



# Global Behavior of an SEIRS Epidemic Model with Time Delays

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**Abstract**—This is a study of dynamic behavior of an SEIRS epidemic model with time delays. It is shown that disease-free equilibrium is globally stable if the reproduction number is not greater than one. When the reproduction number is greater than 1, it is proved that the disease is uniformly persistent in the population, and explicit formulae are obtained by which the eventual lower bound of the fraction of infectious individuals can be computed. Local stability of endemic equilibrium is also discussed. © 2002 Elsevier Science Ltd. All rights reserved.

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## 1. INTRODUCTION

The stability of epidemic models has been studied in many papers. But most of them are concerned with local stability of equilibria. The fraction of papers that obtain global stability of these models is relatively low, especially the models with time delays. Beretta and Takeuchi [1,2] studied the global stability of an SIR epidemic model with time delays. Recently, Takeuchi, Ma and Beretta [3] considered a delayed SIR epidemic model with finite incubation time. They proved that the endemic equilibrium is globally stable if the length of incubation time is small.

Cooke and Driessche in [4] proposed an SEIRS model with two time delays. They presented local stability analysis of disease-free equilibrium and obtained sufficient conditions for global stability of this equilibrium. They also considered local stability of endemic equilibrium of the model for two cases, the SEIS model and the SIRS model. By neglecting disease-related death rates in the SEIRS model in [4], we obtain the following SEIRS epidemic model:

$$\begin{aligned}s' &= b - \lambda s(t)i(t) + \beta i(t - \tau) - bs(t), \\ i' &= \lambda \alpha s(t - \omega)i(t - \omega) - (\gamma + b)i(t),\end{aligned}\tag{1.1}$$

where  $s$  is the fraction of susceptible individuals,  $i$  is the fraction of infectious individuals,  $b$  is the birth rate and death rate of the population,  $\lambda$  is the average number of adequate contacts of an

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infectious individual per unit time,  $\omega$  is the latent period of the disease,  $\tau$  is the immune period of the population,  $\gamma$  is the recovery rate of infectious individuals,  $\alpha = \exp(-b\omega)$ , and  $\beta = \gamma e^{-b\tau}$ . Suppose  $e(t)$  is the fraction of exposed individuals. Since the population is divided into susceptible individuals, exposed individuals, and infectious individuals, we have  $s(t) + e(t) + i(t) = 1$ .

If  $\tau = 0$  and  $R_0 = \lambda\alpha/(\gamma + b) > 1$ , it is shown in [4] that (1.1) has a unique positive equilibrium which is locally stable. If  $\tau > 0$ , a sufficient condition for global stability of the disease-free equilibrium was obtained in [4], but local stability and global stability of endemic equilibrium of (1.1) were untouched there. In this paper, we will study the asymptotic behavior of (1.1) in the case where  $\tau > 0$ . We will give a new proof that  $R_0 \leq 1$  implies that the disease-free equilibrium is globally stable. By this method, the dynamical behavior of the model at critical case  $R_0 = 1$  is also clear. We will show that  $R_0 > 1$  implies that the disease will be uniformly persistent. When the disease is uniformly persistent, we give explicit formulae by which the eventual lower bound of the fraction of infectious individuals is computed. We also study the local stability of endemic equilibrium of the model.

## 2. MAIN RESULT

Let us begin by stating certain notions and notations that will be used throughout this paper.

Set  $T = \max\{\tau, \omega\}$ . Let  $C$  be the space of continuous functions on  $[-T, 0]$  with uniform norm, and use the notation  $x_t = x(t + \theta)$  for  $\theta \in [-T, 0]$ . Due to our background, we will consider (1.1) in the set

$$D \triangleq \{(\phi_1, \phi_2) : 0 \leq \phi_i(\theta) \leq 1, \theta \in [-T, 0], i = 1, 2\}.$$

Using the fact that  $s(t) + e(t) + i(t) \equiv 1$ , it is easy to show that  $D$  is positively invariant for (1.1).

It is clear that  $(1, 0)$  is the disease-free equilibrium of (1.1). Let us consider global stability of this equilibrium.

