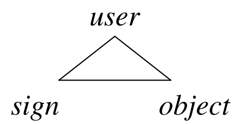
1 Logic, language, thought

- Be advised not to take logic as a model of human thought and reasoning.
- Thought and reasoning are fairly complicated mental processes that do not seem to behave as dictated – at least – by standard logics, and are the subject of psychology, linguistics, artificial intelligence and related fields.
- Laws of reasoning in logic, with some provisos, may be argued to apply to the end-products of thought, idealized as propositions, predicates, quantification, and so on, which are expressed in a formal (for now read as "human-made" or "artificial") language. Logical laws can at best be utilized in justifying these end-products, rather than characterizing how they are discovered or arise in the mind/brain of the thinker.[1]
- This distinction between discovery and justification starts to apply already in mathematics before even approaching "everyday" thinking. Theorems are *discovered* via largely intuitive (read as "not scientifically explained yet") means, they are *justified* (checked for validity) by formal tools of logic.[2]

2 Symbols

- Signs and signification are central concepts in language, logic, and computation.
- An initial and rough conception of signification is a three-part relation:[3]



- Again some initial and rough characterizations:
  - A user uses a sign to **refer** to an object.
  - A sign **denotes** an object.
  - A user has certain **intentions** about an object.

3 Propositions

- Let's characterize what a **proposition** is, indirectly by way of natural language. Our characterization has two parts:

  i. Any expression that you can insert into the contexts "___ is true" or "___ is false" *denotes* a proposition.[4]

  ii. There are no propositions that cannot be denoted as such.

  The following expressions denote propositions:

  The earth revolves around the sun.

  Dünya Güneş etrafında dönüyor.

  John likes Mary.

  If you multiply an even number with an odd number, you obtain an even number.

  while the following do not:

  Around the sun

  Dünya Güneş etrafında dönüyor mu?

  Because John likes Mary

[1] See the introduction to Reichenbach (1947) for some relevant discussion. The classical work dealing mainly with the distinction between human thought and inference on one hand and formal logic on the other is Johnson-Laird (1983)

[2] Although the book itself is advanced for introductory level, (Quine, 1958, §16) discusses this issue.

[3] If you are interested in the study of signs, a good place to start reading is the founders of the science of signs – named **semiotics** or **semiology**: Ferdinand de Saussure's "Course in General Linguistics" and Charles Sanders Peirce' (pronounced like the word *purse*) "Collected Papers". Be warned that Peirce might be quite challenging for beginners, but Saussure is a must read for everyone with a serious interest in language.

[4] The contexts "It is the case that ___" or "It is not the case that ___" will equally do, barring some complications due to quotation. It gets yet more complicated when it comes to Turkish. Can you see why?

If you multiply an even number with an odd number

- In other words, **declarative sentences** denote propositions.
- Now let's assume that every proposition whatsoever has at least one declarative sentence to express it.
- From all we have above it follows that:

  Propositions are objects that can be (said to be) true or false.
- Another way of saying this is that propositions are objects that have **truth values**.
- This much explication of what a proposition is will suffice for our purposes.
- Propositional logic is called so, because its **atomic**[5] symbols refer to objects called **propositions**.
- Can you see what could be the problem with equating declarative sentences with propositions?[6]
- We said that expressions of propositional logic denote propositions, but what are those expressions?

4 Defining a language L_0

- Any specification of a language starts with its alphabet.

  Our alphabet for L_0 – the name we will give to our language – is made up of three sets:

[5]The adjective *atomic* serves to suggest that atomic symbols cannot be broken down to further components. This use of *atomic* is a remnant from a pre-nuclear conception of atoms.

[6]Note to be read after discussing the question. For Willard van Orman Quine, one of the giants of analytical philosophy, what is problematic is not equating propositions with their linguistic expressions but rather the opposite. He objects to maintaining that there are abstract objects called propositions which have their own life independent of the means that express or denote them. Here I chose the "anti-Quinean" exposition, because the discussion of Quine's objection is rather advanced for the current stage. See his "Two dogmas of empiricism" for a starter on this arguably the most central issue of analytical philosophy.

