Level - I

Chapter 7

System of Particles and Rotational Motion

Solutions (Set-1)

SECTION - A

School/Board Exam. Type Questions

Very Short Answer Type Questions :

1. The cross product of two vectors is zero. What is the angle between them?

Sol. Zero. Vectors are parallel to each other

$$|\vec{A} \times \vec{B}| = AB\sin\theta = 0$$

$\Rightarrow$ $\sin\theta = 0$

$\Rightarrow$ $\theta = 0°$

2. Name the physical quantity, which is conserved when no external torque acts on a body.

Sol. Angular momentum (L)

3. What is the moment of inertia of a right circular solid of radius *r* and length *l* about its axis?

Sol. $I = \frac{1}{2}MR^2$

4. Define moment of inertia of a rigid body.

Sol. The moment of inertia of a rigid body about a given axis of rotation is the sum of the product of the masses of the various particles and square of their perpendicular distance from the axis of rotation.

$$I = \sum_{i=1}^{n} m_1 r_i^2$$

5. Two equal masses are placed at a distance *r* from each other. Where does the centre of mass of the system lie?

Sol. The centre of mass of the system lie at the mid-point of the line forming the particles.

6. Give the expression for the total kinetic energy of a rigid body rolling down an inclined plane without slipping.

Sol. K.E. = $K.E_{transt}$ + $K.E_{rot}$ = $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

7. What is the relation between moment of inertia (I), angular momentum (L) and angular velocity (ω)?

Sol. $L = I\omega$

8. Where does the centre of mass of a uniform triangular lamina lie?

Sol. At the centroid of the triangle.

9. Which rule is used to determine the direction of torque?

Sol. Right hand screw rule.

10. Which physical quantity is expressed by the rate of change of angular velocity?

Sol. Angular acceleration.

Short Answer Type Questions :

11. What will be the effect on the duration of the day, if the ice on the polar caps of the earth melts.

Sol. When the ice on the polar caps melt the moment of inertia I increases because the mass concentrated near the axis of rotation spreads out. As no external torque is acting on the system.

$$\therefore\ L = I\omega = I\left(\frac{2\pi}{T}\right) = \text{constant}$$

ω decreases, hence T increases, so the duration of the day will increase

12. Two discs A and B have same mass and thickness. The density of their material is d_1 and d_2 respectively. What is the ratio of their moments of inertia about central axis?

Sol. Moment of inertia of disc $I = \frac{MR^2}{2}$

Let the thickness be x and radius be r_1 and r_2 as the masses are equal

$$\therefore\quad \pi r_1^2\, d_1 = \pi r_2^2\, d_2$$

$$\frac{r_1^2}{r_2^2} = \frac{d_2}{d_1}$$

$$\frac{I_1}{I_2} = \frac{\frac{1}{2}Mr_1^2}{\frac{1}{2}Mr_2^2} = \frac{d_2}{d_1}$$

13. Two bodies of masses 3 kg and 6 kg are moving with velocities 2 m/s and 5 m/s towards each other. Calculate the velocity of their centre of mass.

Sol. $v_{cm} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2}$

as they are moving in opposite directions therefore,

$$v_{cm} = \frac{m_1v_1 + m_2v_2}{m_1 + m_2} = \frac{3\times 2 - 6\times 5}{9} = \frac{6-30}{9} = -\frac{24}{9} = -\frac{8}{3}\ \text{m/s}$$

– ve sign shows that the velocity of the C.M is in the direction of the velocity of 6 kg mass.

14. The moment of inertia of two bodies A and B are 4 kg m^2 and 9 kg m^2 respectively. If they have the same rotational K.E. then calculate the ratio of their angular momenta.

Sol. $K.E_{rotation} = \frac{1}{2}I\omega^2$

$K.E_A = \frac{1}{2}I_A\omega_A^2$, $K.E_B = \frac{1}{2}I_B\omega_B^2$

$K.E_A = K.E_B$

$\frac{1}{2}I_A\omega_A^2 = \frac{1}{2}I_B\omega_B^2$

$4\omega_A^2 = 9\omega_B^2$

$\frac{\omega_A}{\omega_B} = \sqrt{\frac{9}{4}} = \frac{3}{2}$

Angular momentum (L) = $I\omega$

$\frac{L_A}{L_B} = \frac{I_A\omega_A}{I_B\omega_B} = \frac{4}{9}\times\frac{3}{2} = \frac{2}{3}$

15. Calculate the torque that will increase the angular velocity of a solid disc of mass 10 kg and radius 0.5 m from zero to 10 rpm in 4 second.

Sol. Moment of inertia of solid disc $I = \frac{Mr^2}{2} = \frac{10\times 0.5\times 0.5}{2}$

$= 1.25$ kg m^2

Angular acceleration $\alpha = \frac{\omega - \omega_0}{t}$

$\omega_0 = 0$

$\omega = 10$ r.p.m. $= \frac{2\pi\times 10}{60} = \frac{\pi}{3}$

$t = 4$ second

$\alpha = \frac{\frac{\pi}{3}}{4} = \frac{\pi}{12}$

Torque (τ) = $I\alpha = 1.25 \times \frac{\pi}{12}$

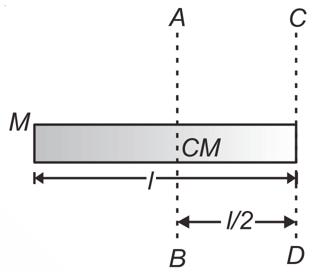
$= 0.327$ Nm

16. What is the moment of inertia of a thin rod of mass m and length l about an axis passing through its centre. Also calculate the MI of the rod about a parallel axis passing through its end.

Sol. Moment of inertia of a rod about an axis passing through its centre of mass is given by

$$I_{AB} = \frac{Ml^2}{12}$$

according to parallel axis theorem



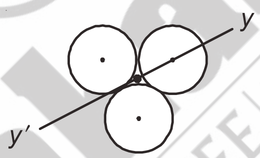
$$I_{CD} = I_{AB} + m\left(\frac{l}{2}\right)^2$$

$$= I_{AB} + \frac{ml^2}{4} = \frac{ml^2}{12} + \frac{ml^2}{4}$$

$$= \frac{ml^2 + 3ml^2}{12} = \frac{4ml^2}{12}$$

$$I_{CD} = \frac{ml^2}{3}$$

17. Three rings each of mass m and radius r are arranged as shown in the figure given below. Calculate the moment of inertia of the system about an axis yy' as shown in figure.



Sol. Moment of inertia I_{CM} of the ring about an axis passing through its CM and parallel to the plane is

$$I_{CM} = \frac{mr^2}{2}$$

according to the parallel axis theorem, the moment of inertia of one ring about yy' is given by

$$I_{yy'} = I_{CM} + mr^2$$

$$= \frac{mr^2}{2} + mr^2$$

$$= \frac{3}{2}mr^2$$

There are 3 rings so the total moment of inertia of the system about yy' is

$$I = \frac{3}{2}mr^2 \times 2 + \frac{1}{2}mr^2$$

$$= \frac{7mr^2}{2}$$

18. A solid sphere of mass 1 kg and radius 2 cm is rotating at a rate of 200 rpm. Calculate the torque required to stop it in 2π revolution.

Sol. Moment of inertia of solid sphere is given by $I = \frac{2}{5}mr^2 = \frac{2}{5}\times 1\times\frac{2}{100}\times\frac{2}{100}=\frac{8}{5}\times 10^{-4}\text{ kg m}^2$

Using, $\omega^2 - \omega_0^2 = 2\alpha\,\theta$

$\theta = 2\pi \times 2\pi = 4\pi^2$

$\omega_0 = 200 \text{ rpm} = \frac{200\times 2\pi}{60}=\frac{400\pi}{60}$

$\omega = 0$

$\alpha = \frac{200}{36}$ rad s^{-2}

$\tau = I\alpha$

$= \frac{8}{5}\times 10^{-4}\times\frac{200}{36}$

$= \frac{16}{5\times 36}\times 10^{-2} = \frac{4}{45}\times 10^{-2}$

$= 8.88 \times 10^{-4}$ Nm

19. Calculate the angular momentum of a fan having moment of inertia 3 kg m^2 and moving with an angular speed of 10 rad s^{-1}. What is the change in the angular momentum of the fan, if torque of 20 N-m acts on it for 3 second?

Sol. $L = I\omega$, $\omega = 10$ rad s^{-1}, $I = 3$kg m^2

$L = 3 \times 10 = 30$ kg m^{2}s^{-1}

$\tau = 20$ Nm, $t = 3$ second

$dL = L_2 - L_1 = \tau \times t = 20 \times 3 = 60$ kg m^{2}s^{-1}

20. If the moment of inertia of a uniform circular ring about its diameter is 100 g cm^2, calculate the moment of inertia of a uniform disc of the same mass and radius, about an axis passing through its centre perpendicular to its plane.

Sol. Moment of inertial of the ring about its diameter I_{ring} is given by

$I_{ring} = \frac{MR^2}{2} = 100$

$MR^2 = 200$

Moment of inertia of the disc of the same mass M and radius R about an axis passing through its centre perpendicular to its plane is given by

$I_{disc} = \frac{MR^2}{2} = \frac{200}{2} = 100\text{ g cm}^2$

21. A solid sphere rolls down an inclined plane without slipping. Calculate the ratio of the kinetic energy of translation to the kinetic energy of rotation.

