A Generalization of the Kermack-McKendrick Deterministic Epidemic Model*

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ABSTRACT

In this paper the Kermack-McKendrick deterministic model is generalized, introducing an interaction term in which the dependence upon the number of infectives occurs via a nonlinear bounded function which may take into account saturation phenomena for large numbers of infectives. An extension of the well-known threshold theorem is obtained, after a stability analysis of the equilibrium points of the system. A numerical example is carried out in detail.

1. INTRODUCTION

Deterministic models for communicable diseases have been introduced in a systematic way by Kermack and McKendrick [8, 9] (from now on KMK), who obtained a well-known threshold result for the model they proposed. (A detailed analysis of their model can be found in [7] and [14].)

Since then a large literature has grown about both deterministic and stochastic models. An up to date selection of papers on the subject is given in the book by N. T. J. Bailey [1].

In the KMK model, the population is divided into three disjoint classes of individuals:

(S), The susceptible class, i.e., the class of those individuals who are capable of contracting the disease and becoming infective,

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(I), the infective class, i.e., the class of those individuals who are capable of transmitting the disease to others,

(R), the removed class, i.e., the class of those individuals who have had the disease and are dead, or have recovered and are permanently immune, or are isolated.

If we denote by $S(t), I(t), R(t)$ the number of individuals in classes $(S), (I), (R)$ respectively at time t, the spread of infection is presumed to be governed by the following system of differential equations:

$$\begin{aligned}\frac{dS}{dt} &= -kSI,\\ \frac{dI}{dt} &= kSI - \gamma I,\\ \frac{dR}{dt} &= \gamma I,\end{aligned} \tag{1}$$

with initial conditions

$$S(0) = S_0 > 0, \qquad I(0) = I_0 > 0, \qquad R(0) = 0, \tag{2}$$

such that $S_0 + I_0 = N$, where N is the total size of the population. Here $k > 0$ is known as the infection rate, and $\gamma > 0$ as the removal rate.

A threshold theorem has been found by Kermack and McKendrick [8, 9] (see also [1] and [7]) for the system (1) (the KMK model), according to which all the points on the S-axis are equilibrium points, the points on the right of γ/k are unstable points, and the points on the left of γ/k are stable points. In addition to this, the KMK threshold theorem states that if $S_0 < \gamma/k$, then $I(t)$ decreases to zero as t tends to infinity; if $S_0 > \gamma/k$, then $I(t)$ first increases up to a maximum value reached on the line $S = \gamma/k$ and then decreases to zero as t tends to infinity. The number of susceptibles $S(t)$ is always a decreasing function of t, and reaches a limiting value $S(\infty)$ always greater than zero.

It is clear from this that the KMK threshold theorem involves only the initial number S_0 of susceptibles, being independent of the initial number I_0 of infectives.

Interesting modifications of the model (1), which introduce the possibility of temporary immunization of recovered individuals, have been studied in [7] (see also [11]).

In the model (1) the interaction term is a linearly increasing function of the number of infectives. While this might be true for a small number of infectives, it appears quite unrealistic that for large I this can still hold. Actually the number of contacts of a susceptible per unit time cannot always increase linearly with I.

Much more realistic appears the introduction of an interaction term of the form $g(I)\cdot S$, where the dependence on the number of infectives occurs only via a nonlinear bounded map g eventually tending to a "saturation level" c. [See the example proposed in Sec. 6 and Fig. 1(a).]

The introduction of such a function g would also allow the possibility of introducing some "psychological" effects: for a very large number of infectives the infection force $g(I)$ may decrease as I increases [see Fig. 1(b)], because in the presence of a very large number of infectives the population may tend to reduce the number of contacts per unit time. These ideas were suggested to the authors after a study of the cholera epidemic spread in Bari in 1973 [3, 4].

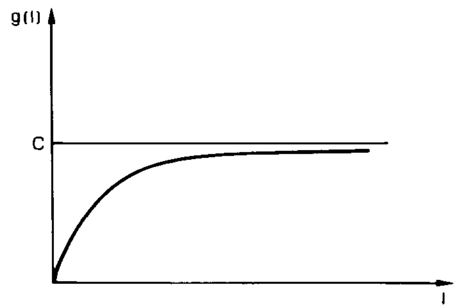
A diffusive analogue of the model (1), with the suggested modification in the interaction term, has been studied in [2].

Here we would like to point out how the results of the KMK nondiffusive model (1) are modified after the introduction of the function g in the interaction term. (As is well known, from this analysis indications come which are useful in studying the asymptotic behavior of the corresponding diffusive model.)

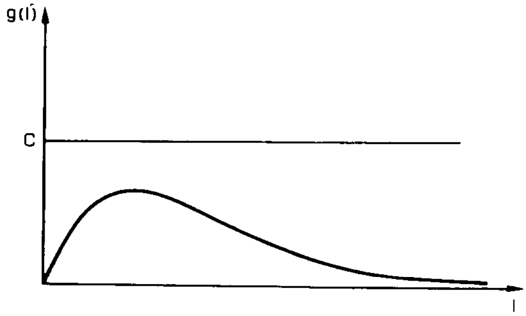
Mathematically the modified model appears as

$$\begin{aligned} \frac{dS}{dt} &= -g(I)S, \\ \frac{dI}{dt} &= g(I)S-\gamma I, \qquad t>0, \\ \frac{dR}{dt} &= \gamma I, \end{aligned} \tag{3}$$

with the same initial conditions (2).



