Contents

1 Sets

1.1 Basic notions

- A fundamental activity in science is to characterize classes[1] of objects. Physics: body, electrical charge, spin; Biology: living cell, protein folding; Sociology: identity, social class; all these are classes of concrete or abstract objects. Any examples from Cognitive Science?

- The mathematical construct **set** provides the clearest model of this activity.

- Listing the objects belonging to it is the most basic and intuitive way of specifying a set. You simply list the objects within curly braces, separated by commas. Here is an example:

$$\{\text{“Muş”}, \text{“Van”}\}$$

- You can pick a symbol to name your set, say A:

$$A = \{\text{“Muş”}, \text{“Van”}\}$$

[1] We use “class” in a non-technical, non-set-theoretic sense.

- When an object x belongs to a set X, we say that x is an **element** or **member** of X, symbolically,[2]

$$x \in X$$

so

$$\text{“Van”} \in A$$

- Non-membership is stated via ‘$\notin$’:

$$\text{“Seattle”} \notin A$$

- For any object x and any set Y, exactly one of the following holds: $x \in Y$, $x \notin Y$. There are no cases in between.

- An object may belong to more than one set. For instance the word “Muş” is an element of many other sets in addition to A above, say the following set B:

$$B = \{2, \text{“Muş”}, \text{“Artvin”}, \text{“France”}\}$$

- Our example set B illustrates that there is no requirement that the elements of a set should be “similar” in an intuitive sense. B has different kinds of elements, and this is perfectly OK. However in almost all mathematical uses of sets there will be a unifying property.

- We said the same object can belong to more than one set, however, an object may belong to a particular set only once. Therefore $\{\text{“Muş”}, \text{“Van”}, \text{“Muş”}\}$ is no different than $\{\text{“Muş”}, \text{“Van”}\}$. In other words, repetitions do not count in a set.

- The order of elements does not matter in sets. Absolutely no difference between $\{\text{“Muş”}, \text{“Van”}\}$ and $\{\text{“Van”}, \text{“Muş”}\}$.

[2] Note that the case of letters is important in mathematics and programming.

- The set with no elements is **the empty set**, denoted as '$\emptyset$'. We say *the* empty set, because there is one and only one empty set.[3]

- Sets can have sets as elements. Therefore sets themselves are objects. The following is a legitimate set:

$$C = \{2, \{3,4\}, \text{“Muş”}, \{\text{“Artvin”}, \text{“France”}\}\}$$

When you need to be more specific in your use of "object", you can call objects that are not sets, **atomic** objects, and the rest, **non-atomic** objects.

Exercise 1.1.
Imagine you are running an experiment, in which you will obtain some measurements from trials. You run a trial and you obtain an integer, you run another one, you have another integer. You don't know in advance the range of possible values for measurements, they can be any integer. Imagine further that you need to accumulate measurements somewhere, so that you can sum them up, or do other things, after you finish experimenting. You also want to keep which measurement comes from which trial. For this task, you are allowed to use sets and integers, and nothing else. How would you do this?

- So far, so good with representing sets by listing their elements. This is, fortunately, not the only way of representing sets. You can characterize a set by giving a property that uniquely identifies the elements of that set. For instance, the set A above could have been characterized as the set of Turkish city names with less than four letters. This would be equivalent to listing the elements that fulfil the given criterion.

- The listing method is sometimes called "definition by extension", and the common property method is called "definition by intension" (Russell 1919:12).[4]

- Definition by intension (aka[5] intensional definition) is "superior" to extensional definition. The superiority comes from the fact that all the sets defined by extension can be defined by intension as well. However, the converse does not hold, namely you cannot give an extensional definition for every set you defined by intension.

  First, there are sets where you may not identify all the elements belonging to it. Examples?

  Second, there are sets where you *cannot* identify all the elements belonging to it, because regardless of how many elements you list, there will always be elements that you have left out, actually infinitely many of them. These are infinite sets.

- Both problems are addressed in the same way, and it is actually very seldom that you list the elements of a set. Instead, as we already started to see above, one provides a **decision procedure** or a number of **membership criteria**, which gives you a "yes" or "no" answer, for any given object, according to whether it is the element of the set or not.

  Here are two very common **infinite** sets, the set of natural numbers and the set of integers:

$$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$$

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

- In some cases membership criteria is left implicit. Here is a finite set of that sort:

$$K = \{2, 4, 6, \ldots, 90\}$$

[3] Why this is so will get clarified in Section 1.2, where we discuss equality.