Sol. Let the mass and radius of the solid sphere be *m* and *r* respectively

K.E. of translation = $\frac{1}{2}mv^2$ ...(i)

where *v* is the linear velocity of the solid sphere with which it is moving

K.E. of rotation = $\frac{1}{2}I\omega^2$ ...(ii)

Moment of inertia of solid sphere = $\frac{2}{5}mr^2$

and $\omega = \frac{v}{r}$

Substituting these values in (ii), we get

K.E. of rotation = $\frac{1}{2}\times\frac{2}{5}mr^2\times\frac{v^2}{r^2} = \frac{1}{5}mr^2$ ...(iii)

Dividing (i) by (iii) we get

$$\frac{\text{K.E. of translation}}{\text{K.E. of rotation}} = \frac{\frac{1}{2}mv^2}{\frac{1}{5}mv^2} = \frac{5}{2}$$

22. Differentiate between the centre of mass and centre of gravity of a rigid body.

Sol. The centre of mass (CM) of a body is a point where the entire mass of the body is supposed to be concentrated whereas the centre of gravity (CG) of a body is a point where the whole weight of the body is supposed to be concentrated. Centre of mass does not have to do anything with the gravity. It depends only on the distribution of mass of the body.

23. Define translational and rotational equilibrium.

Sol. (i) The total forces, *i.e.* the vector sum of the forces acting on the rigid body is zero,

$$\vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n = \sum_{i=1}^{n}\vec{F}_i = 0.$$

If the total force on the body is zero, then the total linear momentum of the body does not change with time. So the above equation gives the condition for the translational equilibrium of the body.

(ii) If the total torque acting on the rigid body vanishes *i.e.* the vector sum of the torques on the rigid body is zero,

$$\vec{\tau}_1 + \vec{\tau}_2 + ...\vec{\tau}_n = \sum_{i=1}^{n}\vec{\tau} = 0.$$

The total angular momentum of the body does not change with time. So the above equation gives the condition for the rotational equilibrium of the body.

24. Do the comparison between translational motion and rotational motion by making a list of variables of translational motion and their analogue rotational variable.

Sol.

S. No.	Linear Motion	Rotational motion about a fixed axis
1.	Displacement x	Angular displacement θ
2.	Velocity v = $\frac{dx}{dt}$	Angular velocity $\omega = \frac{d\theta}{dt}$
3.	Acceleration a = $\frac{dv}{dt}$	Angular acceleration $\alpha = \frac{d\omega}{dt}$
4.	Mass M	Moment of inertia I
5.	Force $F = Ma$	Torque $\tau = I\alpha$
6.	Work $dW = Fds$	Work $dW = \tau d\theta$
7.	Kinetic energy $K = \frac{Mv^2}{2}$	Kinetic energy $K = \frac{I\omega^2}{2}$
8.	Power $P = Fv$	Power $P = \tau\omega$
9.	Linear momentum $P = Mv$	Angular momentum $L = I\omega$
10.	Equations of translatory motion $v = u + at$ $s = ut + \frac{1}{2}at^2$ $v^2 - u^2 = 2as$ where the symbols have their usual meaning.	Equations of rotational motion $\omega = \omega_0 + \alpha t$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega^2 - \omega_0^2 = 2\alpha\theta$ where the symbols have their usual meaning.
11.	Linear momentum is conserved if no external force acts on the system.	Angular momentum is conserved if no external torque acts on the system.

25. Obtain an expression for the instantaneous power of a particle in terms of torque (τ) and angular velocity (ω).

Sol. Power is the rate of work done $P = \frac{d\omega}{dt}$

Small work done by the torque τ acting on the particle in producing angular displacement $d\theta$ is given by

$dW = \tau d\theta$

$$P = \frac{\tau d\theta}{dt}$$

$\frac{d\theta}{dt}$ is the rate of change of angular displacement, which is the angular velocity hence

$$\boxed{P = \tau\omega}$$

26. What will be the effect on the angular momentum of a particle, if the kinetic energy of rotation of the particle is halved keeping its frequency same.

Sol. $E_k = \frac{1}{2}I\omega^2; I = \frac{L}{\omega}$

So $E_k = \frac{1}{2}\frac{L}{\omega}\omega^2 = \frac{1}{2}L\omega$

$\Rightarrow L = \frac{2E_k}{\omega}$

$E'_k = \frac{1}{2}L'\omega'$

$\omega' = \omega$ as frequency is same

$E_k' = \frac{E_k}{2}$

So $L' = \frac{\frac{2E_K}{2}}{\omega} = \frac{1}{2}L$

angular momentum becomes $\frac{1}{2}$

27. Explain why a dancer on ice spins faster when she folds her arms?

Sol. On folding arms, r decrease hence as

$I = MR^2$

therefore, moment of inertia decreases on folding the arms.

$L = I\omega$ is constant as no external torque is acting

$\therefore \quad I_1\omega_1 = I_2\omega_2$

as I decreases, ω (increases) so as to maintain the same angular momentum. Hence the dancer spins faster on folding her arms.

28. Explain why a bomb at rest when explodes, its centre of mass remains at rest.

Hint. Based on the conservation of linear momentum.

Sol. Based on the conservation of linear momentum.

29. Calculate a unit vector perpendicular to both $\vec{A}$ and $\vec{B}$. $\vec{A} = \left(\hat{i} + 2\hat{j} - \hat{k}\right)$, $\vec{B} = \left(\hat{i} - \hat{j} + \hat{k}\right)$

Sol. $\vec{C}$ is the vector which is given by $\left(\vec{A} \times \vec{B}\right)$. $\vec{C}$ lies in the plane perpendicular to both $\vec{A}$ and $\vec{B}$.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i}\,(2-1) + \hat{j}\,(-1-1) + \hat{k}\,(-1-2) = \hat{i} - 2\hat{j} - 3\hat{k}$$

$\vec{C} = \hat{i} - 2\hat{j} - 3\hat{k}$

a unit vector $\hat{n}$ in the direction of $\vec{C}$ is given by

$$\hat{n} = \frac{\vec{C}}{|\vec{C}|} = \frac{\hat{i} - 2\hat{j} - 3\hat{k}}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}}\left(\hat{i} - 2\hat{j} - 3\hat{k}\right)$$

30. A solid sphere of mass 10 kg and radius 1m rotates about its axis with angular velocity of 200 rad s^{-1}. Calculate the angular momentum of the solid sphere about an axis passing through its diameter.

Sol. $I = \frac{2}{5}Mr^2 = \frac{2}{5} \times 10 \times 1 = 4 \text{ kg m}^2$

$L = I\omega = 4 \times 200 = 800 \text{ kg m}^2 \text{ s}^{-1}$

Long Answer Type Questions :

31. Explain the equilibrium of a rigid body.

Sol. (i) The total forces, *i.e.* the vector sum of the forces acting on the rigid body is zero,

$$\vec{F}_1 + \vec{F}_2 + ... + \vec{F}_n = \sum_{i=1}^{n} \vec{F}_i = 0.$$

If the total force on the body is zero, then the total linear momentum of the body does not change with time. So the above equation gives the condition for the translational equilibrium of the body.

(ii) If the total torque acting on the rigid body vanishes *i.e.* the vector sum of the torques on the rigid body is zero,

$$\vec{\tau}_1 + \vec{\tau}_2 + ...\vec{\tau}_n = \sum_{i=1}^{n} \vec{\tau} = 0.$$

The total angular momentum of the body does not change with time. So the above equation gives the condition for the rotational equilibrium of the body.

32. Explain the concept of angular momentum and obtain the expression for the rectangular components of angular momentum.

Sol. Angular momentum is given by

$$\vec{L} = \vec{r} \times \vec{P}$$

We can obtain the three rectangular components of angular moment *i.e.* L_x, L_y and L_z as we did in the case of Torque.

$$\vec{L} = L_x\hat{i} + L_y\hat{j} + L_z\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{P} = P_x\hat{i} + P_y\hat{j} + P_z\hat{k}$$

Using determinant method, we get

$$\left(L_x\hat{i} + L_y\hat{j} + L_z\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ P_x & P_y & P_z \end{vmatrix} = \left(yP_z - zP_y\right)\hat{i} + \left(zP_x - xP_z\right)\hat{j} + \left(xP_y - yP_x\right)\hat{z}$$

Comparing the components of both sides, we get

$$\boxed{\begin{aligned} L_x &= \left(yP_z - zP_y\right) \quad &...(i) \\ L_y &= \left(zP_x - xP_z\right) \quad &...(ii) \\ L_z &= \left(xP_y - yP_x\right) \quad &...(iii) \end{aligned}}$$

And from the above three equations we get

if the particle is moving in $y - z$ plane it has the angular momentum. Component only in x-direction *i.e.* $L_x \neq 0$, $L_y \neq L_z = 0$

Similarly if the particle is moving in x-z plane $L_y \neq 0$, $L_x = L_z = 0$ and if the particle is moving in x-y plane only $L_z \neq 0$, $L_x = L_y = 0$.

33. Obtain an expression for the position vector of the centre of mass of a two particle system in one, two and three dimensions.