(a)



(b)

FIG. 1 (a) An asymptotically saturating g is illustrated. (b) A function g which takes into account "psychological" effects is illustrated.

In this paper a unique positive solution of the system (3), (2) is shown to exist at any time $t>0$, with a suitable choice of the class of functions to which g may belong. Furthermore, by means of a qualitative phase-space analysis of the system (3), (2) we show that a threshold result can be given which is a generalization of the above stated KMK threshold theorem.

For a particular choice of the function g [of the kind illustrated in Fig. 1(a)] we give the behavior of the trajectories of the system (3), (2) in the phase plane (S,I) (see Figs. 3, 4, 5).

2. THE MATHEMATICAL MODEL

On the basis of the discussion in the introduction, the mathematical model describing our epidemic appears to be given by the following set of differential equations:

$$\begin{aligned} \frac{dS}{dt} &= -g(I)S, \\ \frac{dI}{dt} &= g(I)S-\gamma I, \qquad t>0, \\ \frac{dR}{dt} &= \gamma I, \end{aligned} \tag{3}$$

where $S(t), I(t), R(t)$ denote respectively the number of susceptibles, the number of infectives, and the number of removed individuals (because of death, recovery, etc.; γ, the removal rate, is a positive real quantity).

Here $g:\mathbb{R}_+\to\mathbb{R}_+$ is a continuous bounded function which takes into account the "saturation" phenomenon or the other "psychological" effects extensively discussed in the introduction.

Hence, it will be assumed to satisfy the following conditions:

(i) $\forall x \in \mathbb{R}_+ : g(x) \geqslant 0$.

(ii) $g(0)=0$.

(iii) $\exists c \in \mathbb{R}_+ - \{0\}$ s.t. $\forall x \in \mathbb{R}_+ : g(x) \leqslant c$.

(iv) $g' : R_+ \to \mathbb{R}$, the derivative of g, exists and is bounded on any compact interval of $\mathbb{R}_+$, with $g'(0) > 0$.

(v) $\forall x \in \mathbb{R}_+ : g(x) \leqslant g'(0)x$, where $\mathbb{R}_+ \doteq [0, +\infty[$.

The system (3) has to be supplemented by a set of initial conditions of the following type:

$$S(0)=S_0 \geqslant 0, \qquad I(0)=I_0 \geqslant 0, \qquad R(0)=0 \tag{4}$$

such that

$$S_0 + I_0 = N.$$

Remark 1. We observe that summing the three equations in (3), we have

$$\frac{d}{dt}[S(t)+I(t)+R(t)]=0.$$

Hence, at any time $t \geqslant 0$ belonging to the interval of existence of the solution, we have

$$S(t)+I(t)+R(t)=S_0+I_0=N. \tag{5}$$

3. POSITIVITY, GLOBAL EXISTENCE, AND UNIQUENESS OF THE SOLUTION OF THE SYSTEM (3), (4)

Thanks to conditions (i)–(iv) on g, the system (3), (4) admits a unique continuously differentiable solution $W(t)=(S(t),I(t),R(t))'$ on some finite interval of time $[0,\bar{t}] \subset \mathbb{R}_+$.

Now in order to prove the global existence and uniqueness of the solution (i.e., at any time $t \in \mathbb{R}_+$), we will show that the following lemma holds.

LEMMA 1

If $\mathbf{W}(t)=(S(t),I(t),R(t))'$ *is a continuous solution of the system* (3), (4) *in some interval* $J \subset \mathbb{R}_+$, *then*

$$\forall t \in J \quad : \quad S(t) \geqslant 0, \quad I(t) \geqslant 0, \quad R(t) \geqslant 0. \tag{6}$$

In particular if $S_0>0$, $I_0>0$, *then*

$$\forall t\in J-\{0\} \quad : \quad S(t)>0, \quad I(t)>0, \quad R(t)>0. \tag{7}$$

Proof. Consider the first two equations in (3):

$$\begin{aligned} \frac{dS}{dt} &= -g(I)S, \\ \frac{dI}{dt} &= g(I)S-\gamma I, \end{aligned} \qquad t>0, \tag{8}$$

which are independent of the third one, subject to the initial conditions

$$S(0)=S_0>0, \qquad I(0)=I_0>0. \tag{9}$$

Observe that if $S_0=0$, $I_0=N$, the solution of system (8) is given by

$$S(t)=0; \qquad I(t)=Ne^{-\gamma t}, \quad t\geqslant 0.$$

Hence the I-axis is a trajectory for our system.

We recall that, fixing an initial point $\mathbf{Z}_0=(S_0,I_0)'$, if $\mathbf{Z}(t)=(S(t),I(t))'$ is the corresponding solution, the associated trajectory (in the future) is the subset

$$\Gamma(\mathbf{Z}_0)=\{\mathbf{z}\in\mathbb{R}^2 | \exists t\in\mathbb{R}_+ \text{ s.t. } \mathbf{z}=(S(t),I(t))', \mathbf{Z}_0=(S(0),I(0))'\}. \tag{10}$$

Furthermore, all the points of the S-axis $(S_0,0)'\in\mathbb{R}_+\times\{0\}$ are equilibrium states for the system (see Sec. 4); if we choose one of them as the initial point, we have

$$\forall t\geqslant 0 \quad : \quad S(t)=S_0, \quad I(t)=0. \tag{11}$$

In this case, for any of these initial points the trajectory coincides with the point itself. The S-axis is a union of trajectories.

