MATHEMATICS

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MODULE-4

CHAPTER 1

PERMUTATION & COMBINATION

Chapter Contents 01

IMPORTANT NOTES

CHAPTER 1 PERMUTATION & COMBINATION

1. FUNDAMENTAL PRINCIPLE OF COUNTING (counting without actual counting) :

If an event A can occur in 'm' different ways and another event B can occur in 'n' different ways, then the total number of different ways of -

(a) simultaneous occurrence of both events in a definite order is $m \times n$. This can be extended to any number of events (known as multiplication principle).

(b) happening exactly one of the events is $m + n$ (known as addition principle).

Example : There are 15 IITs in India and let each IIT has 10 branches, then the IITJEE topper can select the IIT and branch in $15 \times 10 = 150$ number of ways.

Example : There are 15 IITs & 20 NITs in India, then a student who cleared both IITJEE & AIEEE exams can select an institute in $(15 + 20) = 35$ number of ways.

Illustrations

Illustration 1 : A college offers 6 courses in the morning and 4 in the evening. The possible number of choices with the student if he wants to study one course in the morning and one in the evening is-

(A) 24 (B) 2 (C) 12 (D) 10

Solution : The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways.

Hence the total number of ways $6 \times 4 = 24$. **Ans.(A)**

Illustration 2 : A college offers 6 courses in the morning and 4 in the evening. The number of ways a student can select exactly one course, either in the morning or in the evening-

(A) 6 (B) 4 (C) 10 (D) 24

Solution : The student has 6 choices from the morning courses out of which he can select one course in 6 ways.

For the evening course, he has 4 choices out of which he can select one in 4 ways.

Hence the total number of ways $6 + 4 = 10$. **Ans. (C)**

Do yourself - 1

(i) There are 3 ways to go from A to B, 2 ways to go from B to C and 1 way to go from A to C. In how many ways can a person travel from A to C ?

(ii) There are 2 red balls and 3 green balls. All balls are identical except colour. In how many ways can a person select two balls ?

Greatest Integer

For every real number x, there exist an unique integer k such that $k \le x < k + 1$. Then k is called integral part or greatest integer or floor of x.

It is usually denoted by $\lfloor x \rfloor$ or $[x]$

Here are some example

x	-2.1	$-\sqrt{2}$	3	$\sqrt{5}$	π	2	100	$-\sqrt{70}$
$\lfloor x \rfloor$	-3	-2	3	2	3	2	100	-9

Note :

$x - 1 < \lfloor x \rfloor \le x$

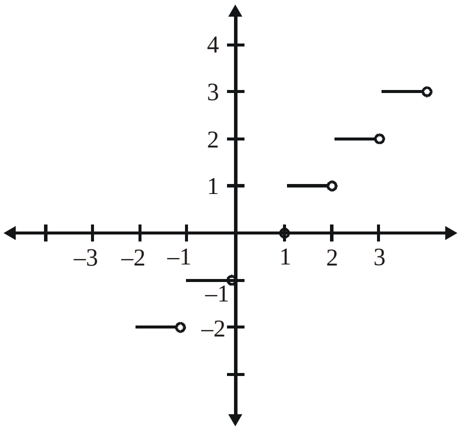
$\lfloor x \rfloor = x \Leftrightarrow x$ is integer

$\lfloor x \rfloor = n \Leftrightarrow x \in [n, n + 1),\ n \in I$

For $n \in I$, $\lfloor x + n \rfloor = n + \lfloor x \rfloor$

$$\lfloor x \rfloor + \lfloor -x \rfloor = \begin{cases} 0, & x \in I \\ -1, & x \notin I \end{cases}$$

Graph of y = [x]



Illustrations

Illustration 3 : Solve following

(i) $4[x] - 8 = 0$ (ii) $3\left[-\dfrac{x}{3}\right] + 9 = 0$

(iii) $[|x|] = 2$ (iv) $|[x]| = 2$ (v) $\left[\dfrac{5+x}{2}\right] + \left[\dfrac{3+x}{2}\right] = -9$

Here [.] denotes greatest integer function.

Solution : (i) $4[x] = 8 \Rightarrow [x] = 2 \Rightarrow x \in [2,3)$

(ii) $3\left[-\frac{x}{3}\right]+9=0 \Rightarrow \left[-\frac{x}{3}\right]=-3 \Rightarrow -3 \le -\frac{x}{3} < -2 \Rightarrow 9 \ge x > 6 \Rightarrow x \in (6,9]$

(iii) $[|x|] = 2 \Rightarrow 2 \le |x| < 3 \Rightarrow x \in (-3,-2] \cup [2,3]$

(iv) $|[x]| = 2 \Rightarrow [x] = -2$ or $2 \Rightarrow x \in [-2,-1) \cup [2,3)$

(v) $\left[\frac{5+x}{2}\right]+\left[\frac{3+x}{2}\right]=-9 \Rightarrow \left[2+\frac{1+x}{2}\right]+\left[1+\frac{1+x}{2}\right]=-9$

$\Rightarrow 2+\left[\frac{1+x}{2}\right]+1+\left[\frac{1+x}{2}\right]=-9 \Rightarrow 2\left[\frac{1+x}{2}\right]=-12 \Rightarrow \left[\frac{1+x}{2}\right]=-6$

$\Rightarrow -6 \le \frac{x+1}{2} < -5 \Rightarrow -12 \le x+1 < -10 \Rightarrow x \in [-13,-11)$

2. FACTORIAL NOTATION :

(i) A Useful Notation : $n!$ (factorial n) $= n.(n-1).(n-2)\ldots\ldots..3.\ 2.\ 1$; $n! = n.\ (n-1)!$ where $n \in N$

(ii) $0! = 1! = 1$

(iii) Factorials of negative integers are not defined.

(iv) $n!$ is also denoted by $\underline{|n}$

(v) $(2n)! = 2^n.n!\ [1.\ 3.\ 5.\ 7\ldots\ldots..(2n-1)]$

(vi) Prime factorisation of n! : Let p be a prime number and n be a positive integer, then exponent of p in n! is denoted by E_p (n!) and is given by

$$E_p(n!) = \left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\left[\frac{n}{p^3}\right]+\ldots..+\left[\frac{n}{p^k}\right]$$

where, $p^k \le n < p^{k+1}$ and [x] denotes the integral part of x.

If we isolate the power of each prime contained in any number n, then n can be written as

$n = 2^{\alpha_1}.3^{\alpha_2}.5^{\alpha_3}.7^{\alpha_4}\ldots.$, where α_i are whole numbers.

Illustrations

Illustration 4 : Find the exponent of 6 in 50!

Solution: $E_2(50!) = \left[\frac{50}{2}\right] + \left[\frac{50}{4}\right] + \left[\frac{50}{8}\right] + \left[\frac{50}{16}\right] + \left[\frac{50}{32}\right] + \left[\frac{50}{64}\right]$ (where [ ] denotes integral part)

$E_2(50!) = 25 + 12 + 6 + 3 + 1 + 0 = 47$

$E_3(50!) = \left[\frac{50}{3}\right] + \left[\frac{50}{9}\right] + \left[\frac{50}{27}\right] + \left[\frac{50}{81}\right]$

$E_3(50!) = 16 + 5 + 1 + 0 = 22$

$\Rightarrow$ 50! can be written as $50! = 2^{47}.\ 3^{22}$.........

Therefore exponent of 6 in 50! = 22 **Ans.**

3. PERMUTATION & COMBINATION :

(a) Permutation : Each of the arrangements in a definite order which can be made by taking some or all of the things at a time is called a PERMUTATION. In permutation, order of appearance of things is taken into account; when the order is changed, a different permutation is obtained.

Generally, it involves the problems of arrangements (standing in a line, seated in a row), problems on digit, problems on letters from a word etc.

${}^nP_r$ denotes the number of permutations of n **different** things, taken r at a time $(n \in N, r \in W, r \le n)$

$${}^nP_r = n(n-1)(n-2) \ldots\ldots\ldots\ldots (n-r+1) = \frac{n!}{(n-r)!}$$

Note :

(i) ${}^nP_n = n!$, ${}^nP_0 = 1$, ${}^nP_1 = n$

(ii) Number of arrangements of n **distinct** things taken all at a time = n!

(iii) ${}^nP_r$ is also denoted by A_r^n or P(n,r).

(b) Combination :

Each of the groups or selections which can be made by taking some or all of the things without considering the order of the things in each group is called a COMBINATION.

Generally, involves the problem of selections, choosing, distributed groups formation, committee formation, geometrical problems etc.

^nC_r denotes the number of combinations of n different things taken r at a time $(n \in N, r \in W, r \leq n)$

$$^nC_r = \frac{n!}{r!(n-r)!}$$

Note :

(i) ^nC_r is also denoted by $\binom{n}{r}$ or C (n, r).

(ii) $^nP_r = {}^nC_r \cdot r!$

Illustrations

Illustration 5 : If *a* denotes the number of permutations of (x + 2) things taken all at a time, *b* the number of permutations of x things taken 11 at a time and *c* the number of permutations of (x – 11) things taken all at a time such that *a = 182 bc*, then the value of x is

(A) 15 (B) 12 (C) 10 (D) 18

Solution : $^{x+2}P_{x+2} = a \Rightarrow a = (x+2)!$

$^xP_{11} = b \Rightarrow b = \frac{x!}{(x-11)!}$

and $^{x-11}P_{x-11} = c \Rightarrow c = (x-11)!$

$\because a = 182bc$

$(x+2)! = 182\frac{x!}{(x-11)!}(x-11)! \Rightarrow (x+2)(x+1) = 182 = 14 \times 13$

$\therefore x+1 = 13 \quad \Rightarrow x = 12$ **Ans. (B)**

Illustration 6 : A box contains 5 different red and 6 different white balls. In how many ways can 6 balls be drawn so that there are atleast two balls of each colour ?

Solution : The selections of 6 balls, consisting of atleast two balls of each colour from 5 red and 6 white balls, can be made in the following ways

Red balls (5)	White balls (6)	Number of ways
2	4	$^5C_2 \times {}^6C_4 = 150$
3	3	$^5C_3 \times {}^6C_3 = 200$
4	2	$^5C_4 \times {}^6C_2 = 75$

Therefore total number of ways = 425 **Ans.**

Illustration 7 : How many 4 letter words can be formed from the letters of the word 'ANSWER' ? How many of these words start with a vowel ?

Solution : Number of ways of arranging 4 different letters from 6 different letters are

$${}^{6}C_{4}4! = \frac{6!}{2!} = 360.$$

There are two vowels (A & E) in the word 'ANSWER'.

Total number of 4 letter words starting with A : A _ _ _ = ${}^{5}C_{3}3! = \frac{5!}{2!} = 60$

Total number of 4 letter words starting with E : E _ _ _ = ${}^{5}C_{3}3! = \frac{5!}{2!} = 60$

$\therefore$ Total number of 4 letter words starting with a vowel = 60 + 60 = 120. **Ans.**

Illustration 8 : If all the letters of the word 'RAPID' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RAPID'.

Solution : First of all, arrange all letters of given word alphabetically : 'ADIPR'

Total number of words starting with A _ _ _ _ = 4! = 24

Total number of words starting with D _ _ _ _ = 4! = 24

Total number of words starting with I _ _ _ _ = 4! = 24

Total number of words starting with P _ _ _ _ = 4! = 24

Total number of words starting with RAD _ _ = 2! = 2

Total number of words starting with RAI _ _ = 2! = 2

Total number of words starting with RAPD _ = 1

Total number of words starting with RAPI _ = 1

$\therefore$ Rank of the word RAPID = 24 + 24 + 24 + 24 + 2 + 2 + 1 + 1 = 102 **Ans.**

Do yourself - 2

(i) Find the exponent of 10 in ${}^{75}C_{25}$.

(ii) If ${}^{10}P_{r} = 5040$, then find the value of r.

(iii) Find the number of ways of selecting 4 even numbers from the set of first 100 natural numbers.

(iv) If all letters of the word 'RANK' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'RANK'.

(v) How many words can be formed using all letters of the word 'LEARN' ? In how many of these words vowels are together ?

(vi) Sketch the graph of

(a) $y = [2x]$ (b) $y = \left[\dfrac{x}{3}\right]$ (c) $y = [-x]$

Here [.] denotes greatest integer function.

(vii) Solve following

(a) $\left\lfloor \dfrac{x}{3} \right\rfloor + 2 = 0$ (b) $3\left\lfloor \dfrac{x}{3} \right\rfloor - 2 = 0$ (c) $\left\lfloor \dfrac{|x|}{3} \right\rfloor = 10$

(d) $\left|\left\lfloor -\dfrac{x}{3} \right\rfloor\right| = 4$ (e) $\left\lfloor \dfrac{-x-1}{2} \right\rfloor + \left\lfloor \dfrac{5-x}{2} \right\rfloor = 3$

4. PROPERTIES OF ^nP_r AND ^nC_r :

(a) The number of permutation of n different objects taken r at a time, when p particular objects are always to be included is $r!.^{n-p}C_{r-p}$ $(p \le r \le n)$

(b) The number of permutations of n different objects taken r at a time, when repetition is allowed any number of times is n^r.

(c) Following properties of ^nC_r should be remembered :

(i) $^nC_r = {}^nC_{n-r}$; $^nC_0 = {}^nC_n = 1$ (ii) $^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(iii) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ (iv) $^nC_0 + {}^nC_1 + {}^nC_2 + \ldots\ldots\ldots\ldots + {}^nC_n = 2^n$

(v) $^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1}$

(vi) ^nC_r is maximum when $r = \dfrac{n}{2}$ if n is even & $r = \dfrac{n-1}{2}$ or $r = \dfrac{n+1}{2}$, if n is odd.

(d) The number of combinations of n different things taking r at a time,

(i) when p particular things are always to be included = $^{n-p}C_{r-p}$

(ii) when p particular things are always to be excluded = $^{n-p}C_r$

(iii) when p particular things are always to be included and q particular things are to be excluded = $^{n-p-q}C_{r-p}$

Illustrations

Illustration 9 : There are 6 pockets in the coat of a person. In how many ways can he put 4 pens in these pockets?

(A) 360 (B) 1296 (C) 4096 (D) none of these

Solution : First pen can be put in 6 ways.

Similarly each of second, third and fourth pen can be put in 6 ways.

Hence total number of ways = $6 \times 6 \times 6 \times 6 = 1296$ **Ans.(B)**

Illustration 10 : A delegation of four students is to be selected from a total of 12 students. In how many ways can the delegation be selected, if-

(a) all the students are equally willing ?

(b) two particular students have to be included in the delegation ?

(c) two particular students do not wish to be together in the delegation ?

(d) two particular students wish to be included together only ?

(e) two particular students refuse to be together and two other particular students wish to be together only in the delegation ?

Solution : (a) Formation of delegation means selection of 4 out of 12.

Hence the number of ways = ${}^{12}C_4 = 495$.

(b) If two particular students are already selected. Here we need to select only 2 out of the remaining 10. Hence the number of ways = ${}^{10}C_2 = 45$.

(c) The number of ways in which both are selected = 45. Hence the number of ways in which the two are not included together = 495 – 45 = 450

(d) There are two possible cases

(i) Either both are selected. In this case, the number of ways in which the selection can be made = 45.

(ii) Or both are not selected. In this case all the four students are selected from the remaining ten students. This can be done in ${}^{10}C_4 = 210$ ways.

Hence the total number of ways of selection = 45 + 210 = 255

(e) We assume that students A and B wish to be selected together and students C and D do not wish to be together. Now there are following 6 cases.

(i)	(A, B, C) selected,	(D) not selected
(ii)	(A, B, D) selected,	(C) not selected
(iii)	(A, B) selected,	(C, D) not selected
(iv)	(C) selected,	(A, B, D) not selected
(v)	(D) selected,	(A, B, C) not selected
(vi)	A, B, C, D not selected	

For (i) the number of ways of selection = ${}^8C_1 = 8$

For (ii) the number of ways of selection = ${}^8C_1 = 8$

For (iii) the number of ways of selection = ${}^8C_2 = 28$

For (iv) the number of ways of selection = ${}^8C_3 = 56$

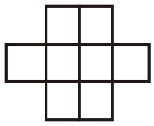
For (v) the number of ways of selection = ${}^8C_3 = 56$

For (vi) the number of ways of selection = ${}^8C_4 = 70$

Hence total number of ways = 8 + 8 + 28 + 56 + 56 + 70 = 226.

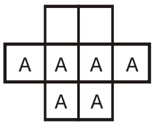
Ans.

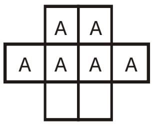
Illustration 11: In the given figure of squares, 6 A's should be written in such a manner that every row contains at least one 'A'. In how many number of ways is it possible ?



(A) 24 (B) 25 (C) 26 (D) 27

Solution : There are 8 squares and 6 'A' in given figure. First we can put 6 'A' in these 8 squares by ${}^8C_6$ number of ways.

(I) 

(II) 

According to question, atleast one 'A' should be included in each row. So after subtracting these two cases, number of ways are = $({}^8C_6 - 2) = 28 - 2 = 26$. **Ans. (C)**

Illustration 12: There are three coplanar parallel lines. If any p points are taken on each of the lines, the maximum number of triangles with vertices at these points is :

(A) $3p^2(p-1)+1$ (B) $3p^2(p-1)$ (C) $p^2(4p-3)$ (D) none of these

Solution : The number of triangles with vertices on different lines = ${}^pC_1 \times {}^pC_1 \times {}^pC_1 = p^3$

The number of triangles with two vertices on one line and the third vertex on any one of the other two lines = ${}^3C_1\,\{{}^pC_2 \times {}^{2p}C_1\} = 6p.\dfrac{p(p-1)}{2}$

So, the required number of triangles = $p^3 + 3p^2(p-1) = p^2(4p-3)$ **Ans. (C)**

Illustration 13: There are 10 points in a row. In how many ways can 4 points be selected such that no two of them are consecutive ?

Solution : Total number of remaining non-selected points = 6

. . . . . .

Total number of gaps made by these 6 points = 6 + 1 = 7

If we select 4 gaps out of these 7 gaps and put 4 points in selected gaps then the new points will represent 4 points such that no two of them are consecutive.

x . . **x** . **x** . . **x** .

Total number of ways of selecting 4 gaps out of 7 gaps = ${}^{7}C_{4}$

Ans.

In general, total number of ways of selection of r points out of n points in a row such that no two of them are consecutive : ${}^{n-r+1}C_{r}$

Do yourself - 3

(i) Find the number of ways of selecting 5 members from a committee of 5 men & 2 women such that all women are always included.

(ii) Out of first 20 natural numbers, 3 numbers are selected such that there is exactly one even number. How many different selections can be made ?

(iii) How many four letter words can be made from the letters of the word 'PROBLEM'. How many of these start as well as end with a vowel ?

5. FORMATION OF GROUPS :

(a) (i) The number of ways in which (m + n) different things can be divided into two groups such that one of them contains m things and other has n things, is $\dfrac{(m+n)!}{m!\ n!}(m \neq n)$.

(ii) If m = n, it means the groups are equal & in this case the number of divisions is $\dfrac{(2n)!}{n!\ n!\ 2!}$.

As in any one way it is possible to interchange the two groups without obtaining a new distribution.

(iii) If 2n things are to be divided equally between two persons then the number of ways :

$\dfrac{(2n)!}{n!\ n!\ (2!)} \times 2!$.

(b) (i) Number of ways in which (m + n + p) different things can be divided into three groups containing m, n & p things respectively is : $\frac{(m+n+p)!}{m!\ n!\ p!}$, $m \neq n \neq p$.

(ii) If m = n = p then the number of groups = $\frac{(3n)!}{n!\ n!\ n!\ 3!}$.

(iii) If 3n things are to be divided equally among three people then the number of ways in which it can be done is $\frac{(3n)!}{(n!)^3}$.

(c) In general, the number of ways of dividing n distinct objects into ℓ groups containing p objects each and m groups containing q objects each is equal to $\frac{n!}{(p!)^{\ell}(q!)^m \ell! m!}$

Here $\ell p + mq = n$

Illustrations

Illustration 14 : In how many ways can 15 students be divided into 3 groups of 5 students each such that 2 particular students are always together ? Also find the number of ways if these groups are to be sent to three different colleges.

Solution : Here first we seperate those two particular students and make 3 groups of 5,5 and 3 of the remaining 13 so that these two particular students always go with the group of 3 students.

$\therefore$ Number of ways $= \frac{13!}{5!5!3!}.\frac{1}{2!}$.

Now if these groups are to be sent to three different colleges, total number of

ways $= \frac{13!}{5!5!3!}.\frac{1}{2!}.3!$

Ans.

Illustration 15 : Find the number of ways of dividing 52 cards among 4 players equally such that each gets exactly one Ace.

Solution : Total number of ways of dividing 48 cards (Excluding 4Aces) in 4 groups $= \frac{48!}{(12!)^4 4!}$

Now, distribute exactly one Ace to each group of 12 cards. Total number of ways

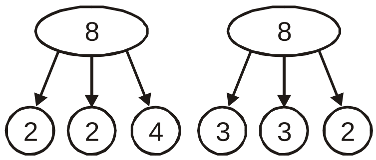
$= \frac{48!}{(12!)^4 4!} \times 4!$

Now, distribute these groups of cards among four players

$= \frac{48!}{(12!)^4 4!} \times 4!4! = \frac{48!}{(12!)^4} \times 4!$ **Ans.**

Illustration 16 : In how many ways can 8 different books be distributed among 3 students if each receives at least 2 books ?

Solution : If each receives at least two books, then the division trees would be as shown below:



(i) (ii)

The number of ways of division for tree in figure (i) is $\left[\dfrac{8!}{(2!)^2 4!2!}\right]$.

The number of ways of division for tree in figure (ii) is $\left[\dfrac{8!}{(3!)^2 2!2!}\right]$.

The total number of ways of distribution of these groups among 3 students

is $\left[\dfrac{8!}{(2!)^2 4!2!}+\dfrac{8!}{(3!)^2 2!2!}\right]\times 3!$. **Ans.**

Do yourself - 4

(i) Find the number of ways in which 16 constables can be assigned to patrol 8 villages, 2 for each.

(ii) In how many ways can 6 different books be distributed among 3 students such that none gets equal number of books and each gets atleast one book ?

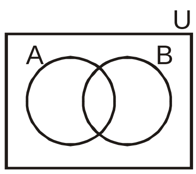
(iii) n different toys are to be distributed among n children. Find the number of ways in which these toys can be distributed so that exactly one child gets no toy.

6. PRINCIPLE OF INCLUSION AND EXCLUSION :

In the Venn's diagram (i), we get

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A' \cap B') = n(U) - n(A \cup B)$



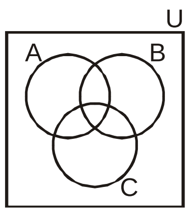
(i)

In the Venn's diagram (ii), we get

$n(A \cup B \cup C)$

$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

$n(A' \cap B' \cap C') = n(U) - n(A \cup B \cup C)$



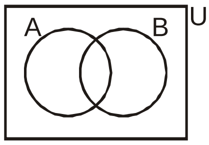
(ii)

In general, we have $n(A_1 \cup A_2 \cup .... \cup A_n)$

$$= \sum n(A_i) - \sum_{i \neq j} n(A_i \cap A_j) + \sum_{i \neq j \neq k} n(A_i \cap A_j \cap A_k) + ... + (-1)^n \sum n(A_1 \cap A_2 \cap ... \cap A_n)$$

Illustrations

Illustration 17 : Find the number of permutations of letters a,b,c,d,e,f,g taken all at a time if neither 'beg' nor 'cad' pattern appear.



Solution : The total number of permutations without any restrictions; n(U) = 7!

b e g a c d f

Let A be the set of all possible permutations in which 'beg' pattern always appears : n(A) = 5!

c a d b e f g

Let B be the set of all possible permutations in which 'cad' pattern always appears : n(B) = 5!

c a d b e g f

$n(A \cap B)$: Number of all possible permutations when both 'beg' and 'cad' patterns appear.

$n(A \cap B) = 3!$.

Therefore, the total number of permutations in which 'beg' and 'cad' patterns do not appear

$n(A' \cap B') = n(U) - n(A \cap B) = n(U) - n(A) - n(B) + n(A \cap B)$

$= 7! - 5! - 5! + 3!$. **Ans.**

Do yourself - 5

(i) Find the number of n digit numbers formed using first 5 natural numbers, which contain the digits 2 & 4 essentially.

7. PERMUTATIONS OF ALIKE OBJECTS :

Case-I : Taken all at a time -

The number of permutations of n things taken all at a time : when p of them are similar of one type, q of them are similar of second type, r of them are similar of third type and the remaining $n - (p + q + r)$ are all different is : $\dfrac{n!}{p!\ q!\ r!}$.

Illustrations

Illustration 18 : In how many ways the letters of the word "ARRANGE" can be arranged without altering the relative position of vowels & consonants.

Solution : The consonants in their positions can be arranged in $\frac{4!}{2!} = 12$ ways.

The vowels in their positions can be arranged in $\frac{3!}{2!} = 3$ ways

$\therefore$ Total number of arrangements = 12 × 3 = 36

Ans.

Illustration 19 : How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?

(A) 17 (B) 18 (C) 19 (D) 20

Solution : There are 4 odd digits (1, 1, 3, 3) and 4 odd places (first, third, fifth and seventh).

At these places the odd digits can be arranged in $\frac{4!}{2!2!} = 6$ ways

Then at the remaining 3 places, the remaining three digits (2, 2, 4) can be arranged in $\frac{3!}{2!} = 3$ ways

$\therefore$ The required number of numbers = 6 × 3 = 18. **Ans. (B)**

Illustration 20 :

(a) How many permutations can be made by using all the letters of the word HINDUSTAN ?

(b) How many of these permutations begin and end with a vowel ?

(c) In how many of these permutations, all the vowels come together ?

(d) In how many of these permutations, none of the vowels come together ?

(e) In how many of these permutations, do the vowels and the consonants occupy the same relative positions as in HINDUSTAN ?

Solution :

(a) The total number of permutations = Arrangements of nine letters taken all at a time = $\frac{9!}{2!} = 181440$.

(b) We have 3 vowels and 6 consonants, in which 2 consonants are alike. The first place can be filled in 3 ways and the last in 2 ways. The rest of the places can be filled in $\frac{7!}{2!}$ ways.

Hence the total number of permutations = $3 \times 2 \times \frac{7!}{2!} = 15120$.

(c) Assume the vowels (I, U, A) as a single letter. The letters (IUA), H, D, S, T, N, N can be arranged in $\frac{7!}{2!}$ ways. Also IUA can be arranged among themselves in 3! = 6 ways.

Hence the total number of permutations = $\frac{7!}{2!} \times 6 = 15120$.

(d) Let us divide the task into two parts. In the first, we arrange the 6 consonants as shown below in $\frac{6!}{2!}$ ways.

$\times C \times C \times C \times C \times C \times C \times$ (Here C stands for a consonant and $\times$ stands for a gap between two consonants)

Now 3 vowels can be placed in 7 places (gaps between the consonants) in ${}^7C_3.3! = 210$ ways.

Hence the total number of permutations $= \frac{6!}{2!} \times 210 = 75600$.

(e) In this case, the vowels can be arranged among themselves in $3! = 6$ ways.

Also, the consonants can be arranged among themselves in $\frac{6!}{2!}$ ways.

Hence the total number of permutations $= \frac{6!}{2!} \times 6 = 2160$.

Ans.

Illustration 21 : If all the letters of the word 'PROPER' are arranged in all possible manner as they are in a dictionary, then find the rank of the word 'PROPER' .

Solution : First of all, arrange all letters of given word alphabetically : EOPPRR

Total number of words starting with-

E _ _ _ _ _ $= \frac{5!}{2!2!} = 30$

O _ _ _ _ _ $= \frac{5!}{2!2!} = 30$

PE _ _ _ _ $= \frac{4!}{2!} = 12$

PO _ _ _ _ $= \frac{4!}{2!} = 12$

PP _ _ _ _ $= \frac{4!}{2!} = 12$

PRE _ _ _ $= 3! = 6$

PROE _ _ $= 2! = 2$

PROPER $= 1 = 1$

Rank of the word PROPER = 105 **Ans.**

Case-II : Taken some at a time

Illustration 22 : Find the total number of 4 letter words formed using four letters from the word "PARALLELOPIPED'.

Solution : Given letters are PPP, LLL, AA, EE, R, O, I, D.

Cases	No.of ways of selection	No.of ways of arrangements	Total
All distinct	${}^8C_4$	${}^8C_4 \times 4!$	1680
2 alike, 2 distinct	${}^4C_1 \times {}^7C_2$	${}^4C_1 \times {}^7C_2 \times \frac{4!}{2!}$	1008
2 alike, 2 other alike	${}^4C_2$	${}^4C_2 \times \frac{4!}{2!2!}$	36
3 alike, 1 distinct	${}^2C_1 \times {}^7C_1$	${}^2C_1 \times {}^7C_1 \times \frac{4!}{3!}$	56
		Total	2780

Ans.

Illustration 23 : Find the number of all 6 digit numbers such that all the digits of each number are selected from the set {1,2,3,4,5} and any digit that appears in the number appears at least twice.

Solution :

Cases	No.of ways of selection	No.of ways of arrangements	Total
All alike	${}^5C_1$	${}^5C_1 \times 1$	5
4 alike + 2 other alike	${}^5C_2 \times 2!$	${}^5C_2 \times 2 \times \frac{6!}{2!4!}$	300
3 alike + 3 other alike	${}^5C_2$	${}^5C_2 \times \frac{6!}{3!3!}$	200
2 alike + 2 other alike +2 other alike	${}^5C_3$	${}^5C_3 \times \frac{6!}{2!2!2!}$	900
		Total	1405

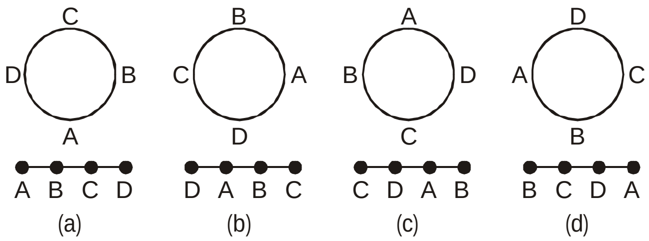
Ans.

Do yourself - 6

(i) In how many ways can the letters of the word 'ALLEN' be arranged ? Also find its rank if all these words are arranged as they are in dictionary.

(ii) How many numbers greater than 1000 can be formed from the digits 1, 1, 2, 2, 3 ?

8. CIRCULAR PERMUTATION :



Let us consider that persons A,B,C,D are sitting around a round table. If all of them (A,B,C,D) are shifted by one place in anticlockwise order, then we will get Fig.(b) from Fig.(a). Now, if we shift A,B,C,D in anticlockwise order, we will get Fig.(c). Again, if we shift them, we will get Fig.(d) and in the next time, Fig.(a).

Thus, we see that if 4 persons are sitting at a round table, they can be shifted four times and the four different arrangements, thus obtained will be the same, because anticlockwise order of A,B,C,D does not change.

But if A,B,C,D are sitting in a row and they are shifted in such an order that the last occupies the place of first, then the four arrangements will be different.

Thus, if there are 4 things, then for each circular arrangement number of linear arrangements is 4.

Similarly, if n different things are arranged along a circle, for each circular arrangement number of linear arrangements is n.

Therefore, the number of linear arrangements of n different things is n × (number of circular arrangements of n different things). Hence, the number of circular arrangements of n different things is -

1/n × (number of linear arrangements of n different things) $= \dfrac{n!}{n} = (n-1)!$

Therefore note that :

(i) The number of circular permutations of n different things taken all at a time is : $(n-1)!$.

If clockwise & anti-clockwise circular permutations are considered to be same, then it is : $\dfrac{(n-1)!}{2}$.

(ii) The number of circular permutations of n different things taking r at a time distinguishing clockwise & anticlockwise arrangements is : $\dfrac{{}^nP_r}{r}$

Illustrations

Illustration 24 : In how many ways can 5 boys and 5 girls be seated at a round table so that no two girls are together?

(A) $5! \times 5!$ (B) $5! \times 4!$ (C) $\frac{1}{2}(5!)^2$ (D) $\frac{1}{2}(5! \times 4!)$

Solution : Leaving one seat vacant between two boys, 5 boys may be seated in 4! ways. Then at remaining 5 seats, 5 girls sit in 5! ways. Hence the required number of ways = $4! \times 5!$

Ans. (B)

Illustration 25 : The number of ways in which 7 girls can stand in a circle so that they do not have same neighbours in any two arrangements ?

(A) 720 (B) 380 (C) 360 (D) none of these

Solution : Seven girls can stand in a circle by $\frac{(7-1)!}{2!}$ number of ways, because there is no difference in anticlockwise and clockwise order of their standing in a circle.

$\therefore \quad \frac{(7-1)!}{2!} = 360$ **Ans. (C)**

Illustration 26 : The number of ways in which 20 different pearls of two colours can be set alternately on a necklace, there being 10 pearls of each colour, is

(A) $9! \times 10!$ (B) $5(9!)^2$ (C) $(9!)^2$ (D) none of these

Solution : Ten pearls of one colour can be arranged in $\frac{1}{2}.(10-1)!$ ways. The number of arrangements of 10 pearls of the other colour in 10 places between the pearls of the first colour = 10!

$\therefore$ The required number of ways $= \frac{1}{2} \times 9! \times 10! = 5\,(9!)^2$ **Ans. (B)**

Illustration 27 : A person invites a group of 10 friends at dinner. They sit

(i) 5 on one round table and 5 on other round table,

(ii) 4 on one round table and 6 on other round table.

Find the number of ways in each case in which he can arrange the guests.

Solution : (i) The number of ways of selection of 5 friends for first table is ${}^{10}C_5$. Remaining 5 friends are left for second table.

The total number of permutations of 5 guests at a round table is 4!. Hence, the total number of arrangements is ${}^{10}C_5 \times 4! \times 4! = \frac{10!4!4!}{5!5!} = \frac{10!}{25}$

(ii) The number of ways of selection of 6 guests is ${}^{10}C_6$.

The number of ways of permutations of 6 guests on round table is 5!. The number of permutations of 4 guests on round table is 3!

Therefore, total number of arrangements is : ${}^{10}C_6 5! \times 3! = \frac{(10)!}{6!4!} 5!3! = \frac{(10)!}{24}$ **Ans. (B)**

Do yourself - 7

(i) In how many ways can 3 men and 3 women be seated around a round table such that all men are always together ?

(ii) Find the number of ways in which 10 different diamonds can be arranged to make a necklace.

(iii) Find the number of ways in which 6 persons out of 5 men & 5 women can be seated at a round table such that 2 men are never together.

(iv) In how many ways can 8 persons be seated on two round tables of capacity 5 & 3.

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9. TOTAL NUMBER OF COMBINATIONS :

(a) Given n different objects , the number of ways of selecting atleast one of them is, ${}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots\ldots + {}^nC_n = 2^n - 1$. This can also be stated as the total number of combinations of n distinct things.

(b) (i) Total number of ways in which it is possible to make a selection by taking some or all out of p + q + r +......things, where p are alike of one kind, q alike of a second kind, r alike of third kind & so on is given by : $(p + 1)(q + 1)(r + 1)\ldots\ldots -1$.

(ii) The total number of ways of selecting one or more things from p identical things of one kind, q identical things of second kind, r identical things of third kind and n different things is given by :

$(p + 1)(q + 1)(r + 1)\, 2^n - 1$.

Illustrations

Illustration 28 : A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P. A subset Q of A is again chosen. The number of ways of choosing P and Q so that $P \cap Q = \phi$ is :-

(A) $2^{2n} - {}^{2n}C_n$ (B) 2^n (C) $2^n - 1$ (D) 3^n

Solution : Let $A = \{a_1, a_2, a_3, \ldots\ldots a_n\}$. For $a_i \in A$, we have the following choices :

(i) $a_i \in P$ and $a_i \in Q$ (ii) $a_i \in P$ and $a_i \notin Q$

(iii) $a_i \notin P$ and $a_i \in Q$ (iv) $a_i \notin P$ and $a_i \notin Q$

Out of these only (ii), (iii) and (iv) imply $a_i \notin P \cap Q$. Therefore, the number of ways in which none of $a_1, a_2, \ldots . a_n$ belong to $P \cap Q$ is 3^n.

