pi}{2}, \frac{3\pi}{2}, 4\pi, 7\pi\right\}$
(II)	Y	(Q)	an arithmetic progression
(III)	Z	(R)	NOT an arithmetic progression
(IV)	W	(S)	$\supseteq \left\{\frac{\pi}{6}, \frac{7\pi}{6}, \frac{13\pi}{6}\right\}$
		(T)	$\supseteq \left\{\frac{\pi}{3}, \frac{2\pi}{3}, \pi\right\}$
		(U)	$\supseteq \left\{\frac{\pi}{6}, \frac{3\pi}{4}\right\}$

8. Which of the following is the only CORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (II), (R), (S) (2) (I), (P), (R) (3) (II), (Q), (T) (4) (I), (Q), (U)

9. Which of the following is the only CORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (IV), (Q), (T) (2) (IV), (P), (R), (S) (3) (III), (R), (U) (4) (III), (P), (Q), (U)

ANSWER KEY

EXERCISE (O-1)

1. C	**2.** D	**3.** C	**4.** C	**5.** C	**6.** B	**7.** A	**8.** C
9. C	**10.** B	**11.** C	**12.** B	**13.** C	**14.** D	**15.** C	**16.** C
17. D	**18.** B	**19.** D	**20.** A	**21.** B	**22.** A	**23.** D	**24.** A
25. C	**26.** A	**27.** B	**28.** D	**29.** A	**30.** C		

EXERCISE (O-2)

1. A,B,C,D	**2.** B,C	**3.** A,B	**4.** B,C	**5.** A,C
6. A,B,C,D	**7.** B,C	**8.** A,C,D	**9.** A,C	**10.** B,C
11. C	**12.** C	**13.** A	**14.** B	**15.** B

16. **(A)→(P); (B)→(P); (C)→(Q); (D)→(R)**

17. **(A) → (S), (B) → (P), (C) → (T), (D) → (Q)**

EXERCISE (S-1)

1. $0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{6}$ & π

2. 10 solutions

3. $30°, 45°, 90°, 135°, 150°$

4. $\frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$

5. $x = n - \frac{1}{4}$, $n \in I$; sum = 5025

6. $-\sqrt{2} \le y \le \sqrt{2}$; $\frac{\pi}{2}, \pi$

7. $2n\pi + \frac{\pi}{4}$ or $(2n+1)\pi - \tan^{-1}2$, $n \in I$

8. $x = 2n\pi + \frac{\pi}{6}$, $n \in I$

9. $x = \frac{n\pi}{7} - \frac{\pi}{84}$ or $x = \frac{n\pi}{4} + \frac{7\pi}{48}$, $n \in I$

10. $\alpha - 2\pi$; $\alpha - \pi$, α, $\alpha + \pi$, where $\tan\alpha = \frac{2}{3}$

EXERCISE (JM)

1. 1	**2.** 1	**3.** 3	**4.** 4	**5.** 1	**6.** 1	**7.** 1	**8.** 3
9. 1	**10.** 1	**11.** 1	**12.** 2	**13.** 8.00			

EXERCISE (JA)

1. 3	**2.** 7	**3.** A,C,D	**4.** D	**5.** 8	**6.** C	**7.** 0.5
8. 3	**9.** 2					

CHAPTER 4

SOLUTIONS OF TRIANGLE

Chapter 04 Contents

IMPORTANT NOTES

CHAPTER 4

SOLUTIONS OF TRIANGLE

SOLUTIONS OF TRIANGLE

The process of calculating the sides and angles of triangle using given information is called solution of triangle.

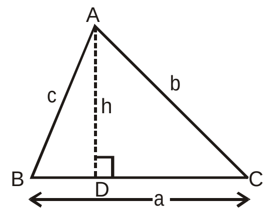
In a ΔABC, the angles are denoted by capital letters A, B and C and the length of the sides opposite these angle are denoted by small letter a, b and c respectively.

1. SINE FORMULAE :

In any triangle ABC

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\lambda=\frac{abc}{2\Delta}=2R$$

where R is circumradius and Δ is area of triangle.



Illustrations

Illustration 1 : Angles of a triangle are in 4 : 1 : 1 ratio. The ratio between its greatest side and perimeter is

(A) $\frac{3}{2+\sqrt{3}}$ (B) $\frac{\sqrt{3}}{2+\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2-\sqrt{3}}$ (D) $\frac{1}{2+\sqrt{3}}$

Solution : Angles are in ratio 4 : 1 : 1.

$\Rightarrow$ angles are 120°, 30°, 30°.

If sides opposite to these angles are a, b, c respectively, then a will be the greatest side.

Now from sine formula $\frac{a}{\sin 120°}=\frac{b}{\sin 30°}=\frac{c}{\sin 30°}$

$$\Rightarrow \frac{a}{\sqrt{3}/2}=\frac{b}{1/2}=\frac{c}{1/2}$$

$$\Rightarrow \frac{a}{\sqrt{3}}=\frac{b}{1}=\frac{c}{1}=k \text{ (say)}$$

then $a=\sqrt{3}k$, perimeter $=(2+\sqrt{3})k$

$\therefore$ required ratio $=\frac{\sqrt{3}k}{(2+\sqrt{3})k}=\frac{\sqrt{3}}{2+\sqrt{3}}$ **Ans. (B)**

Illustration 2 : In triangle ABC, if b = 3, c = 4 and $\angle B = \pi/3$, then number of such triangles is -

(A) 1 (B) 2 (C) 0 (D) infinite

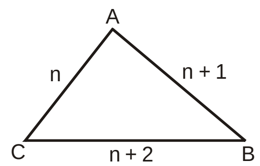
Solution : Using sine formulae $\frac{\sin B}{b} = \frac{\sin C}{c}$

$$\Rightarrow \quad \frac{\sin \pi/3}{3} = \frac{\sin C}{4} \quad \Rightarrow \quad \frac{\sqrt{3}}{6} = \frac{\sin C}{4} \quad \Rightarrow \quad \sin C = \frac{2}{\sqrt{3}} > 1 \text{ which is not possible.}$$

Hence there exist no triangle with given elements. **Ans. (C)**

Illustration 3 : The sides of a triangle are three consecutive natural numbers and its largest angle is twice the smallest one. Determine the sides of the triangle.

Solution : Let the sides be n, n + 1, n + 2 cms.



i.e. AC = n, AB = n + 1, BC = n + 2

Smallest angle is B and largest one is A.

Here, $\angle A = 2\angle B$

Also, $\angle A + \angle B + \angle C = 180°$

$\Rightarrow \quad 3\angle B + \angle C = 180° \quad \Rightarrow \angle C = 180° - 3\angle B$

We have, sine law as,

$$\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1} \quad \Rightarrow \quad \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180 - 3B)}{n+1}$$

$$\Rightarrow \quad \underset{\text{(i)}}{\frac{\sin 2B}{n+2}} = \underset{\text{(ii)}}{\frac{\sin B}{n}} = \underset{\text{(iii)}}{\frac{\sin 3B}{n+1}}$$

from (i) and (ii);

$$\frac{2\sin B\cos B}{n+2} = \frac{\sin B}{n} \quad \Rightarrow \quad \cos B = \frac{n+2}{2n} \qquad \text{......... (iv)}$$

and from (ii) and (iii);

$$\frac{\sin B}{n} = \frac{3\sin B - 4\sin^3 B}{n+1} \quad \Rightarrow \quad \frac{\sin B}{n} = \frac{\sin B(3 - 4\sin^2 B)}{n+1}$$

$$\Rightarrow \quad \frac{n+1}{n} = 3 - 4(1 - \cos^2 B) \qquad \text{.......... (v)}$$

from (iv) and (v), we get

$$\frac{n+1}{n} = -1 + 4\left(\frac{n+2}{2n}\right)^2 \quad \Rightarrow \quad \frac{n+1}{n} + 1 = \left(\frac{n^2 + 4n + 4}{n^2}\right)$$

$$\Rightarrow \quad \frac{2n+1}{n}=\frac{n^2+4n+4}{n^2} \quad \Rightarrow \quad 2n^2 + n = n^2 + 4n + 4$$

$$\Rightarrow \quad n^2 - 3n - 4 = 0 \quad \Rightarrow \quad (n-4)(n+1) = 0$$

$n = 4$ or -1

where $n \neq -1$

$\therefore \quad n = 4$. Hence the sides are 4, 5, 6 **Ans.**

Do yourself - 1

(i) If in a ΔABC, $\angle A = \frac{\pi}{6}$ and $b : c = 2 : \sqrt{3}$, find $\angle B$.

(ii) Show that, in any ΔABC : $a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0$.

(iii) If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, show that a^2, b^2, c^2 are in A.P.

2. COSINE FORMULAE :

(a) $\cos A = \frac{b^2+c^2-a^2}{2bc}$ **(b)** $\cos B = \frac{c^2+a^2-b^2}{2ca}$ **(c)** $\cos C = \frac{a^2+b^2-c^2}{2ab}$

or $a^2 = b^2 + c^2 - 2bc \cos A$

Illustrations

Illustration 4 : In a triangle ABC, if $B = 30°$ and $c = \sqrt{3}\,b$, then A can be equal to -

(A) 45° (B) 60° (C) 90° (D) 120°

Solution : We have $\cos B = \frac{c^2+a^2-b^2}{2ca} \Rightarrow \frac{\sqrt{3}}{2} = \frac{3b^2+a^2-b^2}{2\times\sqrt{3}b\times a}$

$\Rightarrow \quad a^2 - 3ab + 2b^2 = 0 \Rightarrow (a - 2b)(a - b) = 0$

$\Rightarrow \quad$ Either $a = b \Rightarrow A = 30°$

or $\quad a = 2b \Rightarrow a^2 = 4b^2 = b^2 + c^2 \Rightarrow A = 90°$. **Ans. (C)**

Illustration 5 : In a triangle ABC, $(a^2 - b^2 - c^2) \tan A + (a^2 - b^2 + c^2) \tan B$ is equal to -

(A) $(a^2 + b^2 - c^2) \tan C$ (B) $(a^2 + b^2 + c^2) \tan C$

(C) $(b^2 + c^2 - a^2) \tan C$ (D) none of these

Solution : Using cosine law :

The given expression is equal to $-2\,bc \cos A \tan A + 2\,ac \cos B \tan B$

$$= 2abc\left(-\frac{\sin A}{a}+\frac{\sin B}{b}\right)=0$$ **Ans. (D)**

Do yourself - 2

(i) If a : b : c = 4 : 5 : 6, then show that $\angle C = 2\angle A$.

(ii) In any ΔABC, prove that

(a) $\dfrac{\cos A}{a}+\dfrac{\cos B}{b}+\dfrac{\cos C}{c}=\dfrac{a^2+b^2+c^2}{2abc}$

(b) $\dfrac{b^2}{a}\cos A+\dfrac{c^2}{b}\cos B+\dfrac{a^2}{c}\cos C=\dfrac{a^4+b^4+c^4}{2abc}$

3. PROJECTION FORMULAE :

(a) $b\cos C + c\cos B = a$ **(b)** $c\cos A + a\cos C = b$ **(c)** $a\cos B + b\cos A = c$

Illustrations

Illustration 6 : In a ΔABC, $c\cos^2\dfrac{A}{2}+a\cos^2\dfrac{C}{2}=\dfrac{3b}{2}$, then show a, b, c are in A.P.

Solution : Here, $\dfrac{c}{2}(1+\cos A)+\dfrac{a}{2}(1+\cos C)=\dfrac{3b}{2}$

$\Rightarrow$ $a + c + (c\cos A + a\cos C) = 3b$

$\Rightarrow$ $a + c + b = 3b$ {using projection formula}

$\Rightarrow$ $a + c = 2b$

which shows a, b, c are in A.P.

Do yourself - 3

(i) In a ΔABC, if $\angle A=\dfrac{\pi}{4}$, $\angle B=\dfrac{5\pi}{12}$, show that $a+c\sqrt{2}=2b$.

