BioSystems 98 (2009) 43–50

Contents lists available at ScienceDirect

BioSystems

journal homepage: www.elsevier.com/locate/biosystems

Optimal treatment of an *SIR* epidemic model with time delay

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ARTICLE INFO

Article history:
Received 5 December 2008
Received in revised form 4 April 2009
Accepted 13 May 2009

JEL classification:
92D25
49J15
93D20

Keywords:
Epidemic model
Time-delay
Optimality
Existence
Numerical simulation
Ebola outbreak

ABSTRACT

In this paper the optimal control strategies of an *SIR* (susceptible–infected–recovered) epidemic model with time delay are introduced. In order to do this, we consider an optimally controlled *SIR* epidemic model with time delay where a control means treatment for infectious hosts. We use optimal control approach to minimize the probability that the infected individuals spread and to maximize the total number of susceptible and recovered individuals. We first derive the basic reproduction number and investigate the dynamical behavior of the controlled *SIR* epidemic model. We also show the existence of an optimal control for the control system and present numerical simulations on real data regarding the course of Ebola virus in Congo. Our results indicate that a small contact rate(probability of infection) is suitable for eradication of the disease (Ebola virus) and this is one way of optimal treatment strategies for infectious hosts.

1. Introduction

Mathematical modeling of population dynamics is a fast growing research area, which has been playing important roles in discovering the relation between species and their interaction. The basic and important concern for mathematical models in epidemiology is qualitative analysis; the persistence, permanence, asymptotic stability, and the existence and uniqueness for the models. Many influential results related in these topics have been established and can be found in many articles and books. An epidemic model for the spread of infectious disease was first introduced by Kermack and Mckendrick (1927). In their model the populations are subdivided into three classes; the susceptible, infected, and recovered populations. They assumed that susceptible populations in a fixed total population become infected by contact with infected individuals, infected individuals either die or recover at a constant rate. The model consists of three ordinary differential equations (ODEs) which represent the rate of change in their respective population.

Extensions of the Kermack–McKendrick model to populations in which the individuals have mobility in an environment have also been studied. In the classical epidemiological model (Brauer and Castillo-Chavez, 2001), a population of total size (N) was divided into susceptible individuals (S), infected individuals (I), and recovered individuals (R). The relation between these three categories leads to the classical *SIR* model. Several epidemic models on theoretical developments are given in Milner and Pugliese (1999), Linda and Amy (2000), Tuckwell and Toubiana (2007) and Zaman et al. (2007, 2008).

In recent years, some mathematical models incorporating delayed effects have been studied. Smith in (1994) and Smith and Thieme in (1990) derived a scalar delayed differential equation for the population with immature and mature age classes. The maturation period was regarded as a time delay. Using the same idea, a system of delayed differential equations for mature population in a patchy environment has been proposed in So et al. (2001). More recent studies consider an epidemic model with density dependence to describe disease transmission in variable population size, which can be found in (see, Cooke et al., 1999; Hethcote and van den Driessche, 2000; Ma et al., 2003; Bachar and Dorfmayr, 2004). Takeuchi et al. (2000) and Ma et al. (2002)studied the *SIR* infectious disease model in which an infectious disease is transmitted by a vector after an incubation time. In their models, they assumed that the birth and the death rate are all constant so the dynamics of the total population may be simple. In order to investigate population dynamics for the model with more biological meanings, it should be considered that birth and death rates are density dependent. In

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doi:10.1016/j.biosystems.2009.05.006

this point, an *SIR* epidemic model with density dependent birth and death rates with the incubation time was formulated by Yoshida and Hara (2007). They analyzed transmission dynamics for the epidemic *SIR* model with time delay and studied the global stability of the *SIR* model. On the other hand, recently, Zaman et al. (2008) studied the stability and optimal vaccination of a controlled *SIR* epidemic model without time delays. In this project we are interested in combining and improving on the results in Zaman et al. (2008) and Yoshida and Hara (2007).

In this paper, we consider a controlled *SIR* epidemic model with time delay to prevent the spread of diseases by using optimal treatment strategies. In order to do this, we first introduce a control variable representing the optimal treatment for infectious hosts and set an optimal control system for the *SIR* epidemic model. Then we derive the basic reproduction number and investigate the dynamical behavior of the controlled *SIR* epidemic model. Moreover, we show the existence of an optimal control for this control problem and the infection in a community dies out by using the possible optimal control treatment. We also analyze the optimal control and optimality system using optimal control techniques. For numerical simulation, we fit data from Ebola a hemorrhagic fever outbreaks Congo (1995), where we show that the basic reproduction number is less than unity, so the infection in the community dies out by using control treatment strategy. Furthermore, our optimal control strategies reduce infected individuals and increase the total number of susceptible and recovered individuals. From these results, our optimal control system can be used by epidemic researchers to realistically simulate the stochastic dynamics of Ebola epidemics in order to study the effect of control intervention.

The paper is organized as follows. In Section 2, a description of an *SIR* epidemic model and the corresponding objective functional are given. We derive the basic reproduction number for the control system and show the existence. Then, we introduce the optimal control techniques to find the optimal solution of the dynamics system. We report our numerical results obtain from real data and analyze in detail the dynamical behaviors of the control processes in Section 3. Finally, we describe some conclusions in Section 4.