Ans.(D)

Illustration 29 : There are 3 books of mathematics, 4 of science and 5 of english. How many different collections can be made such that each collection consists of-

(i) one book of each subject ?

(ii) at least one book of each subject ?

(iii) at least one book of english ?

Solution : (i) ${}^3C_1 \times {}^4C_1 \times {}^5C_1 = 60$

(ii) $(2^3-1)(2^4-1)(2^5-1) = 7 \times 15 \times 31 = 3255$

(iii) $(2^5-1)(2^3)(2^4) = 31 \times 128 = 3968$ **Ans.**

Illustration 30 : Find the number of groups that can be made from 5 red balls, 3 green balls and 4 black balls, if at least one ball of all colours is always to be included. Given that all balls are identical except colours.

Solution : After selecting one ball of each colour, we have to find total number of combinations that can be made from 4 red. 2 green and 3 black balls. These will be $(4 + 1)(2 + 1)(3 + 1) = 60$ **Ans.**

Do yourself - 8

(i) There are p copies each of n different books. Find the number of ways in which atleast one book can be selected ?

(ii) There are 10 questions in an examination. In how many ways can a candidate answer the questions, if he attempts atleast one question.

10. DIVISORS :

Let $N = p^a . q^b . r^c$ .... where p, q, r........ are distinct primes & a, b, c....... are natural numbers then :

(a) The total numbers of divisors of N including 1 & N is = $(a + 1)(b + 1)(c + 1)$.......

(b) The sum of these divisors is

$$= (p^0 + p^1 + p^2 + ...+ p^a)(q^0 + q^1 + q^2 + ...+ q^b)(r^0 + r^1 + r^2 + ...+ r^c)...$$

(c) Number of ways in which N can be resolved as a product of two factor is =

$$\frac{1}{2}(a+1)(b+1)(c+1)...... \quad \text{if N is not a perfect square}$$

$$\frac{1}{2}\left[(a+1)(b+1)(c+1)......+1\right] \quad \text{if N is a perfect square}$$

(d) Number of ways in which a composite number N can be resolved into two factors which are relatively prime (or coprime) to each other is equal to 2^{n-1} where n is the number of different prime factors in N.

Note :

(i) Every natural number except 1 has atleast 2 divisors. If it has exactly two divisors then it is called a prime. System of prime numbers begin with 2. All primes except 2 are odd.

(ii) A number having more than 2 divisors is called composite. 2 is the only even number which is not composite.

(iii) Two natural numbers are said to be relatively prime or coprime if their HCF is one. For two natural numbers to be relatively prime, it is not necessary that one or both should be prime. It is possible that they both are composite but still coprime, eg. 4 and 25.

(iv) 1 is neither prime nor composite however it is co-prime with every other natural number.

(v) Two prime numbers are said to be twin prime numbers if their non-negative difference is 2 (e.g.5 & 7, 19 & 17 etc).

(vi) All divisors except 1 and the number itself are called proper divisors.

Illustrations

Illustration 31: Find the number of proper divisors of the number 38808. Also find the sum of these divisors.

Solution : **(i)** The number $38808 = 2^3 . 3^2 . 7^2 . 11$

Hence the total number of divisors (excluding 1 and itself i.e.38808)

$= (3 + 1)(2 + 1)(2 + 1)(1 + 1) - 2 = 70$

(ii) The sum of these divisors

$=(2^0 + 2^1 + 2^2 + 2^3)(3^0 + 3^1 + 3^2)(7^0 + 7^1 + 7^2)(11^0 + 11^1) - 1 - 38808$

$= (15)(13)(57)(12) - 1 - 38808 = 133380 - 1 - 38808 = 94571.$ **Ans.**

Illustration 32: In how many ways the number 18900 can be split in two factors which are relative prime (or coprime) ?

Solution: Here $N = 18900 = 2^2 . 3^3 . 5^2 . 7^1$

Number of different prime factors in $18900 = n = 4$

Hence number of ways in which 18900 can be resolved into two factors which are relative prime (or coprime) $= 2^{4-1} = 2^3 = 8$. **Ans.**

Illustration 33: Find the total number of proper factors of the number 35700. Also find

(i) sum of all these factors,

(ii) sum of the odd proper divisors,

(iii) the number of proper divisors divisible by 10 and the sum of these divisors.

Solution: $35700 = 5^2 \times 2^2 \times 3^1 \times 7^1 \times 17^1$

The total number of factors is equal to the total number of selections from (5,5), (2,2), (3), (7) and (17), which is given by $3 \times 3 \times 2 \times 2 \times 2 = 72$.

These include 1 and 35700. Therefore, the number of proper divisors (excluding 1 and 35700) is $72 - 2 = 70$

(i) Sum of all these factors (proper) is :

$(5^\circ + 5^1 + 5^2)(2^\circ + 2^1 + 2^2)(3^\circ + 3^1)(7^\circ + 7^1)(17^\circ + 17^1) - 1 - 35700$

$= 31 \times 7 \times 4 \times 8 \times 18 - 1 - 35700 = 89291$

(ii) The sum of odd proper divisors is :

$(5^\circ + 5^1 + 5^2)(3^\circ + 3^1)(7^\circ + 7^1)(17^\circ + 17^1) - 1$

$= 31 \times 4 \times 8 \times 18 - 1 = 17856 - 1 = 17855$

(iii) The number of proper divisors divisible by 10 is equal to number of selections from (5,5), (2,2), (3), (7), (17) consisting of at least one 5 and at least one 2 and 35700 is to be excluded and is given by $2 \times 2 \times 2 \times 2 \times 2 - 1 = 31$.

Sum of these divisors is :

$(5^1 + 5^2)(2^1 + 2^2)(3^\circ + 3^1)(7^\circ + 7^1)(17^\circ + 17^1) - 35700$

$= 30 \times 6 \times 4 \times 8 \times 18 - 35700 = 67980$ **Ans.**

Do yourself - 9

(i) Find the number of ways in which the number 94864 can be resolved as a product of two factors.

(ii) Find the number of different sets of solution of $xy = 1440$.

11. TOTAL DISTRIBUTION :

(a) **Distribution of distinct objects :** Number of ways in which n distinct things can be distributed to p persons if there is no restriction to the number of things received by them is given by : p^n

(b) **Distribution of alike objects :** Number of ways to distribute n alike things among p persons so that each may get none, one or more thing(s) is given by ${}^{n+p-1}C_{p-1}$.

Illustrations

Illustration 34: In how many ways can 5 different mangoes, 4 different oranges & 3 different apples be distributed among 3 children such that each gets alteast one mango ?

Solution : 5 different mangoes can be distributed by following ways among 3 children such that each gets atleast 1 :

3 1 1

2 2 1

Total number of ways : $\left(\frac{5!}{3!1!1!2!}+\frac{5!}{2!2!2!}\right)\times 3!$

Now, the number of ways of distributing remaining fruits (i.e. 4 oranges + 3 apples) among 3 children = 3^7 (as each fruit has 3 options).

$\therefore$ Total number of ways $=\left(\frac{5!}{3!2!}+\frac{5!}{(2!)^3}\right)\times 3!\times 3^7$ **Ans.**

Illustration 35: In how many ways can 12 identical apples be distributed among four children if each gets atleast 1 apple and not more than 4 apples.

Solution : Let x,y,z & w be the number of apples given to the children.

$\Rightarrow$ x + y + z + w = 12

Giving one-one apple to each

Now, x + y + z + w = 8 ....(i)

Here, $0 \le x \le 3,\ 0 \le y \le 3,\ 0 \le z \le 3,\ 0 \le w \le 3$

$x = 3 - t_1,\ y = 3 - t_2,\ z = 3 - t_3,\ w = 3 - t_4.$

Putting value of x, y, z, w in equation (i)

Put $12 - 8 = t_1 + t_2 + t_3 + t_4$

$\Rightarrow t_1 + t_2 + t_3 + t_4 = 4$

(Here max. value that t_1, t_2, t_3 & t_4 can attain is 3, so we have to remove those cases when any of t_i getting value 4)

$= {}^7C_3 -$ (all cases when atleast one is 4)

$= {}^7C_3 - 4 = 35 - 4 = 31$ **Ans.**

Illustration 36: Find the number of non negative integral solutions of the inequation $x + y + z \le 20$.

Solution : Let w be any number ($0 \le w \le 20$), then we can write the equation as :

$x + y + z + w = 20$ (here $x, y, z, w \ge 0$)

Total ways $= {}^{23}C_3$ **Ans.**

Illustration 37: Find the number of integral solutions of $x + y + z + w < 25$, where $x > -2$, $y > 1$, $z \ge 2$, $w \ge 0$.

Solution : Given $x + y + z + w < 25$

$x + y + z + w + v = 25$ ....(i)

Let $x = -1 + t_1$, $y = 2 + t_2$, $z = 2 + t_3$, $w = t_4$, $v = 1 + t_5$ where ($t_1, t_2, t_3, t_4 \ge 0$)

Putting value of x, y, z, w, v in equation (i)

$\Rightarrow t_1 + t_2 + t_3 + t_4 + t_5 = 21.$

Number of solutions $= {}^{25}C_4$ **Ans.**

Illustration 38: Find the number of positive integral solutions of the inequation $x + y + z \ge 150$, where $0 < x \le 60$, $0 < y \le 60$, $0 < z \le 60$.

Solution : Let $x = 60 - t_1$, $y = 60 - t_2$, $z = 60 - t_3$ (where $0 \le t_1 \le 59$, $0 \le t_2 \le 59$, $0 \le t_3 \le 59$)

Given $x + y + z \ge 150$

or $x + y + z - w = 150$ (where $0 \le w \le 147$) ....(i)

Putting values of x, y, z in equation (i)

$60 - t_1 + 60 - t_2 + 60 - t_3 - w = 150$

$30 = t_1 + t_2 + t_3 + w$

Total solutions $= {}^{33}C_3$ **Ans.**

Illustration 39: Find the number of positive integral solutions of $xy = 12$

Solution : $xy = 12$

$xy = 2^2 \times 3^1$

(i) 3 has 2 ways either 3 can go to x or y

(ii) 2^2 can be distributed between x & y as distributing 2 identical things between 2 persons

(where each person can get 0, 1 or 2 things). Let two person be ℓ_1 & ℓ_2

$\Rightarrow \quad \ell_1 + \ell_2 = 2$

$\Rightarrow \quad {}^{2+1}C_1 = {}^3C_1 = 3$

So total ways $= 2 \times 3 = 6$.

Alternatively :

$xy = 12 = 2^2 \times 3^1$

$x = 2^{a_1}3^{a_2}$ $\quad 0 \le a_1 \le 2$

$\quad 0 \le a_2 \le 1$

$y = 2^{b_1}3^{b_2}$ $\quad 0 \le b_1 \le 2$

$\quad 0 \le b_2 \le 1$

$2^{a_1+b_1}3^{a_2+b_2} = 2^2 3^1$

$\Rightarrow a_1 + b_1 = 2 \rightarrow {}^3C_1$ ways

$a_2 + b_2 = 1 \rightarrow {}^2C_1$ ways

Number of solutions $= {}^3C_1 \times {}^2C_1 = 3 \times 2 = 6$ **Ans.**

Illustration 40 : Find the number of solutions of the equation xyz = 360 when **(i)** $x,y,z \in N$ **(ii)** $x,y,z \in I$

Solution : **(i)** $xyz = 360 = 2^3 \times 3^2 \times 5$ $(x,y,z \in N)$

$x = 2^{a_1}3^{a_2}5^{a_3}$ (where $0 \le a_1 \le 3$, $0 \le a_2 \le 2$, $0 \le a_3 \le 1$)

$y = 2^{b_1}3^{b_2}5^{b_3}$ (where $0 \le b_1 \le 3$, $0 \le b_2 \le 2$, $0 \le b_3 \le 1$)

$z = 2^{c_1}3^{c_2}5^{c_3}$ (where $0 \le c_1 \le 3$, $0 \le c_2 \le 2$, $0 \le c_3 \le 1$)

$\Rightarrow 2^{a_1}3^{a_2}5^{a_3}.2^{b_1}3^{b_2}5^{b_3}.2^{c_1}3^{c_2}5^{c_3} = 2^3 \times 3^2 \times 5^1$

$\Rightarrow 2^{a_1+b_1+c_1}.3^{a_2+b_2+c_2}.5^{a_3+b_3+c_3} = 2^3 \times 3^3 \times 5^1$

$\Rightarrow a_1 + b_1 + c_1 = 3 \rightarrow {}^5C_2 = 10$

$a_2 + b_2 + c_2 = 2 \rightarrow {}^4C_2 = 6$

$a_3 + b_3 + c_3 = 1 \rightarrow {}^3C_2 = 3$

Total solutions = 10 × 6 × 3 = 180.

(ii) If $x,y,z \in I$ then, (a) all positive (b) 1 positive and 2 negative.

Total number of ways $= 180 + {}^3C_2 \times 180 = 720$ **Ans.**

Do yourself - 10

(i) In how many ways can 12 identical apples be distributed among 4 boys. (a) If each boy receives any number of apples. (b) If each boy receives atleast 2 apples.

(ii) Find the number of non-negative integral solutions of the equation x + y + z = 10.

(iii) Find the number of integral solutions of x + y + z = 20, if $x \ge -4$, $y \ge 1$, $z \ge 2$

12. DEARRANGEMENT :

There are n letters and n corresponding envelopes. The number of ways in which letters can be placed in the envelopes (one letter in each envelope) so that no letter is placed in correct envelope is $n!\left[1-\frac{1}{1!}+\frac{1}{2!}+.....+\frac{(-1)^n}{n!}\right]$

Proof : n letters are denoted by 1,2,3,........,n. Let A_i denote the set of distribution of letters in envelopes (one letter in each envelope) so that the i^{th} letter is placed in the corresponding envelope. Then,

$n(A_i) = 1 \times (n-1)!$ [since the remaining n–1 letters can be placed in n –1 envelops in (n–1)! ways]

Then, $n(A_i \cap A_j)$ represents the number of ways where letters i and j can be placed in their corresponding envelopes. Then,

$$n(A_i \cap A_j) = 1 \times 1 \times (n-2)!$$

Also $n(A_i \cap A_j \cap A_k) = 1 \times 1 \times 1 \times (n-3)!$

Hence, the required number is

$$n(A_1' \cup A_2' \cup ..... \cup A_n') = n! - n(A_1 \cup A_2 \cup .... \cup A_n)$$

$$= n! - \left[\sum n(A_i) - \sum n(A_i \cap A_j) + \sum n(A_i \cap A_j \cap A_k) + ... + (-1)^n \sum n(A_i \cap A_2 ..... \cap A_n)\right]$$

$$= n! - [{}^nC_1(n-1)! - {}^nC_2(n-2)! + {}^nC_3(n-3)! + ...+ (-1)^{n-1} \times {}^nC_n 1]$$

$$= n! - \left[\frac{n!}{1!(n-1)!}(n-1)! - \frac{n!}{2!(n-2)!}(n-2)! + ... + (-1)^{n-1}\right] = n!\left[1-\frac{1}{1!}+\frac{1}{2!}+........+\frac{(-1)^n}{n!}\right]$$

Illustrations

Illustration 41: A person writes letters to six friends and addresses the corresponding envelopes. In how many ways can the letters be placed in the envelopes so that

(i) all the letters are in the wrong envelopes.

(ii) at least two of them are in the wrong envelopes.

Solution : (i) The number of ways is which all letters be placed in wrong envelopes

$$= 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right) = 720\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}-\frac{1}{120}+\frac{1}{720}\right)$$

$= 360 - 120 + 30 - 6 + 1 = 265.$

(i) The number of ways in which at least two of them in the wrong envelopes

$$= {}^6C_4 \,.\, 2!\left(1-\frac{1}{1!}+\frac{1}{2!}\right)+{}^6C_3 \,.\, 3!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}\right)+{}^6C_2 \,.\, 4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)$$

$$+ {}^6C_1 \,.\, 5!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}\right)+{}^6C_0\, 6!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\frac{1}{6!}\right)$$

$= 15 + 40 + 135 + 264 + 265 = 719.$ **Ans.**

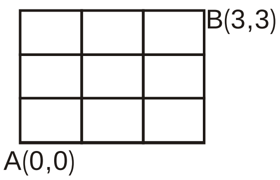
Do yourself - 11

(i) There are four balls of different colours and four boxes of colours same as those of the balls. Find the number of ways in which the balls, one in each box, could be placed in such a way that a ball does not go to box of its own colour.

Illustrations

Miscellaneous Illustrations :

Illustration 42: In how many ways can a person go from point A to point B if he can travel only to the right or upward along the lines (Grid Problem) ?



Solution : To reach the point B from point A, a person has to travel along 3 horizontal and 3 vertical strips. Therefore, we have to arrange 3H and 3V in a row. Total number of

ways $= \frac{6!}{3!3!} = 20$ ways **Ans.**

Illustration 43: Find sum of all numbers formed using the digits 2,4,6,8 taken all at a time and no digit being repeated.

Solution : All possible numbers = 4! = 24

If 2 occupies the unit's place then total numbers = 6

Hence, 2 comes at unit's place 6 times.

Sum of all the digits occuring at unit's place

$= 6 \times (2 + 4 + 6 + 8)$

Same summation will occur for ten's, hundred's & thousand's place. Hence required sum

$= 6 \times (2 + 4 + 6 + 8) \times (1 + 10 + 100 + 1000) = 133320$ **Ans.**

Illustration 44: Find the sum of all the numbers greater than 1000 using the digits 0,1,2,2.

Solution :

(i) When 1 is at thousand's place, total numbers formed will be $= \frac{3!}{2!} = 3$

(ii) When 2 is at thousand's place, total numbers formed will be = 3! = 6

(iii) When 1 is at hundred's, ten's or unit's place then total numbers formed will be-

Thousand's place is fixed i.e. only the digit 2 will come here, remaining two places can be filled in 2! ways.

So total numbers = 2!

(iv) When 2 is at hundred's, ten's or unit's place then total numbers formed will be-

Thousand's place has 2 options and other two places can be filled in 2 ways.

So total numbers = 2 × 2 = 4

$\text{Sum} = 10^3(1\times 3 + 2\times 6) + 10^2(1\times 2 + 2\times 4) + 10^1(1\times 2 + 2\times 4) + (1\times 2 + 2\times 4)$

$= 15\times 10^3 + 10^3 + 10^2 + 10$

$= 16110$ **Ans.**

Illustration 45 : Find the number of positive integral solutions of $x + y + z = 20$, if $x \neq y \neq z$.

Solution : $x \geq 1$

$y = x + t_1 \qquad t_1 \geq 1$

$z = y + t_2 \qquad t_2 \geq 1$

$x + x + t_1 + x + t_1 + t_2 = 20$

$3x + 2t_1 + t_2 = 20$

(i) $x = 1$ $\qquad 2t_1 + t_2 = 17$

$t_1 = 1,2 \,.........\, 8 \Rightarrow 8$ ways

(ii) $x = 2$ $\qquad 2t_1 + t_2 = 14$

$t_1 = 1,2 \,.........\, 6 \Rightarrow 6$ ways

(iii) $x = 3$ $\qquad 2t_1 + t_2 = 11$

$t_1 = 1,2 \,.........\, 5 \Rightarrow 5$ ways

(vi) $x = 4$ $\qquad 2t_1 + t_2 = 8$

$t_1 = 1,2,3 \Rightarrow 3$ ways

(v) $x = 5$ $\qquad 2t_1 + t_2 = 5$

$t_1 = 1, 2 \Rightarrow 2$ ways

Total = 8 + 6 + 5 + 3 + 2 = 24

But each solution can be arranged by 3! ways.

So total solutions = 24 × 3! = 144. **Ans.**

Illustration 46: A regular polygon of 15 sides is constructed. In how many ways can a triangle be formed using the vertices of the polygon such that no side of triangle is same as that of polygon ?

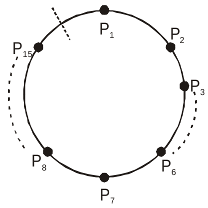
Solution : Select one point out of 15 point, therefore total number of ways = ${}^{15}C_1$

Suppose we select point P_1. Now we have to choose 2 more point which are not consecutive.

since we can not select P_2 & P_{15}.

Total points left are 12.

Now we have to select 2 points out of 12 points



which are not consecutive

Total ways = ${}^{12-2+1}C_2 = {}^{11}C_2$

Every select triangle will be repeated 3 times.

So total number of ways $= \dfrac{{}^{15}C_1 \times {}^{11}C_2}{3} = 275$

Alternative :

First of all let us cut the polygon between points P_1 & P_{15}. Now there are 15 points on a straight line and we have to select 3 points out of these, such that the selected points are not consecutive.

x O y O z O w

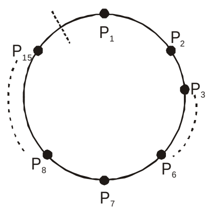
Here bubbles represents the selected points,

x represents the number of points before first selected point,

y represents the number of points between Ist & IInd selected point,

z represents the number of points between IInd & IIIrd selected point

and w represents the number of points after IIIrd selected point.



$x + y + z + w = 15 - 3 = 12$

here $x \geq 0$, $y \geq 1$, $z \geq 1$, $w \geq 0$

Put $y = 1 + y'$ & $z = 1 + z'$ $(y' \geq 0, z' \geq 0)$

$\Rightarrow x + y' + z' + w = 10$

Total number of ways = ${}^{13}C_3$

These selections include the cases when both the points P_1 & P_{15} are selected. We have to remove those cases. Here a represents number of points between P_1 & 3^{rd} selected point & b represents number of points between 3^{rd} selected point and P_{15}

$\Rightarrow a + b = 15 - 3 = 12 \quad (a \geq 1, b \geq 1)$

put $a = 1 + t_1$ & $b = 1 + t_2$

$t_1 + t_2 = 10$

Total number of ways $= {}^{11}C_1 = 11$

Therefore required number of ways $= {}^{13}C_3 - {}^{11}C_1 = 286 - 11 = 275$ **Ans.**

Illustration 47: Find the number of ways in which three numbers can be selected from the set $\{5^1, 5^2, 5^3, ...5^{11}\}$ so that they form a G.P.

Solution : Any three selected numbers which are in G.P. have their powers in A.P.

Set of powers is $= \{1,2,.........6,7,.....11\}$

By selecting any two numbers from $\{1,3,5,7,9,11\}$, the middle number is automatically fixed. Total number of ways $= {}^6C_2$

Now select any two numbers from $\{2,4,6,8,10\}$ and again middle number is automatically fixed. Total number of ways $= {}^5C_2$

$\therefore$ Total number of ways are $= {}^6C_2 + {}^5C_2 = 15 + 10 = 25$ **Ans.**

ANSWER KEY

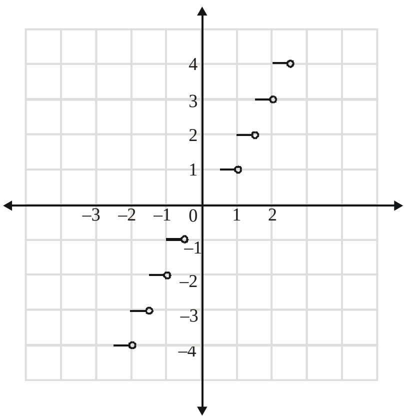
Do yourself-1

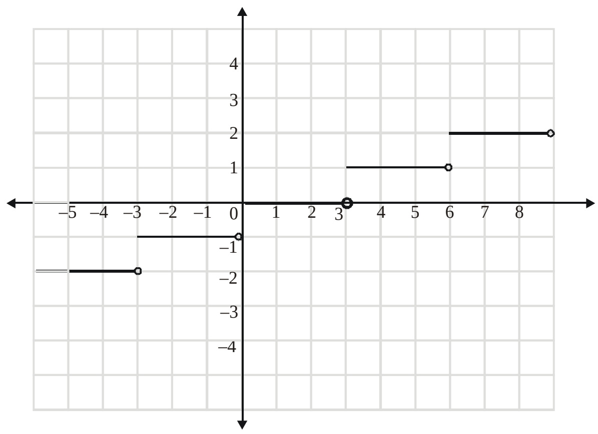
(i) 7 **(ii)** 3

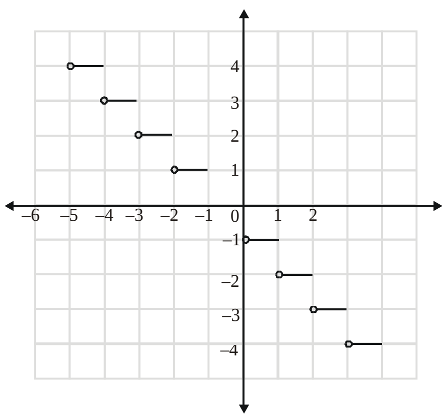
Do yourself-2

(i) 0 **(ii)** $r = 4$ **(iii)** ${}^{50}C_4$ **(iv)** 20 **(v)** 120, 48

(vi) **Sketch the graph of**

(a)


(b)


(c)


(vii) (a) $x \in [-6,-3)$ (b) $x \in \phi$ (c) $x \in (-33,-30] \cup [30,33)$

(d) $x \in (-15,-12] \cup (9,12]$ (e) $x \in (-3,-1]$

Do yourself-3

(i) 10 **(ii)** 450 **(iii)** 840, 40

Do yourself-4

(i) $\dfrac{16!}{(2!)^8 8!} \times 8!$ **(ii)** 360 **(iii)** ${}^{n}C_2.n!$

Do yourself-5

(i) $5^n - 4^n - 4^n + 3^n$

Do yourself-6

(i) 60, 6^{th} **(ii)** 60

Do yourself-7

(i) 36 **(ii)** $\frac{9!}{2} = 181440$ **(iii)** 5400 **(iv)** 2688

Do yourself-8

(i) $(p + 1)^n - 1$ **(ii)** $2^{10} - 1$

Do yourself-9

(i) 23 **(ii)** 36

Do yourself-10

(i) (a) ${}^{15}C_3$ (b) ${}^{7}C_3$ **(ii)** ${}^{12}C_2$ **(iii)** ${}^{23}C_2$

Do yourself-11

(i) 9

EXERCISE (O-1)

ONLY ONE CORRECT :

1. The number of different seven digit numbers that can be written using only three digits 1, 2 & 3 under the condition that the digit 2 occurs exactly twice in each number is

(A) 672 (B) 640 (C) 512 (D) none

2. How many of the 900 three digit numbers have at least one even digit?

(A) 775 (B) 875 (C) 450 (D) 750

3. Number of natural numbers between 100 and 1000 such that at least one of their digits is 7, is

(A) 225 (B) 243 (C) 252 (D) none

4. Number of 4 digit numbers of the form N = abcd which satisfy following three conditions :

(i) $4000 \le N < 6000$ (ii) N is multiple of 5 (iii) $3 \le b < c \le 6$

is equal to

(A) 12 (B) 18 (C) 24 (D) 48

5. Consider the five points comprising of the vertices of a square and the intersection point of its diagonals. How many triangles can be formed using these points?

(A) 4 (B) 6 (C) 8 (D) 10

6. The number of ways in which 5 different books can be distributed among 10 people if each person can get at most one book is :

(A) 252 (B) 10^5 (C) 5^{10} (D) ${}^{10}C_5.5!$

7. A student has to answer 10 out of 13 questions in an examination . The number of ways in which he can answer if he must answer atleast 3 of the first five questions is :

(A) 276 (B) 267 (C) 80 (D) 1200

8. A question paper on mathematics consists of twelve questions divided into three parts A, B and C, each containing four questions. In how many ways can an examinee answer five questions, selecting atleast one from each part.

(A) 624 (B) 208 (C) 1248 (D) 2304

9. 5 Indian & 5 American couples meet at a party & shake hands . If no wife shakes hands with her own husband & no Indian wife shakes hands with a male, then the number of hand shakes that takes place in the party is :

(A) 95 (B) 110 (C) 135 (D) 150

10. The kindergarten teacher has 25 kids in her class . She takes 5 of them at a time, to zoological garden as often as she can, without taking the same 5 kids more than once. Then the number of visits, the teacher makes to the garden exceeds that of a kid by :

(A) ${}^{25}C_5 - {}^{24}C_5$ (B) ${}^{24}C_5$ (C) ${}^{24}C_4$ (D) none

11. Number of cyphers at the end of ${}^{2002}C_{1001}$ is

(A) 0 (B) 1 (C) 2 (D) 200

12. Three vertices of a convex n sided polygon are selected. If the number of triangles that can be constructed such that none of the sides of the triangle is also the side of the polygon is 30, then the polygon is a

(A) Heptagon (B) Octagon (C) Nonagon (D) Decagon

13. There are 12 books on Algebra and Calculus in our library , the books of the same subject being different. If the number of selections each of which consists of 3 books on each topic is greatest then the number of books of Algebra and Calculus in the library are respectively:

(A) 3 and 9 (B) 4 and 8 (C) 5 and 7 (D) 6 and 6

14. Out of seven consonants and four vowels, the number of words of six letters, formed by taking four consonants and two vowels is (Assume that each ordered group of letter is a word):

(A) 210 (B) 462 (C) 151200 (D) 332640

15. The number of six digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6 & 7 so that digits do not repeat and the terminal digits are even is :

(A) 144 (B) 72 (C) 288 (D) 720

16. If the letters of the word "VARUN" are written in all possible ways and then are arranged as in a dictionary, then the rank of the word VARUN is :

(A) 98 (B) 99 (C) 100 (D) 101

17. A new flag is to be designed with six vertical strips using some or all of the colours yellow, green, blue and red. Then, the number of ways this can be done such that no two adjacent strips have the same colour is -

(A) 12×81 (B) 16×192 (C) 20×125 (D) 24×216

18. Number of 5 digit numbers which are divisible by 5 and each number containing the digit 5, digits being all different is equal to k(4!), the value of k is

(A) 84 (B) 168 (C) 188 (D) 208

19. A 5 digit number divisible by 3 is to be formed using the numerals 0, 1, 2, 3, 4 & 5 without repetition. The total number of ways this can be done is :

(A) 3125 (B) 600 (C) 240 (D) 216

20. The number of n digit numbers which consists of the digits 1 & 2 only if each digit is to be used atleast once, is equal to 510 then *n* is equal to

(A) 7 (B) 8 (C) 9 (D) 10

21. If *m* denotes the number of 5 digit numbers if each successive digits are in their descending order of magnitude and *n* is the corresponding figure, when the digits are in their ascending order of magnitude then $(m - n)$ has the value

(A) ${}^{10}C_4$ (B) ${}^{9}C_5$ (C) ${}^{10}C_3$ (D) ${}^{9}C_3$

22. A rack has 5 different pairs of shoes. The number of ways in which 4 shoes can be chosen from it, so that there will be no complete pair is :

(A) 1920 (B) 200 (C) 110 (D) 80

23. Number of ways in which 8 people can be arranged in a line if A and B must be next each other and C must be somewhere behind D, is equal to

(A) 10080 (B) 5040 (C) 5050 (D) 10100

24. An old man while dialing a 7 digit telephone number remembers that the first four digits consists of one 1's, one 2's and two 3's. He also remembers that the fifth digit is either a 4 or 5 while has no memorising of the sixth digit, he remembers that the seventh digit is 9 minus the sixth digit. Maximum number of distinct trials he has to try to make sure that he dials the correct telephone number, is

(A) 360 (B) 240 (C) 216 (D) none

25. Number of 5 digit numbers divisible by 25 that can be formed using only the digits

1, 2, 3, 4, 5 & 0 taken five at a time is

(A) 2 (B) 32 (C) 42 (D) 52

26. Let P_n denotes the number of ways of selecting 3 people out of 'n' sitting in a row, if no two of them are consecutive and Q_n is the corresponding figure when they are in a circle. If $P_n - Q_n = 6$, then 'n' is equal to :

(A) 8 (B) 9 (C) 10 (D) 12

27. Number of 7 digit numbers the sum of whose digits is 61 is :

(A) 12 (B) 24 (C) 28 (D) none

28. In a unique hockey series between India & Pakistan, they decide to play on till a team wins 5 matches. The number of ways in which the series can be won by India, if no match ends in a draw is :

(A) 126 (B) 252 (C) 225 (D) none

29. There are 100 different books in a shelf. Number of ways in which 3 books can be selected so that no two of which are neighbours is

(A) ${}^{100}C_3 - 98$ (B) ${}^{97}C_3$ (C) ${}^{96}C_3$ (D) ${}^{98}C_3$

30. Let P_n denotes the number of ways in which three people can be selected out of 'n' people sitting in a row, if no two of them are consecutive. If, $P_{n+1} - P_n = 15$ then the value of 'n' is :

(A) 7 (B) 8 (C) 9 (D) 10

MATCH THE COLUMN :

31.

Column-I	**Column-II**
(A) Number of increasing permutations of *m* symbols are there from the *n* set numbers $\{a_1, a_2, \ldots, a_n\}$ where the order among the numbers is given by $a_1 < a_2 < a_3 < \ldots a_{n-1} < a_n$ is	(P) n^m
(B) There are *m* men and *n* monkeys. Number of ways in which every monkey has a master, if a man can have any number of monkeys	(Q) ${}^mC_n$
(C) Number of ways in which *n* red balls and $(m-1)$ green balls can be arranged in a line, so that no two red balls are together, is (balls of the same colour are alike)	(R) ${}^nC_m$
(D) Number of ways in which '*m*' different toys can be distributed in '*n*' children if every child may receive any number of toys, is	(S) m^n

32. Number of permutations of 1, 2, 3, 4, 5, 6, 7, 8 and 9 taken all at a time, such that the digit

1 appearing somewhere to the left of 2

3 appearing to the left of 4 and

5 somewhere to the left of 6, is

(*e.g.* 815723946 would be one such permutation)

(A) $9 \cdot 7!$ (B) $8!$ (C) $5! \cdot 4!$ (D) $8! \cdot 4!$

33. Seven different coins are to be divided amongst three persons . If no two of the persons receive the same number of coins but each receives atleast one coin & none is left over, then the number of ways in which the division may be made is

(A) 420 (B) 630 (C) 710 (D) none

34. Number of ways in which 9 different toys be distributed among 4 children belonging to different age groups in such a way that distribution among the 3 elder children is even and the youngest one is to receive one toy more, is :

(A) $\frac{(5!)^2}{8}$ (B) $\frac{9!}{2}$ (C) $\frac{9!}{3!\,(2!)^3}$ (D) none

35. A gentleman invites a party of $m + n$ $(m \neq n)$ friends to a dinner & places m at one table T_1 and n at another table T_2 , the table being round . If not all people shall have the same neighbour in any two arrangement, then the number of ways in which he can arrange the guests, is

(A) $\frac{(m+n)!}{4\ mn}$ (B) $\frac{1}{2}\frac{(m+n)!}{mn}$ (C) $2\frac{(m+n)!}{mn}$ (D) none

36. A team of 8 students goes on an excursion, in two cars, of which one can seat 5 and the other only 4. If internal arrangement inside the car does not matter then the number of ways in which they can travel, is

(A) 91 (B) 182 (C) 126 (D) 3920

37. Let m denote the number of ways in which 4 different books are distributed among 10 persons, each receiving none or one only and let n denote the number of ways of distribution if the books are all alike. Then :

(A) $m = 4n$ (B) $n = 4m$ (C) $m = 24n$ (D) none

38. There are $(p + q)$ different books on different topics in Mathematics. $(p \neq q)$

If L = The number of ways in which these books are distributed between two students X and Y such that X get p books and Y gets q books.

M = The number of ways in which these books are distributed between two students X and Y such that one of them gets p books and another gets q books.