Sol. In one dimension, the centre of mass is given by

$$x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

In two dimensions

$$x = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}, y = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$$

In three dimensions

$$x = \sum_{i=1}^{n} \frac{m_ix_i}{M}, \ y = \sum_{i=1}^{n} \frac{m_iy_i}{M}, \ z = \sum_{i=1}^{n} \frac{m_iz_i}{M}$$

34. Discuss the concept of torque and obtain the expression for the rectangular components of torque.

Sol. The force applied on a rigid body may rotate it in three dimensions. In that case we shall have three components of torque, which can be obtained by using

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

$$\vec{\tau} = \tau_x\hat{i} + \tau_y\hat{j} + \tau_z\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

Using Determinant method

$$\left(\tau_x\hat{i}+\tau_y\hat{j}+\tau_z\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \left(yF_z - zF_y\right)\hat{i} + \left(zF_x - xF_z\right)\hat{j} + \left(xF_y - yF_x\right)\hat{k}$$

Equating the three rectangular components on two sides, we get

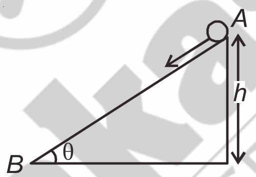
$$\boxed{\begin{aligned} \tau_x &= yF_z - zF_y \quad ...\text{(i)} \\ \tau_y &= zF_x - xF_z \quad ...\text{(ii)} \\ \tau_z &= xF_y - yF_x \quad ...\text{(iii)} \end{aligned}}$$

From the above three equations we can see that if the object is moving in the $y - z$ plane $\tau_x \neq 0$ and $\tau_y = \tau_z = 0$.

Similarly if it is moving in $x - z$ plane $\tau_y \neq 0$ and $\tau_x = \tau_z = 0$ and if it is moving in $x - y$ plane $\tau_z \neq 0$ and $\tau_x = \tau_y = 0$.

35. Obtain the expression for the maximum velocity of a body rolling down an inclined plane.

Sol. Consider an inclined plane of height h, as shown in the figure.



Let a body roll down this inclined plane. From A to B at A, potential energy of the body is mgh as it rolls down (according to the law of conservation of energy). Its potential energy changes into kinetic energy of rotation. So applying here the law of conservation of energy, we get

K.E. = P.E.

$$\Rightarrow \quad \frac{1}{2}mv_{cm}^2\left(1+\frac{k^2}{R^2}\right) = mgh$$

$$\Rightarrow \quad v_{cm}^2 = \frac{2gh}{\left(1+\frac{k^2}{R^2}\right)}$$

$$\Rightarrow \quad \boxed{v_{cm} = \sqrt{\frac{2gh}{\left(1+\frac{k^2}{R^2}\right)}}}$$

The above expression gives the maximum velocity with which the centre of mass of the body reaches the bottom of the inclined plane.

36. Discuss the motion of centre of mass and law of conservation of linear momentum with the help of examples.

Sol. We know that when an object of finite size is thrown with some initial velocity at an angle with the horizontal, it follows a parabolic path. The centre of mass of such an object also follows the parabolic path, even if the object were to disintegrate in mid-air. For example, when a fire cracker projected from *O* explodes in mid-air, the fragments fly off in different directions, describing their own parabolic paths. However, the centre of mass of the cracker would still continue to move along the same parabolic path as shown in the figure given below.

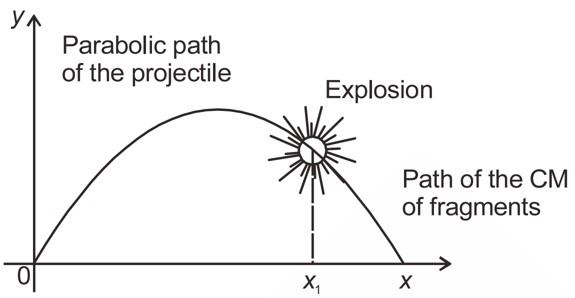


Fig. The centre of mass (CM) of the fragments of the projectile continues along the same parabolic path which it would have followed if there were no explosion

37. Define the terms angular displacement, angular velocity and angular acceleration. Obtain an expression showing the relation between torque and angular momentum.

Sol. **Angular displacement (θ)** is the angle described by the position vector $\vec{r}$ about the axis of rotation.

Angular velocity (ω) is the rate at which angular displacement changes with time i.e. $\omega = \dfrac{d\theta}{dt}$.

Angular acceleration (α) of a rigid body is the rate of change of angular velocity of the body about the given axis of rotation i.e. $\alpha = \dfrac{d\omega}{dt}$

Relation between angular momentum and torque :

$$\frac{d}{dt}\left(\vec{r} \times \vec{P}\right) = \vec{r} \times \frac{d\vec{P}}{dt}$$

$$\frac{d\vec{P}}{dt} = \vec{F}$$

So $\vec{r} \times \dfrac{d\vec{P}}{dt} = \vec{r} \times \vec{F} = \vec{\tau}$

Hence, $\dfrac{d}{dt}\left(\vec{r} \times \vec{P}\right) = \vec{\tau}$

and $\dfrac{d\vec{l}}{dt} = \vec{\tau}$

38. Obtain an expression for the kinetic energy of rotation of a body and then, using that expression define the moment of inertia.

Sol. Linear velocity is angular velocity $\left(\vec{\omega}\right)$, and the relation between $\vec{v}_i$ and $\vec{\omega}$ is given by the relation.

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

The magnitude is given by $\vec{v}_i = \left|\vec{r}_i\right|\left|\vec{\omega}\right|\sin\theta$. The particle of the rigid body moves in a circle of radius r_i and the velocity v_i of the particle is tangent to the circle, so the angle between $\left|\vec{r}\right|$ and $\left|\vec{\omega}\right|$ is 90°.

Hence, $\vec{v}_i = \left|\vec{r}_i\right|\left|\vec{\omega}\right|$ hence equation becomes

$$K_i = \frac{1}{2} m_i r_1^2 \omega^2$$

The total kinetic energy of the body is given by the sum of the kinetic energy K of the body is then given by the sum of the kinetic energies of individual particles

$$K = \sum_{i=1}^{n} K_i = \frac{1}{2} \sum_{i=1}^{n} \left(m_i r_i^2 \omega^2\right)$$

Here, n is the number of particles in the body. And as we know that ω is same for all particles so can be taken out of the summition sign

$$K = \frac{1}{2} \omega^2 \sum_{i=1}^{n} m_i r_i^2$$

$$\Rightarrow K = \frac{1}{2} I \omega^2$$

where $I = \sum_{i=1}^{n} m_i r_i^2$

I is called the moment of inertia of the body about the given axis of rotation.

39. (i) State the theorem of parallel axes.

(ii) A solid sphere and a solid cylinder of the same radius and same mass roll down an inclined plane from the same height h. Which of the two reaches the bottom with maximum velocity?

Sol. (i) Theorem of parallel axis states that the moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axes.

(ii) While rolling down the inclined plane potential energy changes to kinetic energy

Kinetic energy for the rolling is given by

$$\text{K.E.} = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$$

P.E. = mgh

Using the law of conservation of energy, we get

$$mgh = \frac{1}{2}mv^2\left(1+\frac{k^2}{R^2}\right)$$

$$\Rightarrow v^2 = \frac{2gh}{\left(1+\frac{k^2}{R^2}\right)}$$

For solid sphere $k^2 = \frac{2}{5}R^2$

$$\therefore \ v^2 = \frac{2gh}{\left(1+\frac{2}{5}\right)} = \frac{5}{7}\times 2gh$$

$$v_{\text{spin}} = \sqrt{\frac{5}{7}\times 2gh}$$

For solid cylinder, $k^2 = \frac{1}{2}R^2$

$$\therefore \ v^2 = \frac{2gh}{\left(1+\frac{1}{2}\right)} = \frac{2}{3}\times 2gh$$

$$v_{\text{cylinder}} = \sqrt{\frac{2}{3}\times 2gh}$$

$v_{\text{sphere}} > v_{\text{cylinder}}$

Therefore solid sphere reaches the bottom first.

40. (i) State the theorem of perpendicular axis.

(ii) Obtain a relation between torque and moment of inertia of a rigid body.

Sol. (i) Theorems of perpendicular axis for moment of inertia of a planar body states that

$$I_Z = I_X + I_Y$$

where I_X, I_Y and I_Z are the moment of inertia of the body about three mutually perpendicular axis – x-axis, y-axis and z-axis respectively. Here x and y-axis are in the plane of body and z-axis is perpendicular to the plane of body. The three axis intersect at a point. This theorem is applicable for planar bodies only.

(ii) In a perfectly rigid body there is no internal motion therefore, the work done by the external torques is not dissipated and goes on to increase the kinetic energy of the body. The rate of increase of K.E. is given by $\frac{d}{dt}(\text{K.E})$

K.E. of rotation motion K.E. = $\frac{1}{2}I\omega^2$

$$\frac{d}{dt}\left(\frac{I\omega^2}{2}\right) = I\frac{2\omega}{2}\frac{d\omega}{dt}$$

Here, we assume that moment of inertia does not change with time. This means the mass of the body does not change. The body remains rigid and also the axis does not change *i.e.,* its position with respect to the body remains the same.

$$\alpha = \frac{d\omega}{dt}$$

$$\frac{d}{dt}\left(\frac{I\omega^2}{2}\right) = I\omega\alpha$$

Change in kinetic energy is due to work done, and the rate of change of kinetic energy is equal to the rate at which the work is done, by the external torque which given by $\tau\omega$. So on equality the rate of work done and rate of increase of K.E., we get

$$\tau\omega = I\omega\alpha$$

$$\Rightarrow \boxed{\tau = I\alpha}$$

SECTION - B

Model Test Paper

1. What are the dimensions of angular momentum. Give its SI unit. Is it a scalar or a vector?

Sol. The dimensions of angular momentum is $[M^1L^2T^{-1}]$. Its unit is kg m^2s^{-1}. It is a vector quantity.