(ii) In a ΔABC, prove that : (a) $b(a\cos C - c\cos A) = a^2 - c^2$

(b) $2\left(b\cos^2\dfrac{C}{2}+c\cos^2\dfrac{B}{2}\right)=a+b+c$

4. NAPIER'S ANALOGY (TANGENT RULE) :

(a) $\tan\left(\dfrac{B-C}{2}\right)=\dfrac{b-c}{b+c}\cot\dfrac{A}{2}$ **(b)** $\tan\left(\dfrac{C-A}{2}\right)=\dfrac{c-a}{c+a}\cot\dfrac{B}{2}$

(c) $\tan\left(\dfrac{A-B}{2}\right)=\dfrac{a-b}{a+b}\cot\dfrac{C}{2}$

Illustrations

Illustration 7 : In a ΔABC, the tangent of half the difference of two angles is one-third the tangent of half the sum of the angles. Determine the ratio of the sides opposite to the angles.

Solution : Here, $\tan\left(\frac{A-B}{2}\right)=\frac{1}{3}\tan\left(\frac{A+B}{2}\right)$ .... (i)

using Napier's analogy, $\tan\left(\frac{A-B}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$ .... (ii)

from (i) & (ii) ;

$$\frac{1}{3}\tan\left(\frac{A+B}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right) \Rightarrow \frac{1}{3}\cot\left(\frac{C}{2}\right)=\frac{a-b}{a+b}.\cot\left(\frac{C}{2}\right)$$

$\{$as $A + B + C = \pi$ $\therefore \tan\left(\frac{B+C}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)=\cot\frac{C}{2}\}$

$\Rightarrow$ $\frac{a-b}{a+b}=\frac{1}{3}$ or $3a - 3b = a + b$

$2a = 4b$ or $\frac{a}{b}=\frac{2}{1}\Rightarrow\frac{b}{a}=\frac{1}{2}$

Thus the ratio of the sides opposite to the angles is $b : a = 1 : 2$. **Ans.**

Do yourself - 4

(i) In any ΔABC, prove that $\frac{b-c}{b+c}=\frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$

(ii) If ΔABC is right angled at C, prove that : (a) $\tan\frac{A}{2}=\sqrt{\frac{c-b}{c+b}}$ (b) $\sin(A-B)=\frac{a^2-b^2}{a^2+b^2}$

5. HALF ANGLE FORMULAE :

$s=\frac{a+b+c}{2}$ = semi-perimeter of triangle.

(a) (i) $\sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}$ (ii) $\sin\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{ca}}$ (iii) $\sin\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{ab}}$

(b) (i) $\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}}$ (ii) $\cos\frac{B}{2}=\sqrt{\frac{s(s-b)}{ca}}$ (iii) $\cos\frac{C}{2}=\sqrt{\frac{s(s-c)}{ab}}$

(c) (i) $\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\frac{\Delta}{s(s-a)}$ (ii) $\tan\frac{B}{2}=\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}=\frac{\Delta}{s(s-b)}$ (iii) $\tan\frac{C}{2}=\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}=\frac{\Delta}{s(s-c)}$

(d) **Area of Triangle**

$$\Delta=\sqrt{s(s-a)(s-b)(s-c)}=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C=\frac{1}{2}ap_1=\frac{1}{2}bp_2=\frac{1}{2}cp_3,$$

where p_1,p_2,p_3 are altitudes from vertices A,B,C respectively.

Illustrations

Illustration 8 : If in a triangle ABC, CD is the angle bisector of the angle ACB, then CD is equal to-

(A) $\frac{a+b}{2ab}\cos\frac{C}{2}$ (B) $\frac{2ab}{a+b}\sin\frac{C}{2}$ (C) $\frac{2ab}{a+b}\cos\frac{C}{2}$ (D) $\frac{b\sin\angle DAC}{\sin(B+C/2)}$

Solution : $\Delta CAB=\Delta CAD+\Delta CDB$

$$\Rightarrow\quad \frac{1}{2}ab\sin C=\frac{1}{2}b.CD.\sin\left(\frac{C}{2}\right)+\frac{1}{2}a.CD\ \sin\left(\frac{C}{2}\right)$$

$$\Rightarrow\quad CD(a+b)\ \sin\left(\frac{C}{2}\right)=ab\left(2\sin\left(\frac{C}{2}\right)\cos\left(\frac{C}{2}\right)\right)$$

So $\quad CD=\frac{2ab\cos(C/2)}{(a+b)}$

and in ΔCAD, $\frac{CD}{\sin\angle DAC}=\frac{b}{\sin\angle CDA}$ (by sine rule)

$$\Rightarrow\quad CD=\frac{b\sin\angle DAC}{\sin(B+C/2)}$$ **Ans. (C,D)**

Illustration 9 : If Δ is the area and 2s the sum of the sides of a triangle, then show $\Delta\le\frac{s^2}{3\sqrt{3}}$.

Solution : We have, $2s=a+b+c$, $\Delta^2=s(s-a)(s-b)(s-c)$

Now, A.M. $\ge$ G.M.

$$\frac{(s-a)+(s-b)+(s-c)}{3}\ge\{(s-a)(s-b)(s-c)\}^{1/3}$$

or $\quad \frac{3s-2s}{3}\ge\left(\frac{\Delta^2}{s}\right)^{1/3}$

or $\quad \frac{s}{3}\ge\left(\frac{\Delta^2}{s}\right)^{1/3}$

or $\quad \frac{\Delta^2}{s}\le\frac{s^3}{27}\quad\Rightarrow\quad\Delta\le\frac{s^2}{3\sqrt{3}}$ **Ans.**

Do yourself - 5

(i) Given a = 6, b = 8, c = 10. Find

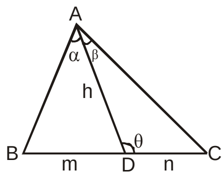
(a) $\sin A$ (b) $\tan A$ (c) $\sin\frac{A}{2}$ (d) $\cos\frac{A}{2}$ (e) $\tan\frac{A}{2}$ (f) Δ

(ii) Prove that in any ΔABC, (abcs) $\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}=\Delta^2$.

6. m-n THEOREM :

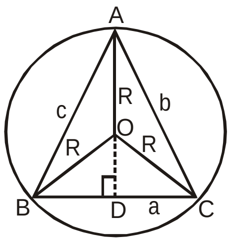
$(m + n)\cot\theta = m\cot\alpha - n\cot\beta$

$(m + n)\cot\theta = n\cot B - m\cot C$.



7. RADIUS OF THE CIRCUMCIRCLE 'R' :

Circumcentre is the point of intersection of perpendicular bisectors of the sides and distance between circumcentre & vertex of triangle is called circumradius 'R'.

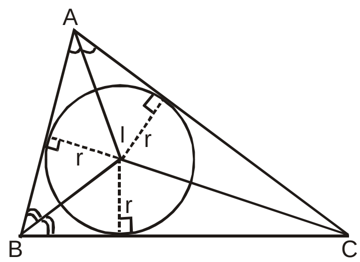


$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C}=\frac{abc}{4\Delta}.$$

8. RADIUS OF THE INCIRCLE 'r' :

Point of intersection of internal angle bisectors is incentre and perpendicular distance of incentre from any side is called inradius 'r'.

$$r=\frac{\Delta}{s}=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2}=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$



$$=a\frac{\sin\frac{B}{2}\sin\frac{C}{2}}{\cos\frac{A}{2}}=b\frac{\sin\frac{A}{2}\sin\frac{C}{2}}{\cos\frac{B}{2}}=c\frac{\sin\frac{B}{2}\sin\frac{A}{2}}{\cos\frac{C}{2}}$$

Illustrations

Illustration 10 : In a triangle ABC, if a : b : c = 4 : 5 : 6, then ratio between its circumradius and inradius is-

(A) $\frac{16}{7}$ (B) $\frac{16}{9}$ (C) $\frac{7}{16}$ (D) $\frac{11}{7}$

Solution : $\frac{R}{r} = \frac{abc}{4\Delta} \Big/ \frac{\Delta}{s} = \frac{(abc)s}{4\Delta^2} \Rightarrow \frac{R}{r} = \frac{abc}{4(s-a)(s-b)(s-c)}$ ....(i)

$\because$ a : b : c = 4 : 5 : 6 $\Rightarrow \frac{a}{4} = \frac{b}{5} = \frac{c}{6}$ = k (say)

$\Rightarrow$ a = 4k, b = 5k, c = 6k

$\therefore \quad s = \frac{a+b+c}{2} = \frac{15k}{2}$, $s - a = \frac{7k}{2}$, $s - b = \frac{5k}{2}$, $s - c = \frac{3k}{2}$

using (i) in these values $\frac{R}{r} = \frac{(4k)(5k)(6k)}{4\left(\frac{7k}{2}\right)\left(\frac{5k}{2}\right)\left(\frac{3k}{2}\right)} = \frac{16}{7}$ **Ans. (A)**

Illustration 11 : If A, B, C are the angles of a triangle, prove that : cosA + cosB + cosC = $1+\frac{r}{R}$.

Solution : cosA + cosB + cosC = $2\cos\left(\frac{A+B}{2}\right).\cos\left(\frac{A-B}{2}\right)+\cos C$

$= 2\sin\frac{C}{2}.\cos\left(\frac{A-B}{2}\right)+1-2\sin^2\frac{C}{2} = 1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\sin\left(\frac{C}{2}\right)\right]$

$= 1+2\sin\frac{C}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right] \quad \left\{\because \frac{C}{2} = 90° - \left(\frac{A+B}{2}\right)\right\}$

$= 1+2\sin\frac{C}{2}.2\sin\frac{A}{2}.\sin\frac{B}{2} = 1+4\sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2}$

$= 1+\frac{r}{R}$ {as, r = 4R sin A/2 . sinB/2 . sinC/2}

$\Rightarrow$ cosA + cosB + cosC = $1+\frac{r}{R}$. Hence proved

Do yourself - 6

(i) If in ΔABC, a = 3, b = 4 and c = 5, find

(a) Δ (b) R (c) r

(ii) In a ΔABC, show that :

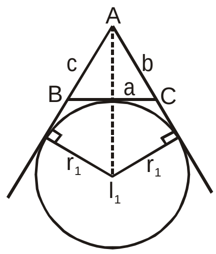
(a) $\frac{a^2-b^2}{c} = 2R\sin(A-B)$ (b) $r\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{\Delta}{4R}$ (c) $a+b+c = \frac{abc}{2Rr}$

(iii) Let Δ & Δ' denote the areas of a Δ and that of its incircle. Prove that

$$\Delta : \Delta' = \left(\cot\frac{A}{2}.\cot\frac{B}{2}.\cot\frac{C}{2}\right) : \pi$$

9. RADII OF THE EX-CIRCLES :

Point of intersection of two external angles and one internal angle bisectors is excentre and perpendicular distance of excentre from any side is called exradius. If r_1 is the radius of escribed circle opposite to $\angle A$ of ΔABC and so on, then -



(a) $$r_1 = \frac{\Delta}{s-a} = s\tan\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{a\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}}$$

(b) $$r_2 = \frac{\Delta}{s-b} = s\tan\frac{B}{2} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} = \frac{b\cos\frac{A}{2}\cos\frac{C}{2}}{\cos\frac{B}{2}}$$

(c) $$r_3 = \frac{\Delta}{s-c} = s\tan\frac{C}{2} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} = \frac{c\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}}$$

I_1, I_2 and I_3 are taken as ex-centre opposite to vertex A, B, C repsectively.

Illustrations

Illustration 12 : Value of the expression $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}$ is equal to -

(A) 1 (B) 2 (C) 3 (D) 0

Solution : $\frac{(b-c)}{r_1}+\frac{(c-a)}{r_2}+\frac{(a-b)}{r_3}$

$$\Rightarrow \quad (b-c)\left(\frac{s-a}{\Delta}\right)+(c-a)\left(\frac{s-b}{\Delta}\right)+(a-b).\left(\frac{s-c}{\Delta}\right)$$

$$\Rightarrow \quad \frac{(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)}{\Delta}$$

$$=\frac{s(b-c+c-a+a-b)-[ab-ac+bc-ba+ac-bc]}{\Delta}=\frac{0}{\Delta}=0$$

Thus, $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$ **Ans. (D)**

Illustration 13 : If $r_1 = r_2 + r_3 + r$, prove that the triangle is right angled.