Basic symbols:

$$P = \{p, q, r, s, p_1, q_1, r_1, s_1, \dots\} \tag{1}$$

Connective symbols:

$$C_1 = \{-\} \tag{2}$$
$$C_2 = \{\wedge, \vee\} \tag{3}$$

Parentheses: $\{(,)\}$

- We can also collect the parts of our alphabet under a single set:

$$\Sigma = P \cup C_1 \cup C_2 \cup \{(,)\} \tag{4}$$

- Let's call an **expression**, any finite sequence of (possibly repeated) elements from Σ. The following are some example expressions:

$$p(\wedge r_{4291841} \vee - \qquad p - p \qquad -)) \qquad \wedge \vee \wedge \qquad q_{345}\wedge$$

- It is not hard to see that there is no bound to the expressions we can form this way. Call this non-finite set of all the possible finite expressions formed by putting together a selection of symbols from Σ in a specific order Σ^*.
- Usually, we are interested in expressions fulfilling certain criteria – a subset of the set of all possible expressions. We distinguish these special expressions as the grammatical sentences of the language we are interested in.
- Now we can define the grammatical expressions (sentences of **well-formed formulas**) of our language L_0. And we will see that $L_0 \subset \Sigma^*$.

- As you might have already realized, we identify a language with the set of its grammatical sentences.

Definition 4.1
Well-formed formulas of L_0.

i. $\alpha \in L_0$, if $\alpha \in P$.
ii. if $\alpha \in L_0$, so is $(\gamma\alpha)$, for $\gamma \in C_1$.
iii. if α, $\beta \in L_0$, then so is $(\alpha\gamma\beta)$, for $\gamma \in C_2$.
iv. Nothing else is in L_0.

- An analytic procedure for defining the notion "sentence of L_0". We start from a complex expression and go down.

Definition 4.2 (wff's of L_0, "top-down")

i. α is a wff if $\alpha \in P$.
ii. $(-\alpha)$ is a wff iff α is a wff.
iii. $(\alpha \wedge \beta)$ is a wff iff α and β are wff's.
iv. $(\alpha \vee \beta)$ is a wff iff α and β are wff's.
v. Any expression that falls out of the above is not a wff.

- Here is a synthetic way of specifying our sentences. We start with the simplest expressions and go up.

Definition 4.3 (L_0, "bottom-up")
An expression is a wff (or belong to L_0) iff it is built from the elements of P by applying a finite number of the following operations:[7]

$$f_-(\alpha) = (-\alpha) \tag{5}$$
$$f_\wedge(\alpha, \beta) = (\alpha \wedge \beta) \tag{6}$$
$$f_\vee(\alpha, \beta) = (\alpha \vee \beta) \tag{7}$$

- This could be put more formally:

Definition 4.4 (L_0, "bottom-up")
An expression α is a wff iff there exists a sequence of expressions ordered in increasing length:

$$\alpha_1, \alpha_2, \ldots, \alpha_n$$

such that $\alpha = \alpha_n$, and for any α_i for $i \leq n$, either

i. $\alpha_i \in P$;
ii. or there exists $j, k < i$ such that $\alpha_i = f_\wedge(\alpha_j, \alpha_k)$ or $\alpha_i = f_\vee(\alpha_j, \alpha_k)$
iii. or there exists a $j < i$ such that $\alpha_i = f_-(\alpha_j)$,
where f_-, $f_\wedge$, and $f_\vee$ are as defined in Definition 4.3.

- All these definitions not only define what the wff's of L_0 are, but also their structure.
- This is best observed over **derivation** (or construction) trees.

Example 4.5 (Derivation trees)
Draw the derivation tree of $((p \wedge q) \vee ((-r) \vee (q \wedge (-s))))$

[7] What is the difference between the parentheses on the left and right side of the equalities?