2. Which physical quantity is represented by the rate of change of angular momentum?

Sol. Torque $(\tau) = \frac{dL}{dt}$

3. Where does the centre of mass of a system having two particles lie if one particle is heavier than the other?

Sol. The centre of mass will be closer to the heavier particle.

4. Where does the centre of mass of a uniform circular ring lie?

Sol. The centre of mass of a uniform circular ring lies at the centre of the ring.

5. Give two factors on which moment of inertia of a body depends.

Sol. Moment of inertia of a body depends on the

(i) Axis of rotation.

(ii) Mass of the body.

6. A dancer stretches her hands out. What will be the change in her angular velocity?

Sol. Angular velocity (ω) decreases because angular momentum is conserved $L = I\omega$, On stretching hands I increases hence, ω decreases.

7. Explain why the moment of inertia of a solid disc is smaller than that of a ring of the same mass and size.

Sol. Because the entire mass of the ring is at its periphery *i.e.* at the maximum distance from the centre where as the mass of the disc is distributed from the centre to the rim.

8. The angular momentum of a system is conserved whose moment of inertia is decreased. What will be the effect on the rotational K.E. of the system?

Sol. $L = I\omega$ = constant

I is decreased, so ω increases

$K.E_{rotation} = \frac{1}{2} I\omega^2$

as ω increases, K.E. increases, hence the kinetic energy of the rotation of the system will increase.

9. State the theorem of perpendicular axes.

Sol. According to the theorem of perpendicular axes, the moment of inertia of a planar body (lamina) about an axis perpendicular to its plane is equal to the sum of its moments of inertia about two perpendicular axes concurrent with perpendicular axis and lying in the plane of the body.

10. The position vector of a particle is $\left(2\hat{i} - 3\hat{j} + \hat{k}\right)$ m. Calculate the $\hat{j}^{\text{th}}$ component of torque of force $\left(\hat{i} - 3\hat{j} + \hat{k}\right)N$ acting on the particle.

Sol. $\vec{r} = \left(2\hat{i} - 3\hat{j} + \hat{k}\right)\text{m}$

$\vec{F} = \left(\hat{i} - 3\hat{j} + \hat{k}\right)\text{N}$

$$\vec{\tau} = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 1 \\ 1 & -3 & 1 \end{vmatrix} = \hat{i}\,(-3+3) - \hat{j}\,(2-1) + \hat{k}\,(-6+3) = 0\hat{i} - \hat{j} - 3\hat{k}$$

$\vec{\tau} = -\hat{j} - 3\hat{k}$

So, the $\hat{j}^{\text{th}}$ component of the torque is – 1.

11. Calculate the angular momentum of a particle having uniform circular motion if the frequency of the particle is doubled and the moment of inertia is halved.

Sol. $L = I\omega$

$\omega = 2\pi\nu$

$\nu' = 2\nu$,

$\omega' = 2\omega$

and $I' = \frac{I}{2}$, $L' = I'\omega'$

$L' = \frac{I}{2} \times 2\omega = I\omega = L$

Hence, the angular momentum of the particle remains the same.

Short Answer Type Questions :

12. A flywheel rotating at the rate of 100 rpm slows down at a constant rate of 2 rad s^{-2}. Calculate the time it will take to stop and the rotation it will make before coming to rest.

Sol. $\omega = 0$, $\omega_0 = 100$ r.p.m. $= \dfrac{2\pi \times 100}{60} = \dfrac{20}{6}\pi$

$\alpha = 2$ rad s^{-2}, $t = ?$, $\theta = ?$

Using $\omega = \omega_0 + \alpha t$

$$t = \frac{\omega - \omega_0}{\alpha} = \frac{-\frac{20}{6}\pi}{-2} = \frac{20}{12}\pi = \frac{5}{3}\pi \text{ s}$$

Using, $\omega^2 - \omega_0^2 = 2\alpha\,\theta$

$$\left(\frac{20}{6}\pi\right)^2 = 2 \times 2\theta$$

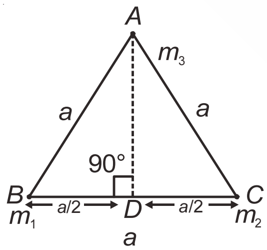
$$\theta = \frac{400\,\pi^2}{36 \times 4} = \frac{100}{36}\pi^2$$

In one rotation, the angular displacement is 2π

Number of rotations $= \dfrac{100\pi^2}{36} \times \dfrac{1}{2\pi} = \dfrac{50}{36}\pi$

13. Three point masses m_1, m_2 and m_3 are located at the vertices of an equilateral triangle of length *a.* Calculate the moment of inertia of the system about an axis along the altitude of the triangle passing through m_3.

Sol.



ABC is an equilateral triangle, axis of rotation is *AD*

$$I = \sum_{i_1=1}^{n} m_i r_i^2$$

$$I = m_1\left(\frac{a}{2}\right)^2 + m_2\left(\frac{a}{2}\right)^2 + m_3(0) \qquad [\because m_3 \text{ lies on the axis of rotation } i.e.\ AD \therefore r = 0]$$

$$\boxed{I = \frac{a^2}{4}(m_1 + m_2)}$$

14. A uniform circular ring of mass 3 kg and radius 20 cm is rotating about an axis passing through its diameter with an angular speed of 200 rpm with the help of a motor. When the motor is switched off, calculate the torque that must be applied to the ring to bring it to rest in 20 revolutions.

Sol. The angular displacement of the ring in 1 revolution = 2π.

So, the angular displacement of the ring in 20 revolutions is $20 \times 2\pi = 40\pi$

Using, $\omega - \omega_0^2 = 2\alpha\theta$

$$\theta = 40\pi \text{ , } \omega_0 = 200 \text{ rpm} = \frac{2\pi \times 200}{60} = \frac{40}{6}\pi \text{ rad s}^{-1}$$

$\omega = 0$

$\alpha = ?$

$$-\left(\frac{40}{6}\pi\right)^2 = 2\alpha 40\pi$$

$$\alpha = \frac{40 \times 40\pi^2}{6 \times 6} \times \frac{1}{2 \times 40\pi}$$

$$= -\frac{20}{36}\pi \text{ rad s}^{-2}$$

Negative sign show that the ring is slowing down $\tau = I\alpha$

Moment of inertia of the ring about an axis passing through its diameter is given by $I = \dfrac{MR^2}{2}$

M = 3 kg, R = 20 cm

$$I = \frac{3}{2} \times \frac{20}{100} \times \frac{20}{100} = \frac{6}{100} \text{ kg m}^2$$

$$\tau = I\alpha = \frac{6}{100} \times \frac{20}{36}\pi = 0.1047 \text{ Nm}$$

15. Shows that the vector $\vec{A} = 2\hat{i} - 3\hat{j} - \hat{k}$ and $\vec{B} = -6\hat{i} + 9\hat{j} + 3\hat{k}$ are parallel.

Sol. $$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & -1 \\ -6 & 9 & 3 \end{vmatrix}$$

$$= (-9+9)\,\hat{i} - \hat{j}\,(6-6) + \hat{k}\,(18-18)$$

$$= 0$$

Since the cross product of $\vec{A}$ and $\vec{B}$ is zero. Therefore, these are parallel vectors.

16. A solid cylinder is rolling on a frictionless plane surface about its axis. Find the ratio of its (i) rotational energy to its total energy and (ii) translational energy to rotational energy.

Sol. Rotational energy K.E. = $\frac{1}{2}I\omega^2$

Translational K.E = $\frac{1}{2}mv^2$

Total energy = $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

for solid cylinder I = $\frac{mr^2}{2}$, $\omega = \frac{v}{r}$

So K.E$_{rot.}$ = $\frac{1}{2} \times \frac{mr^2}{2} \times \frac{v^2}{r^2} = \frac{mv^2}{4}$

So the total K.E. = $\frac{mv^2}{2} + \frac{mv^2}{4} = \frac{3}{4}mv^2$

(i) The ratio of rotational energy to total energy = $\frac{\frac{mv^2}{4}}{\frac{3}{4}mv^2} = 1:3$

(ii) The ratio of translation energy to rotational energy = $\frac{\frac{1}{2}mv^2}{\frac{mv^2}{4}} = 2:1$

17. Explain how by using the principle of conservation of angular momentum a cat is able to land on its feet after a fall.

Sol. While falling a cat stretches its body alongwith the tail so that its moment of inertia (I) increases. As no external torque is acting therefore, its angular momentum is conserved *i.e.* $L = I\omega$ = constant, as I increases, ω decreases and this helps it to land gently on its feet.

Long Answer Type Questions

18. State the law of conservation of angular momentum and prove that torque is given by the product of moment of inertia and angular acceleration and it is the rate of change of angular momentum.

OR

Define torque. Discuss its significance in rotational motion and derive an expression for it.

Sol. (i) If the external forces acting on the system of particles is zero

$\Rightarrow \vec{\tau}_{ext.} = 0$, hence

$\frac{d\vec{L}}{dt}$ = 0

or $\vec{L}$ = constant

Thus, if the total external torque on a system of particles is zero, then the total angular momentum of the system is conserved.

