ALLEN CAREER INSTITUTE OVERSEAS

MATHEMATICS

INDEX

MODULE-1

ALLEN CAREER INSTITUTE OVERSEAS

CHAPTER 1

FUNDAMENTAL OF MATHEMATICS

Chapter 01 Contents

IMPORTANT NOTES

CHAPTER 1 FUNDAMENTAL OF MATHEMATICS

1. SET THEORY

1.1 Sets : A set is a collection of well defined objects which are distinct from each other. Sets are generally denoted by capital letters A, B, C, ... etc. and the elements of the set by *a, b, c*,... etc.

If a is an element of a set A, then we write $a \in A$ (a belongs to A).

If a is **not** an element of a set A, then we write $a \notin A$ (a does not belong to A).

Ex. The collection of vowels in english alphabet is a set containing the elements *a, e, i, o, u*.

1.2 Methods to write a Set

(i) **Roster Method :** In this method a set is described by listing elements, separated by commas and enclosing them by curly brackets.

Ex. The set of vowels of English Alphabet may be described as $\{a, e, i, o, u\}$.

(ii) **Set Builder Form :** In this case we write down a property or rule or proposition, which gives us all the element of the set.

$A = \{x : P(x)\}$

Ex. $A = \{x : x \in \mathbb{N} \text{ and } x = 2n \text{ for } n \in \mathbb{N}\}$

i.e. $A = \{2, 4, 6, ...\}$

Ex. $B = \{x^2 : x \in \mathbb{N}\}$

i.e. $B = \{1, 4, 9, ...\}$

Illustrations

Illustration-1 : The set $A = \{x : x \in \mathbb{R}, x^2 = 16 \text{ and } 2x = 6\}$ is equal to

(A) ϕ (B) $\{14, 3, 4\}$ (C) $\{3\}$ (D) $\{4\}$

Solution : $x^2 = 16 \Rightarrow x = \pm 4$

$2x = 6 \Rightarrow x = 3$

There is no value of x which satisfies both the above equations.

Thus, $A = \phi$

Hence, (A) is the correct answer.

1.3 Cardinality of a Finite Set :

The number of elements in a finite set is called the cardinality of the set A and is denoted |A| or n(A). It is also called cardinal number of the set.

Ex. $A = \{a, b, c, d\} \Rightarrow n(A) = 4$

1.4 Types of Sets

(i) **Null set or Empty set :** A set having no element in it is called an Empty set or a Null set or Void set. It is denoted by ϕ or $\{\ \}$

Ex. $A = \{x \in \mathbb{N} : 5 < x < 6\} = \phi$

A set consisting of atleast one element is called a non-empty set or a non-void set.

(ii) **Singleton set :** A set consisting of a single element is called a singleton set.

Ex. set $\{0\}$, is a singleton set

(iii) **Finite Set :** A set which has only finite number of elements is called a finite set.

Ex. $A = \{a, b, c\}$

(iv) **Infinite set :** A set which has an infinite number of elements is called an infinite set.

Ex. $A = \{1, 2, 3, 4, ...\}$ is an infinite set

(v) **Subset :** Let A and B be two sets, if every element of A is an element of B, then A is called a subset of B. If A is a subset of B, we write $A \subseteq B$

Ex : $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6, 7\} \Rightarrow A \subseteq B$

The symbol "$\Rightarrow$" stands for "implies"

Note : $(x \in A \Rightarrow x \in B) \Leftrightarrow A \subseteq B$

(vi) **Proper subset :** If A is a subset of B and $A \neq B$ then A is a proper subset of B and we write $A \subset B$

Note :

- Every set is a subset of itself i.e. $A \subseteq A$ for all A
- Empty set ϕ is a subset of every set
- The total number of subsets of a finite set containing n elements is 2^n

(vii) **Universal set :** A set consisting of all possible elements which occur in the discussion is called a Universal set and is denoted by U

Note : All sets are contained in the universal set

Ex. If A = {1, 2, 3}, B = {2, 4, 5, 6}, C = {1, 3, 5, 7} then U = {1, 2, 3, 4, 5, 6, 7} can be taken as the Universal set.

(viii) Power set : Let A be any set. The set of all subsets of A is called power set of A and is denoted by P(A).

Note :

- If $A = \phi$ then P(A) has one element.
- Power set of a given set is always non empty

Illustrations

Illustration-2: If A = {1, 2} then find its power set.

Solution : A = {1, 2} then P(A) = {ϕ, {1}, {2}, {1, 2}}

Illustration-3 : If A = {x, y}, then the power set of A is-

(A) $\{x^y, y^x\}$

(B) {ϕ, x, y}

(C) {ϕ, {x} {2y}}

(D) {ϕ, {x}, {y}, {x, y}}

Solution : Clearly P(A) = set of all subsets of A

= {ϕ, {x}, {y}, {x, y}}

$\therefore$ (D) holds.

1.5 Some Operation on Sets

(i) Union of two sets : $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Ex. A = {1, 2, 3}, B = {2, 3, 4} then $A \cup B$ = {1, 2, 3, 4}

Note : $x \in (A \cup B) \Leftrightarrow x \in A$ or $x \in B$

(ii) Intersection of two sets : $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Ex. A = {1, 2, 3}, B = {2, 3, 4} then $A \cap B$ = {2, 3}

Note :

- $x \in (A \cap B) \Leftrightarrow x \in A$ and $x \in B$
- If $A \cap B = \phi$, then A, B are disjoint sets.

Ex. If A = {1, 2, 3}, B = {7, 8, 9} then $A \cap B = \phi$

Illustrations

Illustration-4: If $aN = \{ax : x \in N\}$, then the set $6N \cap 8N$ is equal to-

(A) 8N (B) 48N (C) 12N (D) 24N

Solution : $6N = \{6, 12, 18, 24, 30, ...\}$

$8N = \{8, 16, 24, 32, ...\}$

$\therefore\ 6N \cap 8N = \{24, 48, ...\} = 24N$

Short cut Method

$6N \cap 8N = 24N$ [24 is the L.C.M. of 6 and 8]

(iii) Difference of two sets : $A - B$ or $A \backslash B = \{x : x \in A \text{ and } x \notin B\}$

Ex. $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$; $A - B = \{1\}$

Note : $x \in (A - B) \Leftrightarrow x \in A$ and $x \notin B$

(iv) Complement of a set : A' or A^c or $\bar{A} = \{x : x \notin A \text{ but } x \in U\} = U - A$

Ex. $U = \{1, 2, ..., 10\}$, $A = \{1, 2, 3, 4, 5\}$ then $A' = \{6, 7, 8, 9, 10\}$

Note :

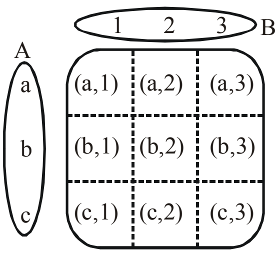
- $x \in A \Leftrightarrow x \notin A^c$
- $A \cap A' = \phi$ $\therefore$ A, A' are disjoint.
- $A \cup A' = U$
- $(A')' = A$.

(v) Cartesian Product of two sets

Cartersian product of two sets A and B, denoted as $A \times B$, is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$

$A \times B = \{(a,b) | a \in A \text{ and } b \in B\}$

Ex : Cartersian product $A \times B$ when $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$ is represented in the square grid



Note : $n(A \times B) = n(A) \times n(B)$

Illustrations

Illustration-5: Find $A \times B$ when A = {xlx is prime number less than 5} and B = {1,2,3}

Solution : A = {2,3}

B = {1,2,3}

$A \times B$ = {(2,1), (2,2), (2,3), (3,1), (3,2),(3,3)}

1.6 Equality of two Sets :

Two sets A and B are said to be equal if every element of A is an element of B, and every element of B is an element of A.

If sets A and B are equal, we write A = B and if A and B are not equal, then $A \neq B$

Ex. A = {1, 2, 6, 7} and B = {6, 1, 2, 7} $\Rightarrow$ A = B

Note : $A \subseteq B$ and $B \subseteq A \Leftrightarrow A = B$

Illustrations

Illustration-6 : Find the pairs of equal sets (if any), give reasons:

A = {0}, B = {x : x > 15 and x < 5},

C = {x : x – 5 = 0}, D = {x : x^2 = 25},

E ={x : x is an integral positive root of the equation $x^2 - 2x - 15 = 0$}.

Solution : Since $0 \in A$ and 0 does not belong to any of the sets B, C, D and E, it follows that,

$A \neq B$, $A \neq C$, $A \neq D$, $A \neq E$.

Since B = ϕ but none of the other sets are empty. Therefore $B \neq C$, $B \neq D$ and $B \neq E$.

Also C = {5} but $-5 \in D$, hence $C \neq D$.

Since E = {5},C = E. Further, D = {–5, 5} and E = {5}, we find that, $D \neq E$.

Thus, the only pair of equal sets is C and E.

Illustration-7 : Show that $A \cup B = A \cap B$ implies A = B

Solution : Let $a \in A$. Then $a \in A \cup B$. Since $A \cup B = A \cap B$, $a \in A \cap B$. So $a \in B$.

Therefore, $A \subseteq B$, Similarly, if $b \in B$, then $b \in A \cup B$. Since

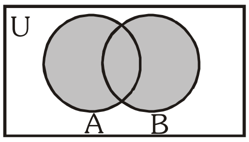
$A \cup B = A \cap B$, $b \in A \cap B$. So, $b \in A$. Therefore, $B \subseteq A$.

Thus, A = B

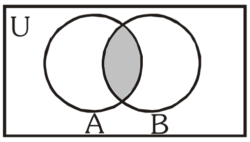
1.7 Venn Diagram

Venn diagram is a diagram representing sets pictorially as circles or closed curves within an enclosing rectangle/ a closed curve (the universal set), common elements of the sets being represented by intersections of the circles/closed curves.

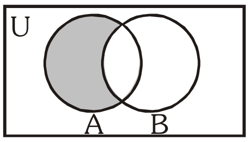
See the following Examples :



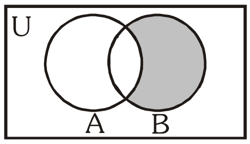
$A \cup B$



$A \cap B$

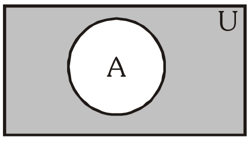


$A - B$

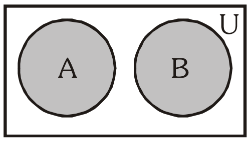


$B - A$

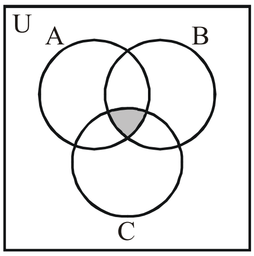
Clearly $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$



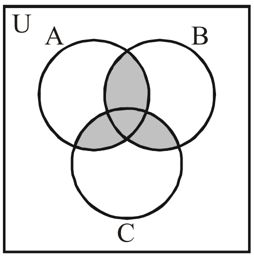
A'



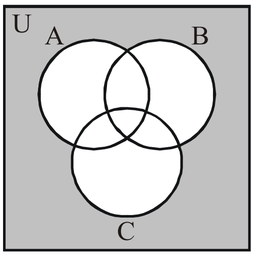
Disjoint Sets



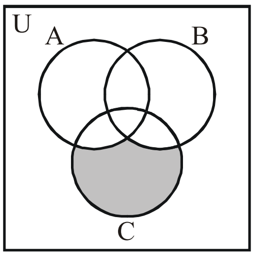
$(A \cap B \cap C)$



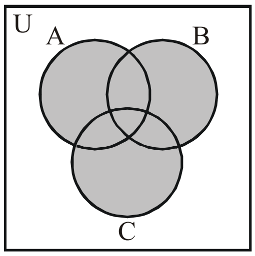
$(A \cap B) \cup (B \cap C) \cup (C \cap A)$



$(A \cup B \cup C)'$



$C - (A \cup B)$



$(A \cup B \cup C)$

Do yourself-1

1. Check whether the following statements are true or false :

(a) $A \cap \phi = \phi$ **(b)** $A \cap U = A$ **(c)** $A \cup \phi = A$ **(d)** $A \cup U = U$

(e) $A \cap B \subseteq A$ **(f)** $A \cap B \subseteq B$ **(g)** $A \subseteq A \cup B$ **(h)** $B \subseteq A \cup B$

(i) $A \subseteq B \Rightarrow A \cap B = A$ **(j)** $A \subseteq B \Rightarrow A \cup B = B$

1.8 Some Important Laws

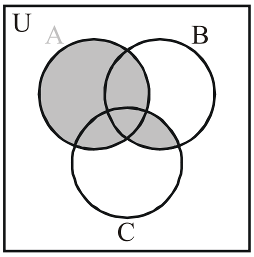
(i) **Commutative Law :** $A \cup B = B \cup A$; $A \cap B = B \cap A$

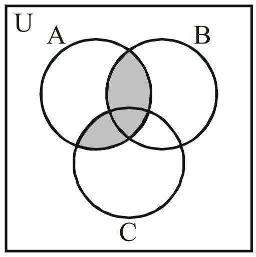
(ii) **Associative Law :** $(A \cup B) \cup C = A \cup (B \cup C)$;$(A \cap B) \cap C = A \cap (B \cap C)$

(iii) **Distributive Law :**

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;

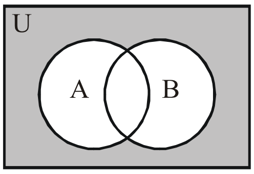
(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$



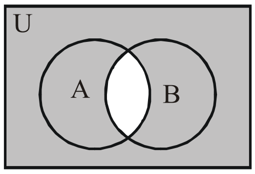


(iv) **De-Morgan's Law :**

(a) $(A \cup B)' = A' \cap B'$;



(b) $(A \cap B)' = A' \cup B'$



Do yourself-2

1. If A and B are any two sets, then verify the following using venn diagram or otherwise :

(i) $A - B = A \cap B'$

(ii) $B - A = B \cap A'$

(iii) $A - B = A \Leftrightarrow A \cap B = \phi$

(iv) $(A - B) \cup B = A \cup B$

(v) $(A - B) \cap B = \phi$

(vi) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

(vii) $A - (B \cup C) = (A - B) \cap (A - C)$

(viii) $A - (B \cap C) = (A - B) \cup (A - C)$

1.9 Some Important Results on Cardinality of Sets

If A, B and C are finite sets, and U be the finite universal set, then

(i) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B) \Leftrightarrow$ A, B are disjoint sets.

(iii) $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

Illustrations

Illustration-8 : In a group of 1000 people, there are 750 who can speak Hindi and 400 who can speak Bengali. How many can speak Hindi only ?How many can speak Bengali ? How many can speak both Hindi and Bengali ?

Solution : Let A and B be the sets of persons who can speak Hindi and Bengali respectively.

then $n(A \cup B) = 1000$, $n(A) = 750$, $n(B) = 400$.

Number of persons whose can speak both Hindi and Bengali

$= n(A \cap B) = n(A) + n(B) - n(A \cup B)$

$= 750 + 400 - 1000 = 150$

Number of persons who can speak Hindi only

$= n(A - B) = n(A) - n(A \cap B) = 750 - 150 = 600$

Number of persons who can speak Bengali only

$= n(B - A) = n(B) - n(A \cap B) = 400 - 150 = 250$

Illustration-9 : Each person in a group of 80 can speak either Hindi or English or both. If 55 persons can speak English and 40 can speak both, find the number of persons who can speak Hindi.

Solution : Let E = set of persons who can speak English.

H = set of persons who can speak Hindi.

$n(E) = 55$, $n(H) = x$, $n(E \cap H) = 40$, $n(H \cup E) = 80$

Using $n(H \cup E) = n(H) + n(E) - n(H \cap E)$

$\Rightarrow 80 = x + 55 - 40$

$\Rightarrow x = 80 - 55 + 40 = 65$

Alternate :

Using the Venn diagram

$n(U) = n(E - x) + n(x) + n(H - x)$

$80 = (55 - 40) + 40 + n(H) - 40$

$\Rightarrow n(H) = 80 - 15 = 65$

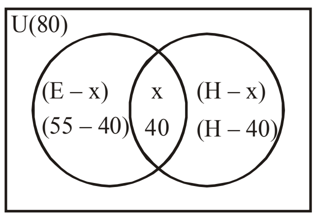


Illustration-10 : A group of members know at least one of the two languages, Hindi or Kannada. In the group, 150 members know Hindi and 80 members know Kannada and 70 of them know both. How many members are there in the group ?

Solution : Let H = set of persons who know Hindi.

K = set of persons who know Kannada.

$n(H \cap K)$ = the number of persons who know both Hindi and Kannada is 70.

$n(H \cup K) = n(H) + n(K) - n(H \cap K)$

$= 150 + 80 - 70 = 160$

Illustration-11 : In a survey of 25 students, it was found that 15 had taken mathematics, 12 had taken physics, and 11 had taken chemistry, 5 had taken mathematics and chemistry, 9 had taken mathematics and physics, 4 had taken physics and chemistry, and 3 had taken all the three subjects. Find the number of students who had taken.

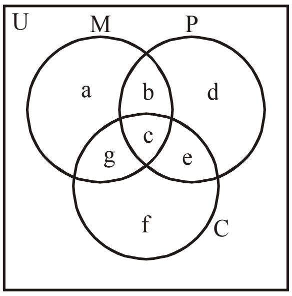
(A) Only mathematics

(B) Mathematics and physics, but not chemistry

(C) At least one of the three subjects

(D) Only one of the three subjects

Solution : Let M denote the set of students who had taken mathematics. P the set of students who had taken physics and C the set of students who had taken chemistry. In the Venn diagram, let a, b, c, d, e, f, g denote the number of students in the respective regions.



Now, $n(M \cap P \cap C) = c = 3$

$n(M \cap C) = g + c = 5 \Rightarrow g = 2$

$n(M \cap P) = b + c = 9 \Rightarrow b = 6$

$n(P \cap C) = e + c = 4 \Rightarrow e = 1$

$n(M) = a + b + g + c = 15 \Rightarrow a = 4$

$n(P) = b + c + d + e = 12 \Rightarrow d = 2$

$n(C) = e + f + g + c = 11 \Rightarrow f = 5$

(a) The number of students who had taken only mathematics = a = 4.

(b) The number of students who had taken mathematics and physics, but not chemistry = b=6

(c) The number of students who has taken at least one of the three subjects = a + b + c + d + e + f + g = 23.

(d) The number of students who had taken only one of the three subjects = a + d + f = 11.

Do yourself-3

1. Verify the following using Venn diagram.

(i) $n(A - B) = n(A) - n(A \cap B)$ i.e. $n(A - B) + n(A \cap B) = n(A)$

(ii) $n(A' \cup B') = n((A \cap B)') = n(U) - n(A \cap B)$

(iii) $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

2. If A and B are two sets, then $A \cap (A \cup B)'$ is equal to-

(A) A (B) B (C) ϕ (D) none of these

3. If A is any set, then-

(A) $A \cup A' = \phi$ (B) $A \cup A' = U$ (C) $A \cap A' = U$ (D) none of these

4. If A, B be any two sets, then $(A \cup B)'$ is equal to-
(A) $A' \cup B'$ (B) $A' \cap B'$ (C) $A \cap B$ (D) $A \cup B$

5. Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}, A = {1, 2, 5}, B = {6, 7} then $A \cap B'$ is-
(A) B' (B) A (C) A' (D) B.

6. If A and B are two sets, then $A \cup B = A \cap B$ iff-
(A) $A \subseteq B$ (B) $B \subseteq A$ (C) A = B (D) none of these

7. Two sets A, B are disjoint iff-
(A) $A \cup B = \phi$ (B) $A \cap B \neq \phi$ (C) $A \cap B = \phi$ (D) None of these

8. Which of the following is a null set ?
(A) {0} (B) $\{x : x > 0 \text{ or } x < 0\}$
(C) $\{x : x^2 = 4 \text{ or } x = 3\}$ (D) $\{x : x^2 + 1 = 0, x \in \mathbb{R}\}$

9. If A = {2, 4, 5}, B = {7, 8, 9} then $n(A \times B)$ is equal to-
(A) 6 (B) 9 (C) 3 (D) 0

10. Which set is the subset of all given sets ?
(A) {1, 2, 3, 4, ...} (B) {1} (C) {0} (D) { }

11. If $Q = \left\{x : x = \frac{1}{y}, \text{ where } y \in \mathbb{N}\right\}$, then-
(A) $0 \in Q$ (B) $1 \in Q$ (C) $2 \in Q$ (D) $\frac{2}{3} \in Q$

12. $A = \{x : x \neq x\}$ represents-
(A) {0} (B) { } (C) {1} (D) {x}

2. THEORY OF NUMBERS

2.1 Types of Numbers :

(i) **Natural numbers :** The counting numbers 1, 2, 3, 4, ... are called natural numbers. The set of natural numbers is denoted by $\mathbb{N}$. Thus $\mathbb{N}$ = {1, 2, 3, 4, ...}

(ii) **Whole numbers :** Natural numbers including zero are called whole numbers. The set of whole numbers is denoted by **W** or $\mathbb{N}_0$. Thus **W** = {0, 1, 2, 3, 4, ...}

(iii) **Integers :** The numbers ... –3, –2, –1, 0, 1, 2, 3, ... are called integers and the set is denoted by **I** or $\mathbb{Z}$. Thus (**I** or $\mathbb{Z}$) = {... –3, –2, –1, 0, 1, 2, 3, ...}

Note :

- Natural numbers are also called positive integers
  (some time denoted by $\mathbf{I}^+$ or $\mathbb{Z}^+$) = {1, 2, 3, ...}
- Whole numbers are also called non-negative integers
  (denoted by **W** or $\mathbf{I}_0^+$ or $\mathbb{Z}_0^+$) = {0, 1, 2, 3, ...}

- The set of negative integers, $\mathbf{I}^-$ or $\mathbb{Z}^-$ = {..., –3, –2, –1}
- The set of non positive integers in $\mathbf{I}_0^-$ or $\mathbb{Z}_0^-$ = {..., –3, –2, –1, 0}
- Zero is neither positive nor negative but 0 is a member of the set of non negative integers as well as of the set of non positive integers.

(iv) **Even integers :** Integers which are divisible by 2 are called even integers.
e.g. 0, ±2, ±4, ...

(v) **Odd integers :** Integers which are not divisible by 2 are called as odd integers.
e.g. ±1, ±3, ±5, ±7, ...

(vi) **Prime number :** Natural number having exactly two positive divisors i.e. 1 and itself are called prime numbers.
e.g. 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, ...

(vii) **Composite number :** Let 'a' be a natural number, 'a' is said to be composite if it has atleast three distinct positive divisors.
Note :
- 1 is the only natural number that is neither a prime number nor a composite number.
- 2 is the only prime number which is even.
- Numbers which are not prime are composite numbers (except 1).
- '4' is the smallest composite number.

(viii) **Co-prime number :** Two natural numbers (not necessarily prime) are coprime, if their H.C.F (Highest common factor) is 1.
e.g. (1, 2), (1, 3), (3, 4), (3, 10), (3, 8), (5, 6), (7, 8) (15, 16) etc.
These numbers are also called relatively prime numbers.
Note :
- Two prime numbers are always co-prime but converse need not be true.
- Two consecutive natural numbers are always co-prime numbers.
- Two consecutive odd natural numbers are always co-prime numbers.

(ix) **Rational numbers :** The numbers, which can be reduced in the form p/q where p, q $\in \mathbb{Z}$ and q $\neq$ 0 and H.C.F (p, q) = 1, are called rational numbers. '$\mathbb{Q}$' represents their set.
Note :
- All integers are rational numbers with q = 1
- When numbers are expressed in reduced form of $\frac{p}{q}$, q $\neq$ 1, the rational numbers are called fractions.
- Rational numbers when represented in decimal form are either 'terminating' or 'non-terminating but repeating'.
  e.g., 5/4 = 1.25 (terminating)
  5/3 = 1.6666 ... or $1.\overline{6}$ or $1.\dot{6}$ (non terminating but repeating)
- $0.\overline{9} = 0.9999... = 1$

Illustrations

Illustration-12: Express the following rational numbers in the form of p/q, (where p, q $\in \mathbb{Z}$)

(i) $0.1\overline{2}$ (ii) $1.5\overline{23}$

Solution: **(i)** Let x = 0.1222...

$10x = 1.\overline{2}$ ... (i)

$100x = 12.\overline{2}$ ... (ii)

$\Rightarrow$ $90x = 11$

$\Rightarrow$ $x = \frac{11}{90}$ (so x is a rational number)

(ii) Let $x = 1.5\overline{23}$

$10x = 15.\overline{23}$

$1000x = 1523.\overline{23}$

$990x = 1508$

$\Rightarrow$ $x = \frac{1508}{990} = \frac{754}{495}$ (so x is a rational number)

(x) **Irrational numbers :** Numbers, which cannot be represented in p/q form as above. In decimal representation, they are neither terminating nor repeating.

e.g., $\sqrt{2}, (15)^{1/3}, \pi$ etc.

Note :

- $\pi \neq \frac{22}{7}$. Infact $\pi < \frac{22}{7}$

$\left(\frac{22}{7} = 3.142857...\right)$ is only an approximate value of π in terms of rational numbers, taken for the sake of convenience.

Actually $\pi = 3.14159265359...$

(xi) **Real numbers :** All rational and irrational numbers taken together form the set of real numbers, represented by $\mathbb{R}$. This is the largest set in the real world of numbers.

Note :

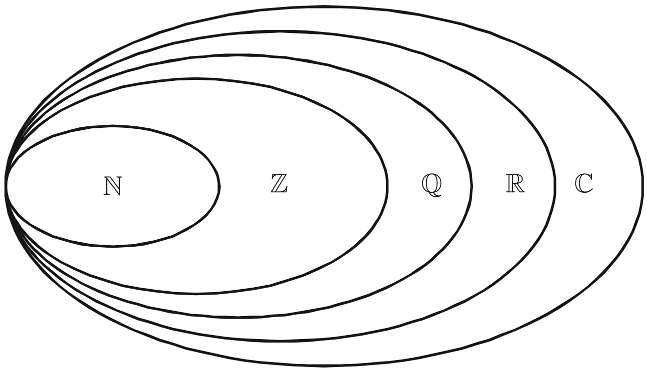
- Division by 0 is not defined.
- Integers are rational number, but converse need not be true.
- Sum of a rational number and an irrational number is always an irrational number.

  e.g. $5 + \sqrt{6}$
- The product of a non zero rational number and an irrational number will always be an irrational number.
- If $a \in \mathbb{Q}$ and $b \notin \mathbb{Q}$ then ab = rational number, only if a = 0.
- Sum, difference, product and quotient of two irrational numbers need not be an irrational number or we can say, result may be a rational number also.
- Sum, difference, product and quotient of two rational numbers is always a rational number.

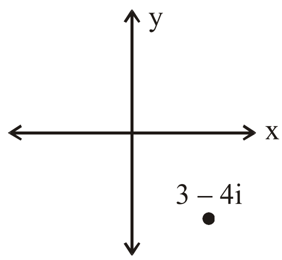
(xii) Complex number : A number z of the form a + ib is called complex number, where a, b $\in \mathbb{R}$ and i stands for $\sqrt{-1}$. Here 'a' is called real part of z denoted by Re(z) and 'b' is called imaginary part of z denoted by Im(z).

The set of complex number is represented by $\mathbb{C}$.

It may be noted that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$



Complex numbers can be represented on the **complex plane,** in the same way as a point (x, y) is plotted on the Cartesian plane. We can plot the number x + iy by taking the real part 'x' as the horizontal coordinate and the imaginary part 'y' as the vertical coordinate. For example in the adjacent diagram, the point 3 – 4i is shown on the complex plane.



When graphing complex numbers, the horizontal axis is often referred as the **real axis** and the vertical axis as the **imaginary axis**.

The complex plane also gives us a way to visualize the magnitude of a complex number.

The magnitude of a complex number is its distance from the origin when plotted on the complex plane. We use |x + iy| to denote the distance between x + iy and the origin when plotted on the complex plane. So, for example, we have $|3-4i| = \sqrt{3^2+(-4)^2} = 5$. More generally, we have $|x+iy| = \sqrt{x^2+y^2}$

Conjugate of z :

If z = a + ib, a, b $\in \mathbb{R}$, then conjugate of z (denoted as $\bar{z}$), $\bar{z}$ = a – ib.

Illustrations

Illustration-13 : Find all complex numbers z when

(i) $\bar{z} = z$ (ii) $\bar{z} = -z$

Solution : (i) Let z = a + i b, $\forall$ a, b $\in \mathbb{R}$

$z = \bar{z} \Rightarrow a + ib = a - ib \Rightarrow 2ib = 0$

$\Rightarrow b = 0$, so $z = a$ where $a \in \mathbb{R}$

(ii) for $\bar{z} = -z \Rightarrow a - ib = -a - ib \Rightarrow 2a = 0$

$\Rightarrow a = 0$, so $z = ib$, $\forall$ b $\in \mathbb{R}$

Numbers to Remember :

Number	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Square	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400
Cube	8	27	64	125	216	343	512	729	1000	1331	1728	2197	2744	3375	4096	4913	5832	6859	8000
Sq. Root	1.41	1.73	2	2.24	2.45	2.65	2.83	3	3.16	← rounded upto two places of decimal									

Note :

- Square of a real number is always non negative. (i.e. $x^2 \geq 0$)
- Square root of a positive number is always positive. e.g. $\sqrt{4} = 2$

Do yourself-4

1. If $a\sqrt{2} + b = 3\sqrt{2} + 4$, find the integral value of a, b and justify your answer.
2. Express the following in form of p/q, where p, q $\in \mathbb{Z}$ and q $\neq$ 0

   (i) $0.\overline{18}$ (ii) $0.1\overline{6}$ (iii) $0.4\overline{23}$

3. Prove that $\sqrt{2}$ is an irrational number.
4. Identify rational, irrational among x + y, x – y, xy and x/y. When x and y(xy $\neq$ 0)

   (i) are both rational

   (ii) are both irrational

   (iii) one is rational other is irrational

5. $\sqrt{a} + \sqrt{b}$ and a – b are rational numbers. Prove that $\sqrt{a}, \sqrt{b}$, a and b all are rational.
6. Check x = 0.101001000100001... is rational or irrational, where the number of zeroes between units increased by 1.

2.2 Divisibility Test

Divisibility of	Test
2	The digit at the unit's place of the number is divisible by 2.
3	The sum of digits of the number is divisible by 3.
4	The last two digits of the number together are divisible by 4.
5	The digit at the unit's place is either 0 or 5.
6	The digit at the unit's place of the number is divisible by 2 & the sum of all digits of the number is divisible by 3.
8	The last 3 digits of the number all together are divisible by 8.
9	The sum of all digits is divisible by 9.
11	The difference between the sum of the digits at even places and the sum of digits at odd places is 0 or multiple of 11. e.g. 1298, 1221, 123321, 12344321, 1234554321, 123456654321

Illustrations

Illustration-14: Consider a number N = 2 1 P 5 3 Q 4

(i) Number of ordered pairs (P, Q) so that the number 'N' is divisible by 9, is

(A) 11 (B) 12 (C) 10 (D) 8

(ii) Number of values of Q so that the number 'N' is divisible by 8, is

(A) 4 (B) 3 (C) 2 (D) 6

Solution : (i) Sum of digits = P + Q + 15

'N' is divisible by 9 if

$P + Q + 15 = 18,\ 27$

$\Rightarrow P + Q = 3$ .... (i) or $P + Q = 12$ .... (ii)

From equation (i)

$$\left.\begin{array}{ll} P = 0, & Q = 3 \\ P = 1, & Q = 2 \\ P = 2, & Q = 1 \\ P = 3, & Q = 0 \end{array}\right\} \text{Number of ordered pairs is 4}$$

From equation (ii)

$$\left.\begin{array}{ll} P = 3, & Q = 9 \\ P = 4, & Q = 8 \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ P = 8, & Q = 4 \\ P = 9, & Q = 3 \end{array}\right\} \text{Number of ordered pairs is 7}$$

Total number of ordered pairs is 11

(ii) N is divisible by 8 if $\Rightarrow Q = 0, 4, 8$

Number of values of Q is 3

Do yourself-5

1. If x, y $\in \mathbb{N}$, and $x \cdot y = 20$ then find all possible ordered pairs (x, y).

2. Find all (x, y) where x, y $\in \mathbb{N}$ such that

(i) $\frac{1}{x}+\frac{1}{y}=1$ **(ii)** $xy + 5x = 4y + 38$

3. Let x and y be positive integers such that $\frac{1}{x}+\frac{1}{y}=\frac{1}{7}$. Find all (x,y).

4. How many integers in between 500 to 2020 (both inclusive) are multiple of 3 or 7?

5. How many integers in between 1000 to 2020 (both inclusive) are divisible by 5 or 7 but not divisible by 35 ?

Paragraph for Q.6 to Q.8

Consider the number N = 7 7 4 9 5 8 P 9 6 Q

6. If P = 2 and the number N is divisible by 3, then find the number of possible values of Q.

7. If N is divisible by 4, then find the number of possible ordered pairs (P, Q).

8. If N is divisible by 8 and 9 both, then find the number of possible ordered pairs (P,Q).

3. INTERVALS

Intervals are basically subsets of $\mathbb{R}$ and are very much important in mathematics as you will get to know shortly. If there are two numbers a, b $\in \mathbb{R}$ such that a < b, we can define four types of intervals as follows :

3.1 Closed Intervals

All numbers 'x' between a and b including both numbers is written in closed interval. It is denoted by [ ]. **i.e.** $a \le x \le b$ or $x \in [a,b]$ or $\{x : x \in \mathbb{R} \text{ and } a \le x \le b\}$

Graphical Representation :

3.2 Open Intervals

All numbers 'x' between a and b excluding both numbers is written in open interval. It is denoted by] [or ( ). **i.e.** $a < x < b$ or $x \in \,]a, b[$ or $x \in (a, b)$ or $\{x : x \in \mathbb{R} \text{ and } a < x < b\}$

Graphical Representation :

3.3 Open-Closed Intervals

All numbers 'x' between a and b including b and excluding a is written in open - closed interval. It is denoted by $]a, b]$ or $(a, b]$ or $a < x \le b$ or $\{x : x \in \mathbb{R} \text{ and } a < x \le b\}$

Graphical Representation :

3.4 Closed-Open Intervals

All numbers 'x' between a and b including a but excluding b is written in closed-open interval. It is denoted by $[a, b[$ or $[a, b)$ or $a \le x < b$ or $\{x : x \in \mathbb{R} \text{ and } a \le x < b\}$

Graphical Representation :

3.5 The Infinite Intervals are Defined as follows :

- $(a, \infty) = \{x : x > a\}$
- $[a, \infty) = \{x : x \ge a\}$
- $(-\infty, b] = \{x : x \le b\}$
- $(-\infty, b) = \{x : x < b\}$

Note : $x \in \{1, 2\}$ denotes some particular values of x, i.e. $x = 1, 2$

If there is no value of x, then we can say $x \in \phi$ (Null set)

Intervals are particularly important in solving inequalities.

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Illustrations

Illustration-15 : Represent following sets on the number line

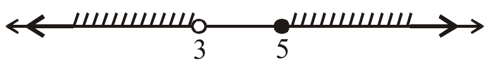
(i) $(-\infty, 3) \cup [5, \infty)$ (ii) $x \le 5$ or $x > 7$

(iii) $(-\infty, -2) \cup (-3, 5]$ (iv) $(-2, 5] \cap [-3, 2)$

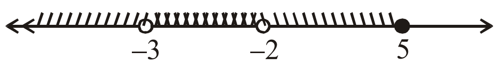
(v) $-1 \le x \le 3$ and $-2 < x$ (vi) $[-2, 10) - (1, 5]$

(vii) $(-1, 2) - (0, 3]$

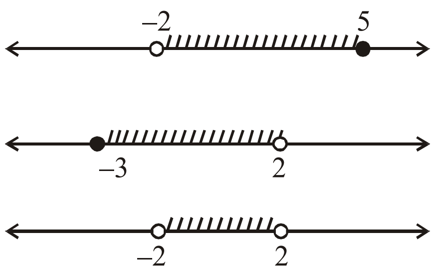
Solution :

(i) 

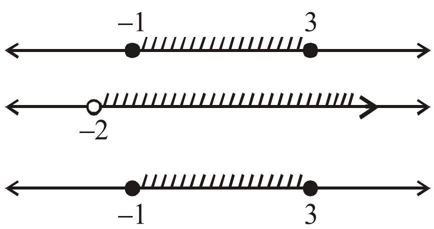
(ii) 

(iii) 

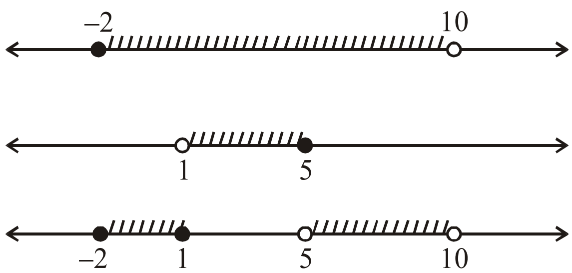
Union of given two sets is $(-\infty, 5]$

(iv) 

The intersection of given two sets is $(-2, 2)$.

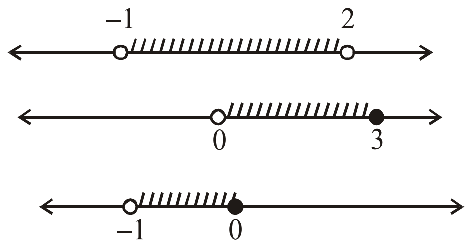
(v) 

The intersection of given two sets is $[-1, 3]$

(vi) 

Difference of given two sets is $[-2, 1] \cup (5, 10)$

(vii)



Difference of given two sets is (–1, 0].

Do yourself-6

Represent following sets on the number line

(i) $(-3, 2] \cup [5, 10) \cup (-1, 4)$

(ii) $(A \cup B) \cap C$ when

$A = (-\infty, -2)$, $B = [4, 10]$ & $C = [-10, 5)$

(iii) $[-3, 5] \cap (-2, 6) \cap [-4, 4)$

(iv) $(A \cup B) - C$ when

$A = (-\infty, -5)$, $B = [4, 10]$ & $C = (-10, 10)$

4. INDICES AND SURDS

4.1 Indices

Definition of Indices :

If 'a' is any non zero real or imaginary number and 'm' is a positive integer, then $a^m = a.\ a.\ a....a$ (m times). Here a is called the base and m is the index, power or exponent.

Law of indices :

(i) $a^0 = 1, (a \neq 0)$

(ii) $a^{-m} = \dfrac{1}{a^m}, (a \neq 0)$

(iii) $a^m.a^n = a^{m+n}$

(iv) $\dfrac{a^m}{a^n} = a^{m-n}, a \neq 0$

(v) $(a^m)^n = a^{mn} = (a^n)^m$

(vi) $\sqrt[q]{a^p} = a^{p/q}$, $q \in \mathbb{N}$ and $q \geq 2$

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(vii) If $x = y$, then $a^x = a^y$, but the converse may not be true. e.g. : $(1)^6 = (1)^8$, but $6 \neq 8$

For $a^x = a^y$ we have following possibilities

- If $a \neq \pm 1, 0$, then $x = y$
- If $a = 1$, then x, y may be any real number
- If $a = -1$, then x, y may be both even or both odd
- If $a = 0$, then x, y may be any positive real number

But if we have to solve the equations like $\left(f(x)\right)^{g(x)} = \left(f(x)\right)^{h(x)}$ (i.e same base,different indices) then we have to solve :

(a) $f(x) = 1$ (b) $f(x) = -1$ (c) $f(x) = 0$ (d) $g(x) = h(x)$

Verification should be done in (b) and (c) cases

Illustrations

Illustration-16 : Solve $(4-x)^{x^3-4x} = (4-x)^{(x^2-4)(x-1)}$

Solution : **Case-1 :** $4 - x = 1 \Rightarrow x = 3$

Case-2 : $4 - x = -1 \Rightarrow x = 5$

for $x = 5$, $x^3 - 4x$ is odd

and $(x^2 - 4)(x - 1)$ is even

so $x = 5$ does not satisfy the given equation

Case-3 : $4 - x = 0 \Rightarrow x = 4$

For $x = 4$, $x^3 - 4x > 0$ and $(x^2 - 4)(x - 1) > 0$

$\Rightarrow x = 4$ satisfies the given equation

Case-4 : $x^3 - 4x = (x^2 - 4)(x - 1)$

$\Rightarrow x^3 - 4x = x^3 - x^2 - 4x + 4$

$\Rightarrow x^2 - 4 = 0, \Rightarrow x = \pm 2$ which satisfies the given equation.

From case-1, case-2, case-3 and case-4, solutions set of the given equation is $\{-2, 2, 3, 4\}$

(viii) If $a^x = b^x$ then consider the following cases :

- If $a \neq \pm b$, then $x = 0$
- If $a = b \neq 0$, then x may have any real value for which a^x is well defined.
- If $a = -b \neq 0$, then x is even.
- If $a = b = 0$, then x can be any positive real.

If we have to solve the equation of the form $[f(x)]^{h(x)} = [g(x)]^{h(x)}$, i.e., same index, different bases, then we have to solve :

(a) $f(x) = g(x)$ (b) $f(x) = -g(x)$ (c) $h(x) = 0$

Verification should be done in (a), (b) and (c) cases.

Illustrations

Illustration-17 : Solve $\left(x^2-4\right)^{2x} = \left(x^2+2x\right)^{2x}$.

Solution : **Case-1 :** when $x = 0$

Base : $x^2 + 2x = 0$, so $x = 0$ does not satisfy given equation

Case-2 : $x^2 - 4 = x^2 + 2x \neq 0$

$\Rightarrow x \in \phi$

Case-3 : $x^2 - 4 = -x^2 - 2x \neq 0 \Rightarrow 2x^2 + 2x - 4 = 0$

$x^2 + x - 2 = 0 \Rightarrow x = 1$

satisfies the given equation

Case-4 : $x^2 - 4 = x^2 + 2x \Rightarrow x = -2$

$x = -2$ does not satisfy the given equation.

From all above cases, $x \in \{1\}$

Illustration-18 : If $a^x = b$, $b^y = c$, $c^z = a$, prove that $xyz = 1$, where a,b,c are distinct numbers.

Solution : We have, $a^{xyz} = (a^x)^{yz}$

$\Rightarrow a^{xyz} = (b)^{yz}$ $[\because a^x = b]$

$\Rightarrow a^{xyz} = (b^y)^z$

$\Rightarrow a^{xyz} = c^z$ $[\because b^y = c]$

$\Rightarrow a^{xyz} = a$ $[\because c^z = a]$

$\therefore a^{xyz} = a^1$

$\Rightarrow xyz = 1$

4.2 Surds

If 'x' is a rational number, which is not the n^{th} power ($n \in \mathbb{N}\setminus\{1\}$) of any rational number, then the number $x^{1/n}$ usually denoted by $\sqrt[n]{x}$ is called surd. The sign '$\sqrt[n]{\ }$' is called the radical sign. The number in the angular part of the sign, i.e., 'n' is called order of the surd. In case of n = 2 the expression $\sqrt[2]{x}$, simply written as $\sqrt{x}$.

Note :

- If $\sqrt[n]{x}$ is a surd then $-\left(\sqrt[n]{x}\right)$ is also a surd.
- Every surd is an irrational number (but every irrational number is not a surd).
- To rationalize the denominator of a fraction of the form $\frac{a}{\sqrt{b}}$, multiply the numerator and denominator of the fraction by $\sqrt{b} \Rightarrow \frac{a}{\sqrt{b}} = \frac{a}{\sqrt{b}}.\frac{\sqrt{b}}{\sqrt{b}} = \frac{a\sqrt{b}}{\sqrt{b^2}} = \frac{a\sqrt{b}}{b}$.

Eg.

(a) $\sqrt{3}$ is a surd and $\sqrt{3}$ is an irrational number.

(b) $\sqrt[3]{5}$ is surd and $\sqrt[3]{5}$ is an irrational number.

(c) π is an irrational number, but it is not a surd.

4.2.1 Conjugate of a Surd

If two binomial surds (surds containing two terms such as $2+\sqrt{3}$, $2\sqrt{5}-\sqrt{7}$ etc) are such that only the sign connecting the individual terms are different, then they are said to be conjugate of each other. If these surds are quadratic, then their product would always be rational. So in case of a binomial quadratic surd, we use its conjugate as its rationalizing factor.

Ex. Conjugate of $3\sqrt{2}+\sqrt{5}$ is $3\sqrt{2}-\sqrt{5}$ or $-3\sqrt{2}+\sqrt{5}$

Illustrations

Illustration-19 : Rationalize the denominator of $\frac{1}{3\sqrt{2}+\sqrt{5}}$

Solution : A conjugate of $3\sqrt{2}+\sqrt{5}$ is $3\sqrt{2}-\sqrt{5}$

Therefore multiplying the conjugate in the numerator and denominator of the given fraction.

$$\frac{3\sqrt{2}-\sqrt{5}}{\left(3\sqrt{2}+\sqrt{5}\right)\left(3\sqrt{2}-\sqrt{5}\right)}$$

$$=\frac{3\sqrt{2}-\sqrt{5}}{\left(3\sqrt{2}\right)^2-\left(\sqrt{5}\right)^2}$$

$$=\frac{3\sqrt{2}-\sqrt{5}}{18-5}$$

$$=\frac{3\sqrt{2}-\sqrt{5}}{13}$$

Illustration-20 : Using the fact that $\left(\sqrt{x}+\sqrt{y}\right)^2=x+y+\sqrt{xy}$ to find the square root of $7+2\sqrt{10}$. If this number can be expressed in the form $\sqrt{a}+\sqrt{b}$, where $a \le b$, find the value of $b-a$.

(A) 3 (B) 4 (C) 0 (D) 2

Solution : Let $\sqrt{7+2\sqrt{10}}=\sqrt{a}+\sqrt{b}$, where $a,b \in \mathbb{Q}$

$\Rightarrow 7+2\sqrt{10}=a+b+2\sqrt{ab}$

$\Rightarrow a+b=7$ and $\sqrt{ab}=\sqrt{10}$

$(b-a)^2=(a+b)^2-4ab$

$=49-40=9$

$\Rightarrow \sqrt{(b-a)^2}=3 \Rightarrow b-a=3$ (since $b > a$)

OR

$$7+2\sqrt{10}=7+2\cdot\sqrt{2}\cdot\sqrt{5}=\left(\sqrt{2}\right)^2+\left(\sqrt{5}\right)^2+2\cdot\sqrt{2}\cdot\sqrt{5}=\left(\sqrt{2}+\sqrt{5}\right)^2$$

So $\sqrt{7+2\sqrt{10}}=\sqrt{2}+\sqrt{5}$ $\therefore b-a=3$

Illustration-21 : If $x = \frac{1}{2+\sqrt{3}}$, find the value of $x^3 - x^2 - 11x + 4$.

Solution : as $x = \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{2-\sqrt{3}}{(2)^2-(\sqrt{3})^2}$

$x = \frac{2-\sqrt{3}}{4-3} = 2-\sqrt{3}$

$x - 2 = -\sqrt{3}$, squaring both sides, we get

$(x-2)^2 = \left(-\sqrt{3}\right)^2 \Rightarrow x^2 + 4 - 4x = 3$

$\Rightarrow x^2 - 4x + 1 = 0$

Now $x^3 - x^2 - 11x + 4$

$= x^3 - 4x^2 + x + 3x^2 - 12x + 4$

$= x(x^2 - 4x + 1) + 3(x^2 - 4x + 1) + 1 = x \times 0 + 3(0) + 1 \quad = 0 + 0 = 0 = 1$

Illustration-22 : If $x = 3 - 2\sqrt{2}$, find $x^2 + \frac{1}{x^2}$

Solution : We have, $x = 3 - 2\sqrt{2}$.

$\therefore \frac{1}{x} = \frac{1}{3-2\sqrt{2}} = \frac{1}{3-2\sqrt{2}} \times \frac{3+2\sqrt{2}}{3+2\sqrt{2}}$

$= \frac{3+2\sqrt{2}}{(3)^2-(2\sqrt{2})^2} = \frac{3+2\sqrt{2}}{9-8} = 3+2\sqrt{2}$

Thus, $x^2 + \frac{1}{x^2} = (3-2\sqrt{2})^2 + (3+2\sqrt{2})^2$

$= 2((3)^2 + (2\sqrt{2})^2) = 2(9+8) = 34$

Illustration-23 : Rationalise the denominator of $\frac{1}{\sqrt{3}-\sqrt{2}-1}$

Solution : $\frac{1}{\sqrt{3}-\sqrt{2}-1} = \frac{1}{\sqrt{3}-\sqrt{2}-1} \times \frac{\sqrt{3}+\sqrt{2}+1}{\sqrt{3}+\sqrt{2}+1}$

$= \frac{\sqrt{3}+\sqrt{2}+1}{(\sqrt{3}-\sqrt{2}-1)(\sqrt{3}+\sqrt{2}+1)} = \frac{\sqrt{3}+\sqrt{2}+1}{(\sqrt{3})^2-(\sqrt{2}+1)^2} = \frac{\sqrt{3}+\sqrt{2}+1}{-2\sqrt{2}}$

$= -\left(\frac{\sqrt{6}+\sqrt{2}+2}{4}\right)$

Illustration-24 : Evaluate the following

(i) $\left(\sqrt[3]{64}\right)^{\frac{-1}{2}}$ (ii) $\left(\frac{121}{169}\right)^{-3/2}$

Solution : (i) $\left(\sqrt[3]{64}\right)^{\frac{-1}{2}} = \left[(64)^{\frac{1}{3}}\right]^{\frac{-1}{2}} = (64)^{\frac{1}{3}\times\frac{-1}{2}} = (64)^{\frac{-1}{6}}$

$= (2^6)^{\frac{-1}{6}} = 2^{6\times\left(\frac{-1}{6}\right)} = 2^{-1} = \frac{1}{2}$

(ii) $\left(\frac{11\times 11}{13\times 13}\right)^{-3/2} = \left(\frac{11^2}{13^2}\right)^{-3/2} = \left(\frac{11}{13}\right)^{2\times\frac{-3}{2}} = \left(\frac{11}{13}\right)^{-3} = \left(\frac{13}{11}\right)^{3} = \frac{2197}{1331}$

Illustration 25 : Simplify $\left[\sqrt[3]{\sqrt[6]{a^9}}\right]^4\left[\sqrt[6]{\sqrt[3]{a^9}}\right]^4$

(A) a^{16} (B) a^{12} (C) a^8 (D) a^4

Solution. $a^{9(1/6)(1/3)4}.a^{9(1/3)(1/6)4} = a^2.a^2 = a^4$.

