

The Darwin Instability

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Last updated: January 14, 2020

Literature

5.1, *Evolutionary Processes in Binary and Multiple Stars*, P. Eggleton

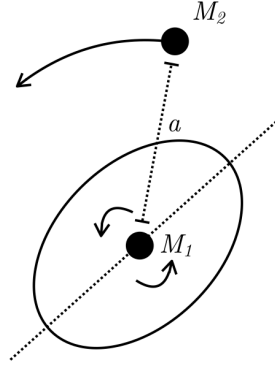


Figure 1: Diagram showing a point mass M_2 with a companion of mass M_1 with an exaggerated tidal bulge. The axis of the bulge is shown by a dotted line, which, in the case of co-rotation, would pass through M_2 .

Consider a circular but unsynchronised orbit of a binary with component masses M_1 and M_2 and separation a . We denote the moment of inertia of M_1 by I_1 , assumed to be constant, the spin angular momentum of M_1 by J_1 , and the spin angular velocity of M_1 by Ω_1 . We consider the simplifying case of the spin of M_2 being small, such that its moment of inertia may be neglected ($I_1\Omega_1 \gg I_2\Omega_2$).

Ignoring winds, the total angular momentum of the binary orbit is therefore

$$J = I_1\Omega_1 + \mu a^2\Omega_{\text{orb}}, \quad (1)$$

where $1/\mu = 1/M_1 + 1/M_2$ and μ is the reduced mass.

We consider the scenario shown in Figure 1, where $\Omega_1 < \Omega_{\text{orb}}$, such that the bulge of M_1 lags behind M_2 . Tidal torques act to transfer angular momentum from the orbit into the spin of M_1 by tightening the orbit. This increases Ω_1 to match Ω_{orb} . But with a tighter orbit, Ω_{orb} also increases. A natural question that arises is whether the binary ever reaches synchronisation:

- If $\dot{\Omega}_1 > \dot{\Omega}_{\text{orb}}$, the spin-up of M_1 is faster than the spin-up of the shrinking orbit. The system approaches synchronisation, $\Omega_{\text{orb}} - \Omega_1 \rightarrow 0^+$.
- If $\dot{\Omega}_1 < \dot{\Omega}_{\text{orb}}$, the shrinking orbit causes the orbit to spin up too quickly for M_1 to catch up, and $\Omega_{\text{orb}} - \Omega_1$ increases catastrophically. This implies the separation shrinks

catastrophically (on a dynamical time), ending with a merger. This is called the *Darwin instability*.

Claim 1. *With this setup, we have $J_1 \frac{\dot{\Omega}_1}{\Omega_1} = \frac{1}{3} J_{\text{orb}} \frac{\dot{\Omega}_{\text{orb}}}{\Omega_{\text{orb}}}$.*

Proof. By conservation of total angular momentum J ,

$$0 = \dot{J} = \underbrace{I_1 \dot{\Omega}_1}_{=J_1 \frac{\dot{\Omega}_1}{\Omega_1}} + 2\mu a \dot{a} \Omega_{\text{orb}} + \underbrace{\mu a^2 \dot{\Omega}_{\text{orb}}}_{J_{\text{orb}} \frac{\dot{\Omega}_{\text{orb}}}{\Omega_{\text{orb}}}}. \quad (2)$$

In the second term, rewrite \dot{a} in terms of $\dot{\Omega}_{\text{orb}}$ by noting that $\dot{\Omega}_{\text{orb}}/\Omega_{\text{orb}} = -(3/2)(\dot{a}/a)$. We then obtain the required expression by collecting like terms. \square

A Corollary of the Claim is that requiring $\dot{\Omega}_1 > \dot{\Omega}_{\text{orb}}$ for stability gives the stability criterion on the angular momenta:

$$\boxed{\text{The system is Darwin stable} \iff J_{\text{orb}} > 3J_1,} \quad (3)$$

i.e. A synchronised orbit is achieved as long as M_1 has sufficiently small spin (less than a third of the orbit). The stability criterion may also be recast as a condition on the separation:

$$\boxed{\text{The system is Darwin stable} \iff a > a_{\text{Darwin}} = \sqrt{\frac{3I_1}{\mu}}.} \quad (4)$$

Application to Binary Orbits

A binary that starts off satisfying Equation 4 may eventually shrink below a_{Darwin} due the evolution of M_1 . Because $I_1 \sim M_1 R_1^2$, the radial expansion of M_1 during its evolution may cause a_{Darwin} to grow and supersede a , leading to the Darwin instability. However, if M_1 fills its Roche lobe before this occurs, then the Darwin instability never becomes relevant. Therefore, a binary never becomes Darwin unstable if its separation exceeds the maximally-realisable a_{Darwin} , which is a_{Darwin} evaluated at the moment M_1 fills its Roche lobe, $x_L(q)a$: $a > a_{\text{Darwin}}(R_1 = x_L(q)a)$, where $q = M_2/M_1$ is the mass ratio. Defining $I_1 = kM_1 R_1^2$, this becomes

$$\frac{1}{3} \frac{a^2}{1+q} > k x_L(q)^2 a^2. \quad (5)$$

The separation a^2 cancels out, so we find a criterion that is independent of a and only depends on q and k . We may use the Eggleton (1983a) analytical approximation of the Roche radius,

$$x_L(q) = \frac{0.49q^{2/3}}{0.6q^{2/3} + \ln(1 + q^{1/3})}. \quad (6)$$

Solving Equation 5 numerically, we find the following critical mass ratios:

- For a $n = 3$ polytrope, which is a reasonable description of a main sequence star, it may be shown that $k = 0.076$ (approximately a fifth that for a uniform sphere), upon which one finds the critical “Darwin” mass ratio $q_D \approx 12$, above which the binary encounters the Darwin instability and plunges in on a dynamical time. Tides do not have enough time to synchronise and circularise the orbit, meaning the binary may proceed to Roche-Lobe overflow with an unsynchronised and eccentric orbit. The likely outcome is merger.
- For a $n = 3/2$ polytrope, a reasonable description of a red giant or the convective core of a giant, we find $q_D \approx 5$.