[4] "Intension" and "extension" are important concepts that will come up again in the future. But for now we need them simply as names to call two different ways of characterizing sets. Do not try to memorize them.

[5] 'Also known as.'

and here is one which is infinite:

$$K = \{2,4,6,\ldots\}$$

- Another common method is **predicate** (or **set-builder**) notation, which has some variants:

$$\{x \,|\, x \in \mathbb{N} \text{ and } 7 < x < 11\} \tag{1}$$

$$\{x \in \mathbb{N} \,|\, 7 < x < 11\} \tag{2}$$

$$\{x+y \,|\, x,y \in \mathbb{N} \text{ and } 7 < x < 11 \text{ and } 1 < y < 4\} \tag{3}$$

- You can also think of a set as a **rule**. A rule that dictates what belongs to it and what does not.

  Here is a more thorough specification of membership in the set of natural numbers – don't worry if you don't understand this fully now:

Definition 1.2.
What is a natural number?

i. 0 is a natural number (written $0 \in \mathbb{N}$);
ii. if $n \in \mathbb{N}$, then $n+1 \in \mathbb{N}$;
iii. If something is not a natural number according to (i) or (ii), then it is not a natural number.

- For any set A, the number of elements in it is called the **cardinality** of A, depicted as $|A|$.

Exercise 1.3.
What is the cardinality of the following sets:

a. $\{1,\{2,\{3,\{4\}\}\}\}$
b. $\emptyset$
c. $\{\emptyset\}$
d. $\{\emptyset,1\}$
e. $\{\emptyset,0,1\}$

1.2 Equality, subset, set operations

Equality:

Two sets are equal if and only if they have the same elements.

Subset:

For any sets A and B,
A is a **subset** of (or included in) B, written $A \subseteq B$, if and only if each element in A is also in B.

For any sets A and B,
A is a **proper subset** of B, written $A \subset B$, if and only if each element in A is also in B and there is at least one $a \in B$, such that $a \notin A$.

- Given any set A, is $A \subseteq A$?
- Given any set A, what can you say about $\emptyset \subseteq A$?
- Given the notion of equality and subsethood, can you see why the following holds?

  For any sets A and B,
  A is **equal** to B, written $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

Set operations:

Union:

Given two sets A and B,
The union of A and B, written $A \cup B$, is the set of objects that are elements of at least one of A and B.

Intersection:

Given two sets A and B,
The intersection of A and B, written $A \cap B$, is the set of objects that are elements of both A and B.

Difference:

Given two sets A and B,
The difference of A and B, written $A - B$, is the set of objects that are elements of A but not B.

Exercise 1.4.
Define $A \cup B$, $A \cap B$ and $A - B$ in predicate notation.

- Two sets are **disjoint** if they have no element in common, their intersection is $\emptyset$.

- Here are some properties of set operations:

Idempotency:	$A \cup A = A$
	$A \cap A = A$
Commutativity:	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity:	$A \cap (B \cap C) = (A \cap B) \cap C$
	$A \cup (B \cup C) = (A \cup B) \cup C$
Distributivity:	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
DeMorgan's Laws:	$A - (B \cup C) = (A - B) \cap (A - C)$
	$A - (B \cap C) = (A - B) \cup (A - C)$

- No need to memorize these, they all follow from the basic definitions we had above. Do not try to memorize anything – except perhaps some names – in this course, it is not the right way to learn mathematics.

Exercise 1.5.

*a. Write the distributivity property by using variables over set operations.

*b. Show that DeMorgan's Laws hold. Concentrate on what is required for two sets to be equal.

- It is possible to take the union or intersection of more than two sets. Given a set of sets A:

$$\bigcup A = \{a \,|\, a \in B \text{ for some } B \in A\}$$
$$\bigcap A = \{a \,|\, a \in B \text{ for each } B \in A\}$$

1.3 Power set and partition

Power set:

For any set A,
The **power set** of A, written $\mathscr{P}(A)$, or $\mathrm{Pow}(A)$, is the set of all subsets of A.

Partition:

For any set A,
A **partition** of A, is a set $\Pi \subseteq \mathscr{P}(A)$ such that:

i. $\emptyset \notin \Pi$;
ii. any two distinct members of Π are disjoint;
iii. $\bigcup \Pi = A$.