N = The number of ways in which these books are divided into two groups of p books and q books then,

(A) $L = M = N$ (B) $L = 2M = 2N$ (C) $2L = M = 2N$ (D) $L = M = 2N$

39. Number of ways in which 7 green bottles and 8 blue bottles can be arranged in a row if exactly 1 pair of green bottles is side by side, is (Assume all bottles to be alike except for the colour).

(A) 84 (B) 360 (C) 504 (D) none

40. There are 10 red balls of different shades & 9 green balls of identical shades. Then the number of arranging them in a row so that no two green balls are together is

(A) $(10!) \cdot {}^{11}P_9$ (B) $(10!) \cdot {}^{11}C_9$ (C) $10!$ (D) $10!\ 9!$

41. There are 2 identical white balls, 3 identical red balls and 4 green balls of different shades. The number of ways in which they can be arranged in a row so that atleast one ball is separated from the balls of the same colour, is :

(A) $6(7! - 4!)$ (B) $7(6! - 4!)$ (C) $8! - 5!$ (D) none

42. Number of ways in which 5 A's and 6 B's can be arranged in a row which reads the same backwards and forwards, is

(A) 6 (B) 8 (C) 10 (D) 12

43. Number of ways in which four different toys and five indistinguishable marbles can be distributed between Amar, Akbar and Anthony, if each child receives atleast one toy and one marble, is

(A) 42 (B) 100 (C) 150 (D) 216

44. There are counters available in x different colours. The counters are all alike except for the colour. The total number of arrangements consisting of y counters, assuming sufficient number of counters of each colour, if no arrangement consists of all counters of the same colour is :

(A) $x^y - x$ (B) $x^y - y$ (C) $y^x - x$ (D) $y^x - y$

45. The number of ways in which 8 distinguishable apples can be distributed among 3 boys such that every boy should get atleast 1 apple & atmost 4 apples is $K \cdot {}^7P_3$ where K has the value equal to

(A) 14 (B) 66 (C) 44 (D) 22

46. There are six periods in each working day of a school. Number of ways in which 5 subjects can be arranged if each subject is allotted at least one period and no period remains vacant is

(A) 210 (B) 1800 (C) 360 (D) 3600

47. Number of positive integral solutions satisfying the equation $(x_1 + x_2 + x_3)(y_1 + y_2) = 77$, is

(A) 150 (B) 270 (C) 420 (D) 1024

48. There are counters available in 3 different colours (atleast four of each colour). Counters are all alike except for the colour. If 'm' denotes the number of arrangements of four counters if no arrangement consists of counters of same colour and 'n' denotes the corresponding figure when every arrangement consists of counters of each colour, then :

(A) $m = 2n$ (B) $6m = 13n$ (C) $3m = 5n$ (D) $5m = 3n$

49. One hundred management students who read at least one of the three business magazines are surveyed to study the readership pattern. It is found that 80 read Business India, 50 read Business world, and 30 read Business Today. Five students read all the three magazines. How many read exactly two magazines?

(A) 50 (B) 10 (C) 95 (D) 65

50. A person writes letters to his 5 friends and addresses the corresponding envelopes. Number of ways in which the letters can be placed in the envelope, so that atleast two of them are in the wrong envelopes,is,

(A) 1 (B) 2 (C) 118 (D) 119

EXERCISE (O-2)

ONLY ONE CORRECT :

1. A committee of 5 is to be chosen from a group of 9 people. Number of ways in which it can be formed if two particular persons either serve together or not at all and two other particular persons refuse to serve with each other, is

(A) 41 (B) 36 (C) 47 (D) 76

2. There are m points on a straight line AB & n points on the line AC none of them being the point A. Triangles are formed with these points as vertices, when

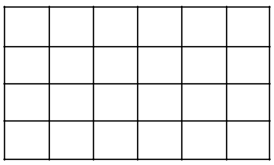
(i) A is excluded (ii) A is included. The ratio of number of triangles in the two cases is:

(A) $\frac{m+n-2}{m+n}$ (B) $\frac{m+n-2}{m+n-1}$ (C) $\frac{m+n-2}{m+n+2}$ (D) $\frac{m(n-1)}{(m+1)(n+1)}$

3. There are 10 straight lines in a plane, such that no 3 are concurrent and no 2 are parallel to each other. If points of intersection of above lines are joined, then maximum number of lines thus formed are (including old lines) -

(A) 610 (B) 620 (C) 630 (D) 640

4. Number of rectangles in the grid shown which are not squares is



(A) 160 (B) 162 (C) 170 (D) 185

5. Given 11 points, of which 5 lie on one circle, other than these 5, no 4 lie on one circle. Then the maximum number of circles that can be drawn so that each contains atleast three of the given points is :

(A) 216 (B) 156 (C) 172 (D) none

6. The number of ways of choosing a committee of 2 women & 3 men from 5 women & 6 men, if Mr. A refuses to serve on the committee if Mr. B is a member & Mr. B can only serve, if Miss C is the member of the committee, is

(A) 60 (B) 84 (C) 124 (D) none

7. Product of all the even divisors of N = 1000, is

(A) $32 \cdot 10^2$ (B) $64 \cdot 2^{14}$ (C) $64 \cdot 10^{18}$ (D) $128 \cdot 10^6$

8. Two classrooms A and B having capacity of 25 and (n–25) seats respectively.A_n denotes the number of possible seating arrangements of room 'A', when 'n' students are to be seated in these rooms, starting from room 'A' which is to be filled up full to its capacity. If $A_n - A_{n-1} = 25!\,({}^{49}C_{25})$ then 'n' equals -

(A) 50 (B) 48 (C) 49 (D) 51

MORE THAN ONE ARE CORRECT :

9. Lines $y = x + i$ & $y = -x + j$ are drawn in x – y plane such that $i \in \{1,2,3,4\}$ & $j \in \{1,2,3,4,5,6\}$. If m represents the total number of squares formed by the lines and n represents the total number of triangles formed by the given lines & x-axis, then correct option/s is/are-

(A) $m + n = 50$ (B) $m - n = 2$ (C) $m + n = 48$ (D) $m - n = 4$

10. The combinatorial coefficient C(n, r) is equal to

(A) number of possible subsets of r members from a set of n distinct members.

(B) number of possible binary messages of length n with exactly r 1's.

(C) number of non decreasing 2-D paths from the lattice point (0, 0) to (r, n).

(D) number of ways of selecting r things out of n different things when a particular thing is always included plus the number of ways of selecting 'r' things out of n, when a particular thing is always excluded.

11. There are 10 questions, each question is either True or False. Number of different sequences of incorrect answers is also equal to

(A) Number of ways in which a normal coin tossed 10 times would fall in a definite order if both Heads and Tails are present.

(B) Number of ways in which a multiple choice question containing 10 alternatives with one or more than one correct alternatives, can be answered.

(C) Number of ways in which it is possible to draw a sum of money with 10 coins of different denominations taken some or all at a time.

(D) Number of different selections of 10 indistinguishable things taken some or all at a time.

12. Number of ways in which 3 numbers in A.P. can be selected from 1, 2, 3, ... n is :

(A) $\left(\frac{n-1}{2}\right)^2$ if n is even

(B) $\frac{n(n-2)}{4}$ if n is odd

(C) $\frac{(n-1)^2}{4}$ if n is odd

(D) $\frac{n(n-2)}{4}$ if n is even

13. The combinatorial coefficient ${}^{n-1}C_p$ denotes

(A) the number of ways in which *n* things of which p are alike and rest different can be arranged in a circle.

(B) the number of ways in which *p* different things can be selected out of *n* different thing if a particular thing is always excluded.

(C) number of ways in which *n* alike balls can be distributed in *p* different boxes so that no box remains empty and each box can hold any number of balls.

(D) the number of ways in which (n – 2) white balls and p black balls can be arranged in a line if black balls are separated, balls are all alike except for the colour.

14. In a certain strange language, words are written with letters from the following six-letter alphabet : A, G, K, N, R, U. Each word consists of six letters and none of the letters repeat. Each combination of these six letters is a word in this language. The word "KANGUR" remains in the dictionary at,

(A) 248^{th} (B) 247^{th} (C) 246^{th} (D) 253^{rd}

15. Six people are going to sit in a row on a bench. A and B are adjacent, C does not want to sit adjacent to D. E and F can sit anywhere. Number of ways in which these six people can be seated, is

(A) 200 (B) 144 (C) 120 (D) 56

16. Six married couple are sitting in a room. Find the number of ways in which 4 people can be selected so that :

(A) they do not form a couple

(B) they form exactly one couple

(C) they form at least one couple

(D) they form atmost one couple

17. The number of three digit numbers having only two consecutive digits identical is :

(A) 153 (B) 162 (C) 180 (D) 161

18. Number of 3 digit numbers in which the digit at hundredth's place is greater than the other two digit is

(A) 285 (B) 281 (C) 240 (D) 204

19. All possible three digits even numbers which can be formed with the condition that if 5 is one of the digit, then 7 is the next digit is :

(A) 5 (B) 325 (C) 345 (D) 365

Paragraph for Question Nos. 20 to 22

16 players P_1, P_2, P_3,.......P_{16} take part in a tennis tournament. Lower suffix player is better than any higher suffix player. These players are to be divided into 4 groups each comprising of 4 players and the best from each group is selected for semifinals.

20. Number of ways in which 16 players can be divided into four equal groups, is

(A) $\frac{35}{27}\prod_{r=1}^{8}(2r-1)$ (B) $\frac{35}{24}\prod_{r=1}^{8}(2r-1)$ (C) $\frac{35}{52}\prod_{r=1}^{8}(2r-1)$ (D) $\frac{35}{6}\prod_{r=1}^{8}(2r-1)$

21. Number of ways in which they can be divided into 4 equal groups if the players P_1, P_2, P_3 and P_4 are in different groups, is :

(A) $\frac{(11)!}{36}$ (B) $\frac{(11)!}{72}$ (C) $\frac{(11)!}{108}$ (D) $\frac{(11)!}{216}$

22. Number of ways in which these 16 players can be divided into four equal groups, such that when the best player is selected from each group, P_6 is one among them, is (k) $\frac{12!}{(4!)^3}$. The value of k is :

(A) 36 (B) 24 (C) 18 (D) 20

23. Consider the word W = MISSISSIPPI

(a) If N denotes the number of different selections of 5 letters from the word W = MISSISSIPPI then N belongs to the set

(A) {15, 16, 17, 18, 19} (B) {20, 21, 22, 23, 24}

(C) {25, 26, 27, 28, 29} (D) {30, 31, 32, 33, 34}

(b) Number of ways in which the letters of the word W can be arranged if atleast one vowel is separated from rest of the vowels

(A) $\frac{8!\cdot 161}{4!\cdot 4!\cdot 2!}$ (B) $\frac{8!\cdot 161}{4\cdot 4!\cdot 2!}$ (C) $\frac{8!\cdot 161}{4!\cdot 2!}$ (D) $\frac{8!}{4!\cdot 2!}\cdot\frac{165}{4!}$

(c) If the number of arrangements of the letters of the word W if all the S's and P's are separated is $(K)\left(\frac{10!}{4!\cdot 4!}\right)$, then K equals -

(A) $\frac{6}{5}$ (B) 1 (C) $\frac{4}{3}$ (D) $\frac{3}{2}$

24. The maximum number of permutations of 2n letters in which there are only a's & b's, taken all at a time is given by :

(A) ${}^{2n}C_n$

(B) $\frac{2}{1}\cdot\frac{6}{2}\cdot\frac{10}{3}\ldots\ldots\frac{4n-6}{n-1}\cdot\frac{4n-2}{n}$

(C) $\frac{n+1}{1}\cdot\frac{n+2}{2}\cdot\frac{n+3}{3}\cdot\frac{n+4}{4}\ldots\ldots\frac{2n-1}{n-1}\cdot\frac{2n}{n}$

(D) $\frac{2^n\cdot[1.3.5\ldots\ldots(2n-3)(2n-1)]}{n!}$

25. Number of ways in which the letters of the word 'B U L B U L' can be arranged in a line in a definite order is also equal to the

(A) number of ways in which 2 alike Apples and 4 alike Mangoes can be distributed in 3 children so that each child receives any number of fruits.

(B) Number of ways in which 6 different books can be tied up into 3 bundles, if each bundle is to have equal number of books.

(C) coefficient of $x^2y^2z^2$ in the expansion of $(x + y + z)^6$.

(D) number of ways in which 6 different prizes can be distributed equally in three children.

26. Which of the following statements are correct?

(A) Number of words that can be formed with 6 only of the letters of the word "CENTRIFUGAL" if each word must contain all the vowels is $3 \cdot 7!$

(B) There are 15 balls of which some are white and the rest black. If the number of ways in which the balls can be arranged in a row, is maximum then the number of white balls must be equal to 7 or 8. Assume balls of the same colour to be alike.

(C) There are 12 things, 4 alike of one kind, 5 alike and of another kind and the rest are all different. The total number of combinations is 240.

(D) Number of selections that can be made of 6 letters from the word "COMMITTEE" is 35.

MATCH THE COLUMN:

27.

	Column-I		**Column-II**
(A)	Four different movies are running in a town. Ten students go to watch these four movies. The number of ways in which every movie is watched by atleast one student, is (Assume each way differs only by number of students watching a movie)	(P)	11
(B)	Consider 8 vertices of a regular octagon and its centre. If T denotes the number of triangles and S denotes the number of straight lines that can be formed with these 9 points then the value of (T – S) equals	(Q)	36
(C)	In an examination, 5 children were found to have their mobiles in their pocket. The Invigilator fired them and took their mobiles in his possession. Towards the end of the test, Invigilator randomly returned their mobiles. The number of ways in which at most two children did not get their own mobiles is	(R)	52
(D)	The product of the digits of 3214 is 24. The number of 4 digit natural numbers such that the product of their digits is 12, is	(S)	60
(E)	The number of ways in which a mixed double tennis game can be arranged from amongst 5 married couple if no husband & wife plays in the same game, is	(T)	84

28. A guardian with 6 wards wishes everyone of them to study either Law or Medicine or Engineering. Number of ways in which he can make up his mind with regard to the education of his wards if every one of them be fit for any of the branches to study, and atleast one child is to be sent in each discipline is :

(A) 120 (B) 216 (C) 729 (D) 540

EXERCISE (S-1)

1. Four visitors A, B, C & D arrive at a town which has 5 hotels. In how many ways can they disperse themselves among 5 hotels, if 4 hotels are used to accommodate them.

2. There are 6 roads between A & B and 4 roads between B & C.

(i) In how many ways can one drive from A to C by way of B ?

(ii) In how many ways can one drive from A to C and back to A, passing through B on both trips ?

(iii) In how many ways can one drive the circular trip described in (ii) without using the same road more than once.

3. (i) Find the number of four letter word that can be formed from the letters of the word HISTORY. (each letter to be used atmost once)

(ii) How many of them contain only consonants?

(iii) How many of them begin & end in a consonant?

(iv) How many of them begin with a vowel?

(v) How many contain the letters Y?

(vi) How many begin with T & end in a vowel?

(vii) How many begin with T & also contain S?

(viii) How many contain both vowels?

4. If repetitions are not permitted

(i) How many 3 digit numbers can be formed from the six digits 2, 3, 5, 6, 7 & 9 ?

(ii) How many of these are less than 400 ?

(iii) How many are even ?

(iv) How many are odd ?

(v) How many are multiples of 5 ?

5. How many two digit numbers are there in which the tens digit and the units digit are different and odd ?

6. Every telephone number consists of 7 digits. How many telephone numbers are there which do not include any other digits but 2 , 3 , 5 & 7 ?

7. (a) In how many ways can four passengers be accommodated in three railway carriages, if each carriage can accommodate any number of passengers.

(b) In how many ways four persons can be accommodated in 3 different chairs if each person can occupy only one chair.

8. How many odd numbers of five distinct digits can be formed with the digits 0,1,2,3,4 ?

9. Number of ways in which 7 different colours in a rainbow can be arranged if green is always in the middle.

10. Find the number of ways in which the letters of the word "MIRACLE" can be arranged if vowels always occupy the odd places.

11. A letter lock consists of three rings each marked with 10 different letters. Find the number of ways in which it is possible to make an unsuccessful attempts to open the lock.

12. (i) Prove that : ${}^{n}P_{r} = {}^{n-1}P_{r} + r.\, {}^{n-1}P_{r-1}$

(ii) If ${}^{20}C_{r+2} = {}^{20}C_{2r-3}$, find ${}^{12}C_{r}$.

(iii) Prove that : ${}^{n-1}C_{3} + {}^{n-1}C_{4} > {}^{n}C_{3}$ if $n > 7$.

(iv) Find r if ${}^{15}C_{3r} = {}^{15}C_{r+3}$.

13. Find the number of ways in which two squares can be selected from an 8 by 8 chess board of size 1×1 so that they are not in the same row and in the same column.

14. There are 10 seats in a double decker bus, 6 in the lower deck and 4 on the upper deck. Ten passengers board the bus, of them 3 refuse to go to the upper deck and 2 insist on going up. The number of ways in which the passengers can be accommodated is _____. (Assume all seats to be duly numbered)

15. In a certain algebraical exercise book there are 4 examples on arithmetical progressions, 5 examples on permutation-combination and 6 examples on binomial theorem . Number of ways a teacher can select for his pupils atleast one but not more than 2 examples from each of these sets, is _______.

16. In how many ways can a team of 6 horses be selected out of a stud of 16, so that there shall always be 3 out of A B C A′ B′ C′, but never A A′, B B′ or C C′ together.

17. There are 2 women participating in a chess tournament. Every participant played 2 games with the other participants. The number of games that the men played between themselves exceeded by 66 as compared to the number of games that the men played with the women. Find the number of participants & the total numbers of games played in the tournament.

18. Each of 3 committees has 1 vacancy which is to be filled from a group of 6 people. Find the number of ways the 3 vacancies can be filled if ;

(i) Each person can serve on atmost 1 committee.

(ii) There is no restriction on the number of committees on which a person can serve.

(iii) Each person can serve on atmost 2 committees.

19. Find the number of ways in which 3 distinct numbers can be selected from the set $\{3^1, 3^2, 3^3, ... 3^{100}, 3^{101}\}$ so that they form a G.P.

20. Find the number of ways in which letters of the word VALEDICTORY be arranged so that the vowels may never be separated.

21. An examination paper consists of 12 questions divided into parts A & B.
Part-A contains 7 questions & Part-B contains 5 questions. A candidate is required to attempt 8 questions selecting atleast 3 from each part. In how many maximum ways can the candidate select the questions ?

22. 5 boys & 4 girls sit in a straight line. Find the number of ways in which they can be seated if 2 girls are together & the other 2 are also together but separate from the first 2.

23. During a draw of lottery, tickets bearing numbers 1, 2, 3,......, 40, 6 tickets are drawn out & then arranged in the descending order of their numbers. In how many ways, it is possible to have 4^{th} ticket bearing number 25.

24. Find the number of distinct natural numbers upto a maximum of 4 digits and divisible by 5, which can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 each digit not occuring more than once in each number.

25. A firm of Chartered Accountants in Bombay has to send 10 clerks to 5 different companies, two clerks in each. Two of the companies are in Bombay and the others are outside. Two of the clerks prefer to work in Bombay while three others prefer to work outside. In how many ways can the assignment be made if the preferences are to be satisfied.

26. Find the number of permutations of the word "AUROBIND" in which vowels appear in an alphabetical order.

27. Define a 'good word' as a sequence of letters that consists only of the letters A, B and C and in which A never immediately followed by B, B is never immediately followed by C, and C is never immediately followed by A. If the number of n-letter good words are 384, find the value of n.

28. In how many different ways a grandfather along with two of his grandsons and four grand daughters can be seated in a line for a photograph so that he is always in the middle and the two grandsons are never adjacent to each other.

29. If as many more words as possible be formed out of the letters of the word "DOGMATIC" then find the number of words in which the relative order of vowels and consonants remain unchanged.

30. There are 10 different books in a shelf. Find the number of ways in which 3 books can be selected so that exactly two of them are consecutive.

31. In how many ways can you divide a pack of 52 cards equally among 4 players. In how many ways the cards can be divided in 4 sets, 3 of them having 17 cards each & the 4^{th} with 1 card.

32. Find the number of ways in which the letters of the word 'KUTKUT' can be arranged so that no two alike letters are together.

33. How many 6 digits odd numbers greater than 60,0000 can be formed from the digits 5, 6, 7, 8, 9, 0 if

(i) repetitions are not allowed **(ii)** repetitions are allowed.

34. In how many other ways can the letters of the word **MULTIPLE** be arranged;

(i) without changing the order of the vowels

(ii) keeping the position of each vowel fixed &

(iii) without changing the relative order/position of vowels & consonants.

Paragraph for Question 35 & 36

Consider the number N = 2910600.

On the basis of above information, answer the following questions :

35. Total number of divisors of N, which are divisible by 15 but not by 36 are-

(A) 92 (B) 94 (C) 96 (D) 98

36. Total number of ways, in which the given number can be split into two factors such that their highest common factor is a prime number is equal to-

(A) 16 (B) 32 (C) 48 (D) 64

37. (a) How many divisors are there of the number x = 21600. Find also the sum of these divisors.

(b) In how many ways the number 7056 can be resolved as a product of 2 factors.

(c) Find the number of ways in which the number 300300 can be split into 2 factors which are relatively prime.

(d) Find the number of positive integers that are divisors of atleast one of the numbers 10^{10} ; 15^7 ; 18^{11}.

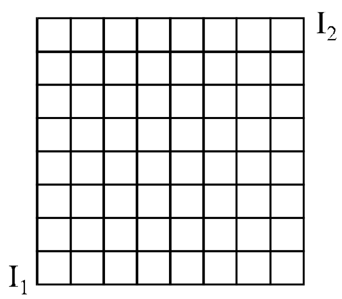
SUBJETIVE :

38. A committee of 10 members is to be formed with members chosen from the faculties of Arts, Economics, Education, Engineering, Medicine and Science. Number of possible ways in which the faculties representation be distributed on this committee, is ________.

(Assume every department contains more than 10 members).

39. If x_1,x_2,x_3 are the whole numbers and gives remainders 0,1,2 respectively, when divided by 3 then total number of different solutions of the equation $x_1 + x_2 + x_3 = 33$ are k, then $\frac{k}{11}$ is equal to

40. On the normal chess board as shown, I_1 & I_2 are two insects which starts moving towards each other. Each insect moving with the same constant speed . Insect I_1 can move only to the right or upward along the lines while the insect I_2 can move only to the left or downward along the lines of the chess board. Find the total number of ways the two insects can meet at same point during their trip.



41. Determine the number of paths from the origin to the point (9, 9) in the cartesian plane which never pass through (5, 5) in paths consisting only of steps going 1 unit North and 1 unit East.

42. There are 20 books on Algebra & Calculus in our library. Prove that the greatest number of selections each of which consists of 5 books on each topic is possible only when there are 10 books on each topic in the library.

EXERCISE (JM)

1. There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is - **[AIEEE-2010]**

(1) 3 (2) 36 (3) 66 (4) 108

2. **Statement - 1 :** The number of ways of distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^9C_3$. **[AIEEE-2011]**

Statement - 2 : The number of ways of choosing any 3 places from 9 different places is ${}^9C_3$.

(1) Statement-1 is true, Statement-2 is false.

(2) Statement-1 is false, Statement-2 is true

(3) Statement-1 is true, Statement-2 is true; Statement-2 is a correct explanation for Statement-1

(4) Statement-1 is true, Statement-2 is true; Statement-2 is **not** a correct explanation for Statement-1.

3. There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then : **[AIEEE-2011]**

(1) $N > 190$ (2) $N \leq 100$ (3) $100 < N \leq 140$ (4) $140 < N \leq 190$

4. Assuming the balls to be identical except for difference in colours, the number of ways in which one or more balls can be selected from 10 white, 9 green and 7 black balls is - **[AIEEE-2012]**

(1) 879 (2) 880 (3) 629 (4) 630

5. Let A and B be two sets containing 2 elements and 4 elements respectively. The number of subsets of $A \times B$ having 3 or more elements is **[JEE (Main)-2013]**

(1) 256 (2) 220 (3) 219 (4) 211

6. Let T_n be the number of all possible triangles formed by joining vertices of an n-sided regular polygon. If $T_{n+1} - T_n = 10$, then the value of n is : **[JEE (Main)-2013]**

(1) 7 (2) 5 (3) 10 (4) 8

7. The number of points, having both co-ordinates as integers, that lie in the interior of the triangle with vertices (0, 0), (0, 41) and (41, 0) is : **[JEE (Main)-2015]**

(1) 820 (2) 780 (3) 901 (4) 861

8. Let A and B be two sets containing four and two elements respectively. Then the number of subsets of the set $A \times B$, each having at least three elements is : **[JEE (Main)-2015]**

(1) 275 (2) 510 (3) 219 (4) 256

9. The number of integers greater than 6000 that can be formed, using the digits 3,5,6,7 and 8 without repetition, is : **[JEE (Main)-2015]**

(1) 120 (2) 72 (3) 216 (4) 192

10. If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary; then the position of the word SMALL is :

[JEE (Main)-2016]

(1) 58^{th} (2) 46^{th} (3) 59^{th} (4) 52^{nd}

11. A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is : **[JEE (Main)-2017]**

(1) 484 (2) 485 (3) 468 (4) 469

12. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf so that the dictionary is always in the middle. The number of such arrangements is- **[JEE(Main)-2018]**

(1) less than 500 (2) at least 500 but less than 750

(3) at least 750 but less than 1000 (4) at least 1000

13. Let S = {1,2,3, ..., 100}. The number of non-empty subsets A of S such that the product of elements in A is even is :- **[JEE(Main)-2019]**

(1) $2^{50}(2^{50}–1)$ (2) $2^{100}–1$ (3) $2^{50}–1$ (4) $2^{50}+1$

14. All possible numbers are formed using the digits 1, 1, 2, 2, 2, 2, 3, 4, 4 taken all at a time. The number of such numbers in which the odd digits occupy even places is : **[JEE(Main)-2019]**

(1) 175 (2) 162 (3) 160 (4) 180

15. The number of four-digit numbers strictly greater than 4321 that can be formed using the digits 0,1,2,3,4,5 (repetition of digits is allowed) is : **[JEE(Main)-2019]**

(1) 288 (2) 306 (3) 360 (4) 310

16. The number of 6 digit numbers that can be formed using the digits 0, 1, 2, 5, 7 and 9 which are divisible by 11 and no digit is repeated, is : **[JEE(Main)-2019]**

(1) 36 (2) 60 (3) 48 (4) 72

17. Suppose that 20 pillars of the same height have been erected along the boundary of a circular stadium. If the top of each pillar has been connected by beams with the top of all its non-adjacent pillars, then the total number beams is : **[JEE(Main)-2019]**

(1) 210 (2) 190 (3) 170 (4) 180

18. The number of ways of choosing 10 objects out of 31 objects of which 10 are identical and the remaining 21 are distinct, is : **[JEE(Main)-2019]**

(1) 2^{20} (2) $2^{20} - 1$ (3) $2^{20} + 1$ (4) 2^{21}

19. Total number of 6-digit numbers in which only and all the five digits 1, 3, 5, 7 and 9 appear, is:

[JEE(Main)-2020]

(1) $\frac{5}{2}(6!)$ (2) 5^6 (3) $\frac{1}{2}(6!)$ (4) $6!$

20. The number of 4 letter words (with or without meaning) that can be formed from the eleven letters of the word 'EXAMINATION' is _______. **[JEE(Main)-2020]**

21. If a,b and c are the greatest value of ${}^{19}C_p, {}^{20}C_q$ and ${}^{21}C_r$ respectively, then **[JEE(Main)-2020]**

(1) $\frac{a}{11} = \frac{b}{22} = \frac{c}{21}$ (2) $\frac{a}{10} = \frac{b}{11} = \frac{c}{21}$ (3) $\frac{a}{10} = \frac{b}{11} = \frac{c}{42}$ (4) $\frac{a}{11} = \frac{b}{22} = \frac{c}{42}$

22. An urn contains 5 red marbles, 4 black marbles and 3 white marbles. Then the number of ways in which 4 marbles can be drawn so that at the most three of them are red is .... **[JEE(Main)-2020]**

EXERCISE (JA)

1. Let S = {1,2,3,4}. The total number of unordered pairs of disjoint subsets of S is equal to - **[JEE 10, 5M, –2M]**

(A) 25 (B) 34 (C) 42 (D) 41

2. The total number of ways in which 5 balls of different colours can be distributed among 3 persons so that each person gets at least one ball is - **[JEE 2012, 3M, –1M]**

(A) 75 (B) 150 (C) 210 (D) 243

Paragraph for Question 3 and 4

Let a_n denotes the number of all n-digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n = the number of such n-digit integers ending with digit 1 and c_n = the number of such n-digit integers ending with digit 0.

3. The value of b_6 is **[JEE 2012, 3M, –1M]**

(A) 7 (B) 8 (C) 9 (D) 11

4. Which of the following is correct ? **[JEE 2012, 3M, –1M]**

(A) $a_{17} = a_{16} + a_{15}$ (B) $c_{17} \neq c_{16} + c_{15}$ (C) $b_{17} \neq b_{16} + c_{16}$ (D) $a_{17} = c_{17} + b_{16}$

5. Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. The number of such distinct arrangements (n_1,n_2,n_3,n_4,n_5) is **[JEE(Advanced)-2014, 3]**

6. Let $n \geq 2$ b an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is **[JEE(Advanced)-2014, 3]**

7. Six cards and six envelopes are numbered 1, 2, 3, 4, 5, 6 and cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 in always placed in envelope numbered 2. Then the number of ways it can be done is - **[JEE(Advanced)-2014, 3(–1)]**

(A) 264 (B) 265 (C) 53 (D) 67

8. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue. Then the value of $\frac{m}{n}$ is **[JEE (Advanced) 2015, 4M, –0M]**

9. A debate club consists of 6 girls and 4 body. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 member) for the team. If the team has to include at most one boy, then the number of ways of selecting the team is **[JEE(Advanced)-2016, 3(–1)]**

(A) 380 (B) 320 (C) 260 (D) 95

10. Words of length 10 are formed using the letters A, B, C, D, E, F, G, H, I, J. Let x be the number of such words where no letter is repeated; and let y be the number of such words where exactly one letter is repeated twice and no other letter is repeated. Then $\frac{y}{9x} =$

[JEE(Advanced)-2017, 3]

11. Let S = {1, 2, 3,.....,9}. For k = 1,2, ..., 5, let N_k be the number of subsets of S, each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$

[JEE(Advanced)-2017, 3(–1)]

(A) 125 (B) 252 (C) 210 (D) 126

12. The number of 5 digit numbers which are divisible by 4, with digits from the set {1, 2, 3, 4, 5} and the repetition of digits is allowed, is ______ **[JEE(Advanced)-2018, 3(0)]**

13. In a high school, a committee has to be formed from a group of 6 boys M_1, M_2, M_3, M_4, M_5, M_6 and 5 girls G_1, G_2, G_3, G_4, G_5.

(i) Let α_1 be the total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 boy and 2 girls.

(ii) Let α_2 be the total number of ways in which the committe can be formed such that the committee has at least 2 members, and having an equal number of boys and girls.

(iii) Let α_3 be the total number of ways in which the committe can be formed such that the committee has 5 members, at least 2 of them being girls.

(iv) Let α_4 be the total number of ways in which the committee can be formed such that the commitee has 4 members, having at least 2 girls and such that both M_1 and G_1 are **NOT** in the committee together.

LIST-I	**LIST-II**
P. The value of α_1 is	**1.** 136
Q. The value of α_2 is	**2.** 189
R. The value of α_3 is	**3.** 192
S. The value of α_4 is	**4.** 200
	5. 381
	6. 461

The correct option is :-

(A) **P → 4; Q → 6, R → 2; S → 1** (B) **P → 1; Q → 4; R → 2; S → 3**

(C) **P → 4; Q → 6, R → 5; S → 2** (D) **P → 4; Q → 2; R → 3; S → 1**

[JEE(Advanced)-2018, 3(–1)]

14. Five person A,B,C,D and E are seated in a ciruclar arrangement. If each of them is given a hat of one of the three colours red, blue and green ,then the number of ways of distributing the hats such that the persons seated in adjacent seats get different coloured hats is

[JEE(Advanced)-2019, 3(0)]

ANSWER KEY

EXERCISE (O-1)

1. A	**2.** A	**3.** C	**4.** C	**5.** C	**6.** D	**7.** A	**8.** A
9. C	**10.** B	**11.** B	**12.** C	**13.** D	**14.** C	**15.** D	**16.** C
17. A	**18.** B	**19.** D	**20.** C	**21.** B	**22.** D	**23.** B	**24.** B
25. C	**26.** C	**27.** C	**28.** A	**29.** D	**30.** B		
31. (A) R; (B) S; (C) Q; (D) P			**32.** A	**33.** B	**34.** C	**35.** A	**36.** C
37. C	**38.** C	**39.** C	**40.** B	**41.** A	**42.** C	**43.** D	**44.** A
45. D	**46.** B	**47.** C	**48.** B	**49.** A	**50.** D		

EXERCISE (O-2)

1. A	**2.** A	**3.** D	**4.** A	**5.** B	**6.** C	**7.** C	**8.** A
9. A,B	**10.** A,B,D	**11.** B,C	**12.** C,D	**13.** B,D	**14.** A	**15.** B	
16. 240, 240, 255, 480		**17.** B	**18.** A	**19.** D	**20.** A	**21.** C	**22.** D
23. (a) C ; (b) B ; (c) B		**24.** A,B,C,D		**25.** A,C,D	**26.** A,B,D		
27. (A) T; (B) R; (C) P; (D) Q; (E) S			**28.** D				

EXERCISE (S-1)

1. 120 **2.** (i) 24 ; (ii) 576; (iii) 360

3. (i) 840; (ii) 120; (iii) 400; (iv) 240; (v) 480; (vi) 40; (vii) 60; (viii) 240

4. (i) 120; (ii) 40; (iii) 40; (iv) 80; (v) 20

5. 20	**6.** 4^7	**7.** (a) 3^4 ; (b) 24	**8.** 36	**9.** 720	**10.** 576
11. 999	**12.** (ii) 792; (iv) r = 3		**13.** 1568		**14.** 172800
15. 3150	**16.** 960	**17.** 13, 156	**18.** 120, 216, 210		**19.** 2500
20. 967680	**21.** 420	**22.** 43200	**23.** ${}^{24}C_2 \cdot {}^{15}C_3$		**24.** 1106
25. 5400	**26.** ${}^8C_4 \cdot 4!$	**27.** n = 8	**28.** 528		**29.** 719
30. 56	**31.** $\frac{52!}{(13!)^4}$; $\frac{52!}{3!(17!)^3}$		**32.** 30		**33.** 240, 15552
34. (i) 3359; (ii) 59; (iii) 359			**35.** C		**36.** C
37. (a) 72; 78120; (b) 23; (c) 32 ; (d) 435			**38.** 3003		**39.** 6
40. 12870	**41.** 30980				

EXERCISE (JM)

1. 4	**2.** 3	**3.** 2	**4.** 1	**5.** 3	**6.** 2	**7.** 2	
8. 3	**9.** 4	**10.** 1	**11.** 2	**12.** 4	**13.** 1	**14.** 4	
15. 4	**16.** 2	**17.** 3	**18.** 1	**19.** **1**	**20.** **2454**	**21.** 4	**22.** 490.00

EXERCISE (JA)

1. D	**2.** B	**3.** B	**4.** A	**5.** 7	**6.** 5	**7.** C
8. 5	**9.** A	**10.** 5	**11.** D	**12.** 625	**13.** C	**14.** 30.00

CHAPTER 2

BINOMIAL THEOREM

Chapter Contents 02

IMPORTANT NOTES

CHAPTER 2

BINOMIAL THEOREM

1. BINOMIAL EXPRESSION :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : $x - y,\ xy + \frac{1}{x}, \frac{1}{z} - 1, \frac{1}{(x-y)^{1/3}} + 3$ etc.