It is well known that no trajectory may intersect another trajectory in a finite time. Hence, if $S(0)>0$, $I(0)>0$, the corresponding trajectory may not go in a finite time outside of the positive cone

$$D_+^{2\,*}=\{\mathbf{z}\in\mathbb{R}^2 | z_1>0, z_2>0\}.$$

Going back to system (3), we can write for the third equation

$$\forall t\in J \quad : \quad R(t)=R_0+\gamma\int_0^t I(x)\,dx, \tag{12}$$

from which

$$\forall t\in J \quad : \quad R(t)\geqslant 0. \tag{13}$$

If, in particular, $I_0 > 0$ then

$$\forall t \in J - \{0\} \quad : \quad R(t) > 0. \tag{14}$$

This concludes the proof of the lemma. ■

Remark 3. The value of Lemma 1 is evident: it states that the quantities $S(t), I(t), R(t)$ always belong to the "physical" cone of $\mathbb{R}^3$.

Thanks to Lemma 1, (5) and Remark 1 we can now state the following lemma.

LEMMA 2

$$\forall t \in J \quad : \quad \|\mathbf{W}(t)\| \equiv |S(t)| + |I(t)| + |R(t)| = S(t) + I(t) + R(t) = N. \tag{15}$$

Lemma 2 expresses the uniform boundedness of the solution $\mathbf{W}(t)$ for $t \in J$.

Due to Lemma 2 and the remarks at the beginning of this section, it is easy to show the following theorem [5, Chapter 3].

THEOREM 1

A unique continuously differentiable solution $\mathbf{W}(t) = (S(t), I(t), R(t))'$ *exists for the system* (3), (4) *at any time* $t \in \mathbb{R}_+$ *and is such that*:

$$\forall t \in \mathbb{R}_+ \quad : \quad \mathbf{W}(t) \in D_+, \quad \|\mathbf{W}(t)\| = N. \tag{16}$$

If $S_0 > 0$, $I_0 > 0$, *then* $\forall t > 0 : \mathbf{W}(t) \in D_+^*$, *where*

$$D_+ = \{\mathbf{w} \in \mathbb{R}^3 | w_1 \geqslant 0,\ w_2 \geqslant 0,\ w_3 \geqslant 0\}$$

and

$$D_+^* = \{\mathbf{w} \in \mathbb{R}^3 | w_1 > 0,\ w_2 > 0,\ w_3 > 0\}.$$

The system (8), (9) is independent of the third equation in (3), (4); hence from Theorem 1 we can extract the following.

COROLLARY

A unique continuously differentiable solution $\mathbf{Z}(t) = (S(t), I(t))'$ *exists for the system* (8), (9) *at any time* $t \in \mathbb{R}_+$ *and is such that*:

$$\forall t \in \mathbb{R}_+ \quad : \quad \mathbf{Z}(t) \in D_+^2, \quad \|\mathbf{Z}(t)\|_2 = |S(t)| + |I(t)| \leqslant N. \tag{17}$$

If $\mathbf{Z}(0) \in D_+^{2*}$, then $\forall t > 0 : \mathbf{Z}(t) \in D_+^{2*}$ where

$$D_+^2 = \{\mathbf{z} \in \mathbb{R}^2 | z_1 \geqslant 0,\ z_2 \geqslant 0\}.$$

Remark 4. Thanks to (17), if $\mathbf{Z}(t)=(S(t),I(t))'$ is the solution of the system (8), (9), its trajectory is always contained in the triangle

$$T=\{\mathbf{z}\in\mathbb{R}^2 | z_1\geqslant 0,\ z_2\geqslant 0,\ z_1+z_2\leqslant N\}.$$

4. QUALITATIVE ANALYSIS IN THE PHASE PLANE (S,I)

For the time being, we will limit our analysis to the system (8), (9), which, as already stated, is independent of the third equation in (3), (4). We can rewrite the system (8) in the form

$$\frac{d}{dt}\mathbf{Z}(t)=\mathbf{f}(\mathbf{Z}(t)), \qquad t>0 \tag{18}$$

where

$$\mathbf{f}(\mathbf{z})=(f_1(\mathbf{z}),f_2(\mathbf{z}))'=(-g(z_2)z_1, g(z_2)z_1-\gamma z_2)', \qquad \mathbf{z}\in T. \tag{19}$$

We look for the equilibrium points of the system (8), (9) in the "physical" triangle T. As is well known, they are those points $\mathbf{z}^{\mathrm{eq}}\in T$ such that

$$\mathbf{f}(\mathbf{z}^{\mathrm{eq}})=\mathbf{0}. \tag{20}$$

We find that all the points on the positive S-axis,

$$\mathbf{z}^{\mathrm{eq}}=(S_1,0)', \qquad S_1\in[0,N], \tag{21}$$

are equilibrium points for the system (18) in T. The stability character of these equilibrium points can be studied via a linearization technique which is based on the analysis of the Jacobian matrix of $\mathbf{f}$ at the equilibrium points themselves. In our case this matrix proves to be

$$J_{\mathbf{f}}(\mathbf{z}^{\mathrm{eq}})=\begin{pmatrix} 0 & -g'(0)S_1 \\ 0 & g'(0)S_1-\gamma \end{pmatrix}, \qquad S_1\in[0,N]. \tag{22}$$

It is then clear that if $\mathbf{z}^{\mathrm{eq}}\equiv(S_1,0)'$ is such that $S_1>\rho^*\equiv\gamma/g'(0)$, then it will be an unstable point [12, Chapter 1], but nothing can be said for the time being about the points $\mathbf{z}^{\mathrm{eq}}$ for which $S_1\leqslant\rho^*$, because of the presence of a zero eigenvalue [13, Chapter 5].