Illustration 26 : Simplify $a\left(\frac{\sqrt{a}+\sqrt{b}}{2b\sqrt{a}}\right)^{-1} + b\left(\frac{\sqrt{a}+\sqrt{b}}{2a\sqrt{b}}\right)^{-1}$

Solution. The given expression is equal to

$$a\left(\frac{2b\sqrt{a}}{\sqrt{a}+\sqrt{b}}\right) + b\left(\frac{2a\sqrt{b}}{\sqrt{a}+\sqrt{b}}\right)$$

$$= \frac{2ab}{\sqrt{a}+\sqrt{b}}\left(\sqrt{a}+\sqrt{b}\right) = 2ab$$

Illustration 27 : Evaluate $\sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{7+\sqrt{48}}}}$

Solution. $\sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{7+\sqrt{48}}}}$

$$= \sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{4+3+2\sqrt{12}}}} = \sqrt{3+\sqrt{3}+\sqrt{2+\sqrt{3}+\sqrt{4}+\sqrt{3}}}$$

$$= \sqrt{3+\sqrt{3}+\sqrt{3}+1} = \sqrt{4+2\sqrt{3}} = \sqrt{3}+1 = \sqrt{3}+1$$

Illustration 28 : Find rational numbers a and b, such that $\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = a + b\sqrt{5}$

Solution. $\frac{4+3\sqrt{5}}{4-3\sqrt{5}} \times \frac{4+3\sqrt{5}}{4+3\sqrt{5}} = a + b\sqrt{5}$

$$\frac{61+24\sqrt{5}}{-29} = a + b\sqrt{5}$$

$$a = -\frac{61}{29}, b = -\frac{24}{29}$$

Do yourself-7

Only One Option is Correct for Q.1 to Q.5

1. $\left(\left((625)^{-\frac{1}{2}}\right)^{-\frac{1}{4}}\right)^2 =$

(A) 4 (B) 5 (C) 2 (D) 3

2. $\left(5\left(8^{\frac{1}{3}} + 27^{\frac{1}{3}}\right)^3\right)^{\frac{1}{4}} =$

(A) 3 (B) 6 (C) 5 (D) 4

3. $\left\{\sqrt[4]{\left(\frac{1}{x}\right)^{-12}}\right\}^{-2/3} =$

(A) $\frac{1}{x^2}$ (B) $\frac{1}{x^4}$ (C) $\frac{1}{x^3}$ (D) $\frac{1}{x}$

4. $\frac{\sqrt{x^3} \times \sqrt[3]{x^5}}{\sqrt[5]{x^3}} \times \sqrt[30]{x^{77}} =$

(A) $x^{76/15}$ (B) $x^{78/15}$ (C) $x^{79/15}$ (D) $x^{77/15}$

5. If $\sqrt[4]{\sqrt[3]{x^2}} = x^k$, then k =

(A) $\frac{2}{6}$ (B) 6 (C) $\frac{1}{6}$ (D) 7

6. Simplify :

(i) $\frac{\left(5\sqrt{3}+\sqrt{50}\right)\left(5-\sqrt{24}\right)}{\left(\sqrt{75}-5\sqrt{2}\right)}$ (ii) $\frac{3\sqrt{2}}{\sqrt{6}+\sqrt{3}}-\frac{4\sqrt{3}}{\sqrt{6}+\sqrt{2}}+\frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}}$

(iii) $\frac{\sqrt{\left[6+2\sqrt{3}+2\sqrt{2}+2\sqrt{6}\right]-1}}{\sqrt{5+2\sqrt{6}}}$ (iv) $\frac{2.3^{n+1}-7.3^{n-1}}{3^{n+1}+2\left(\frac{1}{3}\right)^{1-n}}$

(v) $\left(\frac{1}{3}\right)^{-10}.27^{-3}+\left(\frac{1}{5}\right)^{-4}.(25)^{-2}+\left(64^{-\frac{1}{9}}\right)^{-3}$

7. If $\frac{\left(2^{n+1}\right)^m\left(2^{2n}\right)2^n}{\left(2^{m+1}\right)^n\left(2^{2m}\right)}=1$, then find the value of $\frac{m}{n}$

8. Find rational numbers a and b, such that $\frac{2+3\sqrt{5}}{1-3\sqrt{5}}=a+b\sqrt{5}$

9. The square root of $11+\sqrt{112}$ is $a+\sqrt{b}$, $a,b \in \mathbb{N}$ then $b-a$ is

10. $\sqrt{11+\sqrt{21}}=\sqrt{\frac{a}{b}}+\sqrt{\frac{c}{d}}$, where a, b, c and d are natural numbers with gcd(a, b) = gcd(c, d) = 1.

Find a + b + c + d.

11. For $x \neq 0$ find the value of $\left(\frac{x^{\ell}}{x^m}\right)^{\ell^2+\ell m+m^2}.\left(\frac{x^m}{x^n}\right)^{m^2+mn+n^2}.\left(\frac{x^n}{x^{\ell}}\right)^{n^2+n\ell+\ell^2}$

12. For $a^x = (x+y+z)^y$, $a^y = (x+y+z)^z$, $a^z = (x+y+z)^x$, then find the values of x,y and z. Where $a > 0$ and $a \neq 1$.

5. FACTORIZATIONS

5.1 $a^2 - b^2 = (a - b)(a + b)$

Illustrations

Illustration-29 : $(3x - y)^2 - (2x - 3y)^2$

Solution : Use $a^2 - b^2 = (a - b)(a + b)$

$(3x - y)^2 - (2x - 3y)^2 = (3x - y + 2x - 3y)(3x - y - 2x + 3y) = (5x - 4y)(x + 2y)$

5.2 Factorising the Quadratic expression

Illustrations

Illustration-30 : $x^2 + 6x - 187$

Solution : $x^2 + 6x - 187 = x^2 + 17x - 11x - 187$

$= x(x + 17) - 11(x + 17)$

$= (x + 17)(x - 11)$

5.3 Factorisation by converting the given expression into a perfect square.

Illustrations

Illustration-31 : $9x^4 - 10x^2 + 1$

Solution : $9x^4 - 10x^2 + 1 = (3x^2)^2 - 2\cdot 3x^2 + 1 - 4x^2$

$= (3x^2 - 1)^2 - (2x)^2$

$= (3x^2 - 1 - 2x)(3x^2 - 1 + 2x)$

$= (x - 1)(3x + 1)(x + 1)(3x - 1)$

5.4 $a^3 \pm b^3 \equiv (a \pm b)(a^2 \mp ab + b^2)$

Illustrations

Illustration-32 : $a^6 - b^6$

Solution : $a^6 - b^6 = (a^2)^3 - (b^2)^3$

$= (a^2 - b^2)(a^4 + a^2b^2 + b^4)$

$= (a - b)(a + b)(a^2 - ab + b^2)(a^2 + ab + b^2)$

5.5 Using Factor Theorem :

Illustrations

Illustration-33 : $x^3 - 13x - 12$

Solution : As $x = -1$ makes given expression 0, $x + 1$ is a factor

$$\begin{array}{r|l} x+1 & x^3 - 13x - 12 \;(\; x^2 - x - 12 \\ & \underline{x^3 + x^2} \\ & -x^2 - 13x - 12 \\ & \underline{-x^2 - x} \\ & -12x - 12 \\ & \underline{-12x - 12} \\ & 0 \end{array}$$

$\therefore\ x^3 - 13x - 12 = (x + 1)(x^2 - x - 12)$

$= (x + 1)(x - 4)(x + 3)$

5.6 $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ac)$

Illustrations

Illustration-34 : $8x^3 + y^3 + 27z^3 - 18xyz$

Solution : $8x^3 + y^3 + 27z^3 - 18xyz = (2x)^3 + (y)^3 + (3z)^3 - 3(2x)(y)(3z)$

$= (2x + y + 3z)(4x^2 + y^2 + 9z^2 - 2xy - 6xz - 3yz)$

5.7 Cyclic Expression and its Factorization :

An expression is said to be cyclic with regard to the variables $x_1, x_2, ..., x_n$ arranged in this order, when it is unchanged by changing x_1 into x_2, x_2 into x_3, x_3 into x_4, ..., x_n into x_1.

For three variables

E(x, y, z) is cyclic if $E(x, y, z) = E(y, z, x) = E(z, x, y)$

Ex. 1. $x + y + z$ is a cyclic expression

2. $x^2 + y^2 + z^2 + xy + yz + zx$ is a cyclic expression

3. $E(x, y, z) = x(y - z) + y(z - x) + z(x - y)$ is a cyclic expression because

$\Rightarrow E(y, z, x) = y(z - x) + z(x - y) + x(y - z)$

and $E(z, x, y) = z(x - y) + x(y - z) + y(z - x)$

$\Rightarrow E(x, y, z) = E(y, z, x) = E(z, x, y)$

Theorem :

If E(x, y, z) is a cyclic expression and $x - y$ is a factor of E(x, y, z) then $y - z$ and $z - x$ are also factors of E(x, y, z).

Illustrations

Illustration-35 : Factorize $x^2(y-z)+y^2(z-x)+z^2(x-y)$

Solution : $x^2(y-z)+x(z^2-y^2)+yz(y-z)$

$= (y-z)\,(x^2 - x(z+y)+yz)$

$= (y-z)\,(x^2 - xz - xy + yz)$

$= (y-z)\,(x(x-z) - y\,(x-z))$

$= (y-z)\,(x-z)\,(x-y)$

$= -(x-y)\,(y-z)\,(z-x)$

OR

Let $E(x, y, z) = x^2(y-z)+y^2(z-x)+z^2(x-y)$

when $y = x$, $E(x, y, z) = x^2(x-z)+x^2(z-x)+z^2(x-x) = x^3 - x^2z + x^2z - x^3 = 0$

$\Rightarrow x-y$ is factor of $E(x, y, z)$

So, $y-z$ and $z-x$ are also factors of $E(x, y, z)$

$E(x, y, z) = A(x-y)(y-z)(z-x)$ ...(i)

Since degree of $E(x, y, z)$ is 3 so A is constant. We can find A by substituting values of x, y & z in(i)

Let $x = 0, y = 1, z = 2$

$E(0, 1, 2) = A(-1)(-1)(2)$

$\Rightarrow 1(2-0)+2^2(0-1) = 2A \Rightarrow A = -1$

$E(x, y, z) = -(x-y)(y-z)(z-x)$

5.8 Important Algebraic Identities

- $xy + ax + by + ab = (x+a)(y+b)$
- $x^2 + 2xy + y^2 = (x+y)^2$
- $x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = (x+y+z)^2$
- $x^2 - y^2 = (x-y)(x+y)$
- $x^4 + x^2 + 1 = (x^2+1)^2 - x^2 = (x^2+x+1)(x^2-x+1)$
- $x^3 - y^3 = (x-y)(x^2+xy+y^2)$
- $x^3 + y^3 = (x+y)(x^2-xy+y^2)$
- $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + y^{n-1}),\ n \in \mathbb{N}$
- $x^{2n+1} + y^{2n+1} = (x+y)(x^{2n} - x^{2n-1}\,y + x^{2n-2}\,y^2 - \ldots + y^{2n}),\ n \in \mathbb{N}$
- $x^3 + y^3 + z^3 - 3xyz = (x+y+z)\,(x^2+y^2+z^2-xy-yz-zx)$

$$=\frac{1}{2}(x+y+z)\left\{(x-y)^2+(y-z)^2+(z-x)^2\right\}$$

- $x^3 + y^3 + z^3 = 3xyz$ if $x+y+z=0$ or $x=y=z$
- $x^3 + y^3 + z^3 + 3(x+y)(y+z)(z+x) = (x+y+z)^3$

Problem Solving Strategies :

- When facing a problem with gigantic numbers, try replacing them with smaller numbers and look for a pattern. You can often prove your pattern works and solve the problem by substituting variable expressions for the numbers.

Illustrations

Illustration-36 : Compute $\sqrt{2022\times2020\times2018\times2016+16}$ without using calculator.

Solution : Let x = 2019

$$\sqrt{2022\times2020\times2018\times2016+16}$$

$$=\sqrt{(x+3)(x+1)(x-1)(x-3)+16}$$

$$=\sqrt{(x^2-9)(x^2-1)+16}$$

$$=\sqrt{x^4-10x^2+25}=\sqrt{(x^2-5)^2}=x^2-5$$

$= 2019^2 - 5 = (2000)^2 + (19)^2 + 2 \times 2000 \times 19 - 5$

$= 4000000 + 361 + 76000 - 5$

= 4,076, 356

- Focusing on what makes a problem tricky helps identify what strategies might be best to solve the problem.
- If you have a problem that involves expressions of the form $a+b$ and $a-b$, where a and/or b involve square roots, consider finding a way to multiply the expressions to get rid of the square roots.

Illustration-37 : Suppose $x=z-\sqrt{z^2-5}$ and $5y=z+\sqrt{z^2-5}$.

Find x when y = 2/3.

Solution : $x = z-\sqrt{z^2-5}$ and $5y = z+\sqrt{z^2-5}$, multiplying both equations

we get $5xy = z^2 - (z^2 - 5) = 5$

$\Rightarrow xy = 1 \Rightarrow x = \dfrac{3}{2}$.

- When making a substitution, take some time to look for the substitution that simplifies your work the most.
- If we have the product of two variables added to linear terms with both variables, such as mn + 3m + 5n, then there is a constant we can add that will allow us to factor. For example, adding 15 to mn + 3m + 5n gives us mn + 3m + 5n + 15 = (m + 5)(n + 3).

Illustration-38 : Find all integral solutions of x and y when $xy - y - 2x = 3$.

Solution : $xy - y - 2x = 3$

$\Rightarrow xy - y - 2x + 2 = 5$

$\Rightarrow y(x-1) - 2(x-1) = 5$

$\Rightarrow (x-1)(y-2) = 5$

$x-1$	$y-2$	(x, y)
5	1	(6,3)
1	5	(2,7)
–5	–1	(–4,1)
–1	–5	(0,–3)

All possible (x, y) and (6, 3), (2, 7), (0, –3) and (–4, 1)

- Guessing has a long and glorious history in mathematics and science. It is often a very important first step in many discoveries. Don't be afraid to guess! But make sure you test your guesses – a guess itself is only a first step.

Illustration-39 : Find all pairs of positive integers m and n such that m^2 is 105 greater than n^2.

Solution : Turning the words into math is easy :

$m^2 = n^2 + 105.$

$\Rightarrow (m-n)(m+n) = 105.$

$(m-n)(m+n) = 1.105 = 3.35 = 5.21 = 7.15$

Because m and n are positive, we know that m – n is smaller than m + n, so we only have these four cases to consider :

$m - n = 1$	$m - n = 3$	$m - n = 5$	$m - n = 7$
$m + n = 105$	$m + n = 35$	$m + n = 21$	$m + n = 15$

Each of these systems of equations gives us a solution (m, n). Adding the equations in the first case gives us 2m = 106, so m = 53. Substitution then gives n = 52. Similarly, we can work through each of the other three cases to find the four solutions (m, n) = (53, 52); (19, 16); (13, 8); (11, 4).

Illustration-40 : The number 7,999,999,999 has two prime factors. Find them.

Solution : Let 7,999,999,999 = 8,000,000,000 – 1

$= (2000)^3 - 1^3 = (2000 - 1)(2000^2 + 2000 + 1)$

$= (1999)(4002001)$

According to question, 7,999,999,999 has two prime factors, they must be 1999 and 4002001.

Illustration-41 : Factor $x^4 + 4y^4$.

Solution : $x^4 + 4y^4 = (x^2)^2 + (2y^2)^2 - 2(x^2)(2y^2) + (2xy)^2$

$= (x^2 + 2y^2)^2 - (2xy)^2$

$= (x^2 + 2y^2 - 2xy)(x^2 + 2y^2 + 2xy)$

Do yourself-8

1. Factorize following expressions

(i) $x^4 - y^4$ (ii) $9a^2 - (2x - y)^2$ (iii) $4x^2 - 9y^2 - 6x - 9y$

2. Factorize following expressions

(i) $8x^3 - 27y^3$ (ii) $8x^3 - 125y^3 + 2x - 5y$

3. Factorize following expressions

(i) $x^2 + 3x - 40$ (ii) $x^2 - 3x - 40$ (iii) $x^2 + 5x - 14$

(iv) $x^2 - 3x - 4$ (v) $x^2 - 2x - 3$ (vi) $3x^2 - 10x + 8$

(vii) $12x^2 + x - 35$ (viii) $3x^2 - 5x + 2$ (ix) $3x^2 - 7x + 4$

(x) $7x^2 - 8x + 1$ (xi) $2x^2 - 17x + 26$ (xii) $3a^2 - 7a - 6$

(xiii) $14a^2 + a - 3$

4. Factorize following expressions

(i) $a^2 - 4a + 3 + 2b - b^2$ (ii) $x^4 + 324$ (iii) $x^4 - y^2 + 2x^2 + 1$

(iv) $4a^4 - 5a^2 + 1$ (v) $4x^4 + 81$ (vi) $1 + x^4 + x^8$

5. **Factorize following expressions**

(i) $x^3 - 6x^2 + 11x - 6$ **(ii)** $2x^3 + 9x^2 + 10x + 3$

(iii) $2x^3 - 9x^2 + 13x - 6$ **(iv)** $x^6 - 7x^2 - 6$

(v) $(x + y + z)^3 - x^3 - y^3 - z^3$

6. **(i)** **Factorize the expressions** $8a^6 + 5a^3 + 1$

(ii) Show that $(x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x)$.

7. **Factorize following expressions**

(i) $(x + 1)(x + 2)(x + 3)(x + 4) - 15$

(ii) $4x(2x + 3)(2x - 1)(x + 1) - 54$

(iii) $(x - 3)(x + 2)(x + 3)(x + 8) + 56$

5.9 Miscellaneous Algebraic Manipulations

Illustrations

Illustration-42 : Suppose $x + \frac{1}{x} = 5$ find **(i)** $x^2 + \frac{1}{x^2}$ **(ii)** $x^4 - \frac{1}{x^4}$

Solution : **(i)** Given : $x + \frac{1}{x} = 5$

By squaring both sides $\left(x + \frac{1}{x}\right)^2 = (5)^2 \Rightarrow x^2 + \frac{1}{x^2} + 2 = 25 \Rightarrow x^2 + \frac{1}{x^2} = 23$

(ii) $\left(x - \frac{1}{x}\right)^2 = \left(x + \frac{1}{x}\right)^2 - 4 = 21 \Rightarrow x - \frac{1}{x} = \pm\sqrt{21}$

$x^2 - \frac{1}{x^2} = \pm 5\sqrt{21}$

from (i) we have $x^2 + \frac{1}{x^2} = 23$

So $x^4 - \frac{1}{x^4} = \pm 23 \times 5\sqrt{21} = \pm 115\sqrt{21}$

Illustration-43 : Simplify $\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}$

Solution : Let $x=\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}$

By squaring both sides

$$x^2=\left(\sqrt{6+\sqrt{11}}\right)^2+\left(\sqrt{6-\sqrt{11}}\right)^2+2\sqrt{36-11}$$

= 12 + 10 = 22

Since x is positive, so $x=\sqrt{22}$.

OR

$$x=\sqrt{6+\sqrt{11}}+\sqrt{6-\sqrt{11}}=\frac{1}{\sqrt{2}}\left(\sqrt{12+2\sqrt{11}}+\sqrt{12-2\sqrt{11}}\right)$$

$$=\frac{1}{\sqrt{2}}\left(\sqrt{\left(\sqrt{11}+1\right)^2}+\sqrt{\left(\sqrt{11}-1\right)^2}\right)=\frac{1}{\sqrt{2}}\left(\sqrt{11}+1+\sqrt{11}-1\right)$$

$$x=\sqrt{22}$$

Illustration-44 : If $\left(a+\frac{1}{a}\right)^2=3$, then $a^3+\frac{1}{a^3}$ equals :

(A) $6\sqrt{3}$ (B) $3\sqrt{3}$ (C) 0 (D) $6\sqrt{3}$

Solution : $a+\frac{1}{a}=\pm\sqrt{3}$

$$a^3+\frac{1}{a^3}=\left(a+\frac{1}{a}\right)^3-3\left(a+\frac{1}{a}\right)=\pm 3\sqrt{3}\mp 3\sqrt{3}=0$$

Illustration-45 : If $x=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ and $y=\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, then find x^3+y^3.

Solution : $x=\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}=\left(\sqrt{3}-\sqrt{2}\right)^2=5-2\sqrt{6}$

$y=5+2\sqrt{6}$

$x^3+y^3=(x+y)(x^2-xy+y^2)$

$=(x+y)[(x+y)^2-3xy]=10\times[100-3]=970$

Do yourself-9

1. Suppose $a+\frac{1}{a}=3$:

(a) Find $a^2 + \frac{1}{a^2}$ **(b)** Find $a^4 + \frac{1}{a^4}$ **(c)** Find $a^3 + \frac{1}{a^3}$

2. If $x+\frac{1}{x}=2$, then prove that : $x^2+\frac{1}{x^2}=x^4+\frac{1}{x^4}=x^8+\frac{1}{x^8}$.

3. If $2x + 3y + 4z = 0$, then prove that $8x^3 + 27y^3 + 64z^3 = 72xyz$.

4. I'm thinking of two numbers. The sum of my numbers is 14 and the product of my numbers is 46. What is the sum of the squares of my numbers ?

5. Simplify $\sqrt{7-\sqrt{13}}-\sqrt{7+\sqrt{13}}$.

6. Simplify the expression $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$.

7. If x, y, z are all different real numbers, then prove that

$$\frac{1}{(x-y)^2}+\frac{1}{(y-z)^2}+\frac{1}{(z-x)^2}=\left(\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}\right)^2.$$

8. Solve for x :

(a) $\sqrt{2x+1}=\sqrt{x}+1$ **(b)** $\sqrt{x^2-1}=x-3$ **(c)** $\sqrt{x+1}+2\sqrt{2x-3}=-3$

6. POLYNOMIAL IN ONE VARIABLE

An algebraic expression of the form $p(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ is called a polynomial function in 'x' where $a_i(i = 0, 1, 2, ..., n)$ is a constant which belongs to the set of real numbers and sometimes to the set of complex numbers and n is a whole number.

- a_i is the coefficient of x^i for i = 1, 2, 3, ..., n and a_0 is constant term of p(x).
- If $a_n \neq 0$, then a_nx^n is called leading term and a_n is called **leading co-efficient.**
- If $a_n = 1$, then polynomial is called **monic polynomial.**
- If $a_n \neq 0$, then **degree of the polynomial** is n.
- $f(x) = a_0$ is called **constant polynomial.** Its degree is 0, if $a_0 \neq 0$. If $a_0 = 0$ the polynomial $f(x)$ is called **ZERO polynomial.** Its degree is defined as $-\infty$ to preserve following two properties listed below. Some people prefer not to define degree of zero polynomial.

If $f(x)$ is a polynomial of degree n and g(x) is a polynomial of degree m then

1. $f(x) \pm g(x)$ is a polynomial of degree $\leq \max\{n, m\}$
2. $f(x) \times g(x)$ is a polynomial of degree m + n.

6.1 Types of Polynomials (w.r.t. Degree)

Degree of the polynomial in one variable is the largest exponent of the variable. For example, the degree of the polynomial $3x^7 - 4x^6 + x + 9$ is 7 and the degree of the polynomial $5x^6 - 4x^2 - 6$ is 6.

Polynomials classified by degree

Degree	Name	General form	Example
undefined or $-\infty$	Zero polynomial	0	0
0	(Non-zero) constant polynomial	$a;\ (a \neq 0)$	4
1	Linear polynomial	$ax + b;\ (a \neq 0)$	$x + 1$
2	Quadratic polynomial	$ax^2 + bx + c;\ (a \neq 0)$	$x^2 + 1$
3	Cubic polynomial	$ax^3 + bx^2 + cx + d;\ (a \neq 0)$	$x^3 + 1$

Usually, a polynomial of degree n, for n greater than 3, is called a polynomial of degree n, although the phrases quartic polynomial for degree 4 and quintic polynomial for degree 5 are sometimes used.

Note :

Polynomials having only one term are called monomials. E.g. 2, 2x, $7y^5$, $12t^7$ etc. Polynomials having exactly two dissimilar terms are called binomials. E.g. $p(x) = 2x + 1$, $r(y) = 2y^7 + 5y^6$ etc. Polynomials having exactly three distinct terms are called trinomials.

E.g. $p(x) = 2x^2 + x + 6$, $q(y) = 9y^6 + 4y^2 + 1$ etc.

6.2 Division in Polynomials

Consider two polynomials P(x) & d(x) with d(x) being not identically zero and degree of d(x) $\leq$ degree of P(x) then there exists unique polynomials Q(x) and r(x) such that

$P(x) = Q(x) . d(x) + r(x)$

Here P(x) is called as dividend,

d(x) is called as divisor,

Q(x) is called as quotient,

& r(x) is called as remainder with degree of r(x) < degree of d(x)

Note : If d(x) is a divisor of P(x) then kd(x) will also be a divisor of P(x); $k \in \mathbb{R} - \{0\}$ and d(–x) will be a divisor of P(–x).

6.3 Remainder Theorem

Statement : Let p(x) be a polynomial of degree ≥ 1 and 'a' is any real number. If p(x) is divided by (x – a), then the remainder is p(a).

Illustrations

Illustration-46 : Let p(x) be $x^3 - 7x^2 + 6x + 4$. Divide p(x) with (x – 6) and find the remainder

Solution : Put x = 6 in p(x) i.e. p(6) will be the remainder.

$\therefore$ required remainder be

$p(6) = (6)^3 - 7 \cdot 6^2 + 6 \cdot 6 + 4 = 216 - 252 + 36 + 4 = 256 - 252 = 4$

$$\begin{array}{l}
x - 6\,\big)\; x^3 - 7x^2 + 6x + 4 \;\big(\; x^2 - x \\
\qquad\;\; \underline{\underset{-}{x^3} \underset{+}{-\,6x^2}} \\
\qquad\qquad\quad -x^2 + 6x + 4 \\
\qquad\qquad\quad \underline{\underset{+}{-x^2} \underset{-}{+\,6x}} \\
\qquad \text{Remainder} = 4
\end{array}$$

Thus, p(a) is remainder on dividing p(x) by (x – a).

Remark : (i) p(–a) is remainder on dividing p(x) by (x + a)

[$\because$ x + a = 0 $\Rightarrow$ x = – a]

(ii) $p\left(\frac{b}{a}\right)$ is remainder on dividing p(x) by (ax – b)

[$\because$ ax – b = 0 $\Rightarrow$ x = b/a]

(iii) $p\left(\frac{-b}{a}\right)$ is remainder on dividing p(x) by (ax + b)

[$\because$ ax + b = 0 $\Rightarrow$ x = –b/a]

(iv) $p\left(\frac{b}{a}\right)$ is remainder on dividing p(x) by (b – ax)

[$\because$ b – ax = 0 $\Rightarrow$ x = b/a]

Illustration-47 : Find the remainder when

$x^3 - ax^2 + 6x - a$ is divided by x – a

Solution : Let $p(x) = x^3 - ax^2 + 6x - a$

$p(a) = a^3 - a(a)^2 + 6(a) - a$

$= a^3 - a^3 + 6a - a = 5a$

So, by the Remainder theorem, remainder = 5a

6.4 Factor Theorem

Statement : Let f(x) be a polynomial of degree ≥ 1 and a be any real constant such that f(a) = 0, then (x – a) is a factor of f(x). Conversely, if (x – a) is a factor of f(x), then f(a) = 0.

Proof : By Remainder theorem, if f(x) is divided by (x – a), the remainder will be f(a). Let q(x) be the quotient. Then, we can write,

$f(x) = (x - a) \times q(x) + f(a)$ (∵ Dividend = Divisor × Quotient + Remainder)

If f(a) = 0, then $f(x) = (x - a) \times q(x)$

Thus, (x – a) is a factor of f(x).

Converse Let (x – a) is a factor of f(x).

Then we have a polynomial q(x) such that $f(x) = (x - a) \times q(x)$

Replacing x by a, we get f(a) = 0.

Hence, proved.

Illustrations

Illustration-48 : Use the factor theorem to determine whether (x – 1) is a factor of f(x) = $2\sqrt{2}x^3 + 5\sqrt{2}x^2 - 7\sqrt{2}$

Solution : By using factor theorem, (x – 1) is a factor of f(x), only when f(1) = 0

$f(1) = 2\sqrt{2}(1)^3 + 5\sqrt{2}(1)^2 - 7\sqrt{2} = 2\sqrt{2} + 5\sqrt{2} - 7\sqrt{2} = 0$

Hence, (x – 1) is a factor of f(x).

6.5 Fundamental Theorem of Algebra

Every polynomial function of degree ≥ 1 has atleast one zero in the complex numbers.

In other words, if we have

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \text{ , } a_i \in \text{complex number } \forall\ i = 0, 1, 2, \ldots, n$$

with n ≥ 1, then there exists atleast one $h \in \mathbb{C}$, such that

$$a_n h^n + a_{n-1}h^{n-1} + \ldots + a_1 h + a_0 = 0.$$

From this, it is easy to deduce that a polynomial function of degree 'n' (≥ 1) has exactly n zeroes.

i.e., $f(x) = a(x - r_1)(x - r_2)\ldots(x - r_n)$

Illustrations

Illustration-49: Find the constants a, b, c such that $(2x^2+3x+7)(ax^2+bx+c)=2x^4+11x^3+9x^2+13x-35$

Solution : **Method : 1**

$(2x^2 + 3x + 7)(ax^2 + bx + c) = 2x^4 + 11x^3 + 9x^2 + 13x - 35$

By comparing coefficient of x^4 from both sides

$2a = 2 \Rightarrow a = 1$

By comparing coefficient of x^3 from both sides

$2b + 3a = 11 \Rightarrow b = 4$

By comparing coefficient of x^2 from both sides

$2c + 3b + 7a = 9 \Rightarrow c = -5$

By comparing coefficient of x from both sides

$3c + 7b = 13$ (b & c satisfy)

By comparing constant

$7c = -35 \quad \Rightarrow c = -5$

So $a = 1, b = 4, c = -5$

Method : 2

Given that $(2x^2 + 3x + 7)(ax^2 + bx + c) = 2x^4 + 11x^3 + 9x^2 + 13x - 35$

$$\Rightarrow \quad ax^2+bx+c=\frac{2x^4+11x^3+9x^2+13x-35}{2x^2+3x+7}$$

$$= x^2 + 4x - 5$$

By comparing

$a = 1, b = 4$ and $c = -5$.

Illustration 50 : Show that $(x-3)$ is a factor of the polynomial $x^3 - 3x^2 + 4x - 12$.

Solution. Let $p(x) = x^3 - 3x^2 + 4x - 12$ be the given polynomial. By factor theorem, $(x - a)$ is a factor of a polynomial p(x) iff $p(a) = 0$. Therefore, in order to prove that $x - 3$ is a factor of p(x), it is sufficient to show that $p(3) = 0$.

Now, $p(x) = x^3 - 3x^2 + 4x - 12$

$\Rightarrow p(3) = 3^3 - 3 \times 3^2 + 4 \times 3 - 12 = 27 - 27 + 12 - 12 = 0$

Hence, $(x - 3)$ is a factor of $p(x) = x^3 - 3x^2 + 4x - 12$.

Illustration 51 : Without actual division prove that $2x^4 - 6x^3 + 3x^2 + 3x - 2$ is exactly divisible by $x^2 - 3x + 2$.

Solution. Let $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$ and $g(x) = x^2 - 3x + 2$ be the given polynomials.

Then $g(x) = x^2 - 3x + 2 = x^2 - 2x - x + 2 = x(x-2) - 1(x-2) = (x-1)(x-2)$

In order to prove that f(x) is exactly divisible by g(x), it is sufficient to prove that $x - 1$ and $x - 2$ are factors of f(x). For this it is sufficient to prove that $f(1) = 0$ and $f(2) = 0$.

Now, $f(x) = 2x^4 - 6x^3 + 3x^2 + 3x - 2$

$\Rightarrow \quad f(1) = 2 \times 1^4 - 6 \times 1^3 + 3 \times 1^2 + 3 \times 1 - 2 \Rightarrow f(1) = 0$

and, $f(2) = 2 \times 2^4 - 6 \times 2^3 \times 2^2 + 3 \times 2 - 2 \Rightarrow f(2) = 0$

Hence, $(x-1)$ and $(x-2)$ are factors of f(x).

$\Rightarrow$ $g(x) = (x-1)(x-2)$ is a factors of f(x).

Hence, f(x) is exactly divisible by g(x).

Illustration 52 : The polynomials $P(x) = kx^3 + 3x^2 - 3$ and $Q(x) = 2x^3 - 5x + k$, when divided by $(x - 4)$ leave the same remainder. The value of k is

(A) 2 (B) 1 (C) 0 (D) –1

Solution : $P(4) = 64k + 48 - 3 = 64k + 45$

$Q(4) = 128 - 20 + k = k + 108$

given $P(4) = Q(4)$

$64k + 45 = k + 108$

$\Rightarrow 63k = 63 \qquad \Rightarrow k = 1 \qquad \Rightarrow$ Option **(B)** is correct

Illustration-53 : Let P(x) be a polynomial such that when P(x) is divided by $x - 19$, the remainder is 99, and when P(x) is divided by $x - 99$, the remainder is 19. What is the remainder when P(x) is divided by $(x - 19)(x - 99)$?

Solution : Because we are dividing by a quadratic, the degree of the remainder is not greater than 1. So, the remainder is $ax + b$, for some constants a and b. Therefore, we have

$$P(x) = (x - 19)(x - 99)Q(x) + ax + b,$$

where Q(x) is the quotient when P(x) is divided by $(x - 19)(x - 99)$. We eliminate the Q(x) term by letting $x = 19$ or by letting $x = 99$. Doing each in turn gives us the system of equations

$$P(19) = 19a + b = 99,$$

$$P(99) = 99a + b = 19.$$

Solving this system of equations gives us $a = -1$ and $b = 118$.

So, the remainder is $-x + 118$.

Illustration-54 : The polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has roots 1, 3, 5 and 7. Determine all the coefficients of $f(x)$.

Solution : By applying factor theorem

$f(x) = (x - 1)(x - 3)(x - 5)(x - 7)$

$= (x^2 - 4x + 3)(x^2 - 12x + 35)$

$= x^2(x^2 - 12x + 35) - 4x(x^2 - 12x + 35) + 3(x^2 - 12x + 35)$

$= x^4 - 16x^3 + 86x^2 - 176x + 105$

So $a = -16$, $b = 86$, $c = -176$ and $c = 105$

Illustration-55 : If $f(x)$ is monic polynomial of degree 6 such that $f(0) = 0$, $f(1) = -1$, $f(2) = -2$, $f(3) = -3$, $f(4) = -4$ and $f(5) = -5$, then find $f(x)$.

Solution : According to question

$f(0) = 0,\ f(1) = -1,\ f(2) = -2,\ \ldots,\ f(5) = -5$

$\Rightarrow f(x) + x = 0$ has the roots $x = 0, 1, 2, \ldots, 5$

$\Rightarrow f(x) + x = x(x - 1)(x - 2)(x - 3)(x - 4)(x - 5)$ (By factor theorem)

$\Rightarrow f(x) = x(x - 1)(x - 2)(x - 3)(x - 4)(x - 5) - x.$

Illustration-56 : If $P(x) = 2x^3 + ax^2 + bx + c$, where $a, b, c \in \mathbb{Z}$. If $P\left(\sqrt{3}\right) = 10 - 2\sqrt{3}$.

Find (i) $P\left(-\sqrt{3}\right)$ (ii) $3a + c$

Solution : $P(x) = 2x^3 + ax^2 + bx + c$

$P\left(\sqrt{3}\right) = 10 - 2\sqrt{3}$

$\Rightarrow 6\sqrt{3} + 3a + b\sqrt{3} + c = 10 - 2\sqrt{3}$

$\Rightarrow (3a + c) + \sqrt{3}(6 + b) = 10 - 2\sqrt{3}$

$\Rightarrow 3a + c = 10$ and $6 + b = -2$ $(a, b, c \in I)$

$3a + c = 10$ and $b = -8$

$P\left(-\sqrt{3}\right) = -6\sqrt{3} + 3a - \sqrt{3}b + c$

$= -6\sqrt{3} + (3a + c) + 8\sqrt{3}$

$P\left(-\sqrt{3}\right) = 2\sqrt{3} + 10$

Illustration-57 : Let $P(x) = x^4 + ax^3 + bx^2 + cx + d$, where a, b, c, d are constants. If $P(1) = 10$, $P(2) = 20$, $P(3) = 30$, then compute $P(4) + P(0)$.

Solution : $P(1) = 10$, $P(2) = 20$ and $P(3) = 30$

we can write these information as

$P(x) = 10x$ for $x = 1, 2, 3$

$\Rightarrow P(x) - 10x = 0$ has roots $x = 1, 2$ and 3

By factor theorem

$P(x) - 10x = (x - \alpha)(x - 1)(x - 2)(x - 3)$

$\Rightarrow P(x) = (x - \alpha)(x - 1)(x - 2)(x - 3) + 10x$

$P(4) + P(0) = (4 - \alpha)(3)(2)(1) + 40 + (-\alpha)(-1)(-2)(-3) + 0$

$= 24 - 6\alpha + 40 + 6\alpha$

$= 64$

Illustration-58 : Let a, b and c are roots of $2x^3 + x^2 + x + 1 = 0$

find (i) $a + b + c$ (ii) abc

Solution : By factor theorem

$2x^3 + x^2 + x + 1 = 2(x - a)(x - b)(x - c)$

$\Rightarrow 2x^3 + x^2 + x + 1 = 2(x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc)$

$\Rightarrow 2x^3 + x^2 + x + 1 = 2x^3 - 2(a + b + c)x^2 + 2(ab + bc + ca)x - 2abc$

By comparing coefficient of x^2 and constant term, we have

$-2(a + b + c) = 1$ and $-2abc = 1$

$\Rightarrow a + b + c = \frac{-1}{2}$ and $abc = \frac{-1}{2}$.

Do yourself-10

1. Determine the remainder when the polynomial $P(x) = x^4 - 3x^2 + 2x + 1$ is divided by $(x - 1)$

2. Find the remainder when $f(x) = 3x^3 + 6x^2 - 4x - 5$ is divided by $(x + 3)$.

3. Determine the value of k for which $x^3 - 6x + k$ may be divisible by $(x - 2)$.

4. Find the value of a, if $(x - a)$ is a factor of $x^3 - a^2x + x + 2$.

5. Find remainder when $f(x) = x^5 - x^3 + 3x^2 + 3x + 1$ is divided by $(x^2 - 1)$.

6. Find the value of ℓ and m if $8x^3 + \ell x^2 - 27x + m$ is divisible by $2x^2 - x - 6$.

7. Find ℓ and m if $2x^3 - (2\ell + 1)x^2 + (\ell + m)x + m$ may be exactly divisible by $2x^2 - x - 3$.

8. f(x) when divided by $x^2 - 3x + 2$ leaves the remainder ax + b. If f(1) = 4 and f(2) = 7, determine a and b.

9. A polynomial in x of the third degree which will vanish when x = 1 & x = – 2 and will have the values 4 & 28 when x = –1 and x = 2 respectively. Find the polynomial

10. If $f(x)$ is polynomial of degree 4 such that $f(1) = 1, f(2) = 2, f(3) = 3, f(4) = 4$ & $f(0) = 1$ find $f(5)$.

7. EQUATIONS REDUCIBLE TO QUADRATIC EQUATIONS

There are certain equations which can be reduced to $ax^2 + bx + c = 0$ by some proper substitution.

7.1 $a(f(x))^2 + b(f(x)) + c = 0$, where $f(x)$ is expression of x

Method of solving : Put $f(x) = y$

Illustrations

Illustration-59 : **(a)** Solve $2^{x+3} + 2^{-x} - 6 = 0$ **(b)** Solve $\left(\frac{x}{x+1}\right)^2 + 6 - 5\left(\frac{x}{x+1}\right) = 0$

Solution : **(a)** Put $2^x = y$

$$8y + \frac{1}{y} - 6 = 0$$

$$8y^2 - 6y + 1 = 0$$

$$(4y - 1)(2y - 1) = 0$$

$$y = \frac{1}{4}, \frac{1}{2}$$

$\therefore\ 2^x = 2^{-2}$ and $2^x = 2^{-1}$ $\therefore\ x = -2, x = -1$

(b) Put $\frac{x}{x+1} = y$

$$y^2 - 5y + 6 = 0$$

$$(y - 2)(y - 3) = 0 \Rightarrow \frac{x}{x+1} = 2 \text{ and } \frac{x}{x+1} = 3$$

7.2 $(x - a)^4 + (x - b)^4 = c$,

Method of Solving : Put $\frac{x - a + x - b}{2} = t \Rightarrow x = \frac{(a+b)}{2} + t$

Illustrations

Illustration-60 : $(x-1)^4 + (x-7)^4 = 272$

Solution : Put $x = \frac{7+1}{2} + t$

$\Rightarrow x = t + 4$

$\Rightarrow (t+3)^4 + (t-3)^4 = 272$

$\Rightarrow 2(t^4 + 6.9t^2 + 81) = 272$

$\Rightarrow t^4 + 54t^2 + 81 = 136$

$\Rightarrow t^4 + 54t^2 - 55 = 0$

$\Rightarrow (t^2-1)(t^2+55) = 0$

$\Rightarrow t^2 = 1$

$\Rightarrow t = \pm 1$

$\Rightarrow x = 5, 3$

7.3 $ma^{2x} + n(ab)^x + rb^{2x} = 0$ (a & b > 0)

Method of Solving : Divide the equation by b^{2x} and put $\left(\frac{a}{b}\right)^x = t$ for $t > 0$

Illustrations

Illustration-60 : $3^{2x+2} + 5.6^x - 4^{x+1} = 0$

Solution : $3^{2x+2} + 5.6^x - 4^{x+1} = 0$

$\Rightarrow 9 \times (3)^{2x} + 5 \times (2 \times 3)^x - 4(2)^{2x} = 0$

$$\Rightarrow 9\left(\frac{3}{2}\right)^{2x} + 5\left(\frac{3}{2}\right)^x - 4 = 0 \qquad ...(1)$$

Let $\left(\frac{3}{2}\right)^x = t$ for $t > 0$

Equation 1 becomes

$9t^2 + 5t - 4 = 0$

$$\Rightarrow t = \frac{-5 \pm \sqrt{25+144}}{18} = -1 \text{ or } \frac{4}{9}$$

$t = -1$ (rejected)

$$t = \frac{4}{9} \Rightarrow \left(\frac{3}{2}\right)^x = \left(\frac{2}{3}\right)^2 = \left(\frac{3}{2}\right)^{-2}$$

$\Rightarrow x = -2$

Solution of the given equation is $x = -2$

7.4 $m\times a^{f(x)} + n\times b^{f(x)} + r = 0$, where $ab = 1$, a & $b > 0$ and $f(x)$ is expression of x

Method of Solving : put $a^{f(x)} = t$, then $b^{f(x)} = \dfrac{1}{t}$

Illustrations

Illustration-61 : Solve $\left(\sqrt{5+2\sqrt{6}}\right)^x + \left(\sqrt{5-2\sqrt{6}}\right)^x = 10$

Solution : Let $a = \sqrt{5+2\sqrt{6}}$ and $b = \sqrt{5-2\sqrt{6}}$

$ab = 1$

Let $a^x = t$

Given equation become

$$t + \frac{1}{t} = 10 \Rightarrow t^2 - 10t + 1 = 0$$

$$\Rightarrow t = \frac{10 \pm \sqrt{96}}{2} = 5 \pm 2\sqrt{6}$$

$$\Rightarrow \left(5+2\sqrt{6}\right)^{x/2} = \left(5+2\sqrt{6}\right) \Rightarrow \frac{x}{2} = 1 \Rightarrow x = 2$$

$$\text{or } \left(5+2\sqrt{6}\right)^{x/2} = \left(5-2\sqrt{6}\right) = \left(5+2\sqrt{6}\right)^{-1}$$

$$\Rightarrow \frac{x}{2} = -1 \Rightarrow x = -2$$

Solutions of the given equation is $x = 2$ or -2.

7.5 $(x + a)(x + b)(x + c)(x + d) + e = 0$ when $b + c = a + d$

Illustrations

Illustration-62 : Solve $x(x + 1)(x + 2)(x + 3) - 8 = 0$

Solution : $x(x + 1)(x + 2)(x + 3) - 8 = 0$

$\Rightarrow x(x + 3)(x + 1)(x + 2) - 8 = 0$

$\Rightarrow (x^2 + 3x)(x^2 + 3x + 2) - 8 = 0$

$\Rightarrow (x^2 + 3x)^2 + 2(x^2 + 3x) - 8 = 0$

$\Rightarrow (x^2 + 3x)^2 + 4(x^2 + 3x) - 2(x^2 + 3x) - 8 = 0$

$\Rightarrow (x^2 + 3x)(x^2 + 3x + 4) - 2(x^2 + 3x + 4) = 0$

$\Rightarrow (x^2 + 3x - 2)(x^2 + 3x + 4) = 0$

$\Rightarrow (x^2 + 3x - 2) = 0$ or $(x^2 + 3x + 4) = 0$

$$\Rightarrow x = \frac{-3 \pm \sqrt{17}}{2} \text{ or } x = \frac{-3 \pm \sqrt{7}\,i}{2}$$

7.6 $(x + a)(x + b)(x + c)(x + d) + ex^2 = 0$, where $ad = bc$.

Method of solving : Divide given equation by x^2 and put $x+\frac{ad}{x}=t$

Illustrations

Illustration-63 : $(x + 1)(x + 2)(x + 3)(x + 6) = 3x^2$

Solution : $(x + 1)(x + 6)\ (x + 2)(x + 3) - 3x^2 = 0$

$\Rightarrow (x^2 + 7x + 6)(x^2 + 5x + 6) - 3x^2 = 0$

$\Rightarrow \left(x+\frac{6}{x}+7\right)\left(x+\frac{6}{x}+5\right)-3=0$

Let $x+\frac{6}{x}=t$

$(t + 7)(t + 5) - 3 = 0$

$\Rightarrow t^2 + 12t + 32 = 0$

$t = -8$ or -4

when $x+\frac{6}{x}=-8 \Rightarrow x^2 + 8x + 6 = 0 \Rightarrow x=-4\pm\sqrt{10}$

$x+\frac{6}{x}=-4 \Rightarrow x^2 + 4x + 6 = 0 \Rightarrow x=-2\pm\sqrt{2}\,i$

Illustration-64 : Solve $(x^2 - 3x)(x^2 - 3x + 2) + 1 = 0$

Solution : Let $x^2 - 3x = y$

$\Rightarrow y(y + 2) + 1 = 0$

$\Rightarrow y^2 + 2y + 1 = 0$

$\Rightarrow (y + 1)^2 = 0$

$\Rightarrow y = -1$

Putting $y = 1$ in $x^2 - 3x = y$

we have $x^2 - 3x = -1$

$\Rightarrow x^2 - 3x + 1 = 0$

$\Rightarrow x=\frac{3\pm\sqrt{5}}{2}$

7.7 $ax^4 + bx^3 + cx^2 + dx + e = 0$, where $a = e$ & $b = \pm d$

Method of solving : Divide given equation by x^2 and put $x + \frac{1}{x} = t$ or $x - \frac{1}{x} = t$ whichever is applicable.

Illustrations

Illustration-65 : Solve $x^4 - 2x^3 + 3x^2 - 2x + 1 = 0$

Solution : $x^4 - 2x^3 + 3x^2 - 2x + 1 = 0$

By dividing x^2 both sides we have

$$x^2 - 2x + 3 - \frac{2}{x} + \frac{1}{x^2} = 0$$

$$\Rightarrow x^2 + \frac{1}{x^2} - 2\left(x + \frac{1}{x}\right) + 3 = 0$$

Let $x + \frac{1}{x} = t$

Above equation become

$t^2 - 2 - 2t + 3 = 0$

$\Rightarrow t^2 - 2t + 1 = 0 \Rightarrow (t - 1)^2 = 0$

$$\Rightarrow t = 1 \quad \Rightarrow x + \frac{1}{x} = 1 \Rightarrow x^2 - x + 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{3}\,i}{2}$$

Roots are $\frac{1 \pm \sqrt{3}\,i}{2}$

7.8 By Guessing Rational Roots of Polynomial.

Illustrations

Illustration-66 : Solve : $x^4 + x^3 - 2x^2 - x + 1 = 0$

Solution : Let $P(x) = x^4 + x^3 - 2x^2 - x + 1$

$P(1) = 0$ and $P(-1) = 0$

$\Rightarrow (x - 1)(x + 1)$ factor of $P(x)$

We can find other factor of $P(x)$ by dividing $x^2 - 1$ from $P(x)$.

$P(x) = (x^2 - 1)(x^2 + x - 1) = 0$

$\Rightarrow x = \pm 1$ or $x^2 + x - 1 = 0$

$$\Rightarrow x = \pm 1 \text{ or } x = \frac{-1 \pm \sqrt{5}}{2}$$

solution of $P(x) = 0$ are $x = \pm 1$ or $\frac{-1 \pm \sqrt{5}}{2}$.

Do yourself-11

Solve the following equations for x :

1. $x^{2/3} + x^{1/3} - 2 = 0$

2. $x^{\frac{2}{5}} - 3x^{\frac{1}{5}} + 2 = 0$

3. $3x^4 - 8x^2 + 4 = 0$

4. $4^x - 3.2^{x+3} + 128 = 0$

5. $x^2 + \frac{1}{x^2} - 5\left(x + \frac{1}{x}\right) + 8 = 0$

6. $2\left(x^2 + \frac{1}{x^2}\right) - 3\left(x - \frac{1}{x}\right) - 4 = 0$

7. $2^{2x+1} - 7 \times 10^x + 5^{2x+1} = 0$

8. $(x - 1)^4 + (x - 5)^4 = 82$

9. $\left(\sqrt{3}+\sqrt{2}\right)^x + \left(\sqrt{3}-\sqrt{2}\right)^x - 2\sqrt{3} = 0$

10. $\left(3+2\sqrt{2}\right)^{x/2} + \left(3-2\sqrt{2}\right)^{x/2} - 34 = 0$

11. $x(x + 1)(x + 2)(x + 3) = 24$

12. $(x + 1)(x + 2)(x + 3)(x + 4) = 120$

13. $(x + 1)(x + 2)^2(x + 4) - 2x^2 = 0$

14. $x^4 - 2x^3 - 2x^2 + 2x + 1 = 0$

15. Match the values of x given in **Column-II** satisfying the exponential equation in Column-I (Do not verify). Remember that for a > 0, the term a^x is always greater than zero $\forall\ x \in R$.

Column-I	Column-II
(A) $5^x - 24 = \frac{25}{5^x}$	**(P)** –3
(B) $(2^{x+1})(5^x) = 200$	**(Q)** –2
(C) $4^{2/x} - 5(4^{1/x}) + 4 = 0$	**(R)** –1
(D) $\frac{2^{x-1}.4^{x+1}}{8^{x-1}} = 16$	**(S)** 0
(E) $4^{x^2+2} - 9\left(2^{x^2+2}\right) + 8 = 0$	**(T)** 1
(F) $5^{2x} - 7^x - 5^{2x}(35) + 7^x(35) = 0$	**(U)** 2
	(V) 3
	(X) None

8. SYSTEM OF EQUATIONS

Observe the following Illustrations :

Illustrations

Illustrations-67 : If x – y = 2 and xy = 24, find the value of $\frac{1}{x}+\frac{1}{y}$.

Solution : $(x + y)^2 = (x - y)^2 + 4xy = 4 + 4(24)$

$\Rightarrow (x + y)^2 = 100$

$\Rightarrow x + y = 10, -10$

$$\therefore \frac{x+y}{xy}=\frac{10}{24}=\frac{5}{12};\frac{x+y}{xy}=-\frac{10}{24}=-\frac{5}{12}$$

Illustrations-68 : If 2x – 3y – z = 0 and x + 3y – 14z = 0, then find $\frac{x^2+3xy}{y^2+z^2}$.

Solution : $\frac{2x}{z}-\frac{3y}{z}=1\ \ \&\ \ \frac{x}{z}+\frac{3y}{z}=14$

Solving $\frac{x}{z}=5;\frac{y}{z}=3$

$$\Rightarrow \frac{\left(\frac{x}{z}\right)^2+\frac{3x}{z}.\frac{y}{z}}{\left(\frac{y}{z}\right)^2+1}=\frac{25+3(5)(3)}{(3)^2+1}=\frac{70}{10}=7$$

Illustration-69 : Given a + b = 20 and $a^3 + b^3 = 800$, find $a^2 + b^2$.

Solution : a + b = 20

$\Rightarrow a^2 + b^2 + 2ab = 400$ .... (1)

$a^3 + b^3 = 800$

$\Rightarrow (a + b)(a^2 - ab + b^2) = 800$

$\Rightarrow a^2 + b^2 - ab = 40$ .... (2)

By adding twice of second equation with first equation.

$3(a^2 + b^2) = 480$

$\Rightarrow a^2 + b^2 = 160.$

Illustrations-70 : x(y + z) = 29, y(z + x) = 26; z(x + y) = 51 find x,y,z.

Solution : $xy + zx = 29$ ...(1)

$yz + xy = 26$ ...(2)

$xz + yz = 51$ ...(3)

$\Rightarrow 2[xy + yz + zx] = 106$

$\Rightarrow xy + yz + zx = 53$

Now : $xy = 2$, $zx = 27$; $yz = 24$

$\Rightarrow x^2y^2z^2 = 24 \times 2 \times 27 = (36)^2$

$\Rightarrow xyz = \pm 36 \therefore x = \pm\frac{3}{2}, y = \pm\frac{4}{3}, z = \pm 18$

Illustrations-71 : If $x^3 + y^3 = 35$; and $x^2y + xy^2 = 30$, then find (x,y).