Solution : We have, $r_1 - r = r_2 + r_3$

$$\Rightarrow \quad \frac{\Delta}{s-a}-\frac{\Delta}{s}=\frac{\Delta}{s-b}+\frac{\Delta}{s-c} \qquad \Rightarrow \quad \frac{s-s+a}{s(s-a)}=\frac{s-c+s-b}{(s-b)(s-c)}$$

$$\Rightarrow \quad \frac{a}{s(s-a)}=\frac{2s-(b+c)}{(s-b)(s-c)} \qquad \{as,\ 2s = a + b + c\}$$

$$\Rightarrow \quad \frac{a}{s(s-a)}=\frac{a}{(s-b)(s-c)} \qquad \Rightarrow \quad s^2 - (b + c)\, s + bc = s^2 - as$$

$$\Rightarrow \quad s(-a + b + c) = bc \qquad \Rightarrow \quad \frac{(b+c-a)(a+b+c)}{2}=bc$$

$$\Rightarrow \quad (b + c)^2 - (a)^2 = 2bc \qquad \Rightarrow \quad b^2 + c^2 + 2bc - a^2 = 2bc$$

$$\Rightarrow \quad b^2 + c^2 = a^2$$

$\therefore \quad \angle A = 90°.$ **Ans.**

Do yourself - 7

(i) In an equilateral ΔABC, $R = 2$, find

(a) r (b) r_1 (c) a

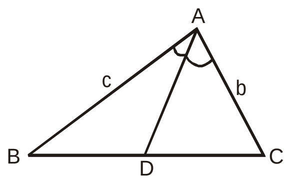
(ii) In a ΔABC, show that

(a) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$ (b) $\frac{1}{4}r^2s^2\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right) = R$

(c) $\sqrt{rr_1r_2r_3} = \Delta$

10. ANGLE BISECTORS & MEDIANS :

An angle bisector divides the base in the ratio of corresponding sides.



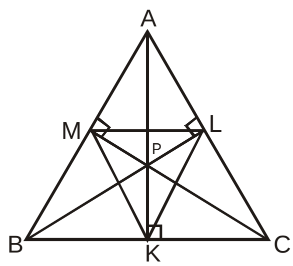
$$\frac{BD}{CD} = \frac{c}{b} \Rightarrow BD = \frac{ac}{b+c} \quad \& \quad CD = \frac{ab}{b+c}$$

If m_a and β_a are the lengths of a median and an angle bisector from the angle A then,

$$m_a = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \text{ and } \beta_a = \frac{2bc\cos\frac{A}{2}}{b+c}$$

Note that $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$

11. ORTHOCENTRE :



(a) Point of intersection of altitudes is orthocentre & the triangle KLM which is formed by joining the feet of the altitudes is called the orthic triangle (special case of pedal triangle).

(b) The distances of the orthocentre from the angular points of the ΔABC are $2R\cos A$, $2R\cos B$, & $2R\cos C$.

(c) The distance of P from sides are $2R\cos B\cos C$, $2R\cos C\cos A$ and $2R\cos A\cos B$.

Do yourself - 8

(i) If x, y, z are the distance of the vertices of ΔABC respectively from the orthocentre, then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$

(ii) If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, prove that

(a) $p_1p_2p_3=\frac{a^2b^2c^2}{8R^3}$ (b) $\Delta=\sqrt{\frac{1}{2}Rp_1p_2p_3}$

(iii) In a ΔABC, AD is altitude and H is the orthocentre prove that AH : DH = (tanB + tanC) : tanA

(iv) In a ΔABC, the lengths of the bisectors of the angle A, B and C are x, y, z respectively. Show that $\frac{1}{x}\cos\frac{A}{2}+\frac{1}{y}\cos\frac{B}{2}+\frac{1}{z}\cos\frac{C}{2}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

12. THE DISTANCES BETWEEN THE SPECIAL POINTS :

(a) The distance between circumcentre and orthocentre is = $R\sqrt{1-8\cos A\cos B\cos C}$

(b) The distance between circumcentre and incentre is $=\sqrt{R^2-2Rr}$

(c) The distance between incentre and orthocentre is $=\sqrt{2r^2-4R^2\cos A\cos B\cos C}$

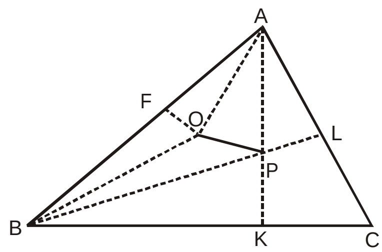
(d) The distances between circumcentre & excentres are

$$OI_1=R\sqrt{1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=\sqrt{R^2+2Rr_1}\ \text{\& so on.}$$

Illustrations

Illustration 14 : Prove that the distance between the circumcentre and the orthocentre of a triangle ABC is $R\sqrt{1-8\cos A\cos B\cos C}$.

Solution : Let O and P be the circumcentre and the orthocentre respectively. If OF is the perpen dicular to AB, we have $\angle OAF = 90° - \angle AOF = 90° - C$. Also $\angle PAL = 90° - C$.
Hence, $\angle OAP = A - \angle OAF - \angle PAL = A - 2(90° - C) = A + 2C - 180°$
$= A + 2C - (A + B + C) = C - B$.



Also $OA = R$ and $PA = 2R\cos A$.
Now in ΔAOP,
$OP^2 = OA^2 + PA^2 - 2OA.\ PA\ \cos OAP$
$= R^2 + 4R^2\cos^2 A - 4R^2\cos A\cos(C-B)$
$= R^2 + 4R^2\cos A[\cos A - \cos(C-B)]$
$= R^2 - 4R^2\cos A[\cos(B+C) + \cos(C-B)] = R^2 - 8R^2\cos A\cos B\cos C$.
Hence $OP = R\sqrt{1-8\cos A\cos B\cos C}$. **Ans.**

13. SOLUTION OF TRIANGLES :

The three sides a,b,c and the three angles A,B,C are called the elements of the triangle ABC. When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

* If the three sides a,b,c are given, angle A is obtained from $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$

or $\cos A = \frac{b^2+c^2-a^2}{2bc}$ .B and C can be obtained in the similar way.

* If two sides b and c and the included angle A are given, then $\tan\frac{B-C}{2} = \frac{b-c}{b+c}\cot\frac{A}{2}$ gives

$\frac{B-C}{2}$. Also $\frac{B+C}{2} = 90° - \frac{A}{2}$, so that B and C can be evaluated. The third side is given by

$a = b\frac{\sin A}{\sin B}$

or $a^2 = b^2 + c^2 - 2bc\cos A$.

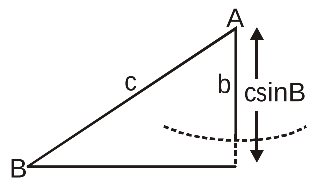
* If two sides b and c and an angle opposite the one of them (say B) are given then

$\sin C = \dfrac{c}{b}\sin B,\ A = 180° - (B+C)$ and $a = \dfrac{b\sin A}{\sin B}$ given the remaining elements.

Case I :

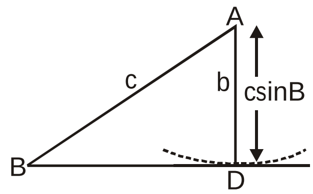
$b < c \sin B$.

We draw the side c and angle B. Now it is obvious from the figure that there is no triangle possible.



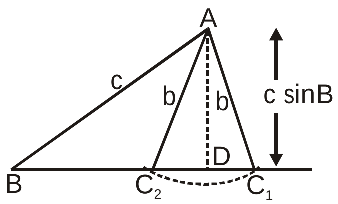
Case II :

$b = c \sin B$ and B is an acute angle, there is only one triangle possible. and it is right-angled at C.



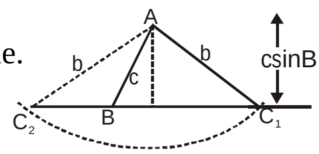
Case III :

$b > c \sin B$, $b < c$ and B is an acute angle, then there are two triangles possible for two values of angle C.



Case IV :

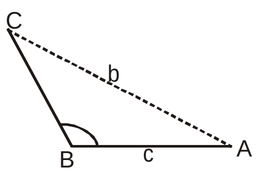
$b > c \sin B$, $c < b$ and B is an acute angle, then there is only one triangle.



Case V :

$b > c \sin B$, $c > b$ and B is an obtuse angle.

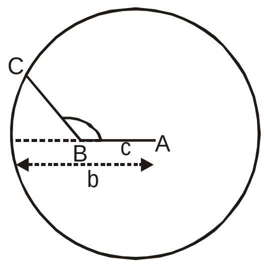
For any choice of point C, b will be greater than c which is a contradication as $c > b$ (given). So there is no triangle possible.



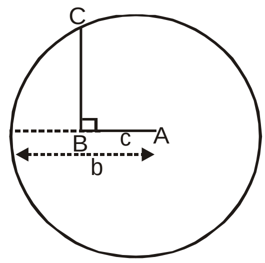
Case VI :

$b > c \sin B$, $c < b$ and B is an obtuse angle.

We can see that the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.



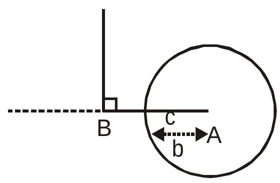
Case VII :



b > c and B = 90°.

Again the circle with A as centre and b as radius will cut the line only in one point. So only one triangle is possible.

Case VIII :



b $\leq$ c and B = 90°.

The circle with A as centre and b as radius will not cut the line in any point. So no triangle is possible.

This is, sometimes, called an ambiguous case.

Alternative Method :

By applying cosine rule, we have cosB = $\dfrac{a^2+c^2-b^2}{2ac}$

$\Rightarrow \quad a^2 - (2c\cos B)a + (c^2 - b^2) = 0 \Rightarrow a = c\cos B \pm\sqrt{(c\cos B)^2-(c^2-b^2)}$

$\Rightarrow \quad a = c\cos B \pm \sqrt{b^2-(c\sin B)^2}$

This equation leads to following cases :

Case-I : If b < csinB, no such triangle is possible.

Case-II: Let b = c sinB. There are further following case :

(a) B is an obtuse angle $\Rightarrow$ cosB is negative. There exists no such triangle.

(b) B is an acute angle $\Rightarrow$ cosB is positive. There exists only one such triangle.

Case-III: Let b > c sin B. There are further following cases :

(a) B is an acute angle $\Rightarrow$ cosB is positive. In this case triangle will exist if and only if

c cosB > $\sqrt{b^2-(c\sin B)^2}$ or c > b $\Rightarrow$ Two such triangle is possible. If c < b, only one such triangle is possible.

(b) B is an obtuse angle $\Rightarrow$ cosB is negative. In this case triangle will exist if and only if

$\sqrt{b^2-(c\sin B)^2}$ > |c cos B| $\Rightarrow$ b > c. So in this case only one such triangle is possible.

If b < c there exists no such triangle.

This is called an ambiguous case.

* If one side a and angles B and C are given, then A = 180° – (B + C), and

$$b = \frac{a\sin B}{\sin A}, c = \frac{a\sin C}{\sin A}.$$

* If the three angles A,B,C are given, we can only find the ratios of the sides a,b,c by using sine rule (since there are infinite similar triangles possible).

Illustrations

Illustration 15 : In the ambiguous case of the solution of triangles, prove that the circumcircles of the two triangles are of same size.

Solution : Let us say b,c and angle B are given in the ambiguous case. Both the triangles will have b and its opposite angle as B. so $\frac{b}{\sin B} = 2R$ will be given for both the triangles.

So their circumradii and therefore their sizes will be same.

Illustration 16 : If a,b and A are given in a triangle and c_1, c_2 are the possible values of the third side, prove that $c_1^2 + c_2^2 - 2c_1c_2 \cos 2A = 4a^2\cos^2 A$.

Solution :

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow c^2 - 2bc\cos A + b^2 - a^2 = 0.$$

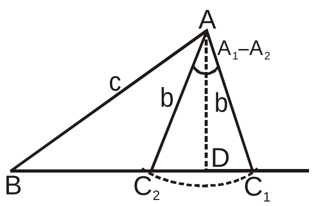
$$c_1 + c_2 = 2bc\cos A \text{ and } c_1c_2 = b^2 - a^2.$$

$$\Rightarrow c_1^2 + c_2^2 - 2c_1c_2\cos 2A = (c_1 + c_2)^2 - 2c_1c_2(1 + \cos 2A)$$

$$= 4b^2\cos^2 A - 2(b^2 - a^2)2\cos^2 A = 4a^2\cos^2 A.$$

Illustration 17 : If b,c,B are given and b < c, prove that $\cos\left(\frac{A_1 - A_2}{2}\right) = \frac{c\sin B}{b}$.