In any case, observe that from (19) the following inequalities can clearly be obtained:

$$\forall (S,I)'\in T^* \quad : \quad \frac{dS}{dt}=f_1(S,I)<0, \tag{23}$$

$$\forall (S,I)'\in T^* \text{ s.t. } S<\frac{\gamma I}{g(I)} \quad : \quad \frac{dI}{dt}=f_2(S,I)<0, \tag{24}$$

$$\forall (S,I)'\in T^* \text{ s.t. } S>\frac{\gamma I}{g(I)} \quad : \quad \frac{dI}{dt}=f_2(S,I)>0; \tag{25}$$

furthermore

$$\forall (S,I)' \in T^* \text{ s.t. } S=\frac{\gamma I}{g(I)} \quad : \quad \frac{dI}{dt}=f_2(S,I)=0,$$

where

$$T^* = \{\mathbf{z} \in T \mid z_1 > 0, z_2 > 0\}.$$

Remark 5. This implies that in the phase plane (S,I) the curve

$$S=\frac{\gamma I}{g(I)} \tag{26}$$

has some kind of "threshold" character. If the initial point $(S_0,I_0)'$ is, in T^*, on the left of this curve, then $I(t)$ is a decreasing function of t; if $(S_0,I_0)'$ is, in T^*, on the right of this curve, then $I(t)$ is an increasing function of t up to a maximum value placed on the curve itself, and then decreases. On the other hand, the number of susceptibles $S(t)$ is always a decreasing function of t.

It is quite evident that this result is a generalization of the analogous result obtained in the KMK model, which was recalled in the introduction.

Observe that, due to (v) in Sec. 2, the curve (26) always lies on the right of the line $S=\rho^*$, which is intersected by it at least at $(\rho^*,0)'$.

Due to (23)–(25) and Remark 5, it can be shown quite easily that the set

$$T_1 = T - \{\mathbf{z}=(S,I)' \mid S > \rho^*, I=0\} \tag{27}$$

is an invariant set for our system; i.e., for any initial point $\mathbf{z}_0=(S_0,I_0)' \in T_1$, the corresponding trajectory $\Gamma(\mathbf{z}_0)$ (see Remark 4) is contained in T_1.

Furthermore a Lyapounov function V can be given for our system, defined in T_1 as follows:

$$V(\mathbf{z})=(S-\rho^*)H(S-\rho^*)+I; \qquad \mathbf{z}=(S,I)' \in T_1, \tag{28}$$

where H is the Heaviside function

$$\begin{aligned} H(x) &= 0 \quad \text{if} \quad x \leqslant 0,\\ &= 1 \quad \text{if} \quad x > 0. \end{aligned}$$

This function $V(\mathbf{z})$ can be shown to be such that

(i) $V(\mathbf{z})=0$ if $\mathbf{z} \in G$,
(ii) $V(\mathbf{z})>0$ if $\mathbf{z} \in T_1 - G$,
(iii) V is strictly decreasing along any trajectory starting in $T_1 - G$,

where $G=([0,\rho^*] \times \{0\}) \cap T$.

Thus, with the invariance of the set T_1, we can state the following.

THEOREM 2 [*6*] *(SEE ALSO* [*10; CHAPTER 2*]*)*

The set G is asymptotically stable with respect to T_1.

It is possible to give more information about the stability of each point in G by means of the following remarks.

From (8) and (9) the following differential equation for the trajectory of our system in the phase plane (S,I) can be obtained:

$$\frac{dI}{dS} = -1 + \frac{\gamma I}{g(I)S}, \qquad 0<S<S_0, \quad I \geqslant 0 \tag{29}$$

with the initial condition

$$I(S_0) = I_0. \tag{30}$$

Due to (v) in Sec. 2, if we set $k = g'(0)$, we have

$$\frac{dI}{dS} \geqslant -1 + \frac{\gamma}{kS}, \qquad 0<S<S_0, \quad I>0.$$

Consider now the differential equation of the trajectory in the KMK model,

$$\frac{d\bar{I}}{dS} = -1 + \frac{\gamma}{kS}, \qquad 0<S<S_0, \quad I \geqslant 0$$

with the same initial condition (30):

$$\bar{I}(S_0) = I_0$$

It is quite easy to show that

$$I(S) \leqslant \bar{I}(S) = N - S + \frac{\gamma}{k}\ln\frac{S}{S_0}, \qquad 0<S<S_0, \quad I \geqslant 0. \tag{31}$$

Remark 6. The RHS of (31) gives the corresponding trajectory of the epidemic for the KMK model [14]. Hence we can state that our trajectory will always lie under this trajectory.

Furthermore, owing to the asymptotic stability of the set G, for t tending to infinity the trajectory of the system starting at any point $(S_0, I_0)' \in T^*$ will end at some point $(S(\infty), 0)' \in G$ such that

$$S(\infty) = \lim_{t\to\infty} S(t) \geqslant \bar{S}(\infty) > 0, \tag{32.1}$$

where we have denoted by $(\bar{S}(\infty), 0)'$ the limit point of the corresponding trajectory in the KMK model. As in the classical KMK model, the spread of the disease does not stop for lack of a susceptible population (see Fig. 2).