Solution : $\frac{x^3+y^3}{x^2y+xy^2} = \frac{(x+y)(x^2-xy+y^2)}{xy(x+y)} = \frac{35}{30} = \frac{7}{6}$

$\Rightarrow 6x^2 - 13xy + 6y^2 = 0$

$\Rightarrow (3x - 2y)(2x - 3y) = 0$

$\Rightarrow 3x = 2y$ or $2x = 3y$

$\Rightarrow \frac{x}{2} = \frac{y}{3} = k$ or $\frac{x}{3} = \frac{y}{2} = k_1$

Illustrations-72 : If x,y and z are real numbers such that $x^2 + y^2 + 2z^2 - 4x - 2z + 2yz + 5 = 0$, find the value of $(x - y - z)$.

Solution : $x^2 + y^2 + 2z^2 - 4x + 2z - 2yz + 5 = 0$

$\Rightarrow x^2 - 4x + 4 + y^2 + z^2 - 2yz + z^2 + 2z + 1 = 0$

$\Rightarrow (x - 2)^2 + (y - z)^2 + (z + 1)^2 = 0$

$\Rightarrow x = 2, y = z, z = -1$

$\therefore x - y - z = 2 + 1 + 1 = 4$

Illustrations-73 : $xy = 12$, $yz = 15$, $zx = 20$ find $x + y + z$

Solution : $(xyz)^2 = (3 \times 4 \times 5)^2$

$\Rightarrow xy\,z = 3 \times 4 \times 5$

$\Rightarrow z = 5, y = 3, x = 4$

$\Rightarrow x + y + z = 5 + 3 + 4 = 12$

Do yourself-12

Solve the following systems of equations :

1. $\begin{cases} x^2 - y^2 = 16, \\ x + y = 8 \end{cases}$

2. $\begin{cases} x - y = 1, \\ x^3 - y^3 = 7 \end{cases}$

3. $x + y = 2$ and $x^3 + y^3 = 56$; $x, y \in \mathbb{R}$, then find x and y.

4. $\begin{cases} x^2 + y^2 + 6x + 2y = 0, \\ x + y + 8 = 0 \end{cases}$

5. $\begin{cases} \dfrac{x}{y} - \dfrac{y}{x} = \dfrac{5}{6}, \\ x^2 - y^2 = 5 \end{cases}$

6. $\begin{cases} \dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{13}{6}, \\ xy = 5 \end{cases}$

7. $\begin{cases} \dfrac{1}{x+1} + \dfrac{1}{y} = \dfrac{1}{3}, \\ \dfrac{1}{(x+1)^2} - \dfrac{1}{y^2} = \dfrac{1}{4} \end{cases}$

8. $\begin{cases} x^2 + y^2 = 25 - 2xy \\ y(x+y) = 10 \end{cases}$

9. $\begin{cases} \dfrac{1}{y-1} - \dfrac{1}{y+1} = \dfrac{1}{x}, \\ y^2 - x - 5 = 0 \end{cases}$

10. $\begin{cases} 2xy + y^2 - 4x - 3y + 2 = 0, \\ xy + 3y^2 - 2x - 14y + 16 = 0 \end{cases}$

11. $\begin{cases} x^3 + y^3 = 7, \\ xy(x+y) = -2 \end{cases}$

12. $\begin{cases} x^4 + y^4 = 82, \\ xy = 3 \end{cases}$

9. INEQUALITIES

9.1 Basic Rules :

- If $a > b$ and $b > c$, then $a > c$.
- If $x > y$, then $x + c > y + c$ for any real number c. Additionally, if $a > b$, then $x + a > y + b$.
- If $x > y$ and $a > 0$, then $xa > ya$.
- If we multiply or divide an inequality by a negative number, we must reverse the sign. For example, if $x > y$ and $a < 0$, then $xa < ya$.
- If $x > y > 0$ and $a > b > 0$, then $xa > yb$.
- If $x > y$ and x and y have the same sign (positive or negative), then $\dfrac{1}{x} < \dfrac{1}{y}$.

- If $x > y \geq 0$, then for any positive real number a, we have $x^a > y^a$.

  In particular, if $0 < a < b$, then $\sqrt[n]{a} < \sqrt[n]{b}$ for all positive integral values of $n > 1$.

  E.g. $\sqrt[4]{2} < \sqrt[4]{7}$, $\sqrt[3]{3} < \sqrt[3]{5}$, $\sqrt[5]{10} < \sqrt[5]{13}$ etc.

  If two simple surds of different orders viz. $\sqrt[n]{a}$ and $\sqrt[m]{b}$ have to be compared, they have to be expressed as surds of the same order i.e. LCM of n and m .

  Ex. compare $\sqrt[4]{6}$ and $\sqrt[3]{5}$,

  we express both as the surds of 12th order.

  $\therefore \sqrt[4]{6} = \sqrt[12]{6^3}$ and $\sqrt[3]{5} = \sqrt[12]{5^4}$. As $6^3 < 5^4 \Rightarrow \sqrt[4]{6} < \sqrt[3]{5}$

Illustrations

Illustration-74 : Solve :

(i) $7 - 3x \geq 8 + 2x$ (ii) $\frac{8-x}{7} \geq 4$ (iii) $\frac{1}{x} \geq 5$ (iv) $\frac{4}{x+1} \leq 2$

Solution : (i) $7 - 3x \geq 8 + 2x \Rightarrow 7 - 8 \geq 2x + 3x$

$\Rightarrow \frac{-1}{5} \geq x \Rightarrow x \in \left(-\infty, -\frac{1}{5}\right]$

(ii) $\frac{8-x}{7} \geq 4 \Rightarrow 8 - x \geq 28 \Rightarrow -x \geq 20$

$\Rightarrow x \leq -20 \Rightarrow x \in (-\infty, -20]$

(iii) $\frac{1}{x} \geq 5$

$\Rightarrow \infty > \frac{1}{x} \geq 5 \quad \Rightarrow 0 < x \leq \frac{1}{5} \quad \Rightarrow x \in \left(0, \frac{1}{5}\right]$

(iv) $\frac{4}{x+1} \leq 2$

$\Rightarrow -\infty < \frac{4}{x+1} < 0$ or $0 < \frac{4}{x+1} \leq 2$ $\quad$ ($\frac{4}{x+1} = 0$ is not possible)

$\Rightarrow 0 > \frac{x+1}{4} > -\infty$ or $\infty > \frac{x+1}{4} \geq \frac{1}{2}$

$\Rightarrow x + 1 < 0$ or $2 \leq x + 1$

$\Rightarrow x < -1$ or $1 \leq x$

$\Rightarrow x \in (-\infty, -1) \cup [1, \infty)$

Do yourself-13

1. Determine which of the following statements is true or false. If it is false, provide an example that shows the statement is false. Assume *a, b, c, x* and *y* are real numbers.

(a) If $a \le b$ and $b \le c$, then $a < c$.
(b) If $a \ge b \ge a$, then $a = b$.
(c) If $a > b$, then $ac > bc$.
(d) If $a > b$ and $c \le 0$, then $ac \le bc$.
(e) If $x + a \ge y + a$, then $x \ge y$.
(f) If $x + a \ge y + b$, then $x \ge y$ and $a \ge b$.

2. Which fraction is larger ?

(a) $\frac{13}{17}$ or $\frac{17}{21}$ (b) $\frac{31}{35}$ or $\frac{37}{41}$

3. Which of the following numbers is largest : $2^{36}, 3^{30}, 5^{18}, 6^{12}, 7^{8}, 8^{4}$? (No calculators!)

4. What values of x satisfy the inequality, $7 - 3x < x - 1 \le 2x + 9$?

5. Which of the following is greater $5+\sqrt{3}, 3+\sqrt{14}$? (Without calculating the values of $\sqrt{3}, \sqrt{14}$)

9.2 Trivial and Sum of squares (SOS) Inequality

The square of any real number is non-negative. So if x is real, then $x^2 \ge 0$. This is known as Trivial inequality. Equality holds only if x = 0.

Sum of squares (SOS) of real numbers is non negative. That is $\Sigma x_i^2 \ge 0$. This is know as SOS inequality.

Equality holds if $x_i = 0 \ \forall\ i$

Ex. $x, y, z \in \mathbb{R}$ and $x^2 + y^2 + z^2 = 0 \Rightarrow x = y = z = 0$.

Note :

- $f(x) = [g(x)]^{2n}$ where $n \in \mathbb{N} \Rightarrow f(x) \ge 0$
- $f(x) = [g(x)]^{1/2n}$, $n \in \mathbb{N}$, $g(x) \ge 0 \Rightarrow f(x) \ge 0$

Illustrations

Illustration-75 : If $a, b, c \in \mathbb{R}$ and $a^2 + b^2 + c^2 - ab - bc - ca = 0$, prove that $a = b = c$.

Solution :

$a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 0$

$\Rightarrow (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ac + a^2) = 0$

$\Rightarrow (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$\Rightarrow a - b = b - c = c - a = 0$

$\Rightarrow a = b = c$.

Illustration-76 : For $x,y \in \mathbb{R}$, find the all possible values (range) of expression $4x^2 + 9y^2 - 12x + 6y$.

Solution : If $E(x,y) = 4x^2 + 9y^2 - 12x + 6y$

$= (2x)^2 - 2(2x) \times 3 + (3)^2 + (3y)^2 + 2(3y) + (1)^2 - 10$

$= (2x - 3)^2 + (3y + 1)^2 - 10$

By sum of square (SOS), $(2x - 3)^2 + (3y + 1)^2 \geq 0$

$\Rightarrow E(x,y) = (2x - 3)^2 + (3y + 1)^2 - 10 \geq 0 - 10$

So Range of $E(x,y) = [-10, \infty)$

Illustration-77 : Find all ordered pairs of real numbers (x,y) such that $(4x^2 + 4x + 3)(y^2 - 6y + 13) = 8$

Solution : Rewriting the equation

$((2x + 1)^2 + 2)\,((y - 3)^2 + 4) = 8$

from the SOS, $(2x + 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$,

so we have

$(2x + 1)^2 + 2 \geq 2$ and $(y - 3)^2 + 4 \geq 4$

$\Rightarrow ((2x + 1)^2 + 2)\,((y - 3)^2 + 4) \geq 8$

Equality holds only if

$$(2x + 1)^2 = (y - 3)^2 = 0 \Rightarrow (x, y) = \left(-\frac{1}{2}, 3\right)$$

Illustration-78 : If $b^2 - 4ac < 0$ and $a > 0$, then show that $ax^2 + bx + c > 0\ \forall\ x \in \mathbb{R}$

Solution : $ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x\right] + c$

$$= a\left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2}\right] + c$$

$$= a\left[x + \frac{b}{2a}\right]^2 - \left(\frac{b^2 - 4ac}{4a}\right) > 0\ \ \forall\ x \in \mathbb{R}.\ \text{Hence proved}$$

9.3 Mean :

In any collection of data a specific value between two extremes (minimum/maximum) is called a mean of the data.

For Two Variables :

Let x, y be two **positive** real numbers with $x \le y$.

- The **Arithmetic Mean** (**A**) of x and y is $\frac{x+y}{2}$

  Observe $x \le \frac{x+y}{2} \le y$

- The **Geometric Mean** (**G**) of x and y is $\sqrt{xy}$

  Observe $x \le \sqrt{xy} \le y$

We have $x \le G \le A \le y$

Equality holds only when $x = y$.

Proof For two positive numbers x and y, the Trivial inequality gives us $\left(\sqrt{x}-\sqrt{y}\right)^2 \ge 0$

$$\Rightarrow x+y-2\sqrt{xy} \ge 0$$

$$\Rightarrow \frac{x+y}{2} \ge \sqrt{xy} \Rightarrow A \ge G$$

Note :

- $x+\frac{1}{x} \ge 2 \ \forall\ x > 0$ and $x+\frac{1}{x} \le -2 \ \forall\ x < 0$

Illustrations

Illustration : 79 Find all possible values (range) of the expression $x+\frac{4}{x}$, when $x \in \mathbb{R}-\{0\}$.

Sol. When $x > 0$, $\frac{x+\frac{4}{x}}{2} \ge \sqrt{x \times \frac{4}{x}} = 2$ (By $A \ge G$)

$\Rightarrow x+\frac{4}{x} \ge 4$,

So, when $x > 0$, range $= [4,\infty)$ ...(1)

for $x < 0$, $\frac{x+\frac{4}{x}}{2} \le -\sqrt{x \times \frac{4}{x}}$

$\Rightarrow x+\frac{4}{x} \le -4$ ...(2)

From (1) and (2), range $= \left(-\infty,-4\right] \cup \left[4,\infty\right)$

Do yourself-14

1. Find the minimum value of $\frac{x^4+8}{x^2}$.

2. For x < 0, find the maximum value of $\frac{3x^2+12}{x}$.

3. If a, b $\in \mathbb{R}^+$, find minimum possible value of $(a+b)\left(\frac{1}{a}+\frac{1}{b}\right)$.

10. RATIO AND PROPORTION

If a and b be two quantities of the same kind, then their ratio is a : b; which may be denoted by the fraction $\frac{a}{b}$ (This may be an integer or fraction)

In the ratio a : b, a is the first term (Antecedent) and b is the second term (Consequent)

A ratio may represented in a number of ways e.g. $\frac{a}{b}=\frac{ma}{mb}=\frac{na}{nb}=\ldots$ where m, n, ... are non-zero numbers.

Let a, b, c, d be positive integers now to compare two ratios a : b and c : d we use following :

- (a : b) > (c : d) if ad > bc
- (a : b) = (c : d) if ad = bc
- (a : b) < (c : d) if ad < bc

To compare two or more ratio, reduce them to common denominator.

Note :

- If a > b > 0 and x > 0, then $\frac{a}{b}>\frac{a+x}{b+x}$, e.g. $\frac{41}{40}>\frac{45}{44}$
- If 0 < a < b and x > 0, then $\frac{a}{b}<\frac{a+x}{b+x}$

Illustrations

Illustration-80 : What term must be added to each term of the ratio 5 : 37 to make it equal to 1 : 3 ?

Solution : Let x be added to each term of the ratio 5 : 37.

Then $\frac{x+5}{x+37}=\frac{1}{3}$ $\Rightarrow$ 3x + 15 = x + 37 i.e. x = 11

Illustration-81 : If x : y = 3 : 4; find the ratio of 7x – 4y : 3x + y

Solution : $\frac{x}{y}=\frac{3}{4} \Rightarrow x=\frac{3}{4}y$

Now $\frac{7x-4y}{3x+y}=\frac{7.\frac{3}{4}y-4y}{3.\frac{3}{4}y+y}$ (putting the value of x)

$=\frac{5y}{13y}=\frac{5}{13}$ i.e. 5 : 13

10.1 Proportion :

When two ratios are equal, then the four quantities compositing them are said to be proportional.

so, if $\frac{a}{b}=\frac{c}{d}$, then it is written as a : b = c : d or a : b :: c : d

Where 'a' and 'd' are known as extremes and 'b and c' are known as means.

(i) An important property of proportion : Product of extremes = product of means.

(ii) If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$ then each is equal to $\frac{a+c+e}{b+d+f}$

(iii) If a : b = c : d, then b : a = d : c (Invertando)

i.e. $\frac{a}{b}=\frac{c}{d}$ $\Rightarrow$ $\frac{b}{a}=\frac{d}{c}$

(iv) If a : b = c : d, then a : c = b : d (Alternando)

i.e. $\frac{a}{b}=\frac{c}{d}$ $\Rightarrow$ $\frac{a}{c}=\frac{b}{d}$

(v) If a : b = c : d, then $\frac{a+b}{b}=\frac{c+d}{d}$ (Componendo)

i.e. $\frac{a}{b}=\frac{c}{d}$ $\Rightarrow$ $\frac{a}{b}+1=\frac{c}{d}+1$

(vi) If a : b = c : d, then $\frac{a-b}{b}=\frac{c-d}{d}$ (Dividendo)

i.e. $\frac{a}{b}=\frac{c}{d}$ $\Rightarrow$ $\frac{a}{b}-1=\frac{c}{d}-1$

(vii) If a : b = c : d, then $\frac{a+b}{a-b}=\frac{c+d}{c-d}$ (Componendo and dividendo)

i.e. $\frac{a}{b}=\frac{c}{d} \Rightarrow \frac{a}{b}+1=\frac{c}{d}+1 \Rightarrow \frac{a+b}{b}=\frac{c+d}{d}$ ... (1)

$\frac{a}{b}-1=\frac{c}{d}-1 \Rightarrow \frac{a-b}{b}=\frac{c-d}{d}$ ... (2)

Dividing equation (1) by (2) we obtain

$$\frac{a+b}{a-b}=\frac{c+d}{c-d}$$

Illustrations

Illustration-82 : If $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ show that $\frac{x^3+a^3}{x^2+a^2}+\frac{y^3+b^3}{y^2+b^2}+\frac{z^3+c^3}{z^2+c^2}=\frac{(x+y+z)^3+(a+b+c)^3}{(x+y+z)^2+(a+b+c)^2}$

Solution : $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=k$ (constant)

x = ak; y = bk; z = ck

substituting these values of x, y, z in the given expression, we obtain

$$\text{L.H.S.} = \frac{a^3k^3+a^3}{a^2k^2+a^2}+\frac{b^3k^3+b^3}{b^2k^2+b^2}+\frac{c^3k^3+c^3}{c^2k^2+c^2}$$

$$=\frac{a^3(k^3+1)}{a^2(k^2+1)}+\frac{b^3(k^3+1)}{b^2(k^2+1)}+\frac{c^3(k^3+1)}{c^2(k^2+1)}=\frac{(k^3+1)}{(k^2+1)}.(a+b+c)$$

$$\text{Now R.H.S} = \frac{(ak+bk+ck)^3+(a+b+c)^3}{(ak+bk+ck)^2+(a+b+c)^2}$$

$$=\frac{k^3(a+b+c)^3+(a+b+c)^3}{k^2(a+b+c)^2+(a+b+c)^2}$$

$$=\frac{(k^3+1)(a+b+c)^3}{(k^2+1)(a+b+c)^2}=\frac{(k^3+1)}{(k^2+1)}.(a+b+c) = \text{L.H.S}$$

Illustration-83 : If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}$, prove that

$(ab + bc + cd + de)^2 = (a^2 + b^2 + c^2 + d^2)(b^2 + c^2 + d^2 + e^2)$

Solution : $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}$, then we have

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\frac{\sqrt{(a^2+b^2+c^2+d^2)}}{\sqrt{(b^2+c^2+d^2+e^2)}}=k \quad \text{(say)}$$

i.e. $a = bk$ $\quad \therefore\ ab = b^2k$

$b = ck$ $\quad \therefore\ bc = c^2k$

$c = dk$ $\quad \therefore\ cd = d^2k$

$d = ek$ $\quad \therefore\ de = e^2k$

so, $(a^2 + b^2 + c^2 + d^2) = k^2(b^2 + c^2 + d^2 + e^2)$ ... (i)

Now L.H.S. $= (ab + bc + cd + de)^2$

$= (kb^2 + kc^2 + kd^2 + ke^2)^2$

$= k^2(b^2 + c^2 + d^2 + e^2)^2$

$= k^2(b^2 + c^2 + d^2 + e^2)(b^2 + c^2 + d^2 + e^2)$

$= (a^2 + b^2 + c^2 + d^2)(b^2 + c^2 + d^2 + e^2) =$ R.H.S (by use of (i))

Illustration-84 : Solve the equation $\frac{3x^4+x^2-2x-3}{3x^4-x^2+2x+3}=\frac{5x^4+2x^2-7x+3}{5x^4-2x^2+7x-3}$

Solution : $\frac{3x^4+x^2-2x-3}{3x^4-x^2+2x+3}=\frac{5x^4+2x^2-7x+3}{5x^4-2x^2+7x-3}$

Applying componendo and dividendo, we have

$$\frac{3x^4}{x^2-2x-3}=\frac{5x^4}{2x^2-7x+3}$$

or $3x^4(2x^2 - 7x + 3) - 5x^4(x^2 - 2x - 3) = 0$

or $x^4[6x^2 - 21x + 9 - 5x^2 + 10x + 15] = 0$

or $x^4(x^2 - 11x + 24) = 0$

$\therefore\ x = 0$ or $x^2 - 11x + 24 = 0$

$\therefore\ x = 0$ or $(x - 8)(x - 3) = 0$

$\therefore\ x = 0, 8, 3$

Do yourself : 15

1. If $\frac{a}{b}=\frac{2}{3}$ and $\frac{b}{c}=\frac{4}{5}$, then find value of $\frac{a+b}{b+c}$

2. If $\frac{a}{b}=\frac{3}{5}$ and $\frac{b}{c}=\frac{7}{13}$, then find the value of a : b : c

3. If sum of two numbers is *s* and their quotient is $\frac{p}{q}$. Find number.

4. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then find the value of $\frac{2a^4b^2+3a^2c^2-5e^4f}{2b^6+3b^2d^2-5f^5}$ in terms of a and b.

5. If x : a = y : b = z : c, then show that $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$.

11. SIGN-SCHEME (WAVY CURVE) METHOD

Given $f(x)$ and g(x) are polynomials.

To solve the inequalities of the type $\frac{f(x)}{g(x)} * 0$, where '*' can be >, ≥, < or ≤, we take the following steps :

(i) Find all the roots of f(x) = 0 and g(x) = 0

(ii) Write all these roots on the real line in increasing order of values.

(iii) Check the sign of the expression $\frac{f(x)}{g(x)}$ at some x greater than the largest root. If it is positive, put + sign in rightmost interval. In case of negative, put -ve sign in rightmost interval and while moving from right to left change sign in accordance with step (iv).

(iv) If a root occurs even number of times, then sign of expression will be same on both sides of the root and if a root occurs odd number of times, then sign of the expression will be different on both sides of the root.

(v) Write the answer according to need of the question.

Note :

- We don't give equality sign on '± ∞' in the solution as they are two improper points of number line.
- We can't take zeroes of denominator in the final answer as at these points expression is not defined (because division by '0' is not defined).
- In case of ≥ 0 or ≤ 0, zeroes of numerator will be part of the answer provided they are not appearing in denominator also.
- Do not cross multiply the terms in the inequalities

Illustrations

Illustration-85 : Find the solution of $-x^2 + 6x + 7 \geq 0$

Solution : $-x^2 + 6x + 7 \geq 0$

$\Rightarrow x^2 - 6x - 7 \leq 0$

$\Rightarrow (x + 1)\,(x - 7) \leq 0$

$\Rightarrow x \in [-1,7]$

Illustration-86 : Find the solution of $2x^2 - 7x - 3 \geq 0$

Solution : $2x^2 - 7x - 3 \geq 0$

$\Rightarrow (2x - 1)\,(x - 3\,) \geq 0$

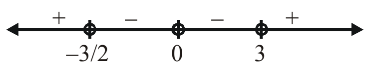
$$x \in \left(-\infty, \frac{1}{2}\right] \cup [3, \infty)$$

Illustration-87 : Solve $2x^4 > 3x^3 + 9x^2$

Solution : $2x^4 - 3x^3 - 9x^2 > 0$

$x^2(2x^2 - 3x - 9) > 0$

$x^2(2x + 3)\,(x - 3) > 0$



$$x \in \left(-\infty, -\frac{3}{2}\right) \cup (3, \infty)$$

Illustration-88 : Find the solution of the inequalities $\dfrac{(x-1)(x-2)}{(x-3)} \geq 0$

Solution : $x - 1 = 0,\ x - 2 = 0,\ x - 3 = 0$

$\Rightarrow x = 1, 2, 3$

Since $x - 3 \neq 0,\ x \neq 3$

so, $x \in [1, 2] \cup (3, \infty)$

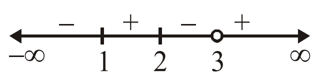
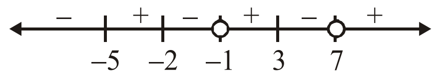


Illustration-89 : If $f(x) = \dfrac{x^3+4x^2-11x-30}{x^2-6x-7}$ then find x such that

(i) $f(x) > 0$ **(ii)** $f(x) < 0$.

Solution : Given $f(x) = \dfrac{(x-3)(x+2)(x+5)}{(x+1)(x-7)}$



(i) $f(x) > 0 \Rightarrow x \in (-5, -2) \cup (-1, 3) \cup (7, \infty)$

(ii) $f(x) < 0 \Rightarrow x \in (-\infty, -5) \cup (-2, -1) \cup (3, 7)$

Illustration 90: Solve for real x : $\dfrac{1}{x+1} + \dfrac{2}{x+2} > \dfrac{3}{x+3}$

Solution : $\dfrac{3x+4}{(x+1)(x+2)} > \dfrac{3}{x+3}$

$$\Rightarrow \frac{3x+4}{(x+1)(x+2)} - \frac{3}{x+3} > 0$$

$$\Rightarrow \frac{4x+6}{(x+1)(x+2)(x+3)} > 0$$

so, $x \in (-\infty, -3) \cup \left(-2, -\dfrac{3}{2}\right) \cup (-1, \infty)$

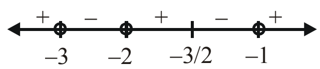
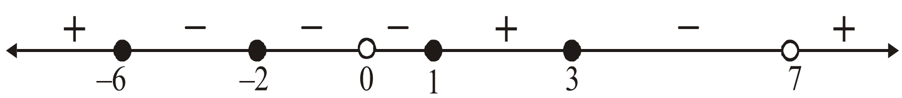


Illustration 91: Let $f(x) = \dfrac{(x-1)^3(x+2)^4(x-3)^5(x+6)}{x^2(x-7)^3}$. Solve the following inequality

(i) $f(x) > 0$ (ii) $f(x) \geq 0$ (iii) $f(x) < 0$ (iv) $f(x) \leq 0$

Solution : We mark on the number line zeroes of numerator of expression : 1, –2, 3 and –6 (with black circles) and the zeroes of denominator 0 and 7 (with white circles), isolate the double points : –2 and 0 and draw the wavy curve :



From graph, we get

(i) $x \in (-\infty, -6) \cup (1, 3) \cup (7, \infty)$

(ii) $x \in (-\infty, -6] \cup \{-2\} \cup [1, 3] \cup (7, \infty)$

(iii) $x \in (-6, -2) \cup (-2, 0) \cup (0, 1) \cup (3, 7)$

(iv) $x \in [-6, 0) \cup (0, 1] \cup [3, 7)$

Do yourself-16

Solve following inequalities over the set of real numbers :

1. $(x-1)^2 (x+1)^3 (x-4) < 0$

2. $\dfrac{6x-5}{4x+1} < 0$

3. $\dfrac{(x-1)(x+2)^2}{-1-x} < 0$

4. $\dfrac{(2x-1)(x-1)^2(x-2)^3}{(x-4)^4} > 0$

5. $\dfrac{(x-1)^2(x+1)^3}{x^4(x-2)} \le 0$

6. $\dfrac{(x-2)^2(1-x)(x-3)^3(x-4)^2}{(x+1)} \le 0$

7. $\dfrac{x^2+4x+4}{2x^2-x-1} < 0$

8. $\dfrac{x^3(x-2)(5-x)}{(x^2-4)(x+1)} > 0$

9. $\dfrac{(2-x^2)(x-3)^3}{(x+1)(x^2-3x-4)} \ge 0$

10. $x^4 - 5x^2 + 4 < 0$

11. $\dfrac{15-4x}{x^2-x-12} < 4$

12. $\dfrac{x^2+1}{4x-3} > 2$

13. $\dfrac{1}{x-2} + \dfrac{1}{x-1} > \dfrac{1}{x}$

14. $(x-2)(x+3) \ge 0$

15 $\dfrac{x}{x+1} > 2$

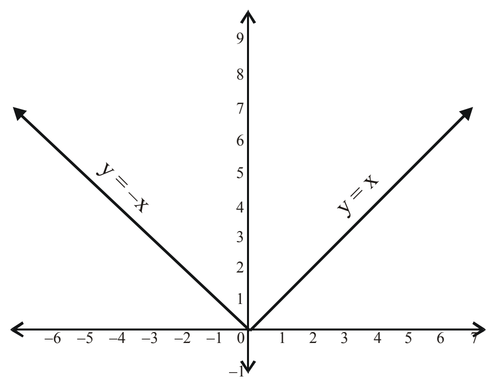
16. $\dfrac{3x-1}{4x+1} \le 0$

12. MODULUS AND MODULUS EQUATIONS

For any real number x, modulus or absolute value of x is denoted by |x| and is defined as

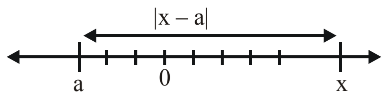
$$|x| = \begin{cases} x, & \text{if } \; x \ge 0 \\ -x, & \text{if } \; x < 0 \end{cases}$$

- Graph of y = |x|



Note :

- $|x| = |-x| \geq 0$
- Geometrically $|x|$ is distance of real number x from zero along the real number line
- More generally $|x - a|$ is distance between 'x' and 'a' on the number line.



- $|x| = \sqrt{x^2}$
- $|xy| = |x|\,|y|$

Illustrations

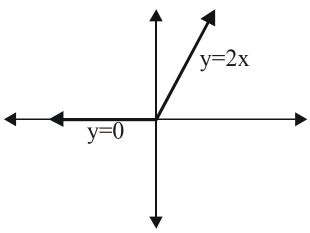
Illustration-92 : Sketch the graph of following equation and also find all possible values (Range) of y

(i) $y = |x| + x$ (ii) $y = |x - 2| + x + 1$

(iii) $y = |x - 3| - |x|$ (iv) $y = 2|x - 2| - |x + 1|$

Soluiton : (i) $y = |x| + x = \begin{cases} x + x & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$

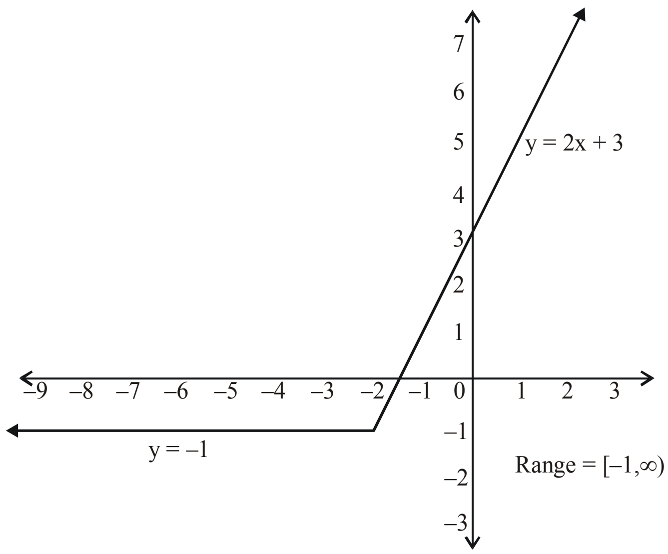
$= \begin{cases} 2x & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$



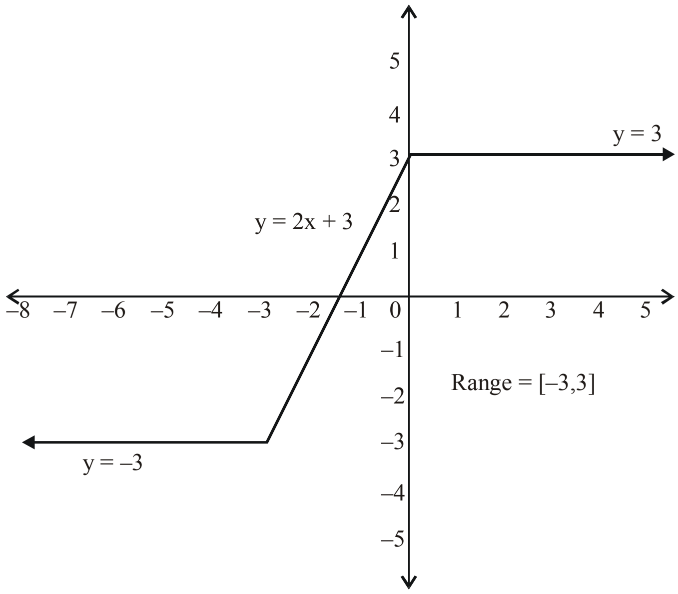
From graph we can find all possible values (range) of y which is $[0,\infty)$

(ii) $y = |x + 2| + x + 1 = \begin{cases} x + 2 + x + 1 & , \quad x \geq -2 \\ -x - 2 + x + 1 & , \quad x < -2 \end{cases}$

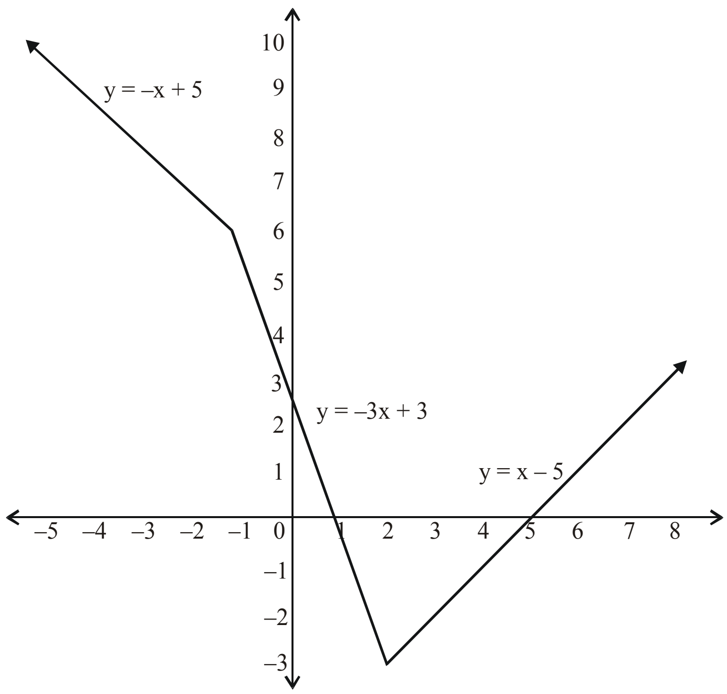
$= \begin{cases} 2x + 3 & , \quad x \geq -2 \\ -1 & , \quad x < -2 \end{cases}$



(iii) $y = |x + 3| - |x| = \begin{cases} x+3-x & , \quad x \geq 0 \\ x+3-(-x) & , \quad x \in [-3,0) \\ -x-3-(-x) & , \quad x < -3 \end{cases} = \begin{cases} 3 & , \quad x \geq 0 \\ 2x+3 & , \quad x \in [-3,0) \\ -3 & , \quad x < -3 \end{cases}$



(iv) $y = 2|x-2| - |x+1| = \begin{cases} 2(x-2)-(x+1) & , \quad x \geq 2 \\ 2(-x+2)-(x+1) & , \quad x \in [-1,2] \\ 2(-x+2)+(x+1) & , \quad x < -1 \end{cases}$

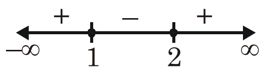


Range = [–3,∞)

Illustration-93 : If $|x - 1||x - 2| = -(x^2 - 3x + 2)$, then find the interval in which x lies ?

Solution : $|(x - 1)(x - 2)| = -(x - 2)(x - 1)$

$\Rightarrow (x - 1)(x - 2) \leq 0$



$\Rightarrow 1 \leq x \leq 2$

Do yourself-17

(1) Sketch the graph of following

(i) $y = |x - 2|$ (ii) $y = |x| - 2$ (iii) $y = 5 - |x|$

Solve for x

(2) $|2x + 5| = 2$ **(3)** $|2x - 5| = 7$ **(4)** $|x - 3| = -1$

(5) $|2x - 3| + 4 = 2$ **(6)** $\left|\frac{3x+4}{3}\right| = 7$ **(7)** $|x^2 - 3x + 2| = 2$

(8) $|2x - 3| = |3x + 5|$ **(9)** $2|x + 3| = 3|x - 4|$ **(10)** $|x^2 + x + 1| = |x^2 + x + 2|$

(11) $|x^2 - 4x + 3| = |x^2 - 5x + 4|$ **(12)** $|x - 6| + |x - 3| = 1$ **(13)** $|2x - 1| + |2x + 3| = 6$

(14) $|3x + 5| + |4x + 7| = 12$ **(15)** $|x| + |x + 1| + |x + 2| = 3$

(16) $|2x - 3| + |2x + 1| + |2x + 5| = 12$

(17) $|x| - 2|x + 1| + 3|x + 2| = 0$ **(18)** $|x| - x = 0$

(19) $|x^2 + 3x + 2| + x + 1 = 0$ **(20)** $x^2 + 3|x| + 2 = 0$

(21) $|x^2 + 1| - x^2 - 1 = 0$

ALLEN CAREER INSTITUTE OVERSEAS

Useful Mathematical Symbols for Reference

Symbol	How it is read
$\forall$	For all...
$\exists$	There exists...
$\wedge$	and
$\vee$	or
$<$	is less than
$>$	is greater than
$\leq$	is less than or equal to
$\geq$	is greater than or equal to
$\nless$	is not less than
$\ngtr$	is not greater than
$\in$	belongs to
$\notin$	does not belong to
$\mid$, $:$, s.t.	such that
$\Rightarrow$	implies (If... then...)
$\Leftrightarrow$	implies and implied by (if and only if / iff)
$!$	factorial
$\sqrt{\ }$	The square root of
$\sqrt[n]{\ }$	n^{th} root, $n \in N$, $n \geq 2$
$\sum$	The summation of
$\prod$	The product of

Greek Letters (Capital, Small) and there pronounciations

Symbol (Capital, Small)	How it is read	Symbol (Capital, Small)	How it is read
A,α	alpha	N,ν	nu
B,β	beta	Ξ,ξ	xi
Γ,γ	gamma	O,o	omicron
Δ,δ	delta	Π,π	pi
E,ε	epsilon	P,ρ	rho
Z,ζ	zeta	Σ,σ	sigma
H,η	eta	T,τ	tau
Θ,θ	theta	Y,υ	upsilon
I,ι	iota	Φ,ϕ	phi
K,κ	kappa	X,χ	chi
Λ,λ	lambda	Ψ,ψ	psi
M,μ	mu	Ω,ω	omega

EXERCISE (O-1)

Straight Objective Type

1. If A and B be any two sets, then $(A \cap B)'$ is equal to-

(A) $A' \cap B'$ (B) $A' \cup B'$ (C) $A \cap B$ (D) $A \cup B$

2. Let A and B be two sets in the universal set. Then $A - B$ equals-

(A) $A \cap B'$ (B) $A' \cap B$ (C) $A \cap B$ (D) none of these

3. If $A \subseteq B$, then $A \cap B$ is equal to-

(A) A (B) B (C) A' (D) B'

4. If A and B are any two sets, then $A \cup (A \cap B)$ is equal to-

(A) A (B) B (C) A' (D) B'

5. Which of the following statements is true ?

(A) $3 \subseteq \{1, 3, 5\}$ (B) $3 \in \{1, 3, 5\}$ (C) $\{3\} \in \{1, 3, 5\}$ (D) $\{3, 5\} \in \{1, 3, 5\}$

6. Which of the following is a null set ?

(A) $A = \{x : x > 1 \text{ and } x < 1]$ (B) $B = \{x : x + 3 = 3\}$

(C) $C = \{\phi\}$ (D) $D = \{x : x \geq 1 \text{ and } x \leq 1\}$

7. If A and B are not disjoint, then $n(A \cup B)$ is equal to-

(A) $n(A) + n(B)$ (B) $n(A) + n(B) - n(A \cap B)$

(C) $n(A) + n(B) + n(A \cap B)$ (D) $n(A).n(B)$

8. $\sqrt{6+\sqrt{6+\sqrt{6+\sqrt{6+...}}}} =$

(A) 3 (B) 2 (C) 1 (D) ±3

9. If $x = 8-\sqrt{60}$, then $\frac{1}{2}\left[\sqrt{x}+\frac{2}{\sqrt{x}}\right] =$

(A) $\sqrt{5}$ (B) $\sqrt{3}$ (C) $2\sqrt{5}$ (D) $2\sqrt{3}$

10. If $\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = a+b\sqrt{5}$, a, b are rational numbers, then (a, b) =

(A) $\left(\frac{61}{29}, \frac{-24}{29}\right)$ (B) $\left(\frac{-61}{29}, \frac{24}{29}\right)$ (C) $\left(\frac{61}{29}, \frac{24}{29}\right)$ (D) $\left(\frac{-61}{29}, \frac{-24}{29}\right)$

11. The square root $5+2\sqrt{6}$ is :

(A) $\sqrt{3}+2$ (B) $\sqrt{3}-\sqrt{2}$ (C) $\sqrt{2}-\sqrt{3}$ (D) $\sqrt{3}+\sqrt{2}$

12. If $\dfrac{4}{2+\sqrt{3}+\sqrt{7}} = \sqrt{a}+\sqrt{b}-\sqrt{c}$, then which of the following can be true -

(A) a = 1, b = 4/3, c = 7/3 (B) a = 1, b = 2/3, c = 7/9

(C) a = 2/3, b = 1, c = 7/3 (D) a = 7/9, b = 4/3, c = 1

13. The numerical value of $\left(x^{1/(a-b)}\right)^{1/(a-c)} \times \left(x^{1/(b-c)}\right)^{1/(b-a)} \times \left(x^{1/(c-a)}\right)^{1/(c-b)}$ is (a, b, c are distinct real numbers)

(A) 1 (B) 8 (C) 0 (D) None

14. $(1^3 + 2^3 + 3^3 + 4^3)^{-3/2} =$

(A) 10^{-3} (B) 10^{-2} (C) 10^{-4} (D) 10^{-1}

15. $\left(7^{\left(-\frac{1}{2}\right)} \times 5^2\right)^2 \div \sqrt{25^3} =$

(A) $\dfrac{5}{7}$ (B) $\dfrac{7}{5}$ (C) 35 (D) $-\dfrac{5}{7}$

16. $(2d^2e^{-1})^3 \times \left(\dfrac{d^3}{e}\right)^{-2} =$

(A) $8e^{-2}$ (B) $8e^{-3}$ (C) $8e^{-1}$ (D) $8e^{-4}$

17. If $x^y = y^x$ and x = 2y, then the values of x and y are (x, y > 0)

(A) x = 4, y = 2 (B) x = 3, y = 2 (C) x = 1, y = 1 (D) None of these

18. If $(a^m)^n = a^{m^n}$, then express 'm' in the terms of n is (a > 0, a ≠ 0, m > 1, n > 1)

(A) $n^{\left(\frac{1}{n-1}\right)}$ (B) $n^{\left(\frac{1}{n+1}\right)}$ (C) $n^{\left(\frac{1}{n}\right)}$ (D) None

19. If $a = x + \dfrac{1}{x}$, then $x^3 + x^{-3} =$

(A) $a^3 + 3a$ (B) $a^3 - 3a$ (C) $a^3 + 3$ (D) $a^3 - 3$

20. If $\left(\sqrt[3]{4}\right)^{2x+\frac{1}{2}} = \dfrac{1}{32}$, then x =

(A) –2 (B) 4 (C) –6 (D) –4

21. If $(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10$, then all possible values of x are

(A) –2, 2 (B) $\sqrt{2}$, $-\sqrt{2}$ (C) 2, $+\sqrt{2}$ (D) 2, –2, $\sqrt{2}$, $-\sqrt{2}$

22. If $3^{2x^2}-2.3^{x^2+x+6}+3^{2(x+6)}=0$ then the value of x is

(A) – 2 (B) 3 (C) Both (A) and (B) (D) None of these

23. How many integers in between 100 to 1500 (both inclusive) are multiples of 5 or 11 ?

(A) 408 (B) 26 (C) 382 (D) 380

24. Square root of $4+\sqrt{15}$ is equal to

(A) $\dfrac{\sqrt{3}+\sqrt{5}}{2}$ (B) $\sqrt{\dfrac{3}{2}}+\sqrt{\dfrac{5}{2}}$ (C) $\sqrt{\dfrac{5}{2}}-\sqrt{\dfrac{3}{2}}$ (D) None of these

25. If A = {(x, y) | xy = 8 and x, y $\in \mathbb{Z}$}, then n(A) =

(A) 4 (B) 8 (C) 12 (D) 16

26. The set of all real numbers x that satisfy $\dfrac{x-2}{x+2}>\dfrac{2x-3}{4x-1}$

(A) $(-\infty,-2)\cup\left(\dfrac{1}{4},\dfrac{3}{2}\right)\cup(2,\infty)$ (B) $\left(-2,\dfrac{1}{4}\right)\cup\left(\dfrac{3}{2},2\right)$

(C) $(-\infty,-2)\cup\left(\dfrac{1}{4},1\right)\cup(4,\infty)$ (D) $\left(-2,\dfrac{1}{4}\right)\cup(1,4)$

27. The set of all real numbers x that satisfy $\dfrac{x^2-4}{x^2-5x+6}\le 0$

(A) [–2, 3] (B) [–2, 3)

(C) (–∞, –2] ∪ (3, ∞) (D) None of these

28. If $\dfrac{(x+3)^2(x-1)^9(x+1)^5}{(x-3)(x-5)^4(x-6)^5}\le 0$, then number of possible integral values of x is -

(A) 6 (B) 3 (C) 4 (D) 5

29. If $\left(\sqrt{2}+1\right)^x+\left(\sqrt{2}-1\right)^x-2\sqrt{2}=0$, then sum of all possible values of x is

(A) 0 (B) 1 (C) 2 (D) 3

30. If $x^2 + y^2 + 4z^2 - 6x - 2y - 4z + 11 = 0$, then xyz is equal to

(A) 3/2 (B) 4 (C) 6 (D) 3

EXERCISE (O-2)

Straight objective Type

1. In a college of 300 students every student reads 5 newspapers and every newspaper is read by 60 students. The number of newspapers is

(A) at least 30 (B) at most 20 (C) exactly 25 (D) none of these

2. If $x + y = 1$ and $x^2 + y^2 = 2$ then the value of $x^4 + y^4$ equals

(A) 7 (B) 6 (C) $\frac{7}{2}$ (D) $\frac{19}{4}$

3. If $x = 2^{1/3} - 2^{-1/3}$ then the value of $2x^3 + 6x$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

4. If $\frac{\sqrt{3}+4\sqrt{2}}{4\sqrt{2}-\sqrt{3}} = \frac{a+b\sqrt{6}}{c}$, then the value of $a + b + c$ (where $a, b, c \in N$ and are relatively prime)

(A) 70 (B) 72 (C) 50 (D) 40

5. If $x^2 + 4y^2 + z^2 - 2xy - 2yz - zx = 0$ then $x : y : z$ equals

(A) 1 : 2 : 1 (B) 2 : 1 : 2 (C) 1 : 2 : 3 (D) 1 : 1 : 2

More than one correct

6. If $A \subseteq B$ then which of the following option(s) is/are correct ?

(A) $A' \subseteq B'$ (B) $B' \subseteq A'$ (C) $A \cap B' = \phi$ (D) $A' \cap B = \phi$

7. If $x = \sqrt{7\sqrt{7\sqrt{7\sqrt{7...}}}}$ where $x, y > 0$

$y = \sqrt{20+\sqrt{20+\sqrt{20+...}}}$ then which of the following is incorrect.

(A) $x + y = 12$ (B) $x - y = 3$ (C) $x^2 + y^2 = 74$ (D) $x^2 - y^2 = 24$

8. If complex number $-3 + ix^2y$ and $x^2 + y + 4i$ are conjugate of each other. Then real value of x & y can be

(A) $x = 1, y = -4$ (B) $x = -1, y = -4$ (C) $x = 1, y = 4$ (D) $x = -2, y = -3$

9. If $8x^3 + 27x^2 - 9x - 50$ is divided by $(x + 2)$ then remainder is λ then

(A) $\frac{3\lambda+4}{10}$ is equal to 4

(B) If $\lambda = \frac{p}{q}$ then $(p + q)$ is divisible by 13 (where p & q are coprime)

(C) λ is a natural number

(D) $(\lambda - 2)$ is divisible by 3

10. If $f(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$, $a \neq 0$ then which of the following option(s) is(are) correct ?

(A) if $a + b + c + d = 0$ then $x - 1$ is factor of $f(x)$.

(B) if $a + b = c + d$ then $x + 1$ is factor of $f(x)$.

(C) if $a + c = b + d$ then $x + 1$ is factor of $f(x)$.

(D) none of these

11. If $x = a$ and $x = b$ are the two roots of the equation $9^x - 4 \times 3^{x+1} + 27 = 0$ then

(A) $a + b = 3$ (B) $(a - b)^2 = 1$ (C) $\frac{a}{b}+\frac{b}{a}=\frac{5}{2}$ (D) $a + b = 4$

12. The real values of 'x' satisfying the equation $\left(4+\sqrt{15}\right)^{x^2-x-1}+\left(4-\sqrt{15}\right)^{x^2-x-1}=8$ is/are :

(A) 0 (B) 1 (C) –1 (D) –2

Linked Comprehension Type

Paragraph for question no. 13 to 15

If $(5+2\sqrt{6})^{x^2-8}+(5-2\sqrt{6})^{x^2-8}=10, x \in R$

On the basis of above information, answer the following questions :

13. Number of solution(s) of the given equation is/are-

(A) 1 (B) 2 (C) 4 (D) infinite

14. Sum of positive solutions is

(A) 3 (B) $3+\sqrt{7}$ (C) $2+\sqrt{5}$ (D) 2

15. If $x \in (-3,5]$, then number of possible values of x, is-

(A) 1 (B) 2 (C) 3 (D) 4

Paragraph for question no. 16 & 17

A polynomial when divided by $(x - 1)$, $(x + 1)$ and $(x + 2)$ gives remainder 5, 7 and 2 respectively. If $p(x)$ is divided by $(x^2 - 1)$ and $(x - 1)(x + 2)$ gives remainder as another polynomial $R(x)$ and $r(x)$ respectively. Then

16. The value of $R(50)$ is :

(A) 34 (B) –44 (C) 44 (D) 104

17. The value of $r(100)$ is :

(A) 34 (B) 44 (C) 54 (D) 104

Matching list type

18. Match the list

List-I	**List-II**
(P) The units digit of $2^7 \times 3^{10}$ is	(1) 1
(Q) The number of prime factors of $6^{100} \times 15^{200}$ is	(2) 2
(R) Number of solutions of $xy = 35$ in natural numbers with $x > y > 1$ is	(3) 3
(S) Remainder when $x^{100} - 3x^{99} + 2x^{98} + 3x - 2$ is divided by $x - 2$ is equal to	(4) 4

Codes :

(A) P-1, Q-2, R-3, S-4 (B) P-2, Q-3, R-1, S-4

(C) P-2, Q-3, R-2, S-3 (D) P-1, Q-4, R-1, S-2

19. Match the list

List-I	**List-II**
(P) If $x=\sqrt{5}-2$ then value of $2x^3+11x^2+10x+4$ is equal to	(1) 8
(Q) If $x=\sqrt{5}+2$ then value of $2x^3-7x^2-6x+7$	(2) 7
(R) If $\left(\left(\sqrt{11+2\sqrt{30}}\right)-\left(\sqrt{10+4\sqrt{6}}\right)\right)^2=p+\sqrt{q}+\sqrt{r}+\sqrt{s}$ where p, q, r, s are integers & q, r, s are not perfect squares the p + q + r + s is equal to	(3) 161
(S) If $\left(\left(\sqrt{11+2\sqrt{30}}\right)-\left(\sqrt{10+4\sqrt{6}}\right)\right)^2=\alpha-\sqrt{\beta}$ where α, β are integers & β is not a perfect square then $\alpha^2+\beta$ is equal to	(4) 977

Codes :

(A) P-1, Q-2, R-3, S-4
(B) P-2, Q-3, R-1, S-4
(C) P-2, Q-1, R-3, S-4
(D) P-2, Q-1, R-4, S-3

Answer the question question 20 and 21 by appropriately matching the lists based on the information given

List-I	**List-II**
(I) If $2\sqrt{x+5}=x+2$, then integral value of x	(P) 3
(II) The number of real solution of the equation $x^2+5|x|+4=0$, is	(Q) –4
(III) The real solution of the equation $(x-2)^2+|x-2|-2=0$, is	(R) –1
(IV) The number of integral solution(s) $|x+2|=2(3-x)$, is	(S) 0
	(T) 4
	(U) 1

20. Which of the following is only **CORRECT** option ?

(A) I $\rightarrow$ R, U (B) II $\rightarrow$ S, T (C) I $\rightarrow$ Q, S (D) II $\rightarrow$ S

21. Which of the following is only **INCORRECT** option ?

(A) III $\rightarrow$ P, U (B) IV $\rightarrow$ U (C) III $\rightarrow$ Q, T (D) III $\rightarrow$ P, U

Numerical Grid Type

22. If $a - \frac{1}{a} = \frac{1}{2}$ then $\left(4a^2 + \frac{4}{a^2}\right)$ is equal to

23. If polynomial $Ax^3 + 4x^2 + Bx + 5$ leaves same remainder, when divided by $x - 1$ and $x + 2$ respectively then value of $3A + B$ is equal to

24. If $9^x + 6^x = 2.4^x$ then the value of $\frac{x^3 + 64}{5}$

25. If $a + b + c = 6$ & $a^2 + b^2 + c^2 = 14$ and $a^3 + b^3 + c^3 = 36$ then the value of $3\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$

EXERCISE (S-1)

1. An investigator interviewed 100 students to determine their preferences for the three drinks : milk (M), coffee (C) and tea (T). He reported the following : 10 students had all the three drinks M, C and T; 20 had M and C; 30 had C and T; 25 had M and T; 12 had M only; 5 had C only; and 8 had T only. Using a Venn diagram find how many did not take any of the three drinks.