(ii) $\frac{d}{dt}\left(\vec{r}\times\vec{P}\right) = \vec{r}\times\frac{d\vec{P}}{dt}$

$\Rightarrow \frac{d\vec{P}}{dt} = \vec{F}$,

So, $\vec{r}\times\frac{d\vec{P}}{dt} = \vec{r}\times\vec{F}=\vec{\tau}$

Hence, $\frac{d}{dt}\left(\vec{r}\times\vec{P}\right)=\vec{\tau}$

and $\frac{d\vec{l}}{dt}=\vec{\tau}$

OR

(i) **Torque (τ) :** A pair of equal and opposite forces with different lines of action is known as a couple or torque. A couple produces rotation without translation. It can also be defined as the rate of change of angular momentum of a rigid body *i.e.* $\vec{\tau}=\frac{d\vec{L}}{dt}$

(ii) In a perfectly rigid body there is no internal motion therefore, the work done by the external torques is not dissipated and goes on to increase the kinetic energy of the body. The rate of increase of K.E. is given by $\frac{d}{dt}(\text{K.E})$

K.E. of rotational motion K.E. = $\frac{1}{2}I\omega^2$

$$\frac{d}{dt}\left(\frac{I\omega^2}{2}\right) = I\,\frac{2\omega}{2}\frac{d\omega}{dt}$$

Here, we assume that moment of inertia does not change with time. This means the mass of the body does not change. The body remains rigid and also the axis does not change i.e., its position with respect to the body remains the same.

$$\alpha = \frac{d\omega}{dt}$$

$$\frac{d}{dt}\left(\frac{I\omega^2}{2}\right)=I\omega\alpha$$

Change in kinetic energy is due to work done, and the rate of change of kinetic energy is equal to the rate at which the work is done, by the external torque which is given by $\tau\omega$. So on equality, the rate of work done and rate of increase of K.E., we get

$\tau\omega = I\omega\alpha$

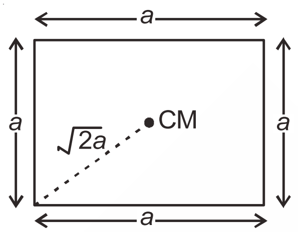
$\boxed{\tau = I\alpha}$

19. State the theorem of parallel axes and using this theorem derive the moment of inertia of a plate of side *a* and mass *m* about an axis perpendicular to its plane and passing through one of its corners.

OR

Define the moment of inertia. A thin uniform rod of mass 6 m is bent to form a regular hexagon of side *a*. What is the moment of inertia of the hexagon about an axis passing through its centre and perpendicular to its plane.

Sol. According to the theorem of parallel axis. The moment of inertia of a body about any axis is equal to the sum of the moment of inertia of the body about a parallel axis passing through its centre of mass and the product of its mass and the square of the distance between the two parallel axis.



The moment of inertia (I) about an axis passing through O.

$$I_{CM} = \frac{m\left(l^2 + b^2\right)}{12}$$

The moment of inertia (I) about an axis passing through O.

$$I = I_{cm} + m\left(\frac{a}{\sqrt{2}}\right)^2$$

$$I = \frac{ma^2}{6} + \frac{ma^2}{2} = \frac{2}{3}ma^2$$

OR

Moment of Inertia : The moment of inertia of a rigid body about a given axis of rotation is the sum of the products of masses of the various particles and squares of their perpendicular distances from the axis of rotation.

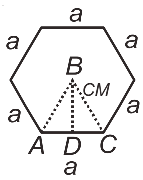
$$I = \sum_{i=1}^{n} m_i r_i^2$$

moment of inertia of a rod about an axis passing through its centre of mass is $\frac{Ml^2}{12}$

Now, rod is bent to form a hexagon of side *a*. A hexagon is formed by equilateral triangles.

So, consider ΔABC altitude, $BD = \frac{\sqrt{3}}{2}a$ which is the distance of the centre of mass of hexagon from the C.M. of the rod AC. The mass of each side of hexagon is 1 kg and moment of inertia is $\frac{ma^2}{12}$, moment of inertia of one side about the centre of mass of the hexagon is given by using the parallel axis theorem

$$I_{CM} = \frac{ma^2}{12} + m\left(\frac{\sqrt{3}}{2}a\right)^2 = \frac{ma^2}{12} + \frac{3ma^2}{4}$$



$$= \frac{ma^2 + 9ma^2}{12} = \frac{10}{12}ma^2$$

m = 1 kg,

So $I_{CM} = \dfrac{10}{12}a^2$

I_{CM} is the moment of inertia of one side about an axis passing C.M. of hexagon. So total moment of inertia of hexagon about this axis is given by

$6 \times \dfrac{10}{12}a^2 = 5a^2$.

So the moment of inertia of cylinder about an axis passing through the centre of mass is $5a^2$.

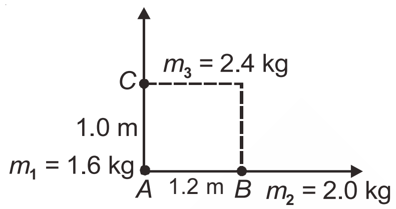
❑ ❑ ❑

Solutions (Set-2)

Objective Type Questions

(Centre of Mass)

1. Three point masses m_1, m_2 and m_3 are placed at the corners of a thin massless rectangular sheet (1.2 m × 1.0 m) as shown. Centre of mass will be located at the point



(1) (0.8, 0.6) m (2) (0.6, 0.8) m (3) (0.4, 0.4) m (4) (0.5, 0.6) m

Sol. Answer (3)

$$x_{cm} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

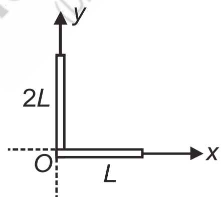
$$y_{cm} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

$$x_{cm} = \frac{(1.6)(0)+(2.4)(0)+2(1.2)}{1.6+2.4+2} = 0.4 \text{ m}$$

$$y_{cm} = \frac{(1.6)(0)+(2.4)1+2(0)}{1.6+2.4+2} = 0.4 \text{ m}$$

So, (x_{cm}, y_{cm}) = (0.4, 0.4) m

2. Figure shows a composite system of two uniform rods of lengths as indicated. Then the coordinates of the centre of mass of the system of rods are



(1) $\left(\frac{L}{2}, \frac{2L}{3}\right)$ (2) $\left(\frac{L}{4}, \frac{2L}{3}\right)$ (3) $\left(\frac{L}{6}, \frac{2L}{3}\right)$ (4) $\left(\frac{L}{6}, \frac{L}{3}\right)$

Sol. Answer (3)

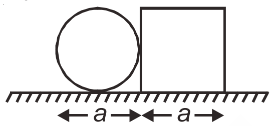
Centre of mass of the uniform rod will lie at its centre

$$x_{cm} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}$$

$$x_{cm} = \frac{m\left(\frac{L}{2}\right) + 2m(0)}{3m}, \quad y_{cm} = \frac{2m(L) + m(0)}{3m}$$

$$x_{cm} = \frac{L}{6} \quad , \quad y_{cm} = \frac{2L}{3}$$

3. A circular plate of diameter '*a*' is kept in contact with a square plate of side *a* as shown. The density of the material and the thickness are same everywhere. The centre of mass of composite system will be



(1) Inside the circular plate (2) Inside the square plate

(3) At the point of contact (4) Outside the system

Sol. Answer (2)

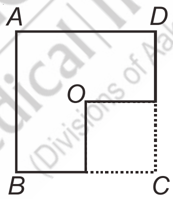
$$x_{cm} = \frac{A_1x_1 + A_2x_2}{A_1 + A_2} \qquad \text{[taking origin at contact point]}$$

$$= \frac{\frac{\pi a^2}{4}\left(\frac{-a}{2}\right) + a^2\left(\frac{a}{2}\right)}{\frac{\pi a^2}{4} + a^2} = \frac{a^3\left(1 - \frac{\pi}{4}\right)}{2a^2\left(a + \frac{\pi}{4}\right)}$$

$$= \frac{a\left(1 - \frac{\pi}{4}\right)}{2\left(1 + \frac{\pi}{4}\right)} > 0$$

$\Rightarrow x_{cm}$ is inside the square plate

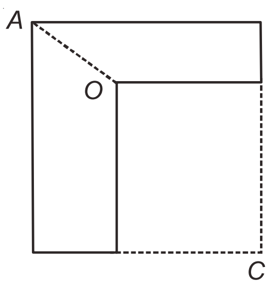
4. From a uniform square plate, one-fourth part is removed as shown. The centre of mass of remaining part will lie on



(1) *OC* (2) *OA* (3) *OB* (4) *OD*

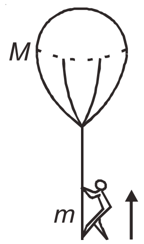
Sol. Answer (2)

Centre of mass will lie on the line of symmetry



OA is the line of symmetry of the remaining part

5. A man of mass m is suspended in air by holding the rope of a balloon of mass M. As the man climbs up the rope, the balloon



(1) Moves upward (2) Moves downward

(3) Remains stationary (4) Cannot say

Sol. Answer (2)

Net external force is zero, and centre of mass of the system is initially at rest. So position of centre of mass will not change. So to have x_{cm} = constant the balloon will move downwards.