Solution : $\angle C_2AC_1$ is bisected by AD.



$$\Rightarrow \quad \text{In } \Delta AC_2D, \cos\left(\frac{A_1 - A_2}{2}\right) = \frac{AD}{AC_2} = \frac{c\sin B}{b}$$

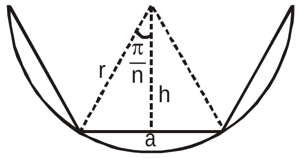
Hence proved.

Do yourself - 9

(i) If b,c,B are given and b<c, prove that $\sin\left(\frac{A_1 - A_2}{2}\right) = \frac{a_1 - a_2}{2b}$

(ii) In a ΔABC, b,c,B (c > b) are gives. If the third side has two values a_1 and a_2 such that $a_1 = 3a_2$, show that $\sin B = \sqrt{\frac{4b^2 - c^2}{3c^2}}$.

14. REGULAR POLYGON :



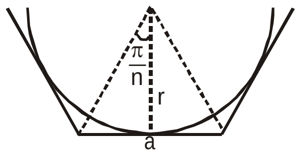
A regular polygon has all its sides equal. It may be inscribed or circumscribed.

(a) Inscribed in circle of radius r :

(i) $a = 2h\ \tan\frac{\pi}{n} = 2r\sin\frac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides inscribed in a circle of radius r are given by $P = 2nr\sin\frac{\pi}{n}$ and $A = \frac{1}{2}nr^2\sin\frac{2\pi}{n}$

(b) Circumscribed about a circle of radius r :



(i) $a = 2r\tan\frac{\pi}{n}$

(ii) Perimeter (P) and area (A) of a regular polygon of n sides circumscribed about a given circle of radius r is given by $P = 2nr\tan\frac{\pi}{n}$ and

$A = nr^2\tan\frac{\pi}{n}$

Do yourself - 10

(i) If the perimeter of a circle and a regular polygon of n sides are equal, then

prove that $\frac{\text{area of the circle}}{\text{area of polygon}} = \frac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}$.

(ii) The ratio of the area of n-sided regular polygon, circumscribed about a circle, to the area of the regular polygon of equal number of sides inscribed in the circle is 4 : 3. Find the value of n.

15. SOME NOTES :

(a) (i) If a cos B = b cos A, then the triangle is isosceles.

(ii) If a cos A = b cos B, then the triangle is isosceles or right angled.

(b) In right angle triangle

(i) $a^2 + b^2 + c^2 = 8R^2$ (ii) $\cos^2 A + \cos^2 B + \cos^2 C = 1$

(c) In equilateral triangle

(i) $R = 2r$ (ii) $r_1 = r_2 = r_3 = \frac{3R}{2}$

(iii) $r : R : r_1 = 1 : 2 : 3$ (iv) area $= \frac{\sqrt{3}a^2}{4}$ (v) $R = \frac{a}{\sqrt{3}}$

(d) (i) The circumcentre lies (1) inside an acute angled triangle (2) outside an obtuse angled triangle & (3) mid point of the hypotenuse of right angled triangle.

(ii) The orthocentre of right angled triangle is the vertex at the right angle.

(iii) The orthocentre, centroid & circumcentre are collinear & centroid divides the line segment joining orthocentre & circumcentre internally in the ratio 2 : 1 except in case of equilateral triangle. In equilateral triangle, all these centres coincide

(e) Area of a cyclic quadrilateral $= \sqrt{(s-a)(s-b)(s-c)(s-d)}$

where a, b, c, d are lengths of the sides of quadrilateral and $s = \frac{a+b+c+d}{2}$.

ANSWERS FOR DO YOURSELF

1 : **(i)** 90°

5 : **(i)** **(a)** $\frac{3}{5}$ **(b)** $\frac{3}{4}$ **(c)** $\frac{1}{\sqrt{10}}$ **(d)** $\frac{3}{\sqrt{10}}$ **(e)** $\frac{1}{3}$ **(f)** 24

6 : **(i)** **(a)** 6 **(b)** $\frac{5}{2}$ **(c)** 1

7 : **(i)** **(a)** 1 **(b)** 3 **(c)** $2\sqrt{3}$

10 : **(ii)** 6

ELEMENTARY EXERCISE

1. Angles A, B and C of a triangle ABC are in A.P. If $\frac{b}{c}=\sqrt{\frac{3}{2}}$, then $\angle A$ is equal to

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{5\pi}{12}$ (D) $\frac{\pi}{2}$

2. If K is a point on the side BC of an equilateral triangle ABC and if $\angle BAK = 15°$, then the ratio of lengths $\frac{AK}{AB}$ is

(A) $\frac{3\sqrt{2}\left(3+\sqrt{3}\right)}{2}$ (B) $\frac{\sqrt{2}\left(3+\sqrt{3}\right)}{2}$ (C) $\frac{\sqrt{2}\left(3-\sqrt{3}\right)}{2}$ (D) $\frac{3\sqrt{2}\left(3-\sqrt{3}\right)}{2}$

3. Let ABC be a right triangle with length of side AB = 3 and hypotenuse AC = 5.

If D is a point on BC such that $\frac{BD}{DC}=\frac{AB}{AC}$, then AD is equal to

(A) $\frac{4\sqrt{3}}{3}$ (B) $\frac{3\sqrt{5}}{2}$ (C) $\frac{4\sqrt{5}}{3}$ (D) $\frac{5\sqrt{3}}{4}$

4. In ΔABC, if a = 2b and A = 3B, then the value of $\frac{c}{b}$ is equal to

(A) 3 (B) $\sqrt{2}$ (C) 1 (D) $\sqrt{3}$

5. If the angle A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of expression $E=\left(\frac{a}{c}\sin 2C+\frac{c}{a}\sin 2A\right)$, is

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

6. The ratio of the sides of a triangle ABC is $1:\sqrt{3}:2$. Then ratio of A : B : C is

(A) 3 : 5 : 2 (B) 1 : $\sqrt{3}$: 2 (C) 3 : 2 : 1 (D) 1 : 2 : 3

7. If in a triangle sin A : sin C = sin (A – B) : sin (B – C), then a^2, b^2, c^2

(A) are in A.P. (B) are in G.P. (C) are in H.P. (D) none of these

8. In a triangle tan A : tan B : tan C = 1 : 2 : 3, then $a^2 : b^2 : c^2$ equals

(A) 5 : 8 : 9 (B) 5 : 8 : 12 (C) 3 : 5 : 8 (D) 5 : 8 : 10

9. In an acute triangle ABC, ∠ABC = 45°, AB = 3 and AC = $\sqrt{6}$. The angle ∠BAC, is

(A) 60° (B) 65° (C) 75° (D) 15° or 75°

10. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. If a = 1, b = 3 and C = 60°, then $\sin^2 B$ is equal to

(A) $\frac{27}{28}$ (B) $\frac{3}{28}$ (C) $\frac{81}{28}$ (D) $\frac{1}{3}$

11. In triangle ABC, If $s = 3+\sqrt{3}+\sqrt{2}$, 3B – C = 30°, A + 2B = 120°, then the length of longest side of triangle is

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 2 (B) $2\sqrt{2}$ (C) $2(\sqrt{3}+1)$ (D) $\sqrt{3}-1$

12. In ΔABC if a = 8, b = 9, c = 10, then the value of $\frac{\tan C}{\sin B}$ is

(A) $\frac{32}{9}$ (B) $\frac{24}{7}$ (C) $\frac{21}{4}$ (D) $\frac{18}{5}$

13. If the sides of a triangle are sin α, cos α, $\sqrt{1+\sin\alpha\cos\alpha}, 0<\alpha<\frac{\pi}{2}$, the largest angle is

(A) 60° (B) 90° (C) 120° (D) 150°

14. In a triangle ABC, ∠A = 60° and b : c = $\left(\sqrt{3}+1\right)$: 2 then (∠B – ∠C) has the value equal to

(A) 15° (B) 30° (C) 22.5 ° (D) 45°

15. In triangle ABC, if $\Delta = a^2 - (b-c)^2$, then tan A =

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\frac{15}{16}$ (B) $\frac{1}{2}$ (C) $\frac{8}{17}$ (D) $\frac{8}{15}$

16. In triangle ABC, if $\cot\frac{A}{2}=\frac{b+c}{a}$, then triangle ABC must be

[Note: All symbols used have usual meaning in ΔABC.]

(A) isosceles
(B) equilateral
(C) right angled
(D) isoceles right angled

17. In ΔABC, if a,b,c (taken in that order) are in A.P. then $\cot\frac{A}{2}\ \cot\frac{C}{2}=$

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 1
(B) 2
(C) 3
(D) 4

18. In a triangle ABC, if a = 6, b = 3 and cos(A – B) = $\frac{4}{5}$, the area of the triangle is

(A) 8
(B) 9
(C) 12
(D) $\frac{15}{2}$

19. ABC is a triangle such that sin (2A + B) = sin (C – A) = – sin (B + 2C) = $\frac{1}{2}$. If A, B, C are in A.P., find A, B, C.

20. In a triangle ABC, if the sides a, b, c are roots of $x^3-11x^2+38x-40=0$. If $\sum\left(\frac{\cos A}{a}\right)=\frac{p}{q}$, then find the least value of (p + q) where p,q $\in$ N.

EXERCISE (O-1)

1. In triangle ABC, if $\sin^3 A + \sin^3 B + \sin^3 C = 3\sin A.\sin B.\sin C$, then triangle is

(A) obtuse angled (B) right angled (C) obtuse right angled (D) equilateral

2. A triangle has vertices A, B and C, and the respective opposite sides have lengths a, b and c. This triangle is inscribed in a circle of radius R. If b = c = 1 and the altitude from A to side BC has length $\sqrt{\frac{2}{3}}$, then R equals

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{3}}{2\sqrt{2}}$

3. In a triangle ABC, if $\angle C = 105°$, $\angle B = 45°$ and length of side AC = 2 units, then the length of the side AB is equal to

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) $\sqrt{2}+1$ (D) $\sqrt{3}+1$

4. In a triangle ABC, if a = 13, b = 14 and c = 15, then angle A is equal to
(All symbols used have their usual meaning in a triangle.)

(A) $\sin^{-1}\frac{4}{5}$ (B) $\sin^{-1}\frac{3}{5}$ (C) $\sin^{-1}\frac{3}{4}$ (D) $\sin^{-1}\frac{2}{3}$

5. In triangle ABC, if AC = 8, BC = 7 and D lies between A and B such that AD = 2, BD = 4, then the length CD equals

(A) $\sqrt{46}$ (B) $\sqrt{48}$ (C) $\sqrt{51}$ (D) $\sqrt{75}$

6. In triangle ABC, if 2b = a + c and A – C = 90°, then sin B equals
[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\frac{\sqrt{7}}{5}$ (B) $\frac{\sqrt{5}}{8}$ (C) $\frac{\sqrt{7}}{4}$ (D) $\frac{\sqrt{5}}{3}$

7. In a triangle ABC, $a^3 + b^3 + c^3 = c^2 (a + b + c)$
(All symbol used have usual meaning in a triangle.)

Statement–1: The value of $\angle C = 60°$.

Statement –2: ΔABC must be equilateral.

(A) Statement–1 is true, statement-2 is true and statement-2 is correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true and statement-2 is NOT the correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

8. A circle is inscribed in a right triangle ABC, right angled at C. The circle is tangent to the segment AB at D and length of segments AD and DB are 7 and 13 respectively. Area of triangle ABC is equal to

(A) 91 (B) 96 (C) 100 (D) 104

9. In triangle ABC, If $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$ then angle C is equal to

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 30° (B) 45° (C) 60° (D) 90°

10. In a triangle ABC, if b = $(\sqrt{3}-1)$ a and $\angle C = 30°$, then the value of (A – B) is equal to

(All symbols used have usual meaning in a triangle.)