2. Suppose $a+\frac{1}{a}=3$. Find the values of

(a) $a^2+\frac{1}{a^2}$ (b) $a^4+\frac{1}{a^4}$ (c) $a^3+\frac{1}{a^3}$

3. Which of the following equation(s) has (have) only unity as the solution.

(A) $2(3^{x+1})-6(3^{x-1})-3^x=9$ (B) $7(3^{x+1})-5^{x+2}=3^{x+4}-5^{x+3}$

4. Which of the following equation (s) has (have) only natural solution(s)

(A) $6.9^{1/x}-13.6^{1/x}+6.\,4^{1/x}=0$ (B) $4^x.\sqrt{8^{x-1}}=4$

5. If $\frac{29}{12}$ can be expressed as $a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}}$, (a, b, c, d $\in \mathbb{N}$) then find the value of $a^3+b^3+c^3+d^3$ is equal to

6. **Factorize the following expressions :**

(i) $(x-y)(x-y-1)-20$ **(ii)** $(x+2y)(x+2y+2)-8$

(iii) $3(4x+5)^2-2(4x+5)-1$ **(iv)** $x^{\frac{2}{3}}+x^{\frac{1}{3}}-2$

(v) x^6-7x^3-8 **(vi)** x^4+15x^2-16

(vii) $(x^2-x-3)(x^2-x-5)-3$ **(viii)** $(x^2+5x+6)(x^2+5x+4)-120$

(ix) $(a^2+1)^2+(a^2+5)^2-4(a^2+3)^2$

(x) $(x-1)^2+(y-1)^2+(z-1)^2+2(x-1)(y-1)+2(y-1)(z-1)+2(z-1)(x-1)$

(xi) $2x^4-x^3-6x^2-x+2$ **(xii)** $(x^4+x^2-4)(x^4+x^2+3)+10$

7. The value of $(4+1)(4^2+1)(4^4+1)(4^8+1)+\frac{1}{3}=\frac{2^{\lambda}}{3}$ where $\lambda \in \mathbb{N}$ then find the value of 'λ' is equal to :

8. Simplify the following expressions :

(a) $\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}$. (b) $\sqrt[3]{18+5\sqrt{13}}+\sqrt[3]{18-5\sqrt{13}}$.

(c) $\sqrt{\frac{1}{\sqrt{2}+1}+\frac{1}{\sqrt{3}+\sqrt{2}}+\frac{1}{\sqrt{4}+\sqrt{3}}+\ldots \text{ upto 99 terms}}$

9. Suppose a and b are constants such that
$(x^3 + bx^2 - 7x + 9)(x^2 + ax + 5) = x^5 + 13x^4 + 38x^3 - 22x^2 + 37x + 45\ \forall\ x \in \mathbb{R}$. Find a and b.

10. If P(x) is a cubic polynomial such that P(1) = 1; P(2) = 2; P(3) = 3 with leading coefficient 3 then find the value of P(4).

11. Find all conditions on a and b ($a, b \in \mathbb{R}$) when ax + b = 4x + 10 has
(i) exactly one solution in 'x'. (ii) no solution in 'x'.
(iii) has exactly two solutions in 'x' (iv) has infinite solutions in x

12. Let x, y, z are real numbers satisfying following equations :
$3x - 2y + 4z = 3$
$x - y + 2z = 10$ and
$2x - 3y + 3z = 5$ then the value of $(2x - 2y + 3z)^2$ is equal to

13. How may ordered pairs of natural numbers (a, x) satisfy ax = a + 4x ?

14. Let $x = \sqrt{3-\sqrt{5}}$ & $y = \sqrt{3+\sqrt{5}}$. If the value of the expression $x - y + 2x^2y + 2xy^2 - x^4y + xy^4$ can be expressed is the form $\sqrt{p} + \sqrt{q}$ (where $p, q \in \mathbb{N}$), then find the value of (p + q)

15. **Solve following Inequalities over the set of real numbers -**

(i) $x^2 + 2x - 3 \le 0$ **(ii)** $x^2 + 6x - 7 \le 0$

(iii) $x^4 - 2x^2 - 63 \le 0$ **(iv)** $\frac{x+1}{(x-1)^2} < 1$

(v) $\frac{x^2 - 7x + 12}{2x^2 + 4x + 5} > 0$ **(vi)** $\frac{(x-1)(x+2)^2}{-1-x} < 0$

(vii) $\frac{x^4 + x^2 + 1}{x^2 - 4x - 5} < 0$ **(viii)** $\frac{x+7}{x-5} + \frac{3x+1}{2} \ge 0$

(ix) $\frac{1}{x+2} < \frac{3}{x-3}$ **(x)** $\frac{14x}{x+1} - \frac{9x-30}{x-4} < 0$

(xi) $\frac{x^2+2}{x^2-1} < -2$ **(xii)** $\frac{5-4x}{3x^2 - x - 4} < 4$

(xiii) $\frac{(x+2)(x^2-2x+1)}{4+3x-x^2} \ge 0$ **(xiv)** $\frac{x^4 - 3x^3 + 2x^2}{x^2 - x - 30} > 0$

(xv) $\frac{2x}{x^2-9} \le \frac{1}{x+2}$ **(xvi)** $\frac{20}{(x-3)(x-4)} + \frac{10}{x-4} + 1 > 0$

JEE-MAINS

1. If A, B and C are three sets such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then :-

(1) $B = C$ (2) $A \cap B = \phi$ (3) $A = B$ (4) $A = C$

2. In a class of 140 students numbered 1 to 140, all even numbered students opted mathematics course, those whose number is divisible by 3 opted Physics course and those whose number is divisible by 5 opted Chemistry course. Then the number of students who did not opt for any of the three courses is :

(1) 102 (2) 42 (3) 1 (4) 38

3. Two newspapers A and B are published in a city. It is known that 25% of the city populations reads A and 20% reads B while 8% reads both A and B. Further, 30% of those who read A but not B look into advertisements and 40% of those who read B but not A also look into advertisements, while 50% of those who read both A and B look into advertisements. Then the percentage of the population who look into advertisement is :-

(1) 12.8 (2) 13.5 (3) 13.9 (4) 13

4. Let $\mathbb{Z}$ be the set of integers. If $A = \left\{x \in \mathbb{Z} : 2^{(x+2)(x^2-5x+6)} = 1\right\}$ and $B = \{x \in \mathbb{Z} : -3 < 2x - 1 < 9\}$, then the number of subsets of the set $A \times B$, is:

(1) 2^{18} (2) 2^{10} (3) 2^{15} (4) 2^{12}

JEE-ADVANCED

1. If X and Y are two sets, then $X \cap (X \cup Y)^c$ equals

(a) X (b) Y (c) ϕ (d) none of these

2. The expression $\frac{12}{3+\sqrt{5}+2\sqrt{2}}$ is equal to

(a) $1-\sqrt{5}+\sqrt{2}+\sqrt{10}$ (b) $1+\sqrt{5}+\sqrt{2}-\sqrt{10}$

(c) $1+\sqrt{5}-\sqrt{2}+\sqrt{10}$ (d) $1-\sqrt{5}-\sqrt{2}+\sqrt{10}$

3. If $x < 0$, $y < 0$, $x + y + \frac{x}{y} = \frac{1}{2}$ and $(x + y)\frac{x}{y} = -\frac{1}{2}$, then x = ... and y = ...

4. The equation $x-\frac{2}{x-1}=1-\frac{2}{x-1}$ has

(A) no root (B) one root (C) two equal roots (D) infinitely many roots

5. If A = {1, 2, 3, 4, 5}, then the number of proper subsets of A is-

(1) 120 (2) 30 (3) 31 (4) 32

6. If A, B, C be three sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then -

(1) A = B (2) B = C (3) A = C (4) A = B = C

7. Sets A and B have 3 and 6 elements respectively. What can be the minimum number of elements in $A \cup B$?

(1) 3 (2) 6 (3) 9 (4) 18

8. Find all real values of x which satisfy $x^2 - 3x + 2 > 0$ and $x^2 - 2x - 4 \le 0$.

9. Find the set of all x for which $\frac{2x}{(2x^2+5x+2)} > \frac{1}{(x+1)}$.

10. The sum of all real roots of the equation $|x - 2|^2 + |x - 2| - 2 = 0$ is ...

11. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is

(A) 4 (B) 1 (C) 2 (D) 3

12. If S is the set of all real x such that $\frac{2x-1}{2x^3+3x^2+x}$ is positive, then S contains

(A) $\left(-\infty,-\frac{3}{2}\right)$ (B) $\left(-\frac{3}{2},-\frac{1}{4}\right)$ (C) $\left(-\frac{1}{4},\frac{1}{2}\right)$ (D) $\left(\frac{1}{2},3\right)$

(E) None of these

13. Let $y=\sqrt{\frac{(x+1)(x-3)}{(x-2)}}$

Find all the real values of x for which y takes real values.

ANSWER KEY

Do yourself-1

All are true

Do yourself-3

2. C **3.** B **4.** B **6.** C **7.** C **8.** D **9.** B **10.** D **11.** B
12. B

Do yourself-4

1. a = 3, b = 4

2. **(i)** $\frac{2}{11}$ **(ii)** $\frac{16}{99}$ **(iii)** $\frac{419}{990}$

4. **(i)** x + y, x – y, xy, x/y are rational numbers.

(ii) x + y, x – y, xy, x/y, all are may or may not be rational.

(iii) x + y, x – y, xy, x/y are irrational numbers.

5. x is irrational number (Non terminating and non-recurring)

Do yourself-5

1. (x,y) ≡ (1,20); (2,10); (4,5); (5,4); (10,2); (20,1)

2. **(i)** (x,y) ≡ (2,2) **(ii)** (x,y) ≡ (5,13); (6,4) ; (7,1)

3. (x,y) ≡ (8,56); (14,14); (56,8)

4. 651 **5.** 293 **6.** 4 **7.** 30 **8.** 3

Do yourself-6

(i) 

(ii) 

(iii) 

(iv) 

Do yourself-7

1.	B	**2.**	C	**3.**	A	**4.**	D	**5.**	C

6. **(i)** 1 **(ii)** 0 **(iii)** 1 **(iv)** 1 **(v)** 7

7. 2 **8.** (a) $-\frac{47}{44}$, (b) $-\frac{9}{44}$ **9.** 5 **10.** 26

11. 1 **12.** $x = y = z = \frac{a}{3}$

Do yourself-8

1. **(i)** $(x - y)(x + y)(x^2 + y)^2$ **(ii)** $(3a - 2x + y)(3a + 2x - y)$

(iii) $(2x + 3y)(2x - 3y - 3)$

2. **(i)** $(2x - 3y)(4x^2 + 6xy + 9y^2)$ **(ii)** $(2x - 5y)[4x^2 + 10xy + 25y^2 + 1]$

3. **(i)** $(x + 8)(x - 5)$ **(ii)** $(x - 8)(x + 5)$ **(iii)** $(x + 7)(x - 2)$

(iv) $(x - 4)(x + 1)$ **(v)** $(x - 3)(x + 1)$ **(vi)** $(3x - 4)(x - 2)$

(vii) $(4x + 7)(3x - 5)$ **(viii)** $(3x - 2)(x - 1)$ **(ix)** $(x - 1)(3x - 4)$

(x) $(7x - 1)(x - 1)$ **(xi)** $(x - 2)(2x - 13)$ **(xii)** $(a - 3)(3a + 2)$

(xiii) $(2a + 1)(7a - 3)$

4. **(i)** $(a - b - 1)(a + b - 3)$ **(ii)** $(x^2 - 6x + 18)(x^2 + 6x + 18)$

(iii) $(x^2 - y + 1)(x^2 + y + 1)$ **(iv)** $(2a^2 - a - 1)(2a^2 + a - 1)$

(v) $(2x^2 - 6x + 9)(2x^2 + 6x + 9)$ **(vi)** $(x^4 - x^2 + 1)(x^4 + x^2 + 1)$

5. **(i)** $(x - 1)(x - 2)(x - 3)$ **(ii)** $(x + 1)(x + 3)(2x + 1)$

(iii) $(x - 1)(x - 2)(2x - 3)$ **(iv)** $(x^2 + 2)(x^2 + 1)\left(x - \sqrt{3}\right)\left(x + \sqrt{3}\right)$

(v) $3(x + y)(y + z)(z + x)$

6. **(i)** $(2a^2 - a + 1)(4a^4 + 2a^3 - a^2 + a + 1)$

7. **(i)** $(x^2 + 5x + 1)(x^2 + 5x + 9)$

(ii) $2(2x^2 + 2x + 3)(4x^2 + 4x - 9)$

(iii) $(x^2 + 5x - 22)(x + 1)(x + 4)$

Do yourself-9

1. (a) 7 (b) 47 (c) 18 **2.** 104 **3.** $-\sqrt{2}$ **4.** 1

8. (a) 0, 4 (b) no solution (c) no solution

Do yourself-10

1. 1 **2.** –20 **3.** 4 **4.** –2 **5.** 4x +3 **6.** $\ell = 2, m = -18$

7. $\ell = -1$, m = –3 **8.** a = 3, b = 1 **9.** $3x^3 + 4x^2 - 5x - 2$ **10.** 6

Do yourself-11

1. 1 or –8 **2.** 1,32 **3.** $\pm\sqrt{2}$, $\pm\sqrt{\frac{2}{3}}$

4. 3 or 4 **5.** $x = 1, x = \frac{3 \pm \sqrt{5}}{2}$ **6.** $x = \pm 1, 2, -\frac{1}{2}$

7. 0,–1 **8.** 2,4 **9.** ±1

10. –4,4 **11.** –4,1 **12.** –6,1

13. $x = -3 \pm \sqrt{5}$ **14.** $\pm 1,\ 1 \pm \sqrt{2}$

15. (A)→(U); (B)→(U); (C)→(T); (D)→(P,Q,R,S,T,U,V);(E)→(R,T); (F)→(S)

Do yourself-12

1. x = 5, y = 3 **2.** x = 2, y = 1 or x = –1, y = –2

3. x = 4, y = –2 or x = –2, y = 4 **4.** x = –6, y = –2 or x = –4, y = –4

5. x = 3, y = 2 or x = –3, y = –2 **6.** x = 5, y = 1 or x = –5, y = –1

7. $x = \frac{11}{13}, y = -\frac{24}{5}$ **8.** x = 3, y = 2 or x = –3, y = –2

9. x = 4, y = ±3 **10.** x = –1, y = 3 or $x \in \mathbb{R}$, y = 2

11. x = 2, y = –1or x = –1, y = 2 **12.** $(x, y) \cong (3,1), (-3,-1), (1,3), (-1,-3)$

Do yourself-13

1. (a) F (b) T (c) F (d) F (e) T (f) F

2. (a) $\frac{17}{21}$ (b) $\frac{37}{41}$ **3.** 3^{30} **4.** $(2,\infty)$

Do yourself-14

1. $4\sqrt{2}$ **2.** -12 **3.** 4

Do yourself-15

1. $\frac{20}{17}$ **2.** 21 : 35 : 65 **3.** $\frac{sp}{p+q}, \frac{sq}{p+q}$ **4.** $\frac{a^4}{b^4}$

Do yourself-16

1. $(-1,4)-\{1\}$

2. $\left(-\frac{1}{4}, \frac{5}{6}\right)$

3. $(-\infty,-2)(-2,-1)\cup(1,\infty)$

4. $\left(-\infty,\frac{1}{2}\right)\cup(2,\infty)-\{4\}$

5. $[-1,2)-\{0\}$

6. $(-1,1]\cup[3,\infty)$

7. $\left(-\frac{1}{2},1\right)$

8. $(-2,-1)\cup(0,5)-\{2\}$

9. $\left[-\sqrt{2},\sqrt{2}\right]\cup[3,4)-\{-1\}$

10. $[-2,-1)\cup(1,2)$

11. $\left(-\infty,\frac{-\sqrt{63}}{2}\right)\cup\left(-3,\frac{\sqrt{63}}{2}\right)\cup(4,\infty)$

12. $\left(\frac{3}{4},1\right)\cup(7,\infty)$

13. $\left(-\sqrt{2},0\right)\cup\left(1,\sqrt{2}\right)\cup(2,\infty)$

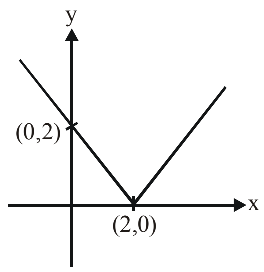
14. $(-\infty,-3]\cup[2,\infty)$

15 $(-2,-1)$

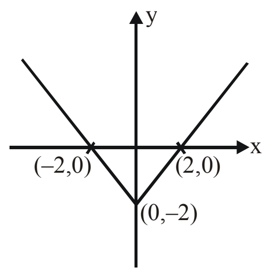
16. $\left(-\frac{1}{4},\frac{1}{3}\right]$

Do yourself-17

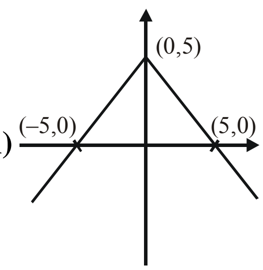
1. **(i)**



(ii)



(iii)



2. $\left\{-\frac{3}{2},-\frac{7}{2}\right\}$ **3.** $\{-1,6\}$ **4.** ϕ **5.** ϕ **6.** $\left\{-\frac{25}{3},\frac{17}{3}\right\}$

7.	$\left\{\frac{3+\sqrt{7}i}{2},0,3\right\}$	8.	$\left\{-8,-\frac{2}{5}\right\}$	9.	$\left\{\frac{6}{5},8\right\}$	10.	$\left\{-1\pm\sqrt{5}i\right\}$
11.	$\left\{1,\frac{7}{2}\right\}$	12.	ϕ	13.	$\{1,-2\}$	14.	$\left\{0,-\frac{24}{7}\right\}$
15.	$\{-2,0\}$	16.	$\left\{-\frac{5}{2},\frac{3}{2}\right\}$	17.	$\{-2\}$	18.	$[0,\infty)$
19.	$\{-3,-1\}$	20.	ϕ	21.	$\mathbb{R}$		

EXERCISE (O-1)

1. B	**2.** A	**3.** A	**4.** A	**5.** B	**6.** A	**7.** B
8. A	**9.** A	**10.** D	**11.** D	**12.** A	**13.** A	**14.** A
15. A	**16.** C	**17.** A	**18.** A	**19.** B	**20.** D	**21.** D
22. C	**23.** C	**24.** B	**25.** B	**26.** C	**27.** D	**28.** D
29. A	**30.** A					

EXERCISE (O-2)

1. C	**2.** C	**3.** C	**4.** B	**5.** D	**6.** B,C	**7.** A,C,D
8. A,B	**9.** A,B,C	**10.** A,C	**11.** A,B,C	**12.** A,B,C	**13.** C	**14.** B
15. C	**16.** B	**17.** D	**18.** B	**19.** D	**20.** D	**21.** C
22. 9.00	**23.** 4	**24.** 12.80	**25.** 5.5			

EXERCISE (S-1)

1. 20 **2.** **(a)** 7 **(b)** 47 **(c)** 18 **3.** A **4.** B **5.** 32

6. **(i)** $(x - y - 5)(x - y + 4)$. **(ii)** $(x + 2y - 2)(x + 2y + 4)$.

(iii) $16(3x + 4)(x + 1)$. **(iv)** $(x^{\frac{1}{3}} + 2)(x^{\frac{1}{3}} - 1)$.

(v) $(x - 2)(x^2 + 2x + 4)(x + 1)(x^2 - x + 1)$.

(vi) $(x^2 + 16)(x - 1)(x + 1)$.

(vii) $(x + 1)(x - 2)(x - 3)(x + 2)$. **(viii)** $(x^2 + 5x + 16)(x + 6)(x - 1)$.

(ix) $-2(a^2 + 1)(a^2 + 5)$. **(x)** $(x + y + z - 3)^2$.

(xi) $(2x - 1)(x - 2)(x + 1)^2$. **(xii)** $(x^2 + 2)(x + 1)(x - 1)(x^4 + x^2 + 1)$.

7. 32 **8.** **(a)** 1 **(b)** 3 **(c)** 3

9. a = 8 and b = 5 **10.** 22

11. (i) $a \neq 4, b \in \mathbb{R}$ (ii) $a = 4, b \neq 10$ (iii) $a, b \neq \phi$ (iv) $a = 4, b = 10$

12. 36 **13.** 3 **14.** 610

15. **(i)** $[-3,1]$ **(ii)** $[-7,1]$

(iii) $[-3, 3]$ **(iv)** $(-\infty, 0) \cup (3, +\infty)$

(v) $(-\infty, 3) \cup (4, +\infty)$ **(vi)** $(-\infty, -2) \cup (-2, -1) \cup (1, +\infty)$

(vii) $(-1, 5)$ **(viii)** $[1, 3] \cup (5, +\infty)$

(ix) $\left(-\frac{9}{2}, -2\right) \cup (3, \infty)$ **(x)** $(-1, 1) \cup (4, 6)$

(xi) $(-1,1)$ **(xii)** $\left(-\infty, -\frac{\sqrt{7}}{2}\right) \cup \left(-1, \frac{\sqrt{7}}{2}\right) \cup \left(\frac{4}{3}, \infty\right)$

(xiii) $(-\infty, -2] \cup (-1, 4)$ **(xiv)** $(-\infty, -5) \cup (1, 2) \cup (6, +\infty)$

(xv) $(-\infty, -3) \cup (-2, 3)$ **(xvi)** $(-\infty, -2) \cup (-1, 3) \cup (4, \infty)$

JEE-MAINS

1. 1 **2.** 4 **3.** 3 **4.** 3

JEE-ADVANCED

1. C **2.** B **3.** $x = -\frac{1}{4}, y = -\frac{1}{4}$ **4.** A **5.** 3

6. 2 **7.** 2 **8.** $x \in \left[1-\sqrt{5}, 1\right) \cup \left(2, 1+\sqrt{5}\right]$

9. $x \in (-2, -1) \cup \left(-\frac{1}{2}, 0\right)$ **10.** 4 **11.** A **12.** A,D **13.** $x \in [-1, 2) \cup [3, \infty)$

CHAPTER 2

LOGARITHM

Chapter Contents 02

IMPORTANT NOTES

CHAPTER 2

LOGARITHM

1. DEFINITION :

The logarithm of a number N to a base 'a' is an exponent indicating the power to which the base 'a' must be raised to obtain the number N. This number is designated as $\log_a N$. (Read it "Log N on base a"). Here N is usually called argument of the Logarithm and 'a' is called base of the Logarithm.

Hence : $\log_a N = x \Leftrightarrow a^x = N \ , \ a > 0 \ , \ a \neq 1$ and $N > 0$

By the definition of logarithm, $\log_2 16$ is the exponent indicating the power to which 2 must be raised in order to obtain 16.

As $2^4 = 16$, hence $\log_2 16 = 4$.

Similarly

Exponential Form		**Logarithmic Form**
$3^5 = 243$	$\Leftrightarrow$	$\log_3 243 = 5$
$5^4 = 625$	$\Leftrightarrow$	$\log_5 625 = 4$
$2^{-3} = \frac{1}{8}$	$\Leftrightarrow$	$\log_2 \frac{1}{8} = -3$
$7^0 = 1$	$\Leftrightarrow$	$\log_7 1 = 0$

Note that the expressions $\log_3(-27)$, $\log_1 16$, $\log_0 5$ and $\log_2 0$ has no sense in real numbers since the equations $3^x = -27$, $1^x = 16$, $0^x = 5$, $2^x = 0$ are absurd for any real x, the reason being obvious that no such exponent x in real number could be found. In general, the expression $\log_a N$ is meaningful if and only if, $a > 0$, $a \neq 1$ and $N > 0$.

The existence and uniqueness of the number $\log_a N$ follows from the properties of exponential functions.

Illustrations

Illustration 1 : If $\log_4 m = 1.5$, then find the value of m.

Solution : $\log_4 m = 1.5 \Rightarrow m = 4^{3/2} \Rightarrow m = 8$

Illustration 2 : If $\log_5 p = a$ and $\log_2 q = a$, then prove that $\frac{p^4q^4}{100} = 100^{2a-1}$

Solution : $\log_5 p = a \Rightarrow p = 5^a$

$\log_2 q = a \Rightarrow q = 2^a$

$$\Rightarrow \frac{p^4q^4}{100} = \frac{5^{4a}.2^{4a}}{100} = \frac{(10)^{4a}}{100} = \frac{(100)^{2a}}{100} = 100^{2a-1}$$

Illustration 3 : The value of N, satisfying $\log_a[1+\log_b\{1+\log_c(1+\log_p N)\}]=0$ is -

(A) 4 (B) 3 (C) 2 (D) 1

Solution : $1+\log_b\{1+\log_c(1+\log_p N)\}=a^0=1$

$\Rightarrow \log_b\{1+\log_c(1+\log_p N)\}=0 \quad \Rightarrow \quad 1+\log_c(1+\log_p N)=1$

$\Rightarrow \log_c(1+\log_p N)=0 \quad \Rightarrow \quad 1+\log_p N=1$

$\Rightarrow \log_p N=0 \quad \Rightarrow \quad N=1$ **Ans. (D)**

Do yourself-1

1 Express the following in logarithmic form :

(a) $81=3^4$ (b) $0.001=10^{-3}$ (c) $2=128^{1/7}$

2. Express the following in exponential form :

(a) $\log_2 32=5$ (b) $\log_{\sqrt{2}} 4=4$ (c) $\log_{10} 0.01=-2$

3. If $\log_{2\sqrt{3}} 1728=x$, then find x.

4. Find the logarithms of the following numbers to the base 2 :

(i) 1 (ii) 2 (iii) 4 (iv) 8 (v) $\frac{1}{2}$

(vi) $\frac{1}{32}$ (vii) $\frac{1}{16}$ (viii) $\sqrt{2}$ (ix) $\sqrt[3]{8}$ (x) $2\sqrt{2}$

(xi) $\frac{1}{\sqrt[5]{2}}$ (xii) $\frac{1}{\sqrt[7]{8}}$

5. Find the logarithms of the following numbers to the base $\frac{1}{2}$:

(i) 1 (ii) $\frac{1}{2}$ (iii) $\frac{1}{8}$ (iv) 16

(v) $\sqrt{2}$ (vi) $\frac{1}{\sqrt{2}}$ (vii) $2\sqrt{2}$ (viii) $\frac{1}{4\sqrt[4]{2}}$

6. Find the logarithms of the following numbers to the base 3 :

(i) 1 (ii) 3 (iii) 9 (iv) 81 (v) $\frac{1}{3}$

(vi) $\sqrt{3}$ (vii) $\frac{1}{3\sqrt{3}}$ (viii) $27\sqrt{3}$ (ix) $\sqrt[7]{9}$

7. Find the logarithms of the following numbers to the base $\frac{1}{3}$:

(i) 1 (ii) $\frac{1}{3}$ (iii) $\frac{1}{9}$ (iv) 3 (v) 9

(vi) 81 (vii) $\sqrt[3]{3}$ (viii) $\frac{1}{\sqrt[7]{3}}$ (ix) $9\sqrt{3}$ (x) $\frac{1}{9\sqrt[4]{3}}$

8. Find the logarithms of the following numbers to the base 5.

(i) 1 (ii) 5 (iii) 25 (iv) 625 (v) $\frac{1}{5}$

(vi) $\frac{1}{25}$ (vii) $\frac{1}{\sqrt{5}}$ (viii) $\sqrt{\sqrt{5}}$ (ix) $5^{\frac{1}{2}}$ (x) $5^{\frac{1}{3}}$

(xi) $\sqrt[4]{5\sqrt[3]{5}}$

9. Find all values of 'a' for which each of the following equalities hold true :

(i) $\log_2 a = 2$ (ii) $\log_a 2 = 1$ (iii) $\log_a 1 = 0$

(iv) $\log_{10}(a(a+3)) = 1$ (v) $\log_{1/3}(a^2 - 1) = -1$ (vi) $\log_2(a^2 - 5) = 2$

(vii) $\log_3 a = 2$ (viii) $\log_{1/3}(a) = 4$ (ix) $\log_{1/3}(a) = 0$

(x) $\log_a(a+2) = 2$ (xi) $\log_3(a^2 + 1) = 1$

10. Evaluate the following :

(i) $\log_5\left(\frac{1}{\sqrt{5}}\right)$ (ii) $\log_{\sqrt{5}+1}(6+2\sqrt{5})$

(iii) $\log_3\left(4\sin^2(x)+4\cos^2(x)-1\right)$ (iv) $\log_{\sqrt{3}-\sqrt{2}}\left(\sqrt{5-2\sqrt{6}}\right)$

2. FUNDAMENTAL LOGARITHMIC IDENTITY :

From the definition of the logarithm of the number N to the base 'a' , we have an identity :

$$a^{\log_a N} = N \;,\; a > 0 \;,\; a \neq 1 \text{ and } N > 0$$

This is known as the **FUNDAMENTAL LOGARITHMIC IDENTITY .**

Note :

Using the basic definition of logarithm we have 3 important deductions :

(a) $\log_a 1 = 0$ i.e. logarithm of unity to any base is zero $(a > 0;\ a \neq 1)$.

(b) $\log_N N = 1$ i.e. logarithm of a number to the same base is 1.

$(N > 0;\ N \neq 1)$

(c) $\log_{\frac{1}{N}} N = -1 = \log_N \frac{1}{N}$ i.e. logarithm of a number to the base as its reciprocal is –1.

$(N > 0;\ N \neq 1)$

Do yourself - 2

1 Find the value of the following :

(a) $\log_{1.4\bar{3}} \dfrac{43}{30}$ **(b)** $\left(\dfrac{1}{2}\right)^{\log_2 5}$

2. If $4^{\log_2 2x} = 36$, then find x.

3. THE PRINCIPAL PROPERTIES OF LOGARITHMS :

If m,n are arbitrary positive numbers where $a > 0$, $a \neq 1$ and x is any real number, then

(a) $\log_a mn = \log_a m + \log_a n$ (b) $\log_a \dfrac{m}{n} = \log_a m - \log_a n$ (c) $\log_a m^x = x \log_a m$

Illustrations

Illustration 4 : Find the value of $2\log\dfrac{2}{5} + 3\log\dfrac{25}{8} - \log\dfrac{625}{128}$

Solution :

$$2\log\frac{2}{5} + 3\log\frac{25}{8} + \log\frac{128}{625}$$

$$= \log\frac{2^2}{5^2} + \log\left(\frac{5^2}{2^3}\right)^3 + \log\frac{2^7}{5^4}$$

$$= \log\frac{2^2}{5^2}.\frac{5^6}{2^9}.\frac{2^7}{5^4} = \log 1 = 0$$

Illustration 5 : If $\log_e x - \log_e y = a$, $\log_e y - \log_e z = b$ & $\log_e z - \log_e x = c$, then find the value of $\left(\dfrac{x}{y}\right)^{b-c} \times \left(\dfrac{y}{z}\right)^{c-a} \times \left(\dfrac{z}{x}\right)^{a-b}$.

Solution :

$$\log_e x - \log_e y = a \Rightarrow \log_e \frac{x}{y} = a \Rightarrow \frac{x}{y} = e^a$$

$$\log_e y - \log_e z = b \Rightarrow \log_e \frac{y}{z} = b \Rightarrow \frac{y}{z} = e^b$$

$$\log_e z - \log_e x = c \Rightarrow \log_e \frac{z}{x} = c \Rightarrow \frac{z}{x} = e^c$$

$$\therefore \left(e^a\right)^{b-c} \times \left(e^b\right)^{c-a} \times \left(e^c\right)^{a-b}$$

$$= e^{a(b-c)+b(c-a)+c(a-b)} = e^0 = 1$$

Illustration 6 : If $a^2 + b^2 = 23ab$, then prove that $\log\frac{(a+b)}{5} = \frac{1}{2}(\log a + \log b)$.

Solution : $a^2 + b^2 = (a + b)^2 - 2ab = 23ab$

$\Rightarrow (a + b)^2 = 25ab \Rightarrow a+b = 5\sqrt{ab}$ ....(i)

Using (i)

L.H.S. $= \log\frac{(a+b)}{5} = \log\frac{5\sqrt{ab}}{5} = \frac{1}{2}\log ab = \frac{1}{2}(\log a + \log b) =$ R.H.S.

Illustration 7 : If $\log_a x = p$ and $\log_b x^2 = q$, then $\log_x \sqrt{ab}$ is equal to (where $a, b, x \in R^+ - \{1\}$)-

(A) $\frac{1}{p}+\frac{1}{q}$ (B) $\frac{1}{2p}+\frac{1}{q}$ (C) $\frac{1}{p}+\frac{1}{2q}$ (D) $\frac{1}{2p}+\frac{1}{2q}$

Solution : $\log_a x = p \Rightarrow a^p = x \Rightarrow a = x^{1/p}$.

similarly $b^q = x^2 \Rightarrow b = x^{2/q}$

Now, $\log_x \sqrt{ab} = \log_x \sqrt{x^{1/p}x^{2/q}} = \log_x x^{\left(\frac{1}{p}+\frac{2}{q}\right)\frac{1}{2}} = \frac{1}{2p}+\frac{1}{q}$

Do yourself - 3

1. Show that $\frac{1}{2}\log 9 + 2\log 6 + \frac{1}{4}\log 81 - \log 12 = 3\log 3$

2. Find the value of following

(i) $\log_{12} 8 + \log_{12}3 + \log_{12}6$ (ii) $\log_5\frac{500}{3} - \log_5\frac{4}{3}$

(iii) $\log_{39}\frac{15}{7} + \log_{39}\frac{13}{3} - \log_{39}\frac{5}{21}$ (iv) $2\log_6 2 + 3\log_6 3 + \log_6 12$

4. BASE CHANGING THEOREM :

Can be stated as "quotient of the logarithm of two numbers is independent of their common base."

Symbolically, $\log_b m = \frac{\log_a m}{\log_a b}$, where $a > 0$, $a \neq 1$, $b > 0$, $b \neq 1$

Note :

(i) $\log_b a . \log_a b = \frac{\log a}{\log b}.\frac{\log b}{\log a} = 1$; hence $\boxed{\log_b a = \frac{1}{\log_a b}}$.

(ii) $a^{\log_b c} = c^{\log_b a}$

(iii) **Base power formula :** $\log_{a^k} m = \frac{1}{k}\log_a m$

(iv) The base of the logarithm can be any positive number other than 1, but in normal practice, only two bases are popular, these are 10 and e(=2.718 approx). Logarithms of numbers to the base 10 are named as 'common logarithm' and the logarithms of numbers to the base e are named as Natural or Napierian logarithm. **We will consider log x as $\log_e x$ or ℓn x.**

(v) Conversion of base e to base 10 & viceversa :

$$\log_e a = \frac{\log_{10} a}{\log_{10} e} = 2.303 \times \log_{10} a\,; \quad \log_{10} a = \frac{\log_e a}{\log_e 10} = \log_{10} e \times \log_e a = 0.434 \log_e a$$

(vi) Some important values : $\log_{10}2 \approx 0.3010$; $\log_{10}3 \approx 0.4771$; $\ell n2 \approx 0.693$, $\ell n10 \approx 2.303$

(vii) The positive real number 'n' is called the antilogarithm of a number 'm' to base 'a' if $\log_a n = m$

Thus, $\log_a n = m \Leftrightarrow n = \text{antilog}_a\, m$

Illustrations

Illustration8 : If a, b, c are distinct positive real numbers different from 1 such that

$(\log_b a \,.\, \log_c a - \log_a a) + (\log_a b \,.\, \log_c b - \log_b b) + (\log_a c \,.\, \log_b c - \log_c c) = 0$,

then abc is equal to -

(A) 0 (B) e (C) 1 (D) none of these

Solution : $(\log_b a \log_c a - 1) + (\log_a b \,.\, \log_c b - 1) + (\log_a c \log_b c - 1) = 0$

$$\Rightarrow \frac{\log a}{\log b}.\frac{\log a}{\log c} + \frac{\log b}{\log a}.\frac{\log b}{\log c} + \frac{\log c}{\log a}.\frac{\log c}{\log b} = 3$$

$\Rightarrow (\log a)^3 + (\log b)^3 + (\log c)^3 = 3\log a \log b \log c$

$\Rightarrow (\log a + \log b + \log c) = 0$ $[\because$ If $a^3 + b^3 + c^3 - 3abc = 0$, then $a + b + c = 0$ if $a \neq b \neq c]$

$\Rightarrow \log abc = \log 1 \Rightarrow abc = 1$

Illustration 9 : Evaluate : $81^{1/\log_5 3} + 27^{\log_9 36} + 3^{4/\log_7 9}$

Solution : $81^{1/\log_5 3} + 27^{\log_9 36} + 3^{4/\log_7 9}$

$= 3^{4\log_3 5} + 3^{\log_3 (36)^{3/2}} + 3^{\log_3 7^2}$ = 625 + 216 + 49 = 890.

Illustration 10 : Show that $\log_4 18$ is an irrational number.

Solution : $\log_4 18 = \log_4(3^2 \times 2) = 2\log_4 3 + \log_4 2 = 2\dfrac{\log_2 3}{\log_2 4} + \dfrac{1}{\log_2 4} = \log_2 3 + \dfrac{1}{2}$

assume the contrary, that this number $\log_2 3$ is rational number.

$\Rightarrow \log_2 3 = \dfrac{p}{q}$. Since $\log_2 3 > 0$ both numbers p and q may be regarded as natural number

$\Rightarrow 3 = 2^{p/q} \quad \Rightarrow \quad 2^p = 3^q$

But this is not possible for any natural number p and q. The resulting contradiction completes the proof.

Illustration 11 : If in a right angled triangle, a and b are the lengths of sides and c is the length of hypotenuse and $c - b \neq 1$, $c + b \neq 1$, then show that

$$\log_{c+b}a + \log_{c-b}a = 2\log_{c+b}a \,.\, \log_{c-b}a.$$

Solution : We know that in a right angled triangle

$c^2 = a^2 + b^2$

$c^2 - b^2 = a^2$ .... (i)

$$\text{LHS} = \frac{1}{\log_a(c+b)} + \frac{1}{\log_a(c-b)} = \frac{\log_a(c-b) + \log_a(c+b)}{\log_a(c+b).\log_a(c-b)}$$

$$= \frac{\log_a(c^2-b^2)}{\log_a(c+b).\log_a(c-b)} = \frac{\log_a a^2}{\log_a(c+b).\log_a(c-b)} \qquad \text{(using (i))}$$

$$= \frac{2}{\log_a(c+b).\log_a(c-b)} = 2\log_{(c+b)}a \,.\, \log_{(c-b)}a = \text{RHS}$$

Do yourself - 4

1. **(i)** Evaluate : $\frac{\log_3 135}{\log_{15} 3} - \frac{\log_3 5}{\log_{405} 3}$ **(ii)** Evaluate : $\log_9 27 - \log_{27} 9$

(iii) Evaluate : $2^{\log_3 5} - 5^{\log_3 2}$

(iv) Evaluate : $\log_3 4 \,.\, \log_4 5 \,.\, \log_5 6 \,.\, \log_6 7 \,.\, \log_7 8 \,.\, \log_8 9$

(v) If $\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} > x$, then x can be -

(A) 2 (B) 3 (C) 3.5 (D) π

(vi) If $\log_a 3 = 2$ and $\log_b 8 = 3$, then $\log_a b$ is -

(A) $\log_3 2$ (B) $\log_2 3$ (C) $\log_3 4$ (D) $\log_4 3$

2. Find value of following

(i) $\log_7 3.\ \log_5 2.\log_3 7.\log_2(125)$ (ii) $25^{\log_5 3}$

(iii) $6^{\log_6 5} + 3^{\log_9 16}$ (vi) $\log_6 4 + \frac{1}{\log_9 6}$

3. Find antilog of $\frac{5}{6}$ to the base 64.

5. LOGARITHMIC EQUATIONS

Illustrations

Illustration12 : $\log_{\frac{1}{2}}\left(\log_2 \sqrt{2}x\right) = 1$, then find x ?

Solution : $\log_{\frac{1}{2}}\left(\log_2 \sqrt{2}x\right) = 1$

$\Rightarrow \log_2\left(\sqrt{2}x\right) = \frac{1}{2}$

$\Rightarrow \sqrt{2}x = 2^{\frac{1}{2}}$

$\Rightarrow x = 1$

Illustration 13 : Solve the equation $2\log_2(\log_2 x)+\log_{1/2}\left(\frac{3}{2}+\log_2 x\right)=1$.

Solution: Let $\log_2 x = t$

$$\Rightarrow 2\log_2(t)+\log_{\frac{1}{2}}\left(\frac{3}{2}+t\right)=1$$

$$\Rightarrow 2\log_2 t-\log_2\left(\frac{3}{2}+t\right)=1$$

$$\Rightarrow \log_2\left(\frac{t^2}{\frac{3}{2}+t}\right)=1$$

$$\Rightarrow \frac{2t^2}{3+2t}=2$$

$\Rightarrow t^2 - 2t - 3 = 0$

$\Rightarrow (t + 1)(t - 3) = 0$

$\Rightarrow t = 3 \because t > 0$

$\Rightarrow x = 8$

Illustration 14 : Solve the equation $\log x^2 - \log(2x) = 3\log 3 - \log 6$.

Solution : $\log x^2 - \log 2x = 3\log 3 - \log 6$

$x > 0 \Rightarrow 2\log x - \log 2 - \log x = 3\log 3 - \log 2 - \log 3 \Rightarrow \log x = 2\log 3$

$\Rightarrow \log x = \log 9$

$\Rightarrow x = 9$

Illustration 15 : Solve the equation $(\log_5 x)^2+\log_5 x+1=\frac{7}{\log_5 x-1}$

Solution : Put $\log_5 x = t$, we get $t^2+t+1=\frac{7}{t-1}$

$(t-1)(t^2+t+1) = 7 \Rightarrow t^3 + t^2 + t - t^2 - t - 1 = 7 \Rightarrow t^3 - 8 = 0$

$\Rightarrow (t-2)(t^2 + 2t + 4) = 0 \Rightarrow t - 2 = 0;\ t^2 + 2t + 4 \neq 0 \Rightarrow t = 2$

Now, $t = \log_5 x$, so $\log_5 x = 2$

$x = 5^2 \Rightarrow x = 25$

Illustration 16 : Solve the equation $|x-1|^{\log_2 x^2 - 2\log_x 4} = (x-1)^7$

Solution : Obviously $x = 2$ is a solution. Since, left side is positive, $x - 1 > 0$.

The equation reduces $\log_2 x^2 - 2\log_x 4 = 7$

$$\Rightarrow 2t - \frac{4}{t} = 7, t = \log_2 x$$

$$\Rightarrow 2t^2 - 7t - 4 = 0 \Rightarrow t = 4, -\frac{1}{2}$$

But $t > 0$ since $x > 1$. $\therefore$ $t = 4$

$\Rightarrow x = 2^4 = 16$

$\therefore$ $x = 2, 16$

Illustration 17 : Solve the equation $x^{\log_3 x^2 + (\log_3 x)^2 - 10} = \frac{1}{x^2}$

Solution : Taking $\log_3$ on both sides, we get

$(2t + t^2 - 10)t = -2t, t = \log_3 x$

$\Rightarrow t(t^2 + 2t - 8) = 0 \Rightarrow t = 0, 2, -4$

$$\Rightarrow x = 1, 9, \frac{1}{81}$$

Illustration 18 : Solve the equation $4^{\log_2 \ell nx} = \ell nx - (\ell nx)^2 + 1$

Solution : $4^{\log_2 \ell nx} = 2^{2\log_2(\ell nx)} = 2^{\log_2(\ell nx)^2} = (\ell nx)^2$

$$\Rightarrow (\ell n\, x)^2 = \ell n\, x - (\ell n\, x)^2 + 1 \Rightarrow 2(\ell n\, x)^2 - \ell n\, x - 1 = 0 \Rightarrow \ell n\, x = 1, -\frac{1}{2}$$

But $\ell nx > 0$

$\therefore$ $\ell nx = 1 \Rightarrow x = e$.

Illustration 19: Solve the equation x : $\log_{x+1}(x^2 + x - 6)^2 = 4$

Solution : We have,

$\log_{x+1}(x^2 + x - 6)^2 = 4 \Rightarrow (x^2 + x - 6)^2 = (x + 1)^4 = (x^2 + 2x + 1)^2$

$\Rightarrow (x^2 + x - 6 - x^2 - 2x - 1)\,(x^2 + x - 6 + x^2 + 2x + 1) = 0$

$\Rightarrow (-x - 7)\,(2x^2 + 3x - 5) = 0 \Rightarrow (x + 7)\,(x - 1)\,(2x + 5) = 0 \Rightarrow x = -7, -5/2, 1$

The values $x = -7$ and $x = -5/2$ are rejected because they make the base $x + 1$ negative

Hence, $x = 1$ is the only solution of the given equation.

Do yourself - 5

1. Find all values of x for which the following equalities hold true.

(i) $\log_2 x^2 = 1$ (ii) $\log_3 x = \log_3(2 - x)$ (iii) $\log_4 x^2 = \log_4 x$

(iv) $\log_{1/2}(2x + 1) = \log_{1/2}(x + 1)$ (v) $\log_{1/3}(x^2 + 8) = -2$

2. Find all the values of x for which the following equalities hold true.

(i) $\log_2 x^2 = 2$ (ii) $\log_{1/4} x^2 = 1$

(iii) $\log_{1/2} x - \log_{1/2}(3 - x) = 0$ (iv) $\log_2(x + 1) - \log_2(2x - 3) = 0$

3. $\log_3(4^x + 15.2^x + 27) - 2\log_3(4.2^x - 3) = 0.$

4. $\dfrac{\log_8\left(8/x^2\right)}{\left(\log_8 x\right)^2} = 3$

5. $\log_x 3.\log_{x/3} 3 + \log_{x/81} 3 = 0.$

6. $1 + 2\log_x 2.\ \log_4(10 - x) = 2/\log_4 x.$

7. $x^{x+1} = x.$

8. $x^{\log_x (x+3)^2} = 16$

9. $\sqrt{x^{\log_{10}\sqrt{x}}} = 10$

10. $3^{\log_a x} + 3.x^{\log_a 3} = 2$

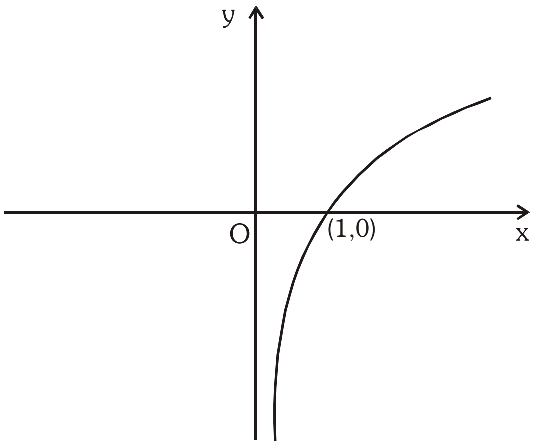
11. $\log_a\left(1 - \sqrt{1 + x}\right) = \log_{a^2}\left(3 - \sqrt{1 + x}\right)$

12. $2\log_8(2x) + \log_8(x^2 + 1 - 2x) = 4/3$

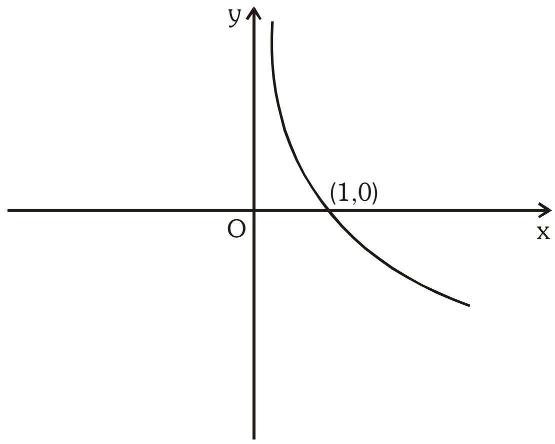
13. $|\log_2 x| = 3$

6. GRAPHS OF LOGARITHMIC FUNCTION AND ITS INEQUALITIES

Graph of $y = \log_a x$:



when a > 1



When 0 < a < 1

(i) $\log_a x < \log_a y \Leftrightarrow \begin{cases} x < y & \text{if} \quad a > 1 \\ x > y & \text{if} \quad 0 < a < 1 \end{cases}$

(ii) If a > 1, then $\log_a x < p \Rightarrow 0 < x < a^p$ and $\log_a x > p \Rightarrow x > a^p$

(iii) If 0 < a < 1, then $\log_a x < p \Rightarrow x > a^p$ and $\log_a x > p \Rightarrow 0 < x < a^p$

Note :

(i) If base of logarithm is greater than on that base 1 then logarithm of greater number is greater. i.e.$\log_2 8 = 3$, $\log_2 4 = 2$ etc. and if base of logarithm is between 0 and 1 then on that base logarithm of greater number is smaller. i.e. $\log_{1/2} 8 = -3$, $\log_{1/2} 4 = -2$ etc.

(ii) It must be noted that whenever the number and the base are on the same side of unity then logarithm of that number to that base is positive, however if the number and the base are located on different side of unity then logarithm of that number to that base is negative.

e.g. $\log_{10} \sqrt[3]{10} = \frac{1}{3}$; $\log_{\sqrt{7}} 49 = 4$; $\log_{\frac{1}{2}}\left(\frac{1}{8}\right) = 3$; $\log_2\left(\frac{1}{32}\right) = -5$; $\log_{10}(0.001) = -3$

Illustrations

Illustration 20 : Solve for x : $x^{\log_5 x} > 5$

Solution : as $x > 0$ (for existance)

now solving inequality

$x^{\log_5 x} > 5$. Taking 'log' with base '5' we have $\log_5 x.\log_5 x > 1$

$\Rightarrow (\log_5 x - 1)(\log_5 x + 1) > 0 \Rightarrow \log_5 x > 1$ or $\log_5 < -1$

$\Rightarrow x > 5$ or $x < 1/5$. Also we must have $x > 0$

Thus, $x \in (0,1/5)$ or $x \in (5,\infty)$

Illustration 21 : Solve for x : $\log_3(2x + 1) < \log_3 5$.

Solution : Checking existance

$2x + 1 > 0 \Rightarrow x > -\dfrac{1}{2}$

Now solving inequality we have $2x + 1 < 5$

$\Rightarrow \quad 2x > -1$ and $2x < 4$

$\Rightarrow \quad x > -1/2$ and $x < 2$,

$\Rightarrow \quad x \in (-1/2,2)$

Illustration22: Solve for x : $(\log_{10}100x)^2 + (\log_{10}10x)^2 + \log_{10}x \le 14$.