2. Optimal Control Techniques in Delay Model

To begin the optimal control procedure, it is necessary to have a model which describes the population dynamics. Yoshida and Hara (2007) considered an *SIR* model with time delay. We use this epidemic model to set our control problem. We may have a population which lacks social structures and where individuals may switch between the Susceptible, Infected, and Recovered (or immune) states according to $S \to I \to R$. We assume that all newborns are susceptible and the number of the total population is denoted by $N(t) = S(t) + I(t) + R(t)$. Then the delayed *SIR* epidemic model which is proposed in Yoshida and Hara (2007) becomes a system of differential equations with time delay:

$$\begin{aligned}
\frac{dS(t)}{dt} &= \left(b - \mu\frac{rN(t)}{K}\right)N(t) - \frac{\beta S(t)I(t-h)}{N(t-h)} - \left(d + (1-\mu)\frac{rN(t)}{K}\right)S(t),\\
\frac{dI(t)}{dt} &= \frac{\beta S(t)I(t-h)}{N(t-h)} - \left(d + (1-\mu)\frac{rN(t)}{K}\right)I(t) - \alpha I(t),\\
\frac{dR(t)}{dt} &= \alpha I(t) - \left(d + (1-\mu)\frac{rN(t)}{K}\right)R(t),
\end{aligned} \tag{1}$$

where $b > 0$, $d > 0$, $\alpha > 0$ and $\beta > 0$ are the birth, death, recovery and contact rate, respectively. $r = b - d$ is the intrinsic growth rate, μ is the convex combination constant, K is the carrying capacity of the population, and h is a non-negative constant which represents a time delay on the infected individuals I and the total individuals N during the spread of diseases. Susceptible individuals acquire infection at a per capita rate $\beta I(t-h)/N(t-h)$. In this model, the incidence rate is $\beta S(t)I(t-h)/N(t-h)$. This incidence rate seems more reasonable than $\beta I(t)/N(t)$ because the force of infection is proportional to $I(t-h)/N(t-h)$ with time delay. Note that in some epidemic models, bilinear incidence rate $\beta S(t)I(t)$ and standard incidence rate $\beta S(t)I(t)/N(t)$ are frequently used. These incidences imply that the contact rate or contact number is constant. Actually, the infection probability per contact is likely influenced by the number of infected individual because more infected individuals can increase infection risk. For instance, during SARS outbreak in 2003, Chinese government did a lot of protection measures and control polices: closing schools, closing restaurants, postponing conferences, isolating infectious etc. These actions greatly reduced the contact number per unit time. The dynamics of the total population N are governed by the following logistic equation:

$$\frac{dN(t)}{dt} = \left(b - \mu\frac{rN(t)}{K}\right)N(t) - \left(d + (1-\mu)\frac{rN(t)}{K}\right)N(t). \tag{2}$$

The birth rate decreases and the death rate increases to its carrying capacity K for $0 < \mu < 1$. The birth and death rate are density independent for $\mu = 0$ and 1, respectively.

Now using the delayed *SIR* epidemic model (1), we will derive an optimal control model to fit our control strategy. The theoretical foundation of optimal control models with underlying dynamics given by ordinary differential equations was developed by Pontryagin and his co-worker in Moscow about 1950 (Kamien and Schwartz, 2000). So by Pontryagin's Maximum Principle, its extension and appropriate numerical methods, we will set an optimal control problem in the time delayed *SIR* epidemic model to control the spread of diseases. The main goal of this problem is to investigate an effective treatment strategy to control infection diseases, which means that we can make an *SIR* epidemic control model which satisfies that the maximum numbers of infected individuals are not larger than that of susceptible individuals and more individuals are recovered after infection.

In order to set an optimal control problem, first, we make the following notational conventions. Let Λ, $T > 0$ be given constant and define the control set:

$$U = \{u(t) \in L^2(0,T) : 0 \le u(t) \le \Lambda, 0 \le t \le T\}, \tag{3}$$

where $u(t)$ is Lebesgue measurable and called a control variable. In this problem, the biological meaning of the control variable is that low levels of the number of infected individuals build by no contact to the susceptible individuals. In case of high contact rate the number of infected individuals increases while the number of susceptible and recovered individuals decreases. Better treatment and low contact rate bring the number of infected individuals to a small level, susceptible individuals begin to build again and more individuals are recovered from infection. Therefore, the probability of infected individuals $I(t)/N(t)$ that an infected individual spread is made by an infectious individual and this is controlled by an optimal control treatment $u(t)$ so that a fraction $u(t)I(t)/N(t)$ of infected individuals are moved from I class to R and S classes. From these facts, our optimal control problem is given by the following.

Find a control $u(t)$ and a triple individual $(S(t), I(t), R(t))$ to minimize the objective functional

$$J_\epsilon(u) = \int_0^T \left[I(t) + \frac{\epsilon u^2(t)}{2}\right] dt \tag{4}$$

subject to the state system

Table 1
The basic reproduction number of well-known diseases.

Disease	Transmission	R_p	Source
Influenza	Airborne droplet	2–3	Mills et al. (2004)
Polio	Fecal-oral route	5–7	May (1983) and Linda (2007)
Measles	Airborne	12–18	May (1983)
Smallpox	Social contact	3–7	Linda (2007) and Zaman et al. (2008)
Mumps	Airborne droplet	4–7	Zaman et al. (2008)
HIV/AIDS	Sexual contact	2–5	Anderson and May (1979)
SARS	Airborne droplet	2–5	Wallinga and Teunis (2004)
Ebola(Congo)	Physical contact	1.83	Chwell et al. (2004)

$$\begin{aligned}
\frac{dS(t)}{dt} &= \left(b-\mu\frac{rN(t)}{K}\right)N(t)+\omega u(t)\frac{I(t)}{N(t)}-\frac{\beta S(t)I(t-h)}{N(t-h)}-\left(d+(1-\mu)\frac{rN(t)}{K}\right)S(t),\\
\frac{dI(t)}{dt} &= \frac{\beta S(t)I(t-h)}{N(t-h)}-u(t)\frac{I(t)}{N(t)}-\left(d+(1-\mu)\frac{rN(t)}{K}\right)I(t)-\alpha I(t),\\
\frac{dR(t)}{dt} &= \alpha I(t)+(1-\omega)u(t)\frac{I(t)}{N(t)}-\left(d+(1-\mu)\frac{rN(t)}{K}\right)R(t)
\end{aligned} \tag{5}$$

with initial conditions

$$S(0)=S_0,\qquad I(0)=I_0,\qquad R(0)=R_0. \tag{6}$$

Here ϵ is a positive constant to keep balance of the size of infected individuals $I(t)$ and $\omega\in[0,1]$.