2. BINOMIAL THEOREM :

The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as **BINOMIAL THEOREM.**

If x, y $\in$ R and n $\in$ N, then :

$$(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots\ldots + {}^nC_rx^{n-r}y^r + \ldots\ldots + {}^nC_ny^n = \sum_{r=0}^{n} {}^nC_rx^{n-r}y^r$$

This theorem can be proved by induction.

Observations :

(a) The number of terms in the expansion is (n+1) i.e. one more than the index.

(b) The sum of the indices of x & y in each term is n.

(c) The binomial coefficients of the terms **(^{n}C$_0$, ^{n}C$_1$.....)** equidistant from the beginning and the end are equal. i.e. $\mathbf{{}^nC_r = {}^nC_{r-1}}$

(d) Symbol $\mathbf{{}^nC_r}$ can also be denoted by $\binom{\mathbf{n}}{\mathbf{r}}$, **C(n, r)** or $\mathbf{A_r^n}$.

Some important expansions :

(i) $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots.. + {}^nC_nx^n$.

(ii) $(1 - x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 + \ldots\ldots.. + (-1)^n \cdot {}^nC_nx^n$.

Note : The coefficient of x^r in $(1 + x)^n = {}^nC_r$ & that in $(1-x)^n = (-1)^r \cdot {}^nC_r$

Illustrations

Illustration 1 : Expand : $(y + 2)^6$.

Solution : ${}^6C_0y^6 + {}^6C_1y^5.2 + {}^6C_2y^4.2^2 + {}^6C_3y^3.2^3 + {}^6C_4y^2.\ 2^4 + {}^6C_5y^1 \cdot 2^5 + {}^6C_6 \cdot 2^6$.

$= y^6 + 12y^5 + 60y^4 + 160y^3 + 240y^2 + 192y + 64$.

Illustration 2 : Write first 4 terms of $\left(1 - \frac{2y^2}{5}\right)^7$

Solution : ${}^7C_0, {}^7C_1\left(-\frac{2y^2}{5}\right), {}^7C_2\left(-\frac{2y^2}{5}\right)^2, {}^7C_3\left(-\frac{2y^2}{5}\right)^3$

Illustration 3 : If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficients of x and x^2 are 3 and – 6 respectively then m is - **[JEE 99]**

(A) 6 (B) 9 (C) 12 (D) 24

Solution :

$$(1 + x)^m (1 - x)^n = \left[1 + mx + \frac{(m)(m-1).x^2}{2} + ...\right]\left[1 - nx + \frac{n(n-1)}{2}x^2 + ...\right]$$

Coefficient of $x = m - n = 3$ ....(i)

Coefficient of $x^2 = -mn + \frac{n(n+1)}{2} + \frac{m(m-1)}{2} = -6$ ....(ii)

Solving (i) and (ii), we get

m = 12 and n = 9.

Do yourself - 1

(i) Expand $\left(3x^2 - \frac{x}{2}\right)^5$ **(ii)** Expand $(y + x)^n$

Pascal's triangle :

$(x+y)^0$	1	1
$(x+y)^1$	$x + y$	1 1
$(x+y)^2$	$x^2 + 2xy + y^2$	1 2 1
$(x+y)^3$	$x^3 + 3x^2y + 3xy^2 + y^3$	1 3 3 1
$(x+y)^4$	$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^3$	1 4 6 4 1

Pascal's triangle

(i) **Pascal's triangle -** A triangular arrangement of numbers as shown. The numbers give the binomial coefficients for the expansion of $(x + y)^n$. The first row is for n = 0, the second for n = 1, etc. Each row has 1 as its first and last number. Other numbers are generated by adding the two numbers immediately to the left and right in the row above.

(ii) Pascal triangle is formed by binomial coefficient.

(iii) The number of terms in the expansion of $(x+y)^n$ is (n + 1) i.e. one more than the index.

(iv) The sum of the indices of x & y in each term is n.

(v) Power of first variable (x) decreases while of second variable (y) increases.

(vi) Binomial coefficients are also called **combinatorial coefficients.**

(vii) Binomial coefficients of the terms equidistant from the begining and end are equal.

(viii) r^{th} term from the beginning in the expansion of $(x + y)^n$ is same as r^{th} term from end in the expansion of $(y + x)^n$.

(ix) r^{th} term from the end in $(x + y)^n$ is **$(n - r + 2)^{th}$ term from the beginning**.

3. IMPORTANT TERMS IN THE BINOMIAL EXPANSION :

(a) General term: The general term or the $(r+1)^{th}$ term in the expansion of $(x+y)^n$ is given by

$T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$

Illustrations

Illustration 4 : Find : (a) The coefficient of x^7 in the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$

(b) The coefficient of x^{-7} in the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$

Also, find the relation between a and b, so that these coefficients are equal.

Solution : (a) In the expansion of $\left(ax^2+\frac{1}{bx}\right)^{11}$, the general term is :

$$T_{r+1} = {}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r.\frac{a^{11-r}}{b^r}.x^{22-3r}$$

putting $22 - 3r = 7$

$\therefore \quad 3r = 15 \quad \Rightarrow \quad r = 5$

$$\therefore \quad T_6 = {}^{11}C_5\frac{a^6}{b^5}.x^7$$

Hence the coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ is ${}^{11}C_5a^6b^{-5}$. **Ans.**

Note that binomial coefficient of sixth term is ${}^{11}C_5$.

(b) In the expansion of $\left(ax-\frac{1}{bx^2}\right)^{11}$, general term is :

$$T_{r+1} = {}^{11}C_r(ax)^{11-r}\left(\frac{-1}{bx^2}\right)^r = (-1)^r\,{}^{11}C_r\frac{a^{11-r}}{b^r}.x^{11-3r}$$

putting $11 - 3r = -7$

$\therefore \quad 3r = 18 \quad \Rightarrow \quad r = 6$

$$\therefore \quad T_7 = (-1)^6.\ {}^{11}C_6\frac{a^5}{b^6}.x^{-7}$$

Hence the coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ is $^{11}C_6a^5b^{-6}$. **Ans.**

Also given :

Coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ = coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$

$\Rightarrow \quad {}^{11}C_5a^6b^{-5} = {}^{11}C_6a^5b^{-6}$

$\Rightarrow \quad ab = 1 \qquad (\because \ {}^{11}C_5 = {}^{11}C_6)$

which is the required relation between a and b. **Ans.**

Illustration 5 : Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Solution : The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r\left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{\frac{1}{6}}\right)^r = {}^{1000}C_r\ 3^{\frac{1000-r}{2}}2^{\frac{r}{2}}$$

The above term will be rational if exponents of 3 and 2 are integers

It means $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, ..., 1000}

Hence, number of rational terms is 501 **Ans.**

(b) Middle term :

The middle term(s) in the expansion of $(x + y)^n$ is (are) :

(i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2}.\ x^{n/2}.\ y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

Important Note :

Middle term has greatest binomial coefficient and if there are 2 middle terms their coefficients will be equal.

$$\Rightarrow {}^nC_r \text{ will be maximum } \begin{cases} \text{When } r=\frac{n}{2} \text{ if n is even} \\ \text{When } r=\frac{n-1}{2} \text{ or } \frac{n+1}{2} \text{ if n is odd} \end{cases}$$

$\Rightarrow$ The term containing greatest binomial coefficient will be middle term in the expansion of $(1 + x)^n$

Illustrations

Illustration 6 : Find the middle term in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$

Solution : The number of terms in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$ is 10 (even). So there are two middle terms.

i.e. $\left(\frac{9+1}{2}\right)^{th}$ and $\left(\frac{9+3}{2}\right)^{th}$ are two middle terms. They are given by T_5 and T_6

$$\therefore \quad T_5 = T_{4+1} = {}^9C_4(3x)^5\left(-\frac{x^3}{6}\right)^4 = {}^9C_4 3^5x^5.\frac{x^{12}}{6^4} = \frac{9.8.7.6}{1.2.3.4}.\frac{3^5}{2^4.3^4}x^{17} = \frac{189}{8}x^{17}$$

$$\text{and} \quad T_6 = T_{5+1} = {}^9C_5(3x)^4\left(-\frac{x^3}{6}\right)^5 = -{}^9C_4 3^4.x^4.\frac{x^{15}}{6^5} = \frac{-9.8.7.6}{1.2.3.4}.\frac{3^4}{2^5.3^5}x^{19} = -\frac{21}{16}x^{19}$$

Ans.

(c) Term independent of x :

Term independent of x does not contain x ; Hence find the value of r for which the exponent of x is zero.

Illustrations

Illustration 7 : The term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is -

(A) 1 (B) $\frac{5}{12}$ (C) ${}^{10}C_1$ (D) none of these

Solution : General term in the expansion is

$${}^{10}C_r\left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10}.\frac{3^{5-r}}{2^{\frac{10-r}{2}}} \quad \text{For constant term, } \frac{3r}{2}=10 \Rightarrow r=\frac{20}{3}$$

which is not an integer. Therefore, there will be no constant term. **Ans. (D)**

Do yourself - 2

(i) Find the 7th term of $\left(3x^2-\frac{1}{3}\right)^{10}$

(ii) Find the term independent of x in the expansion : $\left(2x^2-\frac{3}{x^3}\right)^{25}$

(iii) Find the middle term in the expansion of : (a) $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$ (b) $\left(2x^2-\frac{1}{x}\right)^7$

(d) Numerically greatest term :

Let numerically greatest term in the expansion of $(a+b)^n$ be T_{r+1}.

$$\Rightarrow \begin{cases} |T_{r+1}| \geq |T_r| \\ |T_{r+1}| \geq |T_{r+2}| \end{cases} \text{ where } T_{r+1} = {}^nC_r a^{n-r} b^r$$

Solving above inequalities we get $\dfrac{n+1}{1+\left|\frac{a}{b}\right|}-1 \leq r \leq \dfrac{n+1}{1+\left|\frac{a}{b}\right|}$

Case I : When $\dfrac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer equal to m, then T_m and T_{m+1} will be numerically greatest term.

Case II : When $\dfrac{n+1}{1+\left|\frac{a}{b}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Illustrations

Illustration 8 : Find numerically greatest term in the expansion of $(3-5x)^{11}$ when $x=\frac{1}{5}$

Solution : Using $\frac{n+1}{1+\left|\frac{a}{b}\right|}-1 \le r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$

$$\frac{11+1}{1+\left|\frac{3}{-5x}\right|}-1 \le r \le \frac{11+1}{1+\left|\frac{3}{-5x}\right|}$$

solving we get $2 \le r \le 3$

$\therefore \quad r = 2, 3$

so, the greatest terms are T_{2+1} and T_{3+1}.

$\therefore$ Greatest term (when r = 2)

$T_3 = {}^{11}C_2.3^9\,(-5x)^2 = 55.3^9 = T_4$

From above we say that the value of both greatest terms are equal. **Ans.**

Illustration 9 : Given T_3 in the expansion of $(1-3x)^6$ has maximum numerical value. Find the range of 'x'.

Solution : Using $\frac{n+1}{1+\left|\frac{a}{b}\right|}-1 \le r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$

$$\frac{6+1}{1+\left|\frac{1}{-3x}\right|}-1 \le 2 \le \frac{7}{1+\left|\frac{1}{-3x}\right|}$$

Let |x| = t

$$\frac{21t}{3t+1}-1 \le 2 \le \frac{21t}{3t+1}$$

$$\begin{cases} \dfrac{21t}{3t+1} \le 3 \\ \dfrac{21t}{3t+1} \ge 2 \end{cases} \Rightarrow \begin{cases} \dfrac{4t-1}{3t+1} \le 0 \Rightarrow t \in \left[-\dfrac{1}{3}, \dfrac{1}{4}\right] \\ \dfrac{15t-2}{3t+1} \ge 0 \Rightarrow t \in \left(-\infty, -\dfrac{1}{3}\right] \cup \left[\dfrac{2}{15}, \infty\right) \end{cases}$$

Common solution $t \in \left[\frac{2}{15}, \frac{1}{4}\right] \Rightarrow x \in \left[-\frac{1}{4}, -\frac{2}{15}\right] \cup \left[\frac{2}{15}, \frac{1}{4}\right]$

Do yourself - 3

(i) Find the numerically greatest term in the expansion of $(3-2x)^9$, when $x = 1$.

(ii) In the expansion of $\left(\frac{1}{2}+\frac{2x}{3}\right)^n$ when $x=-\frac{1}{2}$, it is known that 3^{rd} term is the greatest term. Find the possible intgral values of n.

4. PROPERTIES OF BINOMIAL COEFFICIENTS :

$$(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots\ldots\ldots + C_nx^n = \sum_{r=0}^{n} {}^nC_r r^r \text{ ; } n \in N \qquad \ldots.(i)$$

where $C_0, C_1, C_2, \ldots\ldots\ldots\ldots C_n$ are called combinatorial (binomial) coefficients.

(a) The sum of all the binomial coefficients is 2^n.

Put x = 1, in (i) we get

$$C_0 + C_1 + C_2 + \ldots\ldots\ldots\ldots + C_n = 2^n \Rightarrow \sum_{r=0}^{n} {}^nC_r = 0 \qquad \ldots.(ii)$$

(b) Put x=–1 in (i) we get

$$C_0 - C_1 + C_2 - C_3 \ldots\ldots\ldots\ldots + C_n = 0 \Rightarrow \sum_{r=0}^{n} (-1)^r\, {}^nC_r = 0 \qquad \ldots(iii)$$

(c) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.

From (ii) & (iii), $C_0 + C_2 + C_4 + \ldots\ldots\ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots. = 2^{n-1}$

(d) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(e) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(f) ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n}{r}.\dfrac{n-1}{r-1}\,{}^{n-2}C_{r-2} = \ldots\ldots. = \dfrac{n(n-1)(n-2)\ldots\ldots.(n-r+1)}{r(r-1)(r-2)\ldots\ldots\ldots.1}$

(g) ${}^nC_r = \dfrac{r+1}{n+1}.\,{}^{n+1}C_{r+1}$

Illustrations

Illustration 10 : Prove that : ${}^{25}C_{10} + {}^{24}C_{10} + \ldots\ldots + {}^{10}C_{10} = {}^{26}C_{11}$

Solution : LHS $= {}^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + \ldots\ldots\ldots\ldots + {}^{25}C_{10}$

$\Rightarrow \quad {}^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + \ldots\ldots + {}^{25}C_{10}$

$\Rightarrow \quad {}^{12}C_{11} + {}^{12}C_{10} + \ldots\ldots + {}^{25}C_{10}$

$\Rightarrow \quad {}^{13}C_{11} + {}^{13}C_{10} + \ldots\ldots\ldots {}^{25}C_{10}$

and so on. $\therefore$ LHS $= {}^{26}C_{11}$

Aliter :

LHS = coefficient of x^{10} in $\{(1 + x)^{10} + (1 + x)^{11} + \ldots\ldots\ldots\ldots\ldots (1+ x)^{25}\}$

$$\Rightarrow \quad \text{coefficient of } x^{10} \text{ in } \left[(1+x)^{10}\frac{\{1+x\}^{16}-1}{1+x-1}\right]$$

$$\Rightarrow \quad \text{coefficient of } x^{10} \text{ in } \frac{\left[(1+x)^{26}-(1+x)^{10}\right]}{x}$$

$$\Rightarrow \quad \text{coefficient of } x^{11} \text{ in } \left[(1+x)^{26}-(1+x)^{10}\right] = {}^{26}C_{11} - 0 = {}^{26}C_{11}$$

Illustration 11 : A student is allowed to select at most n books from a collection of (2n + 1) books. If the total number of ways in which he can select books is 63, find the value of n.

Solution : Given student selects at most n books from a collection of (2n + 1) books. It means that he selects one book or two books or three books or .... or n books. Hence, by the given condition-

$${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots\ldots\ldots + {}^{2n+1}C_n = 63 \qquad \ldots(i)$$

But we know that

$${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots\ldots + {}^{2n+1}C_{2n+1} = 2^{2n+1} \qquad \ldots(ii)$$

Since ${}^{2n+1}C_0 = {}^{2n+1}C_{2n+1} = 1$, equation (ii) can also be written as

$2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots\ldots + {}^{2n+1}C_n) +$

$({}^{2n+1}C_{n+1} + {}^{2n+1}C_{n+2} + {}^{2n+1}C_{n+3} + \ldots\ldots + {}^{2n+1}C_{2n-1} + {}^{2n+1}C_{2n}) = 2^{2n+1}$

$\Rightarrow 2 + ({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots\ldots\ldots + {}^{2n+1}C_n)$

$\qquad + ({}^{2n+1}C_n + {}^{2n+1}C_{n-1} + \ldots\ldots\ldots + {}^{2n+1}C_2 + {}^{2n+1}C_1) = 2^{2n+1}$

$(\because {}^{2n+1}C_r = {}^{2n+1}C_{2n+1-r})$

$\Rightarrow 2 + 2\,({}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \ldots\ldots + {}^{2n+1}C_n) = 2^{2n+1}$ [from (i)]

$\Rightarrow 2 + 2.63 = 2^{2n+1}$ $\qquad \Rightarrow 1 + 63 = 2^{2n}$

$\Rightarrow 64 = 2^{2n} \Rightarrow 2^6 = 2^{2n}$ $\qquad \therefore 2n = 6$

Hence, n = 3. **Ans.**

Illustration 12 : Prove that :

(i) $C_1 + 2C_2 + 3C_3 + \ldots\ldots + nC_n = n \cdot 2^{n-1}$

(ii) $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots\ldots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

Solution : (i) $\text{L.H.S.} = \sum_{r=1}^{n} r.\,{}^nC_r = \sum_{r=1}^{n} r.\dfrac{n}{r}.\,{}^{n-1}C_{r-1}$

$$= n\sum_{r=1}^{n} {}^{n-1}C_{r-1} = n.\left[{}^{n-1}C_0 + {}^{n-1}C_1 + \ldots\ldots + {}^{n-1}C_{n-1}\right]$$

$$= n \cdot 2^{n-1}$$

Aliter : (Using method of differentiation)

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_nx^n$ ....(A)

Differentiating (A), we get

$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + n.C_nx^{n-1}$.

Put x = 1,

$C_1 + 2C_2 + 3C_3 + \ldots\ldots + n.C_n = n.2^{n-1}$

(ii) $\text{L.H.S.} = \sum_{r=0}^{n} \dfrac{C_r}{r+1} = \dfrac{1}{n+1}\sum_{r=0}^{n}\dfrac{n+1}{r+1}\,{}^nC_r$

$$= \frac{1}{n+1}\sum_{r=0}^{n} {}^{n+1}C_{r+1} = \frac{1}{n+1}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + \ldots\ldots + {}^{n+1}C_{n+1}\right] = \frac{1}{n+1}\left[2^{n+1}-1\right]$$

Aliter : (Using method of integration)

Integrating (A), we get

$$\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots\ldots + \frac{C_nx^{n+1}}{n+1}$$ (where C is a constant)

Put x = 0, we get, $C = -\dfrac{1}{n+1}$

$$\therefore \quad \frac{(1+x)^{n+1}-1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots\ldots + \frac{C_nx^{n+1}}{n+1}$$

Put x = 1, we get $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots\ldots \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

Put x = –1, we get $C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \ldots\ldots = \dfrac{1}{n+1}$

Illustration 13 : If $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$, then prove that $C_1^2 + 2.C_2^2 + 3.C_3^2 + ... + n.C_n^2 = \frac{(2n-1)!}{((n-1)!)^2}$

Solution : $(1 + x)^n = C_0 + C_1x + C_2x^2 + C_2x^3 + ... + C_n x^n$ ....(i)

Differentiating both the sides, w.r.t. x, we get

$n(1 + x)^{n-1} = C_1 + 2C_2x + 3C_2x^2 + ... + n.C_nx^{n-1}$ ....(ii)

also, we have

$(x + 1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ... + C_n$ ....(iii)

Multiplying (ii) & (iii), we get

$(C_1 + 2C_2x + 3C_3x^2 + ... + C_nx^{n-1})(C_0x^n + C_1x^{n-1} + C_2x^{n-2} + ...+ C_n) = n(1 + x)^{2n-1}$

Equating the coefficients of x^{n-1}, we get

$$C_1^2 + 2C_2^2 + 3C_3^2 + ... + n.C_n^2 = n\,.\,{}^{2n-1}C_{n-1} = \frac{(2n-1)!}{((n-1)!)^2}$$ **Ans.**

Illustration 14 : Prove that : $C_0 - 3C_1 + 5C_2 - ....(-1)^n(2n + 1)C_n = 0$

Solution : $T_r = (-1)^r(2r + 1)^nC_r = 2(-1)^r r\,.\,{}^nC_r + (-1)^r\,{}^nC_r$

$$\Sigma T_r = 2\sum_{r=1}^{n}(-1)^r.r.\frac{n}{r}.{}^{n-1}C_{r-1} + \sum_{r=0}^{n}(-1)^r\ {}^nC_r = 2\sum_{r=1}^{n}(-1)^r.{}^{n-1}C_{r-1} + \sum_{r=0}^{n}(-1)^r.{}^nC_r$$

$$= 2\left[{}^{n-1}C_0 - {}^{n-1}C_1 + ...\right] + \left[{}^nC_0 - {}^nC_1 + ...\right] = 0$$

Illustration 15 : Prove that $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - .... + (-1)^n ({}^{2n}C_{2n})^2 = (-1)^n.\ {}^{2n}C_n$

Solution : $(1 - x)^{2n} = {}^{2n}C_0 - {}^{2n}C_1x + {}^{2n}C_2x^2 - ....+(-1)^n\,{}^{2n}C_{2n}x^{2n}$ ....(i)

and $(x + 1)^{2n} = {}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + {}^{2n}C_2x^{2n-2} +...+{}^{2n}C_{2n}$ ....(ii)

Multiplying (i) and (ii), we get

$(x^2 - 1)^{2n} = ({}^{2n}C_0 - {}^{2n}C_1x + ...+ (-1)^n\,{}^{2n}C_{2n}x^{2n})\times({}^{2n}C_0x^{2n} + {}^{2n}C_1x^{2n-1} + ... + {}^{2n}C_{2n})$ ....(iii)

Now, coefficient of x^{2n} in R.H.S.

$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - ...... + (-1)^n ({}^{2n}C_{2n})^2$

$\because$ General term in L.H.S., $T_{r+1} = {}^{2n}C_r(x^2)^{2n-r}(-1)^r$

Putting $2(2n - r) = 2n$

$\therefore$ $r = n$

$\therefore$ $T_{n+1} = {}^{2n}C_nx^{2n}(-1)^n$

Hence coeffiecient of x^{2n} in L.H.S. $= (-1)^n.{}^{2n}C_n$

But (iii) is an identity, therefore coefficient of x^{2n} in R.H.S. = coefficient of x^{2n} in L.H.S.

$\Rightarrow$ $({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - .... + (-1)^n ({}^{2n}C_{2n})^2 = (-1)^n.\ {}^{2n}C_n$

Illustration 16 : Prove that : ${}^{n}C_0 . {}^{2n}C_n - {}^{n}C_1 . {}^{2n-2}C_n + {}^{n}C_2 . {}^{2n-4}C_n + ... = 2^n$

Solution : L.H.S. = Coefficient of x^n in $[{}^{n}C_0(1+x)^{2n} - {}^{n}C_1(1+x)^{2n-2} ......]$

= Coefficient of x^n in $[(1+x)^2 - 1]^n$

= Coefficient of x^n in $x^n(x+2)^n = 2^n$

Illustration 17 : If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ then show that the sum of the products of the C_i's taken two at a time represented by : $\sum\sum_{0\le i<j\le n} C_iC_j$ is equal to $2^{2n-1} - \dfrac{2n!}{2.n!n!}$

Solution : Since $(C_0 + C_1 + C_2 + ... + C_{n-1} + C_n)^2$

$= C_0^2 + C_1^2 + C_2^2 + ... + C_{n-1}^2 + C_n^2 + 2(C_0C_1 + C_0C_2 + C_0C_3 + ... + C_0C_n + C_1C_2 + C_1C_3 + ...$
$+ C_1C_n + C_2C_3 + C_2C_4 + ... + C_2C_n + ... + C_{n-1}C_n)$

$(2^n)^2 = {}^{2n}C_n + 2\sum\sum_{0\le i<j\le n} C_iC_j$

Hence $\sum\sum_{0\le i<j\le n} C_iC_j = 2^{2n-1} - \dfrac{2n!}{2.n!n!}$ **Ans.**

Illustration 18 : If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$ then prove that $\sum\sum_{0\le i<j\le n} (C_i + C_j)^2 = (n-1)\,{}^{2n}C_n + 2^{2n}$

Solution : L.H.S. $\sum\sum_{0\le i<j\le n} (C_i + C_j)^2$

$= (C_0 + C_1)^2 + (C_0 + C_2)^2 + ... + (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + ...$
$+ (C_1 + C_n)^2 + (C_2 + C_3)^2 + (C_2 + C_4)^2 + ... + (C_2 + C_n)^2 + ... + (C_{n-1} + C_n)^2$

$= n(C_0^2 + C_1^2 + C_2^2 + ... + C_n^2) + 2\sum\sum_{0\le i<j\le n} C_iC_j$

$= n.{}^{2n}C_n + 2.\left\{2^{2n-1} - \dfrac{2n!}{2.n!n!}\right\}$ {from Illustration 17}

$= n.{}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n-1).{}^{2n}C_n + 2^{2n}$ = R.H.S.

Do yourself - 4

(i) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n} =$

(A) 2^{n-1} (B) ${}^{2n}C_n$ (C) 2^n (D) 2^{n+1}

(ii) If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, $n \in N$. Prove that

(a) $3C_0 - 8C_1 + 13C_2 - 18C_3 + ...$ upto $(n+1)$ terms $= 0$, if $n \ge 2$.

(b) $2C_0 + 2^2\dfrac{C_1}{2} + 2^3\dfrac{C_2}{3} + 2^4\dfrac{C_3}{4} + ... + 2^{n+1}\dfrac{C_n}{n+1} = \dfrac{3^{n+1}-1}{n+1}$

(c) $C_0^2 + \dfrac{C_1^2}{2} + \dfrac{C_2^2}{3} + ... + \dfrac{C_n^2}{n+1} = \dfrac{(2n+1)!}{((n+1)!)^2}$

5. MULTINOMIAL THEOREM :

Using binomial theorem, we have $(x + a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} a^r$, $n \in N$

$$= \sum_{r=0}^{n} \frac{n!}{(n-r)!r!} x^{n-r} a^r = \sum_{r+s=n} \frac{n!}{r!s!} x^s a^r \text{, where } s + r = n$$

This result can be generalized in the following form.

$$(x_1 + x_2 + \ldots\ldots + x_k)^n = \sum_{r_1+r_2+\ldots+r_k=n} \frac{n!}{r_1!r_2!\ldots r_k!} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}$$

The general term in the above expansion $\frac{n!}{r_1!r_2!r_3!\ldots\ldots r_k!} . x_1^{r_1} x_2^{r_2} x_3^{r_3} \ldots\ldots x_k^{r_k}$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + \ldots\ldots + r_k = n$ because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$

Particular cases :

(i) $(x + y + z)^n = \sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r y^s z^t$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms

(ii) $(x + y + z + u)^n = \sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!} x^p y^q z^r u^s$

There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ terms in the above expansion.

Illustrations

Illustration 19 : Find the coefficient of $x^2 y^3 z^4 w$ in the expansion of $(x - y - z + w)^{10}$

Solution : $(x - y - z + w)^{10} = \sum_{p+q+r+s=10} \frac{n!}{p!q!r!s!} (x)^p (-y)^q (-z)^r (w)^s$

We want to get $x^2y^3z^4w$ this implies that $p = 2, q = 3, r = 4, s = 1$

$\therefore$ Coefficient of $x^2y^3z^4w$ is $\frac{10!}{2!\,3!\,4!\,1!}(-1)^3(-1)^4 = -12600$ **Ans.**

Illustration 20 : Find the total number of terms in the expansion of $(1 + x + y)^{10}$ and coefficient of x^2y^3.

Solution : Total number of terms $= {}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

Coefficient of $x^2y^3 = \dfrac{10!}{2!\times 3!\times 5!} = 2520$ **Ans.**

Illustration 21 : Find the coefficient of x^5 in the expansion of $(2 - x + 3x^2)^6$.

Solution : The general term in the expansion of $(2 - x + 3x^2)^6 = \dfrac{6!}{r!s!t!}2^r(-x)^s(3x^2)^t$,

where r + s + t = 6.

$$= \frac{6!}{r!s!t!}2^r \times (-1)^s \times (3)^t \times x^{s+2t}$$

For the coefficient of x^5, we must have s + 2t = 5.

But, r + s + t = 6,

$\therefore$ s = 5 – 2t and r = 1 + t, where $0 \le r, s, t \le 6$.

Now t = 0 $\Rightarrow$ r = 1, s = 5.

t = 1 $\Rightarrow$ r = 2, s = 3.

t = 2 $\Rightarrow$ r = 3, s = 1.

Thus, there are three terms containing x^5 and coefficient of x^5

$$= \frac{6!}{1!\,5!\,0!}\times 2^1 \times (-1)^5 \times 3^0 + \frac{6!}{2!\,3!\,1!}\times 2^2 \times (-1)^3 \times 3^1 + \frac{6!}{3!\,1!\,2!}\times 2^3 \times (-1)^1 \times 3^2$$

= –12 – 720 – 4320 = –5052. **Ans.**

Illustration 22 : If $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then prove that (a) $a_r = a_{2n-r}$ (b) $\sum_{r=0}^{n-1} a_r = \frac{1}{2}(3^n - a_n)$

Solution : (a) We have

$$\left(1+x+x^2\right)^n = \sum_{r=0}^{2n} a_r x^r \qquad ....(A)$$

Replace x by $\dfrac{1}{x}$

$$\therefore \quad \left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n = \sum_{r=0}^{2n} a_r\left(\frac{1}{x}\right)^r$$

$$\Rightarrow \quad \left(x^2+x+1\right)^n = \sum_{r=0}^{2n} a_r x^{2n-r}$$

$$\Rightarrow \quad \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{2n} a_r x^{2n-r} \qquad \{\text{Using (A)}\}$$

Equating the coefficient of x^{2n-r} on both sides, we get

$$a_{2n-r} = a_r \text{ for } 0 \le r \le 2n.$$

Hence $\quad a_r = a_{2n-r}$.

(b) Putting x=1 in given series, then

$$a_0 + a_1 + a_2 + ...+ a_{2n} = (1+1+1)^n$$

$$a_0 + a_1 + a_2 + ...+ a_{2n} = 3^n \qquad ....(1)$$

But $\quad a_r = a_{2n-r}$ for $0 \le r \le 2n$

$\therefore$ series (1) reduces to

$$2(a_0 + a_1 + a_2 + ...+ a_{n-1}) + a_n = 3^n.$$

$$\therefore \quad a_0 + a_1 + a_2 + ...+ a_{n-1} = \frac{1}{2}(3^n - a_n)$$

Do yourself - 5

(i) Find the coefficient of x^2y^5 in the expansion of $(3 + 2x - y)^{10}$.

6. APPLICATION OF BINOMIAL THEOREM :

Illustrations

Illustration 23 : If $\left(6\sqrt{6}+14\right)^{2n+1} = [N] + F$ and $F = N - [N]$; where [.] denotes greatest integer function, then NF is equal to

(A) 20^{2n+1} (B) an even integer (C) odd integer (D) 40^{2n+1}

Solution : Since $\left(6\sqrt{6}+14\right)^{2n+1} = [N] + F$

Let us assume that $f = \left(6\sqrt{6}-14\right)^{2n+1}$; where $0 \le f < 1$.

Now, $[N] + F - f = \left(6\sqrt{6}+14\right)^{2n+1} - \left(6\sqrt{6}-14\right)^{2n+1}$

$$= 2\left[{}^{2n+1}C_1\left(6\sqrt{6}\right)^{2n}(14) + {}^{2n+1}C_3\left(6\sqrt{6}\right)^{2n-2}(14)^3 + ...\right]$$

$\Rightarrow$ $[N] + F - f$ = even integer.

Now $0 < F < 1$ and $0 < f < 1$

so $-1 < F - f < 1$ and $F - f$ is an integer so it can only be zero

Thus $NF = \left(6\sqrt{6}+14\right)^{2n+1}\left(6\sqrt{6}-14\right)^{2n+1} = 20^{2n+1}$. **Ans. (A,B)**

Illustration 24 : Find the last three digits in 11^{50}.

Solution : Expansion of $(10 + 1)^{50} = {}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + \ldots\ldots + {}^{50}C_{48} 10^2 + {}^{50}C_{49} 10 + {}^{50}C_{50}$

$$= \underbrace{{}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + \ldots\ldots + {}^{50}C_{47} 10^3}_{1000K} + 49 \times 25 \times 100 + 500 + 1$$

$\Rightarrow$ 1000 K + 123001

$\Rightarrow$ Last 3 digits are 001.

Illustration 25 : Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Solution : When 2222 is divided by 7 it leaves a remainder 3.

So adding & subtracting 3^{5555}, we get :

$$E = \underbrace{2222^{5555} - 3^{5555}}_{E_1} + \underbrace{3^{5555} + 5555^{2222}}_{E_2}$$

For E_1 : Now since 2222–3 = 2219 is divisible by 7, therefore E_1 is divisible by 7

($\because$ $x^n - a^n$ is divisible by $x - a$)

For E_2 : 5555 when devided by 7 leaves remainder 4.

So adding and subtracting 4^{2222}, we get :

$E_2 = 3^{5555} + 4^{2222} + 5555^{2222} - 4^{2222}$

$= (243)^{1111} + (16)^{1111} + (5555)^{2222} - 4^{2222}$

Again $(243)^{1111} + 16^{1111}$ and $(5555)^{2222} - 4^{2222}$ are divisible by 7

($\because$ $x^n + a^n$ is divisible by $x + a$ when n is odd)

Hence $2222^{5555} + 5555^{2222}$ is divisible by 7.

Do yourself - 6

(i) Prove that $5^{25} - 3^{25}$ is divisible by 2.

(ii) Find the remainder when the number 9^{100} is divided by 8.

(iii) Find last three digits in 19^{100}.

(iv) Let $R = (8 + 3\sqrt{7})^{20}$ and [.] denotes greatest integer function, then prove that :

(a) [R] is odd (b) $R - [R] = 1 - \dfrac{1}{(8 + 3\sqrt{7})^{20}}$

(v) Find the digit at unit's place in the number $17^{1995} + 11^{1995} - 7^{1995}$.

7. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES :

If $n \in Q$, then $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ \infty$ provided $|x| < 1$.

Note :

(i) When the index n is a positive integer the number of terms in the expansion of $(1+x)^n$ is finite i.e. (n+1) & the coefficient of successive terms are : ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots\ {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1+x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term.

(iii) Following expansion should be remembered ($|x| < 1$).

(a) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots\ \infty$

(b) $(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots\ \infty$

(c) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots\ \infty$

(d) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ \infty$

(e) $(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots\ldots + \frac{(-1)^r(r+1)(r+2)}{2!}x^r + \ldots\ldots$

(f) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots + \frac{(r+1)(r+2)}{2!}x^r + \ldots\ldots$

(iv) The expansions in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x| > 1$ then we may find it convenient to expand in powers of $1/x$, which then will be small.

8. APPROXIMATIONS :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 \ldots\ldots$$

If $\mathbf{x < 1}$, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its square and higher powers may be neglected then $(1+x)^n = 1 + nx$, approximately.