(A) 30° (B) 45° (C) 60° (D) 75°

11. The sides of a triangle are three consecutive integers. The largest angle is twice the smallest one. The area of triangle is equal to

(A) $\frac{5}{4}\sqrt{7}$ (B) $\frac{15}{2}\sqrt{7}$ (C) $\frac{15}{4}\sqrt{7}$ (D) $5\sqrt{7}$

12. The sides a, b, c (taken in that order) of triangle ABC are in A.P.

If $\cos\alpha=\frac{a}{b+c},\cos\beta=\frac{b}{c+a},\cos\gamma=\frac{c}{a+b}$ then $\tan^2\left(\frac{\alpha}{2}\right)+\tan^2\left(\frac{\gamma}{2}\right)$ is equal to

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{3}$

13. AD and BE are the medians of a triangle ABC. If AD = 4, $\angle DAB=\frac{\pi}{6},\angle ABE=\frac{\pi}{3}$, then area of triangle ABC equals

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) $\frac{32}{3}$ (D) $\frac{32}{9}\sqrt{3}$

14. In a triangle ABC, if (a + b + c) (a + b – c) (b + c – a) (c + a – b) = $\frac{8a^2b^2c^2}{a^2+b^2+c^2}$, then the triangle is

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) isosceles (B) right angled (C) equilateral (D) obtuse angled

15. For right angled isosceles triangle, $\frac{r}{R}=$

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\tan\frac{\pi}{12}$ (B) $\cot\frac{\pi}{12}$ (C) $\tan\frac{\pi}{8}$ (D) $\cot\frac{\pi}{8}$

EXERCISE (O-2)

Multiple Correct Answer Type :

1. Given an acute triangle ABC such that $\sin C = \frac{4}{5}$, $\tan A = \frac{24}{7}$ and AB = 50. Then-

(A) centroid, orthocentre and incentre of ΔABC are collinear

(B) $\sin B = \frac{4}{5}$

(C) $\sin B = \frac{4}{7}$

(D) area of $\Delta ABC = 1200$

2. In ΔABC, angle A is 120°, BC + CA = 20 and AB + BC = 21, then

(A) AB > AC (B) AB < AC

(C) ΔABC is isosceles (D) area of $\Delta ABC = 14\sqrt{3}$

3. In which of the following situations, it is possible to have a triangle ABC?
(All symbols used have usual meaning in a triangle.)

(A) $(a + c - b)(a - c + b) = 4bc$ (B) $b^2 \sin 2C + c^2 \sin 2B = ab$

(C) $a = 3, b = 5, c = 7$ and $C = \frac{2\pi}{3}$ (D) $\cos\left(\frac{A-C}{2}\right) = \cos\left(\frac{A+C}{2}\right)$

4. If the lengths of the medians AD,BE and CF of triangle ABC are 6, 8,10 respectively, then-

(A) AD & BE are perpendicular (B) BE and CF are perpendicular

(C) area of $\Delta ABC = 32$ (D) area of $\Delta DEF = 8$

5. In ΔABC, angle A, B and C are in the ratio 1 : 2 : 3, then which of the following is (are) correct?
(All symbol used have usual meaning in a triangle.)

(A) Circumradius of $\Delta ABC = c$ (B) $a : b : c = 1 : \sqrt{3} : 2$

(C) Perimeter of $\Delta ABC = 3 + \sqrt{3}$ (D) Area of $\Delta ABC = \frac{\sqrt{3}}{8} c^2$

6. In a triangle ABC, let BC = 1, AC = 2 and measure of angle C is 30°. Which of the following statement(s) is (are) correct?

(A) $2 \sin A = \sin B$

(B) Length of side AB equals $5 - 2\sqrt{3}$

(C) Measure of angle A is less than 30°

(D) Circumradius of triangle ABC is equal to length of side AB

7. In a triangle ABC, if cos A cos 2B + sin A sin 2B sin C = 1, then

(A) A,B,C are in A.P. (B) B,A,C are in A.P. (C) $\frac{r}{R}=2$ (D) $\frac{r}{R}=\sqrt{2}\sin\frac{\pi}{12}$

8. In a triangle ABC, let $2a^2 + 4b^2 + c^2 = 2a(2b + c)$, then which of the following holds good ?

[**Note:** All symbols used have usual meaning in a triangle.]

(A) $\cos B = \frac{-7}{8}$

(B) $\sin(A - C) = 0$

(C) $\frac{r}{r_1}=\frac{1}{5}$

(D) sin A : sin B : sin C = 1 : 2 : 1

9. In a triangle ABC, which of the following quantities denote the area of the triangle?

(A) $\frac{a^2-b^2}{2}\left(\frac{\sin A\sin B}{\sin(A-B)}\right)$

(B) $\frac{r_1r_2r_3}{\sqrt{\sum r_1r_2}}$

(C) $\frac{a^2+b^2+c^2}{\cot A+\cot B+\cot C}$

(D) $r^2\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cot\frac{C}{2}$

10. Let one angle of a triangle be 60°, the area of triangle is $10\sqrt{3}$ and perimeter is 20 cm. If a > b > c where a, b and c denote lengths of sides opposite to vertices A, B and C respectively, then which of the following is (are) correct?

(A) Inradius of triangle is $\sqrt{3}$

(B) Length of longest side of triangle is 7

(C) Circumradius of triangle is $\frac{7}{\sqrt{3}}$

(D) Radius of largest escribed circle is $\frac{1}{12}$

11. In triangle ABC, let b = 10, c = $10\sqrt{2}$ and $R = 5\sqrt{2}$ then which of the following statement(s) is (are) correct?

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) Area of triangle ABC is 50.

(B) Distance between orthocentre and circumcentre is $5\sqrt{2}$

(C) Sum of circumradius and inradius of triangle ABC is equal to 10

(D) Length of internal angle bisector of ∠ACB of triangle ABC is $\frac{5}{2\sqrt{2}}$

12. In a triangle ABC, if a = 4, b = 8 $\angle C = 60°$, then which of the following relations is (are) correct?

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) The area of triangle ABC is $8\sqrt{3}$

(B) The value of $\sum \sin^2 A = 2$

(C) Inradius of triangle ABC is $\frac{2\sqrt{3}}{3+\sqrt{3}}$

(D) The length of internal angle bisector of angle C is $\frac{4}{\sqrt{3}}$

13. In a triangle ABC, $\angle A = 30°$, b = 6. Let CB_1 and CB_2 are least and greatest integral value of side a for each of which two triangles can be formed. If it is also given angle B_1 is obtuse and angle B_2 is acute angle, then (All symbols used have usual meaning in a triangle.)

(A) $|CB_1 - CB_2| = 1$

(B) $CB_1 + CB_2 = 9$

(C) area of $\Delta B_1CB_2 = 6 + \frac{3}{2}\sqrt{7}$

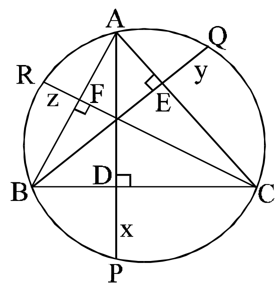
(D) area of $\Delta AB_2C = 6 + \frac{9}{2}\sqrt{3}$

14. Let P be an interior point of ΔABC.

Match the correct entries for the ratios of the Area of ΔPBC : Area of ΔPCA : Area of ΔPAB depending on the position of the point P w.r.t. Δ ABC.

Column-I	**Column-II**
(A) If P is centroid (G)	(P) tanA : tanB : tanC
(B) If P is incentre (I)	(Q) sin2A : sin2B : sin2C
(C) If P is orthocentre (H)	(R) sinA : sinB : sinC
(D) If P is circumcentre	(S) 1 : 1 : 1
	(T) cos A : cosB : cosC

15. As shown in the figure AD is the altitude on BC and AD produced meets the circumcircle of ΔABC at P where DP = x. Similarly EQ = y and FR = z. If a, b, c respectively denotes the sides BC, CA and AB then $\frac{a}{2x} + \frac{b}{2y} + \frac{c}{2z}$ has the value equal to



(A) tanA + tanB + tanC

(B) cotA + cotB + cotC

(C) cosA + cosB + cosC

(D) cosecA + cosecB + cosecC

EXERCISE (S-1)

1. If a,b,c are the sides of triangle ABC satisfying $\log\left(1+\frac{c}{a}\right)+\log a-\log b=\log 2$.

Also $a(1-x^2)+2bx+c(1+x^2)=0$ has two equal roots. Find the value of sinA + sinB + sinC.

2. Given a triangle ABC with sides a = 7, b = 8 and c = 5. If the value of the expression $\left(\sum \sin A\right)\left(\sum \cot\frac{A}{2}\right)$ can be expressed in the form $\frac{p}{q}$ where $p, q \in N$ and $\frac{p}{q}$ is in its lowest form find the value of (p + q).

3. If two times the square of the diameter of the circumcircle of a triangle is equal to the sum of the squares of its sides then prove that the triangle is right angled.

4. With usual notations, prove that in a triangle ABC

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s^2}{\Delta}$$

5. With usual notations, prove that in a triangle ABC

$a\cot A+b\cot B+c\cot C=2(R+r)$

6. With usual notations, prove that in a triangle ABC

$Rr(\sin A+\sin B+\sin C)=\Delta$

7. With usual notations, prove that in a triangle ABC

$$\cot A+\cot B+\cot C=\frac{a^2+b^2+c^2}{4\Delta}$$

8. In acute angled triangle ABC, a semicircle with radius r_a is constructed with its base on BC and tangent to the other two sides. r_b and r_c are defined similarly. If r is the radius of the incircle of triangle ABC then prove that, $\frac{2}{r}=\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}$.

9. If the length of the perpendiculars from the vertices of a triangle A, B, C on the opposite sides are p_1, p_2, p_3 then prove that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$.

10. With usual notations, prove that in a triangle ABC $\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$

11. With usual notations, prove that in a triangle ABC

$$\frac{r_1}{(s-b)(s-c)}+\frac{r_2}{(s-c)(s-a)}+\frac{r_3}{(s-a)(s-b)}=\frac{3}{r}$$

12. With usual notations, prove that in a triangle ABC

$$\frac{abc}{s}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\Delta$$

13. With usual notations, prove that in a triangle ABC

$$\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$$

14. If $r_1 = r + r_2 + r_3$ then prove that the triangle is a right angled triangle.

15. With usual notations, prove that in a triangle ABC $2R\cos A = 2R + r - r_1$

EXERCISE (JM)

1. If 5, 5r, $5r^2$ are the lengths of the sides of a triangle, then r cannot be equal to :

[JEE(Main)-Jan 2019]

(1) $\frac{3}{2}$ (2) $\frac{3}{4}$ (3) $\frac{5}{4}$ (4) $\frac{7}{4}$

2. With the usual notation, in ΔABC, if $\angle A + \angle B = 120°$, $a = \sqrt{3}+1$ and $b = \sqrt{3}-1$, then the ratio $\angle A : \angle B$, is : **[JEE(Main)-Jan 2019]**

(1) 7 : 1 (2) 5 : 3 (3) 9 : 7 (4) 3 : 1

3. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y. If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is : **[JEE(Main)-Jan 2019]**

(1) $\frac{y}{\sqrt{3}}$ (2) $\frac{c}{\sqrt{3}}$ (3) $\frac{c}{3}$ (4) $\frac{3}{2}y$

4. Given $\frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$ for a ΔABC with usual notation. If $\frac{\cos A}{\alpha} = \frac{\cos B}{\beta} = \frac{\cos C}{\gamma}$, then the ordered triad (α, β, γ) has a value :- **[JEE(Main)-Jan 2019]**

(1) (3, 4, 5) (2) (19, 7, 25) (3) (7, 19, 25) (4) (5, 12, 13)

5. If the lengths of the sides of a triangle are in A.P. and the greatest angle is double the smallest, then a ratio of lengths of the sides of this triangle is : **[JEE(Main)-Apr 2019]**

(1) 5 : 9 : 13 (2) 5 : 6 : 7 (3) 4 : 5 : 6 (4) 3 : 4 : 5

6. The angles A, B and C of a triangle ABC are in A.P. and a : b = 1 : $\sqrt{3}$. If c = 4 cm, then the area (in sq. cm) of this triangle is : **[JEE(Main)-Apr 2019]**

(1) $4\sqrt{3}$ (2) $\frac{2}{\sqrt{3}}$ (3) $2\sqrt{3}$ (4) $\frac{4}{\sqrt{3}}$