The objective functional (4) represents the probability of the infected individuals to spread and systemic costs of the possible treatments. Our goal is to minimize the objective functional (4), that is, to minimize the probability that the infected individuals spread and to maximize the total number of susceptible and recovered individuals by using the possible minimal control (treatment) variable $u(t)$. Note that for $\omega=1$, the infected individuals moved to the susceptible class, while for $\omega=0$, the infected individuals moved to the recovered class at rate of control variable $u(t)$. Note that in this optimal problem we consider one control variable (treatment), for more then one control variable we refer the reader to see Lenhart and John (2007) and Zaman et al. (2007).

The basic reproduction number $R_p(t)=\beta/(b+\alpha-\mu r+u(t)/K)$ of the control system (5) is defined as the average number of secondary infections produced when one infected individual is introduced into a host population where everyone is susceptible. The basic reproduction number $R_p(t)$ is dependent on the parameters values and control variable $u(t)$. The control variable $u(t)\in[0,\Lambda]$ (bounded) for all $t\in[0,T]$. The value of $R_p(t)$ decreases when the control variable $u(t)$ increases. For many deterministic epidemiology models, an infection can get started in a fully susceptible population if and only if $R_p>1$. Thus the basic reproduction number R_p is often considered as the threshold quantity that determines when an infection can invade and persist in a new host population (Katri and Ruan, 2004). Estimates of R_p have been obtained for different diseases represents in Table 1. In this model if $R_p\le 1$ then the infection in the community dies out, while if $R_p>1$ then there is a unique positive epidemic equilibrium is given by

$$\begin{aligned}
&(\beta-b-\alpha+\mu r)K>u(t) \quad \text{epidemic},\\
&(\beta-b-\alpha+\mu r)K\le u(t) \quad \text{disease-free}.
\end{aligned} \tag{7}$$

For the unique positive epidemic equilibrium of the system (5) with time delay $h=0$, we set left hand side of the system (5) equal zero to get

$$S_\infty(t)=\frac{K}{R_p},\qquad I_\infty(t)=\frac{(d+r-\mu r)(R_p-1)K}{\beta},$$

$$R_\infty(t)=\frac{\alpha}{d+1-\mu r}I_\infty(t).$$

If the basic reproduction number $R_p(t)<1$, then the infected individual in the unique positive epidemic equilibrium becomes negative which is not biological feasible shows that there is no diseases in the community. Therefore the equilibrium point at $S^*(t)=N(t)=K$ and $I^*(t)=R^*(t)=0$, represents a disease-free stationary solution. In addition, to understand that the time delayed *SIR* model without control is globally asymptotically stable (or not), we refer the reader to see Yoshida and Hara (see, 2007, Theorem 1), while local stability analysis of the *SIR* epidemic model with control can be found in Zaman et al. (2008). In general, epidemic models are used to describe rapid outbreaks that occur in less than 1 year, while endemic models are used for studying diseases over longer periods, during which there is a renewal of susceptible by births or recovery from temporary immunity.

In epidemic dynamics, stability, existence, and optimal control theory are important research topics. At first we will show the existence of solutions for the control system (5). In this control problem, we assume the restriction on the control (treatment) variable such that $0\le u(t)<K$, where $u(t)\ge 0$ for all $t\in[0,T]$. Note that $I(t)$ is a state variable with a control variable $u(t)\in U$. We also assume that $S(\theta)$, $I(\theta)$, $R(\theta)$ are nonnegative continuous functions for $\theta\in[-h,0]$ and $S(0)=S_0\ge 0$, $I(0)=I_0\ge 0$, $R(0)=R_0\ge 0$, respectively and $N(t)\le MN(t-h)$, where M is a positive constant. Then we can rewrite (5) in the following form:

$$\frac{dV(t)}{dt}=AV(t)+F(V(t),\qquad V_h(t))+C(u,V(t)), \tag{8}$$

where

$$V(t)=\begin{bmatrix}S(t)\\ I(t)\\ R(t)\end{bmatrix},$$

$$F(V(t),V_h(t))=\begin{bmatrix}\left(b-\mu\frac{rN(t)}{K}\right)N(t)-\left(1-\mu\frac{rN(t)}{K}\right)S(t)-\frac{\beta S(t)I(t-h)}{N(t-h)}\\ \frac{\beta B(t)I(t-h)}{N(t-h)}-\left(1-\mu\frac{rN(t)}{K}\right)I(t)\\ -\left(1-\mu\frac{rN(t)}{K}\right)R(t)\end{bmatrix}$$

$$A=\begin{bmatrix}b-d & b & b\\ 0 & -(d+\alpha) & 0\\ 0 & \alpha & -d\end{bmatrix},\qquad C(u,V(t))=\begin{bmatrix}\omega u(t)\frac{I(t)}{N(t)}\\ -u(t)\frac{I(t)}{N(t)}\\ (1-\omega)u(t)\frac{I(t)}{N(t)}\end{bmatrix}$$

and $V_h(t)=V(t-h)$. The system (8) is a nonlinear system with a bounded coefficient. We set

$$G(V(t),V_h(t))=AV(t)+F(V(t),V_h(t)). \tag{9}$$

The second term on the right hand side of Eq. (9) satisfies

$$\begin{aligned}
&|F(V_1(t),(V_1)_h(t))-F(V_2(t),(V_2)_h(t))|\le M_1|V_1(t)-V_2(t)|\\
&\quad +M_2|(V_1)_h(t)-(V_2)_h(t)|,
\end{aligned}$$

where M_1 and M_2 are some positive constants, independent of state variables $S(t)$, $I(t)$ and $R(t)\le N(t)$, and

$$|V_1(t)-V_2(t)|=|S_1(t)-S_2(t)|+|I_1(t)-I_2(t)|+|R_1(t)-R_2(t)|,$$

$$\begin{aligned}
&|(V_1)_h(t)-(V_2)_h(t)|=|(S_1)_h(t)-(S_2)_h(t)|+|(I_1)_h(t)-(I_2)_h(t)|\\
&\quad +|(R_1)_h(t)-(R_2)_h(t)|.
\end{aligned}$$