**THEOREM 2.1.** *If  $R_0 \leq 1$ , the disease-free equilibrium  $(1, 0)$  is globally stable.*

**PROOF.** Note that the second equation of (1.1) can be rewritten as

$$\begin{aligned} i' &= \lambda\alpha s(t)i(t) - (\gamma + b)i(t) + \lambda\alpha[s(t - \omega)i(t - \omega) - s(t)i(t)] \\ &= i(t)[\lambda\alpha s(t) - (\gamma + b)] - \lambda\alpha \frac{d}{dt} \int_{t-\omega}^t (s(u)i(u)) du. \end{aligned} \quad (2.1)$$

Suppose  $x_t = (s(t + \theta), i(t + \theta))$ ,  $\theta \in [-T, 0]$ . Let us define

$$V_0(x_t) = i(t) + \lambda\alpha \int_{t-\omega}^t s(\theta)i(\theta) d\theta.$$

The derivative of  $V_0$  along solutions of (1.1) is

$$V_0'(x_t) = i(t)[\lambda\alpha s(t) - (\gamma + b)] = (\gamma + b)i(t)[R_0 s(t) - 1].$$

Let  $S = \{\phi \in D : \dot{V}_0(\phi) = 0\}$  and let  $M$  be the maximal invariant set of in  $S$  with respect to equation (1.1).

If  $R_0 < 1$ , since  $s(t) \leq 1$ , we have

$$V_0'(x_t) \leq (\gamma + b)(R_0 - 1)i(t) \leq 0.$$

It follows from (1.1) that  $M = (1, 0)$ . If  $R_0 = 1$ , we have

$$V_0'(x_t) = (\gamma + b)i(t)[s(t) - 1].$$

It follows from  $s(t) + e(t) + i(t) \equiv 1$  that  $M = (1, 0)$ . Consequently, Theorem 3.1 in [5, p. 143] or Theorem 5.3 in [6, p. 30] imply that the disease-free equilibrium is globally stable in  $D$ . The proof is complete.

REMARK. Theorem 3.2 in [4], applicable in the more general case, states that  $R_0 < e^{-b\omega}$  guarantees that the disease-free equilibrium is globally stable. Reference [4] also pointed out that the disease-free equilibrium is globally stable when  $R_0 < 1$  and  $b < d$  where  $d$  is the natural death rate of the population. Since we consider a special case where the disease-related death rate is neglected, by means of LaSalle invariant principle, we are sure that the disease-free equilibrium is globally stable whenever  $R_0 \leq 1$ .

Let us consider the case where  $R_0 > 1$ .

THEOREM 2.2. Suppose  $R_0 > 1$ . Then there is a positive constant  $\epsilon$  such that each positive solution  $(s(t), i(t))$  of (1.1) satisfies

$$i(t) \geq \epsilon, \quad \text{if } t \text{ is large.}$$

PROOF. Let us consider a positive solution  $(s(t), i(t))$  of (1.1). According to this solution, we define

$$V_1(t) = i(t) + \lambda\alpha \int_{t-\omega}^t s(\theta)i(\theta) d\theta.$$

Then

$$V_1'(t) = i(t)[\lambda\alpha s(t) - (\gamma + b)] = (\gamma + b)i(t)(R_0 s(t) - 1). \quad (2.2)$$

Since  $R_0 > 1$ , we have  $I_1 \equiv (bR_0/\lambda)(1 - 1/R_0) > 0$ . We claim that for any  $t_0 > 0$ , it is impossible that  $i(t) \leq I_1/2$  for all  $t \geq t_0$ . Suppose the contrary. Then there is a  $t_0 > 0$  such that  $i(t) \leq I_1/2$  for all  $t \geq t_0$ . It follows from the first equation of (1.1), that for  $t \geq t_0$ ,

$$s'(t) > b - \left(\lambda \frac{I_1}{2} + b\right) s(t),$$

which implies

$$\begin{aligned} s(t) &> e^{-(\lambda I_1/2 + b)(t-t_0)} \left[ s(t_0) + b \int_{t_0}^t e^{(\lambda I_1/2 + b)(\theta-t_0)} d\theta \right] \\ &> \frac{b}{\lambda I_1/2 + b} \left( 1 - e^{-(\lambda I_1/2 + b)(t-t_0)} \right), \end{aligned} \quad (2.3)$$

where  $0 < s(t_0)$  is used. Since

$$\frac{b}{\lambda I_1/2 + b} = \frac{2}{R_0 + 1},$$

we have

$$s(t) > \frac{2}{R_0 + 1} \left( 1 - e^{-(\lambda I_1/2 + b)(t-t_0)} \right). \quad (2.4)$$

Choose  $T_1 > 0$  such that

$$\frac{1}{4} \left( 1 - \frac{1}{R_0} \right) = e^{-(\lambda I_1/2 + b)T_1}. \quad (2.5)$$

Then (2.4) implies

$$s(t) > \frac{3R_0 + 1}{2R_0(R_0 + 1)} \triangleq \bar{R}, \quad \text{for } t \geq t_0 + T_1. \quad (2.6)$$