On the other hand, the total number of removed individuals, which we suggest to be the correct measure of the danger of the epidemic, is such that

$$R(\infty)=N-S(\infty)-I(\infty)=N-S(\infty)\leqslant N-\bar{S}(\infty). \tag{32.2}$$

Taking into account (32), this means that $R(\infty)$ is always less than the corresponding quantity $\bar{R}(\infty)$ in the KMK model (see Table 1 for the example in Sec. 6).

Consider now any point $\mathbf{z}^{\mathrm{eq}}=(S_1,0)'\in G$. With some tedious calculations it is possible to show, thanks to (31), (32) and Theorem 2, that

$$\forall\varepsilon>0\ \exists\delta>0\ \text{s.t.}$$
$$\forall\mathbf{Z}_0\in T\ :\ \|\mathbf{Z}_0-\mathbf{z}^{\mathrm{eq}}\|_2<\delta\ \Rightarrow\ \forall t>0\ :\ \|\mathbf{Z}(t)-\mathbf{z}^{\mathrm{eq}}\|_2<\varepsilon,$$

which expresses the stability of $\mathbf{z}^{\mathrm{eq}}$ [12, Chapter 1].

We can now state the following.

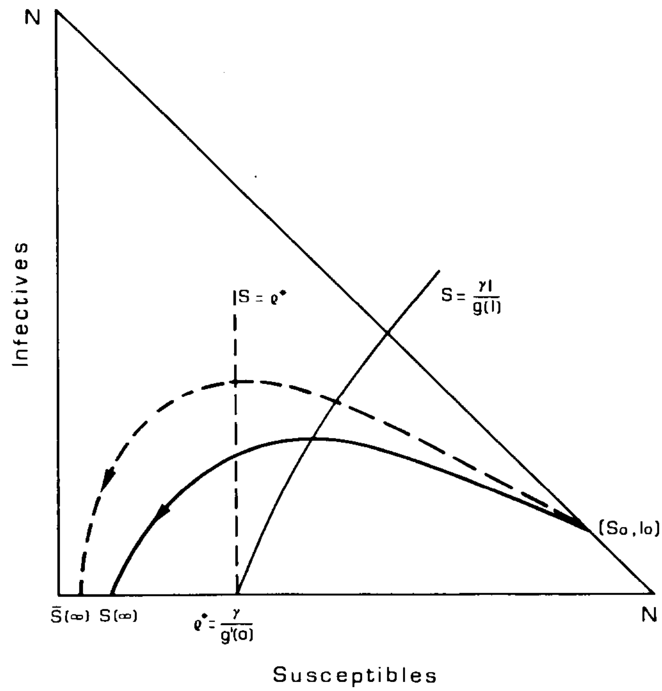


FIG. 2 A qualitative comparison between a KMK trajectory (continuous curve) and the corresponding trajectory in the proposed model (dashed curve). The "threshold curve" $S=\gamma I/g(I)$ is evidenced in comparison with the "threshold line" $S=\rho^*$ of the KMK model.

THEOREM 3

In the phase plane (S,I), all the points $\mathbf{z}^{\mathrm{eq}} \equiv (S_1, 0)' \in [0,N] \times \{0\}$ on the S-axis are critical points for the system. The points for which $S_1 > \rho^* = \gamma / g'(0)$ are unstable points, while the points for which $S_1 \leqslant \rho^*$ are stable.

The set $G = ([0,\rho^*] \times \{0\}) \cap T$ is asymptotically stable.

Remark 7. It is quite clear that if ρ^* is greater than N, then this "threshold" point is out of the "physical" triangle T: all the points in T on the S-axis are stable points.

5. THE MODEL WITH EMIGRATION OF SUSCEPTIBLES

It is quite interesting to see what happens if we introduce an emigration term $-\lambda S$ in the first of the equations (3), with $\lambda > 0$ as small as we like. This corresponds to allowing uninfective susceptibles to go directly to the removed class because of emigration or immunization from the disease.

The system (3) becomes

$$\begin{aligned} \frac{dS}{dt} &= -g(I)S - \lambda S, \\ \frac{dI}{dt} &= g(I)S - \gamma I, \\ \frac{dR}{dt} &= \lambda S + \gamma I, \end{aligned} \tag{33}$$

with the usual initial conditions (4).

In this case all the results listed in Theorem 1 maintain their validity.

If we limit ourselves, as before, to the first two equations, we can see that the only equilibrium point in the positive cone is now $\mathbf{0} = (0,0)'$. Because the Jacobian matrix of the system (33) at $\mathbf{0}$ is

$$J(\mathbf{0}) = \begin{pmatrix} -\lambda & 0 \\ 0 & -\gamma \end{pmatrix},$$

for which both the eigenvalues are real and less than zero, it is clear that this point is always asymptotically stable (see Theorem 1 in [13], Chapter 5).

This means that with the perturbation introduced in (33) even if the parameter $\lambda > 0$ is as small as one likes, the phase plane pattern of the system will change, contracting the asymptotic stability region G to the single point $\mathbf{0} = (0,0)'$.