Here $(S_i)_h(t)=S_i(t-h)$, $(I_i)_h(t)=I_i(t-h)$, and $(R_i)_h(t)=R_i(t-h)$, for i=1, 2. Moreover, we get

$$|G(V_1,V_{1h})-G(V_2,V_{2h})|\le L(|V_1(t)-V_2(t)|+|(V_1)_h(t)-(V_2)_h(t)|),$$

where $L = max\{M_1, M_2, \|A\|\} < \infty$. Thus it follows that the function G is uniformly Lipschitz continuous. From the definition of U and the restriction on $S(t)$, $I(t)$ and $R(t) \geq 0$ we can see that a solution of the system (8) exists (see, Birkhoff and Rota, 1989).

In order to find an optimal solution pair, we consider the optimal control problem (4)–(6). First we should find the Lagrangian and Hamiltonian for the optimal control problem (4)–(6). Actually, the Lagrangian of the optimal problem is given by

$$L(I, u) = I(t) + \frac{\epsilon u^2(t)}{2}.$$

We seek for the minimal value of the Lagrangian. To do this, we set $x(t) = (S(t), I(t), R(t))$ and $x_h(t) = (S_h(t), I_h(t), R_h(t))$, where $S_h(t) := S(t-h), I_h(t) := I(t-h), R_h(t) := R(t-h)$, to define the Hamiltonian $H(t) = H(x, x_h, u, \lambda)(t)$ for the control problem as follows:

$$\begin{aligned} &H(x, x_h, u, \lambda)(t) \\ &= L(I, u) + \lambda_1(t)\left(\left(b - \mu\frac{rN(t)}{K}\right)N(t) + \omega u(t)\frac{I(t)}{N(t)} - \frac{\beta S(t)I(t-h)}{N(t-h)}\right. \\ &\quad \left. - \left(d + (1-\mu)\frac{rN(t)}{K}\right)S(t)\right) + \lambda_2(t)\left(\frac{\beta S(t)I(t-h)}{N(t-h)} - u(t)\frac{I(t)}{N(t)}\right. \\ &\quad \left. - \left(d + (1-\mu)\frac{rN(t)}{K}\right)I(t) - \alpha I(t)\right) + \lambda_3(t)(\alpha I(t) \\ &\quad + (1-\omega)u(t)\frac{I(t)}{N(t)} - \left(d + (1-\mu)\frac{rN(t)}{K}\right)R(t)\right). \end{aligned} \tag{10}$$

Theorem 2.1. *There exists an optimal control $u^* \in U$ such that*

$$J_\epsilon(u^*) = \min_{u \in U} J_\epsilon(u),$$

subject to the control system (5) with the initial condition (6).

Proof. To prove the existence of an optimal control pair we use the result in Lukes (1982). Note that the control and the state variables are nonnegative values. In this minimizing problem, the necessary convexity of the objective functional in $u(t)$ is satisfied. The control variable $u(t) \in U$ is also convex and closed by definition. The optimal system is bounded which determines the compactness needed for the existence of the optimal control. Moreover, the integrand in the functional (4), $I(t) + (\epsilon u^2(t))/2$ is convex on the control set U. In addition, we can easily see that, there exist a constant $\rho > 1$ and positive numbers η_1 and η_2 such that

$$J_\epsilon(u) \geq \eta_2 + \eta_1(|u|^2)^{\rho/2},$$

which completes the existence of an optimal control. □

Our main tool for the study of optimality on the system (5) is given by the Pontryagin Maximum Principle (Kamien and Schwartz, 2000). A necessary condition for optimal control problems can be found in (Halanay, 1968). If we consider $x(t)$ and $x_h(t)$ defined above, then there exists a continuous function $\lambda(t)$ on $[0, T]$ satisfying the following three equations, that is, the state equation

$$x'(t) = H_\lambda(x, x_h, u, \lambda)(t), \tag{11}$$

the optimality condition

$$0 = H_u(x, x_h, u, \lambda)(t), \tag{12}$$

and the adjoint equation

$$-\lambda'(t) = H_x(x, x_h, u, \lambda)(t) + \lambda(t+h)H_{x_h}(x, x_h, u, \lambda)(t), \tag{13}$$

where H_λ, H_u, H_x and H_{x_h} denotes the derivative with respect to λ, u, x and x_h, respectively. Now we apply the necessary conditions to the Hamiltonian H in (10).

Theorem 2.2. *Let $S^*(t)$, $I^*(t)$ and $R^*(t)$ be optimal state solutions associated with the optimal control variable $u^*(t)$ for the optimal control problem (4)–(6). Then there exist adjoint variables $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$ satisfying*

$$\begin{aligned} \lambda_1'(t) &= \lambda_1(t)\left(d + (1+\mu)\frac{rN^*(t)}{K} + (1-\mu)\frac{rS^*(t)}{K} + \frac{\beta I^*(t-h)}{N^*(t-h)} + \omega u^*(t)\frac{I^*(t)}{(N^*(t))^2} - b\right) \\ &\quad + \lambda_2(t)\left((1-\mu)\frac{rI^*(t)}{K} - u^*(t)\frac{I^*(t)}{(N^*(t))^2} - \frac{\beta I^*(t-h)}{N^*(t-h)}\right) \\ &\quad + \lambda_3(t)\left((1-\mu)\frac{rR^*(t)}{K} + (1-\omega)u^*(t)\frac{I^*(t)}{(N^*(t))^2}\right) \\ &\quad + \lambda_1(t+h)(\lambda_2(t) - \lambda_1(t))\frac{\beta S^*(t)I^*(t-h)}{(N^*(t-h))^2}, \end{aligned} \tag{14}$$