It is easy to see  $\bar{R} > 1/R_0$ . Then, by (2.2), we have

$$V_1'(t) > (\gamma + b)i(t) (\bar{R}R_0 - 1), \quad \text{for } t \geq t_0 + T_1. \quad (2.7)$$

Set

$$\underline{i} = \min_{\theta \in [-\omega, 0]} i(t_0 + T_1 + \omega + \theta).$$

We will show that  $i(t) \geq \underline{i}$  for all  $t \geq t_0 + T_1$ . Suppose the contrary. Then there is a  $T_2 \geq 0$  such that  $i(t) \geq \underline{i}$  for  $t_0 + T_1 \leq t \leq t_0 + T_1 + \omega + T_2$ ,  $i(t_0 + T_1 + \omega + T_2) = \underline{i}$ , and  $i'(t_0 + T_1 + \omega + T_2) \leq 0$ . However, the second equation of (1.1) and (2.6) imply, that for  $t = t_0 + T_1 + \omega + T_2$ ,

$$i'(t) \geq [\lambda \alpha s(t - \omega) - (\gamma + b)] \underline{i} > (\gamma + b) [R_0 \bar{R} - 1] \underline{i} > 0.$$

This is a contradiction. Thus,  $i(t) \geq \underline{i}$  for all  $t \geq t_0 + T_1$ . As a consequence, (2.7) leads to

$$V_1'(t) > (\gamma + b) \underline{i} (R_0 \bar{R} - 1), \quad \text{for } t \geq t_0 + T_1,$$

which implies that as  $t \rightarrow \infty$ ,  $V_1(t) \rightarrow \infty$ . This contradicts  $V_1(t) \leq 1 + \lambda \alpha \omega$ . Hence, the claim is proved.

By the claim, we are left to consider two possibilities. First,  $i(t) \geq I_1/2$  for all large  $t$ . Second,  $i(t)$  oscillates about  $I_1/2$  for all large  $t$ . Define

$$I_2 = \frac{I_1}{2} e^{-(\gamma+b)(T_1+\omega)}. \quad (2.8)$$

We hope to show that  $i(t) \geq I_2$  for all large  $t$ . The conclusion is evident in the first case. For the second case, let  $t_1$  and  $t_2$  satisfy

$$\begin{aligned} i(t_1) &= i(t_2) = \frac{I_1}{2}, \\ i(t) &< \frac{I_1}{2}, \quad \text{for } t_1 < t < t_2. \end{aligned}$$

If  $t_2 - t_1 \leq T_1 + \omega$ , since  $i'(t) > -(\gamma + b)i(t)$  and  $i(t_1) = I_1/2$ , it is obvious that  $i(t) \geq I_2$  for  $t_1 < t < t_2$ . If  $t_2 - t_1 \geq T_1 + \omega$ , by the second equation of (1.1), we obtain  $i(t) \geq I_2$  for  $t \in [t_1, t_1 + T_1 + \omega]$ . Then, proceeding exactly as the proof for above claim, we see that  $i(t) \geq I_2$  for  $t_1 + T_1 + \omega \leq t \leq t_2$ . Consequently,  $i(t) \geq I_2$  for  $t \in [t_1, t_2]$ . Since this kind of interval  $[t_1, t_2]$  is chosen in an arbitrary way (we only need  $t_1$  and  $t_2$  to be large), we conclude that  $i(t) \geq I_2$  for all large  $t$  in the second case. In view of our above discussions, the choices of  $T_1$  and  $I_2$  are independent of the positive solution, and we actually have proved that any positive solution of (1.1) satisfies  $i(t) \geq I_2$  for all large  $t$ . The proof is complete.

Theorems 2.1 and 2.2 show that  $R_0$  is the reproduction number of the model. If  $R_0 \leq 1$ , the disease will disappear. If  $R_0 > 1$ , the disease will be uniformly persistent. By the proof of Theorem 2.2, we actually obtain explicit formulae used for computation of an eventual lower bound of infectious fraction.

EXAMPLE. If  $b = 0.2$ ,  $\gamma = 0.4$ ,  $\lambda = 8$ ,  $\omega = 1$ , then  $\alpha = \exp(-b\omega) = 0.8187307531$ ,  $R_0 = 10.91641004$ . By (2.5), we obtain  $T_1 = 1.243974114$ . It follows from (2.8) that  $I_2 = 0.03225055142$ . Thus, when  $t$  is large, the fraction of infectious population is at least 0.03225055142.