The epidemic will always tend to extinction with respect to both infectives and susceptibles. Hence $R(\infty) = N$. [Note that in this case it is not true any more that $R(\infty)$ is a measure of the danger of an epidemic, because a direct transfer from the class (S) to the class (R) is now allowed.]

6. AN EXAMPLE

In order to illustrate in detail the behavior of the trajectories of the system in the phase plane (S,I), and make a comparison with the classical KMK model, we suggest the following example for the function g in (3):

$$g(I)=k\frac{I}{1+(I/\alpha)}, \qquad I\in\mathbb{R}_+, \tag{34}$$

where $k>0$, $\alpha>0$.

It is quite obvious that this function g in (34) satisfies all properties (i)–(v) required in general.

Remark 8. From the form of this function g in (34) we can easily observe that the system (3) gives the KMK model for α tending to infinity.

Now, after introducing this g, the system (3) appears in the form

$$\begin{aligned}
\frac{dS(t)}{dt}&=-k\frac{I}{1+(I/\alpha)}S,\\
\frac{dI(t)}{dt}&=k\frac{I}{1+(I/\alpha)}S-\gamma I, \qquad t>0,\\
\frac{dR(t)}{dt}&=\gamma I,
\end{aligned} \tag{35}$$

subject to the initial conditions (4).

It will be useful to normalize the quantities $S(t)$, $I(t)$ and $R(t)$ by introducing the new variables

$$s(t)=\frac{S(t)}{N}, \qquad i(t)=\frac{I(t)}{N}, \qquad r(t)=\frac{R(t)}{N}, \tag{36}$$

such that

$$s(t)+i(t)+r(t)=s_0+i_0=1. \tag{37}$$

With these new variables, system (35) becomes

$$\left.\begin{aligned}
\frac{ds(t)}{dt}&=-\beta\frac{i(t)}{1+i(t)/\alpha^*}s(t),\\
\frac{di(t)}{dt}&=+\beta\frac{i(t)}{1+i(t)/\alpha^*}s(t)-\gamma i(t),\\
\frac{dr(t)}{dt}&=\gamma i(t),
\end{aligned}\right\} \qquad t>0, \tag{38}$$

with initial conditions

$$s(0)=s_0, \qquad i(0)=i_0, \qquad r(0)=0, \tag{39}$$

where

$$\alpha^*=\frac{\alpha}{N}, \qquad \beta=kN.$$

We can apply to the first two equations in (38) the phase plane analysis of Sec. 4. In this case the "threshold curve" (26) in the phase plane (s,i) is the straight line

$$s=\rho(\alpha^*+i), \tag{40}$$

where

$$\rho=\frac{\gamma}{k\alpha}=\frac{\gamma}{\beta\alpha^*},$$

from which the "threshold point," i.e., the intersection of the "threshold line" with the s-axis, is

$$\mathbf{P}^*\equiv(\rho^*,0), \tag{41}$$

where $\rho^*=\rho\alpha^*=\gamma/\beta$ as in the classical KMK model. In this case it is also possible to obtain explicitly the equations of the trajectories in the phase plane, solving the differential equation

$$\frac{di}{ds}=-1+\rho(\alpha^*+i)\frac{1}{s} \tag{42}$$

obtained from the first two equations in (38).

The general integral of (42) is

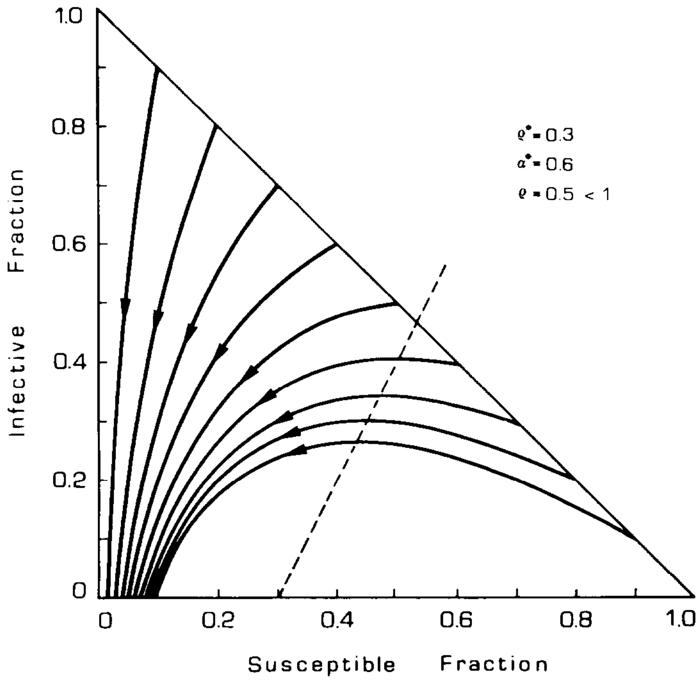
$$i(s)=cs^\rho+\frac{1}{\rho-1}s-\alpha^* \qquad \text{if} \quad \rho\neq 1, \tag{43.1}$$

$$i(s)=cs-s\ln s-\alpha^* \qquad \text{if} \quad \rho=1, \tag{43.2}$$

where $c\in\mathbb{R}$ is the integration constant, which can be determined imposing the initial conditions (39).

The behavior in the phase plane (s,i) of a population is characterized by the values of the parameters ρ^* and α^* $(\rho=\rho^*/\alpha^*)$. Different trajectories for the same population are obtained by varying the initial conditions, i.e., the constant c in (43).