5 Conjunction, alternation, negation

- In the previous section, we established the first half of a formal system. We know which sequences of symbols constitute a sentence of our system. However, we do not yet know what our sentences mean. Let's start with the simplest expressions of L_0.

- We will take our basic symbols to refer to propositions. For instance, we can agree on things like, p stands for the proposition that the snow is white. Given this, we can see that p is interchangeable with declarative sentences that express this proposition. Therefore, we can say

  p is interchangeable with 'The snow is white.'

  p is interchangeable with 'Kar beyazdır.'

  and, so on.

- Sometimes you will see 'means', 'abbreviates', 'stands for', 'equals to', '=', and so on, in place of 'is interchangeable with'.

- When we use a basic symbol of L_0 as part of a sentence of L_0, which possibly consists of only that symbol, we assert that the proposition referred to by the symbol holds. So,

$$p \tag{8}$$

  says that the snow is white.

- The way to assert that a proposition named by a symbol does not hold, or, equivalently, to deny that a proposition named by a symbol, one puts a negation sign in front of it . For instance,

$$(-p) \tag{9}$$

  says that it is not the case that the snow is white.

- In science and other discourses, we are not only interested in what propositions are expressed, but also whether the expressed propositions actually hold or not.

- For any proposition whether it holds or not is indicated by its **truth value**.[8] If a proposition holds, we will say that it is true, and its truth value is 1. If a proposition does not hold, we will say that it is false, and its truth value is 0. Using the first two natural numbers to designate truth values is totally arbitrary, you can pick any two symbols which will not lead to confusion.

- We will assume that, every proposition expressible in our system is either true or false, there is no case in between.

- A sentence consisting only of a basic symbol of L_0 has the same truth value as the proposition it expresses. Therefore, (8) is true, or has the truth value 1, if the snow is actually white, and is false, or has the truth value 0, if it is not the case that snow is white.

- From now on we will directly speak of the truth or falsity of the sentences of L_0, as well as of the propositions that they refer to.

- The negation of a sentence consisting only of a basic symbol has the opposite truth value of the symbol. '-p' is true if and only if 'p' is false.

- Of course, it would not be much interesting to speak about propositions and their denial only one-by-one.

- One way to form complex sentences out of simple ones is **conjunction**. On the meaning side, a conjunction asserts that both conjuncts are true. If at least one of the conjuncts fails to be true, then the conjunction fails to be true as well. If p stands for snow's being white and q stands for Berlin being the capital of France,

$$(p \wedge q) \tag{10}$$

  is true if and only if the snow is white and Berlin is the capital of France.

[8]Note that we are counting on our intuitions here for the meaning of "holding" vs. "not holding". A mathematical rendition of this concept will follow soon.

- Another binary connective is **alternation** (or **disjunction**). It differs from conjunction in being more tolerant about falsehood. The conjunction of two sentences come out true if at least one of the alternates is true.

$$(p \vee q) \tag{11}$$

  is true if and only if the snow is white, or Berlin is the capital of France, or both.

- Note that we use "if and only if" in stating the cases under which conjunction and alternation are true. This is to say that they are false otherwise.

- We would still have a quite impoverished language, if our connectives were to apply only to basic symbols. Fortunately, our connectives are totally blind to the internal makeup of the sentences they connect.[9] All they care, as our definition of sentence given in the previous section makes clear, is that they connect sentences (or **formulas**) of propositional logic, either simple or complex.

- We will use capital symbols P,Q,R,S and their subscripted forms as variables standing for formulas.

- For any formulas P, $Q \in L_0$:

$$\begin{aligned} (P \wedge Q) &\text{ is true, if and only if both } P \text{ and } Q \text{ are true.}\\ (P \vee Q) &\text{ is true, if and only if at least one of } P \text{ and } Q \text{ is true.}\\ (-P) &\text{ is true, if and only if } P \text{ is false.} \end{aligned}$$

- Conjunction and alternation enjoy certain properties familiar from set union and intersection:

$$\begin{gathered} (P \wedge P) = P\\ (P \wedge Q) = (Q \wedge P)\\ ((P \wedge Q) \wedge R) = (P \wedge (Q \wedge R)) \end{gathered}$$

[9]With some abuse of the meaning of "connect", we treat negation as a connective as well, a one-place connective.

  and likewise for alternation.