Solution : Checking existance

$x > 0$

Now solving inequality,

Let $u = \log_{10}x$

$(2 + u)^2 + (1 + u)^2 + u \le 14 \Rightarrow u^2 + 4u + 4 + u^2 + 2u + 1 + u \le 14$

$\Rightarrow 2u^2 + 7u - 9 \le 0 \Rightarrow 2u^2 + 9u - 2u - 9 \le 0$

$\Rightarrow u(2u + 9) - 1(2u + 9) \le 0 \Rightarrow (2u + 9)(u - 1) \le 0$

$\Rightarrow \dfrac{-9}{2} \le u \le 1 \Rightarrow \dfrac{-9}{2} \le \log_{10} x \le 1 \Rightarrow 10^{-\frac{9}{2}} \le x \le 10$

Illustration 23: Solve for x : $\log_3((x+2)(x+4)) + \log_{1/3}(x+2) < \frac{1}{2}\log_{\sqrt{3}} 7$.

Solution : Checking existance,

$(x+2)(x+4) > 0$ and $(x+2) > 0$

$x < -4$ or $x > -2$ and $x > -2$

Now solving inequality.

$$\log_3((x+2)(x+4)) + \log_{1/3}(x+2) < \frac{1}{2}\log_{\sqrt{3}} 7.$$

$\Rightarrow \log_3(x+2)(x+4) - \log_3(x+2) < \log_3 7$

$\Rightarrow \log_3(x+4) < \log_3 7$

$\Rightarrow x + 4 < 7 \Rightarrow x < 3$

$\Rightarrow -2 < x < 3 \Rightarrow \quad x \in (-2,3)$

Illustration 24: Solve for x : $\log_{1/3}\log_4(x^2-5) > 0$

Solution : Checking existence,

(i) $\log_4(x^2-5) > 0 \Rightarrow x^2 - 5 > 1$

$$\Rightarrow \left(x-\sqrt{6}\right)\left(x+\sqrt{6}\right) > 0$$

$$\Rightarrow x \in \left(-\infty, -\sqrt{6}\right) \cup \left(\sqrt{6}, \infty\right)$$

(ii) $x^2 - 5 > 0 \Rightarrow x \in \left(-\infty, -\sqrt{5}\right) \cup \left(\sqrt{5}, \infty\right)$

solving inequality,

$$\log_{\frac{1}{3}} \log_4\left(x^2-5\right) > 0$$

$\Rightarrow \log_4\left(x^2-5\right) < 1$

$\Rightarrow x^2 - 5 < 4 \Rightarrow x^2 - 9 < 0$

$\Rightarrow (x-3)(x+3) < 0$

$\Rightarrow x \in (-3,3)$ ...(iii)

$\therefore$ Answer : (i) $\cap$ (ii) $\cap$ (iii)

$$\Rightarrow x \in \left(-3, -\sqrt{6}\right) \cup \left(\sqrt{6}, 3\right)$$

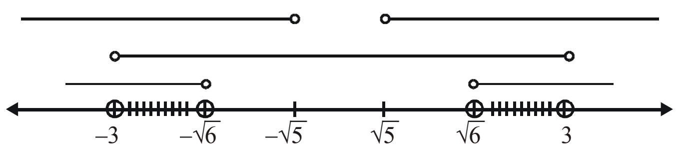


Illustration 25 : Solve for x : $\dfrac{\log_2\left(4x^2-x-1\right)}{\log_2\left(x^2+1\right)}>1$

Solution : Checking existance

(i) $x^2 + 1 \neq 1 \Rightarrow x \neq 0$

(ii) $4x^2 - x - 1 > 0$

Now solving inequality,

$$\frac{\log_2\left(4x^2-x-1\right)}{\log_2\left(x^2+1\right)}>1$$

$\Rightarrow \log_2(4x^2 - x - 1) > \log_2(x^2 + 1) \Rightarrow \log_2(4x^2 - x - 1) - \log_2(x^2 + 1) > 0$

$\Rightarrow \log_2 \dfrac{4x^2-x-1}{x^2+1} > 0$

$\Rightarrow 4x^2 - x - 1 - x^2 - 1 > 0 \Rightarrow 3x^2 - x - 2 > 0 \Rightarrow 3x^2 - 3x + 2x - 2 > 0$

$\Rightarrow 3x(x - 1) + 2(x - 1) > 0 \Rightarrow (x - 1)(3x + 2) > 0 \Rightarrow x < -2/3$ or $x > 1$

$\Rightarrow x \in (-\infty, -2/3)$ or $x \in (1, \infty)$

Note : $x < -\dfrac{2}{3}$ and $x > 1$; $4x^2 - x - 1 > 0$

Illustration 26: Solve for x : $\log_4\left(3^x-1\right)\log_{1/4}\left(\dfrac{3^x-1}{16}\right)\leq\dfrac{3}{4}$

Solution : Checking existance $3^x - 1 > 0 \Rightarrow x > 0$,

Now solving inequality

$$\log_4\left(3^x-1\right)\log_{1/4}\left(\frac{3^x-1}{16}\right)\leq\frac{3}{4}$$

$$\Rightarrow \log_4\left(3^x-1\right).\left[-\log_4\left(3^x-1\right)+\log_4 16\right]\leq\frac{3}{4}$$

$$\Rightarrow \log_4\left(3^x-1\right)\left[-\log_4\left(3^x-1\right)+2\right]\leq\frac{3}{4}\Rightarrow-\left[\log_4\left(3^x-1\right)\right]^2+2\left[\log_4\left(3^x-1\right)\right]\leq\frac{3}{4}$$

Put $\log_4\left(3^x-1\right)=t\Rightarrow -t^2+2t\leq\dfrac{3}{4}$

$\Rightarrow -4t^2 + 8t - 3 \le 0 \Rightarrow 4t^2 - 8t + 3 \ge 0 \Rightarrow 4t^2 - 6t - 2t + 3 \ge 0$

$\Rightarrow 2t(2t - 3) - 1(2t - 3) \ge 0 \Rightarrow (2t - 3)(2t - 1) \ge 0$

$$\Rightarrow \log_4\left(3^x - 1\right) \le \frac{1}{2} \text{ or } \log_4\left(3^x - 1\right) \ge \frac{3}{2}$$

$\Rightarrow 0 < 3^x - 1 \le 4^{1/2}$ or $3^x - 1 \ge 4^{3/2}$

$\Rightarrow 1 < 3^x \le 3$ or $3^x \ge 9$

$\Rightarrow 0 < x \le 1$ or $x \ge 2$

$\Rightarrow x \in (0, 1] \cup [2, \infty)$

Illustration 27: Solve for x : $\log_{1/3}(x^2 - 6x + 18) - 2\log_{1/3}(x - 4) < 0$

Solution : Checking existance

(1) $x^2 - 6x + 18 > 0 \Rightarrow x \in \mathbb{R}$

(2) $x - 4 > 0 \Rightarrow x > 4$

Now solving inequality

$\log_{1/3}(x^2 - 6x + 18) - \log_{1/3}(x - 4)^2 < 0$

$$\Rightarrow \log_{1/3}\frac{\left(x^2 - 6x + 18\right)}{\left(x - 4\right)^2} < 0 \text{ and } 2\log_{1/3}(x - 4) = \log_{1/3}(x - 4)^2$$

only when $x - 4 > 0$, so we get

$$\log_{1/3}\left(\frac{x^2 - 6x + 18}{\left(x - 4\right)^2}\right) < 0 \text{ and } x - 4 > 0 \Rightarrow x > 4 \qquad \text{...(1)}$$

$\Rightarrow x^2 - 6x + 18 > (x - 4)^2 \Rightarrow x^2 - 6x + 18 > x^2 - 8x + 16$

$2x + 2 > 0 \Rightarrow x > -1 \Rightarrow x \in (-1, \infty)$ ...(2)

from equation (1) and (2), we get $x \in (4, \infty)$

Illustration 28: Solve for $x : \log_e(x^2 - 2x - 2) \le 0$

Solution : The values of x satisfying the inequality $\log_e(x^2 - 2x - 2) \le 0$ must be such that

$0 < x^2 - 2x - 2 \le 1$

we have, $x^2 - 2x - 2 > 0 \Rightarrow (x - 1)^2 > 3$

$\Rightarrow |x - 1|^2 > 3 \Rightarrow |x-1| > \sqrt{3} \Rightarrow x-1 > \sqrt{3}$ or $x-1 < -\sqrt{3}$

$\Rightarrow x > 1+\sqrt{3}$ or $x < 1-\sqrt{3}$ ...(1)

Again $x^2 - 2x - 2 \le 1 \Rightarrow x^2 - 2x \le 3 \Rightarrow (x - 1)^2 \le 4 \Rightarrow |x - 1|^2 \le 4$

$\Rightarrow |x - 1| \le 2 \Rightarrow -2 \le x - 1 \le 2 \Rightarrow -1 \le x \le 3$ ...(2)

The value of x satisfying both the inequalities equation (1) and (2) are given by;

Hence, $x \in \left[-1, 1-\sqrt{3}\right) \cup \left(1+\sqrt{3}, 3\right]$

Illustration 29: Solve for x : $\log_x\left(2x - \frac{3}{4}\right) > 2$

Solution : For existanc of logarithm

$2x - \frac{3}{4} > 0$ and $x > 0$ and $x \ne 1$

so, $x \in \left(\frac{3}{8}, \infty\right) - \{1\}$

To find the value of x satisfying the inequality $\log_x[2x - (3/4)] > 2$

Case I. Let $0 < x < 1$

Then, $\log_x[2x - (3/4)] > 2 \Rightarrow [2x - (3/4)] < x^2$

$\Rightarrow x^2 - 2x + (3/4) > 0 \Rightarrow 4x^2 - 8x + 3 > 0 \Rightarrow (2x - 1)(2x - 3) > 0$

$\Rightarrow \left(x - \frac{1}{2}\right)\left[x - (3/2)\right] > 0 \Rightarrow x > 3/2$ or $x < 1/2$

$\Rightarrow x < 1/2$ because we have $0 < x < 1$.

$\therefore$ But for $\log[2x - (3/4)]$ to be meaningful, we must have

$2x - (3/4) > 0 \Rightarrow x > 3/8$

Therefore, if $0 < x < 1$, the values of x satisfying the given inequality are given by :

$3/8 < x < 1/2$

Case II. Let $x > 1$

Then, $\log_x[2x - (3/4)] > 2 \Rightarrow [2x - (3/4)] > x^2$

$\Rightarrow x^2 - 2x + (3/4) < 0 \Rightarrow 4x^2 - 8x + 3 < 0 \Rightarrow (2x - 1)(2x - 3) < 0$

$\Rightarrow \left(x - \frac{1}{2}\right)[x - (3/2)] < 0 \Rightarrow 1/2 < x < 3/2$

But we have $x > 1$

$\therefore$ We must have $1 < x < 3/2$ and obviously these values of x make

$2x - (3/4) > 0$

Thereofre, if $x > 1$, the values of x satisfying the given inequality are given by,
$1 < x < 3/2$

$$x \in \left(\frac{3}{8}, \frac{1}{2}\right) \cup \left(1, \frac{3}{2}\right)$$

Illustration 30 : Solve for x : $\log_{0.5}(x^2 - 5x + 6) \geq -1$

Solution : Checking existance $x^2 - 5x + 6 > 0$,

$\Rightarrow x \in (-\infty, 2) \cup (3, \infty)$

Now solving inequality

$\log_{0.5}(x^2 - 5x + 6) \geq -1 \quad \Rightarrow \quad 0 < x^2 - 5x + 6 \leq (0.5)^{-1}$

$\Rightarrow \quad x^2 - 5x + 6 \leq 2$

$$\begin{cases} x^2 - 5x + 6 > 0 \\ x^2 - 5x + 6 \leq 2 \end{cases} \Rightarrow x \in [1, 2) \cup (3, 4]$$

Hence, solution set of original inequation : $x \in [1,2) \cup (3,4]$

Illustration 31 : Solve for x : $\log_2 x \le \dfrac{2}{\log_2 x - 1}$.

Solution : Let $\log_2 x = t$

$$t \le \frac{2}{t-1} \Rightarrow t - \frac{2}{t-1} \le 0$$

$$\Rightarrow \frac{t^2 - t - 2}{t-1} \le 0 \Rightarrow \frac{(t-2)(t+1)}{(t-1)} \le 0$$

$\Rightarrow$ $t \in (-\infty, -1] \cup (1,2]$

or $\log_2 x \in (-\infty,-1] \cup (1,2]$

or $x \in \left(0, \dfrac{1}{2}\right] \cup (2, 4]$

Illustration 32 : Find all x such that $\log_{1/2} x > \log_{1/3} x$.

Solution : We have $\log_{1/2} x > \log_{1/3} x$.

$\Rightarrow$ $-\log_2 x > -\log_3 x$ $\Rightarrow$ $\log_2 x < \log_3 x$

$$\Rightarrow \log_2 x < \frac{\log_2 x}{\log_2 3}$$

$\Rightarrow$ $\log_2 3 \log_2 x < \log_2 x$ (as $\log_2 3 > 0$)

$\Rightarrow$ $\log_2 x (\log_2 3 - 1) < 0$

Since $\log_2 3 - 1 > 0$, from the latter inequality we obtain $\log_2 x < 0$, hence $x < 1$. But the original inequality is meaningful only when $x > 0$. Therefore all x that satisfy the original inequality lie in the interval $0 < x < 1$.

Answer : $\mathbf{x \in (0, 1)}$

Illustration 33 : Solve the inequation : $\log_{2x+3} x^2 < \log_{2x+3}(2x+3)$

Solution : For existance of logarithm

$$\text{(i)}x^2 > 0 \text{ (ii) } 2x + 3 > 0 \text{ (iii) } 2x + 3 \neq 1 \Rightarrow x \in \left(-\frac{3}{2}, \infty\right) - \{-1, 0\} \quad \text{...(i)}$$

Now solving inequality

Case I : $0 < 2x + 3 < 1 \Rightarrow -\frac{3}{2} < x < -1$

$\because \log_{2x+3} x^2 < \log_{2x+3} 2x+3$

$\Rightarrow x^2 > 2x+3$

$\Rightarrow (x-3)(x+1) > 0$

$\Rightarrow x \in (-\infty, -1) \cup (3, \infty)$; but $-\frac{3}{2} < x < -1$

$\Rightarrow x \in \left(-\frac{3}{2}, -1\right)$ intersection with (i) $\Rightarrow x \in \left(-\frac{3}{2}, -1\right)$

Case II : $2x + 3 > 1 \Rightarrow x > -1$

$\because \log_{2x+3} x^2 < \log_{2x+3} 2x+3$

$\Rightarrow x^2 < 2x+3$

$\Rightarrow (x-3)(x+1) < 0$

$\Rightarrow x \in (-1, 3)$; but $x > -1$

$\Rightarrow x \in (-1, 3)$ intersection with (i) $\Rightarrow x \in (-1, 3) - \{0\}$

$\therefore x \in$ case I $\cup$ case II

$\Rightarrow x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 0) \cup (0, 3)$

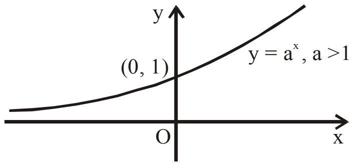
Do yourself-6

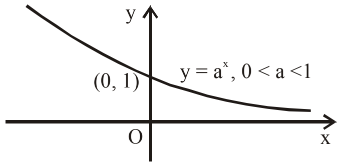
Solve for x :

1. $\log_{0.3}(x^2 + 8) > \log_{0.3}(9x)$ **2.** $\log_7\left(\frac{2x-6}{2x-1}\right) > 0$

3. $\log_{x^2}(2+x) < 1$. **4.** $\log_x \frac{4x+5}{6-5x} < -1$ **.5.** $\log_{1/3}\sqrt{x^2 - 2x} > -\frac{1}{2}$

7. GRAPH OF EXPONENTIAL FUNCTION, ITS EQUATION AND INEQUALITIES





If $a^{f(x)} > b \Rightarrow \begin{cases} f(x) > \log_a b \text{ when } a > 1 \\ f(x) < \log_a b \text{ when } 0 < a < 1 \end{cases}$

Illustrations

Illustratio 34 : Solve the equation $3^x.8^{\frac{x}{x+2}} = 6$.

Solution : Some students solved this equation thus : rewriting it as

$$3^x.2^{\frac{3x}{x+2}} = 3^1.2^1$$

They chose a root x so that the exponents of the respective bases were the same :

$$x = 1, \frac{3x}{x+2} = 1$$

hence the “answer” x = 1.

But this “answer” is incorrect in the sense that only one root of the equation is found and nothing has been said about any other roots. Actually, if the exponents on the appropriate bases are equal, then the products of these powers are equal, however the converse is not in any way implied and is simply incorrect. For instance, the equation

$$3^1.\ 2^1 = 3^2.\ 2^{\log_2(2/3)}$$

is valid, but $1 \neq 2$ and $1 \neq \log_2(2/3)$. Therefore, the foregoing reasoning may lead to a loss of roots, and this is exactly what occurred in the equation at hand.

Taking logarithms of both members of the original to the base 10, we get

$$x\log_{10}3 + \frac{3x}{x+2}\log_{10}2 = \log_{10}6$$

or $\quad x^2\log_{10} 3 + x\,(3 \log_{10} 2 + 2\log_{10} 3 - \log_{10}6) - 2 \log_{10}6 = 0$

We now have to solve this quadratic equation. This can be done using a familiar formula, but we will try to simplify the solution by an ingenious device, since we have already seen, by trial and error, that $x_1 = 1$ is a root of the original equation and, consequently, satisfies the equivalent quadratic equation. For this reason, by Viete’s theorem the second root of the quadratic equation is $x_2 = (-2\log_{10}6)/\log_{10} 3 = -2\log_3 6$ and so the original equation has two roots;

$$x_1 = 1,\ x_2 = 2 \log_3 6.$$

Thus, it is useful to be able to guess a root, but never consider the guessing as the whole solution.

Illustration 35 : $\begin{cases} 2^{y-x}(x+y)=1, \\ (x+y)^{x-y}=2. \end{cases}$

Solution : The domain of definition is x + y > 0.

$$\begin{cases} x+y=\dfrac{1}{2^{y-x}}=2^{x-y}, \\ (x+y)^{x-y}=2. \end{cases}$$

From the first equation we find $x + y = 2^{x-y}$ and substitute it into the second equation.

Then $(2^{x-y})^{x-y}=2 \Rightarrow 2^{(x-y)^2}=2 \Rightarrow (x-y)^2=1$.

The solution of the given system is the solution of the collection of systems

$$\begin{cases} x-y=1, \\ x+y=2, \end{cases} \quad \text{or} \quad \begin{cases} x-y=-1, \\ x+y=\dfrac{1}{2}. \end{cases}$$

Answer : $\left(\dfrac{3}{2},\dfrac{1}{2}\right),\left(-\dfrac{1}{4},\dfrac{3}{4}\right)$.

Illustration 36 : Solve for x : $2^{x+2}>\left(\dfrac{1}{4}\right)^{\frac{1}{x}}$

Solution : We have $2^{x+2} > 2^{-2/x}$. Since the base 2 > 1, we have $x + 2 > -\dfrac{2}{x}$

(the sign of the inequality is retained).

Now $x+2+\dfrac{2}{x}>0 \Rightarrow \dfrac{x^2+2x+2}{x}>0$

$\Rightarrow \dfrac{(x+1)^2+1}{x}>0 \Rightarrow x\in(0,\infty)$

Illustration 37 : Solve for x : $(1.25)^{1-x} < (0.64)^{2(1+\sqrt{x})}$

Solution : We have $\left(\frac{5}{4}\right)^{1-x} < \left(\frac{16}{25}\right)^{2(1+\sqrt{x})}$ or $\left(\frac{4}{5}\right)^{x-1} < \left(\frac{4}{5}\right)^{4(1+\sqrt{x})}$

Since the base $0 < \frac{4}{5} < 1$, the inequality is equivalent to the inequality

$x - 1 > 4(1+\sqrt{x})$

$\Rightarrow \frac{x-5}{4} > \sqrt{x}$

Now, R.H.S. is positive

$\Rightarrow \frac{x-5}{4} > 0 \quad \Rightarrow \quad x > 5 \quad$ ....(i)

we have $\frac{x-5}{4} > \sqrt{x}$

both sides are positive, so squaring both sides

$\Rightarrow \frac{(x-5)^2}{16} > x$ or $\frac{(x-5)^2}{16} - x > 0$

or $x^2 - 26x + 25 > 0$ or $(x-25)(x-1) > 0$

$\Rightarrow x \in (-\infty, 1) \cup (25, \infty)$ ....(ii)

intersection (i) & (ii) gives $x \in (25, \infty)$

Do yourself - 7

Solve for $x \in \mathbb{R}$

1. $4^x - 10.2^{x-1} = 24$

2. $4.2^{2x} - 6^x = 18.3^{2x}$

3. $3^{2x-3} - 9^{x-1} + 27^{2x/3} = 675.$

4. $7^{x+2} - \frac{1}{7}.7^{x+1} - 14.7^{x-1} + 2.7^x = 48$

5. $\left(\frac{5}{3}\right)^{x+1} . \left(\frac{9}{25}\right)^{x^2+2x-11} = \left(\frac{5}{3}\right)^9$

6. $\left(3^{x^2-7.2x+3.9} - 9\sqrt{3}\right)\log(7-x) = 0$

7. $5^{2x} = 3^{2x} + 2.5^x + 2.3^x$

8. $\left(\frac{2}{3}\right)^{\frac{|x|-1}{|x|+1}} > 1$

9. $(1.25)^{1-(\log_2 x)^2} < (0.64)^{2+\log_{\sqrt{2}} x}$

10. $\left(\frac{1}{2}\right)^{\log_3 \log_{1/5}\left(x^2 - \frac{4}{5}\right)} < 1$

11. $4^x < 2^{x+1} + 3$

12. $e^{\frac{x^2+2}{x^2-1}} < \frac{1}{e^2}$

8. CHARACTERISTIC AND MANTISSA :

For any given number N, logarithm can be expressed as $\log_a N$ = Integer + Fraction

The integer part is called characteristic and the fractional part (always taken non negative) is called mantissa. When the value of $\log_{10} N$ is given, then to find digits of 'N' we use only the mantissa part. The characteristic is used only in determining the number of digits in the integral part (if $N \geq 1$) or the number of zeros after decimal & before first non-zero digit in the number (if $0 < N < 1$).

Note :

(i) The mantissa part of logarithm of a number is always non-negative $(0 \leq m < 1)$

(ii) If the characteristic of $\log_{10}N$ be 'C' and $C \geq 0$, then the number of digits in N is (C + 1)

(iii) If the characteristic of $\log_{10}N$ be '– C' and $C > 0$, then there exist (C – 1) zeros after decimal in N.

In summary, if characteristic of $\log_{10}N$ is 'C' then number of digits ($N \geq 1, N \in \mathbb{N}$) or number of zeros after decimal in N ($0 < N < 1$) = $|C + 1|$.

Do yourself - 8

1. Evaluate : $\log_{10}(0.06)^6$

2. Find number of digits in 18^{20}

3. Determine number of cyphers (zeros) between decimal & first significant digit in $\left(\frac{1}{6}\right)^{200}$.

EXERCISE (O-1)

1. If $2^a = 3$ and $9^b = 4$ then value of (ab) is-

(A) 1 (B) 2 (C) 3 (D) 4

2. If $\log_2(4+\log_3(x)) = 3$, then sum of digits of x is -

(A) 3 (B) 6 (C) 9 (D) 18

3. Sum of all the solution(s) of the equation $\log_{10}(x) + \log_{10}(x+2) - \log_{10}(5x+4) = 0$ is-

(A) –1 (B) 3 (C) 4 (D) 5

4. The product of all the solutions of the equation $x^{1+\log_{10} x} = 100000x$ is-

(A) 10 (B) 10^5 (C) 10^{-5} (D) 1

5. If x_1 and x_2 are the roots of equation $e^{3/2}.\ x^{2\ell nx} = x^4$, then the product of the roots of the equation is -

(A) e^2 (B) e (C) $e^{3/2}$ (D) e^{-2}

6. If $\log_2(x^2+1) + \log_{13}(x^2+1) = \log_2(x^2+1)\log_{13}(x^2+1)$, $(x \neq 0)$, then $\log_7(x^2+24)$ is equal to

(A) 1 (B) 2 (C) 3 (D) 4

7. Given $\log_3 a = p = \log_b c$ and $\log_b 9 = \dfrac{2}{p^2}$. If $\log_9\left(\dfrac{a^4b^3}{c}\right) = \alpha p^3 + \beta p^2 + \gamma p + \delta$ $(\forall\ p \in R - \{0\})$, then $(\alpha+\beta+\gamma+\delta)$ equals

(A) 1 (B) 2 (C) 3 (D) 4

8. If $\log_a\left(1-\sqrt{1+x}\right) = \log_{a^2}\left(3-\sqrt{1+x}\right)$, then number of solutions of the equation is-

(A) 0 (B) 1 (C) 2 (D) infinitely many

9. The number of solution(s) of $\sqrt{\log_3(3x^2).\log_9(81x)} = \log_9 x^3$ is-

(A) 0 (B) 1 (C) 2 (D) 3

10. If x_1 & x_2 are the two values of x satisfying the equation $7^{2x^2}-2\left(7^{x^2+x+12}\right)+7^{2x+24}=0$, then $(x_1 + x_2)$ equals-

(A) 0 (B) 1 (C) –1 (D) 7

11. If $x, y \in 2^n$ when $n \in I$ and $1 + \log_x y = \log_2 y$, then the value of $(x + y)$ is

(A) 2 (B) 4 (C) 6 (D) 8

12. If $n \in N$ such that characteristic of n^2 to the base 8 is 2, then number of possible values of n is-

(A) 14 (B) 15 (C) 448 (D) infinite

13. If $x=\log_2\left(\sqrt{56+\sqrt{56+\sqrt{56+\sqrt{56+.......\infty}}}}\right)$, then which of the following statements holds good ?

(A) $x < 0$ (B) $0 < x < 2$ (C) $2 < x < 4$ (D) $3 < x < 4$

14. The greatest value of $(4\log_{10}x - \log_x(.0001))$ for $0 < x < 1$ is-

(A) 4 (B) –4 (C) 8 (D) –8

15. The number $\log_2 7$ is

(A) an integer (B) a rational number

(C) an irrational number (D) a prime number

16. The number of integral solutions of $|\log_5 x^2 - 4| = 2 + |\log_5 x - 3|$ is-

(A) 1 (B) 2 (C) 3 (D) 0

17. If $60^a = 3$ and $60^b = 5$ then the value of $12^{\frac{1-a-b}{2(1-b)}}$ equals

(A) 2 (B) 3 (C) $\sqrt{3}$ (D) $\sqrt{12}$

18. Let ABC be a triangle right angled at C. The value of $\frac{\log_{b+c} a + \log_{c-b} a}{\log_{b+c} a \cdot \log_{c-b} a}$ $(b + c \neq 1, c - b \neq 1)$ equals

(A) 1 (B) 2 (C) 3 (D) 1/2

19. If α and β are the roots of the equation $(\log_2 x)^2 + 4(\log_2 x) - 1 = 0$ then the value of $\log_\beta \alpha + \log_\alpha \beta$ equals

(A) 18 (B) – 16 (C) 14 (D) – 18

20. If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, then x lies in the interval

(A) $(2, \infty)$ (B) $(1, 2)$ (C) $(1, \infty)$ (D) none of these

21. If $\log_{10}\left(\frac{1}{2^x + x - 1}\right) = x(\log_{10} 5 - 1)$, then x =

(A) 4 (B) 3 (C) 2 (D) 1

22. The number of solutions of the equation $\log_{x-3}(x^3 - 3x^2 - 4x + 8) = 3$ is equal to

(A) 4 (B) 3 (C) 2 (D) 1

23. Sum of the roots of the equation $9^{\log_3(\log_2 x)} = \log_2 x - (\log_2 x)^2 + 1$ is equal to

(A) 2 (B) 4 (C) 6 (D) 8

24. If x satisfies the inequality $\log_{25} x^2 + (\log_5 x)^2 < 2$, then $x \in$

(A) $\left(\frac{1}{25}, 5\right)$ (B) (1, 2) (C) (4, 5) (D) (0, 1)

25. If 1, $\log_9(3^{1-x} + 2)$ and $\log_3(4.3^x - 1)$ are in A.P. then x can be

(A) $\log_4 3$ (B) $\log_3 4$ (C) $1 - \log_3 4$ (D) $\log_3 0.25$

EXERCISE (O-2)

Multiple Type From 1 to 7

1. If $\left(\log_{\beta}\alpha\right)^2+\left(\log_{\alpha}\beta\right)^2=79$, $(\alpha>0, \beta>0, \alpha\neq 1, \beta\neq 1)$ then value of $(\log_{\beta}\alpha)+(\log_{\alpha}\beta)$ can be-

(A) 7 (B) –9 (C) 9 (D) –7

2. Which of the following statements is(are) correct ?

(A) $7^{1/7}>(42)^{1/14}>1$

(B) $\log_3(5)\log_7(9)\log_{11}(13)>-2$

(C) $\sqrt{99+70\sqrt{2}}+\sqrt{99-70\sqrt{2}}$ is rational

(D) $\dfrac{1}{\log_4 3}+\dfrac{1}{\log_7 3}>3$

3. For $a>0, \neq 1$ the roots of the equation $\log_{ax}a+\log_x a^2+\log_{a^2x}a^3=0$ can be

(A) $a^{-3/4}$ (B) $a^{-4/3}$ (C) $a^{-1/2}$ (D) none of these

4. The solution of the equation $5^{\log_a x}+5x^{\log_a 5}=3$, $(a>0, a\neq 1)$ is

(A) $a^{-\log_5 2}$ (B) $a^{\log_5 2}$ (C) $2^{-\log_5 a}$ (D) $2^{\log_5 a}$

5. If $2^{x+y}=6^y$ and $3^{x-1}=2^{y+1}$, then the value of $(\log 3-\log 2)/(x-y)$ is

(A) 1 (B) $\log_2 3-\log_3 2$ (C) $\log(3/2)$ (D) $\log 3-\log 2$

6. The equation $x^{(3/4)(\log_2 x)^2+\log_2 x-5/4}=\sqrt{2}$ has

(A) at least one real solution

(B) exactly three real solutions

(C) exactly one irrational solution

(D) complex roots

7. The inequality $-1\leq\left(\dfrac{1}{3}\right)^x<2$ is satisfied by

(A) $x\in[0, 1]$ (B) $x>-\log_3 2$ (C) $x<-\log_3 2$ (D) $x<-1$

Linked Comprehension Type

Paragraph for Question 8 to 10

Let $\log_2 5 = \mathbf{a}$; $\log_5 3 = \mathbf{b}$

8. The value of $\log_{50} 30$ is equal to

(A) $\dfrac{\mathbf{b}+\mathbf{ab}+1}{\mathbf{b}+2}$ (B) $\dfrac{\mathbf{a}+\mathbf{ab}+1}{2\mathbf{a}+1}$ (C) $\dfrac{\mathbf{a}+\mathbf{ab}+1}{\mathbf{a}+2}$ (D) $\dfrac{\mathbf{a}+\mathbf{ab}+\mathbf{b}}{2\mathbf{a}+1}$

9. $\log_2 9 + 3\mathbf{b}.\log_5 9 + \log_2 25 + 3\log_5 9$ lies in the interval given as :

(A) (12,24) (B) (6,12] (C) (24,28) (D) [24,28)

10. Roots of the equation $\log_{10}2.\log_{10}5x^2 - (\log_{10}5 + \log_{10}15.\log_{10}2)x + \log_{10}15 = 0$ are

(A) **a,b** (B) $1 + \mathbf{a}$, $1 + \mathbf{b}$

(C) $\mathbf{a} - 1$, $1 - \mathbf{b}$ (D) $\mathbf{a} + 2$; $\mathbf{b} + 2$

Matrix Match Type

11. In the following matrix match **Column-I** has some quantities and **Column-II** has some comments or other quantities
Match the each element in **Column-I** with corresponding element(s) in **Column-II**

Column-I		Column-II	
(A)	$\log_{\sin \pi/6} e$	(P)	$x^{\log_x 2}$; $x > 0$; $x \neq 1$
(B)	2	(Q)	$2^{\log_5 7}$
(C)	$\log_{\tan \pi/3} \pi$	(R)	negative
(D)	$7^{\log_5 2}$	(S)	positive

Reasoning Type

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True ; Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False.

(D) Statement-1 is False, Statement-2 is True.

12. **Statement-1 :** $\log_{1/5}\dfrac{4x+6}{x}\geq 0$ has no solution for $x\in\left(-\dfrac{3}{2},-1\right)$

and

Statement-2 : $2^{(y-x)}.(x+y)=1,\ (x+y)^{x-y}=2$ has only one pair of solution.

13. **Statement - 1 :** For $3^x+4^x<5^x \Rightarrow x>2$.

and

Statement - 2 : $6^x+6^{x+1}=2^x+2^{x+1}+2^{x+2}$ has no non-zero solutions.

INTEGER TYPE

14. The value of $\log_5 9.\log_2(\sqrt{5}+2).\log_{(\sqrt{5}-2)}5.\log_3\left(\dfrac{1}{4}\right)$ is

15. Number of solutions of $\log_{(1+x)}\left(\dfrac{1-x}{1+x}\right)-2\log_{(1-x)}(1+x)=0$ is

16. Number of solutions of $\log_{x^2+1}(x^2-2)^2=2$ is

17. If x_1, x_2 are the value(s) of x satisfying the equation $\log_2^2(x-2)+\log_2(x-2)\log_2\left(\dfrac{3}{x}\right)-2\log_2^2\left(\dfrac{3}{x}\right)=0$,

then x_1+x_2 is equal to

18. If characteristic of $\log_{10}(0.00006)$ is A & characteristic $\log_3 750$ is B, then A + B is

19. If $\log_{10}2=.3010$, $\log_{10}3=.4771$, then number of digits in $4^8.3^7.5^3$ is 'P', then $\dfrac{P-1}{2}$ is

20. There exist positive integers A, B and C with no common factor greater than 1 such that

$A\log_{200}5+B\log_{200}2=C$, then the value of A + B + C is

21. The minimum possible real x which satisfy the equation, $2\log_2\log_2 x+\log_{1/2}\log_2\left(2\sqrt{2}x\right)=1$.

EXERCISE (S-1)

1. Let A denotes the value of $\log_{10}\left(\frac{ab+\sqrt{(ab)^2-4(a+b)}}{2}\right)+\log_{10}\left(\frac{ab-\sqrt{(ab)^2-4(a+b)}}{2}\right)$

when a = 43 and b = 57 and B denotes the value of the expression $\left(2^{\log_6 18}\right).\left(3^{\log_6 3}\right)$.

Find the value of (A.B).

2. Compute the following : (a) $\log_{1/3}\sqrt[4]{729.\sqrt[3]{9^{-1}}.27^{-4/3}}$ (b) $a^{\frac{\log_b(\log_b N)}{\log_b a}}$

3. Find the square of the sum of the roots of the equation

$\log_3 x.\log_4 x.\log_5 x = \log_3 x.\log_4 x + \log_4 x.\log_5 x + \log_5 x.\log_3 x.$

4. Calculate : $4^{5\log_{4\sqrt{2}}\left(3-\sqrt{6}\right)-6\log_8\left(\sqrt{3}-\sqrt{2}\right)}$

5. Simplify : $\frac{81^{\frac{1}{\log_5 9}}+3^{\frac{3}{\log_{\sqrt{6}} 3}}}{409}.\left(\left(\sqrt{7}\right)^{\frac{2}{\log_{25} 7}}-(125)^{\log_{25} 6}\right)$

6. Simplify : $5^{\log_{1/5}\left(\frac{1}{2}\right)}+\log_{\sqrt{2}}\frac{4}{\sqrt{7}+\sqrt{3}}+\log_{1/2}\frac{1}{10+2\sqrt{21}}$

7. Given that $\log_2 a = s$, $\log_4 b = s^2$ and $\log_{c^2}(8)=\frac{2}{s^3+1}$. Write $\log_2\frac{a^2b^5}{c^4}$ as a function of 's'

(a,b,c>0, c≠1).

8. Find the value of $49^{(1-\log_7 2)}+5^{-\log_5 4}$.

9. Prove that $\frac{\log_2 24}{\log_{96} 2}-\frac{\log_2 192}{\log_{12} 2}=3$.

10. Prove that $a^x - b^y = 0$ where $x=\sqrt{\log_a b}$ & $y=\sqrt{\log_b a}$, a >0 , b > 0 & a , b ≠ 1.

11. Solve the following equations :

i. $\log_{x-1}3 = 2$

ii. $\log_4(2\log_3(1+\log_2(1+3\log_3 x))) = \frac{1}{2}$

iii. $\log_3(1 + \log_3(2^x - 7)) = 1$

iv. $\log_3(3^x - 8) = 2 - x$

v. $\frac{\log_2(9-2^x)}{3-x} = 1$

vi. $\log_{5-x}(x^2 - 2x + 65) = 2$

vii. $\log_{10}5 + \log_{10}(x + 10) - 1 = \log_{10}(21x - 20) - \log_{10}(2x - 1)$

viii. $x^{1+\log_{10}x} = 10x$

ix. $2(\log_x\sqrt{5})^2 - 3\log_x\sqrt{5} + 1 = 0$

x. $3 + 2\log_{x+1}3 = 2\log_3(x + 1)$

12. Solve the inequality. Where ever base is not given take it as 10.

(i) $(\log 100\,x)^2 + (\log 10\,x)^2 + \log x \le 14$

(ii) $\log_{1/2}(x+1) > \log_2(2-x)$.

(iii) $\log_x 2 \,.\, \log_{2x}2 \,.\, \log_2 4x > 1$.

(iv) $\log_{1/5}(2x^2 + 5x + 1) < 0$.

(v) $\log_x \frac{4x+5}{6-5x} < -1$

EXERCISE (JA)

1. Number of solutions of $\log_4(x-1) = \log_2(x-3)$ is **[JEE 2001 (Screening)]**

(A) 3 (B) 1 (C) 2 (D) 0

2. Let (x_0, y_0) be the solution of the following equations **[JEE 2011, 3 (–1)]**

$$(2x)^{\ln 2} = (3y)^{\ln 3}$$

$$3^{\ln x} = 2^{\ln y}$$

Then x_0 is

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) 6

3. The value of $6+\log_{\frac{3}{2}}\left(\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\ldots\ldots}}}\right)$ is **[JEE 2012, 4M]**

4. If $3^x = 4^{x-1}$, then $x =$ **[JEE-Advanced 2013, 4, (–1)]**

(A) $\frac{2\log_3 2}{2\log_3 2-1}$ (B) $\frac{2}{2-\log_2 3}$ (C) $\frac{1}{1-\log_4 3}$ (D) $\frac{2\log_2 3}{2\log_2 3-1}$

5. The value of $\left((\log_2 9)^2\right)^{\frac{1}{\log_2(\log_2 9)}}\times\left(\sqrt{7}\right)^{\frac{1}{\log_4 7}}$ is ______ **[JEE(Advanced)-2018]**

ANSWER KEY

Do yourself - 1

1 **(a)** $\log_3 81 = 4$ **(b)** $\log_{10}(0.001) = -3$ **(c)** $\log_{128} 2 = 1/7$

2. **(a)** $32 = 2^5$ **(b)** $4 = (\sqrt{2})^4$ **(c)** $0.01 = 10^{-2}$

3. 6

4.	(i) 0	(ii) 1	(iii) 2	(iv) 3	(v) –1
	(vi) –5	(vii) –4	(viii) 1/2	(ix) 1	(x) 3/2
	(xi) –1/5	(xii) –3/7			
5.	(i) 0	(ii) 1	(iii) 3	(iv) –4	(v) –1/2
	(vi) 1/2	(vii) –3/2	(viii) 9/4		
6.	(i) 0	(ii) 1	(iii) 2	(iv) 4	(v) –1
	(vi) 1/2	(vii) –3/2	(viii) 7/2	(ix) 2/7	
7.	(i) 0	(ii) 1	(iii) 2	(iv) –1	(v) –2
	(vi) –4	(vii) –1/3	(viii) 1/7	(ix) –5/2	(x) 9/4
8.	(i) 0	(ii) 1	(iii) 2	(iv) 4	(v) –1
	(vi) –2	(vii) –1/2	(viii) 1/4	(ix) 1/2	(x) 1/3
	(xi) 1/3				
9.	(i) 4	(ii) 2	(iii) $a > 0,\ a \neq 1$	(iv) $-5, 2$	(v) $-2, 2$
	(vi) –3,3	(vii) 9	(viii) 1/81	(ix) 1	(x) 2
	(xi) $\sqrt{2}, -\sqrt{2}$				
10.	(i) $\frac{-1}{2}$	(ii) 2	(iii) 1	(iv) 1	

Do yourself - 2

1 **(a)** 1 **(b)** $\frac{1}{5}$ **2.** 3

Do yourself - 3

2. (i) 2 (ii) 3 (iii) 1 (iv) 4

Do yourself - 4

1. **(i)** 3 **(ii)** 5/6 **(iii)** 0 **(iv)** 2 **(v)** (A) **(vi)** (C)
2. (i) 3 (ii) 9 (iii) 9 (vi) 2
3. 32

Do yourself - 5

1. (i) $\sqrt{2},-\sqrt{2}$ (ii) 1 (iii) 1 (iv) 0 (v) 1,–1
2. (i) x = ±2 (ii) $x=\pm\frac{1}{2}$ (iii) $x=\frac{3}{2}$ (iv) x = 4
3. $\log_2 3$ 4. $\frac{1}{8},2$ 5. $9,\frac{1}{9}$ 6. 2,8 7. 1,0
8. ϕ 9. $100,\frac{1}{100}$ 10. $2^{-\log_3 a}$ 11. ϕ 12. 2 13. $x=8,\frac{1}{8}$

Do yourself - 6

1. $x \in (1,8)$
2. $x \in (-\infty, 1/2)$
3. $(-2,-1)\cup(-1,0)\cup(0,1)\cup(2,\infty)$
4. $\left(\frac{1}{2},1\right)$
5. $(-1,0)\cup(2,3)$

Do yourself - 7

1. x = 3 2. x = –2 3. x = 3 4. x = 0 5. $x=\frac{-7}{2},2$
6. $x=\frac{1}{5},6$ 7. x = 1 8. $x \in (-1,1)$ 9. $\left(0,\frac{1}{2}\right)\cup(32,\infty)$
10. $\left(-1,-\frac{2}{\sqrt{5}}\right)\cup\left(\frac{2}{\sqrt{5}},1\right)$ 11. $(-\infty, \log_2 3)$ 12. $(-1,0)\cup(0,1)$

Do yourself - 8

1. $\overline{8}.6686$ 2. 26 3. 155

EXERCISE (O-1)

1. A	**2.** C	**3.** C	**4.** D	**5.** A	**6.** B	**7.** C	**8.** A
9. B	**10.** B	**11.** D	**12.** B	**13.** C	**14.** D	**15.** C	**16.** A
17. A	**18.** B	**19.** D	**20.** A	**21.** D	**22.** D	**23.** A	**24.** A
25. C							

EXERCISE (O-2)

1. B,C	**2.** A,B,D	**3.** B,C	**4.** A,C	**5.** C,D	**6.** A,B,C,D	**7.** A,B	
8. B	**9.** A	**10.** B	**11.** **(A)$\rightarrow$(R); (B)$\rightarrow$(P,S); (C)$\rightarrow$(S); (D)$\rightarrow$(Q,S)**				
12. C	**13.** B	**14.** 4	**15.** 0	**16.** 2	**17.** 9	**18.** 1	**19.** 5
20. 6	**21.** x = 8						

EXERCISE (S-1)

1. 12 **2.** **(a)** -1 **(b)** $\log_b N$ **3.** 3721 **4.** 9 **5.** 1 **6.** 6

7. $2s + 10s^2 - 3(s^3 + 1)$ **8.** $\frac{25}{2}$

11. **i.** $\{1+\sqrt{3}\}$ **ii.** $\{3\}$ **iii.** $\{4\}$ **iv.** $\{2\}$ **v.** $\{0\}$ **vi.** $\{-5\}$

vii. $\{3/2, 10\}$ **viii.** $\{10^{-1}, 10\}$ **ix.** $\{\sqrt{5}, 5\}$ **x.** $\{-(3-\sqrt{3})/3, 8\}$

12 **(i)** $\frac{1}{\sqrt{10^9}} \le x \le 10$ **(ii)** $-1 < x < \frac{1-\sqrt{5}}{2}$ or $\frac{1+\sqrt{5}}{2} < x < 2$

(iii) $2^{-\sqrt{2}} < x < 2^{-1}$; $1 < x < 2^{\sqrt{2}}$ **(iv)** $(-\infty, -2.5) \cup (0, \infty)$ **(v)** $\frac{1}{2} < x < 1$

EXERCISE (JA)

1. B **2.** C **3.** 4 **4.** A,B,C **5.** 8

CHAPTER 3

RELATION

Chapter 03 Contents

IMPORTANT NOTES

CHAPTER 3

RELATION

INTRODUCTION :

Let A and B be two sets. Then a relation R from A to B is a subset of $A \times B$.

thus, R is a relation from A to B $\Leftrightarrow R \subseteq A \times B$.

Ex. If A = {1, 2, 3} and B = {a, b, c}, then R = {(1, b), (2, c), (1, a), (3, a)} being a subset of $A \times B$, is a relation from A to B. Here (1, b), (2, c), (1, a) and (3, a) $\in$ R, so we write 1 Rb, 2Rc, 1Ra and 3Ra. But (2, b) $\notin$ R, so we write 2 $\not R$ b

Total Number of Realtions : Let A and B be two non-empty finite sets consisting of m and n elements respectively. Then $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn}.

Domain and Range of a relation : Let R be a relation from a set A to a set B. Then the set of all first components or coordinates of the ordered pairs belonging to R is called to domain of R, while the set of all second components or coordinates of the ordered pairs in R is called the range of R.

Thus, $\quad$ Dom (R) = {a : (a, b) $\in$ R}

and, $\quad$ Range (R) = {b : (a, b) $\in$ R}

It is evident from the definition that the domain of a relation from A to B is a subset of A and its range is a subset of B.

Ex. Let A = {1, 3, 5, 7} and B = {2, 4, 6, 8} be two sets and let R be a relation from A to B defined by the phrase "$(x, y) \in R \Leftrightarrow x > y$". Under this relation R, we have

3R2, 5R2, 5R4, 7R2, 7R4 and 7R6

i.e. R = {(3, 2), (5, 2), (5, 4), (7, 2), (7, 4), (7, 6)}

$\therefore$ Dom (R) = {3, 5, 7} and Range (R) = {2, 4, 6}

Inverse Relation : Let A, B be two sets and let R be a relation from a set A to a set B. Then the inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by

$$R^{-1} = \{(b, a) : (a, b) \in R\}$$

Clearly, $\quad (a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$

Also, $\quad Dom(R) = Range(R^{-1})$ and $Range (R) = Dom (R^{-1})$

Ex.1 Let A be the set of first ten natural numbers and let R be a relation on A defined by $(x, y) \in R \Leftrightarrow x + 2y = 10$, i.e. $R = \{(x, y) : x \in A, y \in A \text{ and } x + 2y = 10\}$. Express R and R^{-1} as sets of ordered pairs. Determine also (i) domain of R and R^{-1} (ii) range of R and R^{-1}

Sol. We have $(x, y) \in R \Leftrightarrow x + 2y = 10 \Leftrightarrow y = \dfrac{10-x}{2}$, $x, y \in A$

where $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Now, $x = 1 \Rightarrow y = \dfrac{10-1}{2} = \dfrac{9}{2} \notin A$.

This shows that 1 is not related to any element in A. Similarly we can observe. that 3, 5, 7, 9 and 10 are not related to any element of A under the defined relation

Further we find that :

For $x = 2$, $y = \dfrac{10-2}{2} = 4 \in A$ $\therefore (2, 4) \in R$

For $x = 4$, $y = \dfrac{10-4}{2} = 3 \in A$ $\therefore (4, 3) \in R$

For $x = 6$, $y = \dfrac{10-6}{2} = 2 \in A$ $\therefore (6, 2) \in R$

For $x = 8$, $y = \dfrac{10-8}{2} = 1 \in A$ $\therefore (8, 1) \in R$

Thus, $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$

$\Rightarrow$ $R^{-1} = \{(4, 2), (3, 4), (2, 6), (1, 8)\}$

Clearly, $\text{Dom}(R) = \{2, 4, 6, 8\} = \text{Range}(R^{-1})$

and, $\text{Range}(R) = \{4, 3, 2, 1\} = \text{Dom}(R^{-1})$

TYPES OF RELATIONS :

In this section we intend to define various types of relations on a given set A.

Void Relation : Let A be a set. Then $\phi \subseteq A \times A$ and so it is a relation on A. This relation is called the void or empty relation on A.

Universal Relation : Let A be a set. Then $A \times A \subseteq A \times A$ and so it is a relation on A. This relation is called the universal relation on A.

Identity Relation : Let A be a set. Then the relation $I_A = \{(a, a) : a \in A\}$ on A is called the identity relation on A.

In other words, a relation I_A on A is called the identity relation if every element of A is related to itself only.

Ex. The relation $I_A = \{(1, 1), (2, 2), (3, 3)\}$ is the identity relation on set $A = \{1, 2, 3\}$. But relations $R_1 = \{(1, 1), (2, 2)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$ are not identity relations on A, because $(3, 3) \notin R_1$ and in R_2 element 1 is related to elements 1 and 3.

Reflexive Relation : A relation R on a set A is said to be reflexive if every element of A is related to itself.

Thus, R on a set A is not reflexive if there exists an element $A \in A$ such that $(a , a) \notin R$.

Ex. Let $A = \{1, 2, 3\}$ be a set. Then $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A. But $R_1 = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$ is not a reflexive relation on A, because $2 \in A$ but $(2, 2) \notin R_1$.

Note : Every Identity relation is reflexive but every reflexive ralation is not identity.

Symmetric Relation : A relation R on a set A is said to be a symmetric relation iff

$(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

i.e. $a R b \Rightarrow bRa$ for all $a, b, \in A$.

Ex. Let L be the set of all lines in a plane and let R be a relation defined on L by the rule $(x, y) \in R \Leftrightarrow x$ is perpendicular to y. Then R is a symmetric relation on L, because $L_1 \perp L_2 \Rightarrow L_2 \perp L_1$ i.e. $(L_1, L_2) \in R \Rightarrow (L_2, L_1) \in R$.

Ex. Let $A = \{1, 2, 3, 4\}$ and Let R_1 and R_2 be realtion on A given by $R_1 = \{(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)\}$ and $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$. Clearly, R_1 is a symmetric relation on A. However, R_2 is not so, because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$

Transitive Relation : Let A be any set. A relation R on A is said to be a transitive relation iff

$(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$

i.e. $a R b$ and $b R c \Rightarrow a R c$ for all $a, b, c \in A$

Ex. On the set N of natural numbers, the relation R defined by x R y $\Rightarrow$ x is less than y is transitive, because for any x, y, z $\in$ N

$x < y$ and $y < z \Rightarrow x < z \Rightarrow x R y$ and $y R z \Rightarrow x R z$

Ex. Let L be the set of all straight lines in a plane. Then the realtion 'is parallel to' on L is a transitive relation, because from any $\ell_1, \ell_2, \ell_3 \in L$.

$$\ell_1 || \ell_2 \text{ and } \ell_2 || \ell_3 \Rightarrow \ell_1 || \ell_3$$

Antisymmetric Relation : Let A be any set. A relation R on set A is said to be an antisymmetric relation iff

$(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$

Ex. Let R be a relation on the set N of natural numbers defined by

$x R y \Leftrightarrow$ 'x divides y' for all $x, y \in N$

This relation is an antisymmetric relation on N. Since for any two numbers $a, b \in N$

$$a|b \text{ and } b|a \Rightarrow a = b \quad \text{i.e. } a R b \text{ and } b R a \Rightarrow a = b$$

Equivalence Relation : A relation R on a set A is said to be an equivalence relation on A iff

(i) it is reflexive i.e. $(a, a) \in R$ for all $a \in A$

(ii) it is symmetric i.e. $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b \in A$

(iii) it is transitive i.e. $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Ex. Let R be a relation on the set of all lines in a plane defined by $(\ell_1, \ell_2) \in R \Leftrightarrow$ line ℓ_1 is parallel to line ℓ_2. R is an equivalence relation.

Note : It is not neccessary that every relation which is symmetric and transitive is also reflexive.