EXERCISE (JA)

1. Let ABC and ABC′ be two non-congruent triangles with sides AB = 4, AC = AC′ = $2\sqrt{2}$ and angle B = 30°. The absolute value of the difference between the areas of these triangles is

[JEE 2009, 5]

2. (a) If the angle A,B and C of a triangle are in an arithmetic progression and if a,b and c denote the length of the sides opposite to A,B and C respectively, then the value of the expression $\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$, is -

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

(b) Consider a triangle ABC and let a,b and c denote the length of the sides opposite to vertices A,B and C respectively. Suppose a = 6, b = 10 and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to

(c) Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a,b and c denote the lengths of the sides opposite to A,B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is/are **[JEE 2010, 3+3+3]**

(A) $-(2+\sqrt{3})$ (B) $1+\sqrt{3}$ (C) $2+\sqrt{3}$ (D) $4\sqrt{3}$

3. Let PQR be a triangle of area Δ with a = 2, b = $\frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals **[JEE 2012, 3M, –1M]**

(A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$

4. In a triangle PQR, P is the largest angle and cos P = $\frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)

[JEE(Advanced) 2013, 3, (–1)]

(A) 16 (B) 18 (C) 24 (D) 22

5. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is a third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is - **[JEE(Advanced)-2014, 3(–1)]**

(A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$

6. In a triangle XYZ, let x,y,z be the lengths of sides opposite to the angles X,Y,Z, respectively and 2s = x + y + z. If $\frac{s-x}{4}=\frac{s-y}{3}=\frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then-

(A) area of the triangle XYZ is $6\sqrt{6}$ **[JEE(Advanced)-2016, 4(–2)]**

(B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$

(C) $\sin\frac{X}{2}\sin\frac{Y}{2}\sin\frac{Z}{2}=\frac{4}{35}$

(D) $\sin^2\left(\frac{X+Y}{2}\right)=\frac{3}{5}$

7. In a triangle PQR, let $\angle PQR = 30°$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE ?

(A) $\angle QPR = 45°$ **[JEE(Advanced)-2018, 4(–2)]**

(B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120°$

(C) The radius of the incircle of the triangle PQR is $10\sqrt{3}-15$

(D) The area of the circumcircle of the triangle PQR is 100π.

8. In a non-right-angled triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at O. If $p=\sqrt{3}$, q = 1, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

[JEE(Advanced)-2019, 4(–1)]

(1) Area of $\Delta SOE=\frac{\sqrt{3}}{12}$ (2) Radius of incircle of $\Delta PQR=\frac{\sqrt{3}}{2}(2-\sqrt{3})$

(3) Length of $RS=\frac{\sqrt{7}}{2}$ (4) Length of $OE=\frac{1}{6}$

ANSWER KEY

ELEMENTARY EXERCISE

1. C	**2.** C	**3.** B	**4.** D	**5.** D	**6.** D	**7.** A	**8.** A
9. C	**10.** A	**11.** C	**12.** A	**13.** C	**14.** B	**15.** D	**16.** C
17. C	**18.** B	**19.** 45°,60°,75°		**20.** 25			

EXERCISE (O-1)

1. D	**2.** D	**3.** D	**4.** A	**5.** C	**6.** C	**7.** C	**8.** A
9. C	**10.** C	**11.** C	**12.** D	**13.** D	**14.** B	**15.** C	

EXERCISE (O-2)

1. A,B,D	**2.** A,D	**3.** B,C	**4.** A,C,D	**5.** B,D	**6.** A,C,D	**7.** B,D
8. B,C	**9.** A,B,D	**10.** A,C	**11.** A,B,C	**12.** A,B	**13.** A,B,C,D	

14. (A) S; (B) R; (C) P; (D) Q   **15.** A

EXERCISE (S-1)

1. $\frac{12}{5}$ **2.** 107

EXERCISE (JM)

1. 4	**2.** 1	**3.** 2	**4.** 3	**5.** 3	**6.** 3

EXERCISE (JA)

1. 4   **2.** (a) D, (b) 3, (c) B   **3.** C   **4.** B,D   **5.** B   **6.** A,C,D

7. B,C,D   **8.** 2,3,4

CHAPTER 5

HEIGHT AND DISTANCES

Chapter Contents 05

IMPORTANT NOTES

CHAPTER 5

HEIGHT AND DISTANCES

1. INTRODUCTION :

One of the important application of trigonometry is in finding the height and distance of the point which are not directly measurable. This is done with the help of trigonometric ratios.

2. ANGLES OF ELEVATION AND DEPRESSION :

Let OP be a horizontal line in the vertical plane in which an object R is given and let OR be joined.

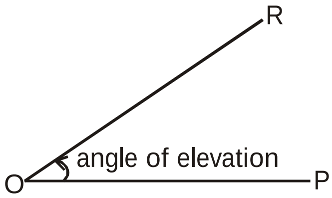


Fig. (a)

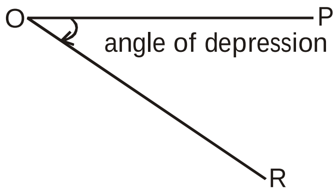


Fig. (b)

In Fig. (a), where the object R is above the horizontal line OP, the angle POR is called the angle of elevation of the object R as seen from the point O. In Fig. (b) where the object R is below the horizontal line OP, the angle POR is called the angle of depression of the object R as seen from the point O.

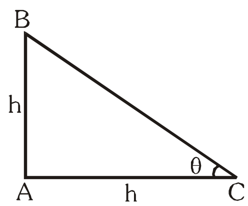
Remark :

Unless stated to the contrary, it is assumed that the height of the observer is neglected, and that the angles of elevation are measured from the ground.

Ex.1 Find the angle of elevation of the sum when the length of shadow of a vertical pole is equal to its height.

Sol. Let height of the pole AB = h and

length of the shadow of the Pole (AC) = h



In ΔABC $\tan\theta = \frac{AB}{AC} = \frac{h}{h} = 1 \Rightarrow \tan\theta = 1$

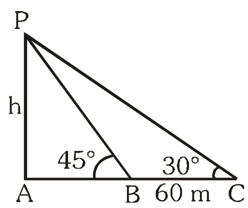
$\Rightarrow \tan\theta = \tan 45° \Rightarrow \theta = 45°$

Ex.2 The shadow of the tower standing on a level ground is found to be 60 metres longer when the sun's altitude is 30° than when it is 45°. The height of the tower is-

(1) 60 m (2) $30(\sqrt{3}-1)$m (3) $60\sqrt{3}$ m (4) $30(\sqrt{3}+1)$ m.

Sol.(4) $AC = h\cot 30° = \sqrt{3}h$

$AB = h\cot 45° = h$



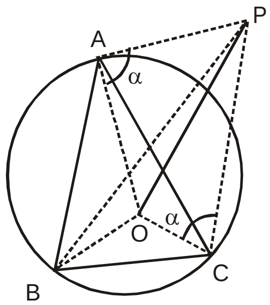
$\therefore$ $BC = AC - AB = h(\sqrt{3}-1) \Rightarrow 60 = h(\sqrt{3}-1)$

$\therefore$ $h = \frac{60}{\sqrt{3}-1} = \frac{60(\sqrt{3}+1)}{3-1} = 30(\sqrt{3}+1)$

Ex.3 The angle of elevation of the tower observed from each of the three point A,B,C on the ground, forming a triangle is the same angle α . If R is the circum - radius of the triangle ABC, then the height of the tower is -

(1) R sin α (2) R cos α (3) R cot α (4) R tan α

Sol.(4) The tower makes equal angles at the vertices of the triangle, therefore foot of the tower is at the circumcentre.



From Δ OCP, OP is perpendicular to OC.

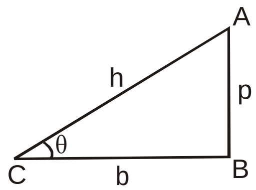
$\angle OCP = \alpha$

so $\tan\alpha = \dfrac{OP}{OA} \Rightarrow OP = OA\tan\alpha$

$OP = R\tan\alpha$

3. SOME USEFUL RESULTS :

- In a triangle ABC,

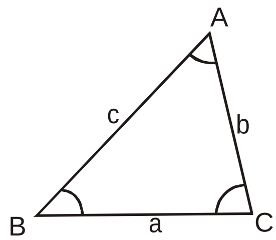


$\sin\theta = \dfrac{p}{h}$, $\cos\theta = \dfrac{b}{h}$, $\tan\theta = \dfrac{P}{b}$

- In any triangle ABC,

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ [By sine rule]

or cosine formula



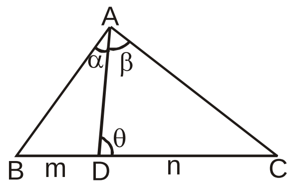
i.e. $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$; $\cos B = \dfrac{a^2+c^2-b^2}{2ac}$, $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

- In any triangle ABC

if BD : DC = m : n and $\angle BAD = \alpha$

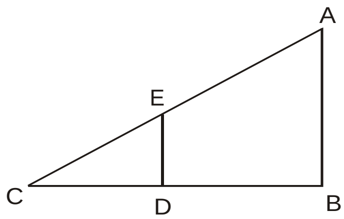
$\angle CAD = \beta$ and $\angle ADC = \theta$,

then $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$



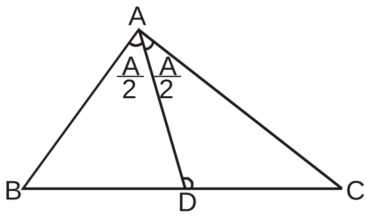
- In a triangle ABC, if DE $\|$ AB

then, $\dfrac{AB}{DE} = \dfrac{BC}{DC}$



- In a triangle the internal bisector of an angle divides the opposite side in the ratio of the arms of the angle

$\therefore \dfrac{BD}{DC} = \dfrac{AB}{AC}$



- In an isosceles triangle the median is perpendicular to the base

SOLVED EXAMPLES

Ex.1 A tower subtends an angle of 30° at a point on the same level as its foot, and at a second point h m above the first, the depression of the foot of tower is 60°. The height of the tower is.

(1) h m (2) 3h m (3) $\sqrt{3}$ h m (4) $\frac{h}{3}$ m.

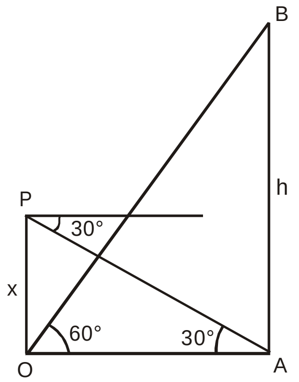
Sol.(4) Let OP be the tower of height x.,A the point on the same level as the foot O of the tower and B be the point h m above A (see Fig.) Then ∠ AOB = 60° and ∠ PAO = 30°. From right-angled triangle AOP,

we have

$$OA = x \cot 30°$$

and from right-angled triangle OAB, we have

$$OA = h \cot 60°$$



Therefore, from (1) and (2), we get

$$x \cot 30° = h \cot 60°$$

$$\sqrt{3}\,x = \frac{1}{\sqrt{3}}h \Rightarrow x = \frac{1}{3}h$$

Ex.2 At a point on level ground, the angle of elevation of a vertical tower is found to be such that its tangent is $\frac{5}{12}$. On walking 192 metres towards the tower, the tangent of the angle of elevation is $\frac{3}{4}$. Find the height of the tower.

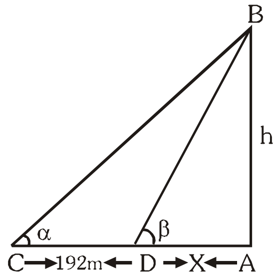
Sol. Let AB be the tower and let the angle of elevation of its top at C be α. Let D be a point at a distance of 192 metres from C such that the angle of elevation of the top of the tower at D be β.

Let h be the height of the tower and AD = x,

It is given that $\tan\alpha = \frac{5}{12}$ and $\tan\beta = \frac{3}{4}$.

In ΔABC, we have

$$\tan\alpha = \frac{AB}{AC} \Rightarrow \tan\alpha = \frac{h}{192+x} \Rightarrow \frac{5}{12} = \frac{h}{192+x} \qquad \text{..... (i)}$$



In ΔABD, we have

$$\tan\beta = \frac{AB}{AD} \Rightarrow \tan\beta = \frac{h}{x} \Rightarrow \frac{3}{4} = \frac{h}{x} \qquad \text{..... (ii)}$$

We have to find h. This means that we have to eliminate x from (i) and (ii).