Exercise 1.6.

a. Define proper subsethood in terms of '$\subseteq$', you can negate it as '$\not\subseteq$' if you need to.
b. Given $A \not\subseteq B$, what can you tell about A?
c. Give a partition of $A = \emptyset$.

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2 Relations

- Up to now we saw objects and their collections, sets. Mathematics extensively deals with **relations** between objects.

- We have already seen some relations; one is the membership relation designated with '$\in$'. It relates objects with sets that they are members of, say, $a \in \{a,b,c\}$. Membership is a **two-place** (or **binary**) relation, since it relates two things – the technical term for things related is **relata**, and **relatum** for singular. Generally there can be n-ary relations. Some examples?

  The mathematical way of representing binary relations is – rather weirdly you may find – to form the set of pairs of related objects. For instance, 'less than' relation is the set of pairs of numbers, where the first number in each pair is less than the second.

  Therefore, in order to represent relations, we need a way to represent pairs.

- Now comes a new type of object:

  Tuple:

  $$(o_1, o_2, \ldots, o_n)$$

  Order and repetition matter: $(a,a,b) \neq (a,b) \neq (b,a)$.

  Terminology: "ordered pair" for 2-tuple, "ordered triple" for 3-tuple, and so on.

- The Cartesian **product** of two sets:

  $$A \times B = \{(a,b) | a \in A, b \in B\}$$

  generally, the product of n sets:

  $$A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i\}$$

Exercise 2.1.

a. Give the product of $\{1,2,3\}$ and $\{a,b,c\}$.
b. Give the product of $\{a,b,c\}$ and $\emptyset$.
*c. Think of a way to represent ordered pairs as sets.

- A **binary relation** on sets A and B is a subset of $A \times B$.

  For instance $\{(a,\{a\}),(b,\{a\})\}$ is a binary relation on $\{a,b,c\}$ and $\{\{a\},\{b\},c\}$.

- Generally an n-ary relation on sets $A_1,\ldots,A_n$ is a subset of $A_1 \times \ldots \times A_n$.

- There is no requirement that A_is be distinct, when they are the same, abbreviate $A_1 \times \ldots \times A_n$ as A^n.

For instance ‘less than’ is a subset of $\mathbb{N}^2$, namely:

$$\{(i,j) \,|\, i,j \in \mathbb{N} \text{ and } i < j\}$$

- The **domain** of a relation $R \subseteq A \times B$:

$$\{a \,|\, \text{there is a } b \in B \text{ such that } (a,b) \in R\}$$

- The **range** of a relation $R \subseteq A \times B$:

$$\{b \,|\, \text{there is an } a \in A \text{ such that } (a,b) \in R\}$$

- Any binary relation $R \subseteq A \times B$ has an **inverse** $R^{-1} \subseteq B \times A$ defined as:

$$R^{-1} = \{(b,a) \,|\, (a,b) \in R\}$$

2.1 Some properties of binary relations

- Certain types of relations are of special interest due to way they are structured.
- Let’s focus on binary relations on a single set, $R \subseteq A \times A$.

Reflexivity:

A relation $R \subseteq A \times A$ is **reflexive** if and only if for each $x \in A$, $(x,x) \in R$.

A relation is **nonreflexive**, if it is not reflexive.

A relation $R \subseteq A \times A$ is **irreflexive**, if for each $(x,y) \in R$, $x \neq y$.

Symmetry:

A relation $R \subseteq A \times A$ is **symmetric** if and only if for each $(x,y) \in R$, (y,x) is also in R.

A relation $R \subseteq A \times A$ is **nonsymmetric** if and only if for some $(x,y) \in R$, (y,x) is not in R.

A relation $R \subseteq A \times A$ is **asymmetric** if and only if for each $(x,y) \in R$, (y,x) is not in R.

A relation $R \subseteq A \times A$ is **anti-symmetric** if and only if whenever (x,y) and (y,x) are in R, then $x = y$.

Transitivity:

A relation $R \subseteq A \times A$ is **transitive** if and only if whenever (x,y) and (y,z) are in R, then (x,z) is also in R.

A relation is **nontransitive**, if it is not transitive.

A relation $R \subseteq A \times A$ is **intransitive** if and only if for no pair (x,y) and (y,z) in R, (x,z) is in R.