$$\begin{aligned} \lambda_2'(t) &= \lambda_1(t)\left(2\mu\frac{rN^*(t)}{K} + (1-\mu)\frac{rS^*(t)}{K} - \omega u^*(t)\frac{S^*(t) + R^*(t)}{(N^*(t))^2} - b\right) \\ &\quad + \lambda_2(t)\left(\alpha + d + (1-\mu)(N^*(t) + I^*(t))\frac{r}{K} + u^*(t)\frac{(S^*(t) + R^*(t))}{(N^*(t))^2}\right) \\ &\quad + \lambda_3(t)\left((1-\mu)\frac{rR^*(t)}{K} - (1-\omega)u^*(t)\frac{(S^*(t) + R^*(t))}{(N^*(t))^2} - \alpha\right) - 1 \\ &\quad + \lambda_2(t+h)(\lambda_1(t) - \lambda_2(t))\frac{\beta(S^*(t-h) + R^*(t-h))}{(N^*(t-h))^2}, \end{aligned} \tag{15}$$

$$\begin{aligned} \lambda_3'(t) &= \lambda_1(t)\left((1-\mu)\frac{rS^*(t)}{K} + 2\mu\frac{rN^*(t)}{K} + \omega u^*(t)\frac{I^*(t)}{(N^*(t))^2} - b\right) \\ &\quad + \lambda_2(t)\left((1-\mu)\frac{rI^*(t)}{K} - u^*(t)\frac{I^*(t)}{(N^*(t))^2}\right) \\ &\quad + \lambda_3(t)\left(d + (1-\mu)(N^*(t) + R^*(t))\frac{r}{K} + (1-\omega)u^*(t)\frac{I^*(t)}{(N^*(t))^2}\right) \\ &\quad + \lambda_3(t+h)(\lambda_2(t) - \lambda_1(t))\frac{\beta S^*(t)I^*(t-h)}{(N^*(t-h))^2}, \end{aligned} \tag{16}$$

with transversality conditions (or boundary conditions)

$$\lambda_i(T) = 0, i = 1, 2, 3. \tag{17}$$

Furthermore, the optimal control is given as follows:

$$u^*(t) = \max\left\{\min\left\{(-\omega\lambda_1(t) + \lambda_2(t) - (1-\omega)\lambda_3(t))\frac{I^*(t)}{\epsilon N^*(t)}, \Lambda\right\}, 0\right\}. \tag{18}$$

Proof. To determine the adjoint equations and the transversality conditions we use the Hamiltonian (10). By using the adjoint Eq. (13) and differentiating the Hamiltonian (10) with respect to $x(t)$, and $x_h(t)$, with setting $x(t) = x^*(t)$, and $x_h(t) = x_h^*(t)$, we obtain

$$\begin{aligned} -\lambda_1'(t) &= H_{S^*}(t) + \lambda_1(t+h)H_{S_h^*}(t), \\ -\lambda_2'(t) &= H_{I^*}(t) + \lambda_2(t+h)H_{I_h^*}(t), \\ -\lambda_3'(t) &= H_{R^*}(t) + \lambda_3(t+h)H_{R_h^*}(t). \end{aligned}$$

Substituting the corresponding derivatives in the above inequalities and by rearranging we obtain the adjoint Eqs. (14)–(16). By the optimality condition (12) we have

$$\left.\frac{\partial H}{\partial u}\right|_{u(t)=u^*(t)} = \epsilon u^*(t) + \omega\lambda_1(t)\frac{I^*(t)}{N^*(t)} - \lambda_2(t)\frac{I^*(t)}{N^*(t)} + (1-\omega)\lambda_3(t)\frac{I^*(t)}{N^*(t)} = 0,$$

which implies

$$u^*(t) = (-\omega\lambda_1(t) + \lambda_2(t) - (1-\omega)\lambda_3(t))\frac{I^*(t)}{\epsilon N^*(t)}.$$

Using the property of control set in (3) we obtain

$$u^*(t)=\begin{cases}0 & \text{if } (-\omega\lambda_1(t)+\lambda_2(t)-(1-\omega)\lambda_3(t))\dfrac{I^*(t)}{\epsilon N^*(t)}\le 0,\\ (-\omega\lambda_1(t)+\lambda_2(t)-(1-\omega)\lambda_3(t))\dfrac{I^*(t)}{\epsilon N^*(t)} & \text{if } 0<(-\omega\lambda_1(t)+\lambda_2(t)-(1-\omega)\lambda_3(t))\dfrac{I^*(t)}{\epsilon N^*(t)}<\Lambda,\\ \Lambda & \text{if } (-\omega\lambda_1(t)+\lambda_2(t)-(1-\omega)\lambda_3(t))\dfrac{I^*(t)}{\epsilon N^*(t)}\ge\Lambda.\end{cases}$$

This can be rewritten in compact notation

$$u^*(t)=\max\left\{\min\left\{(-\omega\lambda_1(t)+\lambda_2(t)-(1-\omega)\lambda_3(t))\frac{I^*(t)}{\epsilon N^*(t)},\Lambda\right\},0\right\}.$$

□

Here we call formula (18) for u^* the characterization of the optimal control. The optimal control and the state are found by solving the optimality system, which consists of the state system (5), the adjoint Eqs. (14)–(16), boundary conditions (6) and (17), and the characterization of the optimal control (18). To solve the optimality system we use the initial and transversality conditions together with the characterization of the optimal control pair $u^*(t)$ given by (18).