From (ii), we have $3x = 4h \Rightarrow x = \frac{4h}{3}$

Substituting this value of x in (i), we get

$\frac{5}{12} = \frac{h}{192 + 4h/3} \Rightarrow 5\left(192 + \frac{4h}{3}\right) = 12h$

$\Rightarrow \quad 5(576 + 4h) = 36h \Rightarrow 2880 + 20h = 36h$

$\Rightarrow \quad 16h = 2880 \Rightarrow h = \frac{2880}{16} = 180$

Hence, height of tower = 180 metres.

Ex.3 Let α be the solution of $16^{\sin^2\theta} + 16^{\cos^2\theta} = 10$ in $(0, \pi/4)$. If the shadow of a vertical pole is $\frac{1}{\sqrt{3}}$ of its height, then then the altitude of the sun is-

(1) α (2) $\frac{\alpha}{2}$ (3) 2α (4) $\frac{\alpha}{3}$

Sol. We have $16^{\sin^2\theta} + 16^{\cos^2\theta} = 10$

$\Rightarrow \quad 16^{\sin^2\theta} + 16^{1-\sin^2\theta} = 10 \Rightarrow x + \frac{16}{x} = 10,$ where $x = 16^{\sin^2\theta}$

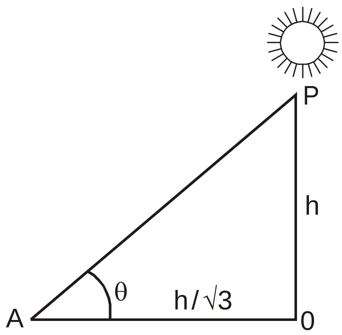
$\Rightarrow \quad x = 2, 8 \Rightarrow 16^{\sin^2\theta} = 2, 8$

$\Rightarrow \quad 2^{4\sin^2\theta} = 2, 2^3$

$\Rightarrow \quad 4\sin^2\theta = 2, 3$

$\Rightarrow \quad \sin^2\theta = \frac{1}{2}, \left(\frac{\sqrt{3}}{2}\right)^2 \Rightarrow \theta = \frac{\pi}{6}, \frac{\pi}{3}$

$\therefore \quad \alpha = \frac{\pi}{6}$



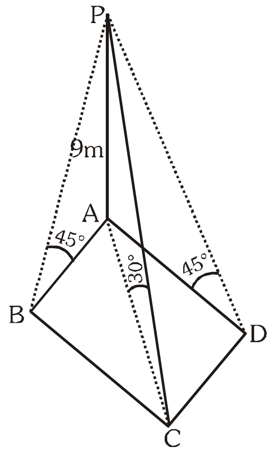
Let the altitude of the sun be θ. Then,

$\tan\theta = \frac{h}{\frac{h}{\sqrt{3}}} \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3} \Rightarrow \theta = 2\alpha$

Ex.4 A vertical lamp-post of height 9 metres stands at the corner of a rectangular field. The angle of elevation of its top from the farthest corner is 30°, while from another corner it is 45°. The area of the field is-

(1) $81\sqrt{2}\ m^2$ (2) $9\sqrt{2}\ m^2$ (3) $81\sqrt{3}\ m^2$ (4) $9\sqrt{3}\ m^2$

Sol. Let AP be the lamp-post of 9 m standing at corner A of the rectangular field ABCD.



In Δ's BAP and CAP, we have

$$\tan 45° = \frac{PA}{BA} \text{ and } \tan 30° = \frac{PA}{AC}$$

$\Rightarrow$ BA = 9 m and AC = $9\sqrt{3}$ m

$\therefore$ $BC = \sqrt{AC^2 - AB^2} = \sqrt{243 - 81} = \sqrt{162} = 9\sqrt{2}$ m

Hence, area of the field = AB × BC = $9 \times 9\sqrt{2}\ m^2 = 81\sqrt{2}\ m^2$

Ex.5 A vertical tower stands on a horizontal plane and is surmounted by a vertical flag staff of height h. At a point on the plane, the angle of elevation of the bottom and the top of the flag staff are α and β respectively. Prove that the height of tower is $\dfrac{h\tan\alpha}{\tan\beta - \tan\alpha}$.

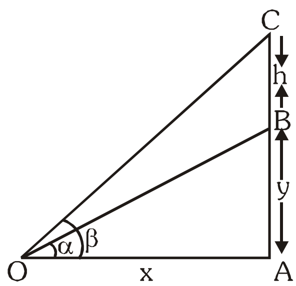
Sol. Let AB be the tower and BC be the flag staff. Let O be a point on the plane containing the foot of the tower such that the angles of elevation of the bottom B and top C of the flagstaff at O are α and β respectively. Let OA = x metres, AB = y metres and BC = h metres.

In ΔOAB, we have

$$\cot\alpha = \frac{OA}{AB} \Rightarrow \cot\alpha = \frac{x}{y}$$

$\Rightarrow x = y\cot\alpha$ ...(i)

In ΔOAC, we have



$$\cot\beta = \frac{x}{y+h}$$

$\Rightarrow x = (y + h)\cot\beta$ ...(ii)

Equating the values of x from (i) and (ii), we get

$y \cot \alpha = (y + h) \cot \beta$

$\Rightarrow \quad y \cot \alpha - y \cot \beta = h \cot \beta$

$\Rightarrow \quad y (\cot \alpha - \cot \beta) = h \cot \beta$

$$\Rightarrow \quad y = \frac{h \cot \beta}{\cot \alpha - \cot \beta} \Rightarrow y = \frac{h / \tan \beta}{\frac{1}{\tan \alpha} - \frac{1}{\tan \beta}} = \frac{h \tan \alpha}{\tan \beta - \tan \alpha}$$

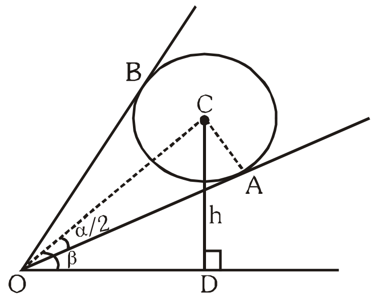
Ex.6 A spherical ball of diameter-δ subtends an angle α at the eye of an observer when the elevation of its centre is β. Prove that the height of the centre of the ball is $\frac{1}{2} \delta \sin \beta \operatorname{cosec} \left(\frac{\alpha}{2}\right)$.

Sol. O is the position of eye.

As is clear from figure, from ΔODC,

$$OC = \frac{h}{\sin \beta}$$

From ΔOAC,



$$\sin \frac{\alpha}{2} = \frac{CA}{OC} = \frac{\frac{\delta}{2}}{h / \sin \beta}$$

$$\Rightarrow \quad h = \frac{1}{2} \delta \sin \beta . \operatorname{cosec} \frac{\alpha}{2} .$$

CHECK YOUR GRASP EXERCISE-1

1. The angle of elevation of the top of a tower from a point 20 metre away from its base is 45°. Then height of the tower is-

(1) 10 m (2) 20 m (3) 40 m (4) $20\sqrt{3}$ m

2. At a point 15 metre away from the base of a 15 metre high house, the angle of elevation of the top is-

(1) 45° (2) 30° (3) 60° (4) 90°

3. An aeroplane flying at a height 300 metre above the ground passes vertically above another plane at an instant when the angles of elevation of the two planes from the same point on the ground are 60° and 45° respectively. Then the height of the lower plane from the ground in metres is-

(1) $100\sqrt{3}$ (2) $100/\sqrt{3}$ (3) 50 (4) $150(\sqrt{3}+1)$.

4. If the elevation of the sun is 30° then the length of the shadow cast by a tower of 150 metres height is-

(1) $75\sqrt{3}$m (2) $200\sqrt{3}$m (3) $150\sqrt{3}$m (4) None of these

5. From the top of the cliff 300 metres heigh, the top of a tower was observed at an angle of depression 30° and from the foot of the tower the top of the cliff was observed at an angle of elevation 45°. The height of the tower is -

(1) $50(3-\sqrt{3})$m (2) $200(3-\sqrt{3})$m (3) $100(3-\sqrt{3})$m (4) None of these

6. The angles of elevation of the top of a tower at the top and the foot of a pole of height 10 m are 30° and 60° respectively. The height of the tower is-

(1) 10 m (2) 15 m (3) 20 m (4) None of these

7. A person, standing on the bank of a river, observes that the angle subtended by a tree on the opposite bank is 60°; when he retreates 20 m from the bank, he finds the angle to be 30°. The height of the tree and the breadth of the river are -

(1) $10\sqrt{3}$ m, 10 m (2) 10 m ; $10\sqrt{3}$ m (3) 20 m, 30m (4) None of these

8. From the top of a light house 60 m high with its base at sea level the angle of depression of a boat is 15°. The distance of the boat from the light house is-

(1) $60\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$m (2) $60\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$m (3) $30\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$m (4) $30\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$m

9. The angle of elevation of the sun when the length of the shadow of a pole is $\sqrt{3}$ times the height of the pole will be-

(1) 30° (2) 60° (3) 90° (4) 45°

10. If the length of the shadow of a vertical pole on the horizontal ground is equal to its height, find the angle of elevation of the sun -

(1) 60° (2) 30° (3) 45° (4) 90°

11. The angle of elevation of the top of a tower from a point 20m away from its base is 60°. The height of the tower is -

(1) 10 m (2) $20/\sqrt{3}$ m (3) 40 m (4) $20\sqrt{3}$ m

12. A flag staff on the top of the tower 80 meter high, subtends an angle $\tan^{-1}\left(\frac{1}{9}\right)$ at a point on the ground 100 meters away from the foot of the tower. Find the height of the flag-staff -

(1) 20 m (2) 30 m (3) 25 m (4) 35 m

13. A person walking along a straight road observes that a two points 1 km apart, the angles of elevation of a pole in front of him are 30° and 75°. The height of the pole is -

(1) $250(\sqrt{3}+1)$ m (2) $250(\sqrt{3}-1)$ m (3) $225(\sqrt{2}-1)$ m (4) $225(\sqrt{2}+1)$ m

14. An observer in a boat finds that the angle of elevation of a tower standing on the top of a cliff is 60° and that of the top of cliff is 30°. If the height of the tower be 60 meters, then the height of the cliff is-

(1) 30 m (2) $60\sqrt{3}$ m (3) $20\sqrt{3}$ m (4) None of these

15. ABCD is a square plot. The angle of elevation of the top of a pole standing at D from A and C is 30° and that from B is θ, then tan θ is equal to -

(1) $\sqrt{6}$ (2) $1/\sqrt{6}$ (3) $\sqrt{3}/\sqrt{2}$ (4) $\sqrt{2}/\sqrt{3}$

16. The angle of elevation of a ladder against a wall is 58° and the length of foot of the ladder is 9.6 m from the wall. Then the length of the ladder is - [cos 58° = 0.5299]

(1) 18.11 m (2) 16.11 m (3) 17.11 m (4) 19.11 m

17. From the top of a tower, the angle of depression of a point P on the ground is 30°. If the distance of the point P from the tower be 24 meters then height of the tower is.