Figures 3(a, b) and 4(a, b) illustrate the behavior of trajectories (43.1) for different values of the parameters ρ^*, α^*. In Fig. 3 these are such that ρ is less than one, while in Fig. 4 ρ is greater than one.



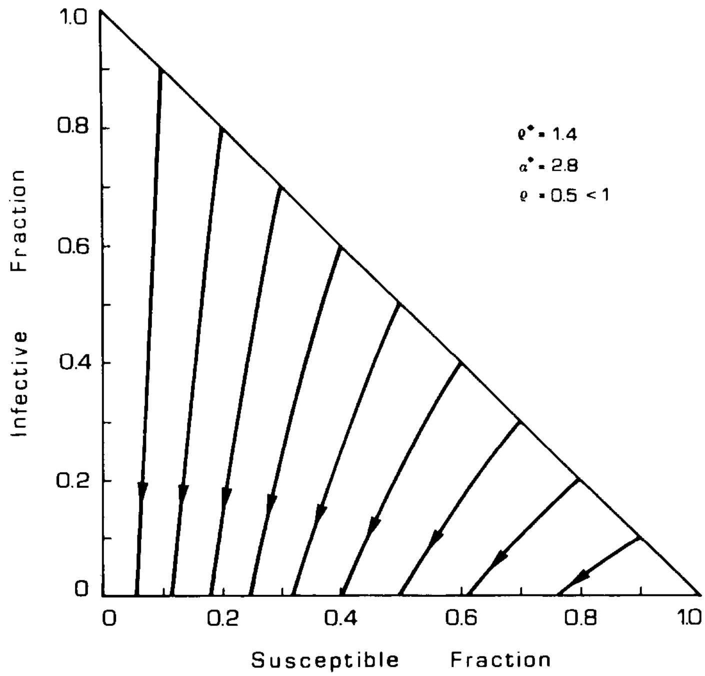


FIG. 3 Typical (s,i) plane portraits for $\rho^* < 1$ (a) and $\rho^* > 1$ (b), with $\rho < 1$. The dashed line in (a) denotes the "threshold curve" (40).

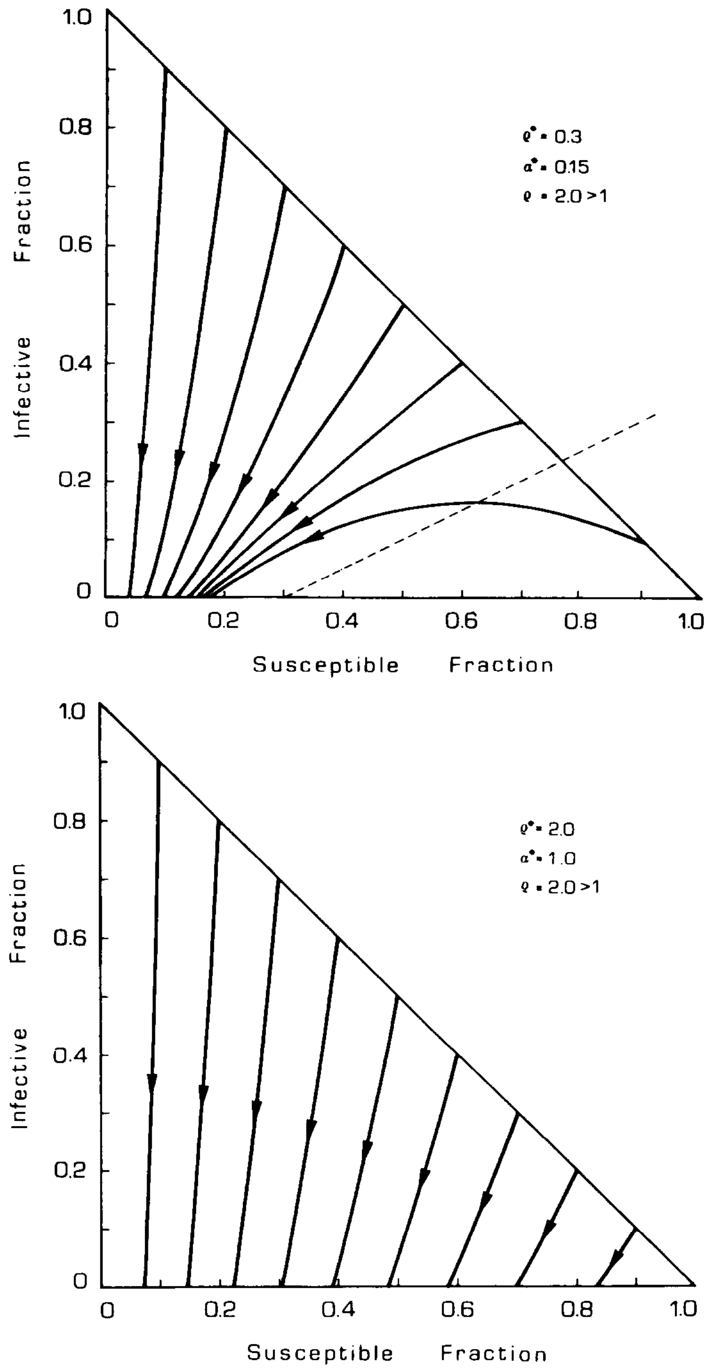


FIG. 4 Typical (s,i) plane portraits for $\rho^* < 1$ (a) and $\rho^* > 1$ (b), with $\rho > 1$. The dashed line in (a) denotes the "threshold curve" (40).