In addition, the second derivative of the Lagrangian with respect to u is positive, which shows that the optimal problem is minimum at control $u^*(t)$. By substituting the value of $u^*(t)$ in the control system (5) we get the following system

$$\begin{aligned}
\frac{dS^*(t)}{dt} &= \left(b-\mu\frac{rN^*(t)}{K}\right)N^*(t)-\frac{\beta S^*(t)I^*(t-h)}{N^*(t-h)}-\left(d+(1-\mu)\frac{rN^*(t)}{K}\right)S^*(t)\\
&\quad+\omega\max\left\{\min\left\{(-\omega\lambda_1(t)+\lambda_2(t)-(1-\omega)\lambda_3(t))\frac{I^*(t)}{\epsilon N^*(t)}\Lambda\right\},0\right\}\frac{I^*(t)}{N^*(t)},\\
\frac{dI^*(t)}{dt} &= \frac{\beta S^*(t)I^*(t-h)}{N^*(t-h)}-\left(d+(1-\mu)\frac{rN^*(t)}{K}\right)I^*(t)-\alpha I^*(t)\\
&\quad-\max\left\{\min\left\{(-\omega\lambda_1(t)+\lambda_2(t)-(1-\omega)\lambda_3(t))\frac{I^*(t)}{\epsilon N^*(t)},\Lambda\right\},0\right\}\frac{I^*(t)}{N^*(t)},\\
\frac{dR^*(t)}{dt} &= \alpha I^*(t)-\left(d+(1-\mu)\frac{rN^*(t)}{K}\right)R^*(t)\\
&\quad+(1-\omega)\max\left\{\min\left\{(-\omega\lambda_1(t)+\lambda_2(t)-(1-\omega)\lambda_3(t))\frac{I^*(t)}{\epsilon N^*(t)},\Lambda\right\},0\right\}\frac{I^*(t)}{N^*(t)}
\end{aligned}\tag{19}$$

with the Hamiltonian H^* at $(S^*, I^*, R^*, S_h^*, I_h^*, R_h^*, u^*, \lambda_1, \lambda_2, \lambda_3)$

$$\begin{aligned}
H^* &= I^*(t)+\frac{1}{2}\left[\epsilon\left(\max\left\{\min\left\{(-\omega\lambda_1(t)+\lambda_2(t)-(1-\omega)\lambda_3(t))\frac{I^*(t)}{\epsilon N^*(t)},\Lambda\right\},0\right\}\right)^2\right]\\
&\quad+\lambda_1(t)\left[\left(b-\mu\frac{rN^*(t)}{K}\right)N^*(t)-\frac{\beta S^*(t)I^*(t-h)}{N^*(t-h)}-\left(d+(1-\mu)\frac{rN^*(t)}{K}\right)S^*(t)+\omega\max\left\{\min\left\{(-\omega\lambda_1(t)+\lambda_2(t)-(1-\omega)\lambda_3(t))\frac{I^*(t)}{\epsilon N^*(t)}\Lambda\right\},0\right\}\frac{I^*(t)}{N^*(t)}\right]\\
&\quad+\lambda_2(t)\left[-\alpha I^*(t)+\frac{\beta S^*(t)I^*(t-h)}{N^*(t-h)}-\left(d+(1-\mu)\frac{rN^*(t)}{K}\right)I^*(t)+\max\left\{\min\left\{(-\omega\lambda_1(t)+\lambda_2(t)-(1-\omega)\lambda_3(t))\frac{I^*(t)}{\epsilon N^*(t)}\Lambda\right\},0\right\}\frac{I^*(t)}{N^*(t)}\right]\\
&\quad+\lambda_3(t)\left[\alpha I^*(t)-\left(d+(1-\mu)\frac{rN^*(t)}{K}\right)R^*(t)+(1-\omega)\max\left\{\min\left\{(-\omega\lambda_1(t)+\lambda_2(t)-(1-\omega)\lambda_3(t))\frac{I^*(t)}{\epsilon N^*(t)}\Lambda\right\},0\right\}\frac{I^*(t)}{N^*(t)}\right].
\end{aligned}\tag{20}$$

To find out the optimal control and state variable we will numerically solve the above system (19) and (20). We consider in next section, the real data obtained from the Congo Ebola hemorrhagic fever outbreaks in 1995, for numerical simulation.

3. Numerical Simulations

In order to present some numerical results which make the point that introducing both a time delay and an optimal control into an *SIR* model can have a profound effect on the solution. System of both ODEs and delay differential equations (DDEs) are used to describe many phenomena of physical interest. In ODEs system contains derivatives which depend on the solution at the present value of the independent variable (time), while DDEs system contain in addition derivatives which depend on the solution at previous times. Despite the obvious similarities between ODEs and DDEs, solutions of DDE problems can differ from solutions for ODE problems in several striking, and significant, ways (Baker et al., 1995). This accounts in part for the lack of several general-purposes software for solving DDEs. Here we obtain the optimality system from the state and adjoint equations. The optimal control problem strategy is obtained by solving the optimal system, which consists of six ordinary differential equations and boundary conditions.

For numerical simulations we consider the time delayed *SIR* model (1), the control system (5), the initial conditions (6), the adjoint Eqs. (14)–(16), and the transversality conditions (17) by using a MATLAB code **dde23** which is based on a standard Runge–Kutta scheme. We consider the real data obtained from the Congo Ebola hemorrhagic fever outbreaks in 1995 include the identification dates of the causative agent and data sources (Khan et al., 1995). The first outbreak during this period (the fourth in total) occurred in the town of Kikwit, is situated about 500 km southeast of the DRC capital, Kinshasa (Khan et al., 1995). This outbreak began in the Bandundu region, primarily in Kikwit, located on the banks of the Kwilu River. The first case (January 6) involved a 42-year-old male charcoal worker and farmer who died on January 13. The Ebola virus was not identified as the causative agent until May 9. At that time, an international team implemented a control plan that involved active surveillance (identification of cases) and education programs for infected people and their family members. Family members were visited for up to 3 weeks (maximum incubation period) after their last identified contact with a probable case.