6. A man of mass m starts moving on a plank of mass M with constant velocity v with respect to plank. If the plank lies on a smooth horizontal surface, then velocity of plank with respect to ground is

(1) $\frac{Mv}{m+M}$ (2) $\frac{mv}{M}$ (3) $\frac{Mv}{m}$ (4) $\frac{mv}{m+M}$

Sol. Answer (4)

(Motion of Centre of Mass)

7. Two particles A and B initially at rest move towards each other under a mutual force of attraction. At the instant when velocity of A is v and that of B is $2v$, the velocity of centre of mass of the system

(1) v (2) $2v$ (3) $3v$ (4) Zero

Sol. Answer (4)

If $F_{ext} = 0$

v_{cm} is at rest initially so $v_{cm} = 0$

as $F_{ext} = 0$ $\Rightarrow$ $a_{cm} = 0$

8. A shell following a parabolic path explodes somewhere in its flight. The centre of mass of fragments will move in

(1) Vertical direction (2) Any direction (3) Horizontal direction (4) Same parabolic path

Sol. Answer (4)

The path of centre of mass will not change due to internal forces

9. A ball of mass m is thrown upward and another ball of same mass is thrown downward so as to move freely under gravity. The acceleration of centre of mass is

(1) g (2) $\frac{g}{2}$ (3) $2g$ (4) zero

Sol. Answer (1)

$$a_{cm} = \frac{m(-g)+m(-g)}{m+m} \Rightarrow a_{cm} = -g$$

(Torque and Angular Momentum)

10. An angular impulse of 20 Nms is applied to a hollow cylinder of mass 2 kg and radius 20 cm. The change in its angular speed is

(1) 25 rad/s (2) 2.5 rad/s (3) 250 rad/s (4) 2500 rad/s

Sol. Answer (3)

$$20 = 2\left(\frac{1}{2}\right)(2)\left(\frac{20}{100}\right)^2 \Delta\omega \quad \begin{bmatrix}\text{Angular impulse} \\ = \text{Change in angular momentum}\end{bmatrix}$$

$$\Delta\omega = \frac{500}{2} = 250 \text{ rad/s}$$

11. A hollow sphere of mass 1 kg and radius 10 cm is free to rotate about its diameter. If a force of 30 N is applied tangentially to it, its angular acceleration is (in rad/s^2)

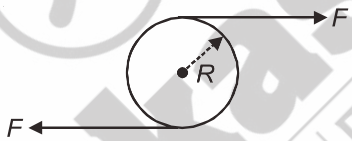
(1) 5000 (2) 450 (3) 50 (4) 5

Sol. Answer (2)

Use $\tau = I\alpha$

$$30\left(\frac{10}{100}\right) = \frac{2}{3}(1)\left(\frac{10}{100}\right)^2 \alpha \Rightarrow \alpha = 450 \text{ rad/s}^2$$

12. Two equal and opposite forces are applied tangentially to a uniform disc of mass M and radius R as shown in the figure. If the disc is pivoted at its centre and free to rotate in its plane, the angular acceleration of the disc is



(1) $\frac{F}{MR}$ (2) $\frac{2F}{3MR}$ (3) $\frac{4F}{MR}$ (4) Zero

Sol. Answer (3)

$$2FR = \frac{1}{2}MR^2\alpha$$

$$\alpha = \frac{4F}{MR}$$

13. A wheel having moment of inertia 4 kg m^2 about its symmetrical axis, rotates at rate of 240 rpm about it. The torque which can stop the rotation of the wheel in one minute is

(1) $\frac{5\pi}{7}$Nm (2) $\frac{8\pi}{15}$Nm (3) $\frac{2\pi}{9}$Nm (4) $\frac{3\pi}{7}$Nm

Sol. Answer (2)

$w = w_0 - \alpha t$

$$\alpha = \frac{w_0}{t} = \frac{2\pi(240)}{60 \times 60}$$

$$\alpha = \frac{8\pi}{60} \text{ rad/s}^2$$

$\tau = I\alpha$

$$= 4\frac{8\pi}{60} = \frac{8\pi}{15} \text{ Nm}$$

14. A force $\vec{F} = (2\hat{i} + 3\hat{j} - 5\hat{k})$ N acts at a point $\vec{r}_1 = (2\hat{i} + 4\hat{j} + 7\hat{k})$ m. The torque of the force about the point $\vec{r}_2 = (\hat{i} + 2\hat{j} + 3\hat{k})\,m$ is

(1) $(17\hat{j} + 5\hat{k} - 3\hat{i})$ Nm

(2) $(2\hat{i} + 4\hat{j} - 6\hat{k})$ Nm

(3) $(12\hat{i} - 5\hat{j} + 7\hat{k})$ Nm

(4) $(13\hat{j} - 22\hat{i} - \hat{k})$ Nm

Sol. Answer (4)

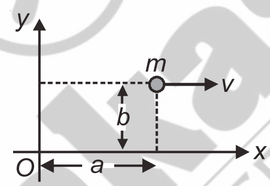
$$\vec{r} = \vec{r}_1 - \vec{r}_2 = (2i + 4j + 7k) - (i + 2j + 3k)$$

$$= \hat{i} + 2\hat{j} + 4\hat{k}$$

$$\vec{F} = 2\hat{i} + 3\hat{j} - 5\hat{k}$$

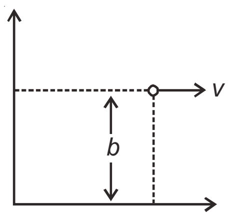
$$\tau = \vec{r} \times \vec{F} = \begin{vmatrix} i & j & k \\ 1 & 2 & 4 \\ 2 & 3 & -5 \end{vmatrix} = -22\hat{i} + 13\hat{j} - \hat{k}$$

15. A particle of mass m is moving with constant velocity v parallel to the x-axis as shown in the figure. Its angular momentum about origin O is



(1) mvb (2) mva (3) $mv\sqrt{a^2 + b^2}$ (4) $mv(a + b)$

Sol. Answer (1)

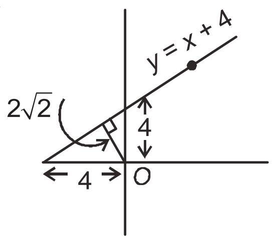


$$|L| = mbv$$

16. A particle of mass 5 kg is moving with a uniform speed $3\sqrt{2}$ in XOY plane along the line $Y = X + 4$. The magnitude of its angular momentum about the origin is

(1) 40 units (2) 60 units (3) Zero (4) $40\sqrt{2}$ units

Sol. Answer (2)

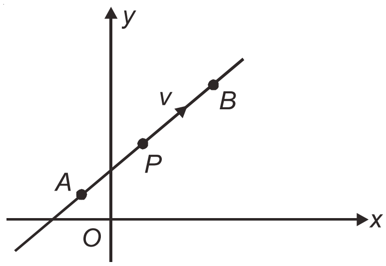


$L = mvr$

$$= (5)\left(3\sqrt{2}\right)\left(2\sqrt{2}\right)$$

= 60 units

17. A particle P is moving along a straight line as shown in the figure. During the motion of the particle from A to B the angular momentum of the particle about O



(1) Increases (2) Decreases

(3) Remains constant (4) First increases and then decreases

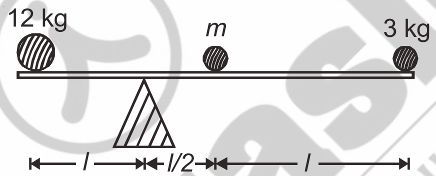
Sol. Answer (3)

$L = mvr_\perp$

$r_\perp$ is constant so L = constant

(Equililbrium of a Rigid Body)

18. For equilibrium of the system, value of mass m should be



(1) 9 kg (2) 15 kg (3) 21 kg (4) 1 kg

Sol. Answer (2)

Net torque = 0 for equilibrium

$$12l = m\left(\frac{l}{2}\right) + 3\left(\frac{3l}{2}\right)$$

$$12\,l - 4.5l = \frac{ml}{2}$$

$$7.5l = \frac{ml}{2}$$

m = 15 kg

(Moment of Inertia)

19. The moment of inertia of a body depends on

(1) The mass of the body

(2) The distribution of the mass in the body

(3) The axis of rotation of the body

(4) All of these

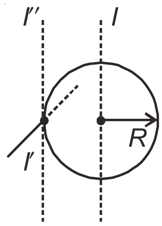
Sol. Answer (4)

$I = mr^2$

20. The moment of inertia of a thin uniform circular disc about one of its diameter is I. Its moment of inertia about an axis tangent to it and perpendicular to its plane is

(1) $\frac{2I}{3}$ (2) $2I$ (3) $\frac{I}{2}$ (4) $6I$

Sol. Answer (4)



$$I = \frac{MR^2}{4}$$

Using parallel axis theorem

$$I'' = \frac{MR^2}{4} + MR^2$$

Now perpendicular axis theorem

$I' = I'' + I$

$$= \frac{MR^2}{4} + MR^2 + \frac{MR^2}{4}$$

$$= \frac{3}{2}MR^2$$

$$= \frac{3}{2}.(4I) = 6I$$

21. The two spheres, one of which is hollow and other solid, have identical masses and moment of inertia about their respective diameters. The ratio of their radii is given by

(1) 5 : 7 (2) 3 : 5 (3) $\sqrt{3} : \sqrt{5}$ (4) 3 : 7

Sol. Answer (3)

$$\frac{2}{3}mr_1^2 = \frac{2}{5}mr_2^2$$

$$\frac{r_1}{r_2} = \sqrt{\frac{3}{5}}$$

22. A circular disc is to be made by using iron and aluminium so that it possesses maximum moment of inertia about geometrical axis. It is possible with

(1) Aluminium at interior and iron surrounding it

(2) Iron at interior surrounded by aluminium

(3) Using iron and aluminium layers in alternate order

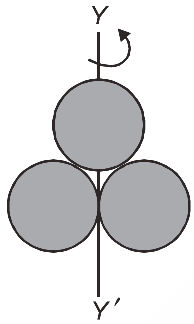
(4) Sheet of iron is used at both external surfaces and aluminium as interior layer

Sol. Answer (1)

Iron is much denser than Aluminium. To have the maximum moment of inertia, material having higher density should be placed farther from the rotational axis.