- Observe that the equalities we list above are on the basis of truth values. There is no way the two equated sentences can differ in their truth value.

- Our definition of L_0 requires the insertion of parenthesis in every step of connection, thereby making every possible sentence ambiguous with regards to what is grouped with what. However, for ease of inspection and writing, we will omit parentheses where no ambiguity arises.

  For instance there is no harm in writing '$P \wedge Q$' instead of '$(P \wedge Q)$', or '$P \wedge Q \wedge R$' instead of '$((P \wedge Q) \wedge R)$'.

- But we need to agree on a convention when omitting parentheses from sentences with negation. For instance if we have '$-P \wedge Q$', we need to make clear the intended scope of negation, do we mean '$(-(P \wedge Q))$' or '$((-P) \wedge Q)$'? We will say that negation **binds more tightly** than conjunction and alternation. In this convention, '$-P \wedge Q$' will mean '$((-P) \wedge Q)$'. In simplifying '$(-(P \wedge Q))$', the furthest we can go is '$-(P \wedge Q)$'

Example 5.1
Let p abbreviate 'John took vitamin C', and q, 'John got flu.'

$p \wedge q$	John took vitamin C and John got flu.
$-p \wedge q$	John did not take vitamin C and John got flu.
$p \wedge -q$	John took vitamin C and John did not get flu.
$-p \wedge -q$	John did not take vitamin C and John did not get flu

What would '$-(p \wedge q)$' or '$-(p \vee q)$' be?

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6 Truth functions, truth tables, valuations

- As our definitions for conjunction, disjunction and negation reveal, the truth value of a formula formed by a certain connective depends entirely on the truth values of the formula(s) being connected and the definition of the connective.

- For instance you can see conjunction as an $f : \{0,1\} \times \{0,1\} \to \{0,1\}$. Likewise for alternation. Negation is an $f : \{0,1\} \to \{0,1\}$, namely $\{(0,1),(1,0)\}$.

- A more pictorial way of representing this is to use a **truth table**.

  For conjunction:

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

- This property makes our connectives **truth functional**.

- Can you think of a non-truth functional connective from natural language, which forms a declarative sentence by connecting two declarative sentences, but the truth value of the resulting sentence does not depend only on the truth values of the connected sentences.

- A **valuation** is a function from the set of propositional symbols to $\{0,1\}$. It tells which of our basic propositions are true and which are false.

  From this perspective, every formula imposes a filter on valuations; it picks a subset of the set of all possible valuations. This subset is the set of valuations that make the formula in question true.

7 Conditional

- Under what circumstances would you consider the person who uttered the following sentence to have kept her word?

  "If I get a job next summer, then I will marry you."

- We add a new connective '$\to$', named **conditional** (or **material conditional**), and which is read "if…then", or "only if". It has the following truth-table:

P	Q	$P \to Q$
1	1	1
1	0	0
0	1	1
0	0	1

- The conditional is redundant, you can define it in terms of the other connectives.

- A related connective, which is again redundant, is '$\leftrightarrow$', named **biconditional**. It is read "if and only if" and comes out true whenever the two components agree in their truth value.

- Both connectives of this section bind less tightly then conjunction and alternation. Therefore '$p \vee q \to r$' is '$((p \vee q) \to r)$'; you can simplify '$(p \vee (q \to r))$' at most as '$p \vee (q \to r)$'.

8 Validity, contradictoriness, consistency

- A formula P is **valid** if and only if it always comes out true regardless of the truth values of its components.

  We designate the validity of P by:

$$\models P$$

  Valid formulas of propositional logic are also called **tautologous**.