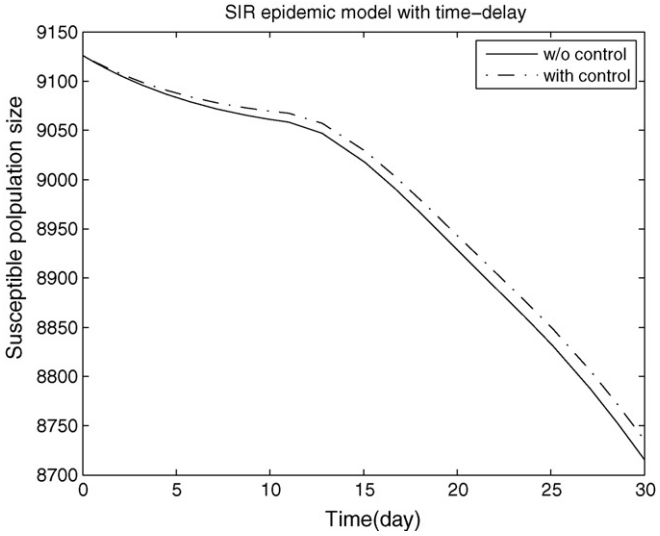

Fig. 1. The plot shows the population of susceptible individuals both with control and without control. The value of $\epsilon = 50$ and $\omega = 0.35$ in the control system are used for numerical simulation.

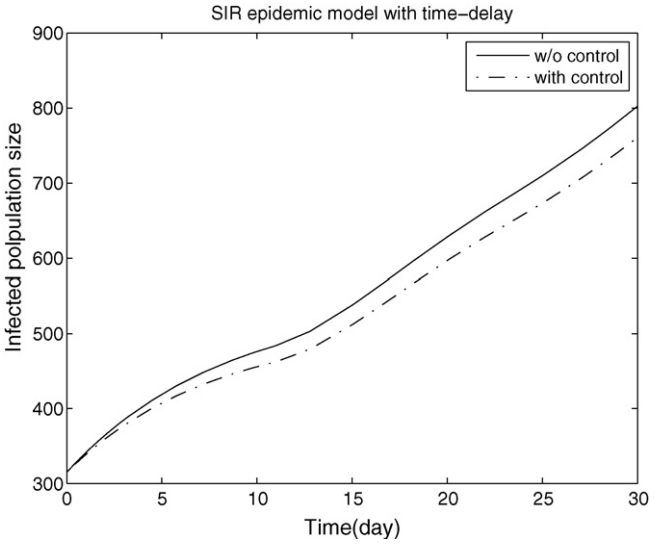

Fig. 2. The plot represents the population of infected individuals both with control and without control. The value of $\epsilon = 50$ and $\omega = 0.35$ in the control system are used for numerical simulation.

Nosocomial transmission (transmission from patients within hospital settings) occurred in Kikwit General Hospital. Transmission was halted through the institution of strict barrier nursing techniques that included the use of protective equipment and special isolation wards. Between January and July 1995, the virus killed 256 of its 315 victims, a mortality rate (81%) characteristic of the Zaire subtype. Daily Ebola cases by date of a symptom onset from March 1 through July 12 are available in Khan et al. (1995). The population density 19 km^{-2} of Congo was estimated in 1995. The first outbreak occurred in the town of Kikwit, situated 500 km southeast with a total population about 9500 people.

The birth and death rate is density independent for $\mu = 0$ and 1, respectively. The birth rate decreases and the death rate increases to its carrying capacity K for $0 < \mu < 1$. Chwell et al. (2004) estimated the infectious period between 3.5 and 10.7 days, so we have chosen $\mu = 0.014$ and $\beta = 0.21$ for our numerical simulation. The incubation period of the Ebola virus is 1–21 days. Therefore, we consider the recovery rate $\alpha = 0.0476$ and time delay $h = 11$ days. The initial growth rate for the Congo, (1995) epidemic is $b = 0.07$ day^{-1}, and the disease mortality rate is 81% that is $d = 0.0123$. Lekone and Finkenstädt (2006) used 10 different initial parameter values for Ebola hemorrhagic model and represented some numerical results. We consider total population $N(0) = 9500$ contain susceptible individuals $S(0) = 9126$, infected individuals $I(0) = 315$, and recovered individuals $R(0) = 59$ with a maximum carrying capacity $K = 10{,}900$ for numerical simulation.

When viewing the graphs, remember that each of the individuals without control is marked by undashed lines. The control individuals are marked by dash-dotted lines. In Fig. 1, we plot the susceptible individual in two systems (1) and (5). The numerical results show that the number of susceptible individuals increase after the optimal control treatment and small number of individuals are infected from the Ebola virus. Fig. 2, represents the population of infected individuals in the two systems (1) without control and (5) with control. The population of infected individuals without the optimal control treatment is more sharply increasing than the individuals with control treatment and becomes very large, so we need treatment to control the infected individuals. In Fig. 3, the recovered individuals increase rapidly while the infected (Fig. 2) individuals slightly increase. The solid line denotes that there are more infected individuals when the control treatment is not implemented to the infected individual. The time frame used for this algorithm is 30 days. Our numerical results show that on day 30 (the last day of treatment) there remains a small number of infected individuals than before the optimal control. Thus, the rate of infected individuals decreases after the control treatment and so more individuals are recovered (Fig. 3) after infection.

The control variable for two different value of time delay $h = 0.5$ and $h = 11$ are represented in Figs. 4 and 5, respectively. Fig. 4 represents that in case of small time delay we need a large value of optimal control variable. While for large time delay the control variable decreases represents in Fig. 5. Which make insure that by introducing both a time delay and an optimal control into an *SIR* model can have a profound effect on the elimination of infectious diseases. Actually, the number of infected individuals after the treatment (control) is smaller than that of infected individuals before the treatment. Furthermore, our numerical results obtain from real data regarding the course of the Ebola virus in Congo represent that the optimal control techniques provide effective tools on population dynamics.