SOLVED EXAMPLES

Ex.1 Three relation R_1, R_2 and R_3 are defined on set A = {a, b, c} as follows :

(i) R_1 {a, a), (a, b), (a, c), (b, b), (b, c), (c, a), (c, b), (c, c)}

(ii) R_2 {(a, b), (b, a), (a, c), (c, a)}

(iii) R_3{(a, b), (b, c), (c, a)}

Find whether each of R_1, R_2 and R_3 is reflexive, symmetric and transitive.

Sol. (i) Reflexive : Clearly, (a, a), (b, b), (c, c) $\in R_1$. So, R_1 is reflexive on A.

Symmetric : We observe that (a, b) $\in R_1$ but (b, a) $\notin R_1$. So, R_1 is not symmetric on A.

Transitive : We find that (b, c) $\in R_1$ and (c, a) $\in R_1$ but (b, a) $\notin R_1$. So, R is not transitive on A.

(ii) Reflexive : Since (a, a), (b, b) and (c, c) are not in R_2. So, it is not a reflexive realtion on A.

Symmetric : We find that the ordered pairs obtained by interchanging the components of ordered pairs in R_2 are also in R_2. So, R_2 is a symmetric relation on A.

Transitive : Clearly (c, a) $\in R_2$ and (a, b) $\in R_2$ but (c, b) $\notin R_2$. So, it is not a transitive relation on R_2.

(iii) Reflexive : Since non of (a, a), (b, b) and (c, c) is an element of R_3. So, R_3 is not reflexive on A.

Symmetric : Clearly, (b, c) $\in R_3$ but (c, b) $\notin R_3$. so, is not symmetric on A.

Transitive : Clearly, (b, c) $\in R_3$ and (c, a) $\in R_3$ but (b, a) $\notin R_3$. So, R_3 is not transitive on A.

Ex.2 Prove that therelation R on the set Z of all integers defined by

$$(x, y) \in R \Leftrightarrow x - y \text{ is divisible by } n$$

is an equivalence relation on Z.

Sol. We observe the following properties

Reflexivity : For any a $\in$ N, we have

$$a - a = 0 = 0 \times n \Rightarrow a - a \text{ is divisible by } n \Rightarrow (a, a) \in R$$

Thus, (a, a) $\in$ R for all a $\in$ Z

So, R is reflexive on Z

symmetry : Let (a, b) $\in$ R. Then,

(a, b) $\in R \Rightarrow$ (a – b) is divisible by n

$\Rightarrow$ a – b = np for some p $\in$ Z

$\Rightarrow$ b – a = n(–p)

$\Rightarrow$ b – a is divisible by n $\quad [\because\ p \in Z \Rightarrow -p \in Z]$

$\Rightarrow$ (b, a) $\in$ R

Thus, $(a, b) \in R \Rightarrow (b, a) \in R$ for all $a, b, \in Z$

So, R is symmetric on Z.

Transitivity : Let $a, b, c \in Z$ such that $(a, b) \in R$ and $(b, c) \in R$. Then,

$(a, b) \in R \Rightarrow (a - b)$ is divisible by n

$\Rightarrow$ $a - b = np$ for some $p \in Z$

$(b, c) \in R \Rightarrow (b - c)$ is divisible by n

$\Rightarrow b - c = nq$ for some $q \in Z$

$\therefore$ $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow$ $a - b = np$ and $b - c - nq$

$\Rightarrow$ $(a - b) + (b - c) = np + nq$

$\Rightarrow$ $a - c = n(p + q)$

$\Rightarrow$ $a - c$ is divisible by n [$\because$ $p, q \in Z \Rightarrow p + q = Z$]

$\Rightarrow$ $(a, c) \in R$

thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$ for all $a, b, c \in Z$. so, R is transitive realtion in Z.

Ex.3 Shw that the relation is congruent to' on the set of all triangles in a plane is an equivalence relation.

Sol. Let S be the set of all triangles in a plane and let R be the relation on S defined by $(\Delta_1, \Delta_2) \in R$ $\Leftrightarrow$ triangle Δ_1 is congruent to triangle Δ_2. We observe the following properties.

Reflexivity : For each triangle $\Delta \in S$, we have

$\Delta \cong \Delta \Rightarrow (\Delta, \Delta) \in R$ for all $\Delta \in S \Rightarrow R$ is reflexive on S

Symmetry : Let $\Delta_1, \Delta_2 \in S$ such that $(\Delta_1, \Delta_2) \in R$. Then, $(\Delta_1, \Delta_2) \in R \Rightarrow \Delta_1 \cong \Delta_2 \Rightarrow \Delta_2 \cong \Delta_1 \Rightarrow$ $(\Delta_2, \Delta_1) \in R$

So, R is symmetric on S

Transitivity : Let $\Delta_1, \Delta_2, \Delta_3 \in S$ such that $(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R$. Then,

$(\Delta_1, \Delta_2) \in R$ and $(\Delta_2, \Delta_3) \in R \Rightarrow \Delta_1 \cong \Delta_2$ and $\Delta_2 \cong \Delta_3 \Rightarrow \Delta_1 \cong \Delta_3 \Rightarrow (\Delta_1, \Delta_3) \in R$

So, R is transitive on S.

Hence, R being reflexive, symmetric and transitive, is an equivalence relation on S.

EXERCISE-I (CHECK YOUR GRASP)

1. If R is a relation from a finite set A having m elements to a finite set B having n elements, then the number of relations from A to B is-

(1) 2^{mn} (2) $2^{mn}-1$ (3) 2mn (4) m^n

2. In the set A = {1, 2, 3, 4, 5}, a relation R is defined by R = {(x, y) | x, y ∈ A and x < y}. Then R is-

(1) Reflexive (2) Symmetric (3) Transitive (4) None of these

3. For real numbers x and y, we write $x\ R\ y \Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then the relation R is-

(1) Reflexive (2) Symmetric (3) Transitive (4) none of these

4. Let X = {1, 2, 3, 4} and Y = {1, 3, 5, 7, 9}. Which of the following is relations from X to Y-

(1) $R_1 = \{(x, y) \mid y = 2 + x, x \in X, y \in Y\}$

(2) $R_2 = \{(1, 1), (2, 1), (3, 3), (4, 3), (5, 5)\}$

(3) $R_3 = \{(1, 1), (1, 3), (3, 5), (3, 7), (5, 7)\}$

(4) $R_4 = \{(1, 3), (2, 5), (2, 4), (7, 9)\}$

5. Let L denote the set of all straight lines in a plane. Let a relation R be defined by $\alpha\ R\ \beta \Leftrightarrow \alpha \perp \beta$, $\alpha, \beta \in L$. Then R is-

(1) Reflexive (2) Symmetric (3) Transitive (4) none of these

6. Let R be a relation defined in the set of real numbers by $a\ R\ b \Leftrightarrow 1 + ab > 0$. Then R is-

(1) Equivalence relation (2) Transitive

(3) Symmetric (4) Anti-symmetric

7. Which one of the following relations on R is equivalence relation-

(1) $x\ R_1 y \Leftrightarrow |x| = |y|$ (2) $x\ R_2 y \Leftrightarrow x \geq y$

(3) $x\ R_3 y \Leftrightarrow x \mid y$ (4) $x\ R_4 y \Leftrightarrow x < y$

8. Two points P and Q in a plane are related if OP = OQ, where O is a fixed point. This relation is-

(1) Reflexive but not symmetric (2) Symmetric but not transitive

(3) An equivalence relation (4) none of these

9. The relation R defined in A = {1, 2, 3} by a R b if $|a^2 - b^2| \leq 5$. Which of the following is false-

(1) R ={(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 2)

(2) $R^{-1} = R$

(3) Domain of R = {1, 2, 3}

(4) Range of R = {5}

10. Let a relation R is the set N of natural numbers be defined as $(x, y) \in R$ if and only if $x^2 - 4xy + 3y^2 = 0$ for all $x, y \in N$. The relation R is-

(1) Reflexive (2) Symmetric

(3) Transitive (4) An equivalence relation

11. Let A = {2, 3, 4, 5} and let R = {(2, 2), (3, 3), (4, 4), (5, 5), (2, 3), (3, 2), (3, 5), (5, 3)} be a relation in A. Then R is-

(1) Reflexive and transitive (2) Reflexive and symmetric

(3) Reflexive and antisymmetric (4) none of these

12. If A = {2, 3} and B = {1, 2}, then $A \times B$ is equal to-

(1) {(2, 1), (2, 2), (3, 1), (3, 2)} (2) {(1, 2), (1, 3), (2, 2), (2, 3)}

(3) {(2, 1), (3, 2)} (4) {(1, 2), (2, 3)}

13. Let R be a relation over the set $N \times N$ and it is defined by $(a, b) R (c, d) \Rightarrow a + d = b + c$. Then R is-

(1) Reflexive only (2) Symmetric only

(3) Transitive only (4) An equivalence relation

14. Let N denote the set of all natural numbers and R be the relation on $N \times N$ defined by $(a, b) R (c, d)$ if $ad(b + c) = bc(a + d)$, then R is-

(1) Symmetric only (2) Reflexive only

(3) Transitive only (4) An equivalence relation

15. If A = {1, 2, 3}, B = {1, 4, 6, 9} and R is a relation from A to B defined by 'x is greater than y'. Then range of R is-

(1) {1, 4, 6, 9} (2) {4, 6, 9} (3) {1} (4) none of these

16. Let L be the set of all straight lines in the Euclidean plane. Two lines ℓ_1 and ℓ_2 are said to be related by the relation R if ℓ_1 is parallel to ℓ_2. Then the relation R is-

(1) Reflexive (2) Symmetric (3) Transitive (4) Equivalence

17. A and B are two sets having 3 and 4 elements respectively and having 2 elements in common. The number of relations which can be defined from A to B is-

(1) 2^5 (2) $2^{10} - 1$ (3) $2^{12} - 1$ (4) 2^{12}

18. For $n, m \in N$, $n|m$ means that n is a factor of m, the relation $|$ is-

(1) reflexive and symmetric (2) transitive and symmetric

(3) reflexive, transitive and symmetric (4) reflexive, transitive and not symmetric

19. Let $R = \{(x, y) : x, y \in A, x + y = 5\}$ where A = {1, 2, 3, 4, 5} then

(1) R is not reflexive, symmetric and not transitive

(2) R is an equivalence relation

(3) R is reflexive, symmetric but not transitive

(4) R is not reflexive, not symmetric but transitive

20. Let R be a relation on a set A such that $R = R^{-1}$ then R is-

(1) reflexive (2) symmetric (3) transitive (4) none of these

21. Let $x, y \in I$ and suppose that a relation R on I is defined by x R y if and only if $x \le y$ then

(1) R is partial order ralation (2) R is an equivalence relation

(3) R is reflexive and symmetric (4) R is symmetric and transitive

22. Let R be a relation from a set A to a set B, then-

(1) $R = A \cup B$ (2) $R = A \cap B$ (3) $R \subseteq A \times B$ (4) $R \subseteq B \times A$

23. Given the relation R = = {(1, 2), (2, 3)} on the set A = {1, 2, 3}, the minimum number of ordered pairs which when added to R make it an equivalence relation is-

(1) 5 (2) 6 (3) 7 (4) 8

24. Let $P = \{(x, y) \mid x^2 + y^2 = 1, x, y \in R\}$ Then P is-

(1) reflexive (2) symmetric (3) transitive (4) anti-symmetric

25. Let X be a family of sets and R be a relation on X defined by 'A is disjoint from B'. Then R is-

(1) reflexive (2) symmetric (3) anti-symmetric (4) transitive

26. In order that a relation R defined in a non-empty set A is an equivalence relation, it is sufficient that R

(1) is reflexive (2) is symmetric

(3) is transitive (4) possesses all the above three properties

27. If R is an equivalence relation in a set A, then R^{-1} is-

(1) reflexive but not symmetric (2) symmetric but not transitive

(3) an equivalence relation (4) none of these

28. Let A = {p, q, r}. Which of the following is an equivalence relation in A ?

(1) $R_1 = \{(p, q), (q, r), (p, r), (p, p)\}$

(2) $R_2 = \{(r, q)\ (r, p), (r, r), (q, q)\}$

(3) $R_3 = \{(p, p), (q, q), (r, r), (p, q)\}$

(4) none of these

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EXERCISE-2 (PREVIOUS YEAR QUESTIONS)

1. Let R = {(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)} be a releation on the set A = {1, 2, 3, 4}. The relation R is- **[AIEEE - 2004]**

(1) transitive (2) not symmetric

(3) reflexive (4) a function

2. Let R = {(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)} be relation on the set A = {3, 6, 9, 12). The relation is- **[AIEEE - 2005]**

(1) rflexive and transitive only

(2) reflexive only

(3) an equilvalence relation

(4) reflexive and symmetric only

3. Let W denote the words in the English dictionary. Define the relation R by :

$R = \{(x, y) \in W \times W |$ the words x and y have at least one letter in common}. Then R is-

(1) reflexive, symmetric and not transitive **[AIEEE - 2006]**

(2) reflexive, symmetric and transitive

(3) reflexive, not symmetric and transtive

(4) not reflexive, symmetric and transitive

4. Consider the following relations :-

R = {(x, y) | x, y are real numbers and x = wy for some rational number w} ;

$S = \{(\frac{m}{n}, \frac{p}{q}) \mid$ m, n, p and q are integers such that $n, q \neq 0$ and qm = pn}.

Then : **[AIEEE - 2010]**

(1) R is an equivalence relation but S is not an equivalence relation

(2) Neither R nor S is an equivalence relation

(3) S is an equivalence relation but R is not an equivalence relation

(4) R and S both are equivalence relations

5. Let R be the set of real numbers. **[AIEEE - 2011]**

Statement-1:

A = {(x, y) ∈ R × R : y – x is an integer} is an equivalence relation on R.

Statement-2:

B = {(x, y)∈ R × R : x = αy for some rational number α} is an equivalence relation on R.

(1) Statement-1 is true, Statement-2 is false.

(2) Statement-1 is false, Statement-2 is true

(3) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(4) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.

6. Let Z be the set of integers.

If $A=\left\{x \in Z: 2^{(x+2)\left(x^{2}-5x+6\right)}=1\right\}$

and $B=\{x \in Z:-3<2x-1<9\}$,

then the number of subsets of the set A × B, is: **[JEE(Main) Jan-19]**

(1) 2^{18} (2) 2^{10} (3) 2^{15} (4) 2^{12}

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EXERCISE-I (ANSWER KEY)

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	1	3	1	1	2	3	1	3	4	1	2	1	4	4	3
Que.	16	17	18	19	20	21	22	23	24	25	26	27	28		
Ans.	4	4	4	1	2	1	3	3	2	2	4	3	4		

EXERCISE-II (ANSWER KEY)

Que.	1	2	3	4	5	6
Ans.	2	1	1	3	1	3

CHAPTER 4

QUADRATIC EQUATION

Chapter Contents 04

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IMPORTANT NOTES

QUADRATIC EQUATION

1. INTRODUCTION :

The algebraic expression of the form $ax^2 + bx + c,\ a \neq 0$ is called a quadratic expression, because the highest order term in it is of second degree. Quadratic equation means, $ax^2 + bx + c = 0$. In general whenever one says zeroes of the expression $ax^2 + bx + c$, it implies roots of the equation $ax^2 + bx + c = 0$, unless specified otherwise.

A quadratic equation has exactly two roots which may be real (equal or unequal) or imaginary.

2. SOLUTION OF QUADRATIC EQUATION & RELATION BETWEEN ROOTS & CO-EFFICIENTS :

(a) The general form of quadratic equation is $ax^2 + bx + c = 0,\ a \neq 0$.

The roots can be found in following manner :

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0 \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2} = 0$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} \quad \Rightarrow \quad \boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This expression can be directly used to find the two roots of a quadratic equation.

(b) The expression $b^2 - 4\,ac \equiv D$ is called the discriminant of the quadratic equation.

(c) If $\alpha\ \&\ \beta$ are the roots of the quadratic equation $ax^2 + bx + c = 0$, then :

(i) $\alpha + \beta = -\,b/a$ (ii) $\alpha\beta = c/a$ (iii) $|\alpha - \beta| = \dfrac{\sqrt{D}}{|a|}$

(d) A quadratic equation whose roots are $\alpha\ \&\ \beta$ is $(x - \alpha)(x - \beta) = 0$ i.e.

$x^2 - (\alpha + \beta)\,x + \alpha\beta = 0$ i.e. x^2 – (sum of roots) x + product of roots = 0.

Illustrations

Illustration 1 : If α, β are the roots of a quadratic equation $x^2 - 3x + 5 = 0$, then the equation whose roots are $(\alpha^2 - 3\alpha + 7)$ and $(\beta^2 - 3\beta + 7)$ is -

(A) $x^2 + 4x + 1 = 0$ (B) $x^2 - 4x + 4 = 0$

(C) $x^2 - 4x - 1 = 0$ (D) $x^2 + 2x + 3 = 0$

Solution : Since α, β are the roots of equation $x^2 - 3x + 5 = 0$

So $\alpha^2 - 3\alpha + 5 = 0$

$\beta^2 - 3\beta + 5 = 0$

$\therefore$ $\alpha^2 - 3\alpha = -5$

$\beta^2 - 3\beta = -5$

Putting in $(\alpha^2 - 3\alpha + 7)$ & $(\beta^2 - 3\beta + 7)$ ....(i)

$-5 + 7, -5 + 7$

$\therefore$ 2 and 2 are the roots.

$\therefore$ The required equation is $x^2 - 4x + 4 = 0$. **Ans. (B)**

Illustration 2 : If α and β are the roots of $ax^2 + bx + c = 0$, find the value of $(a\alpha + b)^{-2} + (a\beta + b)^{-2}$.

Solution : We know that $\alpha + \beta = -\dfrac{b}{a}$ & $\alpha\beta = \dfrac{c}{a}$

$$(a\alpha + b)^{-2} + (a\beta + b)^{-2} = \frac{1}{(a\alpha+b)^2} + \frac{1}{(a\beta+b)^2}$$

$$= \frac{a^2\beta^2 + b^2 + 2ab\beta + a^2\alpha^2 + b^2 + 2ab\alpha}{(a^2\alpha\beta + ba\beta + ba\alpha + b^2)^2} = \frac{a^2(\alpha^2+\beta^2) + 2ab(\alpha+\beta) + 2b^2}{(a^2\alpha\beta + ab(\alpha+\beta) + b^2)^2}$$

($\alpha^2 + \beta^2$ can always be written as $(\alpha + \beta)^2 - 2\alpha\beta$)

$$= \frac{a^2\left[(\alpha+\beta)^2 - 2\alpha\beta\right] + 2ab(\alpha+\beta) + 2b^2}{(a^2\alpha\beta + ab(\alpha+\beta) + b^2)^2} = \frac{a^2\left[\dfrac{b^2 - 2ac}{a^2}\right] + 2ab\left(-\dfrac{b}{a}\right) + 2b^2}{\left(a^2\dfrac{c}{a} + ab\left(-\dfrac{b}{a}\right) + b^2\right)^2} = \frac{b^2 - 2ac}{a^2c^2}$$

Alternatively :

If α & β are roots of $ax^2 + bx + c = 0$

then, $a\alpha^2 + b\alpha + c = 0$

$$\Rightarrow \quad a\alpha + b = -\frac{c}{\alpha}$$

same as $a\beta + b = -\frac{c}{\beta}$

$$\therefore \quad (a\alpha + b)^{-2} = (\alpha\beta + b)^{-2} = \frac{\alpha^2}{c^2} + \frac{\beta^2}{c^2}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{c^2}$$

$$= \frac{(-b/a)^2 - 2(c/a)}{c^2}$$

$$= \frac{b^2 - 2ac}{a^2c^2}$$

Do yourself - 1

1. Find the roots of following equations :

(a) $x^2 + 3x + 2 = 0$ (b) $x^2 - 8x + 16 = 0$ (c) $x^2 - 2x - 1 = 0$

2. Find the roots of the equation $a(x^2 + 1) - (a^2 + 1)x = 0$, where $a \neq 0$.

3. Solve : $\frac{6-x}{x^2-4} = 2 + \frac{x}{x+2}$

4. If the roots of $4x^2 + 5k = (5k + 1)x$ differ by unity, then find the values of k.

5. Form a quadratic equation whose roots are the numbers $\frac{1}{10-\sqrt{72}}$ and $\frac{1}{10+6\sqrt{2}}$.

6. For what values of a is the sum of the roots of the equation $x^2 + (2 - a - a^2)x - a^2 = 0$ equal to zero ?

7. For what values of a is the ratio of the roots of the equation $ax^2 - (a + 3)x + 3 = 0$ equal to 1.5?

8. The roots x_1 and x_2 of the equation $x^2 + px + 12 = 0$ are such that $x_2 - x_1 = 1$. Find p.

9. Find k in the equation $5x^2 - kx + 1 = 0$ such that the difference between the roots of the equation is unity.

10. Find p in the equation $x^2 - 4x + p = 0$ if it is know that the sum of the squares of its roots is equal to 16.

11. For what values of a is the difference between the roots of the equation
$2x^2 - (a + 1)x + (a - 1) = 0$ equal to their product ?

12. Express $x_1^3 + x_2^3$ in terms of the coefficients of the equation $x^2 + px + q = 0$, where x_1 and x_2 are the roots of the equation.

13. Assume that x_1 and x_2 are roots of the equation $3x^2 - ax + 2a - 1 = 0$. Calculate $x_1^3 + x_2^3$.

14. Without solving the equation $3x^2 - 5x - 2 = 0$, find the sum of the cubes of its roots.

3. NATURE OF ROOTS :

(a) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b , c $\in \mathbb{R}$ & $a \neq 0$ then ;

$$x = \frac{-b \pm \sqrt{D}}{2a}$$

(i) $D > 0 \Leftrightarrow$ roots are real & distinct (unequal).

(ii) $D = 0 \Leftrightarrow$ roots are real & coincident (equal)

(iii) $D < 0 \Leftrightarrow$ roots are imaginary.

(iv) If p + i q is one root of a quadratic equation, then the other root must be the conjugate p – i q & vice versa. $(p, q \in \mathbb{R}\ \&\ i = \sqrt{-1})$.

(b) Consider the quadratic equation $ax^2 + bx + c = 0$ where a, b, c $\in \mathbb{Q}$ & $a \neq 0$ then ;

(i) If D is a perfect square, then roots are rational.

(ii) If $\alpha = p + \sqrt{q}$ is one root in this case, (where p is rational & $\sqrt{q}$ is an irrational) then other root will be $p - \sqrt{q}$.

Illustration 3 : If roots of the equation $(a-b)x^2 + (c-a)x + (b-c) = 0$ are equal, then

(A) $2b = a + c$ (B) $b = \frac{2ac}{a+c}$ (C) $b^2 = ac$ (D) none of these

Solution : $(a-b)x^2 + (c-a)x + (b-c) = 0$

As roots are equal so

$B^2 - 4AC = 0 \Rightarrow (c-a)^2 - 4(a-b)(b-c) = 0 \quad \Rightarrow (c-a)^2 - 4ab + 4b^2 + 4ac - 4bc = 0$

$\Rightarrow \quad (c-a)^2 + 4ac - 4b(c+a) + 4b^2 = 0 \Rightarrow \quad (c+a)^2 - 2\,.\,(2b)(c+a) + (2b)^2 = 0$

$\Rightarrow \quad [c + a - 2b]^2 = 0 \quad \Rightarrow \quad c + a - 2b = 0 \Rightarrow \quad c + a = 2b$

Alternative method :

$\because$ Sum of the coefficients = 0

Hence one root is 1 and other root is $\frac{b-c}{a-b}$.

Given that both roots are equal, so

$1 = \frac{b-c}{a-b} \Rightarrow \quad a - b = b - c \quad \Rightarrow \quad 2b = a + c$

Illustration 4 : If equation $\frac{x^2 - bx}{ax - c} = \frac{k-1}{k+1}$ has roots equal in magnitude & opposite in sign, then the value of k is -

(A) $\frac{a+b}{a-b}$ (B) $\frac{a-b}{a+b}$ (C) $\frac{a}{b}+1$ (D) $\frac{a}{b}-1$

Solution : Let the roots are α & $-\alpha$.

given equation is

$(x^2 - bx)(k + 1) = (k - 1)(ax - c) \quad \{\text{Considering, } x \neq c/a \ \& \ k \neq -1\}$

$\Rightarrow \quad x^2(k+1) - bx(k+1) = ax\,(k-1) - c(k-1)$

$\Rightarrow \quad x^2(k+1) - bx(k+1) - ax\,(k-1) + c(k-1) = 0$

Now sum of roots = 0 $\quad (\because \ \alpha - \alpha = 0)$

$\therefore \quad b(k+1) + a(k-1) = 0 \Rightarrow \quad k = \frac{a-b}{a+b}$ **Ans. (B)**

Illustration 5 : If the coefficient of the quadratic equation are rational & the coefficient of x^2 is 1, then find the equation one of whose roots is $\sqrt{2}-1$.

Solution : Irrational roots always occur in conjugational pairs.

Hence if one root is $(-1+\sqrt{2})$, the other root will be $(-1-\sqrt{2})$. Equation is

$(x-(-1+\sqrt{2}))\,(x-(-1-\sqrt{2})) = 0 \quad \Rightarrow \quad x^2 + 2x - 1 = 0$

Illustration 6 : Find all the integral values of a for which the quadratic equation $(x-a)(x-10)+1=0$ has integral roots.

Solution : Here the equation is $x^2-(a+10)x+10a+1=0$. Since integral roots will always be rational it means D should be a perfect square.

From (i) $D = a^2 - 20a + 96$.

$\Rightarrow \quad D = (a-10)^2 - 4 \Rightarrow \quad 4 = (a-10)^2 - D$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a-10)^2 = 4$ and $D = 0$.

$\Rightarrow \quad (a-10) = \pm 2 \quad \Rightarrow \quad a = 12, 8$ **Ans.**

Illustration 7 : If D_1 and D_2 are the disrcriminant of two Quadratic Equations with real coefficients respectively then comment upon the nature of the roots of the Quadratic Equations under the following conditions

(i) $D_1 + D_2 \geq 0$ (ii) $D_1 + D_2 < 0$ (iii) $D_1\ D_2 < 0$ (iv) $D_1\ D_2 > 0$ (v) $D_1\ D_2 = 0$

Solution :

(i) If D_1 and D_2 are discriminant of two quadratic equations and $D_1 + D_2 \geq 0$, then at least one of D_1 and $D_2 \geq 0$

$\Rightarrow$ At least one of the equation has real roots.

(ii) $D_1 + D_2 < 0 \Rightarrow$ At least one of D_1 and $D_2 < 0$

$\Rightarrow$ At least one of the equation has imaginary roots.

(iii) If $D_1\ D_2 < 0$. $\Rightarrow (D_1 > 0$ and $D_2 < 0)$ or $(D_1 < 0$ and $D_2 > 0)$.

Then one of the equations has real root and other equation has imaginary roots.

(iv) $D_1\ D_2 > 0$

Case I : If $D_1 > 0$ and $D_2 > 0$, then both the equations has real roots

Case II : If $D_1 < 0$ and $D_2 < 0$, then both the equations has non real roots

(v) If $D_1\ D_2 = 0$, then $D_1 = 0$ or $D_2 = 0$

$\Rightarrow$ at least of equation has equal roots.

Illustration 8 : Let $a > 0$, $b > 0$ and $c > 0$. Then both the roots of the equation $ax^2 + bx + c = 0$ can

(a) be real and negative (b) have negative real parts

(c) be rational numbers (d) none of these

Solution : (abc)

We have $D = b^2 - 4ac$.

If $D \geq 0$, then the roots of the equation are given by $x = \dfrac{-b \pm \sqrt{D}}{2a}$

As $D = b^2 - 4ac < b^2$ ($\because a > 0, c > 0$), it follows that the roots of the quadratic equation are negative.

In case $D < 0$, then the roots of the equation are given by $x = \dfrac{-b \pm i\sqrt{-D}}{2a}$

which have negative real parts. ($\because$ Both a and b are positive)

Roots will be rational numbers when $\dfrac{b}{a}, \dfrac{c}{a} \in \mathbb{Q}$ and $\dfrac{D}{4a^2}$ is perfect square of some rational number.

4. SYMMETRIC EXPRESSIONS OF α AND β.

Let a and b be the roots of the equation $ax^2 + bx + c = 0$ such that

$f(\alpha, \beta) = f(\beta, \alpha)$ then $f(\alpha, \beta)$ denotes symmetric expression of the roots.

e.g. $f(\alpha,\beta) = \alpha^2\beta + \alpha\beta^2$

It is to be noted that every symmetric expression in α,β can be expressed in terms of two symmetric expression $\alpha + \beta$ and $\alpha\beta$

Do yourself - 2

1. For the following equations, find the nature of the roots (real & distinct, real & coincident or imaginary).
   (a) $x^2 - 6x + 10 = 0$
   (b) $x^2 - (7 + \sqrt{3})x + 6(1 + \sqrt{3}) = 0$
   (c) $4x^2 + 28x + 49 = 0$
2. Prove that roots of $(x - a)(x - b) = h^2$ are always real.
3. For what values of a does the equation $9x^2 - 2x + a = 6 - ax$ posses equal roots ?
4. Find the value of k for which the equation $(k - 1)x^2 + (k + 4)x + k + 7 = 0$ has equal roots.
5. Find the values of a for which the roots of the equation $(2a - 5)x^2 - 2(a - 1)x + 3 = 0$ are equal.
6. For what values of 'a' does the quadratic equation $x^2 + (2a\sqrt{a^2 - 3})x + 4 = 0$ possess equal roots ?

7. Find the least integral value of k for which the equation $x^2 - 2(k+2)x + 12 + k^2 = 0$ has two different real roots.

8. If ℓ, m are real and $\ell \neq m$, then show that the roots of $(\ell - m)x^2 - 5(\ell + m)x - 2(\ell - m) = 0$ are real and unequal.

9. For what values of m does the equation $x^2 - x + m = 0$ possess no real roots ?

10. For what values of c does the equation $(c-2)x^2 + 2(c-2)x + 2 = 0$ possess no real roots ?

11. Find integral values of k for which the quadratic equation $(k-12)x^2 + 2(k-12)x + 2 = 0$ possess no real roots ?

12. For what values of m does the equation $x^2 - x + m^2 = 0$ possess no real roots ?

13. For what values of m does the equation $mx^2 - (m+1)x + 2m - 1 = 0$ possess no real roots ?

14. If $2+\sqrt{3}$ is a root of the equation $x^2 + bx + c = 0$, where $b, c \in \mathbb{Q}$, find b, c.

15. Find the value of $x^3 + x^2 - x + 22$ when $x = 1 + 2i$.

16. Find the value of $x^3 - 3x^2 - 8x + 15$ when $x = 3 + i$.

17. Consider $f(x) = x^2 + bx + c$.

(a) Find c if $x = 0$ is a root of $f(x) = 0$.

(b) Find c if $\alpha, \dfrac{1}{\alpha}$ are roots of $f(x) = 0$.

(c) Comment on sign of b & c, if $\alpha < 0 < \beta$ & $|\beta| > |\alpha|$, where α, β are roots of $f(x) = 0$.

18. Which of the following expressions in α, β will denote the symmetric functions in α,β. Give proper reasoning.

(i) $f(\alpha, \beta) = \alpha^2 - \beta$ (ii) $f(\alpha, \beta) = \alpha^2\beta + \alpha\beta^2$

(iii) $f(\alpha, \beta) = \alpha^3 + \beta^3$ (iv) $f(\alpha, \beta) = \dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}$

19. If α,β are the roots of the equation $ax^2 + bx + c = 0$, find the value of

(A) $\dfrac{1}{\alpha^2} + \dfrac{1}{\beta^2}$ (B) $\alpha^4\beta^7 + \alpha^7\beta^4$ (C) $\left(\dfrac{\alpha}{\beta} - \dfrac{\beta}{\alpha}\right)^2$

20. Calculate $\dfrac{1}{x_1^3} + \dfrac{1}{x_2^3}$, where x_1 and x_2 are roots of the equation $2x^2 - 3ax - 2 = 0$.

21. For what value of a is the difference between the roots of the equation $(a-2)x^2-(a-4)x-2=0$ equal to 3 ?

22. Find the value of a for which one root of the equation $x^2+(2a-1)x+a^2+2=0$ is twice as large as the other.

23. For what values of a is the ratio of the roots of the equation $x^2+ax+a+2=0$ equal to 2 ?

24. For what values of a do the roots x_1 and x_2 of the equation $x^2-(3a+2)x+a^2=0$ satisfy the relation $x_1=9x_2$? Find the roots.

25. Find a such that one of the roots of the equation $x^2-\frac{15}{4}x+a=0$ is the square of the other.

26. Find all the values of a for which the sum of the roots of the equation $x^2-2a(x-1)-1=0$ is equal to the sum of the squares of its roots.

27. Find the coefficients of the equation $x^2+px+q=0$ such that its roots are equal to p and q.

5. TRANSFORMATION OF THE EQUATION :

Let $ax^2+bx+c=0$ be a quadratic equation with two roots α and β. If we have to find an equation whose roots are $f(\alpha)$ and $f(\beta)$, i.e. some expression in α & β, then this equation can be found by finding α in terms of y. Now as α satisfies given equation, put this α in terms of y directly in the equation.

$$y = f(\alpha)$$

By transformation, $\alpha = g(y)$

$$a(g(y))^2+b(g(y))+c=0$$

This is the required equation in y.

Illustration 9 : If the roots of $ax^2+bx+c=0$ are α and β, then find the equation whose roots are :

(a) $\frac{-2}{\alpha},\frac{-2}{\beta}$ (b) $\frac{\alpha}{\alpha+1},\frac{\beta}{\beta+1}$ (c) $\alpha^2,\ \beta^2$

Solution : (a) $\frac{-2}{\alpha},\frac{-2}{\beta}$

put, $y=\frac{-2}{\alpha} \Rightarrow \alpha=\frac{-2}{y}$

$$a\left(-\frac{2}{y}\right)^2+b\left(\frac{-2}{y}\right)+c=0 \Rightarrow cy^2-2by+4a=0$$

Required equation is $cx^2-2bx+4a=0$

(b) $\dfrac{\alpha}{\alpha+1}, \dfrac{\beta}{\beta+1}$

put, $y = \dfrac{\alpha}{\alpha+1} \Rightarrow \alpha = \dfrac{y}{1-y}$

$$\Rightarrow \quad a\left(\frac{y}{1-y}\right)^2 + b\left(\frac{y}{1-y}\right) + c = 0 \Rightarrow (a + c - b)y^2 + (-2c + b)y + c = 0$$

Required equation is $(a + c - b)\,x^2 + (b - 2c)\,x + c = 0$

(c) α^2, β^2

put $y = \alpha^2 \Rightarrow \alpha = \sqrt{y}$

$ay + b\sqrt{y} + c = 0$

$b^2y = a^2y^2 + c^2 + 2acy$

$\Rightarrow \quad a^2y^2 + (2ac - b^2)\,y + c^2 = 0$

Required equation is $a^2x^2 + (2ac - b^2)\,x + c^2 = 0$

Illustration 10 : If the roots of $ax^3 + bx^2 + cx + d = 0$ are α, β, γ then find equation whose roots are $\dfrac{1}{\alpha\beta}, \dfrac{1}{\beta\gamma}, \dfrac{1}{\gamma\alpha}$.

Solution : Put $y = \dfrac{1}{\alpha\beta} = \dfrac{\gamma}{\alpha\beta\gamma} = -\dfrac{a\gamma}{d} \quad (\because \alpha\beta\gamma = -\dfrac{d}{a})$

Put $x = -\dfrac{dy}{a}$

$$\Rightarrow \quad a\left(-\frac{dy}{a}\right)^3 + b\left(-\frac{dy}{a}\right)^2 + c\left(-\frac{dy}{a}\right) + d = 0$$

Required equation is $d^2x^3 - bdx^2 + acx - a^2 = 0$

Do yourself - 3

1. If α, β are the roots of $ax^2 + bx + c = 0$, then find the equation whose roots are

(a) $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ (b) $\frac{1}{a\alpha+b}, \frac{1}{a\beta+b}$ (c) $\alpha+\frac{1}{\beta}, \beta+\frac{1}{\alpha}$

2. If α and β are the roots of $x^2 + px + q = 0$, form the equation whose roots are $(\alpha - \beta)^2$ and $(\alpha + \beta)^2$.

3. If α, β are roots of $x^2 - px + q = 0$, then find the quadratic equation whose root are $(\alpha^2-\beta^2)(\alpha^3-\beta^3)$ and $\alpha^2\beta^3+\alpha^3\beta^2$.

6. EQUATION VS IDENTITY :

An equation which is true for every value of the variable within the domain is called an identity , for example : $5(a-3) = 5a - 15$, $(a+b)^2 = a^2 + b^2 + 2ab$ for all $a, b \in R$.

Note : An equation of degree ≤ 2 cannot have three or more roots & if it has , it becomes an identity.

If $ax^2 + bx + c = 0$ is an identity $\Leftrightarrow a = b = c = 0$

Illustration 11 : If the equation $(\lambda^2-5\lambda+6)x^2+(\lambda^2-3\lambda+2)x+(\lambda^2-4)=0$ has more than two roots, then find the value of λ ?

Solution : As the equation has more than two roots so it becomes an identity. Hence

$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda = 2, 3$

and $\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1, 2$

and $\lambda^2 - 4 = 0 \Rightarrow \lambda = 2, -2$

So $\lambda = 2$ **Ans. $\lambda = 2$**

7. COMMON ROOTS OF TWO QUADRATIC EQUATIONS :

(a) Atleast one common root.

Let α be the common root of $ax^2 + bx + c = 0$ & $a'x^2 + b'x + c' = 0$ then

$a\alpha^2 + b\alpha + c = 0$ & $a'\alpha^2 + b'\alpha + c' = 0$.

By Cramer's Rule $\frac{\alpha^2}{bc'-b'c} = \frac{\alpha}{a'c-ac'} = \frac{1}{ab'-a'b}$

Therefore, $\alpha = \frac{ca'-c'a}{ab'-a'b} = \frac{bc'-b'c}{a'c-ac'}$

So the condition for a common root is $(ca' - c'a)^2 = (ab' - a'b)(bc' - b'c)$.

(b) If both roots are same then $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

Illustration 12: Find p and q such that $px^2 + 5x + 2 = 0$ and $3x^2 + 10x + q = 0$ have both roots in common.

Solution : $a_1 = p,\ b_1 = 5,\ c_1 = 2$ and $a_2 = 3,\ b_2 = 10,\ c_2 = q$

We know that :

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow \frac{p}{3} = \frac{5}{10} = \frac{2}{q} \Rightarrow p = \frac{3}{2};\ q = 4$$

Illustration 13 : Find the possible value(s) of a for which the equations $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have atleast one common root.

Solution : Let α is a common root

then $\alpha^2 + a\alpha + 1 = 0$

& $\alpha^2 + \alpha + a = 0$

by cramer's rule

$$\frac{\alpha^2}{a^2 - 1} = \frac{\alpha}{1 - a} = \frac{1}{1 - a}$$

$\Rightarrow (1 - a)^2 = (a^2 - 1)(1 - a)$

$\Rightarrow a = 1, -2$

Illustration 14 : If the equations $x^2 - ax + b = 0$ and $x^2 - cx + d = 0$ have one root in common and second equation has equal roots, prove that $ac = 2(b + d)$

Solution : The equation $x^2 - cx + d = 0$ has equal roots.

$\Rightarrow D = 0 \Rightarrow D = c^2 - 4d = 0$ ...(i)

Also, the equal roots are $x = -\dfrac{b}{2a}$ (for $ax^2 + bx + c = 0$ having equal roots)

$\Rightarrow x = \dfrac{c}{2}$ is the equal root of this equation.

Now this should be the common root.

$\therefore x = \dfrac{c}{2}$ will satisfy the first equation

$$\Rightarrow \frac{c^2}{4} - a\left(\frac{c}{2}\right) + b = 0 \quad \Rightarrow c^2 + 4b = 2ac$$

$\Rightarrow 4d + 4b = 2ac$ [Using (i)]

$\Rightarrow 2(d + b) = ac$

Hence $ac = 2(b + d)$

Illsutration 15 : If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a root in common, find the relation between a, b and c.

Solution : Solve the two equations :

$ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$

$$\frac{x^2}{ba - c^2} = \frac{-x}{a^2 - bc} = \frac{1}{ac - b^2}$$

$\Rightarrow (a^2 - bc)^2 = (ba - c^2)(ac - b^2)$

Simplify to get : $a(a^3 + b^3 + c^3 - 3abc) = 0$

$\Rightarrow a = 0$ or $a^3 + b^3 + c^3 = 3abc$

This is the relation between a, b and c.

Illsutration 16 : Find the conditions on a, b, c and d such that equations

$2ax^3 + bx^2 + cx + d = 0$ and $2ax^2 + 3bx + 4c = 0$ have a common root.

Solution : Let 'α' be a common root of the given equations,

then $2a\alpha^3 + b\alpha^2 + c\alpha + d = 0$ ...(1)

and $2a\alpha^2 + 3b\alpha + 4c = 0$ ...(2)

Multiply (2) by α and then subtract (1) from it, to get

$2b\alpha^2 + 3c\alpha - d = 0$ ...(3)

Now (2) and (3) are quadratic having a common root α, so

$$\frac{\alpha^2}{3bd - 12c^2} = \frac{\alpha}{8bc + 2ad} = \frac{1}{6ac - 6b^2}, \quad \alpha^2 = \frac{bd + 4c^2}{2b^2 - 2ac}, \quad \alpha = \frac{4bc + ad}{3ac - 3b^2}$$

Eliminating α from these two equations, we get

$(4bc + ad)^2 = \frac{9}{2}(bd + 4c^2)(b^2 - ac)$, which is the required condition.

Illustration 17 : The values of 'a' for which the equations

$x^3 - 6x^2 + (6 + a)x - 6 = 0$ and $x^2 - ax + 4 = 0$ have a common root.

Solution : Let 'α' is the common root.

$\alpha^3 - 6\alpha^2 + (6 + a)\alpha - 6 = 0$

$\alpha^2 - a\alpha + 4 = 0$

Adding

$\alpha^3 - 5\alpha^2 + 6\alpha - 2 = 0$

$\Rightarrow \alpha = 1$ is one solution of above equation.

$$\begin{array}{r|l} & \quad \alpha^2-4\alpha+2 \\ \hline \alpha-1 & \alpha^3-5\alpha^2+6\alpha-2 \\ & \alpha^3-\alpha^2 \\ \hline & -4\alpha^2+6\alpha-2 \\ & -4\alpha^2+6\alpha \\ \hline & 2\alpha-2 \\ & 2\alpha-2 \\ \hline & 0 \end{array}$$

$\Rightarrow \quad (\alpha - 1)(\alpha^2 - 4\alpha + 2) = 0$

$\Rightarrow \quad (\alpha-1)\{\alpha-(2-\sqrt{2})\}\{\alpha-(2+\sqrt{2})\}=0$

$\alpha = 1,\ \alpha=2-\sqrt{2}$ or $\alpha=2+\sqrt{2}$

This common root can be 1, $2-\sqrt{2},\ 2+\sqrt{2}$.

We can obtain values of 'a'.

For $\alpha = 1$, $1 - a + 4 = 0 \Rightarrow a = 5$

For $\alpha=2-\sqrt{2}$, $(6-4\sqrt{2})-a(2-\sqrt{2})+4=0$

$\Rightarrow \quad a(2-\sqrt{2})=10-4\sqrt{2}$

$\Rightarrow \quad a=\dfrac{10-4\sqrt{2}}{2-\sqrt{2}}=\dfrac{(10-4\sqrt{2})(2+\sqrt{2})}{(4-2)}$

$\Rightarrow \quad a=(5-2\sqrt{2})(2+\sqrt{2})$

$\Rightarrow \quad a=10+5\sqrt{2}-4\sqrt{2}-4$

$\Rightarrow \quad a=6+\sqrt{2}$

For $\alpha = 2+\sqrt{2} \Rightarrow \left(6+4\sqrt{2}\right) - a\left(2+\sqrt{2}\right) + 4 = 0$

$\Rightarrow \quad a = \dfrac{10+4\sqrt{2}}{2+\sqrt{2}} = \dfrac{\left(10+4\sqrt{2}\right)\left(2-\sqrt{2}\right)}{2}$

$\Rightarrow \quad a = \left(5+2\sqrt{2}\right)\left(2-\sqrt{2}\right)$

$\Rightarrow \quad a = 10 - 5\sqrt{2} + 4\sqrt{2} - 4$

$\Rightarrow \quad a = 6 - \sqrt{2}$

Thus possible values of 'a' are 5, $6 \pm \sqrt{2}$.

Illustration 18 : If each pair of the following three equations

$x^2 + ax + b = 0,\ x^2 + cx + d = 0,\ x^2 + ex + f = 0$ has exactly one root in common, then show that $(a + c + e)^2 = 4(ac + ce + ea - b - d - f)$.

Solution : Given equations are

$x^2 + ax + b = 0$ ...(1)

$x^2 + cx + d = 0$ ...(2)

$x^2 + ex + f = 0$ ...(3)

Let α, β be the roots of (1), β, γ be the roots of (2) and γ, α be the roots of (3).

$\therefore\ \alpha + \beta = -a,\ \alpha\beta = b$ ...(4)

$\beta + \gamma = -c,\ \beta\gamma = d$ ...(5)

$\gamma + \alpha = -e,\ \gamma\alpha = f$ ...(6)

LHS $= (a + c + e)^2$

$= (-\alpha - \beta - \beta - \gamma - \gamma - \alpha)^2$ {from (4), (5), (6)}

$= 4(\alpha + \beta + \gamma)^2$ ...(7)

RHS $= 4(ac + ce + ea - b - d - f)$

$= 4\{(\alpha + \beta)(\beta + \gamma) + (\beta + \gamma)(\gamma + \alpha) + (\alpha + \beta)(\gamma + \alpha) - \alpha\beta - \beta\gamma - \gamma\alpha\}$ {from (4), (5), (6)}

$= 4(\alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha)$

$= 4(\alpha + \beta + \gamma)^2$ ...(8)

From (7) and (8),

$(a + c + e)^2 = 4(ac + ce + ea - b - d - f)$

Do yourself - 4

1. For what values of a, the equation
$(a + 3)(a - 1)x^2 + (a - 1)(4a + 3)x + (a^2 - 3a + 2) = 0$ has more than two roots.

2. If $a.\frac{(x-b)(x-c)}{(a-b)(a-c)} + b.\frac{(x-c)(x-a)}{(b-c)(b-a)} + c.\frac{(x-a)(x-b)}{(c-a)(c-b)} = x$ then prove that it is an identity.

3. If $x^2 + bx + c = 0$ & $2x^2 + 9x + 10 = 0$ have both roots in common, find b & c.

4. If $x^2 - 7x + 10 = 0$ & $x^2 - 5x + c = 0$ have a common root, find c.

5. Show that $x^2 + (a^2 - 2)x - 2a^2 = 0$ and $x^2 - 3x + 2 = 0$ have exactly one common root for all $a \in R$.

6. For what values of a do the equations $x^2 + ax + 1 = 0$ and $x^2 + x + a = 0$ have exactly one root in common ?

7. Given two quadratic equations $x^2 - x + m = 0$ and $x^2 - x + 3m = 0$, $m \neq 0$. Find the value of m for which one of the roots of the second equation is equal to double the root of the first equation.

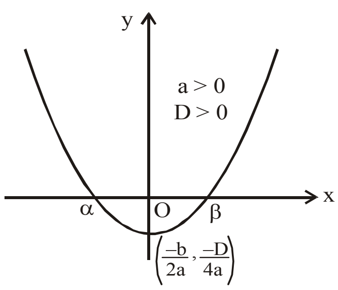
8. QUADRATIC EXPRESSION AND IT'S GRAPHS :

Consider the quadratic expression, $y = ax^2 + bx + c$, $a \neq 0$ & $a, b, c \in R$ then ;

(a) The graph between x, y is always a parabola. If a > 0 then the shape of the parabola is concave upwards & if a < 0 then the shape of the parabola is concave downwards.

(b) The graph of $y = ax^2 + bx + c$ can be divided in 6 broad categories which are as follows : (Let the real roots of quadratic equation $ax^2 + bx + c = 0$ be α & β where $\alpha \leq \beta$).

Fig. 1

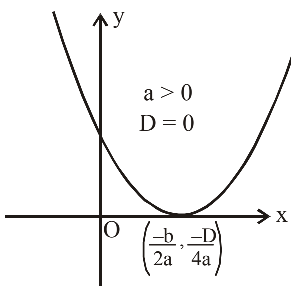


Roots are real & distinct

$ax^2 + bx + c > 0 \ \forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$

$ax^2 + bx + c < 0 \ \forall \ x \in (\alpha, \beta)$

Fig. 2

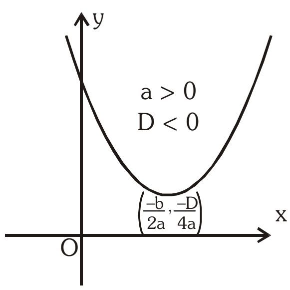


Roots are coincident

$ax^2 + bx + c > 0 \ \forall \ x \in R - \{\alpha\}$

$ax^2 + bx + c = 0$ for $x = \alpha = \beta$

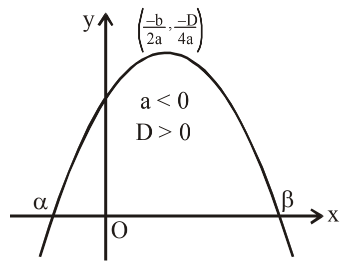
Fig.3



Roots are complex conjugate

$ax^2 + bx + c > 0 \ \forall \ x \in R$

Fig. 4

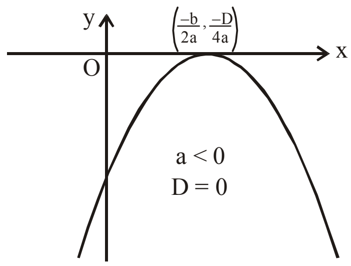


Roots are real & distinct

$ax^2 + bx + c > 0 \ \forall \ x \in (\alpha, \beta)$

$ax^2 + bx + c < 0 \ \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$

Fig. 5

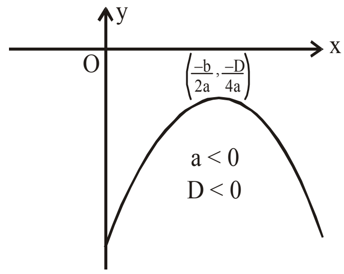


Roots are coincident

$ax^2 + bx + c < 0 \ \forall \ x \in R - \{\alpha\}$

$ax^2 + bx + c = 0$ for $x = \alpha = \beta$

Fig.6



Roots are complex conjugate

$ax^2 + bx + c < 0 \ \forall \ x \in R$

Important Note :

(i) The quadratic expression $ax^2 + bx + c > 0$ for each $x \in R \Rightarrow a > 0, D < 0$ & vice-versa (Fig. 3)

(ii) The quadratic expression $ax^2 + bx + c < 0$ for each $x \in R \Rightarrow a < 0, D < 0$ & vice-versa (Fig. 6)

9. INEQUALITIES (rational, irrational and modulus)

9.1 Rational inequalities (When numerator or denomerator contains irreducible quadratic factor)

Illustration 19: Find x such that $3x^2 - 7x + 6 < 0$

Solution : $f(x) = 3x^2 - 7x + 6$

$D = 49 - 72 < 0$ and $a > 0$

$\Rightarrow 3x^2 - 7x + 6 > 0 \ \forall \ x \in R \Rightarrow x \in \phi$

Illustration 20: Find x such that $2x^2 + 4x + 9 > 0$

Solution : $f(x) = 2x^2 + 4x + 9$

$a > 0$ and $D < 0$

$\Rightarrow 2x^2 + 4x + 9 > 0 \ \forall \ x \in R$

$\Rightarrow x \in (-\infty, \infty)$

Illustration-21 : Solve for x : $\dfrac{(x^2-1)^2 + 1}{x^2 - 1} \geq 2$.

Solution : Recall :

The sum of any real number except zero and its reciprocal should not lie in (–2, 2).