Connectedness:

A relation $R \subseteq A \times A$ is **connected** if and only if for each x, $y \in A$ where $x \neq y$, either (x,y), (y,x) or both are in R.

- Our final special type of relation is **equivalence relation**, which is reflexive, symmetric and transitive.

Exercise 2.2.

a. State the properties of the following relations defined over set of humans: ‘spouse’, ‘ancestor’, ‘sister’, ‘sibling’, ‘father’, ‘child’, ‘admire’, ‘identical’, ‘older than’, ‘older than or at the same age as’, ‘has the same hight as’, ‘has the same biological father’, ‘has the same cousin’.

b. Give other examples for each type of relation from $\mathbb{N}^2$ and/or H^2, where H is the set of humans.

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3 Functions

- A **function** is a special type of relation.
- A function from set A to set B is a relation $R \subseteq A \times B$, such that for each $a \in A$ there is exactly one pair in R with a as the first component.

Exercise 3.1.
Is $\{(a,\{a\}),(b,\{a\}),(c,c)\}$ a function from $\{a,b,c\}$ to $\{\{a\},\{b\},c\}$?

- Letters f, g, h are usually reserved for representing functions.
- You can think of a function as a mapping from a set to another, written $f : A \to B$.

  For an $a \in A$, $f(a) \in B$ is called the **image** of a under f, or simply f of a.

  Given any set $A' \subseteq A$, the image of A' under f:

  $$\{b \mid f(a) = b \text{ for some } a \in A'\}$$

  For any function $f : A \to B$,
  the **domain** of f, denoted by Dom(f), is...
  the **range** of f is denoted by Ran(f), and Ran(f)...

  Seen as a mapping, the condition for functionhood is that the function maps each and every element in its domain to one and only one (= exactly one) element in its range.

Exercise 3.2.
Which of these are functions (where y would be the image of x): 'x is the mother of y', 'x is a child of y', 'x is y years old', 'x is the age of y', 'x is the capital of y', 'the capital of x is y', 'x is the same person as y'?

- When the domain of a function consists of tuples we omit the parentheses around tuples:

  For $f : A_1 \times, \ldots, A_n \to B$, instead of $f((a_1,\ldots,a_n))$ for $a_i \in A_i$, we write $f(x_1,\ldots,x_n)$.

  The objects $a_1,\ldots,a_n$ are called the **arguments** of f. The object $b \in B$ that f maps these arguments to is the **value** of $f(a_1,\ldots,a_n)$.

Exercise 3.3.
Unlike relations, the inverse of a function may not be a function. Why?

- Given two relations Q and R, the **composition** of them, $Q \circ R$ is the relation,

  $$\{(a,c) \mid (a,b) \in R \text{ and } (b,c) \in Q \text{ for some } b\}$$

- The composition of two functions $f : A \to B$ and $g : B \to C$, denoted by $g \circ f$, is a function $h : A \to C$, such that

  $$h(a) = g(f(a)) \text{ for each } a \in A$$

- Some special types of functions:

  Constant functions:
  any function $f : A \to B$ such that for all $a \in A$, $f(a) = c$ for some $c \in B$.

  A function $f : A \to B$ is **onto** B (or simply onto) if Ran(f) $= B$.

  A function $f : A \to B$ is **one-to-one** if for any $a_1, a_2 \in A, f(a_1) \neq f(a_2)$.[6]

  A function $f : A \to B$ is a **bijection** (or **one-to-one correspondence**), if it is one-to-one and onto.

- Let's think of some examples for each type.

Exercise 3.4.
Every equivalence relation defines a partition on the set it is defined over, where each cell of the partition is called an **equivalence class**. Can you see how/why?

[6] What's wrong with this?

A Answers for selected exercises

1.1 Number your trials from 0 to k; put n number of braces around the score of the nth trial and put everything in a set. Another method is to bring together a trial number and its measurement into a set, but putting one of them, say the measurement, in another set. E.g. trial 10 with measurement 20 is represented as $\{10,\{20\}\}$. Putting all such sets in another set you can collect them without losing any information.

1.3 2, 0, 1, 2, 3.

1.4 $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$
$A - B = \{x \mid x \in A \text{ and } x \notin B\}$

*1.5a $A\,\alpha\,(B\,\beta\,C) = (A\,\alpha\,B)\,\beta\,(A\,\alpha\,C)$, for $\alpha,\beta \in \{\cap,\cup\}$ and $\alpha \neq \beta$.