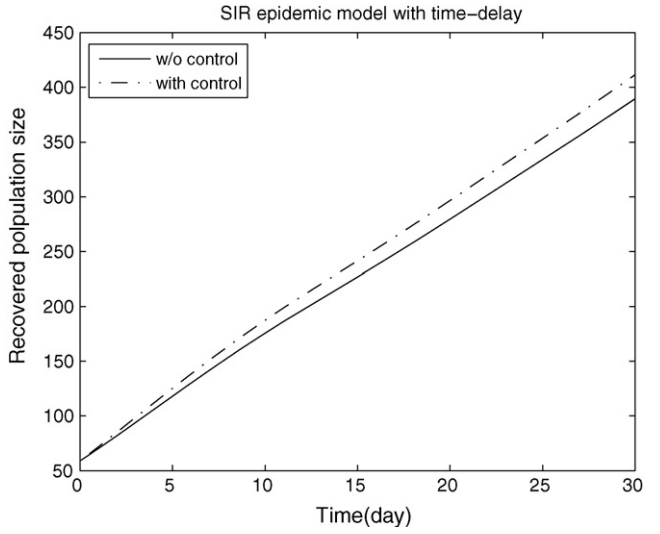

Fig. 3. The plot shows the population of recovered individuals both with control and without control. The value of $\epsilon = 50$ and $\omega = 0.35$ in the control system are used for numerical simulation.

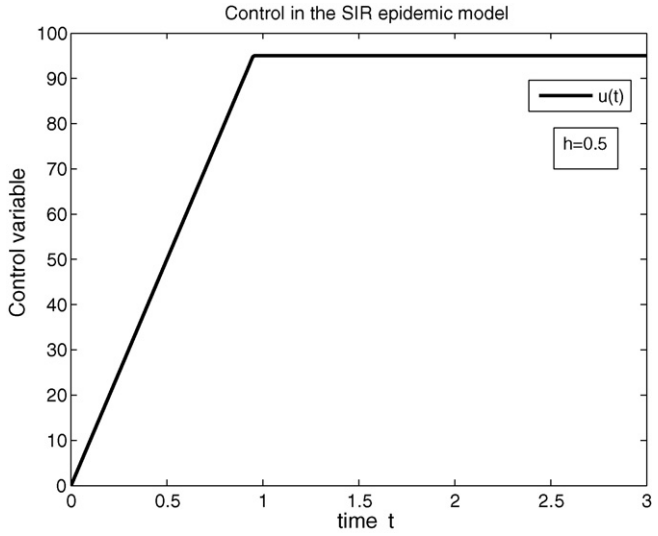

Fig. 4. The plot represents the control variable $u(t)$ for time delay $h = 0.5$. The value of $\epsilon = 50$ and $\omega = 0.35$ in the control system are used for numerical simulation.

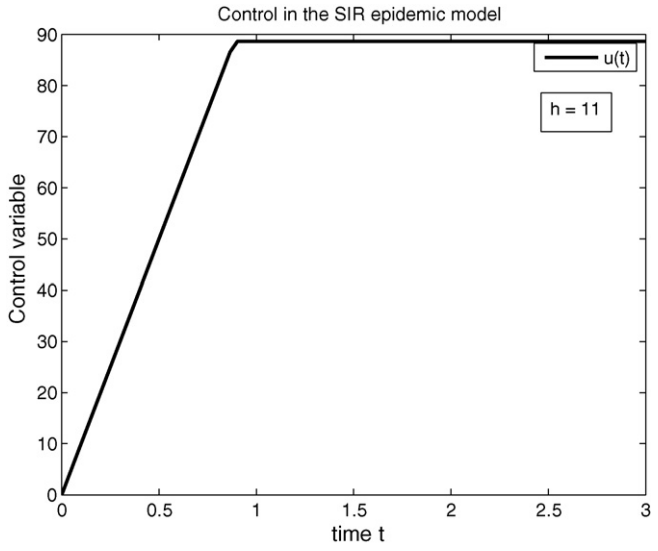

Fig. 5. The plot represents the control variable $u(t)$ for time delay $h = 11$. The value of $\epsilon = 50$ and $\omega = 0.35$ in the control system are used for numerical simulation.

Remark 3.1. Estimates of the basic reproduction number R_p for common SEIR infectious diseases range over 1.8–1.4 for Ebola hemorrhagic fever (Chwell et al., 2004; Lekone and Finkenstädt, 2006). The reproduction number R_p in our proposed model without optimal control is 1.7981 represents in an epidemic, while the reproduction number R_p after our optimal control treatment is 0.7348, show that Ebola infection in the community dies out.

4. Conclusion

We considered in this paper the *SIR* epidemic model with time delay. To prevent the spread of infected individuals, we introduced optimal control techniques by the Pontryagin's Maximum Principle. First, we derived the basic reproduction number from the control system. Then we shown the existence of this control system, where we focus on the application of optimal control theory to minimizing the spread of infection and the optimality was measured by the minimality of the probability of infected individuals. We also derived the necessary conditions for our optimal control problem. Furthermore our numerical simulation and results obtained from the optimal control technique shown that by applying the optimal treatment strategy, one can very effectively reduce the spread of infection as well as the total number of infected individuals. Finally, we presented the efficiency of this optimal control theory. In fact, we considered a special disease, Ebola virus of the Congo (1995) Ebola hemorrhagic fever outbreaks, in a specific community to apply our optimal treatment strategy. We shown that the basic reproduction number from the control system is less than unity so the infection in the community dies out. We believe that this model and the control techniques could be adapted to other epidemics than Ebola virus.

Acknowledgement

This work was supported for two years by Pusan National University Research Grant.

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