This is an approximate value of $(1+x)^n$

Illustrations

Illustration 26 : If x is so small such that its square and higher powers may be neglected then find the approximate value of $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$

Solution : $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}=\frac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}}=\frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2}=\frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right)$

$=\frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right)=1-\frac{x}{8}-\frac{19}{12}x=1-\frac{41}{24}x$ **Ans.**

Illustration 27 : The value of cube root of 1001 upto five decimal places is –

(A) 10.03333 (B) 10.00333 (C) 10.00033 (D) none of these

Solution : $(1001)^{1/3}=(1000+1)^{1/3}=10\left(1+\frac{1}{1000}\right)^{1/3}=10\left\{1+\frac{1}{3}.\frac{1}{1000}+\frac{1/3(1/3-1)}{2!}\frac{1}{1000^2}+.....\right\}$

$= 10\{1 + 0.0003333 - 0.00000011 + ...\} = 10.00333$ **Ans. (B)**

Illustration 28 : The sum of $1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+....\infty$ is -

(A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{3}$ (D) $2^{3/2}$

Solution : Comparing with $1 + nx + \frac{n(n-1)}{2!}x^2+....$

$nx = 1/4$ ....(i)

and $\frac{n(n-1)x^2}{2!}=\frac{1.3}{4.8}$

or $\frac{nx(nx-x)}{2!}=\frac{3}{32} \Rightarrow \frac{1}{4}\left(\frac{1}{4}-x\right)=\frac{3}{16}$ (by (i))

$\Rightarrow \left(\frac{1}{4}-x\right)=\frac{3}{4}\Rightarrow x=\frac{1}{4}-\frac{3}{4}=-\frac{1}{2}$ ....(ii)

putting the value of x in (i)

$n(-1/2) = 1/4 \Rightarrow n = -1/2$

$\therefore$ sum of series $= (1 + x)^n = (1 - 1/2)^{-1/2} = (1/2)^{-1/2} = \sqrt{2}$ **Ans. (A)**

9. EXPONENTIAL SERIES :

(a) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(b) Logarithms to the base 'e' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm.**

(c) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\infty$; where x may be any real or complex number & $e = \lim\limits_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$

(d) $a^x = 1 + \frac{x}{1!}\ell na + \frac{x^2}{2!}\ell n^2 a + \frac{x^3}{3!}\ell n^3 a + ...\infty$, where a > 0

(e) $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ...\infty$

10. LOGARITHMIC SERIES :

(a) $\ell n\,(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$, where $-1 < x \le 1$

(b) $\ell n\,(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + ...\infty$, where $-1 \le x < 1$

Remember : (i) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...\infty = \ell n\,2$ (ii) $e^{\ln x} = x$; for all x > 0

(iii) $\ell n2 = 0.693$ (iv) $\ell n10 = 2.303$

ANSWER KEY

Do yourself-1

(i) $^5C_0x(3x^2)^5+{}^5C_1(3x^2)^4\left(-\frac{x}{2}\right)+{}^5C_2(3x^2)^3\left(-\frac{x}{2}\right)^2+{}^5C_3(3x^2)^2\left(-\frac{x}{2}\right)^3+{}^5C_4(3x^2)^1\left(-\frac{x}{2}\right)^4+{}^5C_5\left(-\frac{x}{2}\right)^5$

(ii) $^nC_0y^n + {}^nC_1y^{n-1}.x + {}^nC_2.y^{n-2}.x^2 + \text{........} + {}^nC_n.x^n$

Do yourself-2

(i) $\frac{70}{3}x^8$; **(ii)** $\frac{25!}{10!\ 5!}2^{15}3^{10}$; **(iii)** (a) –20; (b) $-560x^5$, $280x^2$

Do yourself-3

(i) 4^{th} & 5^{th} i.e. 489888 **(ii)** n = 4, 5, 6

Do yourself-4

(i) C

Do yourself-5

(i) –272160 or $-{}^{10}C_5 \times {}^5C_2 \times 108$

Do yourself-6

(ii) 1 **(iii)** 001 **(v)** 1

EXERCISE (O-1)

[SINGLE CORRECT CHOICE TYPE]

1. If the coefficients of x^7 & x^8 in the expansion of $\left[2+\frac{x}{3}\right]^n$ are equal, then the value of n is :

(A) 15 (B) 45 (C) 55 (D) 56

2. Set of value of r for which, ${}^{18}C_{r-2} + 2.\ {}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ contains :

(A) 4 element (B) 5 elements (C) 7 elements (D) 10 elements

3. If the constant term of the binomial expansion $\left(2x-\frac{1}{x}\right)^n$ is – 160, then n is equal to -

(A) 4 (B) 6 (C) 8 (D) 10

4. The coefficient of x^{49} in the expansion of $(x-1)\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right)....\left(x-\frac{1}{2^{49}}\right)$ is equal to -

(A) $-2\left(1-\frac{1}{2^{50}}\right)$ (B) +ve coefficient of x

(C) –ve coefficient of x (D) $-2\left(1-\frac{1}{2^{49}}\right)$

5. The largest real value for x such that $\sum_{k=0}^{4}\left(\frac{5^{4-k}}{(4-k)!}\right)\left(\frac{x^k}{k!}\right)=\frac{8}{3}$ is -

(A) $2\sqrt{2}-5$ (B) $2\sqrt{2}+5$ (C) $-2\sqrt{2}-5$ (D) $-2\sqrt{2}+5$

6. The expression $[x + (x^3 - 1)^{1/2}]^5 + [x - (x^3 - 1)^{1/2}]^5$ is a polynomial of degree

(A) 5 (B) 6 (C) 7 (D) 8

7. Number of rational terms in the expansion of $\left(\sqrt{2}+\sqrt[4]{3}\right)^{100}$ is :

(A) 25 (B) 26 (C) 27 (D) 28

8. Given $(1 - 2x + 5x^2 - 10x^3)(1 + x)^n = 1 + a_1x + a_2x^2 + ...$ and that $a_1^2 = 2a_2$ then the value of n is-

(A) 6 (B) 2 (C) 5 (D) 3

9. The sum of the co-efficients of all the even powers of x in the expansion of $(2x^2 - 3x + 1)^{11}$ is -

(A) 2.6^{10} (B) 3.6^{10} (C) 6^{11} (D) none

10. Co-efficient of α^t in the expansion of ,

$(\alpha + p)^{m-1} + (\alpha + p)^{m-2}(\alpha + q) + (\alpha + p)^{m-3} (\alpha + q)^2 + ...(\alpha + q)^{m-1}$ where $\alpha \neq -q$ and $p \neq q$ is:

(A) $\dfrac{{}^mC_t\left(p^t - q^t\right)}{p-q}$ (B) $\dfrac{{}^mC_t\left(p^{m-t} - q^{m-t}\right)}{p-q}$ (C) $\dfrac{{}^mC_t\left(p^t + q^t\right)}{p-q}$ (D) $\dfrac{{}^mC_t\left(p^{m-t} + q^{m-t}\right)}{p-q}$

11. Let $\binom{n}{k}$ represents the combination of 'n' things taken 'k' at a time, then the value of the sum

$\binom{99}{97}+\binom{98}{96}+\binom{97}{95}+.......+\binom{3}{1}+\binom{2}{0}$ equals -

(A) $\binom{99}{97}$ (B) $\binom{100}{98}$ (C) $\binom{99}{98}$ (D) $\binom{100}{97}$

[COMPREHENSION TYPE]

Paragraph for question nos. 12 to 14

If $n \in N$ and if $(1 + 4x + 4x^2)^n = \sum\limits_{r=0}^{2n} a_r x^r$, where $a_0, a_1, a_2, ..., a_{2n}$ are real numbers.

12. The value of $2\sum\limits_{r=0}^{n} a_{2r}$, is

(A) $9^n - 1$ (B) $9^n + 1$ (C) $9^n - 2$ (D) $9^n + 2$

13. The value of $2\sum\limits_{r=1}^{n} a_{2r-1}$, is-

(A) $9^n - 1$ (B) $9^n + 1$ (C) $9^n - 2$ (D) $9^n + 2$

14. The value of a_{2n-1} is -

(A) 2^{2n} (B) $n. 2^{2n}$ (C) $(n - 1)2^{2n}$ (D) $(n + 1)2^{2n}$

15. If $n \in N$ & n is even, then $\dfrac{1}{1.(n-1)!}+\dfrac{1}{3!.(n-3)!}+\dfrac{1}{5!.(n-5)!}+.......+\dfrac{1}{(n-1)!1!}=$

(A) 2^n (B) $\dfrac{2^{n-1}}{n!}$ (C) $2^n n!$ (D) none of these

EXERCISE (O-2)

[ONE OR MORE THAN ONE CORRECT CHOICE TYPE]

1. If it is known that the third term of the binomial expansion $\left(x+x^{\log_{10} x}\right)^5$ is 10^6 then x is equal to-

(A) 10 (B) $10^{-5/2}$ (C) 100 (D) 5

2. In the expansion of $\left(x^3+3.2^{-\log_{\sqrt{2}}\sqrt{x^3}}\right)^{11}$

(A) there appears a term with the power x^2

(B) there does not appear a term with the power x^2

(C) there appears a term with the power x^{-3}

(D) the ratio of the co-efficient of x^3 to that of x^{-3} is 1/3

3. In the expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$, the term which does not contain x is-

(A) ${}^{11}C_4-{}^{10}C_3$ (B) ${}^{10}C_7$ (C) ${}^{10}C_4$ (D) ${}^{11}C_5-{}^{10}C_5$

4. Let $(1 + x^2)^2 (1 + x)^n = A_0 + A_1x + A_2x^2 + \ldots\ldots$ If A_0,A_1,A_2 are in A.P. then the value of n is-

(A) 2 (B) 3 (C) 5 (D) 7

5. Consider $E=\left(\sqrt[8]{x}+\sqrt[5]{y}\right)^z=I+f, 0\le f<1$

(A) If x = 5, y = 2, z = 100, then number of irrational terms in expansion of E is 98

(B) If x = 5, y = 2, z = 100, then number of rational terms in expansion of E is 4

(C) If x = 16, y = 1 & z = 6, then I = 197

(D) If x = 16, y = 1 & z = 6, then $f=\left(\sqrt{2}-1\right)^6$

6. Greatest term in the binomial expansion of $(a + 2x)^9$ when a = 1 & $x=\frac{1}{3}$ is :

(A) 3^{rd} & 4^{th} (B) 4^{th} & 5^{th} (C) only 4^{th} (D) only 5^{th}

7. Let $(5+2\sqrt{6})^n=p+f$ where $n,p \in N$ and $0 < f < 1$ then the value of $f^2 - f + pf - p$ is -

(A) a natural number (B) a negative integer (C) a prime number (D) are irrational number

8. If $(9+\sqrt{80})^n = I + f$ where I, n are integers and $0 < f < 1$, then -

(A) I is an odd integer
(B) I is an even integer
(C) $(I + f)(1 - f) = 1$
(D) $1 - f = (9-\sqrt{80})^n$

9. If $\sum_{r=1}^{10} r(r-1)\ {}^{10}C_r = k.\ 2^9$, then k is equal to-

(A) 10
(B) 45
(C) 90
(D) 100

10. The sum $\frac{\binom{11}{0}}{1}+\frac{\binom{11}{1}}{2}+\frac{\binom{11}{2}}{3}+..........+\frac{\binom{11}{11}}{12}$ equals $\left(\text{where } \binom{n}{r} \text{denotes } {}^nC_r\right)$

(A) $\frac{2^{11}}{12}$
(B) $\frac{2^{12}}{12}$
(C) $\frac{2^{11}-1}{12}$
(D) $\frac{2^{12}-1}{12}$

11. **Statement-1 :** The sum of the series ${}^nC_0.\ {}^mC_r + {}^nC_1.\ {}^mC_{r-1} + {}^nC_2.\ {}^mC_{r-2} +......+ {}^nC_r.{}^mC_0$ is equal to ${}^{n+m}C_r$, where ${}^nC_r$'s and ${}^mC_r$'s denotes the combinatorial coefficients in the expansion of $(1 + x)^n$ and $(1 + x)^m$ respectively.

Statement-2 : Number of ways in which r children can be selected out of (n + m) children consisting of n boys and m girls if each selection may consist of any number of boys and girls is equal to ${}^{n+m}C_r$.

(A) Statement-1 is true, statement-2 is true ; statement-2 is a correct explanation for statement-1.
(B) Statement-1 is true, statement-2 is true ; statement-2 is NOT a correct explanation for statement-1.
(C) Statement-1 is true, statement-2 is false.
(D) Statement-1 is false, statement-2 is true.

12. Which of the following statement(s) is/are correct ?

(A) $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+.......+\infty = 4$

(B) Integral part of $\left(9+4\sqrt{5}\right)^n$, $n \in N$ is even.

(C) $({}^nC_0 + {}^nC_1 + {}^nC_2 + ...+ {}^nC_n)^2 = 1 + {}^{2n}C_1 + {}^{2n}C_2 +.....+ {}^{2n}C_{2n}$

(D) $\frac{1}{(3+2x)^2}$ can be expanded as infinite series in ascending powers of x only if $|x| < \frac{2}{3}$.

13. If for $n \in I$, $n > 10$; $1 + (1 + x) + (1 + x)^2 + \ldots\ldots + (1 + x)^n = \sum_{k=0}^{n} a_k . x^k$, $x \neq 0$ then

(A) $\sum_{k=0}^{n} a_k = 2^{n+1}$

(B) $a_{n-2} = \dfrac{n(n+1)}{2}$

(C) $a_p > a_{p-1}$ for $p < \dfrac{n}{2}$, $p \in N$

(D) $(a_9)^2 - (a_8)^2 = {}^{n+2}C_{10}\,({}^{n+1}C_{10} - {}^{n+1}C_9)$

14. Let $P(n) = \sum_{r=0}^{n} \dfrac{(-1)^r r}{r+1}\, {}^nC_r$. Now which of the following holds good ?

(A) $|P_{10}|$ is harmonic mean of $|P_9|$ & $|P_{11}|$

(B) $\sum_{r=5}^{10} P(r)P(r-1) = -\dfrac{6}{55}$

(C) $|P_{10}|$ is arithmetic mean of $|P_9|$ & $|P_{11}|$

(D) $\sum_{r=5}^{10} P(r)P(r-1) = \dfrac{6}{55}$

15. Let $(1 + x)^m = C_0 + C_1x + C_2x^2 + C_3x^3 + \ldots.. + C_mx^m$, where $C_r = {}^mC_r$ and $A = C_1C_3 + C_2C_4 + C_3C_5 + C_4C_6 + \ldots\ldots.+ C_{m-2}C_m$, then -

(A) $A \geq {}^{2m}C_{m-2}$

(B) $A < {}^{2m}C_{m-2}$

(C) $A > C_0^2 + C_1^2 + C_2^2 + \ldots..C_m^2$

(D) $A < C_0^2 + C_1^2 + C_2^2 + \ldots..C_m^2$

EXERCISE (S-1)

1. (a) If the coefficients of $(2r+4)^{th}$, $(r-2)^{th}$ terms in the expansion of $(1+x)^{18}$ are equal, find r.

(b) If the coefficients of the r^{th}, $(r+1)^{th}$ & $(r+2)^{th}$ terms in the expansion of $(1+x)^{14}$ are in AP, find r.

(c) If the coefficients of 2^{nd}, 3^{rd} & 4^{th} terms in the expansion of $(1+x)^{2n}$ are in AP, show that $2n^2 - 9n + 7 = 0$.

2. Find the term independent of x in the expansion of (i) $\left[\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{2x^2}\right]^{10}$ (ii) $\left[\frac{1}{2}x^{1/3}+x^{-1/5}\right]^8$

3. Prove that the ratio of the coefficient of x^{10} in $(1-x^2)^{10}$ & the term independent of x in $\left(x-\frac{2}{x}\right)^{10}$ is 1 : 32.

4. Find the term independent of x in the expansion of $(1+x+2x^3)\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$.

5. Let $(1+x^2)^2 \cdot (1+x)^n = \sum_{K=0}^{n+4} a_K \cdot x^K$. If a_1, a_2 & a_3 are in AP, find n.

6. Let $f(x) = 1 - x + x^2 - x^3 + \ldots\ldots + x^{16} - x^{17} = a_0 + a_1(1+x) + a_2(1+x)^2 + \ldots\ldots + a_{17}(1+x)^{17}$, find the value of a_2.

7. Find the coefficient of x^r in the expression :

$$(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + \ldots\ldots + (x+2)^{n-1}$$

8. Find numerically greatest term in the expansion of :

(i) $(2+3x)^9$ when $x = \frac{3}{2}$ (ii) $(3-5x)^{15}$ when $x = \frac{1}{5}$

9. (a) Show that the integral part in each of the following is odd. $n \in N$

(A) $\left(5+2\sqrt{6}\right)^n$ (B) $\left(8+3\sqrt{7}\right)^n$

(b) Show that the integral part in each of the following is even. $n \in N$

(A) $\left(3\sqrt{3}+5\right)^{2n+1}$ (B) $\left(5\sqrt{5}+11\right)^{2n+1}$

10. Let $N = {}^{2000}C_1 + 2 \cdot {}^{2000}C_2 + 3 \cdot {}^{2000}C_3 + \ldots\ldots + 2000 \cdot {}^{2000}C_{2000}$. Prove that N is divisible by 2^{2003}.

11. Prove the following identities using the theory of permutation where $C_0, C_1, C_2, \ldots\ldots, C_n$ are the combinatorial coefficients in the expansion of $(1 + x)^n$, $n \in N$:

(a) $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = \dfrac{(2n)!}{n!\,n!}$

(b) $C_0 C_1 + C_1 C_2 + C_2 C_3 + \ldots\ldots + C_{n-1} C_n = \dfrac{(2n)!}{(n+1)!\,(n-1)!}$

(c) $C_oC_r + C_1C_{r+1} + C_2C_{r+2} + \ldots\ldots + C_{n-r} C_n = \dfrac{2n!}{(n-r)!\,(n+r)!}$

(d) $\displaystyle\sum_{r=0}^{n-2}({}^nC_r \,.\, {}^nC_{r+2}) = \frac{(2n)!}{(n-2)!(n+2)!}$

(e) ${}^{100}C_{10} + 5.\ {}^{100}C_{11} + 10\ .\ {}^{100}C_{12} + 10\ .\ {}^{100}C_{13} + 5.\ {}^{100}C_{14} + {}^{100}C_{15} = {}^{105}C_{90}$

12. If $C_0, C_1, C_2, \ldots\ldots, C_n$ are the combinatorial coefficients in the expansion of $(1 + x)^n$, $n \in N$, then prove the following :

(a) $C_1 + 2C_2 + 3C_3 + \ldots\ldots + n\ .\ C_n = n\ .\ 2^{n-1}$

(b) $C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n+1)C_n = (n+2)2^{n-1}$

(c) $C_0 + 3C_1 + 5C_2 + \ldots\ldots + (2n+1)C_n = (n+1)\,2^n$

(d) $(C_0+C_1)(C_1+C_2)(C_2+C_3)\ \ldots\ldots\ (C_{n-1}+C_n) = \dfrac{C_0 . C_1 . C_2 \ldots . C_{n-1}(n+1)^n}{n!}$

(e) $1\ .\ C_o^2 + 3\ .\ C_1^2 + 5\ .\ C_2^2 + \ldots\ldots + (2n+1)\,C_n^2 = \dfrac{(n+1)\,(2n)!}{n!\,n!}$

13. Prove that

(a) $\dfrac{C_1}{C_0} + \dfrac{2C_2}{C_1} + \dfrac{3C_3}{C_2} + \ldots\ldots + \dfrac{n.C_n}{C_{n-1}} = \dfrac{n(n+1)}{2}$ (b) $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots\ldots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

(c) $2\ .\ C_o + \dfrac{2^2.C_1}{2} + \dfrac{2^3.C_2}{3} + \dfrac{2^4.C_3}{4} + \ldots\ldots \dfrac{2^{n+1}.C_n}{n+1} = \dfrac{3^{n+1}-1}{n+1}$

(d) $C_o - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \ldots\ldots + (-1)^n\ \dfrac{C_n}{n+1} = \dfrac{1}{n+1}$

14. Given that $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$, find the values of :

(i) $a_0 + a_1 + a_2 + \ldots\ldots + a_{2n}$;

(ii) $a_0 - a_1 + a_2 - a_3 \ldots\ldots + a_{2n}$; (iii) $a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots\ldots + a_{2n}^2$

15. Find the sum of the series $\displaystyle\sum_{r=0}^{n}(-1)^r . {}^nC_r\left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \ldots\ldots\ldots \text{up to m terms}\right]$

16. Find the coefficient of

(a) x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$ (b) x^4 in the expansion of $(2 - x + 3x^2)^6$

17. Find the coefficient of

(a) $x^2\,y^3\,z^4$ in the expansion of $(ax - by + cz)^9$.

(b) $a^2\,b^3\,c^4\,d$ in the expansion of $(a - b - c + d)^{10}$.

EXERCISE (JM)

1. Let $S_1 = \sum_{j=1}^{10} j(j-1)^{10}C_j$, $S_2 = \sum_{j=1}^{10} j^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^{2\,10}C_j$. **[AIEEE-2010]**

Statement–1 : $S_3 = 55 \times 2^9$.

Statement–2 : $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$.

(1) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(2) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.

(3) Statement–1 is true, Statement–2 is false.

(4) Statement–1 is false, Statement–2 is true.

2. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is :- **[AIEEE 2011]**

(1) – 144 (2) 132 (3) 144 (4) – 132

3. If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ is : **[AIEEE 2012]**

(1) a rational number other than positive integers (2) an irrational number

(3) an odd positive integer (4) an even positive integer

4. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-x^{1/2}}\right)^{10}$ is : **[JEE-Main 2013]**

(1) 4 (2) 120 (3) 210 (4) 310

5. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to :- **[JEE(Main)-2014]**

(1) $\left(16, \frac{251}{3}\right)$ (2) $\left(14, \frac{251}{3}\right)$ (3) $\left(14, \frac{272}{3}\right)$ (4) $\left(16, \frac{272}{3}\right)$

6. The sum of coefficients of integral powers of x in the binomial expansion of $\left(1-2\sqrt{x}\right)^{50}$ is : **[JEE(Main)-2015]**

(1) $\frac{1}{2}\left(3^{50}-1\right)$ (2) $\frac{1}{2}\left(2^{50}+1\right)$ (3) $\frac{1}{2}\left(3^{50}+1\right)$ (4) $\frac{1}{2}\left(3^{50}\right)$

7. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is :- **[JEE(Main)-2016]**

(1) 729 (2) 64 (3) 2187 (4) 243

8. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + ... + ({}^{21}C_{10} - {}^{10}C_{10})$ is :- **[JEE(Main)-2017]**

(1) $2^{20} - 2^{10}$ (2) $2^{21} - 2^{11}$ (3) $2^{21} - 2^{10}$ (4) $2^{20} - 2^9$

9. The sum of the co-efficients of all odd degree terms in the expansion of

$\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, $(x > 1)$ is - **[JEE(Main)-2018]**

(1) 0 (2) 1 (3) 2 (4) –1

10. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to : **[JEE(Main)- 2019]**

(1) 14 (2) 6 (3) 4 (4) 8

11. The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is **[JEE(Main)- 2019]**

(1) 12 (2) 15 (3) 10 (4) 14

12. If $\sum_{r=0}^{25}\left\{{}^{50}C_r\cdot{}^{50-r}C_{25-r}\right\}=K\left({}^{50}C_{25}\right)$, then K is equal to : **[JEE(Main)- 2019]**

(1) $2^{25}-1$ (2) $(25)^2$ (3) 2^{25} (4) 2^{24}

13. The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3}+\frac{3}{x}\right)^8$

equals 5670 is : **[JEE(Main)- 2019]**

(1) 6 (2) 8 (3) 0 (4) 4

14. The value of r for which ${}^{20}C_r\,{}^{20}C_0 + {}^{20}C_{r-1}\,{}^{20}C_1 + {}^{20}C_{r-2}\,{}^{20}C_2 + ... {}^{20}C_0\,{}^{20}C_r$ is maximum, is **[JEE(Main)- 2019]**

(1) 20 (2) 15 (3) 11 (4) 10

15. Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1x + a_2x^2 + ... + a_{50}x^{50}$, for all $x\in R$, then $\frac{a_2}{a_0}$ is equal to:-

[JEE(Main)- 2019]

(1) 12.50 (2) 12.00 (3) 12.75 (4) 12.25

16. Let $S_n = 1 + q + q^2 + ... + q^n$ and $T_n=1+\left(\frac{q+1}{2}\right)+\left(\frac{q+1}{2}\right)^2+........+\left(\frac{q+1}{2}\right)^n$, where q is a real number and $q \neq 1$. If ${}^{101}C_1 + {}^{101}C_2.S_1 + ... + {}^{101}C_{101}.S_{100} = \alpha T_{100}$, then α is equal to :-

[JEE(Main)- 2019]

(1) 2^{100} (2) 200 (3) 2^{99} (4) 202

17. The total number of irrational terms in the binomial expansion of $\left(7^{1/5}-3^{1/10}\right)^{60}$ is :

[JEE(Main)- 2019]

(1) 55 (2) 49 (3) 48 (4) 54

18. If some three consecutive in the binomial expansion of $(x + 1)^n$ is powers of x are in the ratio 2 : 15 : 70, then the average of these three coefficient is :- **[JEE(Main)- 2019]**

(1) 964 (2) 625 (3) 227 (4) 232

19. The coefficient of x^{18} in the product $(1+x)(1-x)^{10}(1 + x + x^2)^9$ is : **[JEE(Main)- 2019]**

(1) –84 (2) 84 (3) 126 (4) –126

20. If ${}^{20}C_1 + (2^2)\,{}^{20}C_2 + (3^2)\,{}^{20}C_3 + ... + (20^2){}^{20}C_{20} = A(2^\beta)$, then the ordered pair (A, β) is equal to: **[JEE(Main)- 2019]**

(1) (420, 18) (2) (380, 19) (3) (380, 18) (4) (420, 19)

21. The term independent of x in the expansion of $\left(\frac{1}{60}-\frac{x^8}{81}\right)\cdot\left(2x^2-\frac{3}{x^2}\right)^6$ is equal to :

[JEE(Main)- 2019]

(1) 36 (2) – 108 (3) – 72 (4) – 36

22. The coefficient of x^7 in the expression $(1 + x)^{10} + x(1 + x)^9 + x^2(1 + x)^8 + ... + x^{10}$ is :

[JEE(Main)- 2020]

(1) 120 (2) 330 (3) 210 (4) 420

23. If the sum of the coefficients of all even powers of x in the product

$(1 + x + x^2 + ... + x^{2n})(1 - x + x^2 - x^3 + ... + x^{2n})$ is 61, then n is equal to _________.

[JEE(Main)- 2020]

24. If α and β be the coefficients of x^4 and x^2 respectively in the expansion of

$\left(x+\sqrt{x^2-1}\right)^6+\left(x-\sqrt{x^2-1}\right)^6$, then **[JEE(Main)- 2020]**

(1) $\alpha + \beta = 60$ (2) $\alpha + \beta = -30$ (3) $\alpha - \beta = -132$ (4) $\alpha - \beta = 60$

25. In the expansion of $\left(\frac{x}{\cos\theta}+\frac{1}{x\sin\theta}\right)^{16}$, if ℓ_1 is the least value of the term independent of x when

$\frac{\pi}{8}\le\theta\le\frac{\pi}{4}$ and ℓ_2 is the least value of the term independent of x when $\frac{\pi}{16}\le\theta\le\frac{\pi}{8}$, then the ratio

$\ell_2 : \ell_1$ is equal to : **[JEE(Main)- 2020]**

(1) 1 : 8 (2) 1 : 16 (3) 8 : 1 (4) 16 : 1

26. If $C_r \equiv {}^{25}C_r$ and $C_0 + 5.C_1 + 9.C_2 + ... + (101).C_{25} = 2^{25}.k$, then k is equal to _____.

[JEE(Main)- 2020]

EXERCISE (JA)

1. For r = 0, 1,....,10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to - **[JEE 2010, 5]**

(A) $B_{10} - C_{10}$ (B) $A_{10}\left(B_{10}^2 - C_{10}A_{10}\right)$ (C) 0 (D) $C_{10} - B_{10}$

2. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =

[JEE (Advanced) 2013, 4M, –1M]

3. Coefficient of x^{11} in the expansion of $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$ is -

(A) 1051 (B) 1106 (C) 1113 (D) 1120

[JEE(Advanced)-2014, 3(–1)]

4. The coefficient of x^9 in the expansion of $(1 + x)(1 + x^2)(1 + x^3)\ldots(1 + x^{100})$ is

[JEE 2015, 4M, –0M]

5. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \ldots\ldots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is **[JEE(Advanced)-2016, 3(0)]**

6. Let $X = \left({}^{10}C_1\right)^2 + 2\left({}^{10}C_2\right)^2 + 3\left({}^{10}C_3\right)^2 + \ldots + 10\left({}^{10}C_{10}\right)^2$, where ${}^{10}C_r$, $r \in \{1, 2, \ldots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is _____ . **[JEE(Advanced)-2018, 3(0)]**

7. Suppose $\det\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^nC_k k^2 \\ \sum_{k=0}^{n} {}^nC_k k & \sum_{k=0}^{n} {}^nC_k 3^k \end{bmatrix} = 0$, holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^nC_k}{k+1}$ equals

[JEE(Advanced)-2019, 3(0)]

ANSWER KEY

EXERCISE (O-1)

1.	C	**2.**	C	**3.**	B	**4.**	A	**5.**	A	**6.**	C	**7.**	B
8.	A	**9.**	B	**10.**	B	**11.**	D	**12.**	B	**13.**	A	**14.**	B
15.	B												

EXERCISE (O-2)

1.	A,B	**2.**	B,C,D	**3.**	A,C,D	**4.**	A,B	**5.**	A,C	**6.**	B	**7.**	B
8.	A,C,D	**9.**	B	**10.**	D	**11.**	A	**12.**	A,C	**13.**	B,C,D	**14.**	A,D
15.	B,D												

EXERCISE (S-1)

1. **(a)** $r = 6$ **(b)** $r = 5$ or 9

2. (i) $\frac{5}{12}$ (ii) $T_6 = 7$

4. $\frac{17}{54}$

5. $n = 2$ or 3 or 4

6. 816

7. ${}^{n}C_r(3^{n-r} - 2^{n-r})$

8. **(i)** $T_7 = \frac{7.3^{13}}{2}$ **(ii)** 455×3^{12}

14. **(i)** 3^n **(ii)** 1, **(iii)** a_n

15. $\frac{(2^{mn}-1)}{(2^n-1)(2^{mn})}$

16. **(a)** 990 **(b)** 3660

17. **(a)** $-1260 . a^2b^3c^4$ **; (b)** -12600

EXERCISE (JM)

1.	3	**2.**	1	**3.**	2	**4.**	3	**5.**	4	**6.**	3		

7. **Bonus**

Note : In the problem 'number of terms should be 13 instead of 28', then (1) will be the answer

8.	1	**9.**	3	**10.**	4	**11.**	2	**12.**	3	**13.**	3	**14.**	1
15.	4	**16.**	1	**17.**	4	**18.**	4	**19.**	2	**20.**	1	**21.**	4
22.	**2**	**23.**	30	**24.**	3	**25.**	4	**26.**	51				

EXERCISE (JA)

1.	D	**2.**	6	**3.**	C	**4.**	8	**5.**	5	**6.**	646	**7.**	6.20

CHAPTER 3

PRINCIPLE OF MATHEMATICAL INDUCTION

Chapter Contents 03

IMPORTANT NOTES

CHAPTER 3

PRINCIPLE OF MATHEMATICAL INDUCTION

1. THEOREM-I

If P(n) is a statement depending upon n, then to prove P(n) by induction, we procced as follows:

(i) Verify the validity of P(n) for n = 1

(ii) Assume that P(n) is true for any positive integer m and then using it establish the validity of P(n) for

n = m + 1.

Then P(n) is true for each $n \in N$

2. THEOREM-II

If P(n) is a statement depending upon n but begining with any positive integer k, then to prove P(n) by Induction, we procced as follows :

(i) Verify the validity of P(n) for n = k.

(ii) Assume that the P(n) is true for $n = m \geq k$.

Then using it estabish the validity of P(n) for n = m + 1.

Then P(n) is true for each $n \geq k$

3. SOME USEFUL RESULT BASED ON PRINCIPLE OF MATHEMATICAL INDUCTION :

For any natural number n

(i) $1 + 2 + 3 + ... + n = \Sigma n = \dfrac{n(n+1)}{2}$

(ii) $1^2 + 2^2 + 3^2 + ... + n^2 = \Sigma n^2 = \dfrac{n(n+1)(2n+1)}{6}$

(iii) $1^3 + 2^3 + 3^3 + ... + n^3 = \Sigma n^3 = (\Sigma n)^2 = \left\{\dfrac{n(n+1)}{2}\right\}^2$

(iv) $2 + 4 + 6 + ... + 2n = \Sigma 2n = n(n + 1)$

(v) $1 + 3 + 5 + ... + (2n - 1) = \Sigma(2n - 1) = n^2$

(vi) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + ... + xy^{n-2} + y^{n-1})$

(vii) $x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 + ... - xy^{n-2} + y^{n-1})$

when n is odd positive integer

4. IMPORTANT TIPS :

(i) Product of r successive integers is divisible by r!

(ii) For $x \neq y$, $x^n - y^n$ is divisible by

(a) $x + y$, if n is even (b) $x - y$, if n is even or odd

(iii) $x^n + y^n$ is divisible by

$x + y$, If n is odd

(iv) For solving objective question related to natural numbers we find out the correct alternative by negative examination of this principle. If the given statement is P(n), then by putting n = 1, 2, 3 .... in P(n) we decide the correct answer. We also use the above formulae established by this principle to find the sum of n terms of a given series. For this we first express T_n as a polynomial in n and then for finding S_n, we put Σ before each term of this polynomial and then use above results of Σn, Σn^2, Σn^3 etc.

SOLVED EXAMPLES

Ex.1 Use the principle of mathematical induction to show that $5^{2n+1} + 3^{n+2}.2^{n-1}$ divisible by 19 for all natural numbers n.

Sol. Let $P(n) = 5^{2n+1} + 3^{n+2}.2^{n-1}$

Step I : For n = 1

$P(1) = 5^{2+1} + 3^{1+2}.2^{1-1}$

$= 125 + 27$

$= 152$, which is divisible by 19.

Therefore, the result is true for n = 1.

Step II : Assume that the result is true for n = k, i.e. $P(k) = 5^{2k+1} + 3^{k+2}, 2^{k-1}$ is divisible by 19.

$\Rightarrow P(k) = 19r$, where r is an integer.

Step III : For n = k + 1

$P(k + 1) = 5^{2(k+1)+1} + 3^{k+1+2}.\ 2^{k+1-1}$

$= 5^{2k+3} + 3^{k+3}.2^{k}$

$= 25.5^{2k+1} + 3.3^{k+2}.2.2^{k-1}$

$= 25.5^{2k} + 6.3^{k+2}.2^{k-1}$

Now $25.5^{2k+1} + 6.3^{k+2}2^{k-1} = 25.(5^{2k+1} + 3^{k-2}.2^{k-1}) - 19.3^{k+2}.2^{k-1}$

i.e. $P(k + 1) = 25\ P(k) - 19.3^{k+2}.2^{k-1}$

But we know that P(k) is divisible by 19. Also $19.3^{k+2}.2^{k-1}$ is clearly divisible by 19.

Hence P(k + 1) is divisible by 19. This shows that the result is true for n = k + 1. Hence by the priciniple of mathematical induction, the result is true for all $n \in N$.

Ex.2. Use the principle of mathematical induction to show that 1.3 + 2.4 + ... + n.(n+2)

$= \frac{1}{6}n(n+1)(2n+7)$.

Sol. Let P(n) : $1.3 + 2.4 + ... + n.(n+2) = \frac{1}{6}n(n + 1)(2n + 7)$

Step I : For n = 1

LHS of P(1) $= 1.3 = 3 = \frac{1}{6}.1.2.9 = \frac{1}{6}.1(1+1)(2.1+7)$ = RHS of P(1)

So P(1) is true

Step II : Now assume P(k) is true, for some natural number k, i.e

$1.3 + 2.4 + ... + k.(k + 2) = \frac{1}{6}k(k + 1)(2k + 7)$.