  Some examples:

$$\models p \vee -p$$
$$\models p \to (q \to p)$$

- A formula P is **contradictory**, designated $\not\models P$, if and only if it always comes out false regardless of the truth values of its components.

Exercise 8.1
Can you think of a general method of testing whether a given formula is valid, contradictory, or consistent?

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9 Implication, equivalence and substitution

- A formula P **implies** Q, designated $P \models Q$, if and only if $\models P \rightarrow Q$.
- Two formulas P and Q are **equivalent**, designated $P \equiv Q$, if and only if $\models P \leftrightarrow Q$.
- Given a formula R, $R \equiv R_{[P/Q]}$ iff $P \equiv Q$, where $R_{[P/Q]}$ means the formula formed by substituting P for each occurrence of Q in R.

Exercise 9.1
Given '$p \rightarrow q$', state which of the following imply or are implied by it:

$-p$	q	$-p \vee q$	$q \wedge r$
	$p \rightarrow q \vee r$	$p \vee r \rightarrow q$	
$p \rightarrow q \wedge r$	$(p \rightarrow q) \vee r$	$-q \rightarrow -p$	$p \wedge r \rightarrow q$

---

10 Natural deduction

Natural deduction is a method for **deriving** a formula Q from (possibly empty) set of formulas $\{P_1, P_2, \ldots, P_n\}$, called **premisses**, such that Q is true in case all the premisses are jointly true. The relation of derivability is designated as:

$$P_1, P_2, \ldots, P_n \vdash Q$$

and,

$$P_1, P_2, \ldots, P_n \vdash Q \quad \text{iff} \quad P_1 \wedge P_2 \wedge \ldots \wedge P_n \models Q$$

The method of natural deduction involves a set of **inference rules** and **proof techniques**.

10.1 Inference rules

Simplification:

$$\frac{P \wedge Q}{P} \qquad \frac{P \wedge Q}{Q}$$

Adjunction:

$$\frac{\begin{array}{c}P\\Q\end{array}}{P \wedge Q} \qquad \frac{\begin{array}{c}P\\Q\end{array}}{Q \wedge P}$$

Addition:

$$\frac{P}{P \vee Q} \qquad \frac{Q}{P \vee Q}$$

Modus ponens (MP):

$$\frac{\begin{array}{l}P \rightarrow Q\\P\end{array}}{Q}$$

Modus tollens (MT):

$$\frac{\begin{array}{l}P \rightarrow Q\\-Q\end{array}}{-P}$$

Modus tollendo ponens (MTP):

$$\frac{\begin{array}{c}P \vee Q \\ -P\end{array}}{Q} \qquad \frac{\begin{array}{c}P \vee Q \\ -Q\end{array}}{P}$$

Double negation:

$$\frac{--P}{P} \qquad \frac{P}{--P}$$

Repetition:

$$\frac{P}{P}$$

10.2 Proof techniques

10.2.1 Direct proof

Let's illustrate how a direct proof works over an example,

$$p \wedge q \vdash q \wedge p$$

We start by designating our target – the formula we aim to derive:

In proofs we can pick and add any of our premisses at any point, if we believe it will be useful. Here we do that, and add our only premiss.

1.	*Show* $q \wedge p$	
2.	$p \wedge q$	Prem.

Next we observe that we can apply one of our rules, simplification, to the premiss twice, obtaining p and q. In the ideal case we write the justification of each step.

1.	*Show* $q \wedge p$	
2.	$p \wedge q$	Prem.
3.	p	2 Simp.
4.	q	2 SSimp.

Next we use another rule, adjunction, to form the desired formula

1.	*Show* $q \wedge p$	
2.	$p \wedge q$	Prem.
3.	p	2 Simp.
4.	q	2 Simp.
5.	$q \wedge p$	Adj.

In a direct proof, when we obtain the formula we wanted to derive, we "box" the proof, and cancel the initial *Show*.