Case I :

When $t + \dfrac{1}{t} \geq 2$. This holds $\forall$ t > 0 and equality holds only when t = 1.

Case II :

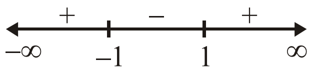
When $t + \dfrac{1}{t} \leq -2$. This holds $\forall$ t < 0 and equality holds if t = –1.

Note : $x^2 - 1 \neq 0,\ x^2 \neq 1 \Rightarrow x \neq \pm 1$

Rewrite the given inequality $\dfrac{(x^2-1)^2+1}{x^2-1} \geq 2$

as $(x^2-1)+\dfrac{1}{(x^2-1)} \geq 2$

$\Rightarrow x^2 - 1 > 0 \Rightarrow (x + 1)(x - 1) > 0$



$\Rightarrow x \in (-\infty, -1) \cup (1, \infty)$

Do yourself- 5

Solve for x ∈ ℝ

1. $x^2 - 3x + 4 > 0$

2. $\dfrac{x}{x^2+2} < 1$

3. $\dfrac{x^2-5x+12}{x^2-4x+5} < 3$

4. $\dfrac{(2x-1)(x+3)(2-x)(1-x)^2}{x^4(x+6)(x-9)(2x^2+4x+9)} < 0$

5. $\dfrac{7x-17}{x^2-3x+4} \geq 1$

6. $\dfrac{x^2+6x-7}{x^2+1} \leq 2$

7. $\dfrac{2x}{x^2-9} \leq \dfrac{1}{x+2}$

8. $\dfrac{x^2-5x+12}{x^2-4x+5} > 3$

9. $\dfrac{x^2-5x+6}{x^2+x+1} < 0$

10 $\dfrac{x^2+4x+4}{2x^2-x-1} > 0$

9.2. IRRATIONAL INEQUALITIES :

While solving Irrational Inequality, first we have to find largest interval of x for which left hand side and right hand side both make sense. This interval is called domain of the Inequality.

Note :

(i) For $\sqrt{f(x)}$ to be well defined $f(x)$ must be greater than or equal to 0.

(ii) $\sqrt{f(x)} \geq 0$ whenever defined.

Type 1 : $\sqrt{f(x)} \geq g(x)$

Case-I : $f(x) \geq 0$ and $g(x) > 0$ ... (1)

Above inequality become

$f(x) \geq g^2(x)$ ... (2)

Answer in this case is intersection of solutions of (1) and (2)

Case-II : $f(x) \geq 0$ and $g(x) \leq 0$.

In this type final answer of the given Inequality is union of case I and case II.

Type 2 : $g(x) \geq \sqrt{f(x)}$

Since $\sqrt{f(x)} \geq 0$ for $\forall$ $f(x) \geq 0$

So g(x) must be non-negative

$g(x) \geq 0$ ... (1)

$f(x) \geq 0$ ... (2)

$g^2(x) \geq f(x)$ ... (3)

Answer of the inequality is intersection of (1), (2) and (3).

Illustration-22 : Solve : $x + 1 \geq \sqrt{5 - x}$

Solution : $5 - x \geq 0$ and $x + 1 \geq 0$

$\Rightarrow x \in [-1, 5]$ ... (1)

For $x \in [-1, 5]$

$x + 1 \geq \sqrt{5 - x}$

$\Rightarrow (x + 1)^2 \geq 5 - x \Rightarrow x^2 + 3x - 4 \geq 0$

$\Rightarrow (x - 1)(x + 4) \geq 0 \Rightarrow x \in (-\infty, -4] \cup [1, \infty)$

$\Rightarrow x \in [1, 5]$

Illustration-23 : Solve : $\sqrt{6+x} \geq 6-x$

Solution :

Case-I $\quad 6 + x \geq 0$ and $6 - x > 0$

$x \in [-6, 6)$

$\sqrt{6+x} \geq 6 - x \Rightarrow 6 + x \geq 36 - 12x + x^2$

$\Rightarrow 0 \geq x^2 - 13x + 30 \Rightarrow (x - 10)(x - 3) \leq 0 \Rightarrow x \in [3, 10]$

$x \in [3, 10] \cap [-6, 6) = [3, 6)$

Case-II $\quad 6 + x \geq 0$ and $6 - x \leq 0 \Rightarrow x \in [6, \infty)$

In this case given Inequality is always true.

From case-I and case-II,

$x \in [3, \infty)$.

Illustration 24 : Solve for x, if $\sqrt{x^2-3x+2} > x-2$

Solution :

$$\left[\begin{array}{c} \left\{\begin{array}{c} x^2-3x+2\geq 0 \\ x-2\geq 0 \\ (x^2-3x+2)>(x-2)^2 \end{array}\right. \\ \text{or} \\ \left\{\begin{array}{c} x^2-3x+2\geq 0 \\ x-2<0 \end{array}\right. \end{array}\right. \Rightarrow \left[\begin{array}{c} \left\{\begin{array}{c} (x-1)(x-2)\geq 0 \\ (x-2)\geq 0 \\ x-2>0 \end{array}\right. \Rightarrow x>2 \\ \text{or} \\ \left\{\begin{array}{c} (x-1)(x-2)\geq 0 \\ x-2<0 \end{array}\right. \Rightarrow x\leq 1 \end{array}\right.$$

Hence, solution set of the original inequation is $x \in R - (1,2]$

Do yourself- 6

Solve for x if

1 $\quad \sqrt{x^2-x} > (x-1)$

2. $\quad \sqrt{\dfrac{x-2}{1-2x}} > -1$

3. $\quad \dfrac{\sqrt{x-3}}{x-2} > 0$

4. $\quad 2\sqrt{x-1} < x$

5. $\quad \sqrt{9x-20} < x$

6. $\quad \sqrt{5-2x} < 6x-1$

7. $\quad \sqrt{2x-x^2} < 5-x$

8. $\quad \sqrt{x+18} < 2-x$

9.3 Modulus inequalities

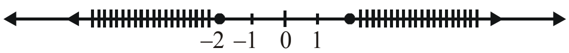
- $|x| \leq a$, if $a \geq 0$
  $\Rightarrow -a \leq x \leq a$
  if $a < 0$, then $x \in \phi$
- $|x| \geq a$
  if $a \geq 0$, then $x \in (-\infty, -a] \cup [a, \infty)$
  if $a < 0$, then $x \in (-\infty, \infty)$
- $|xy| = |x|\,|y|$
- $\big||x| - |y|\big| \leq |x + y| \leq |x| + |y|$
- $|x| + |y| = |x + y| \quad \Leftrightarrow \quad xy \geq 0$
- $|x| + |y| = |x - y| \quad \Leftrightarrow \quad xy \leq 0$

Illustration -25 : Represent x on the number line when

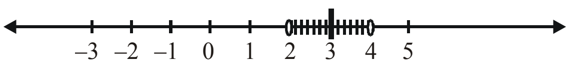
(i) $|x| \geq 2$ (ii) $|x - 3| \leq 1$ (iii) $|5 - x| < 2$

(iv) $1 \leq |x + 2| < 5$ (v) $|2x - 5| < 1$

Solution : (i) $|x| \geq 2 \Rightarrow$ distance of x from O on the number line is greater than or equal to 2.



(ii) $|x - 3| \leq 1 \Rightarrow$ distance of x from 3 is less than or equal to 1.

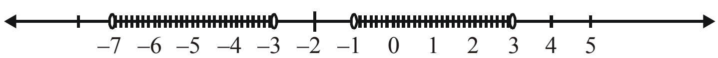


(iii) $|5 - x| < 2 \Rightarrow$ distance of x from 2 is less than 2.

(iv) $1 \leq |x + 2| < 5$

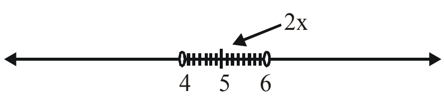
$\Rightarrow 1 \leq |x - (2)| < 5$

$\Rightarrow$ distance of x from –2 is greater than or equal to 1 andl ess than 5.



(v) $|2x - 5| < 1$

$\Rightarrow$ distance of 2x from 5 on the number line is less than or equal to 1.



$\Rightarrow 4 < 2x < 6 \quad \Rightarrow 2 < x < 3$

Illustration -26 : Solve following

(i) $|x^2 - 4x| = x + 1$

(ii) $|x^2 + 1| = |x - 2|$

(iii) $|x^2 - 16| - 8|x - 2| = x(8 - x)$

(iv) $a^2|x + a| + |a^2x + 1| = 1 - a^3$

where a is a real constant.

Solution : **(i) Case-1**

$x^2 - 4x \geq 0$

$\Rightarrow x \in (-\infty,0] \cup [4,\infty)$ ...(1)

$|x^2 - 4x| = x + 1$

$\Rightarrow x^2 - 4x = x + 1 \Rightarrow x^2 - 5x - 1 = 0$

$$\Rightarrow \ x = \frac{5 \pm \sqrt{29}}{2}$$

from (1) and (2)

$$x = \frac{5 \pm \sqrt{29}}{2}$$

Case -2

where $x^2 - 4x < 0$

$\Rightarrow x \in (0,4)$ ...(3)

$|x^2 - 4x| = x + 1$

$\Rightarrow 4x - x^2 = x + 1 \Rightarrow x^2 - 3x + 1 = 0$

$$\Rightarrow \ x = \frac{3 \pm \sqrt{5}}{2} \qquad ...(4)$$

From (3) and (4)

Set of all solutions of the given equation is

$$\left\{\frac{5-\sqrt{29}}{2}, \frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, \frac{5+\sqrt{29}}{2}\right\}$$

(ii) $|x^2 + 1| = |x - 2|$

As $x^2 + 1 = 0\ \forall\ x \in \mathbb{R}$

$\Rightarrow |x^2 + 1| = x^2 + 1,\ \forall\ x \in \mathbb{R}$

Case-1

$x - 2 \geq 0 \quad \Rightarrow x \in [2,\infty)$

$|x^2 + 1| = |x - 2|$

$\Rightarrow x^2 + 1 = x - 2 \Rightarrow x^2 - x + 3 = 0$

$$\Rightarrow \ x = \frac{1 \pm \sqrt{11}i}{2}$$

Case-2

$x - 2 < 0 \Rightarrow x \in (-\infty,2)$

$|x^2 + 1| = |x - 2|$

$\Rightarrow x^2 + 1 = -x + 2 \Rightarrow x^2 + x - 1 = 0$

$$\Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$$

Both x belongs to $(-\infty,2)$

Hence set of solutions is $\left\{\frac{-1 \pm \sqrt{5}}{2}\right\}$

(iii) $|x^2 - 16| - 8|x - 2| = x(8 - x)$

Case 1 : $x \leq -4$

$x^2 - 16 + 8x - 16 = 8x - x^2$

$2x^2 = 32$

Case 2 : $x \in [-4,2]$

$-x^2 + 16 + 8x - 16 = 8x - x^2$

$x = x$ always true

$x \in [-4,2]$

Case 3 : $x \in [2,4]$

$- x^2 + 76 - 8x + 16 = 8x - x^2$

$16x = 32$

$x = 2$

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Case 4 : $x \geq 4$

$x^2 - 16 - 8x + 16 = 8x - x^2$

$2x^2 = 16x$

$x = 8, 0$ (rejected)

$x \in [-4,2] \cup \{8\}$

(iv) $a^2|x + a| + |a^2x + 1| = 1 - a^3 \Rightarrow |a^2x + a^3| + |a^2x + 1| = 1 - a^3$

Let $A = a^2x + a^3$, $B = a^2x + 1$ and $C = 1 - a^3$

Given equation can be written as

$|A| + |B| = C = B - A$

$\Rightarrow B \geq 0$ and $A \leq 0$

$a^2x + 1 \geq 0$ and $a^2x + a^3 \leq 0$

when $a = 0$, $x \in R$

when $a \neq 0$, $x \geq \frac{-1}{a^2}$ and $x \leq -a$ $\Rightarrow x \in \left[-\frac{1}{a^2}, \infty\right) \cap (-\infty, -a]$

Case-1

when $-\frac{1}{a^2} > -a$ $\Rightarrow \frac{1}{a^2} < a \Rightarrow 1 < a^3$ $\Rightarrow a \in (1,\infty)$

$$x \in \left[-\frac{1}{a^2}, \infty\right) \cap (-\infty, -a] = \phi$$

Case-2

when $-a \geq -\frac{1}{a^2} \Rightarrow a \leq \frac{1}{a^2}$ and $a \neq 0$

$\Rightarrow a^3 \leq 1$ and $a \neq 0$ $\Rightarrow a < 1$ and $a \neq 0 \Rightarrow a \in (-\infty, 1] - \{0\}$

$$x \in \left[-\frac{1}{a^2}, \infty\right) \cap (-\infty, -a]$$

$$x \in \left[-\frac{1}{a^2}, -a\right]$$

Solutions of the given equation is

$x \in \mathbb{R}$ when $a = 0$

$x \in \phi$ when $a > 1$

$x \in \left[-\frac{1}{a^2}, -a\right]$ when $a \in (-\infty, 1] - \{0\}$

Illustrations-27 : Let A = {x : x $\in$ R, |x| < 1] ; B = [x : x $\in$ R, |x – 1| $\geq$ 1] and A $\cup$ B = R – D, then the set D is-

(1) [x : 1 < x $\leq$ 2] (2) [x : 1 $\leq$ x < 2] (3) [x : 1 $\leq$ x $\leq$ 2] (4) none of these

Solution : A = [x : x $\in$ R, –1 < x < 1]

B = [x : x $\in$ R : x – 1 $\leq$ –1 or x – 1 $\geq$ 1]

= [x : x $\in$ R : x $\leq$ 0 or x $\geq$ 2]

$\therefore$ A $\cup$ B = R – D

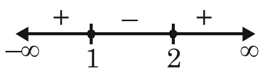
where D = [x : x $\in$ R, 1 $\leq$ x < 2]

Thus (2) is the correct answer.

Illustration-28 : If $|x - 1||x - 2| = -(x^2 - 3x + 2)$, then find the interval in which x lies ?

Solution : $|x - 1||x - 2| = -(x - 2)(x - 1)$

$\Rightarrow (x - 1)(x - 2) \leq 0$



$\Rightarrow 1 \leq x \leq 2$

Illustration-29 : Solve : (i) $||x - 2| - 5| \geq 1$

Solution : $||x - 2| - 5| \geq 1$

$\Rightarrow |x - 2| - 5 \geq 1$ or $|x - 2| - 5 \leq -1$

$\Rightarrow |x - 2| \geq 6$ or $|x - 2| \leq 4$

Case-1

$|x - 2| \geq 6$

$\Rightarrow x - 2 \geq 6$ or $x - 2 \leq -6 \Rightarrow x \geq 8$ or $x \leq -4$

$\Rightarrow x \in (-\infty, -4] \cup [8, \infty)$ ...(1)

Case-2

$|x - 2| \geq 4$

$\Rightarrow -4 \leq x - 2 \leq 4 \Rightarrow -2 \leq x \leq 6$

$\Rightarrow x \in [-2, 6]$ ...(2)

From (1) and (2) $x \in (-\infty, -4] \cup [-2, -6] \cup [8, \infty)$

Illustration 30 : If x satisfies $|x-1| + |x-2| + |x-3| \geq 6$, then

(A) $0 \leq x \leq 4$ (B) $x \leq -2$ or $x \geq 4$

(C) $x \leq 0$ or $x \geq 4$ (D) none of these

Solution :

Case I : $x \leq 1$, then

$1 - x + 2 - x + 3 - x \geq 6 \Rightarrow x \leq 0$

Hence $x \leq 0$ ...(i)

Case II : $1 < x \leq 2$, then

$x - 1 + 2 - x + 3 - x \geq 6 \Rightarrow x \leq -2$

But $1 < x \leq 2 \Rightarrow$ No solution. ...(ii)

Case III : $2 < x \leq 3$, then

$x - 1 + x - 2 + 3 - x \geq 6 \Rightarrow x \geq 6$

But $2 < x \leq 3 \Rightarrow$ No solution. ...(iii)

Case IV : $x > 3$, then

$x - 1 + x - 2 + x - 3 \geq 6 \Rightarrow x \geq 4$

Hence $x \geq 4$ ...(iv)

From (i), (ii), (iii) and (iv) the given inequality holds for $x \leq 0$ or $x \geq 4$.

Illustration 31 : Solve for x : (a) $||x-1|+2| \leq 4$. (b) $\dfrac{x-4}{x+2} \leq \left|\dfrac{x-2}{x-1}\right|$

Solution :

(a) $||x-1|+2| \leq 4 \Rightarrow \quad -4 \leq |x-1| + 2 \leq 4$

$\Rightarrow \quad -6 \leq |x-1| \leq 2$

$\Rightarrow \quad |x-1| \leq 2 \quad \Rightarrow \quad -2 \leq x - 1 \leq 2$

$\Rightarrow \quad -1 \leq x \leq 3 \quad \Rightarrow \quad x \in [-1, 3]$

(b) **Case 1 :** Given inequation will be statisfied for all x such that

$\dfrac{x-4}{x+2} \leq 0 \quad \Rightarrow \quad x \in (-2, 4] - \{1\}$ ....(i)

(Note : {1} is not in domain of RHS)

Case 2 : $\dfrac{x-4}{x+2} > 0 \quad \Rightarrow \quad x \in (-\infty, -2) \cup (4, \infty)$ ....(ii)

Given inequation becomes

$\dfrac{x-2}{x-1} \geq \dfrac{x-4}{x+2}$ or $\dfrac{x-2}{x-1} \leq -\dfrac{x-4}{x+2}$

on solving we get | on solving we get

$x \in \left(-2, 4/5\right) \cup (1, \infty)$ | $x \in (-2, 0] \cup (1, 5/2]$

taking intersection with (ii) we get | taking intersection with (ii) we get

$x \in (4, \infty)$ ....(iii) | $x \in \phi$

Hence, solution of the original inequation : $x \in (-2, \infty) - \{1\}$

(taking union of (i) & (iii)

Illustration 32 : The equation $|x| + \left|\dfrac{x}{x-1}\right| = \dfrac{x^2}{|x-1|}$ is always true for x belongs to

(A) $\{0\}$ (B) $(1, \infty)$ (C) $(-1, 1)$ (D) $(-\infty, \infty)$

Solution : $\dfrac{x^2}{|x-1|} = \left|x + \dfrac{x}{x-1}\right|$

$\because\ |x| + \left|\dfrac{x}{x-1}\right| = \left|x + \dfrac{x}{x-1}\right|$ is true only if $\left(x.\dfrac{x}{x-1}\right) \geq 0 \Rightarrow x \in \{0\} \cup (1, \infty)$.

Ans (A,B)

Do yourself-7

Solve for x $\in \mathbb{R}$

1.	$\lvert x\rvert \geq 10$	**2.**	$\lvert x - 5\rvert \leq 5$	**3.**	$\lvert 2 - x\rvert \leq 4$
4.	$1 \leq \lvert 2x - 3\rvert \leq 10$	**5.**	$\lvert x\rvert < \sqrt{2}$	**6.**	$\lvert x\rvert < 0$
7.	$\lvert 2x\rvert < 3$	**8.**	$\lvert 3x\rvert + 5 < 0$	**9.**	$\lvert 3x - 5\rvert < 2$
10.	$\lvert 1 - x\rvert \leq 2$	**11.**	$\lvert x\rvert > 2$	**12.**	$\lvert 2 - 7x\rvert < 8$
13.	$\lvert 2x\rvert + 4 > 0$	**14.**	$\lvert 2x\rvert - 7 > 0$		

10. MAXIMUM & MINIMUM VALUES OF QUADRATIC EXPRESSIONS :

$y = ax^2 + bx + c$:

We know that $y = ax^2 + bx + c$ takes following form : $y = a\left[\left(x + \dfrac{b}{2a}\right)^2 - \dfrac{(b^2-4ac)}{4a^2}\right]$,

which is a parabola. $\therefore$ vertex $= \left(\dfrac{-b}{2a}, \dfrac{-D}{4a}\right)$

When $a > 0$, y will take a minimum value at vertex ; $x = \dfrac{-b}{2a}$; $y_{min} = \dfrac{-D}{4a}$

When $a < 0$, y will take a maximum value at vertex; $x = \dfrac{-b}{2a}$; $y_{max} = \dfrac{-D}{4a}$.

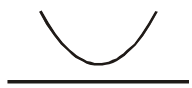
If quadratic expression ax^2+bx+c is a perfect square, then $a > 0$ and $D = 0$

Illustration 33 : The value of the expression $x^2 + 2bx + c$ will be positive for all real x if -

(A) $b^2 - 4c > 0$ (B) $b^2 - 4c < 0$ (C) $c^2 < b$ (D) $b^2 < c$

Solution : As $a > 0$, so this expression will be positive if $D < 0$

so $4b^2 - 4c < 0$



$b^2 < c$

Ans. (D)

Illustration 34 : The minimum value of the expression $4x^2 + 2x + 1$ is -

(A) 1/4 (B) 1/2 (C) 3/4 (D) 1

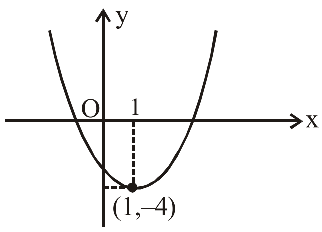
Solution : Since $a = 4 > 0$

therefore its minimum value $= -\dfrac{D}{4a} = \dfrac{4(4)(1)-(2)^2}{4(4)} = \dfrac{16-4}{16} = \dfrac{12}{16} = \dfrac{3}{4}$ **Ans. (C)**

Illustration 35: If $y = x^2 - 2x - 3$, then find the range of y when :

(i) $x \in R$ (ii) $x \in [0,3]$ (iii) $x \in [-2,0]$

Solution : We know that minimum value of y will occur at



$$x = -\frac{b}{2a} = -\frac{(-2)}{2\times 1} = 1$$

$$y_{min} = -\frac{D}{4a} = \frac{-(4+3\times 4)}{4} = -4$$

(i) $x \in R$;

$y \in [-4,\infty)$ **Ans.**

(ii) $x \in [0, 3]$

$f(0) = -3$, $f(1) = -4$, $f(3) = 0$

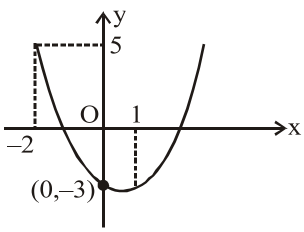
$\because$ $f(3) > f(0)$

$\therefore$ y will take all the values from minimum to f(3).

$y \in [-4, 0]$ **Ans.**

(iii) $x \in [-2, 0]$

This interval does not contain the minimum value of y for $x \in R$.



y will take values from f(0) to f(–2)

$f(0) = -3$

$f(-2) = 5$

$y \in [-3, 5]$ **Ans.**

Illustration 36 : If $ax^2 + bx + 10 = 0$ does not have real & distinct roots, find the minimum value of $5a-b$.

Solution : Either $f(x) \geq 0 \ \forall \ x \in R$ or $f(x) \leq 0 \ \forall \ x \in R$

$\because \quad f(0) = 10 > 0 \Rightarrow f(x) \geq 0 \ \forall \ x \in R$

$\Rightarrow \quad f(-5) = 25a - 5b + 10 \geq 0$

$\Rightarrow \quad 5a - b \geq -2$ **Ans.**

Illustration-37 Find possible values (range) of x^2 if

(i) $x \in [1,3]$ (ii) $x \in [-2,0]$ (iii) $x \in [-1,2]$

(iv) $x \in [-2,1]$ (v) $x \in [-2,2]$

Solution : (i) $1 \leq x \leq 3 \Rightarrow 1 \leq x^2 \leq 9 \Rightarrow x^2 \in [1,9]$

(ii) $-2 \leq x \leq 0 \Rightarrow 4 \geq x^2 \geq 0 \Rightarrow x^2 \in [0,4]$

(iii) $-1 \leq x < 0$ or $0 \leq x \leq 2$

$\Rightarrow 1 \geq x^2 > 0$ or $0 \leq x^2 \leq 4 \Rightarrow 0 \leq x^2 \leq 4 \Rightarrow x^2 \in [0,4]$

(iv) $-2 \leq x < 0$ or $0 \leq x \leq 1$

$\Rightarrow 4 \geq x^2 > 0$ or $0 \leq x^2 \leq 1 \Rightarrow x^2 \in [0,4]$

(v) $-2 \leq x < 0$ or $0 \leq x \leq 2$

$\Rightarrow 4 \geq x^2 > 0$ or $0 \leq x^2 \leq 4 \Rightarrow 0 \leq x^2 \leq 4 \Rightarrow x^2 \in [0,4]$

Illustration-38 : If $x \in \left[-\frac{3}{2}, 1\right]$ then find range of (i) $(2x-1)^2 - 3$ (ii) $4x^3 + 1$

Solution : (i) $[-3,13]$ (ii) $\left[-\frac{25}{2}, 5\right]$

Do yourself - 8

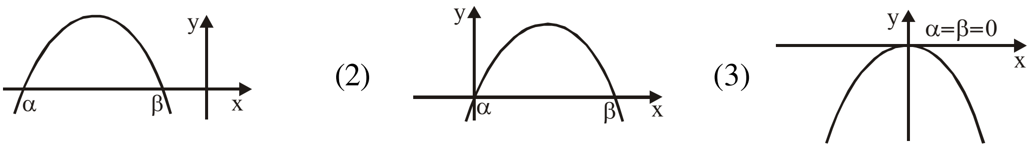
1. Find the minimum value of :

(a) $y = x^2 + 2x + 2$ (b) $y = 4x^2 - 16x + 15$

2. For following graphs of $y = ax^2 + bx + c$ with $a,b,c \in R$, comment on the sign of :

(i) a (ii) b (iii) c (iv) D (v) $\alpha + \beta$ (vi) $\alpha\beta$

(1) (2) (3)



3. Given the roots of equation $ax^2 + bx + c = 0$ are real & distinct, where $a,b,c \in R^+$, then the vertex of the graph will lie in which quadrant.

4. Find the range of 'a' for which :

(a) $ax^2 + 3x + 4 > 0 \quad \forall\ x \in R$ (b) $ax^2 + 4x - 2 < 0 \quad \forall\ x \in R$

5. The trinomial $ax^2 + bx + c$ has no real roots, $a + b + c < 0$. Find the sign of the number c.

6. For what values of a do the graphs of the functions $y = 2ax + 1$ and $y = (a - 6)x^2 - 2$ not intersect?

7. For what values of k is the inequality $x^2 - (k - 3)x - k + 6 > 0$ valid for all real x?

8. For what integral k is the inequality $x^2 - 2(4k - 1)x + 15k^2 - 2k - 7 > 0$ valid for any real x ?

9. For what values of a is the inequality $ax^2 + 2ax + 0.5 > 0$ valid for all real x ?

10. For what least integral k is the expression $(k - 2)x^2 + 8x + k + 4 > 0$ for all values of x ?

11. Find all values of a for which the inequality $(a - 3)x^2 - 2ax + 3a - 6 > 0$ is true for all $x \in R$.

11. MAXIMUM & MINIMUM VALUES OF RATIONAL ALGEBRAIC EXPRESSIONS :

$$y = \frac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}, \frac{1}{ax^2 + bx + c}, \frac{a_1x + b_1}{a_2x^2 + b_2x + c_2}, \frac{a_1x^2 + b_1x + c_1}{a_2x + b_2} :$$

Sometime we have to find range of expression of form $\dfrac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}$.

The following procedure is used :

Step 1 : Equate the given expression to y i.e. $y=\dfrac{a_1x^2+b_1x+c_1}{a_2x^2+b_2x+c_2}$

Step 2 : By cross multiplying and simplifying, obtain a quadratic equation in x.

$(a_1-a_2y)x^2+(b_1-b_2y)x+(c_1-c_2y)=0$

Step 3 : Put Discriminant ≥ 0 and solve the inequality for possible set of values of y.

Illustration : 39 If $x \in [-2, 3]$, find all possible values (Range) of

(i) $\dfrac{1}{x+1}$ (ii) $\dfrac{x+2}{x+4}$ (iii) $\dfrac{2x}{x-1}$

Solution : (i) $-2 \leq x \leq 3 \Rightarrow -1 \leq x+1 \leq 4$

(For $x+1=0$, $\dfrac{1}{x+1}$ is undefined)

so, $-1 \leq x+1 < 0$ or $0 < x+1 \leq 4$

$\Rightarrow -1 \geq \dfrac{1}{x+1} > -\infty$ or $\infty > x+1 \geq \dfrac{1}{4}$

So range of $\dfrac{1}{x+1}$ is $(-\infty, -1] \cup \left[\dfrac{1}{4}, \infty\right)$

(ii) $\dfrac{x+2}{x+4}=1-\dfrac{2}{x+4}$

Given $-2 \leq x \leq 3$

$\Rightarrow 2 \leq x+4 \leq 7 \Rightarrow \dfrac{1}{2} \geq \dfrac{1}{x+4} \geq \dfrac{1}{7}$

$\Rightarrow \dfrac{2}{2} \geq \dfrac{2}{x+4} \geq \dfrac{2}{7} \Rightarrow -1 \leq \dfrac{-2}{x+4} \leq \dfrac{-2}{7}$

$\Rightarrow 0 \leq 1-\dfrac{2}{x+4} \leq \dfrac{5}{7} \Rightarrow$Range $=\left[0, \dfrac{5}{7}\right]$

(iii) $\frac{2x}{x-1}=2+\frac{2}{x-1}$

for $-2 \le x \le 3 \Rightarrow -3 \le x-1 \le 2$

$\Rightarrow -3 \le x-1 < 0$ or $0 < x-1 \le 2$

$$\Rightarrow \frac{-2}{3} \ge \frac{2}{x-1} > -\infty \text{ or } \infty > \frac{2}{x-1} \ge \frac{2}{2}$$

$$\Rightarrow \frac{4}{3} \ge 2+\frac{2}{x-1} \text{ or } 2+\frac{2}{x-1} \ge 3$$

so range = $\left(-\infty, \frac{4}{3}\right] \cup [3, \infty)$

Illustration 40 : For $x \in R$, find the set of values attainable by $\frac{x^2-3x+4}{x^2+3x+4}$.

Solution : Let $y=\frac{x^2-3x+4}{x^2+3x+4}$

$x^2(y-1)+3x(y+1)+4(y-1)=0$

Case- I : $y \neq 1$

For $y \neq 1$ above equation is a quadratic equation.

So for $x \in R$, $D \ge 0$

$\Rightarrow \quad 9(y+1)^2-16(y-1)^2 \ge 0 \quad \Rightarrow \quad 7y^2-50y+7 \le 0$

$\Rightarrow \quad (7y-1)(y-7) \le 0 \quad \Rightarrow \quad y \in \left[\frac{1}{7}, 7\right]-\{1\}$

Case II : when $y = 1$

$\Rightarrow \quad 1=\frac{x^2-3x+4}{x^2+3x+4}$

$\Rightarrow \quad x^2+3x+4=x^2-3x+4$

$\Rightarrow \quad x=0$

Hence $y = 1$ for real value of x.

so range of y is $\left[\frac{1}{7}, 7\right]$

Illustration 41 : Find the values of a for which the expression $\frac{ax^2+3x-4}{3x-4x^2+a}$ assumes all real values for real values of x.

Solution : Let $y=\frac{ax^2+3x-4}{3x-4x^2+a}$

$x^2(a+4y)+3(1-y)x-(4+ay)=0$

If $x \in R$, $D \geq 0$

$\Rightarrow \quad 9(1-y)^2+4(a+4y)(4+ay) \geq 0 \Rightarrow (9+16a)y^2+(4a^2+46)y+(9+16a) \geq 0$

for all $y \in R$, $(9+16a) > 0$ & $D \leq 0$

$\Rightarrow \quad (4a^2+46)^2-4(9+16a)(9+16a) \leq 0 \Rightarrow 4(a^2-8a+7)(a^2+8a+16) \leq 0$

$\Rightarrow \quad a^2-8a+7 \leq 0 \quad \Rightarrow \quad 1 \leq a \leq 7$

$9+16a > 0$ & $1 \leq a \leq 7$

Taking intersection, $a \in [1, 7]$

Now, checking the boundary values of a

For $\quad a = 1$

$$y=\frac{x^2+3x-4}{3x-4x^2+1}=-\frac{(x-1)(x+4)}{(x-1)(4x+1)}$$

$\because \quad x \neq 1 \Rightarrow y \neq -1$

$\Rightarrow \quad a = 1$ is not possible.

if $\quad a = 7$

$$y=\frac{7x^2+3x-4}{3x-4x^2+7}=\frac{(7x-4)(x+1)}{(7-4x)(x+1)} \qquad \because x \neq -1 \Rightarrow y \neq -1$$

So y will assume all real values for some real values of x.

So $\quad a \in (1,7)$

Illustration 42 : Find the values of m so that the inequality : $\left|\frac{x^2+mx+1}{x^2+x+1}\right|<3$ holds for all $x \in R$.

Solution : We know that

$|a| < b \Rightarrow -b < a < b$ (for $b > 0$)

Hence $\left|\frac{x^2+mx+1}{x^2+x+1}\right|<3$. $\Rightarrow -3<\frac{x^2+mx+1}{x^2+x+1}<3$

Case I :

$$\frac{x^2+mx+1}{x^2+x+1}<3$$

$$\Rightarrow \frac{(x^2+mx+1)-3(x^2+x+1)}{x^2+x+1}<0 \Rightarrow \frac{-2x^2+(m-3)x-2}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}<0$$

Multiplying both sides by denominator, we get :

$\Rightarrow -2x^2 + (m-3)x - 2 < 0$ (because denominator is always positive)

$\Rightarrow 2x^2 - (m-3)x + 2 > 0$

A quadratic expression in x is always positive if coefficient of $x^2 > 0$ and $D < 0$.

$\Rightarrow (m-3)^2 - 4(2)(2) < 0 \Rightarrow (m-3)^2 - 4^2 < 0 \Rightarrow (m-7)(m+1) < 0 \Rightarrow m \in (-1, 7)$ ...(i)

Case II :

$$-3<\frac{x^2+mx+1}{x^2+x+1}$$

$$\Rightarrow \frac{(x^2+mx+1)+3(x^2+x+1)}{x^2+x+1}>0 \Rightarrow 4x^2+(m+3)x+4>0$$

For this to be true for all $x \in \mathbb{R}$, coefficient of $x^2 > 0$ and $D < 0$

$\Rightarrow (m+3)^2 - 4(4)(4) < 0$

$\Rightarrow (m+3-8)(m+3+8) < 0$

$\Rightarrow (m-5)(m+11) < 0$

$\Rightarrow m \in (-11, 5)$ ...(ii)

We will combine (i) and (ii) because both must be satisfied.

$\Rightarrow$ The common solution is $m \in (-1, 5)$.

Illustration 43 : Find the values of m for which the expression : $\frac{2x^2-5x+3}{4x-m}$ can take all real values for $x \in R-\left\{\frac{m}{4}\right\}$

Solution : Let $\frac{2x^2-5x+3}{4x-m}=y \Rightarrow 2x^2-(4y+5)x+3+my=0$

$\Rightarrow$ As $x \in R$, discriminant ≥ 0

$\Rightarrow$ $(4y+5)^2-8(3+my)\geq 0$

$\Rightarrow$ $16y^2+(40-8m)y+1\geq 0$ ...(i)

A quadratic in y is non-negative for all values of y if coefficient of y^2 is positive and discriminant ≤ 0.

$\Rightarrow$ $(40-8m)^2-4(16)(1)\leq 0 \Rightarrow (5-m)^2-1\leq 0$

$\Rightarrow$ $(m-5-1)(m-5+1)\leq 0 \Rightarrow (m-6)(m-4)\leq 0$

$\Rightarrow$ $m \in [4, 6]$ but $m = 4,6$ will be rejected as shown in Illustration 41

So for the given expression to take all real values, m should take values : $m \in (4, 6)$.

Doyourself - 9

1. If x is real prove that the expression $\frac{8x-4}{x^2+2x-1}$ where x is real cannot have values between 2 and 4, in its range.

2. Find the range of $\frac{x^2+2x+1}{x^2+2x+7}$, where x is real

3. If x is real, then prove that $\frac{x^2-x+1}{x^2+x+1}$ lies from $\frac{1}{3}$ to 3.

4. If x be real, prove that $\frac{x}{x^2-5x+9}$ must lie between 1 and $-\frac{1}{11}$ (both inclusive).

5. If x be real, prove that $\frac{x^2+34x-71}{x^2+2x-7}$ can have no value between 5 and 9.

6. Find the greatest value of $\frac{x+2}{2x^2+3x+6}$ for real values of x.

12. LOCATION OF ROOTS :

This article deals with an elegant approach of solving problems on quadratic equations when the roots are located / specified on the number line with variety of constraints :

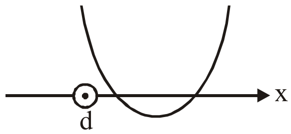
Consider the quadratic equation **$ax^2 + bx + c = 0$ with $a > 0$ and let $f(x) = ax^2 + bx + c$**

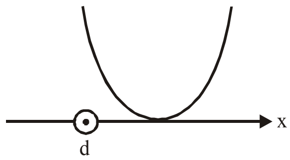
Type-1 :

Both roots of the quadratic equation are greater than a specific number (say d).

The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $-\dfrac{b}{2a} > d$





Note : When both roots of the quadratic equation are less than a specific number d than the necessary and sufficient condition will be :

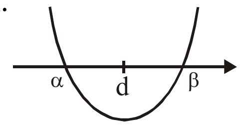
(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $-\dfrac{b}{2a} < d$

Type-2 :

Both roots lie on either side of a fixed number say (d). Alternatively one root is greater than 'd' and other root less than 'd' or 'd' lies between the roots of the given equation.

The necessary and sufficient condition for this are : $f(d) < 0$

Note : Consideration of discriminant is not needed.

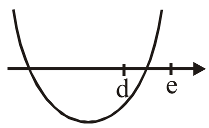


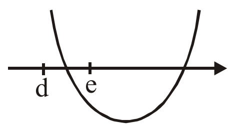
Type-3 :

Exactly one root lies in the interval (d, e).

The necessary and sufficient condition for this are :

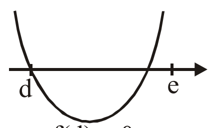
$f(d) \cdot f(e) < 0$



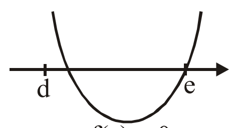


Note : The extremes of the intervals found by given conditions give 'd' or 'e' as the root of the equation.

Hence in this case also check for end points.



f(d) = 0



f(e) = 0

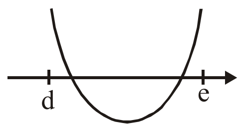
Type-4 :

When both roots are confined between the number d and e (d < e).

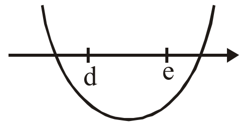
The necessary and sufficient condition for this are :

(i) $D \geq 0$; (ii) $f(d) > 0$; (iii) $f(e) > 0$

(iv) $d < -\dfrac{b}{2a} < e$



Type-5 :



One root is greater than e and the other roots is less than d (d < e).

The necessary and sufficient condition for this are : f(d) < 0 and f(e) < 0

Note : If a < 0 in the quadratic equation $ax^2 + bx + c = 0$ then we divide the whole equation by 'a'.

Now assume $x^2 + \frac{b}{a}x + \frac{c}{a}$ as f(x). This makes the coefficient of x^2 positive and hence above cases are applicable.

Illustration 44 : Let the quadratic equation $ax^2 + bx + c = 0$, $a,b,c \in R$ has real roots then find the conditions such that

(a) roots are equal in magnitude but opposite in sign

(b) one root is zero other is – b/a

(c) roots are reciprocal to each other

(d) roots are of opposite signs

(e) both roots are negative.

(f) both roots are positive.

(g) Greater root in magnitude is negative.

(h) Greater root in magnitude is positive.

(i) one root is 1 and second root is c/a or (–b–a)/a.

Solution

(a) $b = 0, D > 0$ **(b)** $c = 0$ **(c)** $a = c, D \geq 0$

(d) $a.c < 0$ **(e)** $D \geq 0, \frac{b}{a} > 0, \frac{c}{a} > 0$ **(f)** $D \geq 0, \frac{b}{a} < 0, \frac{c}{a} > 0$

(g) $\frac{b}{a} > 0, D \geq 0$ **(h)** $\frac{b}{a} < 0, D \geq 0$ **(i)** $a + b + c = 0$ and $a \neq 0$

Illustration 45 : Find the values of the parameter 'a' for which the roots of the quadratic equation $x^2 + 2(a-1)x + a + 5 = 0$ are

(i) real and distinct (ii) equal

(iii) opposite in sign (iv) equal in magnitude but opposite in sign

(v) positive (vi) negative

(vii) greater than 3 (viii) smaller than 3

(ix) such that both the roots lie in the interval (1, 3)

Solution : Let $f(x) = x^2 + 2(a-1)x + a + 5 = Ax^2 + Bx + C$ (say)

$\Rightarrow$ $A = 1, B = 2(a-1), C = a + 5$.

Also $D = B^2 - 4AC = 4(a-1)^2 - 4(a+5) = 4(a+1)(a-4)$

(i) $D > 0$

$\Rightarrow$ $(a+1)(a-4) > 0 \Rightarrow$ $a \in (-\infty, -1) \cup (4, \infty)$.

(ii) $D = 0$

$\Rightarrow$ $(a+1)(a-4) = 0 \Rightarrow$ $a = -1, 4$.

(iii) This means that 0 lies between the roots of the given equation.

$\Rightarrow$ $f(0) < 0$ and $D > 0$ i.e. $a \in (-\infty, -1) \cup (4, \infty)$

$\Rightarrow$ $a + 5 < 0 \Rightarrow a < -5 \Rightarrow a \in (-\infty, -5)$.

(iv) This means that the sum of the roots is zero

$\Rightarrow$ $-2(a-1) = 0$ and $D > 0$ i.e. $a \in (-\infty, -1) \cup (4, \infty)$ $\Rightarrow$ $a = 1$

which does not belong to $(-\infty, -1) \cup (4, \infty)$

$\Rightarrow$ $a \in \phi$.

(v) This implies that both the roots are greater than zero

$$\Rightarrow \quad -\frac{B}{A} > 0,\ \frac{C}{A} > 0, D \geq 0 \ \Rightarrow\ -(a-1) > 0,\ a + 5 > 0,\ a \in (-\infty, -1] \cup [4, \infty)$$

$\Rightarrow$ $a < 1, -5 < a, a \in (-\infty, -1] \cup [4, \infty) \Rightarrow$ $a \in (-5, -1]$.

(vi) This implies that both the roots are less than zero

$$\Rightarrow \quad -\frac{B}{A} < 0,\ \frac{C}{A} > 0, D \geq 0 \ \Rightarrow\ -(a-1) < 0,\ a + 5 > 0,\ a \in (-\infty, -1] \cup [4, \infty)$$

$\Rightarrow$ $a > 1, a > -5, a \in (-\infty, -1] \cup [4, \infty)$ $\Rightarrow$ $a \in [4, \infty)$.

(vii) In this case

$$-\frac{B}{2a} > 3,\ A.f(3) > 0 \text{ and } D \geq 0.$$

$\Rightarrow$ $-(a-1) > 3, 7a + 8 > 0$ and $a \in (-\infty, -1] \cup [4, \infty)$

$\Rightarrow$ $a < -2, a > -8/7$ and $a \in (-\infty, -1] \cup [4, \infty)$

Since no value of 'a' can satisfy these conditions simultaneously, there can be no value of a for which both the roots will be greater than 3.

(viii) In this case

$$-\frac{B}{2a} < 3, \ A.f(3) > 0 \text{ and } D \geq 0.$$

$\Rightarrow \quad a > -2, a > -8/7$ and $a \in (-\infty, -1] \cup [4, \infty) \Rightarrow \quad a \in (-8/7, -1] \cup [4, \infty)$

(ix) In this case

$$1 < -\frac{B}{2A} < 3, \ A.f(1) > 0, \ A.f(3) > 0, \ D \geq 0.$$

$\Rightarrow \quad 1 < -1(a-1) < 3, \ 3a + 4 > 0, \ 7a + 8 > 0, \ a \in (-\infty, -1] \cup [4, \infty)$

$\Rightarrow \quad -2 < a < 0, \ a > -4/3, \ a > -8/7, \ a \in (-\infty, -1] \cup [4, \infty) \Rightarrow a \in \left(-\frac{8}{7}, -1\right]$

Illustration 46 : Find value of k for which one root of equation $x^2 - (k+1)x + k^2 + k - 8 = 0$ exceeds 2 & other is less than 2.

Solution : $4 - 2(k+1) + k^2 + k - 8 < 0 \quad \Rightarrow \quad k^2 - k - 6 < 0$

$(k-3)(k+2) < 0 \quad \Rightarrow \ -2 < k < 3$

Taking intersection, $k \in (-2, 3)$.

Illustration 47 : Find all possible values of a for which exactly one root of $x^2 - (a+1)x + 2a = 0$ lies in interval (0,3).

Solution : $f(0) \,.\, f(3) < 0$

$\Rightarrow \quad 2a(9 - 3(a+1) + 2a) < 0 \quad \Rightarrow \quad 2a(-a+6) < 0$

$\Rightarrow \quad a(a-6) > 0 \quad \Rightarrow \ a < 0 \text{ or } a > 6$

Checking the extremes.

If $a = 0$, $\quad x^2 - x = 0$

$x = 0, 1$

$1 \in (0, 3)$

If $a = 6$, $\quad x^2 - 7x + 12 = 0$

$x = 3, 4 \qquad$ But $4 \notin (0, 3)$

Hence solution set is

$a \in (-\infty, 0] \cup (6, \infty)$

Illustration 48 : Let $x^2 - (m - 3)x + m = 0$ $(m \in R)$ be a quadratic equation. Find the value of m for which the roots of the equation are

(i) real and distinct

(ii) equal

(iii) not real

(iv) opposite in sign

(v) equal in magnitude but opposite in sign

(vi) positive

(vii) negative

(viii) such that at least one is positive

(ix) one root is smaller than 2 and the other root is greater than 2

(x) both the roots are greater than 2

(xi) both the roots are smallers than 2

(xii) exactly one root lies in the interval (1, 2)

(xiii) both the roots lie in the interval (1, 2)

(xiv) such that at least one root lie in the interval (1, 2)

(xv) one root is greater than 2 and the other root is smaller than 1

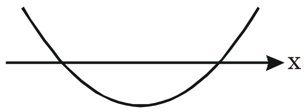
Solution :

Let $f(x) = x^2 - (m - 3)x + m$

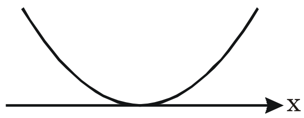
(i) Both the roots are real and distinct $\Rightarrow D > 0$

$\Rightarrow (m - 3)^2 - 4m > 0 \Rightarrow m^2 - 10m + 9 > 0$

$\Rightarrow (m - 1)(m - 9) > 0 \Rightarrow m \in (-\infty, 1) \cup (9, \infty)$

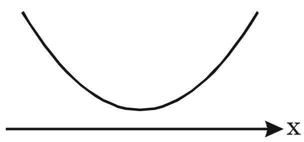


(ii) Both the roots are equal $\Rightarrow D = 0 \Rightarrow m = 9$ or $m = 1$



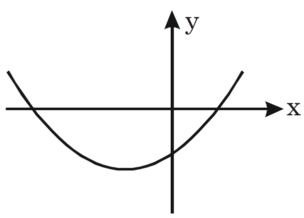
(iii) Both the roots are imaginary $\Rightarrow D < 0$

$(m - 1)(m - 9) < 0 \Rightarrow m \in (1, 9)$



(iv) Both the roots are opposite in sign

$\Rightarrow f(0) < 0 \Rightarrow m < 0 \Rightarrow m \in (-\infty, 0)$

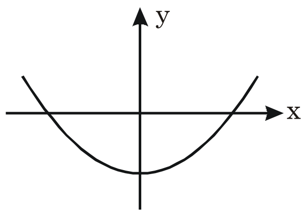


(v) Roots are equal in magnitude but opposite in sign

$\Rightarrow$ sum of roots is zero as well as $D \geq 0$

$\Rightarrow m \in (-\infty, 1] \cup [9, \infty)$ and $m - 3 = 0$ i.e., $m = 3$

$\Rightarrow$ no such m exists

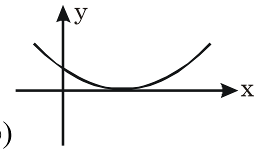


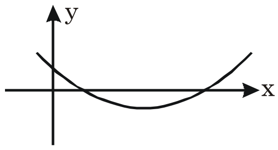
(vi) Both the roots are positive $\Rightarrow D \geq 0$,

both sum and product of roots are positive

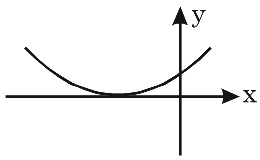
$\Rightarrow m - 3 > 0, m > 0$ and $m \in (-\infty, 1] \cup [9, \infty)$

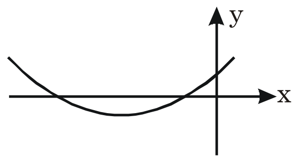
$\Rightarrow m \in [9, \infty)$





(vii) Both the roots are negative

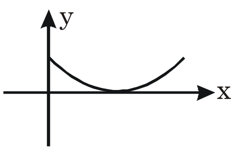


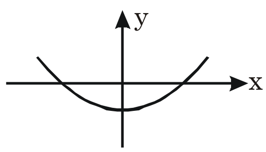


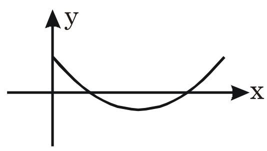
$\Rightarrow D \geq 0$, sum of roots is negative but product of roots is positive.