*b We discuss the exercise deliberately in a rather roundabout way to increase our familiarity with sets and logical inference. Take
$A-(B\cup C) = (A-B)\cap(A-C)$
if there is an a in the left hand side (LHS), it must be something in A but neither in B nor C. The question is is a guaranteed to be in the RHS as well? For a to be in RHS it must be present in both parts of the intersection. Suppose it is not, then it must be missing from at least one of the parts of the intersection. For it to be missing in $A-B$, it must either be not in A or it must be in B. Both possibilities contradict our assumption that a is in LHS. The second way that a is missing from the intersection on RHS, it must be missing from $A-C$. Then in this case it is either not in A or it is in C. Again both possibilities clash with our initial assumption. Therefore there is no way that a is in LHS but not in RHS; it is simply impossible. You can prove that a must be in LHS, if it is in RHS, in a similar fashion; thereby finalizing the proof of the first DeMorgan's Law. The proof for the second would be very similar to the first one.

1.6a A is a proper subset of B if and only if $A \subseteq B$ and $B \nsubseteq A$.

b It has at least one element which is not in B, and therefore it is not the empty set.

c $\emptyset$

2.1a $\{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c)\}$.

b $\emptyset$.

c A first attempt would be having (a,b) equivalent to $\{a,\{b\}\}$. There are cases where this solution would fail. Can you see when? The definitive solution is to have (a,b) equivalent to $\{a,\{a,b\}\}$.

2.2a 'admire' is the only non-reflexive relation; 'identical' and 'has the same hight', 'has the same biological father as' are equivalence relations; 'has the same cousin' is reflexive, symmetric but non-transitive (why is it not intransitive?); 'older than or at the same age as' is reflexive, transitive and anti-symmetric…

3.1 Yes.

3.2 'x is y years old', 'x is the capital of y', 'the capital of x is y', 'x is the same person as y'.

3.3 Take the function in Exercise 3.1 as an example.

*3.4 Let $R \subseteq A \times A$ be an equivalence relation in A. Let $g : A \mapsto \mathscr{P}(A)$ be the function defined as $g(a) = \{x \mid (a,x) \in R\}$, for $a \in A$. We need to show that the range of g, which is some collection of sets $\mathscr{G}$, is a partition of A. **First**, we need to show that $\emptyset \notin \mathscr{G}$. Assume that $\emptyset \in \mathscr{G}$. Then, given the definition of g, there needs to be an $a \in A$ where there is no $b \in A$ such that $(a,b) \in R$. But given that R is an equivalence relation, $(a,a) \in R$, providing such a b, resulting in a contradiction. Therefore $\emptyset \notin \mathscr{G}$. **Second** we need to show that for $G_1,G_2 \in \mathscr{G}$, if $G_1 \neq G_2$, then $G_1 \cap G_2 = \emptyset$. Take two such non-identical sets $G_1,G_2 \in \mathscr{G}$. Assume $G_1 \cap G_2 \neq \emptyset$. Then there must be an $a \in A$ which belongs to both G_1 and G_2. Take any $b \in G_1$, given the definition of g, that R is an equivalence relation and $a \in G_1$, $(a,b),(b,a) \in R$; and as $a \in G_2$, $b \in G_2$ as well. Now pick any $c \in G_2$, by the same reasoning c is also in G_1, establishing that $G_1 = G_2$. This contradicts with what we had about G_1 and G_2 in the beginning. Therefore, whatever two non-identical sets we pick out of $\mathscr{G}$, they are guaranteed to be disjoint. **Finally**, we need to show that

$\bigcup\mathscr{G} = A$. One way this fail to be true is that there is an $a \in A$ which does not belong to any set in $\mathscr{G}$. This is obviously impossible, since given R is an equivalence relation and $a \in A$, $(a,a) \in R$, and therefore $g(a)$ is some set in $\mathscr{G}$ such that $a \in g(a)$. As it is also obvious that there can be no $a \in \bigcup\mathscr{G}$ which is not present in A, we have to conclude that $\bigcup\mathscr{G} = A$. This finishes our proof that an equivalence relation over a set defines a partition of that set. Each member of this partition is called an equivalence class induced by the equivalence relation.

References

Russell, B. (1919). *Introduction to Mathematical Philosophy*. Gearge Allen and Unwin Ltd., London.