Note that large  $\gamma$  or large  $b$  or large  $\omega$  implies  $R_0 < 1$ . The disease will disappear if  $\gamma$ ,  $b$ , or  $\omega$  is large. This means that the disease will be persistent only if all the  $b$ ,  $\omega$ ,  $\gamma$  are not large.

If  $R_0 > 1$ , (1.1) has an endemic equilibrium

$$(s^*, i^*) = \left( \frac{\gamma + b}{\lambda \alpha}, \frac{b(\lambda \alpha - b - \gamma)}{\lambda(\gamma + b - \beta \alpha)} \right).$$

Let us consider the local asymptotical stability of this endemic equilibrium.

**THEOREM 2.3.** Suppose  $R_0 > 1$ . Then the endemic equilibrium  $(s^*, i^*)$  is asymptotically stable if

$$\alpha(-\beta + \lambda) > \omega[\beta\alpha(\lambda\alpha - b - \gamma)(1 + \tau) + (\gamma + b)(\gamma + b - \beta\alpha)]. \quad (2.9)$$

**PROOF.** Linearizing (1.1) at  $(s^*, i^*)$ , we obtain

$$\begin{aligned} \frac{dx(t)}{dt} &= -(b + \lambda i^*)x(t) - \lambda s^* y(t) + \beta y(t - \tau), \\ \frac{dy(t)}{dt} &= -(\gamma + b)y(t) + \lambda \alpha i^* x(t - \omega) + \lambda \alpha s^* y(t - \omega). \end{aligned} \quad (2.10)$$

The characteristic equation of this system is

$$\begin{aligned} H(z) \equiv z^2 + z\gamma + 2zb - z\lambda\alpha s^* \exp(-z\omega) + \lambda i^* z + \lambda i^* \gamma + \lambda i^* b + b\gamma + b^2 \\ - b\lambda\alpha s^* \exp(-z\omega) - \lambda \alpha i^* \exp(-z\omega)\beta \exp(-z\tau) = 0. \end{aligned} \quad (2.11)$$

If  $\tau = 0$ , Theorem 5.2 in [4] shows that all of roots of (2.11) have negative real parts. By the same arguments as those in [4], instability can occur for  $\tau > 0$  only by roots crossing the finite imaginary axis. Setting  $z = 0$ , and using

$$i^* = \frac{b(\lambda\alpha - b - \gamma)}{\lambda(\gamma + b - \beta\alpha)},$$

we obtain

$$H(0) = (1 - s^*)b\lambda\alpha > 0.$$

Thus,  $z = 0$  is not a root of the characteristic equation (2.11). Suppose that (2.11) has purely imaginary root  $z = Ix$ ,  $x > 0$  (here  $I$  is the imaginary unit). Then the imaginary part of the characteristic equation is

$$\begin{aligned} x\gamma + 2xb - x\lambda\alpha s^* \cos(x\omega) + \lambda i^* x + b\lambda\alpha s^* \sin(x\omega) \\ + \lambda \alpha i^* \sin(x\omega)\beta \cos(x\tau) + \lambda \alpha i^* \cos(x\omega)\beta \sin(x\tau) = 0. \end{aligned} \quad (2.12)$$

After dividing by  $x$ , this can be rearranged as

$$\begin{aligned} \gamma + 2b - \lambda\alpha s^* \cos(x\omega) + \lambda i^* + b\lambda\alpha s^* \omega \frac{\sin(x\omega)}{\omega x} \\ + \lambda \alpha i^* \beta \omega \cos(x\tau) \frac{\sin(x\omega)}{\omega x} + \lambda \alpha i^* \cos(x\omega) \beta \tau \frac{\sin(x\tau)}{x\tau} = 0. \end{aligned} \quad (2.13)$$

Notice that

$$|\cos(x\omega)| \leq 1, \quad \frac{\sin(x\tau)}{x\tau} \geq -1, \quad \frac{\sin(x\omega)}{x\omega} \geq -1.$$

We have

$$\gamma + 2b + \lambda i^* - \lambda\alpha s^* - \omega[b\lambda\alpha s^* + \lambda \alpha i^* \beta(1 + \tau)] \leq 0.$$

Since (2.9) is equivalent to

$$\gamma + 2b + \lambda i^* - \lambda\alpha s^* - \omega[b\lambda\alpha s^* + \lambda \alpha i^* \beta(1 + \tau)] > 0,$$

we have a contradiction. Hence, the equilibrium is asymptotically stable. The proof is complete.

This theorem essentially means that small  $\omega$  or large  $\lambda$  implies that the endemic equilibrium is asymptotically stable.

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