(Theorems of Perpendicular Axes and parallel Axes)

23. Three solid spheres each of mass P and radius Q are arranged as shown in fig. The moment of inertia of the arrangement about YY' axis



(1) $\frac{7}{5}PQ^2$ (2) $\frac{14}{5}PQ^2$ (3) $\frac{16}{5}PQ^2$ (4) $\frac{5}{14}PQ^2$

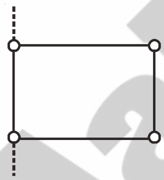
Sol. Answer (3)

$$I = \left(\frac{2}{5}mR^2\right) + \left(\frac{2}{5}mR^2 + mR^2\right) + \left(\frac{2}{5}mR^2 + mR^2\right) = \frac{16}{5}mR^2$$

24. Four spheres of diameter $2a$ and mass M are placed with their centres on the four corners of a square of side b. Then moment of inertia of the system about an axis about one of the sides of the square is

(1) $Ma^2 + 2Mb^2$ (2) Ma^2 (3) $Ma^2 + 4Mb^2$ (4) $\frac{8}{5}Ma^2 + 2Mb^2$

Sol. Answer (4)



$$I = 4\left(\frac{2}{5}Ma^2\right) + 2Mb^2$$

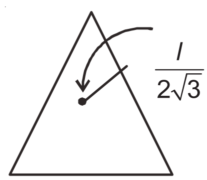
$$= \frac{8}{5}Ma^2 + 2Mb^2$$

25. Three rods each of mass m and length L are joined to form an equilateral triangle as shown in the figure. What is the moment of inertia about an axis passing through the centre of mass of the system and perpendicular to the plane?

(1) $2\ mL^2$ (2) $\frac{mL^2}{2}$ (3) $\frac{mL^2}{3}$ (4) $\frac{mL^2}{6}$

Sol. Answer (2)

Using parallel axis theorem for one rod

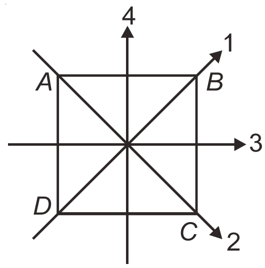


$$I = \frac{ml^2}{12} + m\left(\frac{l}{2\sqrt{3}}\right)^2$$

For all three rods

$$I' = 3I = 3ml^2\left[\frac{1}{12} + \frac{1}{12}\right] = \frac{ml^2}{2}$$

26. The moment of inertia of a thin square plate *ABCD* of uniform thickness about an axis passing through the centre *O* and perpendicular to the plane of the plate is *I*. Which of the following is false?

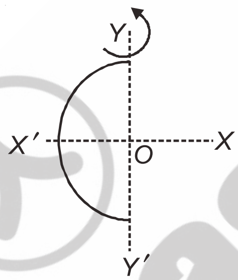


(1) $I = I_1 + I_2$ (2) $I = I_1 + I_3$ (3) $I = I_4 + I_2$ (4) $I = I_1 + I_2 + I_3 + I_4$

Sol. Answer (4)

Using perpendicular axis theorem, $I \neq I_1 + I_2 + I_3 + I_4$

27. A thin wire of length ℓ and mass *m* is bent in the form of a semicircle as shown. Its moment of inertia about an axis joining its free ends will be



(1) $m\ell^2$ (2) Zero (3) $\frac{m\ell^2}{\pi^2}$ (4) $\frac{m\ell^2}{2\pi^2}$

Sol. Answer (4)

$\ell = \pi R \quad \Rightarrow \quad R = \frac{\ell}{\pi}$

$$I = \frac{mr^2}{2}$$

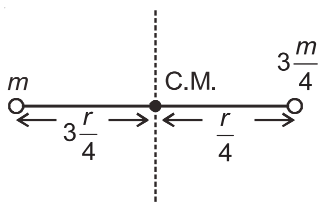
$$I = \frac{1}{2}m\left(\frac{\ell}{\pi}\right)^2 \quad \Rightarrow \quad I = \frac{m\ell^2}{2\pi^2}$$

28. Two point masses *m* and 3*m* are placed at distance *r*. The moment of inertia of the system about an axis passing through the centre of mass of system and perpendicular to the line joining the point masses is

(1) $\frac{3}{5}mr^2$ (2) $\frac{3}{4}mr^2$ (3) $\frac{3}{2}mr^2$ (4) $\frac{6}{7}mr^2$

Sol. Answer (2)

$$I = m\left(\frac{3r}{4}\right)^2 + 3m\left(\frac{r}{4}\right)^2$$



$$= \frac{3}{4}mr^2$$

(Kinematics of Rotational Motion about a Fixed Axis)

29. A wheel starts from rest and attains an angular velocity of 20 radian/s after being uniformly accelerated for 10 s. The total angle in radian through which it has turned in 10 second is

(1) 20 π (2) 40 π (3) 100 (4) 100 π

Sol. Answer (3)

$\omega_f = \omega_i + \alpha t$

$20 = 0 + \alpha(10)$

$\alpha = 2$ rad/s^2

Now, $\theta = \omega_i t + \frac{1}{2}\alpha t^2$

$\theta = \frac{1}{2}(2)(100)$

$= 100$ radian

(Dynamics of Rotational Motion about a Fixed Axis)

30. A meter stick is held vertically with one end on the floor and is allowed to fall. The speed of the other end when it hits the floor assuming that the end at the floor does not slip is (g = 9.8 m/s^2)

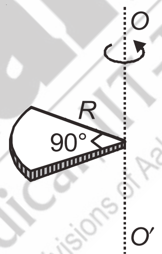
(1) 3.2 m/s (2) 5.4 m/s (3) 7.6 m/s (4) 9.2 m/s

Sol. Answer (2)

$$\frac{mgl}{2} = \frac{1}{2}\frac{ml^2}{3}\omega^2$$

$$\omega^2 = \frac{3g}{\ell} \Rightarrow \omega = \sqrt{\frac{3g}{\ell}} = \sqrt{30}$$

31. A quarter disc of radius R and mass m is rotating about the axis OO' (perpendicular to the plane of the disc) as shown. Rotational kinetic energy of the quarter disc is



(1) $\frac{1}{2}mR^2\omega^2$ (2) $\frac{1}{4}mR^2\omega^2$

(3) $\frac{1}{8}mR^2\omega^2$ (4) $\frac{1}{16}mR^2\omega^2$

Sol. Answer (2)

$$k = \frac{1}{2}I\omega^2$$

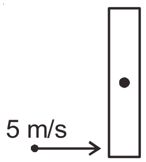
$$= \frac{1}{2}\left(\frac{mr^2}{2}\right)\omega^2$$

$$k = \frac{1}{4}mr^2\omega^2$$

32. A metre stick is pivoted about its centre. A piece of wax of mass 20 g travelling horizontally and perpendicular to it at 5 m/s strikes and adheres to one end of the stick so that the stick starts to rotate in a horizontal circle. Given the moment of inertia of the stick and wax about the pivot is 0.02 kg m^2, the initial angular velocity of the stick is

(1) 1.58 rad/s (2) 2.24 rad/s (3) 2.50 rad/s (4) 5.00 rad/s

Sol. Answer (3)



$L = I\omega$

$$\frac{20}{1000} \times 5 \times \frac{l}{2} = 0.02\,\omega \Rightarrow \omega = 2.5 \text{ rad/s}$$

33. A circular disc of mass 2 kg and radius 10 cm rolls without slipping with a speed 2 m/s. The total kinetic energy of disc is

(1) 10 J (2) 6 J (3) 2 J (4) 4 J

Sol. Answer (2)

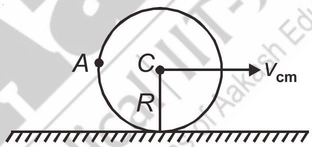
$$k = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}\frac{ml^2}{2}\cdot\frac{v^2}{l^2}$$

$$= \frac{3}{4}(2)(2)^2$$

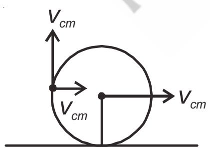
$$= 6 \text{ J}$$

34. In case of pure rolling, what will be the velocity of point A of the ring of radius R ?



(1) v_{cm} (2) $\sqrt{2}\,v_{cm}$ (3) $\frac{v_{cm}}{2}$ (4) $2v_{cm}$

Sol. Answer (2)



$$v_{net} = \sqrt{v_{cm}^2 + v_{cm}^2}$$

$$= v_{cm}\sqrt{2}$$

35. A thin circular ring of mass M and radius r is rotating about its axis with a constant angular velocity ω. Two objects, each of mass m are attached gently to the opposite ends of a diameter of the ring. The ring now rotates with an angular velocity