Now deduce P(k + 1).

LHS of P(k + 1) = 1.3 + 2.4+......+k.(k+2) + (k + 1). (k + 1 + 2)

= (LHS of P(k)) + (k + 1) (k + 3)

= (RHS of P(k)) + (k + 1) (k + 3), (by inductive assumption)

$$= \frac{1}{6}k(k+1)(2k+7)+(k+1)(k+3)$$

$$= \frac{1}{6}(k+1)(k(2k+7)+6(k+3))$$

$$= \frac{1}{6}(k+1)(2k^2+13k+18)$$

$$= \frac{1}{6}(k+1)(k+2)(2k+9)$$

$$= \frac{1}{6}(k+1)(k+1+1)(2(k+1)+7)$$

= RHS of P(k + 1).

So P(k + 1) is true, if P(k) is true.

Hence by induction P(n) is true for all natural numbers n.

Ex.3 Use the principle of mathematical induction to show that for any positive integer number n, $n^3 + 2n$, is divisible by 3.

Sol. Statement P(n) is defined by $n^3 + 2n$ is divisible 3

Step 1 : We first show that P(1) is true. Let n = 1 and calculate $n^3 + 2n$

$1^3 + 2(1) = 3$

Hence P(1) is true.

Step 2 : We now assume that P(k) is true $k^3 + 2k$ is divisible by 3. is equivalent to

$k^3 + 2k = 3M$, where M is a positive integer.

We now consider the algebraic expression $(k + 1)^3 + 2(k + 1)$; expand it and group like terms.

$(k + 1)^3 + 2(k + 1) = k^3 + 3k^2 + 5k + 3$

$= [k^3 + 2k] + [3k^2 + 3k + 3]$

$= 3M + 3[k^2 + k + 1] = 3[M + k^2 + k + 1]$

Hence $(k + 1)^3 + 2(k + 1)$ is also divisible by 3 and therefore statement P(k + 1) is true.

Ex.4 Prove that $3^n > n^2$ for n = 1, n = 2 and use the mathematical induction to prove that $3^n > n^2$ for n, a positive integer greater than 2.

Sol Statement P(n) is defined by

$3^n > n^2$

Step 1 : We first show that P(1) is true. Let n =1 and calculate 3^1 and 1^2 and compare them

$$3^1 = 3$$

$$1^2 = 1$$

3 is greater than 1 and hence P(1) is true.

Let us also show that P(2) is true.

$$3^2 = 9$$

$$2^2 = 4$$

Hence P(2) is also true.

Step 2 : We now assume that P(k) is true

$$3^k > k^2$$

Multiply both sides of the above inequality by 3.

$$3 * 3^k > 3 * k^2$$

The left side is equal to 3^{k+1}. For k > 2, we can write

$$k^2 > 2\,k \text{ and } k^2 > 1$$

We now combine the above inequalities by adding the left hand sides and the right hand sides of the two inequalities.

$$2k^2 > 2k + 1$$

We now add k^2 to both sides of the above inequality to obtain the inequality

$$3k^2 > k^2 + 2k + 1$$

Factor the right side we can write

$$3 * k^2 > (k + 1)^2$$

If $3 * 3^k > 3 * k^2$ and $3 * k^2 > (k + 1)^2$ then

$$3 * 3^k > (k + 1)^2$$

Rewrite the left side as 3^{k+1}

$$3^{k+1} > (k + 1)^2$$

* Which proves that P(k + 1) is true.

Ex.5. Use mathematical induction to prove De Moiver's theorem

$[R\,(\cos t + i \sin t)]^n = R^n(\text{cosx } nt + i \sin nt)$ for n a positive integer.

Sol. **Step 1 :** For n = 1

$[R\,(\cos t + i \sin t)]^1 = R^1(\cos 1.t + i \sin 1.t)$

It can be easily be seen that the two sides are equal.

Step 2 : We now assume that the theorem is true for $n = k$, hence

$[R(\cos t + i\sin t)]^k = R^k(\cos kt + i \sin kt)$

Multiply both sides of the above equation by $R(\cos t + i \sin t)$

$[R (\cos t + i \sin t)]^k R (\cos t + i \sin t) = R^k(\cos kt + i \sin kt) R(\cos t + i \sin t)$

Rewrite the above as follows

$[R (\cos t + i \sin t)]^{k+1} = R^{k+1}[(\cos kt \cos t – \sin kt \sin t) + i(\sin kt \cos t + \cos kt \sin t)]$

Trigonometric identities can be used to write the trigonometric expressions $(\cos kt \cos t – \sin kt \sin t)$ and $(\sin kt \cos t + \cos kt \sin t)$ as follows

$(\cos kt \cos t – \sin kt \sin t) = \cos(kt + t) = \cos(k + 1)t$

$(\sin kt \cos t + \cos kt \sin t) = \sin(kt + t) = \sin(k + 1) t$

Substitute the above into the last equation to obtain.

$[R (\cos t + i \sin t)]^{k+1} = R^{k+1}[\cos (k + 1) t + \sin(k + 1)t]$

It has been established that the theorem is true for $n = 1$ and that if it assumed true for $n = k$ it is true for $n = k + 1$.

EXERCISE-1

1. Let P(n) : $n^2 + n$ is an odd integer. It is seen that truth of P(n) $\Rightarrow$ the truth of P(n + 1). Therefore, P(n) is true for all-

(1) $n > 1$ (2) n (3) $n > 2$ (4) None of these

2. If $n \in N$, then $x^{2n-1} + y^{2n-1}$ is divisible by-

(1) $x + y$ (2) $x - y$ (3) $x^2 + y^2$ (4) $x^2 + xy$

3. If $n \in N$, then $11^{n+2} + 12^{2n+1}$ is divisible by-

(1) 113 (2) 123 (3) 133 (4) None of these

4. If $n \in N$, then $3^{4n+2} + 5^{2n+1}$ is a multiple of-

(1) 14 (2) 16 (3) 18 (4) 20

5. For every positive integer

n, $\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}$ is-

(1) an integer

(2) a rational number which is not an integer

(3) a negative real number

(4) an odd integer

6. Sum of the infinite seriese $\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + ...$ equals-

(1) 1/3 (2) 3 (3) 1/4 (4) 2

7. If $a_k = \frac{1}{k(k+1)}$; then $\left(\sum_{k=1}^{n} a_k\right)^2$ is equal to-

(1) $\frac{n^2}{(n+1)^2}$ (2) $\frac{n^4}{(n+1)^4}$ (3) $\frac{n^2}{n^2+1}$ (4) $\frac{n}{n+1}$

8. The sum of n terms of the series

$\frac{\frac{1}{2}\cdot\frac{2}{2}}{1^3} + \frac{\frac{2}{2}\cdot\frac{3}{2}}{1^3+2^3} + \frac{\frac{3}{2}\cdot\frac{4}{2}}{1^3+2^3+3^3} + ...$ is-

(1) $\frac{1}{n(n+1)}$ (2) $\frac{n}{n+1}$ (3) $\frac{n+1}{n}$ (4) $\frac{n+1}{n+2}$

9. For all $n \in N$, $7^{2n} - 48n - 1$ is divisible by-

(1) 25 (2) 26 (3) 1234 (4) 2304

10. For all positive integral values of n, $3^{2n} - 2n + 1$ is divisible by-

(1) 2 (2) 4 (3) 8 (4) 12

11. The smallest positive integer for which the statement $3^{n+1} < 4^n$ holds is-

(1) 1 (2) 2 (3) 3 (4) 4

12. For positive integer n, $10^{n-2} > 81n$ when-

(1) $n < 5$ (2) $n > 5$ (3) $n \geq 5$ (4) $n > 6$

13. If P is a prime number then $n^p - n$ is divisible by p when n is a

(1) natural number greater than 1

(2) odd number

(3) even number

(4) None of these

14. A student was asked to prove a statement by induction. He proved

(i) P(5) is true and

(ii) Truth of $P(n) \Rightarrow$ truth of $p(n + 1)$, $n \in N$

On the basis of this, he could conclude that P(n) is true for

(1) no $n \in N$ (2) all $n \in N$ (3) all $n \geq 5$ (4) None of these

EXERCISE-2

1. If $x > -1$, then the statement
$P(n) : (1 + x)^n > 1 + nx$ is true for-
(1) all $n \in N$ (2) all $n > 1$
(3) all $n > 1$ and $x \neq 0$ (4) None of these

2. For every positive integral value of n, $3^n, > n^3$ when-
(1) $n > 2$ (2) $n \geq 3$ (3) $n \geq 4$ (4) $n < 4$

3. $P(n) : 3^{2n+2} - 8n - 9$ is divisible by 64, is true for-
(1) all $n \in N \cup \{0\}$ (2) $n \geq 2, n \in N$ (3) $n \in N, n > 2$ (4) None of these

4. If m, n are any two odd positive integer wih $n < m$, then the largest positive integers which divides all the numbers of the type $m^2 - n^2$ is-
(1) 4 (2) 6 (3) 8 (4) 9

5. For all $n \in N$, $\cos\theta \cos2\theta \cos4\theta \ldots \cos2^{n-1}\theta$ equals to
(1) $\frac{\sin 2^n\theta}{2^n \sin\theta}$ (2) $\frac{\sin 2^n\theta}{\sin\theta}$ (3) $\frac{\cos 2^n\theta}{2^n \cos 2\theta}$ (4) $\frac{\cos 2^n\theta}{2^n \sin\theta}$

6. $x(x^{n-1} - na^{n-1}) + a^n(n - 1)$ is divisible by $(x - a)^2$ for-
(1) $n > 1$ (2) $n > 2$ (3) all $n \in N$ (4) None of these

7. For any odd integer $n \geq 1$
$n^3 - (n - 1)^3 + \ldots + (-1)^{n-1}.1^3$ is equal to-
(1) $\frac{1}{4}(n + 1)^2(2n - 1)$ (2) $\frac{1}{4}(n - 1)^2(2n - 1)$
(3) $\frac{1}{2}(n - 1)^2(2n - 1)$ (4) $\frac{1}{2}(n + 1)^2(2n - 1)$

8. If p and q are respectively. The sum and the sum of squares of n successive integers beginning with a, then $nq - p^2$ is-
(1) independent of a (2) independent of n (3) dependent on a (4) None of these

9. The sum of first n terms of the given series
$1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, then sum will be-
[AIEEE-2004]
(1) $\frac{n(n+1)^2}{2}$ (2) $\frac{1}{2}n^2(n + 1)$ (3) $n(n + 1)^2$ (4) None

10. Let $S(k) = 1 + 3 + 5 + \ldots + (2k - 1) = 3 + k^2$, then which of the following is true ? **[AIEEE-2004]**
(1) S(1) is true
(2) $S(k) \Rightarrow S(k + 1)$
(3) $S(k) \nRightarrow S(k + 1)$
(4) Principle of mathematical Induction can be used to prove that formula

11. The sum of n terms of the series

$1 + (1 + a) + (1 + a + a^2) + (1 + a + a^2 + a^3) + \ldots..,$ is-

(1) $\frac{n}{1-a} - \frac{a(1-a^n)}{(1-a)^2}$ (2) $\frac{n}{1-a} + \frac{a(1-a^n)}{(1-a)^2}$ (3) $\frac{n}{1-a} + \frac{a(1+a^n)}{(1-a)^2}$ (4) $-\frac{n}{1-a} + \frac{a(1-a^n)}{(1-a)^2}$

12. **Statement :1** For every natural number $n \geq 2$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \ldots + \frac{1}{\sqrt{n}} > \sqrt{n}$$

Statement –2 : For every natural number $n \geq 2$, $\sqrt{n(n+1)} < n+1$. **[AIEEE-2008]**

(1) Statement –1 is false, Statement –2 is true

(2) Statement–1 is true, Statement–2 is false

(3) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1

(4) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for Statement–1

13. **Statement - 1:** For each natural number n,$(n + 1)^7 – n^7 –1$ is divisible by 7.

Statement - 2: For each natural number n, $n^7 – n$ is divisible by 7. **[AIEEE-2011]**

(1) Statement-1 is false, statement-2 is true.

(2) Statement-1 is true, statement-2 istrue; Statement-2 is correct explanation for statement-1.

(3) Statement-1 is true, statement-2 is true; Statement-2 is not a correct explanation for statement-1.

(4) Statement-1 is true, statement-2 is false.

14. Consider the statement : "P(n): $n^2 – n + 41$ is prime." Then which one of the following is true? **[JEE(Main) Jan-19]**

(1) P(5) is false but P(3) is true (2) Both P(3) and P(5) are false

(3) P(3) is false but P(5) is true (4) Both P(3) and P(5) are true

ANSWER KEY

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Ans.	**4**	**1**	**3**	**1**	**1**	**1**	**1**	**2**	**4**	**1**	**4**	**3**	**1**	**3**

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Ans.	**3**	**3**	**1**	**3**	**1**	**3**	**1**	**1**	**2**	**2**	**1**	**3**	**2**	**4**

CHAPTER 4

MATHEMATICAL REASONING

Chapter Contents 04

IMPORTANT NOTES

CHAPTER 4

MATHEMATICAL REASONING

1. STATEMENT :

A sentence which is either true or false but cannot be both are called a statement. A sentence which is an exclamatory or a wish or an imperative or an interrogative can not be a statement.

If a statement is true then its truth value is T and if it is false then its truth value is F

For ex.

(i) "New Delhi is the capital of India", a true statement

(ii) "3 + 2 = 6", a false statement

(iii) "Where are you going ?" not a statement beasuse

it connot be defined as true or false

Note : A statement cannot be both true and false at a time

2. SIMPLE STATEMENT :

Any statement whose truth value does not depend on other statement are called simple statement

For ex. (i) " $\sqrt{2}$ is an irrational number" (ii) "The set of real number is an infinite set"

3. COMPOUND STATEMENT :

A statement which is a combination of two or more simple statements are called compound statement

Here the simple statements which form a compound statement are known as its sub statements

For ex.

(i) "If x is divisible by 2 then x is even number"

(ii) "ΔABC is equilatral if and only if its three sides are equal"

4. LOGICAL CONNECTIVES :

The words or phrases which combined simple statements to form a compound statement are called logical connectives.

In the following table some possible connectives, their symbols and the nature of the compound statement formed by them

S.N.	Connectives	symbol	use	operation
1.	and	$\wedge$	$p \wedge q$	conjunction
2.	or	$\vee$	$p \vee q$	disjunction
3.	not	$\sim$ or $'$	$\sim p$ or p'	negation
4.	If .... then	$\Rightarrow$ or $\rightarrow$	$p \Rightarrow q$ or $p \rightarrow q$	Implication or conditional
5.	If and only if (iff)	$\Leftrightarrow$ or $\leftrightarrow$	$p \Leftrightarrow q$ or $p \leftrightarrow q$	Equivalence or Bi-conditional

Explanation :

(i) $p \wedge q \equiv$ statement p and q

($p \wedge q$ is true only when p and q both are true otherwise it is false)

(ii) $p \vee q \equiv$ statement p or q

($p \vee q$ is true if at least one from p and q is true i.e. $p \vee q$ is false only when p and q both are false)

(iii) $\sim p \equiv$ not statement p

($\sim p$ is true when p is false and $\sim p$ is false when p is true)

(iv) $p \Rightarrow q \equiv$ statement p then statement q

($p \Rightarrow q$ is false only when p is true and q is false otherwise it is true for all other cases)

(v) $p \Leftrightarrow q \equiv$ statement p if and only if statement q

($p \Leftrightarrow q$ is true only when p and q both are true or false otherwise it is false)

5. TRUTH TABLE :

A table which shows the relationship between the truth value of compound statement S(p, q, r ....) and the truth values of its sub statements p, q, r, ... is said to be truth table of compound statement S

If p and q are two simple statements then truth table for basic logical connectives are given below

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

p	$(\sim p)$
T	F
F	T

Conditional

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$ or $p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note : If the compound statement contain n sub statements then its truth table will contain 2^n rows.

6. LOGICAL EQUIVALENCE :

Two compound statements $S_1(p, q, r...)$ and $S_2(p, q, r ....)$ are said to be logically equivalent or simply equivalent if they have same truth values for all logically possibilities

Two statements S_1 and S_2 are equivalent if they have identical truth table i.e. the entries in the last column of their truth table are same. If statements S_1 and S_2 are equivalent then we write $S_1 \equiv S_2$

For ex. The truth table for $(p \to q)$ and $(\sim p \vee q)$ given as below

p	q	$(\sim p)$	$p \to q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We observe that last two columns of the above truth table are identical hence compound statements $(p \to q)$ and $(\sim p \vee q)$ are equivalent

i.e. $\boxed{p \to q \equiv \sim p \vee q}$

7. TAUTOLOGY AND CONTRADICTION :

(i) Tautology : A statement is said to be a tautology if it is true for all logical possibilities i.e. its truth value always T. it is denoted by t.

For ex. the statement $p \vee \sim (p \wedge q)$ is a tautology

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Clearly, The truth value of $p \vee \sim (p \wedge q)$ is T for all values of p and q. so $p \wedge \sim (p \wedge q)$ is a tautology

(ii) Contradiction : A statement is a contradiction if it is false for all logical possibilities.

i.e. its truth value always F. It is denoted by c.

For ex. The statement $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction

p	q	$\sim p$	$\sim q$	$p \vee q$	$(\sim p \wedge \sim q)$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

Clearly, then truth value of $(p \vee q) \wedge (\sim p \wedge \sim q)$ is F for all value of p and q. So $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.

Note : The negation of a tautology is a contradiction and negation of a contradiction is a tautology

8. DUALITY :

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$

If a compound statement contains the special variable t (tautology) and c (contradiction) then obtain its dual we replaced t by c and c by t in addition to replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

Note :

(i) the connectives $\wedge$ and $\vee$ are also called dual of each other.

(ii) If $S^*(p, q)$ is the dual of the compound statement $S(p, q)$ then

(a) $S^*(\sim p, \sim q) \equiv \sim S(p, q)$ (ii) $\sim S^*(p, q) \equiv S(\sim p, \sim q)$

For ex. The duals of the following statements

(i) $(p \wedge q) \vee (r \vee s)$ (ii) $(p \vee t) \wedge (p \vee c)$

(iii) $\sim(p \wedge q) \vee [p \wedge \sim(q \vee \sim s)]$

are as given below

(i) $(p \vee q) \wedge (r \wedge s)$

(ii) $(p \wedge c) \vee (p \wedge t)$

(iii) $\sim(p \vee q) \wedge [p \vee \sim(q \wedge \sim s)]$

9. CONVERSE, INVERSE AND CONTRAPOSITIVE OF THE CONDITIONAL STATEMENT ($p \rightarrow q$):

(i) Converse : The converse of the conditional statement $p \rightarrow q$ is defined as $q \rightarrow p$

(ii) Inverse : The inverse of the conditional statement $p \rightarrow q$ is defined as $\sim p \rightarrow \sim q$

(iii) Contrapositive : The contrapositive of conditional statement $p \rightarrow q$ is defined as $\sim q \rightarrow \sim p$

10. NEGATION OF COMPOUND STATEMENTS :

If p and q are two statements then

(i) **Negation of conjunction :** $\sim(p \wedge q) \equiv \sim p \vee \sim q$

p	q	~ p	~ q	$(p \wedge q)$	$\sim(p \wedge q)$	$(\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(ii) Negation of disjunction : $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	~ p	~ q	$(p \vee q)$	$(\sim p \vee q)$	$(\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

(iii) Negation of conditional : $\sim(p \to q) \equiv p \wedge \sim q$

p	q	~ q	$(p \to q)$	$\sim(p \to q)$	$(p \wedge \sim q)$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

(iv) Negation of biconditional : $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

we know that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

$\therefore \sim(p \leftrightarrow q) \equiv \sim[(p \to q) \wedge (q \to p)]$

$\equiv \sim(p \to q) \vee \sim(q \to p)$

$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Note : The above result also can be proved by preparing truth table for $\sim(p \leftrightarrow q)$ and $(p \wedge \sim q) \vee (q \wedge \sim p)$

11. ALGEBRA OF STATEMENTS :

If p, q, r are any three statements then the some low of algebra of statements are as follow

(i) Idempotent Laws :

(a) $p \wedge p \equiv p$ (b) $p \vee p \equiv p$

i.e. $p \wedge p \equiv p \equiv p \vee p$

p	$(p \wedge p)$	$(p \vee p)$
T	T	T
F	F	F

(ii) Comutative laws :

(a) $p \wedge q \equiv q \wedge p$ (b) $p \vee q \equiv q \vee p$

p	q	$(p \wedge q)$	$(q \wedge p)$	$(p \vee q)$	$(q \vee p)$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	F	F	T	T
F	F	F	F	F	F

(iii) Associative laws :

(a) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

(b) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

p	q	r	$(p \wedge q)$	$(q \wedge r)$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Similarly we can proved result (b)

(iv) Distributive laws : (a) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ (c) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$

(b) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (d) $p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$

p	q	r	$(q \vee r)$	$(p \wedge q)$	$(p \wedge r)$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Similarly we can prove result (b), (c), (d)

(v) De Morgan Laws : (a) $\sim (p \wedge q) \equiv \sim p \vee \sim q$

(b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim (p \wedge q)$	$(\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Similarly we can proved resulty (b)

(vi) Involution laws (or Double negation laws) : $\sim(\sim p) \equiv p$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

(vii) Identity Laws : If p is a statement and t and c are tautology and contradiction respectively then

(a) $p \wedge t \equiv p$ (b) $p \vee t \equiv t$ (c) $p \wedge c \equiv c$ (d) $p \vee c \equiv p$

p	t	c	$(p \wedge t)$	$(p \vee t)$	$(p \wedge c)$	$(p \vee c)$
T	T	F	T	T	F	T
F	T	F	F	T	F	F

(viii) Complement Laws :

(a) $p \wedge (\sim p) \equiv c$ (b) $p \vee (\sim p) \equiv t$ (c) $(\sim t) \equiv c$ (d) $(\sim c) \equiv t$

p	$\sim p$	$(p \wedge \sim p)$	$(p \vee \sim p)$
T	F	F	T
F	T	F	T

(ix) Contrapositive laws : $p \rightarrow q \equiv \sim q \rightarrow \sim p$

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

12. QUANTIFIED STATEMENTS AND QUANTIFIERS :

The words or phrases "All", "Some", "None", "There exists a" are examples of quantifiers.

A statement containing one or more of these words (or phrases) is a quantified statement.

E.g. (1) All dogs are poodles

(2) Some books have hard covers

(3) There exists an odd number which is prime.

Note : Phrases "There exists a" and "Atleast one" and the word "some" have the same meaning.

NEGATION OF QUANTIFIED STATEMENTS :

(1) '**None**' is the negation of '**at least one**' or '**some**' or '**few**'

Statement : Some dogs are poodles.

Negation : No dogs are poodles.

Similarly negation of '**some**' is '**none**'

(2) The negation of "**some A are B**" or "**There exist A which is B**" is "**No A are (is) B**" or "**There does not exist any A which is B".**

Statement-1 : Some boys in the class are smart

Statement-2 : There exists a boy in the class who is smart

Statement-3 :Alteast one boy in the class is smart

All the three above statements have same meaning as they all indicate "**existence**" of smart boy in the class.

Negation of these statements is

No boy in the class is smart.

or

There does not exist any boy in the class who is smart.

(3) Negation of "**All A are B**" is "**Some A are not B**".

Statement : All boys in the class are smart.

Negation : Some boys in the class are not smart.

or

There exists a boy in the class who is not smart.

SOLVED EXAMPLES

Ex.1 Which of the following is correct for the statements p and q ?

(1) $p \wedge q$ is true when at least one from p and q is true

(2) $p \rightarrow q$ is true when p is true and q is false

(3) $p \leftrightarrow q$ is true only when both p and q are true

(4) $\sim(p \vee q)$ is true only when both p and q are false

Sol.(4) We know that $p \wedge q$ is true only when both p and q are true so option (1) is not correct

we know that $p \rightarrow q$ is false only when p is true and q is false so option (2) is not correct

we know that $p \leftrightarrow q$ is true only when either p and q both are true or both are flase

so option (3) is not correct

we know that $\sim(p \vee q)$ is true only when $(p \vee q)$ is false

i.e. p and q both are false

So option (4) is correct

Ex.2 $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to-

(1) p (2) ~p (3) q (4) ~q

Sol.(2) $\because\ \sim(p \vee q) \vee (\sim p \wedge q) \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$ (By Demorgan Law)

$\equiv \sim p \wedge (\sim q \vee q)$ (By distributive laws)

$\equiv \sim p \wedge t$ (By complement laws)

$\equiv \sim p$ (By Identity Laws)

Ex.3 Which of the following is logically equivalent to $(p \wedge q)$?

(1) $p \rightarrow \sim q$ (2) $\sim p \vee \sim q$ (3) $\sim(p \rightarrow \sim q)$ (4) $\sim(\sim p \wedge \sim q)$

Sol.(3) $\because\ p \rightarrow \sim q \equiv \sim p \vee \sim q \equiv \sim(p \wedge q)$ $(\because\ p \rightarrow q \equiv \sim p \vee q)$

i.e. $\sim(p \rightarrow \sim q) \equiv p \wedge q$

$\because\ \sim p \vee \sim q \equiv \sim(p \wedge q)$

and $\sim(\sim p \wedge \sim q) \equiv p \vee q$

Ex.4 If $p \rightarrow (q \vee r)$ is false, then the truth values of p, q, r respectively are-

(1) T, F, F (2) F, F, F (3) F, T, T (4) T, T, F

Sol.(1) We know $p \rightarrow (q \vee r)$ is false only when p is true and $(q \vee r)$ is false. but $(q \vee r)$ is false only when q and r both are false

Hence truth values of p, q, r are respectively T, F, F

Ex.5 Statement $(p \wedge \sim q) \wedge (\sim p \vee q)$ is

(1) a tautology (2) a contradiction

(3) neither a tautology not a contradiction (4) None of these

Sol.(2) $\because$ $(p \wedge \sim q) \wedge (\sim p \vee q)$

$\equiv (p \wedge \sim q) \wedge \sim(p \wedge \sim q)$ (By Demargon Laws)

$\equiv c$, where c is contradiction (By complement laws)

Ex.6 Negation of the statement $p \rightarrow (q \wedge r)$ is-

(1) $\sim p \rightarrow \sim(q \wedge r)$ (2) $\sim p \vee (q \wedge r)$ (3) $(q \wedge r) \rightarrow p$ (4) $p \wedge (\sim q \vee \sim r)$

Sol.(4) $\sim(p \rightarrow (q \wedge r)) \equiv p \wedge \sim(q \wedge r)$ $(\because \sim(p \rightarrow q) \equiv p \wedge \sim q)$

$\equiv p \wedge (\sim q \vee \sim r)$

Ex.7 If $x = 5$ and $y = -2$ then $x - 2y = 9$. The contrapositive of this statement is-

(1) If $x - 2y \neq 9$ then $x \neq 5$ or $y \neq -2$ (2) If $x - 2y \neq 9$ then $x \neq 5$ and $y \neq -2$

(3) If $x - 2y = 9$ then $x = 5$ and $y = -2$ (4) None of these

Sol.(1) Let p, q, r be the three statements such that

$p : x = 5$, $q : y = -2$ and $r : x - 2y = 9$

Here given statement is $(p \wedge q) \rightarrow r$ and its contrapositive is $\sim r \rightarrow \sim(p \wedge q)$

i.e. $\sim r \rightarrow (\sim p \vee \sim q)$

i.e. if $x - 2y \neq 9$ then $x \neq 5$ or $y \neq -2$

Ex.8 Which of the following is wrong ?

(1) $p \rightarrow q$ is logically equivalent to $\sim p \vee q$

(2) If the $(p \vee q) \wedge (q \vee r)$ is true then truth values of p, q, r are T, F, T respectively

(3) $\sim(p \wedge (q \vee r)) \equiv (\sim p \vee \sim q) \wedge (\sim p \vee \sim r)$

(4) The truth value of $p \wedge \sim(p \vee q)$ is always T

Sol.(4) We know that $p \rightarrow q \equiv \sim p \vee q$

If $(p \vee q) \wedge (q \vee r)$ is true then

$(p \vee q)$ and $(q \vee r)$ both are true.

i.e. truth values of p, q, r may be T, F, T respectively

$\because$ $\sim(p \wedge (q \vee r)) \equiv \sim((p \wedge q) \vee (p \wedge r) \equiv \sim(p \wedge q) \wedge \sim(p \wedge r) \equiv (\sim p \vee \sim q) \wedge (\sim p \vee \sim r)$

If p is true and q is false then $\sim(p \vee q)$ is false i.e. $p \wedge \sim(p \vee q)$ is false

Ex.9 If $S^*(p, q, r)$ is the dual of the compound statement $S(p, q, r)$ and $S(p, q, r) = \sim p \wedge [\sim(q \vee r)]$ then $S^*(\sim p, \sim q, \sim r)$ is equivalent to-

(1) $S(p, q, r)$ (2) $\sim S(\sim p, \sim q, \sim r)$ (3) $\sim S(p, q, r)$ (4) $S^*(p, q, r)$

Sol.(3) $\because \quad S(p, q, r) = \sim p \wedge [\sim(q \vee r)]$

So $S(\sim p, \sim q, \sim r) \equiv \sim(\sim p) \wedge [\sim(\sim q \vee \sim r)] \equiv p \wedge (q \wedge r)$

$S^*(p, q, r) \equiv \sim p \vee [\sim(q \wedge r)]$

$S^*(\sim p, \sim q, \sim r) \equiv p \vee (q \vee r)$

Clearly $S^*(\sim p, \sim q, \sim r) \equiv \sim S(p, q, r)$

Ex.10 The negation of the statement "If a quadrilateral is a square then it is a rhombus"

(1) If a quadrilateral is not a square then is a rhombus it

(2) If a quadrilateral is a square then it is not a rhombus

(3) a quadrilateral is a square and it is not a rhombus

(4) a quadritateral is not a square and it is a rhombus

Sol.(3) Let p and q be the statements as given below

p : a quadrilateral is a square

q : a quadritateral is a rhombus

the given statement is $p \to q$

$\because \quad \sim(p \to q) \equiv p \wedge \sim q$

Therefore the negation of the given statement is a quadrilateral is a square and it is not a rhombus

EXERCISE-1

1. The inverse of the statement $(p \wedge \sim q) \rightarrow r$ is-

(1) $\sim(p \vee \sim q) \rightarrow \sim r$ (2) $(\sim p \wedge q) \rightarrow \sim r$ (3) $(\sim p \vee q) \rightarrow \sim r$ (4) None of these

2. $(\sim p \vee \sim q)$ is logically equivalent to-

(1) $p \wedge q$ (2) $\sim p \rightarrow q$ (3) $p \rightarrow \sim q$ (4) $\sim p \rightarrow \sim q$

3. The equivalent statement of $(p \leftrightarrow q)$ is-

(1) $(p \wedge q) \vee (p \vee q)$ (2) $(p \rightarrow q) \vee (q \rightarrow p)$

(3) $(\sim p \vee q) \vee (p \vee \sim q)$ (4) $(\sim p \vee q) \wedge (p \vee \sim q)$

4. If the compound statement $p \rightarrow (\sim p \vee q)$ is false then the truth value of p and q are respectively-

(1) T, T (2) T, F (3) F, T (4) F, F

5. The statement $(p \rightarrow \sim p) \wedge (\sim p \rightarrow p)$ is-

(1) a tautology (2) a contradiction

(3) neither a tautology nor a contradiction (4) None of these

6. Negation of the statement $(p \wedge r) \rightarrow (r \vee q)$ is-

(1) $\sim(p \wedge r) \rightarrow \sim(r \vee q)$ (2) $(\sim p \vee \sim r) \vee (r \vee q)$

(3) $(p \wedge r) \wedge (r \wedge q)$ (4) $(p \wedge r) \wedge (\sim r \wedge \sim q)$

7. The dual of the statement $\sim p \wedge [\sim q \wedge (p \vee q) \wedge \sim r]$ is-

(1) $\sim p \vee [\sim q \vee (p \vee q) \vee \sim r]$ (2) $p \vee [q \vee (\sim p \wedge \sim q) \vee r]$

(3) $\sim p \vee [\sim q \vee (p \wedge q) \vee \sim r]$ (4) $\sim p \vee [\sim q \wedge (p \wedge q) \wedge \sim r]$

8. Which of the following is correct-

(1) $(\sim p \vee \sim q) \equiv (p \wedge q)$ (2) $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$

(3) $\sim(p \rightarrow \sim q) \equiv (p \wedge \sim q)$ (4) $\sim(p \leftrightarrow q) \equiv (p \rightarrow q) \vee (q \rightarrow p)$

9. The contrapositive of $p \rightarrow (\sim q \rightarrow \sim r)$ is-

(1) $(\sim q \wedge r) \rightarrow \sim p$ (2) $(q \rightarrow r) \rightarrow \sim p$ (3) $(q \vee \sim r) \rightarrow \sim p$ (4) None of these

10. The converse of $p \rightarrow (q \rightarrow r)$ is-

(1) $(q \wedge \sim r) \vee p$ (2) $(\sim q \vee r) \vee p$ (3) $(q \wedge \sim r) \wedge \sim p$ (4) $(q \wedge \sim r) \wedge p$

11. If p and q are two statement then $(p \leftrightarrow \sim q)$ is true when-

(1) p and q both are true (2) p and q both are false

(3) p is false and q is true (4) None of these

12. Statement $(p \wedge q) \rightarrow p$ is-

(1) a tautology (2) a contradiction (3) neither (1) nor (2) (4) None of these

13 If statements p, q, r have truth values T, F, T respectively then which of the following statement is true-

(1) $(p \rightarrow q) \wedge r$ (2) $(p \rightarrow q) \vee \sim r$ (3) $(p \wedge q) \vee (q \wedge r)$ (4) $(p \rightarrow q) \rightarrow r$

14. If statement $p \rightarrow (q \vee r)$ is true then the truth values of statements p, q, r respectively-

(1) T, F, T (2) F, T, F (3) F, F, F (4) All of these

15. Which of the following statement is a contradiction-

(1) $(p \wedge q) \wedge (\sim(p \vee q))$ (2) $p \vee (\sim p \wedge q)$

(3) $(p \rightarrow q) \rightarrow p$ (4) $\sim p \vee \sim q$

16. The negative of the statement "If a number is divisible by 15 then it is divisible by 5 or 3"

(1) If a number is divisible by 15 then it is not divisible by 5 and 3

(2) A number is divisible by 15 and it is not divisible by 5 or 3

(3) A number is divisible by 15 or it is not divisible by 5 and 3

(4) A number is divisible by 15 and it is not divisible by 5 and 3

17. If x = 5 and y = –2 then x – 2y = 9. The contrapositive of this statement is-

(1) If $x - 2y \neq 9$ then $x \neq 5$ or $y \neq -2$ (2) If $x - 2y \neq 9$ then $x \neq 5$ and $y \neq -2$

(3) If $x - 2y = 9$ then $x = 5$ and $y = -2$ (4) None of these

18. The negation of the statement "2 + 3 = 5 and 8 < 10" is-

(1) $2 + 3 \neq 5$ and $8 \nless 10$ (2) $2 + 3 \neq 5$ or $8 > 10$

(3) $2 + 3 \neq 5$ or $8 \geq 10$ (4) None of these

19. For any three simple statement p, q, r the statement $(p \wedge q) \vee (q \wedge r)$ is true when-

(1) p and r true and q is false (2) p and r false and q is true

(3) p, q, r all are false (4) q and r true and p is false

20. Which of the following statement is a tautology-

(1) $(\sim p \vee \sim q) \vee (p \vee \sim q)$ (2) $(\sim p \vee \sim q) \wedge (p \vee \sim q)$

(3) $\sim p \wedge (\sim p \vee \sim q)$ (4) $\sim q \wedge (\sim p \vee \sim q)$

21. Which of the following statement is a contradiction-

(1) $(\sim p \vee \sim q) \vee (p \vee \sim q)$ (2) $(p \rightarrow q) \vee (p \wedge \sim q)$

(3) $(\sim p \wedge q) \wedge (\sim q)$ (4) $(\sim p \wedge q) \vee (\sim q)$

22. The negation of the statement $q \vee (p \wedge \sim r)$ is equivalent to-

(1) $\sim q \wedge (p \rightarrow r)$ (2) $\sim q \wedge \sim(p \rightarrow r)$ (3) $\sim q \wedge (\sim p \wedge r)$ (4) None of these

23. Let Q be a non empty subset of N. and q is a statement as given below :-

q : There exists an even number $a \in Q$.