$\Rightarrow m - 3 < 0, m > 0, m \in (-\infty, 1] \cup [9, \infty) \Rightarrow m \in (0, 1]$

(viii) at least one root is positive





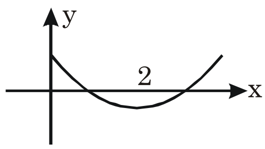


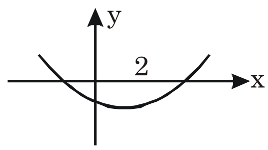
$\Rightarrow$ either one root is positive or both the roots are positive

$\Rightarrow$ union of (iv) and (vi) with $m = 0$

(i.e., one root is zero and in this case other root becomes negative) $\Rightarrow m \in (-\infty, 0) \cup [9, \infty)$

(ix) one root is smaller than 2 and



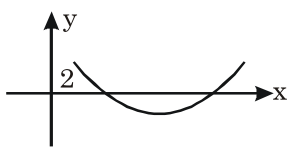


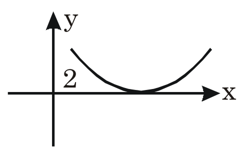
other root is greater than 2 $\Rightarrow$ 2 lies between the roots

$\Rightarrow f(2) < 0 \Rightarrow 4 - 2(m - 3) + m < 0 \Rightarrow m > 10$

(x) Both roots are greater than 2

$\Rightarrow f(2) > 0, D \geq 0, \dfrac{-b}{2a} > 2$

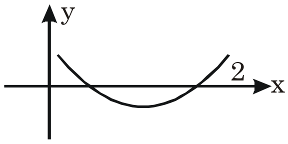


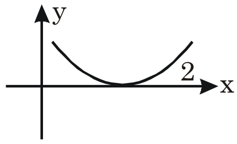


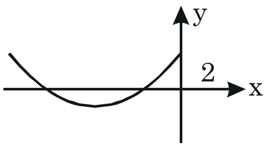
$\Rightarrow m < 10$ and $m \in (-\infty, 1] \cup [9, \infty)$ and $m - 3 > 4$

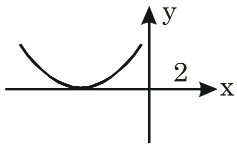
$\Rightarrow m \in [9, 10)$

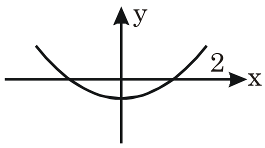
(xi) Both the roots are smaller than 2





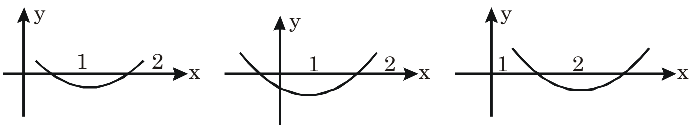






$\Rightarrow f(2) > 0,\ D \geq 0,\ \dfrac{-b}{2a} < 2 \Rightarrow m \in (-\infty, 1]$

(xii) Exactly one root lies between (1, 2)

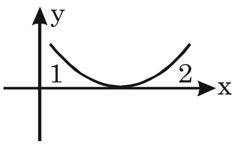


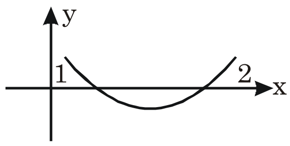
$\Rightarrow f(1).f(2) < 0 \Rightarrow 4(10 - m) < 0 \Rightarrow m \in (10, \infty)$

(xiii) Both roots lie in the interval (1, 2). Then

$f(1) > 0,\ 1 - m + 3 + m > 0 \Rightarrow 4 > 0$

$f(2) > 0,\ 4 - 2m + 6 + m > 0 \Rightarrow m < 10$

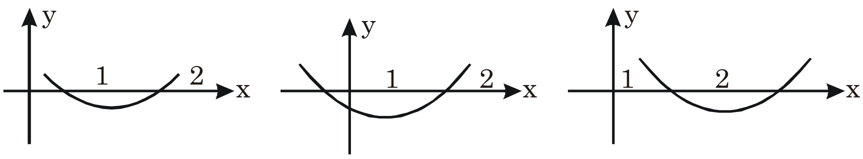




$1 < \dfrac{m-3}{2} < 2 \Rightarrow 5 < m < 7$

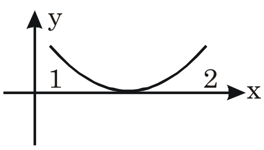
$D \geq 0,\ m \in (-\infty, 1] \cup [9, \infty) \Rightarrow$ no solution

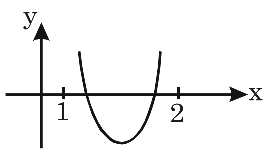
(xiv) **Case I :** Exactly one root lies in (1, 2)



$f(1).f(2) < 0 \Rightarrow m > 10$

Case II : Both roots lie in (1, 2)

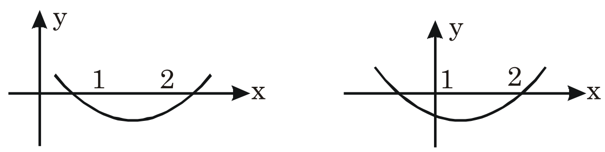




From (xiii) $m \in \phi$

At least one root lies in $(1, 2) \Rightarrow m \in (10, \infty)$

(xv) For one root greater than 2 and other is smaller than 1, conditions are



$f(1) < 0$ ...(1)

and $f(2) < 0$ ...(2)

From (1), $f(1) < 0$, but $f(1) = 4$ which is not possible

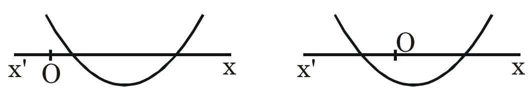
Thus no such 'm' exists.

Illustration 49 : Find the value(s) of 'a' for which $ax^2 + (a - 3)x + 1 < 0$ for at least one positive x.

Solution : Let $f(x) = ax^2 + (a - 3)x + 1$

Case I :

If $a > 0$, then $f(x)$ will be negative only for those values of x, which lie between the roots. From the graph we can see that, $f(x)$ will be less than zero for at least one positive real x, when $f(x) = 0$ has distinct roots and at least one of these roots is positive real root.



For this D > 0, i.e., $(a - 3)^2 - 4a > 0$

$\Rightarrow \quad a < 1$ or $a > 9$ ...(1)

It is easy to see that for figure 1, $a < 3$ and figure 2 is not possible as $f(0) = 1$.

Hence for case-I $0 < a < 1$.

Case II :

If $a < 0$, then $f(x)$ will certainly be negative for infintely many positive x. Thus $a \in (-\infty, 0)$.

Case III :

if $a = 0$, $f(x) = -3x + 1 \Rightarrow f(x) < 0 \ \forall \ x > 1/3$

Hence the required set of values of 'a' is $(-\infty, 1)$

Illustration 50 : Find the values of 'a' for which $4^t - (a-4)2^t + \frac{9}{4}a < 0 \ \forall \ t \in (1, 2)$.

Solution : Let $2^t = x$ and $f(x) = x^2 - (a-4)x + \frac{9}{4}a$

We want

$f(x) < 0 \ \forall \ x \in (2^1, 2^2)$ i.e., $\forall \ x \in (2, 4)$

Since we want $f(x) < 0 \ \forall \ x \in (2, 4)$, one of the roots of $f(x)$ should be less than or equal to 2 and the other must be greater than or equal to 4

i.e., $f(2) \le 0$ and $f(4) \le 0 \Rightarrow a \le -48$ and $a \ge 128/7$, which is not possible.

Hence no such 'a' exists.

Illustration 51 : Find the value(s) of 'a' for which the inequality $\tan^2 x + (a+1)\tan x - (a-3) < 0$, is true for at least one $x \in \left(0, \frac{\pi}{2}\right)$.

Solution : The required condition will be satisfied if

(i) The quadratic expression (quadratic in tanx)

$f(x) = \tan^2 x + (a+1)\tan x - (a-3)$ has positive discriminant, and

(ii) At least one root of $f(x) = 0$ is positive, as $\tan x > 0, \ \forall \ x \in (0, \pi/2)$

For (i) Discriminant $> 0 \Rightarrow (a+1)^2 + 4(a-3) > 0$

$\Rightarrow \quad a > 2\sqrt{5} - 3$ or $a < -\left(2\sqrt{5}+3\right)$

For (ii), we first find the condition, that both the roots of $t^2 + (a+1)t - (a-3) = 0$

$(t = \tan x)$ are non-positive for which

Sum of roots < 0 product of roots ≥ 0

$\Rightarrow -(a+1) < 0$ and $-(a-3) \ge 0 \Rightarrow -1 < a \le 3$

Condition (ii) will be fulfilled if $a \le -1$ or $a > 3$ ...(2)

Required values of a is given by intersection of (1) and (2)

Hence $a \in \left(-\infty, -\left(2\sqrt{5}+3\right)\right) \cup (3, \infty)$

Do yourself - 10

1. If α, β are roots of $7x^2 + 9x - 2 = 0$, find their position with respect to following $(\alpha < \beta)$

   (a) -3 (b) 0 (c) 1

2. If $a > 1$, roots of the equation $(1 - a)x^2 + 3ax - 1 = 0$ are -

   (a) one positive one negative (b) both negative

   (c) both positive (d) both non-real

3. Find the set of value of a for which the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are less than 3.

4. If α, β are the roots of $x^2 - 3x + a = 0$, $a \in R$ and $\alpha < 1 < \beta$, then find the values of a.

5. If α, β are roots of $4x^2 - 16x + \lambda = 0$, $\lambda \in R$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then find the range of λ.

6. For what values of a does the equation $(2 - x)(x + 1) = a$ posses real and positive roots ?

13. GENERAL QUADRATIC EXPRESSION IN TWO VARIABLES :

$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors if ;

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \text{OR} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Illustration 52 : If $x^2 + 2xy + 2x + my - 3$ have two linear factor then m is equal to -

(A) 6, 2 (B) –6, 2 (C) 6, –2 (D) –6, –2

Solution : Here $a = 1, h = 1, b = 0, g = 1, f = m/2, c = -3$

$$\text{So} \quad \Delta = 0 \Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & m/2 \\ 1 & m/2 & -3 \end{vmatrix} = 0$$

$$\Rightarrow -\frac{m^2}{4} - (-3 - m/2) + m/2 = 0 \qquad \Rightarrow -\frac{m^2}{4} + m + 3 = 0$$

$$\Rightarrow m^2 - 4m - 12 = 0 \qquad \Rightarrow m = -2, 6$$

Ans. (C)

Do yourself - 11

1. Find the value of k for which the expression $x^2 + 2xy + ky^2 + 2x + k = 0$ can be resolved into two linear factors.
2. For what values of m will the expression $y^2 + 2xy + 2x + my - 3$ be capable of resolution into two rational factors ?
3. Find the value of m which will make $2x^2 + mxy + 3y^2 - 5y - 2$ equivalent to the product of two linear factors.
4. Find the condition that the expression $\ell x^2 + mxy + ny^2$, $\ell' x^2 + m'xy + n'y^2$ may have a common linear factor.
5. Find the condition that the expression $ax^2 + 2hxy + by^2$, $a'x^2 + 2h'xy + b'y^2$may be respectively divisible by factors of the form y - mx,my + x.
6. If x and y are two real quantities connected by the equation $9x^2 + 2xy + y^2 - 92x - 20y + 244 = 0$, then show that x will lie between 3 and 6. and y between 1 and 10.

14. THEORY OF EQUATIONS :

Let $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ are roots of the equation, $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-1}x + a_n = 0$, where $a_0, a_1, \ldots a_n$ are constants and $a_0 \neq 0$.

$f(x) = a_0(x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \ldots (x - \alpha_n)$

$\therefore \quad a_0x^n + a_1x^{n-1} + \ldots a_{n-1}x + a_n = a_0(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)$

Comparing the coefficients of like powers of x, we get

$$\sum \alpha_i = -\frac{a_1}{a_0} = S_1 \text{ (say)}$$

or $$S_1 = -\frac{\text{coefficient of } x^{n-1}}{\text{coefficient of } x^n}$$

$$S_2 = \sum_{i \neq j} \alpha_i \alpha_j = (-1)^2 \frac{a_2}{a_0}$$

$$S_3 = \sum_{i \neq j \neq k} \alpha_i \alpha_j \alpha_k = (-1)^3 \frac{a_3}{a_0}$$

$\vdots$

$$S_n = \alpha_1 \alpha_2 \ldots \alpha_n = (-1)^n \frac{a_n}{a_0} = (-1)^n \frac{\text{constant term}}{\text{coefficient of } x^n}$$

where S_k denotes the sum of the product of root taken k at a time.

Quadratic equation : If α, β are the roots of the quadratic equation $ax^2 + bx + c = 0$,

then $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$

Cubic equation : If α, β, γ are roots of a cubic equation $ax^3 + bx^2 + cx + d = 0$,

then $\alpha+\beta+\gamma=-\frac{b}{a}$, $\alpha\beta+\beta\gamma+\gamma\alpha=\frac{c}{a}$ and $\alpha\beta\gamma=-\frac{d}{a}$

Note :

(i) If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x-\alpha)$ or $(x-\alpha)$ is a factor of $f(x)$ and conversely.

(ii) Every equation of nth degree ($n \geq 1$) has exactly n root & if the equation has more than n roots, it is an identity.

(iii) If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. **imaginary roots occur in conjugate pairs.**

(iv) If the coefficients in the equation are all rational & $\alpha+\sqrt{\beta}$ is one of its roots, then $\alpha-\sqrt{\beta}$ is also a root where $\alpha,\beta \in Q$ & β is not a perfect square.

(v) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then $f(x) = 0$ must have atleast one real root between 'a' and 'b'.

(vi) Every equation $f(x) = 0$ of degree odd and leading coefficient positive has atleast one real root of a sign opposite to that of its non zero constant term.

Illustration 53: If two roots are equal, find the roots of $4x^3 + 20x^2 - 23x + 6 = 0$.

Solution : Let roots be α, α and β

$\therefore \quad \alpha + \alpha + \beta = -\frac{20}{4} \quad \Rightarrow \quad 2\alpha + \beta = -5 \quad$ .... (i)

$\therefore \quad \alpha \cdot \alpha + \alpha\beta + \alpha\beta = -\frac{23}{4} \quad \Rightarrow \quad \alpha^2 + 2\alpha\beta = -\frac{23}{4}$ & $\alpha^2\beta = -\frac{6}{4}$

from equation (i)

$\alpha^2 + 2\alpha(-5-2\alpha) = -\frac{23}{4} \Rightarrow \alpha^2 - 10\alpha - 4\alpha^2 = -\frac{23}{4} \Rightarrow 12\alpha^2 + 40\alpha - 23 = 0$

$\therefore \quad \alpha = 1/2, -\frac{23}{6}$

when $\alpha = \frac{1}{2}$

$\alpha^2\beta = \frac{1}{4}(-5-1) = -\frac{3}{2}$

when $\alpha = -\frac{23}{6} \Rightarrow \alpha^2\beta = \frac{23\times23}{36}\left(-5-2x\left(-\frac{23}{6}\right)\right) \neq -\frac{3}{2} \quad \Rightarrow \quad \alpha = \frac{1}{2} \qquad \beta = -6$

Hence roots of equation $= \frac{1}{2}, \frac{1}{2}, -6$ **Ans.**

Illustration 54 : If α, β, γ are the roots of $x^3 - px^2 + qx - r = 0$, find :

(i) $\sum \alpha^3$ (ii) $\alpha^2(\beta+\gamma)+\beta^2(\gamma+\alpha)+\gamma^2(\alpha+\beta)$

Solution : We know that $\alpha + \beta + \gamma = p$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$

$\alpha\beta\gamma = r$

(i) $\alpha^3+\beta^3+\gamma^3 = 3\alpha\beta\gamma+(\alpha+\beta+\gamma)\{(\alpha+\beta+\gamma)^2-3(\alpha\beta+\beta\gamma+\gamma\alpha)\}$

$= 3r + p\{p^2 - 3q\} = 3r + p^3 - 3pq$

(ii) $\alpha^2(\beta+\gamma)+\beta^2(\alpha+\gamma)+\gamma^2(\alpha+\beta) = \alpha^2(p-\alpha)+\beta^2(p-\beta)+\gamma^2(p-\gamma)$

$= p(\alpha^2+\beta^2+\gamma^2)-3r-p^3+3pq = p(p^2-2q)-3r-p^3+3pq = pq-3r$

Illustration 55 : If $b^2 < 2ac$ and $a, b, c, d \in \mathbb{R}$, then prove that $ax^3 + bx^2 + cx + d = 0$ has exactly one real root.

Solution : Let α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$

Then $\alpha + \beta + \gamma = -\frac{b}{a}$

$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$

$\alpha\beta\gamma = \frac{-d}{a}$

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}$

$\Rightarrow$ $\alpha^2 + \beta^2 + \gamma^2 < 0$, which is not possible if all α, β, γ are real. So atleast one root is non-real, but complex roots occurs in pair. Hence given cubic equation has two non-real and one real roots.

Illustration 56 : If the sum of two roots of the equation $x^3 - px^2 + qx - r = 0$ is zero, then prove that $pq = r$.

Solution : Let the roots of the given equation by α, β, γ such that $\alpha + \beta = 0$, Then,

$$\alpha+\beta+\gamma=-\frac{(-p)}{1} \Rightarrow \alpha+\beta+\gamma = p \Rightarrow \gamma = p \qquad [\because \alpha+\beta=0]$$

But γ is a root of the given equation. Therefore,

$\gamma^3 - p\gamma^2 + q\gamma - r = 0 \Rightarrow p^3 - p^3 + qp - r = 0 \Rightarrow pq = r$

Illustration 57 : Find the condition that the roots of the equation $x^3 - px^2 + qx - r = 0$ may be in A.P.

Solution : Let the roots of the given equation be $a - d, a, a + d$.

Then,

$$(a-d)+a+(a+d) = \frac{-(-p)}{1} \Rightarrow a = p/3$$

Since a is a root of the given equation. Therefore,

$$a^3 - pa^2 + qa - r = 0 \Rightarrow \frac{p^3}{27} - \frac{p^3}{9} + \frac{qp}{3} - r = 0 \Rightarrow 2p^3 - 9pq + 27r = 0$$

This is the required condition.

Illustration 58 : If $x^2 + x + 1$ is a factor of $ax^3 + bx^2 + cx + d$, then find the real root of $ax^3 + bx^2 + cx + d = 0$.

Solution : $x^2 + x + 1$ is a factor of $ax^3 + bx^2 + cx + d$. So roots of $x^2 + x + 1 = 0$ are also the roots of the equation $ax^3 + bx^2\ cx + d = 0$.

Let third roots of $ax^3 + bx^2 + cx + d$ is α.

Now we can write,

$ax^3 + bx^2 + cx + d = a(x^2 + x + 1)(x - \alpha)$

Comparing constant term on both sides, we get : $d = -a\alpha$

Hence, the real root of $ax^3 + bx^2 + cx + d = 0$ is $-\frac{d}{a}$.

Note : Roots of $x^2 + x + 1 = 0$ are complex

Do yourself - 12

1. Let α, β be two of the roots of the equation $x^3 - px^2 + qx - r = 0$. If $\alpha + \beta = 0$, then show that $pq = r$
2. If two roots of $x^3 + 3x^2 - 9x + c = 0$ are equal, then find the value of c.
3. If α, β, γ be the roots of $ax^3 + bx^2 + cx + d = 0$, then find the value of

(a) $\sum \alpha^2$ (b) $\sum \frac{1}{\alpha}$ (c) $\sum \alpha^2(\beta+\gamma)$

Miscellaneous Illustrations :

Illustrations 59: If α, β are the roots of $x^2 + px + q = 0$, and γ, δ are the roots of $x^2 + rx + s = 0$, evaluate $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$ in terms of p, q, r and s. Deduce the condition that the equations have a common root.

Solution : α, β are the roots of $x^2 + px + q = 0$

$\therefore \quad \alpha + \beta = -p, \alpha\beta = q$ ....(1)

and γ, δ are the roots of $x^2 + rx + s = 0$

$\therefore \quad \gamma + \delta = -r, \quad \gamma\delta = s$ ....(2)

Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta)$

$= [\alpha^2 - \alpha(\gamma + \delta) + \gamma\delta][\beta^2 - \beta(\gamma + \delta) + \gamma\delta]$

$= (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s)$

$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s(\alpha^2 + \beta^2) + sr(\alpha + \beta) + s^2$

$= \alpha^2\beta^2 + r\alpha\beta(\alpha + \beta) + r^2\alpha\beta + s((\alpha + \beta)^2 - 2\alpha\beta)) + sr(\alpha + \beta) + s^2$

$= q^2 - pqr + r^2q + s(p^2 - 2q) + sr(-p) + s^2$

$= (q - s)^2 - rpq + r^2q + sp^2 - prs$

$= (q - s)^2 - rq(p - r) + sp(p - r)$

$= (q - s)^2 + (p - r)(sp - rq)$

For a common root (Let $\alpha = \gamma$ or $\beta = \delta$) ....(3)

then $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = 0$ ....(4)

from (3) and (4), we get

$(q - s)^2 + (p - r)(sp - rq) = 0$

$\Rightarrow \quad (q - s)^2 = (p - r)(rq - sp)$, which is the required condition.

Illustrations 60 : If $(y^2 - 5y + 3)(x^2 + x + 1) < 2x$ for all $x \in R$, then find the interval in which y lies.

Solution : $(y^2 - 5y + 3)(x^2 + x + 1) < 2x, \ \forall \ x \in R$

$\Rightarrow \quad y^2 - 5y + 3 < \dfrac{2x}{x^2 + x + 1}$

Let $\dfrac{2x}{x^2 + x + 1} = P$

$\Rightarrow \quad px^2 + (p - 2)x + p = 0$

(1) Since x is real, $(p - 2)^2 - 4p^2 \geq 0$

$\Rightarrow \quad -2 \leq p \leq \dfrac{2}{3}$

(2) The minimum value of $2x/(x^2 + x + 1)$ is -2.

So, $y^2 - 5y + 3 < -2 \quad \Rightarrow \quad y^2 - 5y + 5 < 0$

$\Rightarrow \quad y \in \left(\dfrac{5 - \sqrt{5}}{2}, \dfrac{5 + \sqrt{5}}{2}\right)$

EXERCISE (O-1)

1. If the roots of the equation $x^2 - 5x + 16 = 0$ are α, β and the roots of the equation $x^2 + px + q = 0$ are $(\alpha^2 + \beta^2)$ and $\frac{\alpha\beta}{2}$, then -

(A) $p = 1$ and $q = 56$ (B) $p = 1$ and $q = -56$

(C) $p = -1$ and $q = 56$ (D) $p = -1$ and $q = -56$

2. If the roots of the equation $x^2 + px + q = 0$ are 8 and 2 and the roots of $x^2 + rx + s = 0$ are 3 and 3, then roots of $x^2 + px + s = 0$ are

(A) –1, –9 (B) 1, 9 (C) 8, 3 (D) None

3. If α and β be the roots of the equation $(x - a)(x - b) = c$ and $c \neq 0$, then roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are-

(A) a and c (B) b and c (C) a and b (D) a + b and b + c

4. If $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$ then the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ is-

(A) 19/3 (B) 25/3 (C) –19/3 (D) none of these

5. The value of a for which one roots of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other is

(A) – 2/3 (B) 1/3 (C) – 1/3 (D) 2/3

6. The number of real solutions of the equation $x^2 - 3|x| + 2 = 0$, is-

(A) 4 (B) 1 (C) 3 (D) 2

7. If $(1 - p)$ is a root of quadratic equation $x^2 + px + (1 - p) = 0$ then its roots are-

(A) 0, – 1 (B) – 1, 1 (C) 0, 1 (D) –1, 2

8. If the roots of the equation $x^2 - bx + c = 0$ be two consecutive integers, then $b^2 - 4c$ equals-

(A) 1 (B) 2 (C) 3 (D) –2

9. If $x^2 - A(x + 1) + C = 0$ has roots x_1 & x_2, then the value of $x_1^2 + x_2^2 + (2 + A)x_1x_2$, is-

(A) AC (B) $A^2 + AC$ (C) $A^2 - AC$ (D) –AC

10. The quadratic $x^2 + ax + b + 1 = 0$ has roots which are positive integers, then $(a^2 + b^2)$ can be equal to-

(A) 50 (B) 17 (C) 29 (D) 53

11. **Statement-1 :** If sum of the roots of quadratic equation $2x^2 + bx + c = 0$ is equal to sum of the squares of the roots, then $b^2 + 2b = 4c$.

Statement-2 : If one root of quadratic equation $x^2 + px + q = 0$, is $4+\sqrt{3}$, then other root is $4-\sqrt{3}$.

(A) Statement-1 is true and Statement-2 is false.

(B) Statement-1 and Statement-2 both are true.

(C) Statement-1 is false and Statement-2 is true.

(D) Statement-1 and Statement-2 both are false.

12. If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, then the value of 'q' is-

(A) 3 (B) 12 (C) 49/4 (D) 4

13. The sum of the values of m for which the quadratic polynomial $P(x) = x^2 + (m + 5)x + (5m + 1)$ is a perfect square ($m \in R$) is

(A) 3 (B) 7 (C) 8 (D) 10

14. If equations $x^2 - 5x + 5 = 0$ and $x^3 + ax^2 + bx + 5 = 0$ have common root, then value of a + b ($a, b \in Q$) is -

(A) 4 (B) –4 (C) 0 (D) can't find

15. If the equation $ax^2 + bx + c = 0$ has distinct real roots, both negative, then-

(A) a,b,c must be of same sign

(B) a,b must be of opposite sign

(C) a,c must be of opposite sign

(D) a,b must be of same sign and opposite to sign of c

16. If $P(x) = x^2 - (2 - p)x + p - 2$ assumes both positive and negative value, then the complete set of values of 'p' is-

(A) $(-\infty,2)$ (B) $(6,\infty)$ (C) $(2,6)$ (D) $(-\infty,2) \cup (6, \infty)$

17. If value of a for which the sum of the squares of the roots of the equation $x^2 - (a - 2)x - a - 1 = 0$ assume the least value is-

(A) 2 (B) 3 (C) 0 (D) 1

18. If $x^2 + 2ax + 10 - 3a > 0$ for all $x \in R$, then

(A) $-5 < a < 2$ (B) $a < -5$ (C) $a > 5$ (D) $2 < a < 5$

19. If the expression $y = 8x - x^2 - 15$ is negative, then x lies in the interval-

(A) $(3,5)$ (B) $(5,50)$ (C) $(3,\infty)$ (D) $(-\infty,3) \cup (5,\infty)$

20. Let $f(x) = ax^2 + bx + 8$ $(a,b \in R)$ be a quadratic polynomial whose graph is symmetric about the line $x = 2$. If minimum value of $f(x)$ is 6, then the value of $2a - b$ is-

(A) 0 (B) 1 (C) 2 (D) 3

21. Let $g(x) = x^2 - (b + 1)x + (b - 1)$, where b is a real parameter. The largest natural number b satisfying $g(x) > -2\ \forall\ x \in R$, is -

(A) 1 (B) 2 (C) 3 (D) 4

22. $y = x^2 - 6x + 5$, $x \in [2,4]$, then-

(A) least value of y is –3 (B) least value of y is 3
(C) greatest value of y is 4 (D) greatest value of y is –3

23. Range of the expression $\frac{16x^2 - 12x + 9}{16x^2 + 12x + 9} : (x \in R)$ is-

(A) $\left[\frac{1}{3}, 3\right]$ (B) $\left(-\infty, \frac{1}{3}\right]$ (C) $[3, \infty)$ (D) R

24. If the roots of equation
$(4p - p^2 - 5)x^2 - (2p - 1)x + 3p = 0$ lie on either side of unity then the number of integral values of p is-

(A) 4 (B) 2 (C) 3 (D) 1

25. If exactly one root of the equation $2^k x^2 - 4^k x + 2^k - 1 = 0$ lies in $[0,1)$, then complete range of k is-

(A) $(-\infty,0]$ (B) $(-\infty,0)$ (C) $(0,\infty)$ (D) $[0,\infty)$

26. If $\forall\ p \in R$ one root of the equation $x^2 + 2px + q^2 - p^2 - 6 = 0$ is less than 1 and other root is greater than 1, then range of q is -

(A) $(-\infty, -2)$ (B) $(-2, 2)$ (C) $\left(-\sqrt{5}, \sqrt{5}\right)$ (D) $(2, \infty)$

27. Let $f(x) = 2x^2 + px + 1$ is given. If $f(x)$ is negative integer for only one real value of x, then product of all possible values of p is -

(A) –3 (B) –16 (C) 5 (D) –7

28. Let r_1, r_2, r_3 be roots of equation $x^3 - 2x^2 + 4x + 5074 = 0$, then the value of $(r_1 + 2)(r_2 + 2)(r_3 + 2)$ is

(A) 5050 (B) –5050 (C) –5066 (D) –5068

29. Let a, b, c are roots of equation $x^3 + 8x + 1 = 0$, then the value of

$$\frac{bc}{(8b+1)(8c+1)} + \frac{ac}{(8a+1)(8c+1)} + \frac{ab}{(8a+1)(8b+1)}$$ is equal to

(A) 0 (B) –8 (C) –16 (D) 16

30. Let $f(x) = x^3 + x + 1$ and P(x) be a cubic polynomial such that $P(0) = -1$ and the roots of $P(x) = 0$ are the squares of the roots of $f(x) = 0$, then value of P(9) is -

(A) 98 (B) 899 (C) 80 (D) 898

EXERCISE (O-2)

1. If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009} =$

(A) -2 (B) -1 (C) 1 (D) 2

2. The sum of all the real values of x satisfying the equation $2^{(x-1)(x^2+5x-50)} = 1$ is :

(A) 16 (B) -5 (C) -4 (D) 14

3. The number of integral values of m for which the equation

$(1 + m^2)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has no real root is :

(A) infinitely many (B) 2 (C) 3 (D) 1

4. Sum of all distinct integral value(s) of α such that equation $x^2 - \alpha x + \alpha + 1 = 0$ has integral roots, is equal to-

(A) 2 (B) 4 (C) 3 (D) None of these

5. For the equation, $3x^2 + px + 3 = 0$, $p > 0$ if one of the roots is square of the other, then p is equal to -

(A) 1/3 (B) 1 (C) 3 (D) 2/3

6. If one root of the equation $x^2 + px + q = 0$ is the square of the other, then

(A) $p^3 + q^2 - q(3p + 1) = 0$ (B) $p^3 + q^2 + q(1 + 3p) = 0$

(C) $p^3 + q^2 + q(3p - 1) = 0$ (D) $p^3 + q^2 + q(1 - 3p) = 0$

7. Let α, β, be the roots of the equation $x^2 - px + r = 0$ and $\frac{\alpha}{2}$, 2β be the roots of the equation $x^2 - qx + r = 0$. Then the value of r is -

(A) $\frac{2}{9}(p-q)(2q-p)$ (B) $\frac{2}{9}(q-p)(2p-q)$ (C) $\frac{2}{9}(q-2p)(2q-p)$ (D) $\frac{2}{9}(2p-q)(2q-p)$

8. Let p, q $\in \mathbb{Q}$. If $2-\sqrt{3}$ is a root of the quadratic equation, $x^2 + px + q = 0$, then :

(A) $q^2 + 4p + 14 = 0$ (B) $p^2 - 4q - 12 = 0$ (C) $q^2 - 4p - 16 = 0$ (D) $p^2 - 4q + 12 = 0$

9. If α and β are the roots of the quadratic equation, $x^2 + x\sin\theta - 2\sin\theta = 0$, $\theta \in \left(0, \frac{\pi}{2}\right)$, then

$\frac{\alpha^{12}+\beta^{12}}{\left(\alpha^{-12}+\beta^{-12}\right)(\alpha-\beta)^{24}}$ is equal to :

(A) $\frac{2^6}{(\sin\theta+8)^{12}}$ (B) $\frac{2^{12}}{(\sin\theta-8)^6}$ (C) $\frac{2^{12}}{(\sin\theta-4)^{12}}$ (D) $\frac{2^{12}}{(\sin\theta+8)^{12}}$

10. If f(x) is a quadratic expression such that f(1) + f(2) = 0, and –1 is a root of f(x) = 0, then the other root of f(x) = 0 is :-

(A) $-\frac{5}{8}$ (B) $\frac{8}{5}$ (C) $-\frac{8}{5}$ (D) $\frac{5}{8}$

11. If the equations $x^2 + bx - 1 = 0$ and $x^2+x+b=0$ have a common root different from –1, then 'b' is equal to :-

(A) $\sqrt{2}$ (B) $\sqrt{3}\,i$ (C) 3i (D) 2

12. Let p(x) be a quadratic polynomial such that p(0) = 1. If p(x) leaves remainder 4 when divided by x – 1 and it leaves remainder 6 when divided by x + 1; then :

(A) p(2) = 19 (B) p(–2) = 19 (C) p(–2) = 11 (D) p(2) = 11

13. If α,β are the roots of the equation $ax^2 + bx + c = 0$, then the roots of the equation $a(2x+1)^2 + b(2x+1)(x-1) + c(x-1)^2 = 0$ are

(A) $\frac{2\alpha+1}{\alpha-1},\frac{2\beta+1}{\beta-1}$ (B) $\frac{2\alpha-1}{\alpha+1},\frac{2\beta-1}{\beta+1}$ (C) $\frac{\alpha+1}{\alpha-2},\frac{\beta+1}{\beta-2}$ (D) $\frac{2\alpha+3}{\alpha-1},\frac{2\beta+3}{\beta-1}$

14. Let a, b, c be the sides of a triangle. No two of them are equal and $\lambda \in R$. If the roots of the equation $x^2 + 2(a+b+c)x + 3\lambda(ab+bc+ca) = 0$ are real then -

(A) $\lambda < \frac{4}{3}$ (B) $\lambda > \frac{5}{3}$ (C) $\lambda \in \left(\frac{1}{3},\frac{5}{3}\right)$ (D) $\lambda \in \left(\frac{4}{3},\frac{5}{3}\right)$

15. The equations $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$, where $b^2 - 4ac \neq 0$ have a common root, then $a^3 + b^3 + c^3$ is equal to $(a \neq 0)$

(A) 3abc (B) abc (C) 0 (D) 1

16. If α & $\beta(\alpha < \beta)$, are the roots of the equation, $x^2 + bx + c = 0$, where $c < 0 < b$, then -

(A) $0 < \alpha < \beta$ (B) $\alpha < 0 < \beta < |\alpha|$ (C) $\alpha < \beta < 0$ (D) $\alpha < 0 < |\alpha| < \beta$

17. If b > a, then the equation, (x – a) (x - b) – 1 = 0, has -

(A) both roots in [a, b] (B) both roots in $(-\infty, a)$

(C) both roots in $[b, \infty)$ (D) one root in $(-\infty, a)$ & the other in $(b, +\infty)$

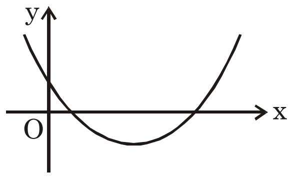
18. The graph of $y = ax^2 + bx + c$ is shown. Which of the following does **NOT** hold good?

(A) $ab^2c^3 > 0$

(B) $ab^3c^2 < 0$

(C) $ab^3c^5 > 0$

(D) $b^2 > 4ac$



19. The sum of the solutions of the equation $\left|\sqrt{x}-2\right|+\sqrt{x}\left(\sqrt{x}-4\right)+2=0,\ (x>0)$ is equal to :

(A) 4 (B) 9 (C) 10 (D) 12

20. The set of all real numbers x for which $x^2 - |x + 2| + x > 0$, is

(A) $(-\infty, -2) \cup (2, \infty)$

(B) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

(C) $(-\infty, -1) \cup (1, \infty)$

(D) $(\sqrt{2}, \infty)$

21. If α,β are roots of equation $3x^2 - 9x - \lambda = 0$, $\lambda \in R$ such that $1 < \alpha < 3$ & $3 < \beta < 5$ then λ lies in

(A) (–6,30) (B) (–6,0) (C) (0,30) (D) Null set

22. The set of values of 'a' for which $f(x) = ax^2 + 2x(1 - a) - 4$ is negative for exactly three integral values of x, is-

(A) (0,2) (B) (0,1] (C) [1,2) (D) $[2,\infty)$

23. If $\alpha,\beta,\gamma,\delta$ are the roots of equation $x^4 - \beta x + 3 = 0$, then the equation whose solutions are

$$\frac{\alpha+\beta+\gamma}{\delta^2},\frac{\alpha+\beta+\delta}{\gamma^2},\frac{\alpha+\gamma+\delta}{\beta^2},\frac{\beta+\gamma+\delta}{\alpha^2}$$ is-

(A) $3x^4 - \beta x^3 - 1 = 0$

(B) $3x^4 - \beta x^3 + 1 = 0$

(C) $3x^4 + \beta x^3 - 1 = 0$

(D) $3x^4 + \beta x^3 + 1 = 0$

[MATCHING COLUMN TYPE]

24. Let $f(x) = \dfrac{x^2-6x+5}{x^2-5x+6}$

Match the expressions/statements in Column I with expressions/statements in Column II.

Coloumn-I	**Coloumn-II**
(A) If $-1 < x < 1$, then f(x) satisfies	(p) $0 < f(x) < 1$
(B) If $1 < x < 2$, then f(x) satisfies	(q) $f(x) < 0$
(C) If $3 < x < 5$, then f(x) satisfies	(r) $f(x) > 0$
(D) If $x > 5$, then f(x) satisfies	(s) $f(x) < 1$

ALLEN CAREER INSTITUTE OVERSEAS

EXERCISE (S-1)

1. α, β are the roots of the equation $K(x^2 - x) + x + 5 = 0$. If K_1 & K_2 are the two values of K for which the roots α, β are connected by the relation $(\alpha/\beta) + (\beta/\alpha) = 4/5$. Find the value of $(K_1/K_2) + (K_2/K_1)$.

2. Let the quadratic equation $x^2 + 3x - k = 0$ has roots a, b and $x^2 + 3x - 10 = 0$ has roots c, d such that modulus of difference of the roots of the first equation is equal to twice the modulus of the difference of the roots of the second equation. If the value of 'k' can be expressed as rational number in the lowest form as m/n then find the value of $(m + n)$.

3. If α, β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$ and $\alpha + \delta$, $\beta + \delta$ are the roots of $Ax^2 + Bx + C = 0$, $(A \neq 0)$ for some constant δ, then prove that, $\frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2}$.

4. If the quadratic equations, $x^2 + bx + c = 0$ and $bx^2 + cx + 1 = 0$ have a common root then prove that either $b + c + 1 = 0$ or $b^2 + c^2 + 1 = bc + b + c$.

5. Find the value of *m* for which the quadratic equations $x^2 - 11x + m = 0$ and $x^2 - 14x + 2m = 0$ may have common root.

6. Let a, b be arbitrary real numbers. Find the smallest natural number 'b' for which the equation $x^2 + 2(a + b)x + (a - b + 8) = 0$ has unequal real roots for all $a \in R$.

7. Find the range of values of a, such that $f(x) = \frac{ax^2 + 2(a+1)x + 9a + 4}{x^2 - 8x + 32}$ is always negative.

8. Consider the quadratic polynomial $f(x) = x^2 - 4ax + 5a^2 - 6a$

(a) Find the smallest positive integral value of 'a' for which f(x) positive for every real x.

(b) Find the largest distance between the roots of the equation $f(x) = 0$

(c) Find the set of values of 'a' for which range of f(x) is $[-8, \infty)$

9. We call 'p' a good number if the inequality $\frac{2x^2+2x+3}{x^2+x+1} \le p$ is satisfied for any real x. Find the smallest integral good number.

10. Number of integral values of 'a' for which $2x^2 - 2ax + a^2 - a - 6 = 0$ has roots of opposite sign is

11. Find all values of p for which the roots of the equation $(p - 3)x^2 - 2px + 5p = 0$ are real and positive.

12. Find all the values of the parameter 'a' for which both roots of the quadratic equation $x^2 - ax + 2 = 0$ belong to the interval (0, 3).

13. At what values of 'a' do all the zeroes of the function $f(x) = (a - 2)x^2 + 2ax + a + 3$ lie on the interval $(-2, 1)$?

14. If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then find k.

15. When $y^2 + my + 2$ is divided by $(y - 1)$ then the quotient is f (y) and the remainder is R_1. When $y^2 + my + 2$ is divided by $(y + 1)$ then quotient is g (y) and the remainder is R_2. If $R_1 = R_2$ then find the value of *m*.

JEE-MAINS

1. Let for $a \neq a_1 \neq 0$, $f(x) = ax^2 + bx + c$, $g(x) = a_1x^2 + b_1x + c_1$ and $p(x) = f(x) - g(x)$.

If $p(x) = 0$ only for $x = -1$ and $p(-2) = 2$, then the value of $p(2)$ is: **[AIEEE-2011]**

(1) 18 (2) 3 (3) 9 (4) 6

2. Sachin and Rahul attempted to solve a quadratic equation. Sachin made a mistake in writing down the constant term and ended up in roots (4, 3). Rahul made a mistake in writing down coefficient of x to get roots (3, 2). The correct roots of equation are: **[AIEEE-2011]**

(1) –4, –3 (2) 6, 1 (3) 4, 3 (4) –6, –1

3. If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in R$, have a common root, then $a : b : c$ is : **[JEE-MAIN-2013]**

(1) 1 : 2 : 3 (2) 3 : 2 : 1 (3) 1 : 3 : 2 (4) 3 : 1 : 2

4. Let α and β be the roots of equation $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$, for $n \geq 1$, then the value of $\dfrac{a_{10} - 2a_8}{2a_9}$ is equal to : **[JEE-MAIN-2015]**

(1) 3 (2) – 3 (3) 6 (4) – 6

5. The sum of all real values of x satisfying the equation $\left(x^2 - 5x + 5\right)^{x^2+4x-60} = 1$ is :-

(1) 5 (2) 3 (3) –4 (4) 6

[JEE-MAIN-2016]

6. Let α and β be two roots of the equation $x^2 + 2x + 2 = 0$, then $\alpha^{15} + \beta^{15}$ is equal to :

(1) 512 (2) –512 (3) –256 (4) 256

[JEE(Main)-2019]

7. The number of all possible positive integral values of α for which the roots of the quadratic equation, $6x^2 - 11x + \alpha = 0$ are rational numbers is : **[JEE(Main)-2019]**

(1) 2 (2) 5 (3) 3 (4) 4

8. Consider the quadratic equation $(c-5)x^2-2cx+(c-4)=0, c\neq5$. Let S be the set of all integral values of c for which one root of the equation lies in the interval (0,2) and its other root lies in the interval (2,3). Then the number of elements in S is : **[JEE(Main)-2019]**

(1) 11 (2) 18 (3) 10 (4) 12

9. If λ be the ratio of the roots of the quadratic equation in x, $3m^2x^2+m(m-4)x+2=0$, then the least value of m for which $\lambda+\frac{1}{\lambda}=1$, is : **[JEE(Main)-2019]**

(1) $2-\sqrt{3}$ (2) $4-3\sqrt{2}$ (3) $-2+\sqrt{2}$ (4) $4-2\sqrt{3}$

10. Let α and β be the roots of the equation $x^2-x-1=0$. If $p_k=(\alpha)^k+(\beta)^k, k\geq1$, then which one of the following statements is not true ? **[JEE (Main)-2020]**

(1) $(p_1+p_2+p_3+p_4+p_5)=26$ (2) $p_5=11$

(3) $p_3=p_5-p_4$ (4) $p_5=p_2\cdot p_3$

11. The least positive value of 'a' for which the equation $2x^2+(a-10)x+\frac{33}{2}=2a$ has real roots is **[JEE(Main)-2020]**

12. If $A=\{x\in\mathbf{R}:|x|<2\}$ and $B=\{x\in\mathbf{R}:|x-2|\geq3\}$; then :

(1) $A\cup B=\mathbf{R}-(2,5)$ (2) $A\cap B=(-2,-1)$

(3) $B-A=\mathbf{R}-(-2,5)$ (4) $A-B=[-1,2)$ **[JEE(Main)-2020]**

JEE-ADVANCED

1. The smallest value of k, for which both the roots of the equation, $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real, distinct and have values at least 4, is **[JEE 2009, 4 (–1)]**

2. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are nonzero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots is **[JEE 2010, 3]**

(A) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$ (B) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

(C) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$ (D) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

3. Let α and β be the roots of $x^2 - 6x - 2 = 0$, with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is **[JEE 2011]**

(A) 1 (B) 2 (C) 3 (D) 4

4. A value of b for which the equations

$$x^2 + bx - 1 = 0$$
$$x^2 + x + b = 0,$$

have one root in common is - **[JEE 2011]**

(A) $-\sqrt{2}$ (B) $-i\sqrt{3}$ (C) $i\sqrt{5}$ (D) $\sqrt{2}$

5. Let S be the set of all non-zero numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals is(are) a subset(s) of S ? **[JEE 2015, 4M, –0M]**

(A) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (B) $\left(-\frac{1}{\sqrt{5}}, 0\right)$ (C) $\left(0, \frac{1}{\sqrt{5}}\right)$ (D) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

PARAGRAPH

Let p,q be integers and let α,β be the roots of the equation, $x^2 - x - 1 = 0$, where $\alpha \neq \beta$. For $n = 0,1,2,\ldots.,$ let $a_n = p\alpha^n + q\beta^n$.

FACT : If a and b are rational numbers and $a + b\sqrt{5} = 0$, then $a = 0 = b$.

6. If $a_4 = 28$, then $p + 2q =$ **[JEE(Advanced)-2017, 3(–1)]**

(A) 14 (B) 7 (C) 12 (D) 21

7. $a_{12} =$

(A) $2a_{11} + a_{10}$ (B) $a_{11} - a_{10}$ (C) $a_{11} + a_{10}$ (D) $a_{11} + 2a_{10}$

[JEE(Advanced)-2017, 3(–1)]

ANSWERS

Do yourself-1

1 **(a)** $-1, -2$; **(b)** 4; **(c)** $1 \pm \sqrt{2}$;

2 $a, \frac{1}{a}$; **3.** $\frac{7}{3}$ **4.** $3, -\frac{1}{5}$

5. $28x^2 - 20x + 1 = 0$ **6.** $a_1 = -2, a_2 = 1$ **7.** $a_1 = 2, a_2 = 9/2$

8. $p = \pm 7$ **9.** $k = \pm 3\sqrt{5}$ **10.** $p = 0$

11. $a = 2$ **12.** $x_1^3 + x_2^3 = 3pq - p^3$ **13.** $x_1^3 + x_2^3 = \frac{a(a^2 - 18a + 9)}{27}$

14. $\frac{215}{27}$

Do yourself-2

1. (a) imaginary; **(b)** real & distinct; **(c)** real & coincident

3. $a = 20 \pm 6\sqrt{5}$ **4.** $k_1 = \frac{-22}{3}, k_2 = 2$ **5.** $a = 4$

6. $a = \pm 2$ **7.** $k = 3$ **9.** For all $m \in \left(\frac{1}{4}, +\infty\right)$

10. For all $c \in [2, 4)$ **11.** $k = 13$

12. For all $m \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$

13. For all $m \in \left(-\infty, -\frac{1}{7}\right) \cup (1, +\infty)$ **14.** $b = -4, c = 1$;

15. 7 **16.** -15

17. **(a)** $c = 0$; **(b)** $c = 1$; **(c)** $b \to$ negative, $c \to$ negative **18.** (ii), (iii) and (iv)

19. (A) $\frac{b^2 - 2ac}{a^2}$ (B) $\frac{bc^4(3ac - b^2)}{a^7}$ (C) $\frac{b^2(b^2 - 4ac)}{a^2c^2}$

20. $\frac{1}{x_1^3} + \frac{1}{x_2^3} = -\frac{27a^3 + 36a}{8}$ **21.** $a_1 = 3/2, a_2 = 3$ **22.** $a = -4$

23. $a_1 = -\frac{3}{2}, a_2 = 6.$ **24.** $\{2, 18\}$ for $a = 6$, $\left\{\frac{2}{19}, \frac{18}{19}\right\}$ for $a = -\frac{6}{19}$

25. $a_1 = -\frac{125}{8}, a_2 = \frac{27}{8}$ **26.** $a_1 = 1/2, a_2 = 1$ **27.** $p_1 = 0, q_1 = 0, p_2 = 1, q_2 = -2$

Do yourself-3

1. **(a)** $c^2y^2 + y(2ac - b^2) + a^2 = 0$; **(b)** $acx^2 - bx + 1 = 0$;
(c) $acx^2 + (a + c)bx + (a + c)^2 = 0$

2. $x^2 - 2(p^2 - 2q)x + p^2(p^2 - 4q) = 0$

3. $x^2 - p(p^4 - 5p^2q + 5q^2)x + p^2q^2(p^2 - 4q)(p^2 - q) = 0$

Do yourself-4

1. 1 **3.** $b = \frac{9}{2}, c = 5$ **4.** $c = 0, c = 6$

6. $a = -2$ **7.** $m = -2$

Do yourself-5

1. $x \in R$ **2.** $x \in R$ **3.** $x \in \left(-\infty, \frac{1}{2}\right) \cup (3, \infty)$

4. $x \in (-6, -3) \cup \left(\frac{1}{2}, 2\right) - \{1\} \cup (9, \infty)$ **5.** $[3,7]$ **6.** $x \in \mathbb{R}$

7. $x \in (-\infty, -3) \cup (-2,3)$ **8.** $\left(\frac{1}{2}, 3\right)$ **9.** $(2, 3)$

10 $(-\infty, -2) \cup \left(-2, -\frac{1}{2}\right) \cup (1, \infty)$

Do yourself-6

1. $x \in R - (0,1]$ **2.** $(1/2, 2]$ **3.** $(3, +\infty)$

4. $[1, 2) \cup (2, +\infty)$ **5.** $[20/9, 4) \cup (5, +\infty)$ **6.** $\left(\frac{1}{2}, \frac{5}{2}\right]$

7. $[0, 2]$ **8.** $x \in [-18, -2)$

Do yourself-7

1. $(-\infty,-10]\cup[10,\infty)$
2. $x\in[0,10]$
3. $x\in[-2,6]$
4. $\left[-\frac{7}{2},1\right]\cup\left[2,\frac{13}{2}\right]$
5. $\left(-\sqrt{2},\sqrt{2}\right)$
6. ϕ
7. $\left(-\frac{3}{2},\frac{3}{2}\right)$
8. ϕ
9. $\left(1,\frac{7}{3}\right)$
10. $[-1,3]$
11. $(-\infty,-2)\cup(2,\infty)$
12. $\left(-\frac{6}{7},\frac{10}{7}\right)$
13. $x\in R$
14. $\left(-\infty,-\frac{7}{2}\right)\cup\left(\frac{7}{2},\infty\right)$

Do yourself-8

1. **(a)** 1 **(b)** -1
2. (1) (i) $a<0$ (ii) $b<0$ (iii) $c<0$ (iv) $D>0$ (v) $\alpha+\beta<0$ (vi) $\alpha\beta>0$
   (2) (i) $a<0$ (ii) $b>0$ (iii) $c=0$ (iv) $D>0$ (v) $\alpha+\beta>0$ (vi) $\alpha\beta=0$
   (3) (i) $a<0$ (ii) $b=0$ (iii) $c=0$ (iv) $D=0$ (v) $\alpha+\beta=0$ (vi) $\alpha\beta=0$
3. Third quadrant
4. (a) $a>9/16$ (b) $a<-2$
5. $c<0$
6. For all $a\in(-6,3)$
7. $(-3,5)$
8. $k=3$
9. $\left[0,\frac{1}{2}\right)$
10. $k=5$
11. For all $a\in(6,+\infty)$

Do yourself-9

1. $(-\infty,2]\cup[4,\infty)$
2. $[0,1)$
3. $\left[\frac{1}{3},3\right]$
4. $\left[-\frac{1}{11},1\right]$
5. $(-\infty,5]\cup[9,\infty)$
6. $\frac{1}{3}$

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Do yourself-10

1. (a) α and β are greater than -3

(b) '0' lies in between the roots α and β.

(c) α and β are less than 1.

2. C **3.** $a < 2$ **4.** $a < 2$ **5.** $12 < \lambda < 16$

6. For all $a \in \left(2, \frac{9}{4}\right]$

Do yourself-11

1. 0, 2 **2.** -2 **3.** ± 7

4. $(n\ell' - n'\ell)^2 = (mn' - m'n)(\ell m' - \ell' m)$

5. $(aa' - bb')^2 = -4(ah' + b'h)(a'h + bh')$

Do yourself-12

2. $-27, 5$

3. **(a)** $\frac{1}{a^2}(b^2 - 2ac)$ **(b)** $-\frac{c}{d}$, **(c)** $\frac{1}{a^2}(3ad - bc)$

EXERCISE (O-1)

1. D	**2.** B	**3.** C	**4.** A	**5.** D	**6.** A	**7.** A	**8.** A
9. A	**10.** A	**11.** A	**12.** C	**13.** D	**14.** B	**15.** A	**16.** D
7. D	**18.** A	**19.** D	**20.** D	**21.** B	**22.** D	**23.** A	**24.** B
25. D	**26.** B	**27.** B	**28.** B	**29.** C	**30.** B		

EXERCISE (O-2)

1. C	**2.** C	**3.** A	**4.** B	**5.** C	**6.** D	**7.** D	**8.** B
9. D	**10.** B	**11.** B	**12.** B	**13.** C	**14.** A	**15.** A	**16.** B
17. D	**18.** C	**19.** C	**20.** B	**21.** D	**22.** C	**23.** D	

24. (A) $\rightarrow$ (p, r, s); (B) $\rightarrow$ (q, s); (C) $\rightarrow$ (q, s); (D) $\rightarrow$ (p, r, s)

EXERCISE (S-1)

1. 254 **2.** 191 **5.** 0 or 24 **6.** **5** **7.** $a \in \left(-\infty, -\frac{1}{2}\right)$

8. **(a)** 7, **(b)** 6, **(c)** 2 or 4 **9.** 4 **10.** 4 **11.** For all $p \in \left[3, \frac{15}{4}\right]$

12. $2\sqrt{2} \le a < \frac{11}{3}$ **13.** $\left(-\infty, -\frac{1}{4}\right) \cup \{2\} \cup (5, 6]$ **14.** $(-\infty, 4)$ **15.** **0**

JEE-MAINS

1. 1 **2.** 2 **3.** 1 **4.** 1 **5.** 2 **6.** 3 **7.** 3 **8.** 1

9. 2 **10.** 4 **11.** 8.00 **12.** 3

JEE-ADVANCED

1. 2 **2.** B **3.** C **4.** B **5.** A,D **6.** C **7.** C

IMPORTANT NOTES