(1) 12 m (2) $8\sqrt{3}$ m (3) $24\sqrt{3}$ m (4) $12\sqrt{3}$ m

18. A tower subtends an angle of 30° at a point on the same level as the foot of the tower. At a second point, h metre above first, point the depression of the foot of the tower is 60°, the horizontal distance of the tower from the points is

(1) h cos 60° (2) (h/3) cot 30° (3) (h/3) cot 60° (4) h cot 30°

19. From a point on the ground 100 m away from the base of a building, the angle of elevation of the top of the building is 60°. Which of the following is the best approximation for the height of the building-

(1) 172 m (2) 173 m (3) 174 m (4) 175 m

20. A kite is flying with the string inclined at 75° to the horizon. If the length of the string is 25 m, the height of the kite is-

(1) $(25/2)(\sqrt{3}-1)$ (2) $(25/4)(\sqrt{3}+1)$ (3) $(25/4)(\sqrt{3}+1)^2$ (4) $(25/4)(\sqrt{6}+\sqrt{2})$

21. If a flagstaff 6 metres high placed on the top of a tower throws a shadow of $2\sqrt{3}$ metres along the ground then the angle (in degrees) that the sun makes with the ground is-

(1) 15° (2) 30° (3) 60° (4) $\tan^{-1} 2\sqrt{3}$

22. A 6-ft tall man finds that the angle of elevation of the top of a 24-ft-high pillar and the angle of depression of its base are complementary angles. The distance of the man from the pillar is-

(1) $2\sqrt{3}$ ft (2) $8\sqrt{3}$ ft (3) $6\sqrt{3}$ ft (4) None of these

23. A flagstaff stands vertically on a pillar, the height of the flagstaff being double the height of the pillar. A man on the ground at a distance finds that both the pillar and the flagstaff subtend equal angles at his eyes. The ratio of the height of the pillar and the distance of the man from the pillar is-

(1) $\sqrt{3}:1$ (2) 1 : 3 (3) $1:\sqrt{3}$ (4) $\sqrt{3}:2$

24. The shadow of a tower of height (1 + $\sqrt{3}$) metre standing on the ground is found to be 2 metre longer when the sun's elevation is 30°, than when the sun's elevation was -

(1) 30° (2) 45° (3) 60° (4) 75°

25. The angle of elevation of the top of an incomplete vertical pillar at a horizontal distance of 50 m from its base is 45°. If the angle of elevation of the top of the complete pillar the same point is to be 60°, then the height of the incomplete pillar is to be increased by-

(1) $50(\sqrt{3}-1)$ m (2) $50(\sqrt{3}+1)$ m (3) 50 m (4) $25\sqrt{2}$ m.

26. A vertical tower stands on a declivity which is inclined at 15° to the horizon. From the foot of the tower a man ascends the declivity for 80 feet and then finds that tower subtends an angle of 30°. The height of the tower is-

(1) $20(\sqrt{6}-\sqrt{2})$ (2) $40(\sqrt{6}-\sqrt{2})$ (3) $40(\sqrt{6}+\sqrt{2})$ (4) None of these

27. AB is a vertical pole. The point A of pole AB is on the level ground. C is the middle point of AB. P is a point on the level ground. The portion BC substends an angle β at P. If AP = n AB, then tan β is equal to-

(1) $\frac{n}{2n^2+1}$ (2) $\frac{n}{n^2-1}$ (3) $\frac{n}{n^2+1}$ (4) None of these

28. The top of a hill observed from the top and bottom of a building of height h is at angles of elevation p and q respectively. The height of the hill is -

(1) $\frac{h\cot q}{\cot q-\cot p}$ (2) $\frac{h\cot p}{\cot p-\cot q}$ (3) $\frac{h\tan p}{\tan p-\tan q}$ (4) None of these

29. A and B are two points 30 m apart in a line on the horizontal plane through the foot of a tower lying on opposite sides of the tower. If the distance of the top of the tower from A and B are 20 m and 15 m respectively, the angle of elevation of the top of the tower at A is-

(1) $\cos^{-1}(43/48)$ (2) $\sin^{-1}(43/48)$ (3) $\cos^{-1}(29/36)$ (4) $\sin^{-1}(29/36)$

30. A vertical pole subtends an angle $\tan^{-1}(1/2)$ at a point P on the ground. The angle subtended by the upper half of the pole at the point P is-

(1) $\tan^{-1}(1/4)$ (2) $\tan^{-1}(2/9)$ (3) $\tan^{-1}(1/8)$ (4) $\tan^{-1}(2/3)$

31. The angle of elevation of the top of a tower standing on a horizontal plane from a point A is α. After walking a distance d towards the foot of the tower, the angle of elevation is found to be β. The height of the tower is-

(1) $\frac{d\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$ (2) $\frac{d\sin\alpha\sin\beta}{\sin(\alpha-\beta)}$ (3) $\frac{d\sin(\beta-\alpha)}{\sin\alpha\sin\beta}$ (4) $\frac{d\sin(\alpha-\beta)}{\sin\alpha\sin\beta}$

32. The angle of elevation of the top of two vertical towers as seen from the middle point of the line joining the foot of the towers are 60° and 30° respectively. The ratio of the height of the towers is-

(1) 2 : 1 (2) $\sqrt{3}:1$ (3) 3 : 2 (4) 3 : 1

33. A person walking along a st. road towards a hill observes at two points, distance $\sqrt{3}$ km, the angles of elevation of the hill to be 30° and 60°. The height of the hill is-

(1) $\frac{3}{2}$ km (2) $\sqrt{\frac{2}{3}}$ km (3) $\frac{\sqrt{3}+1}{2}$ km (4) $\sqrt{3}$ km

34. The length of the shadow of a vertical pole of height h, thrown by the sun's rays at three different moments are h, 2h and 3h. The sum of the angles of elevation of the rays at these three moments is equal to-

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{6}$

35. The upper $\frac{3}{4}$th portion of a vertical pole subtends an an angle $\tan^{-1}\frac{3}{5}$ at a point in the horizontal plane through its foot and at a distance 40 m from the foot. A possible height of the vertical pole is- **[AIEEE-2002]**

(1) 80 m (2) 20 m (3) 40 m (4) 60 m

36. A person standing on the bank of a river observes that the angle of elevation of the top of a tree on the opposite bank of the river is 60° and when he retires 40 meters away from the tree the angle of elevation becomes 30°. The breadth of the river is- **[AIEEE-2004]**

(1) 20 m (2) 30 m (3) 40 m (4) 60 m

37. A tower stands at the centre of a circular park. A and B are two points on the boundary of the park such that AB (=a) subtends an angle of 60° at the foot of the tower, and the angle of elevation of the top of the tower from A or B is 30°. The height of the tower is- **[AIEEE-2007]**

(1) $2a/\sqrt{3}$ (2) $2a\sqrt{3}$ (3) $a/\sqrt{3}$ (4) $a\sqrt{3}$

38. AB is a vertical pole with B at the ground level and A at the top. A man finds that the angle of elevation of the point A from a certain point C on the ground is 60°. He moves away from the pole along the line BC to a point D such that CD = 7 m. From D the angle of elevation of the point A is 45°. Then the height of the pole is- **[AIEEE-2008]**

(1) $\frac{7\sqrt{3}}{2}\frac{1}{\sqrt{3}-1}$m (2) $\frac{7\sqrt{3}}{2}(\sqrt{3}+1)$m (3) $\frac{7\sqrt{3}}{2}(\sqrt{3}-1)$m (4) $\frac{7\sqrt{3}}{2}\frac{1}{\sqrt{3}+1}$m

39. A man standing on a horizontal plane, observes the angle of elevation of the top of a tower to be α. After walking a distance equal to double the height of the tower, the angle of elevation becomes 2α, then α is -

(1) $\frac{\pi}{18}$ (2) $\frac{\pi}{12}$ (3) $\frac{\pi}{6}$ (4) $\frac{\pi}{2}$

40. ABCD is a trapezium such that AB and CD are parallel and BC ⊥ CD. If ∠ADB = θ, BC = p and CD = q, then AB is equal to **[JEE-MAINS-2013]**

(1) $\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$ (2) $\frac{p^2+q^2\cos\theta}{p\cos\theta+q\sin\theta}$ (3) $\frac{p^2+q^2}{p^2\cos\theta+q^2\sin\theta}$ (4) $\frac{(p^2+q^2)\sin\theta}{(p\cos\theta+q\sin\theta)^2}$

41. If the angles of elevation of the top of a tower from three collinear points A, B and C, on a line leading to the foot of the tower, are 30°, 45° and 60° respectively, then the ratio, AB : BC, is : **[JEE(Main)-2015]**

(1) $1:\sqrt{3}$ (2) 2 : 3 (3) $\sqrt{3}:1$ (4) $\sqrt{3}:\sqrt{2}$

42. A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30°. After walking for 10 minutes from A in the same direction, at a point B, he observes that the angle of elevation of the top of the pillar is 60°. Then the time taken (in minutes) by him, form B to reach the pillar, is : **[JEE(Main)-2016]**

(1) 5 (2) 6 (3) 10 (4) 20

43. Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that AP = 2AB. If ∠BPC = β, then tanβ is equal to :- **[JEE(Main)-2017]**

(1) $\frac{4}{9}$ (2) $\frac{6}{7}$ (3) $\frac{1}{4}$ (4) $\frac{2}{9}$

44. PQR is a triangular park with PQ = PR = 200 m. A T.V. tower stands at the mid-point of QR. If the angles of elevation of the top of the tower at P,Q and R are respectively 45°, 30° and 30°, then the height of the tower (in m) is- **[JEE(Main)-2018]**

(1) 50 (2) $100\sqrt{3}$ (3) $50\sqrt{2}$ (4) 100

45. Consider a triangular plot ABC with sides AB=7m, BC=5m and CA=6m. A vertical lamp-post at the mid point D of AC subtends an angle 30° at B. The height (in m) of the lamp-post is : **[JEE(Main) Jan-19]**

(1) $7\sqrt{3}$ (2) $\frac{2}{3}\sqrt{21}$ (3) $\frac{3}{2}\sqrt{21}$ (4) $2\sqrt{21}$

46. If the angle of elevation of a cloud from a point P which is 25 m above a lake be 30° and the angle of depression of reflection of the cloud in the lake from P be 60°, then the height of the cloud (in meters) from the surface of the lake is : **[JEE(Main) Jan-19]**

(1) 42 (2) 50 (3) 45 (4) 60

47. Two vertical poles of heights, 20m and 80m stand a part on a horizontal plane. The height (in meters) of the point of intersection of the lines joining the top of each pole to the foot of the other, from this horizontal plane is : **[JEE(Main) Apr-19]**

(1) 12 (2) 15 (3) 16 (4) 18

48. Two poles standing on a horizontal ground are of heights 5m and 10 m respectively. The line joining their tops makes an angle of 15° with ground. Then the distance (in m) between the poles, is :- **[JEE(Main) Apr-19]**

(1) $\frac{5}{2}(2+\sqrt{3})$ (2) $5(\sqrt{3}+1)$ (3) $5(2+\sqrt{3})$ (4) $10(\sqrt{3}-1)$

49. ABC is a triangular park with AB = AC = 100 metres. A vertical tower is situated at the mid-point of BC. If the angles of elevation of the top of the tower at A and B are $\cot^{-1}\left(3\sqrt{2}\right)$ and $\operatorname{cosec}^{-1}\left(2\sqrt{2}\right)$ respectively, then the height of the tower (in metres) is : **[JEE(Main) Apr-19]**

(1) $10\sqrt{5}$ (2) $\frac{100}{3\sqrt{3}}$ (3) 20 (4) 25

50. A 2 m ladder leans against a vertical wall. If the top of the ladder begins to slide down the wall at the rate 25 cm/ sec., then the rate (in cm/sec.) at which the bottom of the ladder slides away from the wall on the horizontal ground when the top of the ladder is 1 m above the ground is : **[JEE(Main) Apr-19]**

(1) $25\sqrt{3}$ (2) 25 (3) $\frac{25}{\sqrt{3}}$ (4) $\frac{25}{3}$

51. The angle of elevation of the top of vertical tower standing on a horizontal plane is observed to be 45° from a point A on the plane. Let B be the point 30 m vertically above the point A. If the angle of elevation of the top of the tower from B be 30°, then the distance (in m) of the foot of the tower from the point A is: **[JEE(Main) Apr-19]**

(1) $15\left(3-\sqrt{3}\right)$ (2) $15\left(3+\sqrt{3}\right)$ (3) $15\left(1+\sqrt{3}\right)$ (4) $15\left(5-\sqrt{3}\right)$

ANSWER KEY

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	**2**	**1**	**1**	**3**	**3**	**2**	**1**	**2**	**1**	**3**	**4**	**1**	**1**	**1**	**2**
Que.	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Ans.	**1**	**2**	**2**	**2**	**4**	**3**	**3**	**3**	**2**	**1**	**2**	**1**	**2**	**1**	**2**
Que.	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45
Ans.	**1**	**4**	**1**	**1**	**3**	**1**	**3**	**2**	**2**	**1**	**3**	**1**	**4**	**4**	**2**
Que.	46	47	48	49	50	51									
Ans.	**2**	**3**	**3**	**3**	**3**	**2**									