1.	~~*Show*~~ $q \wedge p$	
2.	$p \wedge q$	Prem.
3.	p	2 Simp.
4.	q	2 Simp.
5.	$q \wedge p$	3, 4 Adj.

Although direct proof is conceptually simple, it is seldom adequate on its own.

10.2.2 Conditional proof

When the target formula is a conditional, we assume the antecdent and show that the consequent is derviable under this assumption. Take,

$$-q \rightarrow -r, \quad p \rightarrow r \vdash p \rightarrow q \tag{12}$$

Again we start with a *Show* line.

1.	*Show* $p \to q$

In a conditional proof, we start with assuming the antecedent, p in this case:

1.	*Show* $p \to q$	
2.	p	Asmp.

The aim is to derive the consequent q. In this task, in addition to the premisses, we are allowed to make use of the assumption p.From here on we proceed as in a direct proof of q, namely applying the available rules to the formulas available. It is crucial to observe that we cannot apply MP to $p \to q$ and p. Any formula that has an uncancelled *Show* is UNavailable in a proof. In order to proceed, we take a premiss that we can feed into the MP rule together with p and obtain r.

1.	*Show* $p \to q$	
2.	p	Asmp.
3.	$p \to r$	Prem.
4.	r	2, 3 MP

The rest of the proof proceeds in a similar fashion, eventually obtaining q, boxing the proof and cancelling the *Show*.

1.	~~*Show*~~ $p \to q$	
2.	p	Asmp.
3.	$p \to r$	Prem.
4.	r	2, 3 MP
5.	$--r$	4 DN
6.	$-q \to -r$	Prem.
7.	$--q$	5, 6 MT
8.	q	7 DN

Now we turn to an example that calls for nested *Show*s. Take,

$$p \to (q \to r), \quad p \to (r \to s) \vdash p \to (q \to s) \tag{13}$$

We start our conditional proof by assuming p:

1.	*Show* $p \to (q \to s)$	
2.	p	Asmp.

At this point we have a new goal, proving $q \to s$. If we can do that, then we can conclude that assuming p yields $q \to s$, achieving our initial goal.

1.	~~*Show*~~ $p \to (q \to s)$	
2.	p	Asmp.
3.	*Show* $(q \to s)$	
4.		

In our **subproof** we proceed as in a conditional proof, assuming q and trying to obtain s:

1.	*Show* $p \to (q \to s)$	
2.	p	Asmp.
3.	*Show* $(q \to s)$	
4.	q	Asmp.
5.		

Once we obtain s we box its proof and cancel the *Show* preceeding our interim goal $q \to s$,

1.	*Show* $p \to (q \to s)$	
2.	p	Asmp.
3.	~~*Show*~~ $(q \to s)$	
4.	q	Asmp.
5.	$p \to (q \to r)$	Prem.
6.	$q \to r$	2, 5 MP
7.	r	4, 6 MP
8.	$p \to (r \to s)$	Prem.
9.	$r \to s$	2, 8 MP
10.	s	7, 9 MP

Our initial aim was to see whether we can obtain $q \to s$ under the assumption p (and of course our premisses). Any formula that is preceeded by a cancelled ~~*Show*~~ is available for use in the unboxed parts of our proof. Given that, we see that we derived $q \to s$ on the assumption p. We box the proof and cancel our top-most *Show*:

1.	~~*Show*~~ $p \to (q \to s)$	
2.	p	Asmp.
3.	~~*Show*~~ $(q \to s)$	
4.	q	Asmp.
5.	$p \to (q \to r)$	Prem.
6.	$q \to r$	2, 5 MP
7.	r	4, 6 MP
8.	$p \to (r \to s)$	Prem.
9.	$r \to s$	2, 8 MP
10.	s	7, 9 MP

10.2.3 Indirect proof

An indirect proof starts with assuming the opposite (denial) of what is being tried to be proved. If this assumption leads to a contradiction – having both P and $-P$ for some formula P, we conclude that the formula we denied in the beginning holds. This technique is sometimes called 'proof by contradiction' or *reductio ad absurdum*.