Negation of the statement q will be :-

(1) There is no even number in the set Q. (2) Every $a \in Q$ is an odd number.

(3) (1) and (2) both (4) None of these

24. The statement $\sim(p \to q) \leftrightarrow (\sim p \vee \sim q)$ is-

(1) a tautology (2) a contradiction

(3) neither a tautology nor a contradiction (4) None of these

25. Which of the following is equivalent to $(p \wedge q)$

(1) $p \to \sim q$ (2) $\sim(\sim p \wedge \sim q)$ (3) $\sim(p \to \sim q)$ (4) None of these

26. The dual of the following statement "Reena is healthy and Meena is beautiful" is-

(1) Reena is beaufiful and Meena is healthy (2) Reena is beautiful or Meena is healthy

(3) Reena is healthy or Meena is beutiful (4) None of these

27. If p is any statement, t and c are a tautology and a contradiction respectively then which of the following is not correct-

(1) $p \wedge t \equiv p$ (2) $p \wedge c \equiv c$ (3) $p \vee t \equiv c$ (4) $p \vee c \equiv p$

28. If $S^*(p, q)$ is the dual of the compound statement $S(p, q)$ then $S^*(\sim p, \sim q)$ is equivalent to-

(1) $S(\sim p, \sim q)$ (2) $\sim S(p, q)$ (3) $\sim S^*(p, q)$ (4) None of these

29. If p is any statement, t is a tautology and c is a contradiction then which fo the following is not correct-

(1) $p \wedge (\sim c) \equiv p$ (2) $p \vee (\sim t) \equiv p$

(3) $t \vee c \equiv p \vee t$ (4) $(p \wedge t) \vee (p \vee c) \equiv (t \wedge c)$

30. If p, q, r are simple statement with truth values T, F, T respectively then the truth value of $((\sim p \vee q) \wedge \sim r) \to p$ is-

(1) True (2) False (3) True if r is false (4) True if q is true

31. Which of the following is wrong-

(1) $p \vee \sim p$ is a tautology (2) $\sim(\sim p) \leftrightarrow p$ is a tautology

(3) $p \wedge \sim p$ is a contradiction (4) $((p \wedge p) \to q) \to p$ is a tautology

32. The statement "If $2^2 = 5$ then I get first class" is logically equivalent to-

(1) $2^2 = 5$ and I do not get first class (2) $2^2 = 5$ or I do not get first class

(3) $2^2 \neq 5$ or I get first class (4) None of these

33. If statement $(p \vee \sim r) \rightarrow (q \wedge r)$ is false and statement q is true then statement p is-

(1) true (2) false

(3) may be true or false (4) None of these

34. Which of the following statement are not logically equivalent-

(1) $\sim(p \vee \sim q)$ and $(\sim p \wedge q)$ (2) $\sim(p \rightarrow q)$ and $(p \wedge \sim q)$

(3) $(p \rightarrow q)$ and $(\sim q \rightarrow \sim p)$ (4) $(p \rightarrow q)$ and $(\sim p \wedge q)$

35. Consider the following statements

p : Virat kohli plays cricket.

q : Virat kohli is good at maths

r : Virat kohli is successful.

then negation of the statement "If virat kohli plays cricket and is not good at maths then he is successful" will be :-

(1) $\sim p \wedge (q \wedge r)$ (2) $(\sim p \vee q) \wedge r$ (3) $p \wedge (\sim q \wedge \sim r)$ (4) None of these

36. Let p statement "If 2 > 5 then earth will not rotate" and q be the statement "$2 \not> 5$ or earth will not rotate".

Statement–1 : p and q are equivalent.

Statement–2 : $m \rightarrow n$ and $\sim m \vee n$ are equivalent.

(1) Statement–1 is true, Statement–2 is true; Statement–2 is not the correct explanation of Statement–1.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is false.

(4) Statement–1 is true, Statement–2 is true; Statement–2 is the correct explanation of Statement–1.

37. Which of the following is a tautology :-

(1) $[(\sim p \wedge p) \rightarrow q] \longrightarrow (p \wedge p)$ (2) $[(\sim p \wedge p) \rightarrow q] \longrightarrow (\sim p \rightarrow p)$

(3) $[(\sim p \wedge p) \rightarrow q] \longrightarrow (p \rightarrow p)$ (4) None of these

38. Negation of the statement "No one in the class is fond of music" is :-

(1) everyone in the class is fond of music.

(2) Some of the students in the class are fond of music.

(3) There exists a student in the class who is fond of music.

(4) (2) and (3) both

EXERCISE-2

1. The negation of the statement **[JEE(Main)-2012]**

"If I become a teacher, then I will open a school", is :

(1) I will not become a teacher or I will open a school.

(2) I will become a teacher and I will not open a school.

(3) Either I will not become a teacher or I will not open a school.

(4) Neither I will become a teacher nor I will open a school.

2. Consider : **[JEE(Main)-2013]**

Statement-I : $(p \wedge \sim q) \wedge (\sim p \wedge q)$ is a fallacy.

Statement-II : $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ is a tuatology.

(1) Statement-I is true, Statement-II is true; statement-II is a **correct** explanation for Statement-I.

(2) Statement-I is true, Statement-II is true; statement-II is **not** a correct explanation for Statement-I.

(3) Statement-I is true, Statement-II is false.

(4) Statement-I is false, Statement-II is true.

3. The statement $\sim(p \leftrightarrow \sim q)$ is : **[JEE(Main)-2014]**

(1) equivalent to $p \leftrightarrow q$ (2) equivalent to $\sim p \leftrightarrow q$

(3) a tautology (4) a fallacy

4. The negation of $\sim s \vee (\sim r \wedge s)$ is equivalent to : **[JEE(Main)-2015]**

(1) $s \vee (r \vee \sim s)$ (2) $s \wedge r$ (3) $s \wedge \sim r$ (4) $s \wedge (r \wedge \sim s)$

5. The Boolean Expression $(p\wedge\sim q)\vee q\vee(\sim p\wedge q)$ is equivalent to :- **[JEE(Main)-2016]**

(1) $p\vee\sim q$ (2) $\sim p\wedge q$ (3) $p\wedge q$ (4) $p\vee q$

6. The following statement

$(p \rightarrow q) \rightarrow [(\sim p \rightarrow q) \rightarrow q]$ is : **[JEE(Main)-2017]**

(1) a fallacy (2) a tautology

(3) equivalent to $\sim p \rightarrow q$ (4) equivalent to $p \rightarrow \sim q$

7. The Boolean expression $\sim (p \vee q) \vee (\sim p \wedge q)$ is equivalent to : **[JEE(Main)-2018]**

(1) p (2) q (3) $\sim q$ (4) $\sim p$

8. If the Boolean expression $(p \oplus q) \wedge (\sim p \odot q)$ is equivalent to $p \wedge q$, where $\oplus, \odot \in \{\wedge, \vee\}$, then the ordered pair $(\oplus, \odot)$ is: **[JEE(Main)-19]**

(1) $(\wedge, \vee)$ (2) $(\vee, \vee)$ (3) $(\wedge, \wedge)$ (4) $(\vee, \wedge)$

9. The logical statement $[\sim(\sim p \vee q) \vee (p \wedge r)] \wedge (\sim q \wedge r)$ is equivalent to : **[JEE(Main)-19]**

(1) $(p \wedge r) \wedge \sim q$ (2) $(\sim p \wedge \sim q) \wedge r$ (3) $\sim p \vee r$ (4) $(p \wedge \sim q) \vee r$

10. Consider the following three statements :

P : 5 is a prime number.

Q : 7 is a factor of 192.

R : L.C.M. of 5 and 7 is 35.

Then the truth value of which one of the following statements is true ? **[JEE(Main)-19]**

(1) $(P \wedge Q) \vee (\sim R)$ (2) $(\sim P) \wedge (\sim Q \wedge R)$ (3) $(\sim P) \vee (Q \wedge R)$ (4) $P \vee (\sim Q \wedge R)$

11. If q is false and $p \wedge q \leftrightarrow r$ is true, then which one of the following statements is a tautology?

[JEE(Main)-19]

(1) $(p \vee r) \rightarrow (p \wedge r)$ (2) $p \vee r$ (3) $p \wedge r$ (4) $(p \wedge r) \rightarrow (p \vee r)$

12. Contrapositive of the statement

"If two numbers are not equal, then their squares are not equal." is :- **[JEE(Main)-19]**

(1) If the squares of two numbers are equal, then the numbers are equal.

(2) If the squares of two numbers are equal, then the numbers are not equal.

(3) If the squares of two numbers are not equal, then the numbers are equal.

(4) If the squares of two numbers are not equal, then the numbers are not equal.

13. The contrapositive of the statement "If you are born in India, then you are a citizen of India", is :

(1) If you are born in India, then you are not a citizen of India. **[JEE(Main)-19]**

(2) If you are not a citizen of India, then you are not born in India.

(3) If you are a citizen of India, then you are born in India.

(4) If you are not born in India, then you are not a citizen of India.

14. For any two statements p and q, the negation of the expression $p \vee (\sim p \wedge q)$ is

[JEE(Main)-19]

(1) $p \wedge q$ (2) $p \leftrightarrow q$ (3) $\sim p \vee \sim q$ (4) $\sim p \wedge \sim q$

15. If the truth value of the statement $p \rightarrow (\sim q \vee r)$ is false(F), then the truth values of the statements p, q, r are respectively : **[JEE(Main)-19]**

(1) F, T, T (2) T, F, F (3) T, T, F (4) T, F, T

16. The Boolean expression $\sim(p \Rightarrow (\sim q))$ is equivalent to : **[JEE(Main)-19]**

(1) $(\sim p) \Rightarrow q$ (2) $p \vee q$ (3) $q \Rightarrow \sim p$ (4) $p \wedge q$

17. Let A, B, C and D be four non-empty sets. The contrapositive statement of "If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq C$" is : **[JEE(Main)-20]**

(1) If $A \subseteq C$, then $B \subset A$ or $D \subset B$ (2) If $A \not\subseteq C$, then $A \not\subseteq B$ or $B \not\subseteq D$

(3) If $A \not\subseteq C$, then $A \subseteq B$ and $B \subseteq D$ (4) If $A \not\subseteq C$, then $A \not\subseteq B$ and $B \subseteq D$

18. The logical statement $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to : **[JEE(Main)-20]**

(1) p (2) q (3) ~p (4) ~q

19. Which of the following statements is a tautology? **[JEE(Main)-20]**

(1) $\sim(p \vee \sim q) \rightarrow p \vee q$ (2) $\sim(p \wedge \sim q) \rightarrow p \vee q$

(3) $\sim(p \vee \sim q) \rightarrow p \wedge q$ (4) $p \vee (\sim q) \rightarrow p \wedge q$

20. Which one of the following is a tautology ? **[JEE(Main)-20]**

(1) $P \wedge (P \vee Q)$ (2) $P \vee (P \wedge Q)$

(3) $Q \rightarrow (P \wedge (P \rightarrow Q))$ (4) $(P \wedge (P \rightarrow Q)) \rightarrow Q$

21. If $p \rightarrow (p \wedge \sim q)$ is false, then the truth values of p and q are respectively : **[JEE(Main)-20]**

(1) F, T (2) T, T (3) F, F (4) T, F

22. Negation of the statement : **[JEE(Main)-20]**

$\sqrt{5}$ is an integer or 5 is irrational is :

(1) $\sqrt{5}$ is irrational or 5 is an integer.

(2) $\sqrt{5}$ is not an integer and 5 is not irrational.

(3) $\sqrt{5}$ is an integer and 5 is irrational.

(4) $\sqrt{5}$ is not an integer or 5 is not irrational.

ANSWER KEY

EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Ans.	3	3	4	2	2	4	3	2	1	1	3	1	4	4	1	4	1	3	4	1
Que.	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38		
Ans.	3	1	3	3	3	3	3	2	4	1	4	3	3	4	3	4	3	4		

EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans.	2	2	1	2	4	2	4	1	1	4	4	1	2	4	3
Que.	16	17	18	19	20	21	22								
Ans.	4	2	3	1	4	2	2								

CHAPTER 5

STATISTICS

Chapter Contents 05

IMPORTANT NOTES

CHAPTER 5 STATISTICS

MEASURES OF CENTRAL TENDENCY :

An average value or a central value of a distribution is the value of variable which is representative of the entire distribution, this representative value are called the measures of central tendency.

Generally the following five measures of central tendency.

(a) Mathematical average

(i) Arithmetic mean (ii) Geometric mean (iii) Harmonic mean

(b) Positional average

(i) Median (ii) Mode

1. ARITHMETIC MEAN :

(i) For ungrouped dist. : If $x_1, x_2, \ldots\ldots x_n$ are n values of variate x_i then their A.M. $\overline{x}$ is defined as

$$\overline{x} = \frac{x_1 + x_2 + \ldots.. + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\Rightarrow \Sigma x_i = n\overline{x}$$

(ii) For ungrouped and grouped freq. dist. : If $x_1, x_2, \ldots. x_n$ are values of variate with corresponding frequencies $f_1, f_2, \ldots f_n$ then their A.M. is given by

$$\overline{x} = \frac{f_1x_1 + f_2x_2 + \ldots. + f_nx_n}{f_1 + f_2 + \ldots. + f_n} = \frac{\sum_{i=1}^{n} f_ix_i}{N}, \quad \text{where } N = \sum_{i=1}^{n} f_i$$

Ex.1 Find the A.M. of the following freq. dist.

x_i	5	8	11	14	17
f_i	4	5	6	10	20

Sol. Here $N = \Sigma f_i = 4 + 5 + 6 + 10 + 20 = 45$

$\Sigma f_ix_i = (5 \times 4) + (8 \times 5) + (11 \times 6) + (14 \times 10) + (17 \times 20) = 606$

$$\therefore \quad \overline{x} = \frac{\Sigma f_ix_i}{N} = \frac{606}{45} = 13.47$$

(iii) By short method : If the value of x_i are large, then the calculation of A.M. by using previous formula is quite tedious and time consuming. In such case we take deviation of variate from an arbitrary point a.

Let $d_i = x_i - a$

$$\therefore \quad \overline{x} = a + \frac{\Sigma f_id_i}{N}, \quad \text{where a is assumed mean}$$

(iv) By step deviation method : Sometime during the application of short method of finding the A.M. If each deviation d_i are divisible by a common number h(let)

Let $$u_i = \frac{d_i}{h} = \frac{x_i - a}{h}$$

$\therefore$ $$\bar{x} = a + \left(\frac{\Sigma f_i u_i}{N}\right)h$$

Ex.2 Find the mean of the following freq. dist.

x_i	5	15	25	35	45	55
f_i	12	18	27	20	17	6

Sol. Let assumed mean a = 35, h = 10

here $N = \Sigma f_i = 100$, $u_i = \dfrac{(x_i - 35)}{10}$

$\therefore$ $\Sigma f_i u_i = (12 \times -3) + (18 \times -2) + (27 \times -1) + (20 \times 0) + (17 \times 1) + (6 \times 2) = -70$

$\therefore$ $\bar{x} = a + \left(\dfrac{\Sigma f_i u_i}{N}\right)h = 35 + \dfrac{(-70)}{100} \times 10 = 28$

(v) Weighted mean : If $w_1, w_2, \ldots\ldots w_n$ are the weights assigned to the values $x_1, x_2, \ldots\ldots x_n$ respectively then their weighted mean is defined as

$$\text{Weighted mean} = \frac{w_1x_1 + w_2x_2 + \ldots\ldots + w_nx_n}{w_1 + \ldots\ldots + w_n} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}$$

Ex.3 Find the weighted mean of first n natural numbers when their weights are equal to their squares respectively

Sol. $$\text{Weighted Mean} = \frac{1.1^2 + 2.2^2 + \ldots + n.n^2}{1^2 + 2^2 + \ldots + n^2} = \frac{1^3 + 2^3 + \ldots\ldots + n^3}{1^2 + 2^2 + \ldots + n^2} = \frac{[n(n+1)/2]^2}{[n(n+1)(2n+1)/6]} = \frac{3n(n+1)}{2(2n+1)}$$

(vi) Combined mean : If $\bar{x}_1$ and $\bar{x}_2$ be the means of two groups having n_1 and n_2 terms respectively then the mean (combined mean) of their composite group is given by

$$\text{combined mean} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$$

If there are more than two groups then, $$\text{combined mean} = \frac{n_1\bar{x}_1 + n_1\bar{x}_2 + n_3\bar{x}_3 + \ldots}{n_1 + n_2 + n_3 + \ldots}$$

Ex.4 The mean income of a group of persons is Rs. 400 and another group of persons is Rs. 480. If the mean income of all the persons of these two groups is Rs. 430 then find the ratio of the number of persons in the groups.

Sol. Here $\bar{x}_1 = 400$, $\bar{x}_2 = 480$, $\bar{x} = 430$

$$\because \quad \bar{x} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} \Rightarrow 430 = \frac{400n_1 + 480n_2}{n_1 + n_2}$$

$$\Rightarrow \frac{n_1}{n_2} = \frac{5}{3}$$

(vii) Properties of Arithmetic mean :

- Sum of deviations of variate from their A.M. is always zero i.e. $\Sigma(x_i - \bar{x}) = 0$, $\Sigma f_i(x_i - \bar{x}) = 0$
- Sum of square of deviations of variate from their A.M. is minimum i.e. $\Sigma(x_i - \bar{x})^2$ is minimum
- If $\bar{x}$ is the mean of variate x_i then A.M. of $(x_i + \lambda) = \bar{x} + \lambda$
  A.M. of $(\lambda x_i) = \lambda\bar{x}$
  A.M. of $(ax_i + b) = a\bar{x} + b$ (where λ, a, b are constant)
- A.M. is independent of change of assumed mean i.e. it is not effected by any change in assumed mean.

2. MEDIAN :

The median of a series is the value of middle term of the series when the values are written in ascending order. Therefore median, divided an arranged series into two equal parts.

Formulae of median :

(i) For ungrouped distribution : Let n be the number of variate in a series then

$$\text{Median} = \begin{cases} \left(\dfrac{n+1}{2}\right)^{th} \text{term}, \ (\text{when } n \text{ is odd}) \\ \text{Mean of } \left(\dfrac{n}{2}\right)^{th} \text{ and } \left(\dfrac{n}{2}+1\right)^{th} \text{ terms}, \ (\text{when } n \text{ is even}) \end{cases}$$

(ii) For ungrouped freq. dist. : First we prepare the cumulative frequency (c.f.) column and Find value of N then

$$\text{Median} = \begin{cases} \left(\dfrac{N+1}{2}\right)^{th} \text{term}, \ (\text{when } N \text{ is odd}) \\ \text{Mean of } \left(\dfrac{N}{2}\right)^{th} \text{ and } \left(\dfrac{N}{2}+1\right)^{th} \text{ terms}, \ (\text{when } N \text{ is even}) \end{cases}$$

(iii) For grouped freq. dist : Prepare c.f. column and find value of $\frac{N}{2}$ then find the class which contain value of c.f. is equal or just greater to N/2, this is median class

$$\therefore \text{ Median} = \ell + \frac{\left(\frac{N}{2} - F\right)}{f} \times h$$

where ℓ — lower limit of median class

f — freq. of median class

F — c.f. of the class preceeding median class

h — Class interval of median class

Ex.5 Find the median of following freq. dist.

class	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50
f	8	30	40	12	10

Sol.

class	f_i	c.f.
0 – 10	8	8
10 – 20	30	38
20 – 30	40	78
30 – 40	12	90
40 – 50	10	100

Here $\frac{N}{2} = \frac{100}{2} = 50$ which lies in the value 78 of c.f. hence corresponding class of this c.f. is 20-30 is the median class, so $\ell = 20$, $f = 40$, $F = 38$, $h = 10$

$$\therefore \text{ Median} = \ell + \frac{\left(\frac{N}{2} - F\right)}{f} \times h = 20 + \frac{(50-38)}{40} \times 10 = 23$$

3. MODE :

In a frequency distribution the mode is the value of that variate which have the maximum frequency

Method for determining mode :

(i) For ungrouped dist. : The value of that variate which is repeated maximum number of times

(ii) For ungrouped freq. dist. : The value of that variate which have maximum frequency.

(iii) For grouped freq. dist. : First we find the class which have maximum frequency, this is model calss

$$\therefore \text{ Mode} = \ell + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h$$

where ℓ — lower limit of model class

f_0 — freq. of the model class

f_1 — freq. of the class preceeding model class

f_2 — freq. of the class succeeding model class

h — class interval of model class

Ex. 6 Find the mode of the following frequecy dist

class	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
f_i	2	18	30	45	35	20	6	3

Sol. Here the class 30–40 has maximum freq. so this is the model class

$\ell = 30, f_0 = 45, f_1 = 30, f_2 = 35, h = 10$

$$\therefore \text{ Mode} = \ell + \frac{f_0 - f_1}{2f_0 - f_1 - f_2} \times h = 30 + \frac{45 - 30}{2 \times 45 - 30 - 35} \times 10 = 36$$

4. RELATION BETWEEN MEAN, MEDIAN AND MODE :

In a moderately asymmetric distribution following relation between mean, median and mode of a distribution. It is known as imprical formula.

Mode = 3 Median – 2 Mean

Note (i) Median always lies between mean and mode

(ii) For a symmetric distribution the mean, median and mode are coincide.

5. MEASURES OF DISPERSION :

The dispersion of a statistical distribution is the measure of deviation of its values about the their average (central) value.

It gives an idea of scatteredness of different values from the average value.

Generally the following measures of dispersion are commonly used.

(i) Range (ii) Mean deviation (iii) Variance and standard deviation

(i) Range : The difference between the greatest and least values of variate of a distribution, are called the range of that distribution.

If the distribution is grouped distribution, then its range is the difference between upper limit of the maximum class and lower limit of the minimum class.

$$\text{Also, coefficient of range} = \frac{\text{difference of extreme values}}{\text{sum of extreme values}}$$

Ex.7 Find the range of following numbers 10, 8, 12, 11, 14, 9, 6

Sol. Here greatest value and least value of the distribution are 14 and 6 resp. therefore

Range = 14 – 6 = 8

(ii) Mean deviation (M.D.) : The mean deviation of a distribution is, the mean of absolute value of deviations of variate from their statistical average (Mean, Median, Mode).

If A is any statistical average of a distribution then mean deviation about A is defined as

$$\text{Mean deviation} = \frac{\sum_{i=1}^{n} |x_i - A|}{n} \qquad \text{(for ungrouped dist.)}$$

$$\text{Mean deviation} = \frac{\sum_{i=1}^{n} f_i\,|x_i - A|}{N} \qquad \text{(for freq. dist.)}$$

Note :- is minimum when it taken about the median

$$\text{Coefficient of Mean deviation} = \frac{\text{Mean deviation}}{A}$$

(where A is the central tendency about which Mean deviation is taken)

Ex.8 Find the mean deviation of number 3, 4, 5, 6, 7

Sol. Here n = 5, $\bar{x} = 5$

$\therefore$ Mean deviation $= \dfrac{\Sigma|x_i - \bar{x}|}{n}$

$$= \frac{1}{5}[|3-5| + |4-5| + |5-5| + |6-5| + |7-5|]$$

$$= \frac{1}{5}[2 + 1 + 0 + 1 + 2] = \frac{6}{5} = 1.2$$

Ex.9 Find the mean deviation about mean from the following data

x_i	3	9	17	23	27
f_i	8	10	12	9	5

Sol.

x_i	f_i	$f_i x_i$	$\lvert x_i - \bar{x} \rvert$	$f_i \lvert x_i - \bar{x} \rvert$
3	8	24	12	96
9	10	90	6	60
17	12	204	2	24
23	9	207	8	72
27	5	135	12	60
	$N = 44$	$\Sigma f_i x_i = 660$		$\Sigma f_i \lvert x_i - \bar{x} \rvert = 312$

Mean ($\bar{x}$) = $\dfrac{\Sigma f_i x_i}{N} = \dfrac{660}{44} = 15$

Mean deviation = $\dfrac{\Sigma f_i \lvert x_i - \bar{x} \rvert}{N} = \dfrac{312}{44} = 7.09$

(iii) Variance and standard deviation : The variance of a distribution is, the mean of squares of deviation of variate from their mean. It is denoted by σ^2 or var(x).

The positive square root of the variance are called the standard deviation. It is denoted by σ or S.D.

Hence standard deviation = $+\sqrt{\text{variance}}$

Formulae for variance :

(i) for ungrouped dist. :

$$\sigma_x^2 = \frac{\Sigma(x_i - \bar{x})^2}{n}$$

$$\sigma_x^2 = \frac{\Sigma x_i^2}{n} - \bar{x}^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$$

$$\sigma_d^2 = \frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2, \quad \text{where } d_i = x_i - a$$

(ii) For freq. dist. :

$$\sigma_x^2 = \frac{\Sigma f_i (x_i - \bar{x})^2}{N}$$

$$\sigma_x^2 = \frac{\Sigma f_i x_i^2}{N} - (\bar{x})^2 = \frac{\Sigma f_i x_i^2}{N} - \left(\frac{\Sigma f_i x_i}{N}\right)^2$$

$$\sigma_d^2 = \frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2$$

$$\sigma_u^2 = h^2\left[\frac{\Sigma f_i u_i^2}{N} - \left(\frac{\Sigma f_i u_i}{N}\right)^2\right] \quad \text{where } u_i = \frac{d_i}{h}$$

(iii) Coefficient of S.D. = $\frac{\sigma}{\bar{x}}$

Coefficient of variation = $\frac{\sigma}{\bar{x}} \times 100$ (in percentage)

Note :- $\sigma^2 = \sigma_x^2 = \sigma_d^2 = h^2 \sigma_u^2$

Ex.10 Find the variance of first n natural numbers

Sol. $\sigma^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2 = \frac{\Sigma n^2}{n} - \left(\frac{\Sigma n}{n}\right)^2 = \frac{n(n+1)(2n+1)}{6n} - \left\{\frac{n(n+1)}{2n}\right\}^2 = \frac{n^2-1}{12}$

Ex.11 If $\sum_{i=1}^{18}(x_i - 8) = 9$ and $\sum_{i=1}^{18}(x_i - 8)^2 = 45$, then find the standard deviation of $x_1, x_2, ... x_{18}$

Sol. Let $(x_i - 8) = d_i$

$$\therefore \ \sigma_x = \sigma_d = \sqrt{\frac{\Sigma d_i^2}{n} - \left(\frac{\Sigma d_i}{n}\right)^2} = \sqrt{\frac{45}{18} - \left(\frac{9}{18}\right)^2} = \sqrt{\frac{5}{2} - \frac{1}{4}} = \frac{3}{2}$$

Ex.12 Find the coefficient of variation of first n natural numbers

Sol. For first n natural numbers.

Mean $(\bar{x}) = \frac{n+1}{2}$, S.D.$(\sigma) = \sqrt{\frac{n^2-1}{12}}$

$$\therefore \ \text{coefficient of variance} = \frac{\sigma}{\bar{x}} \times 100 = \sqrt{\frac{n^2-1}{12}} \times \frac{1}{\left(\frac{n+1}{2}\right)} \times 100 = \sqrt{\frac{(n-1)}{3(n+1)}} \times 100$$

6. MEAN SQUARE DEVIATION :

The mean square deviation of a distrubution is the mean of the square of deviations of variate from assumed mean. It is denoted by S^2

Hence $$S^2 = \frac{\Sigma(x_i - a)^2}{n} = \frac{\Sigma d_i^2}{n} \quad \text{(for ungrouped dist.)}$$

$$S^2 = \frac{\Sigma f_i(x_i - a)^2}{N} = \frac{\Sigma f_i d_i^2}{N} \quad \text{(for freq. dist.)}, \quad \text{where } d_i = (x_i - a)$$

7. RELATION BETWEEN VARIANCE AND MEAN SQUARE DEVIATION :

$$\because \quad \sigma^2 = \frac{\Sigma f_i d_i^2}{N} - \left(\frac{\Sigma f_i d_i}{N}\right)^2$$

$$\Rightarrow \sigma^2 = s^2 - d^2, \quad \text{where } d = \bar{x} - a = \frac{\Sigma f_i d_i}{N}$$

$$\Rightarrow s^2 = \sigma^2 + d^2 \Rightarrow s^2 \geq \sigma^2$$

Hence the variance is the minimum value of mean square deviation of a distribution

Ex.13 Determine the variance of the following frequency dist.

class	0 – 2	2 – 4	4 – 6	6 – 8	8 – 10	10 – 12
f_i	2	7	12	19	9	1

Sol. Let a = 7, h = 2

class	x_i	f_i	$u_i = \frac{x_i - a}{h}$	f_iu_i	$f_iu_i^2$
0 – 2	1	2	–3	–6	18
2 – 4	3	7	–2	–14	28
4 – 6	5	12	–1	–12	12
6 – 8	7	19	0	0	0
8 – 10	9	9	1	9	9
10 – 12	11	1	2	2	4
		N = 50		$\Sigma f_iu_i = -21$	$\Sigma f_iu_i^2 = 71$

$$\therefore \quad \sigma^2 = h^2\left[\frac{\Sigma f_iu_i^2}{N} - \left(\frac{\Sigma f_iu_i}{N}\right)^2\right] = 4\left[\frac{71}{50} - \left(\frac{-21}{50}\right)^2\right] = 4[1.42 - 0.1764] = 4.97$$

8. MATHEMATICAL PROPERTIES OF VARIANCE :

- $\text{Var.}(x_i + \lambda) = \text{Var.}(x_i)$

  $\text{Var.}(\lambda x_i) = \lambda^2.\text{Var}(x_i)$

  $\text{Var}(ax_i + b) = a^2.\text{Var}(x_i)$

  where λ, a, b, are constant

- If means of two series containing n_1, n_2 terms are $\bar{x}_1, \bar{x}_2$ and their variance's are σ_1^2, σ_2^2 respectively and their combined mean is $\bar{x}$ then the variance σ^2 of their combined series is given by following formula

$$\sigma^2 = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{(n_1 + n_2)} \qquad \text{where } d_i = \bar{x}_1 - \bar{x},\ d_2 = \bar{x}_2 - \bar{x}$$

i.e.
$$\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1 n_2}{(n_1 + n_2)^2}(\bar{x}_1 - \bar{x}_2)^2$$

SOLVED EXAMPLES

Ex.1 If in an examination different weights are assigned to different subjects Physics (2), Chemistry (1), English (1), Mathematics (2) A student scores 60 in Physics, 70 in Chemistry, 70 in English and 80 in Mathematics, then weighted mean is-

(1) 60 (2) 70 (3) 80 (4) 85

Sol.(2) Weighted mean = $\dfrac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i} = \dfrac{2\times 60 + 1\times 70 + 1\times 70 + 2\times 80}{6} = 70$

Ex.2 The mean of two groups of sizes 200 and 300 are 25 and 10 respectively.Their standard deviation are 3 and 4 respectively. The variance of combined sample of size 500 is-

(1) 64 (2) 65.2 (3) 67.2 (4) 64.2

Sol.(3) Combined mean $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2} = \dfrac{200\times 25 + 300\times 10}{500} = 16$

Here $d_1 = \bar{x}_1 - \bar{x} = 25 - 16 = 9$ and $d_2 = \bar{x}_2 - \bar{x} = 10 - 16 = -6$

We know that $\sigma^2 = \dfrac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2} = \dfrac{200(9+81) + 300(16+36)}{500} = \dfrac{33600}{500} = 67.2$

Ex.3 If the mean of the series $x_1, x_2, \ldots\ldots, x_n$ is $\bar{x}$, then the mean of the series $x_i + 2i$, i = 1, 2, ..., n will be-

(1) $\bar{x} + n$ (2) $\bar{x} + n + 1$ (3) $\bar{x} + 2$ (4) $\bar{x} + 2n$

Sol.(2) As given $\bar{x} = \dfrac{x_1 + x_2 + \ldots.. + x_n}{n}$ ....(1)

If the mean of the series $x_i + 2i$, i = 1, 2, ..., n be $\bar{X}$, then

$$\bar{X} = \frac{(x_1 + 2) + (x_2 + 2.2) + (x_3 + 2.3) + \ldots.. + (x_n + 2.n)}{n}$$

$$= \frac{x_1 + x_2 + \ldots.. + x_n}{n} + \frac{2(1 + 2 + 3 + \ldots.. + n)}{n}$$

$$= \bar{x} + \frac{2n(n+1)}{2n} \qquad \text{from (1)}$$

$$= \bar{x} + n + 1$$

Ex.4 The variance of first 20-natural numbers is-

(1) $\frac{133}{4}$ (2) $\frac{379}{12}$ (3) $\frac{133}{2}$ (4) $\frac{399}{4}$

Sol.(1) $\because \quad \sigma^2 = \frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2$

$$= \frac{1}{20}[1^2 + 2^2 + ... + 20^2] - \left[\frac{1}{20}(1+2+....+20)\right]^2$$

$$= \frac{1}{20}\frac{20\times 21(2\times 20+1)}{6} - \left[\frac{1}{20}\frac{20\times 21}{2}\right]^2 = \frac{7\times 41}{2} - \frac{441}{4} = \frac{133}{4}.$$

In fact, the variance of first n-natural numbers is $\frac{n^2-1}{12}$

Ex.5 The mean of the following freq. table is 50 and $\Sigma f = 120$

class	0 – 20	20 – 40	40 – 60	60 – 80	80 – 100
f	17	f_1	32	f_2	19

the missing frequencies are-

(1) 28, 24 (2) 24, 36 (3) 36, 28 (4) None of these

Sol.(1) $\Sigma f = 120 = 17 + f_1 + 32 + f_2 + 19$

$\Rightarrow f_1 + f_2 = 52$ ...... (1)

and $\Sigma fx = (10 \times 17) + (30 \times f_1) + (50 \times 32) + (70 \times f_2) + (90 \times 19) = 30f_1 + 70f_2 + 3480$

$\therefore \quad \bar{x} = \frac{\Sigma fx}{\Sigma f} \Rightarrow 50 = \frac{30f_1 + 70f_2 + 3480}{120}$

$\Rightarrow 30f_1 + 70f_2 = 2520 \Rightarrow 3f_1 + 7f_2 = 252$ ..... (2)

by (1) and (2) we get $f_1 = 28$, $f_2 = 24$

Ex.6 A student obtained 75%, 80%, 85% marks in three subjects. If the marks of another subject are added then his average marks can not be less than-

(1) 60% (2) 65% (3) 80% (4) 90%

Sol.(1) Total marks obtained from three subjects out of 300 = 75 + 80 + 85 = 240

if the marks of another subject is added then total marks obtained out of 400 is greater than 240

if marks obtained in fourth subject is 0 then

minimum average marks = $\frac{240}{400} \times 100 = 60\%$

Ex.7 The mean and variance of a series containing 5 terms are 8 and 24 respectively. The mean and variance of another series containing 3 terms are also 8 and 24 respectively. The variance of their combined series will be-

(1) 20 (2) 24 (3) 25 (4) 42

Sol.(2) Using $\sigma^2 = \frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1+n_2} + \frac{n_1 n_2}{(n_1+n_2)^2}(\bar{x}_1 - \bar{x}_2)^2 \Rightarrow \sigma^2 = \frac{5(24)+3(24)}{5+3} + \frac{5(3)}{(5+3)^2}(8-8)^2 = 24$

Ex.8 The mean deviation about median from the following data 340, 150, 210, 240, 300, 310, 320, is-

(1) 52.4 (2) 52.5 (3) 52.8 (4) none of these

Sol.(3) Arranging the observations in ascending order of magnitude, we have 150, 210, 240, 300, 310, 320, 340. Clearly, the middle observation is 300. So, median = 300

Calculation of Mean deviation

x_i	$\lvert x_i - 300 \rvert$
340	40
150	150
210	90
240	60
300	0
310	10
320	20
Total	$\sum \lvert x_i - 300 \rvert = 370$

Mean deviation from median = $\frac{1}{7}\sum |x_i - 300| = \frac{370}{7} = 52.8$

Ex.9 Variance of the data given below is

Size of item	3.5	4.5	5.5	6.5	7.5	8.5	9.5
Frequency	3	7	22	60	85	32	8

(1) 1.29 (2) 2.19 (3) 1.32 (4) none of these

Sol.(3) Let the assumed mean be a = 6.5

Calculation of variance

x_i	f_i	$d_i = x_i - 6.5$	$f_i d_i$	$f_i d_i^2$
3.5	3	–3	–9	27
4.5	7	–2	–14	28
5.5	22	–1	–22	22
6.5	60	0	0	0
7.5	85	1	85	85
8.5	32	2	64	128
9.5	8	3	24	72
	$N = \sum f_i = 217$		$\sum f_i d_i = 128$	$\sum f_i d_i^2 = 362$

Here N = 217, $\sum f_i d_i = 128$ and $\sum f_i d_i^2 = 362$

$$\therefore \quad \text{Var}(X) = \left(\frac{1}{N}\sum f_i d_i^2\right) - \left(\frac{1}{N}\sum f_i d_i\right)^2 = \frac{362}{217} - \left(\frac{128}{217}\right)^2 = 1.668 - 0.347 = 1.321$$

Ex.10 If a variable takes the value 0, 1, 2......n with frequencies proportional to the bionomial coefficients $^nC_0, {}^nC_1, \ldots\ldots, {}^nC_n$ then the mean of the distribution is-

(1) $\frac{n(n+1)}{4}$ (2) $\frac{n}{2}$ (3) $\frac{n(n-1)}{2}$ (4) $\frac{n(n+1)}{2}$

Sol.(2) $N = \sum f_i = k\,[{}^nC_0 + {}^nC_1 + \ldots. + {}^nC_n] = k2^n$

$$\sum f_i x_i = k\,[1.{}^nC_1 + 2.\,{}^nC_2 + \ldots. + n\,{}^nC_n] = k\sum_{r=1}^{n} r.{}^nC_r = kn\sum_{r=1}^{n} {}^{n-1}C_{r-1} = kn2^{n-1}$$

Thus $\bar{x} = \frac{1}{2^n}(n\ \ 2^{n-1}) = \frac{n}{2}$.