(1) $\frac{\omega M}{m+M}$ (2) $\frac{\omega(M-2m)}{M+2m}$ (3) $\frac{\omega M}{M+2m}$ (4) $\omega\left(\frac{M+2m}{M}\right)$

Sol. Answer (3)

$I_1\omega_1 = I_2\omega_2$

$Mr^2\omega = (M + 2m)\, r^2\omega'$

$$\omega' = \frac{M\omega}{M+2m}$$

36. A horizontal disc rotating freely about a vertical axis through its centre makes 90 revolutions per minute. A small piece of wax of mass *m* falls vertically on the disc and sticks to it at a distance *r* from the axis. If the number of revolutions per minute reduce to 60, then the moment of inertia of the disc is

(1) mr^2 (2) $\frac{3}{2}mr^2$ (3) $2\,mr^2$ (4) $3\,mr^2$

Sol. Answer (3)

$$\omega_1 = \frac{2\pi 90}{60} = 3\pi \text{ rps}$$

$$\omega_2 = \frac{2\pi 60}{60} = 2\pi \text{ rps}$$

Using angular momentum conservation

$I(1.5) = (I + mr^2)\,(1)$

$$\frac{I}{2} = mr^2$$

$$I = 2mr^2$$

37. If two discs of moment of inertia I_1 and I_2 rotating about collinear axis passing through their centres of mass and perpendicular to their plane with angular speeds ω_1 and ω_2 respectively in opposite directions are made to rotate combinedly along same axis, then the magnitude of angular velocity of the system is

(1) $\frac{I_1\omega_1 + I_2\omega_2}{I_1+I_2}$ (2) $\frac{I_1\omega_1 - I_2\omega_2}{I_1+I_2}$ (3) $\frac{I_1\omega_1 + I_2\omega_2}{\omega_1+\omega_2}$ (4) $\frac{I_1\omega_1 - I_2\omega_2}{\omega_1-\omega_2}$

Sol. Answer (2)

Using angular momentum conservation, $I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\omega$

$$\omega = \frac{I_1\omega_1 + I_2\omega_2}{I_1+I_2}$$

(Angular Momentum in Case of Rotation about a Fixed Axis)

38. The angular momentum of a particle performing uniform circular motion is *L*. If the kinetic energy of partical is doubled and frequency is halved, then angular momentum becomes

(1) $\frac{L}{2}$ (2) $2L$ (3) $\frac{L}{4}$ (4) $4L$

Sol. Answer (4)

$L = I\omega$

$$K = \frac{1}{2}I\omega^2$$

$$2K = \frac{1}{2}I'\left(\frac{\omega}{2}\right)^2$$

$$\frac{1}{2} = \frac{I}{I'}(4)$$

$I' = 8I$

$$L' = (8I)\left(\frac{\omega}{2}\right) = 4I\omega$$

$\Rightarrow L' = 4\ L$

39. If torque acting upon a system is zero, the quantity that remains constant is

(1) Force (2) Linear momentum (3) Angular momentum (4) Angular velocity

Sol. Answer (3)

40. A constant torque acting on a uniform circular wheel changes its angular momentum from A_0 to $4A_0$ in 4 seconds. The magnitude of this torque is

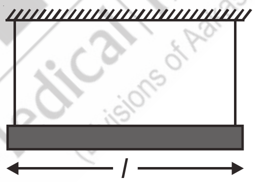
(1) $\frac{3A_0}{4}$ (2) A_0 (3) $4A_0$ (4) $12A_0$

Sol. Answer (1)

$$\tau = \frac{\Delta L}{\Delta t}$$

$$\tau = \frac{4A_0 - A_0}{4} = \frac{3A_0}{4}$$

41. A uniform rod of mass m and length l is suspended by two strings at its ends as shown. When one of the strings is cut, the rod starts falling with an initial angular acceleration



(1) $\frac{g}{l}$ (2) $\frac{g}{2l}$ (3) $\frac{3g}{2l}$ (4) $\frac{3g}{4l}$

Sol. Answer (3)

$\tau = I\alpha$

$$\frac{mgl}{2} = \frac{ml^2}{3}\alpha$$

$$\alpha = \frac{3g}{2l} \quad \text{and} \quad a = \alpha r$$

$$= \frac{l}{2}.\frac{3g}{2l} = \frac{3g}{4}$$

42. Two like parallel forces 20 N and 30 N act at the ends *A* and *B* of a rod 1.5 m long. The resultant of the forces will act at the point

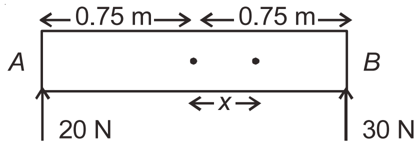
(1) 90 cm from *A* (2) 75 cm from *B* (3) 20 cm from *B* (4) 85 cm from *A*

Sol. Answer (1)

Net torque should be same for the new point

20(0.75) + 30(0.75) = 50(*x*)

Solve for *x*



(Rolling Motion)

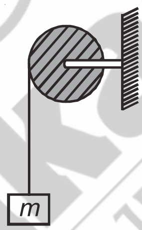
43. A solid sphere, a spherical shell, a ring and a disc of same radius and mass are allowed to roll down an inclined plane without slipping. The one which reaches the bottom first is

(1) Solid sphere (2) Spherical shell (3) Ring (4) Disc

Sol. Answer (1)

Body of smaller $\frac{K^2}{R^2}$ will take less time so solid sphere will reach the ground first.

44. A disc of mass *m* and radius *r* is free to rotate about its centre as shown in the figure. A string is wrapped over its rim and a block of mass *m* is attached to the free end of the string. The system is released from rest. The speed of the block as it descends through a height *h*, is



(1) $\sqrt{2gh}$ (2) $\sqrt{\frac{2}{3}gh}$ (3) $2\sqrt{\frac{gh}{3}}$ (4) $\frac{1}{2}\sqrt{3gh}$

Sol. Answer (3)

Using mechanical energy conservation

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{1}{2}\right)mr^2 . \frac{v^2}{r^2}$$

$$mgh = \frac{3}{4}mv^2$$

$$v^2 = \frac{4gh}{3}$$

$$v = \sqrt{\frac{4gh}{3}}$$

45. When a body is rolling without slipping on a rough horizontal surface, the work done by friction is

(1) Always zero (2) May be zero (3) Always positive (4) Always negative

Sol. Answer (1)

46. A solid spherical ball is rolling without slipping down an inclined plane. The fraction of its total energy associated with rotation is

(1) $\frac{2}{5}$ (2) $\frac{2}{7}$ (3) $\frac{3}{5}$ (4) $\frac{3}{7}$

Sol. Answer (2)

$$K_R = \frac{1}{2}\left(\frac{2}{5}mr^2\right)\frac{v^2}{r^2}$$

$$K_R = \frac{1}{5}mv^2$$

$$K_{Tr} = \frac{1}{2}mv^2$$

$$K_{Total} = \frac{1}{5}mv^2 + \frac{1}{2}mv^2$$

$$= \frac{7}{10}mv^2$$

$$\frac{K_R}{K_{Total}} = \frac{\frac{1}{5}mv^2}{\frac{7}{10}mv^2} = \frac{2}{7}$$

47. A solid cylinder of mass *M* and radius *R* rolls down an inclined plane of height *h* without slipping. The speed of its centre of mass when it reaches the bottom is

(1) $\sqrt{2gh}$ (2) $\sqrt{\frac{4}{3}gh}$ (3) $\sqrt{\frac{3}{4}gh}$ (4) $\sqrt{\frac{4g}{h}}$

Sol. Answer (2)

$$Mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}mv^2 + \frac{1}{2}\frac{ml^2}{2}.\frac{v^2}{l^2}$$

Solving,

$$v = \sqrt{\frac{4gh}{3}}$$

48. An inclined plane makes an angle of 30° with the horizontal. A solid sphere rolling down this inclined plane from rest without slipping has a linear acceleration equal to

(1) $\frac{g}{3}$ (2) $\frac{2g}{3}$ (3) $\frac{5g}{7}$ (4) $\frac{5g}{14}$

Sol. Answer (4)

$\tau = I\alpha$

$$\alpha = \frac{\tau}{I} = \frac{mgr\sin\theta}{\frac{2}{5}mr^2 + mr^2}$$

$$a = \alpha r = \frac{mgr^2\sin\theta}{\frac{2}{5}mr^2 + mr^2}$$

$$a = \frac{5g\sin 30^\circ}{7} = \frac{5g}{14}$$

49. An object slides down a smooth incline and reaches the bottom with velocity v. If same mass is in the form of a ring and it rolls down an inclined plane of same height and angle of inclination, then its velocity at the bottom of inclined plane will be

(1) v (2) $\frac{v}{\sqrt{2}}$ (3) $2v$ (4) $\sqrt{2}\,v$

Sol. Answer (2)

$$v = \sqrt{2gh}$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}mr^2\frac{v^2}{r^2}$$

$$v' = \sqrt{gh}$$

$$v' = \frac{v}{\sqrt{2}}$$

50. A swimmer while jumping into river from a height easily forms a loop in air if

(1) He pulls his arms and legs in (2) He spreads his arms and legs

(3) He keeps himself straight (4) None of these

Sol. Answer (1)

Using angular momentum conservation, by pulling his arms and legs in, moment of inertia will decrease hence ω will increase.

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