Example 10.1
Prove 14 by natural deduction.

$$-p \to q, \quad p \to q \vdash q \tag{14}$$

We start by denying our target q:

1.	*Show* q	
2.	$-q$	Asmp.

From here on our aim is to derive a contradiction. *Any* formula P such that we have both P and $-P$. We achieve this aim as follows:

1.	*Show* q	
2.	$-q$	Asmp.
3.	$p \to q$	Prem.
4.	$-p$	2, 3 TP
5.	$-p \to q$	Prem.
6.	q	4, 5 MP

Our assumption $-q$ has allowed us to derive q, resulting in a contradiction. We can now conclude that the assumption $-q$ is unatainable, and therefore q must hold. This completes the proof. We box and cancel as usual.

1.	~~*Show*~~ q	
2.	$-q$	Asmp.
3.	$p \to q$	Prem.
4.	$-p$	2, 3 MP
5.	$-p \to q$	Prem.
6.	q	4, 5 MP

Example 10.2
Take the following argument:

> Harry is the murderer, only if he was at the apartment around 10pm. The police will find a fingerprint, provided that he was at the apartment around 10pm. It is not the case that if he forgot to wear a glove, the police will find a fingerprint. Therefore, Harry is not the murderer.

Given the symbolization,

m: Harry is the murderer;

r: Harry was at the apartment around 10pm;

p: The police will find a fingerprint;

t: Harry forgot to wear a glove,

the argument will be:[10]

$$m \to r, \quad r \to p \quad -(t \to p) \vdash -m \tag{15}$$

We proceed with an indirect proof:

1.	*Show* $-m$	
2.	$--m$	Asmp.
3.	m	5 DN

The same proof could be started as,

1.	*Show* $-m$	
2.	m	Asmp.

□ leaving the application of DN implicit. Now, the aim is to derive a contradiction. The most basic strategy is to derive some formula that contradicts what we already have in the proof and unused premisses, if there are any. Let's attempt to derive $t \to p$ via a conditional subproof:

1.	*Show* $-m$	
2.	m	Asmp.
3.	~~*Show*~~ $t \to p$	
4.	t	
5.	$m \to r$	Prem.
6.	r	2, 6 MP
7.	$r \to p$	Prem.
8.	p	9, 8 MP

Having proved $t \to p$, we arrived at a contradiction, namely with the premiss $-(t \to p)$:

[10]Note that 'p only if q' is $p \to q$, while 'p provided that q' is $q \to p$.

1.	*Show* $-m$	
2.	m	Asmp.
3.	$-(t \to p)$	Prem.
4.	~~*Show*~~ $t \to p$	
5.	t	
6.	$m \to r$	Prem.
7.	r	2, 6 MP
8.	$r \to p$	Prem.
9.	p	9, 8 MP

which completes our proof as:

1.	~~*Show*~~ $-m$	
2.	m	Asmp.
3.	$-(t \to p)$	Prem.
4.	~~*Show*~~ $t \to p$	
5.	t	
6.	$m \to r$	Prem.
7.	r	2, 6 MP
8.	$r \to p$	Prem.
9.	p	9, 8 MP

□

Self study

Except natural deduction, you can review what we have covered in propositional logic from Partee *et al.* (1990), Chapter 6, up to Section 6.5. Please note some terminological differences: they use "statement logic" for "propositional logic", "logical consequence" for "implication", "disjunction" for "alternation", "contingent" for "consistent", and so on.

Please do not try to memorize Table 6.2. The caption of the table, "Laws of statement logic" is misleading. These are not laws, they are just valid formulas, which are on an equal status with any other valid formula of propositional logic. Try to see and work out why they are valid, and that's it for now.

As for the inductive definitions of the well-formed expressions of L_0 and their top-down analysis and bottom-up generation (Section 4 of the notes), they can be reviewed from the notes. If you do not fully understand them at the moment, don't worry. Give priority to the logic part.

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