Ex.11 The mean and variance of 5 observations of an experiment are 4 and 5.2 respectively. If from these observations three are 1, 2 and 6, then the remaining will be-

(1) 2, 9 (2) 5, 6 (3) 4, 7 (4) 3, 8

Sol.(3) As given $\bar{x} = 4$, n = 5 and $\sigma^2 = 5.2$. If the remaining observations are x_1, x_2 then

$$\sigma^2 = \frac{\sum(x_i - \bar{x})^2}{n} = 5.2$$

$$\Rightarrow \frac{(x_1-4)^2+(x_2-4)^2+(1-4)^2+(2-4)^2+(6-4)^2}{5} = 5.2$$

$$\Rightarrow (x_1-4)^2 + (x_2-4)^2 = 9 \qquad ....(1)$$

Also $\bar{x} = 4 \Rightarrow \frac{x_1+x_2+1+2+6}{5} = 4 \Rightarrow x_1 + x_2 = 11$ ....(2)

from eq.(1), (2) $x_1, x_2 = 4, 7$

Ex.12 The mean deviation of the series a, a + d, a + 2d, ..., a + 2nd from its mean is-

(1) $\frac{n+1}{2n+1}|d|$ (2) $\frac{n(n+1)}{2n+1}|d|$ (3) $\frac{n(n-1)}{2n+1}|d|$ (4) none of these

Sol.(2) Number of terms in the series = 2n + 1

$$\therefore \text{ mean } \bar{x} = \frac{a+(a+d)+(a+2d)+.....+(a+2nd)}{2n+1} = \frac{1}{2n+1}\left[\frac{2n+1}{2}(a+a+2nd)\right] = a+nd$$

Also $\sum|x_i - \bar{x}| = |-nd| + |(1-n)d| + ... + |-d| + 0 + |d| + ... + |nd|$

$$= 2|d|[n+(n-1)+.......+1] = 2|d|\frac{n(n+1)}{2} = n(n+1)|d|$$

$$\therefore \text{ mean deviation from mean } = \frac{\sum|x_i-\bar{x}|}{N} = \frac{n(n+1)}{2n+1}|d|$$

Ex.13 Let $x_1, x_2 ....., x_n$ be values taken by a variable X and $y_1, y_2, ..., y_n$ be the values taken by a variable Y such that $y_i = ax_i + b$, $i = 1, 2, ..., n$. Then-

(1) $Var(Y) = a^2 Var(X)$ (2) $Var(Y) = a^2 Var(X) + b$

(3) $Var(Y) = Var(X) + b$ (4) None of these

Sol.(1) We have,

$$Var(Y) = \frac{1}{n}\sum_{i=1}^{n}(y_i - \bar{y})^2$$

$$[\because y_i = ax_i + b;\ i = 1, 2, ..., n \Rightarrow \bar{Y} = a\bar{X} + b]$$

$$\Rightarrow Var(Y) = \frac{1}{n}\sum_{i=1}^{n}a^2(x_i - \bar{X})^2$$

$$\Rightarrow Var(Y) = a^2\left\{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{X})^2\right\} = a^2 Var(X)$$

Ex.14 The mean square deviation of a set of n observations $x_1, x_2, ..., x_n$ about a point c is defined as $\frac{1}{n}\sum_{i=1}^{n}(x_i - c)^2$ The mean square deviation about –2 and 2 are 18 and 10 respectively, then standard deviation of this set of observations is-

(1) 3 (2) 2 (3) 1 (4) None of these

Sol.(1) $\because \frac{1}{n}\Sigma(x_i + 2)^2 = 18$ and $\frac{1}{n}\Sigma(x_i - 2)^2 = 10$

$\Rightarrow \Sigma(x_i + 2)^2 = 18n$ and $\Sigma(x_i - 2)^2 = 10n$

$\Rightarrow \Sigma(x_i + 2)^2 + \Sigma(x_i - 2)^2 = 28\,n$ and $\Sigma(x_i + 2)^2 - \Sigma(x_i - 2)^2 = 8\,n$

$\Rightarrow 2\Sigma x_i^2 + 8n = 28\,n$ and $8\Sigma x_i = 8n$

$\Rightarrow \Sigma x_i^2 = 10\,n$ and $\Sigma x_i = n$

$\Rightarrow \frac{\Sigma x_i^2}{n} = 10$ and $\frac{\Sigma x_i}{n} = 1$

$$\therefore \sigma = \sqrt{\frac{\Sigma x_i^2}{n} - \left(\frac{\Sigma x_i}{n}\right)^2} = \sqrt{10 - (1)^2} = 3$$

EXERCISE - 1

Arithmetic mean, Weighted mean, Combined mean

1. Mean of the first n terms of the A.P. a, (a + d), (a + 2d), ... is-

(1) $a+\frac{nd}{2}$ (2) $a+\frac{(n-1)d}{2}$ (3) a + (n – 1) d (4) a + nd

2. The A.M. of first n even natural number is -

(1) n(n + 1) (2) $\frac{n+1}{2}$ (3) $\frac{n}{2}$ (4) n + 1

3. The A.M. of $^nC_0$, $^nC_1$, $^nC_2$, ...^nC_n is -

(1) $\frac{2^n}{n}$ (2) $\frac{2^{n+1}}{n}$ (3) $\frac{2^n}{n+1}$ (4) $\frac{2^{n+1}}{n+1}$

4. If the mean of numbers 27, 31, 89, 107, 156 is 82, then the mean of numbers 130, 126, 68, 50, 1 will be-

(1) 80 (2) 82 (3) 75 (4) 157

5. If the mean of n observations x_1, x_2, ...x_n is $\overline{x}$, then the sum of deviations of observations from mean is :-

(1) 0 (2) $n\overline{x}$ (3) $\frac{\overline{x}}{n}$ (4) None of these

6. The mean of 9 terms is 15. if one new term is added and mean become 16, then the value of new term is :-

(1) 23 (2) 25 (3) 27 (4) 30

7. If the mean of first n natural numbers is equal to $\frac{n+7}{3}$, then n is equal to-

(1) 10 (2) 11 (3) 12 (4) none of these

8. The mean of first three terms is 14 and mean of next two terms is 18. The mean of all the five terms is-

(1) 15.5 (2) 15.0 (3) 15.2 (4) 15.6

9. If the mean of five observations x, x + 2 , x + 4, x+ 6 and x + 8 is 11, then the mean of last three obsevations is-

(1) 11 (2) 13 (3) 15 (4) 17

10. The mean of a set of numbers is $\overline{x}$. If each number is decreased by λ, the mean of the new set is-

(1) $\overline{x}$ (2) $\overline{x}+\lambda$ (3) $\lambda-\overline{x}$ (4) $\overline{x}-\lambda$

11. The mean of 50 observations is 36. If its two observations 30 and 42 are deleted, then the mean of the remaining observations is-

(1) 48 (2) 36 (3) 38 (4) none of these

12. In a frequency dist. , if d_i is deviation of variates from a number ℓ and mean = $\ell + \frac{\Sigma f_i d_i}{\Sigma f_i}$, then ℓ is:-

(1) Lower limit (2) Assumed mean

(3) Number of observation (4) Class interval

13. The A.M. of n observation is $\overline{x}$. If the sum of n – 4 observations is K, then the mean of remaining observations is-

(1) $\frac{\overline{x}-K}{4}$ (2) $\frac{n\overline{x}-K}{n-4}$ (3) $\frac{n\overline{x}-K}{4}$ (4) $\frac{n\overline{x}-(n-4)K}{4}$

14. The mean of values $1, \frac{1}{2}, \frac{1}{3}, ...\frac{1}{n}$ which have frequencies 1, 2, 3, ... n resp., is :-

(1) $\frac{2n+1}{3}$ (2) $\frac{2}{n}$ (3) $\frac{n+1}{2}$ (4) $\frac{2}{n+1}$

15. The sum of squares of deviation of variates from their A.M. is always :-

(1) Zero (2) Minimum (3) Maximum (4) Nothing can be said

16. If the mean of following feq. dist. is 2.6, then the value of f is :-

x_i	1	2	3	4	5
f_i	5	4	f	2	3

(1) 1 (2) 3 (3) 8 (4) None of these

17. The weighted mean (W.M.) is computed by the formula ?

(1) W.M. = $\frac{\Sigma x_i}{\Sigma w_i}$ (2) W.M.=$\frac{\Sigma w_i}{\Sigma x_i}$ (3) W.M. = $\frac{\Sigma w_i x_i}{\Sigma x_i}$ (4) W.M.= $\frac{\Sigma w_i x_i}{\Sigma w_i}$

18. The weighted mean of first n natural numbers when their weights are equal to corresponding natural number, is :-

(1) $\frac{n+1}{2}$ (2) $\frac{2n+1}{3}$ (3) $\frac{(n+1)(2n+1)}{6}$ (4) None of these

19. The average income of a group of persons is $\overline{x}$ and that of another group is $\overline{y}$. If the number of persons of both group are in the ratio 4 : 3, then average income of combined group is :-

(1) $\frac{\overline{x}+\overline{y}}{7}$ (2) $\frac{3\overline{x}+4\overline{y}}{7}$ (3) $\frac{4\overline{x}+3\overline{y}}{7}$ (4) None of these

20. In a group of students, the mean weight of boys is 65 kg. and mean weight of girls is 55 kg. If the mean weight of all students of group is 61 kg, then the ratio of the number of boys and girls in the group is :-

(1) 2 : 3 (2) 3 : 1 (3) 3 : 2 (4) 4 : 3

Median, Mode

21. The median of an arranged series of n even observations, will be :-

(1) $\left(\frac{n+1}{2}\right)$th term (2) $\left(\frac{n}{2}\right)$th term

(3) $\left(\frac{n}{2}+1\right)$ th term (4) Mean of $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$ th terms

22. The median of the numbers 6, 14, 12, 8, 10, 9, 11, is :-

(1) 8 (2) 10 (3) 10.5 (4) 11

23. Median of the following freq. dist.

x_i	3	6	10	12	7	15
f_i	3	4	2	8	13	10

(1) 7 (2) 10 (3) 8.5 (4) None of these

24. Median is independent of change of :-

(1) only Origin (2) only Scale

(3) Origin and scale both (4) Neither origin nor scale

25. A series which have numbers three 4's, four 5's, five 6's, eight 7's, seven 8's and six 9's then the mode of numbers is :-

(1) 9 (2) 8 (3) 7 (4) 6

26. Mode of the following freqency distribution

x :	4	5	6	7	8
f :	6	7	10	8	3

(1) 5 (2) 6 (3) 8 (4) 10

27. The mode of the following freq. dist is :-

Class	1–10	11–20	21–30	31–40	41–50
f_i	5	7	8	6	4

(1) 24 (2) 23.83 (3) 27.16 (4) None of these

Symmetric and asymmetric distribution, Range

28. For a normal dist :-

(1) mean = median (2) median = mode

(3) mean = mode (4) mean = median = mode

29. The relationship between mean, median and mode for a moderately skewed distribution is-

(1) mode = median – 2 mean (2) mode = 2 median – mean

(3) mode = 2 median – 3 mean (4) mode = 3 median – 2 mean

30. The range of observations 2, 3, 5, 9, 8, 7, 6, 5, 7, 4, 3 is :-

(1) 6 (2) 7 (3) 5.5 (4) 11

Mean Deviation

31. The mean deviation of a frequency dist. is equal to :-

(1) $\frac{\Sigma d_i}{\Sigma f_i}$ (2) $\frac{\Sigma |d_i|}{\Sigma f_i}$ (3) $\frac{\Sigma f_i d_i}{\Sigma f_i}$ (4) $\frac{\Sigma f_i |d_i|}{\Sigma f_i}$

32. Mean deviation from the mean for the observation –1, 0, 4 is-

(1) $\sqrt{\frac{14}{3}}$ (2) $\frac{2}{3}$ (3) 2 (4) none of these

33. Mean deviation of the observations 70, 42, 63, 34, 44, 54, 55, 46, 38, 48 from median is :-

(1) 7.8 (2) 8.6 (3) 7.6 (4) 8.8

34. Mean deviation of 5 observations from their mean 3 is 1.2, then coefficient of mean deviation is :-

(1) 0.24 (2) 0.4 (3) 2.5 (4) None of these

35. The mean deviation from median is

(1) greater than the mean deviation from any other central value

(2) less than the mean deviation from any other central value

(3) equal to the mean deviation from any other central value

(4) maximum if all values are positive

Variance and Standard Deviation

36. The variate x and u are related by $u = \frac{x-a}{h}$ then correct relation between σ_x and σ_u is :-

(1) $\sigma_x = h\sigma_u$ (2) $\sigma_x = h + \sigma_u$ (3) $\sigma_u = h\sigma_x$ (4) $\sigma_u = h + \sigma_x$

37. The S.D. of the first n natural numbers is-

(1) $\sqrt{\frac{n^2-1}{2}}$ (2) $\sqrt{\frac{n^2-1}{3}}$ (3) $\sqrt{\frac{n^2-1}{4}}$ (4) $\sqrt{\frac{n^2-1}{12}}$

38. The variance of observations 112, 116, 120, 125, 132 is :-

(1) 58.8 (2) 48.8 (3) 61.8 (4) None of these

39. If $\sum_{i=1}^{10}(x_i - 15) = 12$ and $\sum_{i=1}^{10}(x_i - 15)^2 = 18$ then the S.D. of observations x_1, x_2 .... x_{10} is :-

(1) $\frac{2}{5}$ (2) $\frac{3}{5}$ (3) $\frac{4}{5}$ (4) None of these

40. The S.D. of 7 scored 1, 2, 3, 4, 5, 6, 7 is-

(1) 4 (2) 2 (3) $\sqrt{7}$ (4) none of these

41. The variance of series a, a + d, a + 2d, ..., a + 2nd is :-

(1) $\frac{n(n+1)}{2}d^2$ (2) $\frac{n(n+1)}{3}d^2$ (3) $\frac{n(n+1)}{6}d^2$ (4) $\frac{n(n+1)}{12}d^2$

42. Variance is independent of change of-

(1) only origin (2) only scale

(3) origin and scale both (4) none of these

43. If the coefficient of variation and standard deviation of a distribution are 50% and 20 respectively, then its mean is-

(1) 40 (2) 30 (3) 20 (4) None of these

44. If each observation of a dist. whose S.D. is σ, is increased by λ, then the variance of the new observations is -

(1) σ (2) $\sigma+\lambda$ (3) σ^2 (4) $\sigma^2+\lambda$

45. The variance of 2, 4, 6, 8, 10 is-

(1) 8 (2) $\sqrt{8}$ (3) 6 (4) none of these

46. If each observation of a dist., whose variance is σ^2, is multiplied by λ, then the S.D. of the new new observations is-

(1) σ (2) $\lambda\sigma$ (3) $|\lambda|\sigma$ (4) $\lambda^2\sigma$

47. The standard deviation of variate x_i is σ. Then standard deviation of the variate $\frac{ax_i+b}{c}$, where a , b, c are constants is-

(1) $\left(\frac{a}{c}\right)\sigma$ (2) $\left|\frac{a}{c}\right|\sigma$ (3) $\left(\frac{a^2}{c^2}\right)\sigma$ (4) None of these

EXERCISE - 2

1. The A.M. of the series 1, 2, 4, 8, 16, ..., 2^n is-

(1) $\frac{2^n-1}{n}$ (2) $\frac{2^{n+1}-1}{n+1}$ (3) $\frac{2^n-1}{n+1}$ (4) $\frac{2^{n+1}-1}{n}$

2. If the mean of n observations $1^2, 2^2, 3^2, ... n^2$ is $\frac{46n}{11}$, then n is equal to-

(1) 11 (2) 12 (3) 23 (4) 22

3. The weighted mean of first n natural numbers whose weights are equal, is :-

(1) $\frac{n+1}{2}$ (2) $\frac{2n+1}{2}$ (3) $\frac{2n+1}{3}$ (4) $\frac{(2n+1)(n+1)}{6}$

4. The average age of a group of men and women is 30years. If average age of men is 32 and that of women is 27, then the percentage of women in the group is-

(1) 60 (2) 50 (3) 40 (4) 30

5. Mean and median of four numbers a, b, c and d ($b < a < d < c$) is 35 and 25 respectively then the value of $b + c - a - d$ will be :-

(1) 90 (2) 115 (3) 40 (4) 10

6. Variance of the group $\alpha, \alpha + 2, \alpha + 4, \alpha + 6$, ... upto n terms ($\alpha \neq 0$) is :-

(1) $\frac{n^2-1}{12}+2n+\alpha$ (2) $\frac{n^2-1}{3}+\alpha$ (3) $\frac{n^2-1}{3}$ (4) None

7. Product of n positive numbers is unit. The sum of these numbers can not be less than-

(1) 1 (2) n (3) n^2 (4) none of these

8. The A.M. of first n terms of the series

1.3.5, 3.5.7, 5.7.9,....., is-

(1) $3n^3 + 6n^2 + 7n - 1$ (2) $n^3 + 8n^2 + 7n - 1$

(3) $2n^3 + 8n^2 - 7n - 2$ (4) $2n^3 + 8n^2 + 7n - 2$

9. The observations 29, 32, 48, 50, x, x + 2, 72, 78, 84, 95 are arranged in ascending order and their median is 63 then the value of x is :-

(1) 61 (2) 62 (3) 62.5 (4) 63

10. If the mode of a distribution is 18 and the mean is 24, then median is-

(1) 18 (2) 24 (3) 22 (4) 21

11. If the mean and S.D. of n observations $x_1, x_2, \ldots, x_n$ are $\bar{x}$ and σ resp, then the sum of squares of observations is :-

(1) $n(\sigma^2 + \bar{x}^2)$ (2) $n(\sigma^2 - \bar{x}^2)$ (3) $n(\bar{x}^2 - \sigma^2)$ (4) None of these

12. The variance of observations 8, 12, 13, 15, 22, is :-

(1) 21 (2) 21.2 (3) 21.4 (4) None of these

13. If the mean of a set of observations $x_1, x_2, \ldots\ldots, x_{10}$ is 20, then the mean of $x_1 + 4$, $x_2 + 8$, $x_3 + 12, \ldots\ldots, x_{10} + 40$ is-

(1) 34 (2) 42 (3) 38 (4) 40

14. The mean of values 0, 1, 2, ..., n when their weights are 1, ${}^nC_1, {}^nC_2, \ldots, {}^nC_n$, resp., is

(1) $\frac{2^n}{n+1}$ (2) $\frac{n+1}{2}$ (3) $\frac{2^{n+1}}{n(n+1)}$ (4) $\frac{n}{2}$

15. For 15 observations of x, mean and median were found to be 12 and 20 respectively. Later an observation which was 25 found to be wrong then replaced by its correct value 55, then new mean and median will be :-

(1) 14 and 50 respectively (2) 12 and 20 respectively

(3) 14 and 20 respectively (4) Mean is 14 but median can't be determined.

16. If a variable takes the discrete values $\alpha + 4$, $\alpha - \frac{7}{2}$, $\alpha - \frac{5}{2}$, $\alpha - 3$, $\alpha - 2$, $\alpha + \frac{1}{2}$, $\alpha - \frac{1}{2}$, $\alpha + 5 (\alpha > 0)$, then the median of these values-

(1) $\alpha - \frac{5}{4}$ (2) $\alpha - \frac{1}{2}$ (3) $\alpha - 2$ (4) $\alpha + \frac{5}{4}$

17. The S.D. of first n odd natural numbers is :-

(1) $\sqrt{\frac{n^2-1}{2}}$ (2) $\sqrt{\frac{n^2-1}{3}}$ (3) $\sqrt{\frac{n^2-1}{6}}$ (4) $\sqrt{\frac{n^2-1}{12}}$

18. If the sum and sum of squares of 10 observations are 12 and 18 resp., then, The S.D. of observations is :-

(1) $\frac{1}{5}$ (2) $\frac{2}{5}$ (3) $\frac{3}{5}$ (4) $\frac{4}{5}$

19. The mean of n values of a distribution is $\bar{x}$. If its first value is increased by 1 , second by 2, ... then the mean of new values will be-

(1) $\bar{x}+n$ (2) $\bar{x}+n/2$ (3) $\bar{x}+\left(\frac{n+1}{2}\right)$ (4) None of these

20. The mean of the series x_1, x_2, ..., x_n is $\bar{X}$. If x_2 is replaced by λ, then the new mean is-

(1) $\frac{\bar{X}-x_2+\lambda}{n}$ (2) $\frac{n\bar{X}+x_2-\lambda}{n}$ (3) $\frac{(n-1)\bar{X}+\lambda}{n}$ (4) $\frac{n\bar{X}-x_2+\lambda}{n}$

21. The mean square deviation about –1 and +1 of a set of observations are 7 and 3 respectively then standard deviation of the set is :-

(1) $\sqrt{2}$ (2) $\sqrt{3}$ (3) 2 (4) None

22. The mean deviation of the numbers 1, 2, 3, 4, 5 is-

(1) 0 (2) 1.2 (3) 2 (4) 1.4

23. If mean = (3 median – mode) x, then the value of x is-

(1) 1 (2) 2 (3) 1/2 (4) 3/2

24. A man spends equal ammount on purchasing three kinds of pens at the rate 5 Rs/pen, 10 Rs/pen, 20 Rs/pen, then average cost of one pen is :-

(1) 10 Rs (2) $\frac{35}{3}$ Rs (3) $\frac{60}{7}$ Rs (4) None of these

25. The median of 21 observation is 40. if each observations greater than the median are increased by 6, then the median of the observations will be-

(1) 40 (2) 46 (3) 46 + 40/21 (4) 46 – 40/21

26. The coefficient of range of the following distribution 10, 14, 11, 9, 8, 12, 6

(1) 0.4 (2) 2.5 (3) 8 (4) 0.9

27. The S.D. of the following freq. dist. :-

Class	0 – 10	10 – 20	20 – 30	30 – 40
f_i	1	3	4	2

(1) 7.8 (2) 9 (3) 8.1 (4) 0.9

28. The mean of a dist. is 4. if its coefficient of variation is 58%. Then the S.D. of the dist. is:-

(1) 2.23 (2) 3.23 (3) 2.32 (4) None of these

29. The mean of a set of observations is $\overline{x}$. If each observation is divided by α, $(\alpha \neq 0)$ and then is increased by 10, then the mean of the new set is

(1) $\dfrac{\overline{x}}{\alpha}$ (2) $\dfrac{\overline{x}+10}{\alpha}$ (3) $\dfrac{\overline{x}+10\alpha}{\alpha}$ (4) $\dfrac{\alpha\overline{x}+10}{\alpha}$

30. The average age of a teacher and three students is 20 years. If all students are of equal age and the difference between the age of the teacher and that of a student is 20 years, then the age of the teacher is-

(1) 25 years (2) 30 years (3) 35 years (4) 45 years

31. Median of 5 observations i.e.

$$3^{\log_9 4}, 5^{\log_{1/2} 8}, e^{2\ell n 3}, \ell n\left(\frac{1}{e}2\right)+3, e^{2\ell n 3+\frac{1}{\log_4 e}} \text{ :-}$$

(1) 1 (2) 2 (3) 9 (4) 36

32. Median of ${}^{2n}C_0$, ${}^{2n}C_1$, ${}^{2n}C_2$,....., ${}^{2n}C_n$

(when n is even) is-

(1) ${}^{2n}C_{\frac{n-1}{2}}$ (2) ${}^{2n}C_{\frac{n}{2}}$ (3) ${}^{2n}C_{\frac{n+1}{2}}$ (4) None of these

33. The mean deviation from mean of observations 5, 10, 15, 20, ...85 is :-

(1) 43.71 (2) 21.17 (3) 38.7 (4) None of these

34. If standard deviation of variate x_i is 10, then variance of the variate $(50 + 5x_i)$ will be-

(1) 50 (2) 250 (3) 500 (4) 2500

35. The S.D. of the numbers 31, 32, 33, ... 47 is-

(1) $2\sqrt{6}$ (2) $4\sqrt{3}$ (3) $\sqrt{\dfrac{47^2-1}{12}}$ (4) None of these

36. The sum of the squares of deviation of 10 observations from their mean 50 is 250, then coefficient of variation is-

(1) 10% (2) 40% (3) 50% (4) None of these

37. The median and standard deviation (S.D.) of a distribution will be, If each term is increased by 2-

(1) median and S.D. will increased by 2

(2) median will increased by 2 but S.D. will remain same

(3) median will remain same but S.D. will increased by 2

(4) median and S.D. will remain same

38. If $\bar{X}_1$ and $\bar{X}_2$ are the means of two series such that $\bar{X}_1 < \bar{X}_2$ and $\bar{X}$ is the mean of the combined series, then-

(1) $\bar{X} < \bar{X}_1$ (2) $\bar{X} > \bar{X}_2$ (3) $\bar{X}_1 < \bar{X} < \bar{X}_2$ (4) $\bar{X} = \dfrac{\bar{X}_1 + \bar{X}_2}{2}$

39. The median of 19 observations of a group is 30. If two observations with values 8 and 32 are further included, then the median of the new group of 21 observation will be

(1) 28 (2) 30 (3) 32 (4) 34

40. The coefficient of mean deviation from median of observations 40, 62, 54, 90, 68, 76 is :-

(1) 2.16 (2) 0.2 (3) 5 (4) None of these

41. A group of 10 observations has mean 5 and S.D. $2\sqrt{6}$. another group of 20 observations has mean 5 and S.D. $3\sqrt{2}$, then the S.D. of combined group of 30 observations is :-

(1) $\sqrt{5}$ (2) $2\sqrt{5}$ (3) $3\sqrt{5}$ (4) None of these

42. For the values $x_1, x_2 \ldots\ldots x_{101}$ of a distribution $x_1 < x_2 < x_3 < \ldots\ldots < x_{100} < x_{101}$. The mean deviation of this distribution with respect to a number k will be minimum when k is equal to-

(1) x_1 (2) x_{51} (3) x_{50} (4) $\frac{x_1 + x_2 + \ldots\ldots + x_{101}}{101}$

43. In any discrete series (when all the value are not same) the relationship between M.D. about mean and S.D. is-

(1) M.D. = S.D. (2) M.D. > S.D. (3) M.D. < S.D. (4) M.D. ≤ S.D.

44. Median of observations x_i such that$(x_i^2 - 7x_i + 12)(x_i^3 - x_i^2 - 4x_i + 4) = 0$ will be :-

(1) 1 (2) 2 (3) 3 (4) None

EXERCISE - 3

1. All the students of a class performed poorly in Mathematics. The teacher decided to give grace marks of 10 to each of the students. Which of the following statistical measures will not change even after the grace marks were given? **[JEE(Main)-2013]**

(1) mean (2) median (3) mode (4) variance

2. The variance of first 50 even natural numbers is :- **[JEE(Main)-2014]**

(1) $\frac{833}{4}$ (2) 833 (3) 437 (4) $\frac{437}{4}$

3. The mean of the data set comprising of 16 observations is 16. If one of the observation valued 16 is deleted and three new observations valued 3, 4 and 5 are added to the data, then the mean of the resultant data, is : **[JEE(Main)-2015]**

(1) 15.8 (2) 14.0 (3) 16.8 (4) 16.0

4. If the standard deviation of the numbers 2, 3, a and 11 is 3.5, then which of the following is true? **[JEE(Main)-2016]**

(1) $3a^2 - 23a + 44 = 0$ (2) $3a^2 - 26a + 55 = 0$

(3) $3a^2 - 32a + 84 = 0$ (4) $3a^2 - 34a + 91 = 0$

5. If $\sum_{i=1}^{9}(x_i - 5) = 9$ and $\sum_{i=1}^{9}(x_i - 5)^2 = 45$, then the standard deviation of the 9 items $x_1, x_2, ..., x_9$ is- **[JEE(Main)-2018]**

(1) 4 (2) 2 (3) 3 (4) 9

6. 5 students of a class have an average height 150 cm and variance 18 cm^2. A new student, whose height is 156 cm, joined them. The variance (in cm^2) of the height of these six students is: **[JEE(Main)-19]**

(1) 22 (2) 20 (3) 16 (4) 18

7. A data consists of n observations :

$x_1, x_2, ..., x_n$. If $\sum_{i=1}^{n}(x_i + 1)^2 = 9n$ and $\sum_{i=1}^{n}(x_i - 1)^2 = 5n$, then the standard deviation of this data is : **[JEE(Main)-19]**

(1) 5 (2) $\sqrt{5}$ (3) $\sqrt{7}$ (4) 2

8. The mean of five observations is 5 and their variance is 9.20. If three of the given five observations are 1, 3 and 8, then a ratio of other two observations is : **[JEE(Main)-19]**

(1) 4 : 9 (2) 6 : 7 (3) 5 : 8 (4) 10 : 3

9. The outcome of each of 30 items was observed; 10 items gave an outcome $\frac{1}{2} - d$ each, 10 items gave outcome $\frac{1}{2}$ each and the remaining 10 items gave outcome $\frac{1}{2} + d$ each. If the variance of this outcome data is $\frac{4}{3}$ then |d| equals :- **[JEE(Main)-19]**

(1) 2 (2) $\frac{\sqrt{5}}{2}$ (3) $\frac{2}{3}$ (4) $\sqrt{2}$

10. The mean and the variance of five observation are 4 and 5.20, respectively. If three of the observations are 3, 4 and 4; then then absolute value of the difference of the other two observations, is : **[JEE(Main)-19]**

(1) 1 (2) 3 (3) 7 (4) 5

11. The mean and variance of seven observations are 8 and 16, respectively. If 5 of the observations are 2, 4, 10, 12, 14, then the product of the remaining two observations is : **[JEE(Main)-19]**

(1) 40 (2) 49 (3) 48 (4) 45

12. If the standard deviation of the numbers –1, 0, 1, k is $\sqrt{5}$ where k > 0, then k is equal to

(1) $2\sqrt{\frac{10}{3}}$ (2) $2\sqrt{6}$ (3) $4\sqrt{\frac{5}{3}}$ (4) $\sqrt{6}$

13. If for some $x \in R$, the frequency distribution of the marks obtained by 20 students in a test is : **[JEE(Main)-19]**

Marks	2	3	5	7
Frequency	$(x+1)^2$	$2x-5$	x^2-3x	x

then the mean of the marks is :

(1) 2.8 (2) 3.2 (3) 3.0 (4) 2.5

14. If both the mean and the standard deviation of 50 observations $x_1, x_2,...,x_{50}$ are equal to 16, then the mean of $(x_1 - 4)^2, (x_2 - 4)^2,...(x_{50} - 4)^2$ is : **[JEE(Main)-19]**

(1) 525 (2) 380 (3) 480 (4) 400

15. If the mean and variance of eight numbers 3, 7, 9, 12, 13, 20, x and y be 10 and 25 respectively, then $x \cdot y$ is equal to_____ **[JEE(Main)-20]**

16. If the variance of the first n natural numbers is 10 and the variance of the first m even natural numbers is 16, then m + n is equal to ________. **[JEE(Main)-20]**

17. The mean and variance of 20 observations are found to be 10 and 4, respectively. On rechecking, it was found that an observation 9 was incorrect and the correct observation was 11. Then the correct variance is **[JEE(Main)-20]**

(1) 3.99 (2) 3.98 (3) 4.02 (4) 4.01

18. The mean and the standard deviation (s.d.) of 10 observations are 20 and 2 resepectively. Each of these 10 observations is multiplied by p and then reduced by q, where $p \neq 0$ and $q \neq 0$. If the new mean and new s.d. become half of their original values, then q is equal to **[JEE(Main)-20]**

(1) –20 (2) 10 (3) –10 (4) –5

19. Let the observations $x_i (1 \leq i \leq 10)$ satisfy the equations, $\sum_{i=1}^{10}(x_i - 5) = 10$ and $\sum_{i=1}^{10}(x_i - 5)^2 = 40$. If μ and λ are the mean and the variance of the observations, $x_1 - 3, x_2 - 3, ..., x_{10} - 3$, then the ordered pair (μ, λ) is equal to : **[JEE(Main)-20]**

(1) (6, 6) (2) (3, 6) (3) (6, 3) (4) (3, 3)

ANSWER KEY

EXERCISE - 1

Que.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Ans.	2	4	3	3	1	2	2	4	2	4	2	2	3	4	2	1	4	2	3	3
Que.	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
Ans.	4	2	3	4	3	2	2	4	4	2	4	3	2	2	2	1	4	2	2	2
Que.	**41**	**42**	**43**	**44**	**45**	**46**	**47**													
Ans.	2	1	1	3	1	3	2													

EXERCISE - 2

Que.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
Ans.	2	1	1	3	3	3	2	4	2	3	1	2	2	4	3	1	2	3	3	4
Que.	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
Ans.	2	2	3	3	1	1	2	3	3	3	3	2	2	4	1	1	2	3	2	2
Que.	**41**	**42**	**43**	**44**																
Ans.	2	2	3	2																

EXERCISE - 3

Que.	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
Ans.	4	2	2	3	2	2	2	1	4	3	3	2	1	4	54.00	18	1	1	4