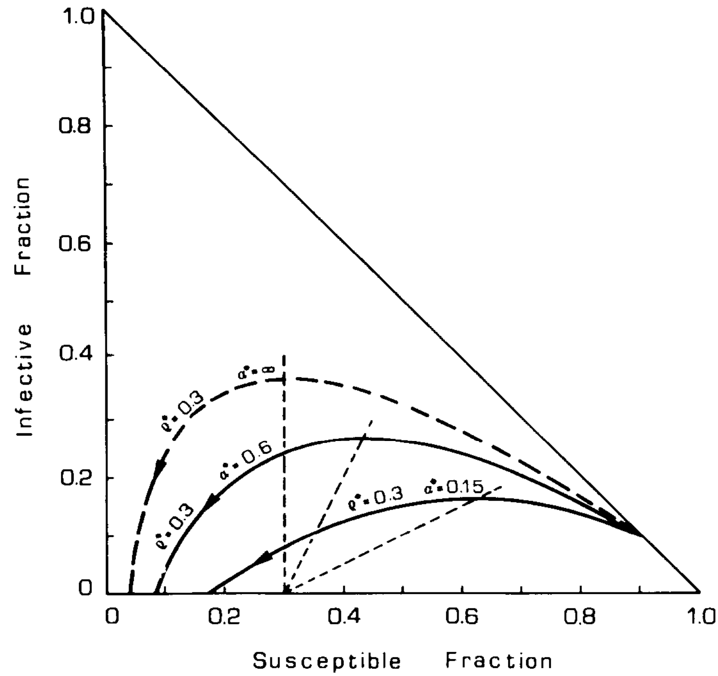


FIG. 5 Trajectories in the (s, i) plane corresponding to the same initial condition and the same value of ρ^* with increasing values of α^*. The curve with $\alpha^* = +\infty$ corresponds to the KMK model.

Remark 9. If we rewrite (40) in the form

$$s = \frac{\gamma}{\beta}\left(1 + \frac{i}{\alpha^*}\right)$$

for $\alpha^* \to \infty$ (which corresponds to the KMK model), we obtain for the "threshold curve" the known result

$$s = \frac{\gamma}{\beta} = \rho^*.$$

Figure 5 illustrates this fact. With the same initial conditions and the same value of ρ^*, three different trajectories are given relative to increasing value of α^*.

In Table 1 a comparison of of $\bar{r}(\infty)$ (the fraction of removed individuals in the KMK model) with $r(\infty)$ (the corresponding quantity in our model) is made, with different values of α^* and the same value of ρ^*.

TABLE 1
Comparison between the Fraction $r(\infty)$ of Removed Individuals with the Corresponding Quantity $\bar{r}(\infty)$ in the KMK Model

	$r(\infty)^a$		
(s_0, i_0)	$\rho=2$ $\alpha^*=0.15$	$\rho=0.5$ $\alpha^*=0.6$	$\bar{r}(\infty)^a$
(0.9,0.1)	0.813	0.906	0.963
(0.8,0.2)	0.820	0.913	0.967
(0.7,0.3)	0.833	0.926	0.971
(0.6,0.4)	0.840	0.933	0.976
(0.5,0.5)	0.860	0.940	0.979
(0.4,0.6)	0.880	0.953	0.984
(0.3,0.7)	0.906	0.960	0.988
(0.2,0.8)	0.933	0.963	0.992
(0.1,0.9)	0.960	0.967	0.996

[a] $\rho^*=0.3$.

7. CONCLUDING REMARKS

Let us conclude with a few comments on the assumptions and the results obtained.

The mathematical assumptions on the function $g(I)$ which appears in the interaction term simply reflect the "physical" ones widely discussed in the introduction.

If the initial number of susceptibles S_0 and the number of infectives I_0 are both strictly positive, then a unique solution exists for our model which is always in the physical cone $S \geqslant 0$, $I \geqslant 0$, $R \geqslant 0$, on the plane $S+I+R=N$, where N is the total population number.

In the phase plane (S,I) the system evolves in the triangle $S+I \leqslant N$. A "threshold curve" is shown to exist in this phase plane which generalizes the "threshold line" of the KMK model. If the initial point $(S_0, I_0)'$ is on the left of this curve, then $I(t)$ is a decreasing function of t; if $(S_0, I_0)'$ is on the right of this curve, then $I(t)$ is an increasing function of t up to a maximum value placed on the curve itself, and then decreases. The number of susceptibles is always a decreasing function of t.

But the epidemic as in the KMK model never stops for lack of susceptibles: the number of infectives always tends to zero, but $S(\infty)$ is always greater than zero.

As a measure of the size of the epidemic we use the value of $R(\infty)$. In our model it is always less than the corresponding KMK value, as expected.

An example is shown in detail corresponding to a particular choice of the function $g(I)$, which contains a parameter α^* such that the KMK model can be reobtained for $\alpha^* \to \infty$.

In Sec. 5 it has been pointed out that the phase space pattern of the system changes with the introduction of an emigration term $(-\lambda S)$ of susceptibles: the region of asymptotic stability $G=([0,\rho^*]\times\{0\})\cap T$ reduces to the single point $\mathbf{0}=(0,0)'$. The epidemic will thus always tend to extinction.

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