Essay A: The Magnetorotational Instability

1 The Viscosity Problem

To be accreted by the central source, a particle's angular momentum must either be removed from the disk by an external torque or redistributed within the disk by an internal torque. This is described by a viscous stress $\boldsymbol{\sigma}$ in the fluid momentum equation

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \nabla \cdot \boldsymbol{\sigma} + \mathbf{F}, \tag{1}$$

where d/dt is the convective derivative and $\mathbf{F}$ includes any other body force. The theory of gas kinetics gives the following expression for $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = \eta\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{u}\right) + \zeta\mathbf{I}\nabla\cdot\mathbf{u}. \tag{2}$$

The first term is traceless and gives rise to *shear viscosity*, whose effect is to transport momentum in a direction orthogonal to the momentum. The second term contains the trace and gives rise to bulk viscosity, whose effect is to transport momentum in the direction along it, and is zero for an incompressible fluid ($\nabla\cdot\mathbf{u} = 0$). Clearly, an accretion disk requires a source of shear viscosity, since accretion requires outward radial angular momentum transport but the main inertial flow is in the azimuthal direction.

1.1 Molecular Viscosity

An obvious and natural source to consider is molecular viscosity, which arises from molecular collisions or friction. We show, however, that its effects on an accretion flow are very small and cannot account for observed accretion rates. An elementary kinetic theory calculation gives the kinematic viscosity $\nu = \eta/\rho = \frac{1}{3}\tilde{v}\tilde{\lambda}$, where $\tilde{v}$ and $\tilde{\lambda}$ are the unspecified velocity and length scales associated with the fluctuations transporting the momentum. The Reynolds number Re is given by the ratio of the viscous terms $\sim |\nabla\cdot\boldsymbol{\sigma}|$ to the inertial terms $\sim |\rho\partial\mathbf{u}/\partial t|$ in the momentum equation (1). Taking the ratio of

$$|\nabla\cdot\boldsymbol{\sigma}| \sim \frac{1}{R}\eta\frac{dv_\phi}{dr} \sim \frac{1}{R}\rho\tilde{v}\tilde{\lambda}\frac{v_\phi}{R} \sim \frac{v_\phi}{R^2}\rho\tilde{v}\tilde{\lambda}, \qquad \left|\rho\frac{\partial\mathbf{u}}{\partial t}\right| \sim \rho\frac{v_\phi^2}{R}, \tag{3}$$

we obtain

$$\boxed{Re \sim \frac{v_\phi}{\tilde{v}}\frac{R}{\tilde{\lambda}}} \tag{4}$$

For molecular viscosity, the relevant velocity and length scales are the sound speed $\tilde{v} \sim c_s$ and the thermal mean free path $\tilde{\lambda} \sim \lambda_{\mathrm{mfp}}$, and so

$$Re_{\mathrm{mol}} \sim \frac{v_\phi}{c_s}\frac{R}{\lambda_{\mathrm{mfp}}}. \tag{5}$$

For hydrogen plasma, $\nu_{\rm mol} \sim c_s \lambda_{\rm mfp} \sim 10^5 {\rm cm}^2 s^{-1}$. Using $v_\phi = \sqrt{GM/R}$, $M \sim 10 M_\odot$, and $R \sim 10^{10}$cm for accretion disks around close black hole binaries, $Re_{\rm mol} \gtrsim 10^{14}$ and so the accretion flow is almost completely uninfluenced by molecular viscosity. From a different perspective, this calculation yields an accretion timescale (setting $Re_{\rm mol} \sim 1$) of 3×10^7yr, and therefore cannot account for the much shorter timescale of X-ray nova outbursts, which is believed to be due to caused by non-steady accretion.

1.2 Turbulent Viscosity

On the other hand, the largeness of $Re_{\rm mol}$ suggests that accretion disk flows are turbulent. Then, the fluctuations required for momentum transport may be provided by turbulent eddies with characteristic velocity $\tilde{v}$ and characteristic size $\tilde{\lambda}$. Then, the Reynolds number associated with this turbulent viscosity may be small for the largest and fastest turbulent eddies as seen from (4).

Performing a Reynolds decomposition of the momentum equation (1) into mean values and small-amplitude fluctuations shows that turbulence gives rise to a stress $\boldsymbol{\tau} = -\rho \langle \tilde{\mathbf{u}} \tilde{\mathbf{u}} \rangle$, called the *Reynolds stress* or *turbulent stress*, where $\tilde{\mathbf{u}}$ is the turbulent fluctuation about the mean flow velocity. Thus, momentum is transported by correlations in velocity fluctuations. In particular, $\rho \langle u_i u_j \rangle$ for $i \neq j$ is a source of shear turbulent viscosity.

Identifying the source of the (turbulent) viscosity to explain observed accretion rates has been a central problem in accretion disk astrophysics for decades until the *magnetorotational instability* (MRI) was identified by Balbus & Hawley (1991) as a promising source.

2 Magnetorotational Instability (Balbus-Hawley Instability)

In the absence of a magnetic field, accretion disks satisfy the Rayleigh stability criterion and so are hydrodynamically stable. A particle executes retrograde epicycles about its mean circular orbit. MHD instability of an accretion disk is demonstrated by performing linear stability analysis of the incompressible ideal MHD equations for axisymmetric perturbations. In a simplified analysis that assumes a constant background magnetic field in the direction of the disk normal and taking the wavenumber of the vertical perturbations to be much larger than that of radial perturbations for a thin disk, $|k_z| \gg |k_r|$, we obtain the dispersion relation

$$\omega^4 - \omega^2 \left(2k^2 v_A^2 + \frac{d\Omega^2}{d\ln r} + 4\Omega^2 \right) + k^2 v_A^2 \left(k^2 v_A^2 + \frac{d\Omega^2}{d\ln r} \right) = 0 \tag{6}$$

where $v_A = B_0/\sqrt{\mu_0 \rho_0}$ is the Alfvén velocity. The MRI is said to exist when $\omega^2 < 0$, which clearly requires

$$\boxed{k^2 v_A^2 < -\frac{d\Omega^2}{d\ln r}.} \tag{7}$$

There always exist a small enough k for which Inequality 7 is satisfied if $d\Omega^2/d\ln r < 0$. Thus, all accretion disks with $d\Omega^2/d\ln r < 0$ are unstable against perturbations above a sufficiently large lengthscale. In particular, it is satisfied for a Keplerian disk, which are therefore always unstable against the MRI (with important caveats discussed later). When the instability criterion is satisfied, Equation 6 can be maximised to find the maximum growth rate

$$|\omega_{\max}| = \frac{1}{2}\left|\frac{d\Omega}{d\ln r}\right|. \tag{8}$$

Remarkably, this is independent of the field strength given it is non-zero. It is therefore incorrect to neglect even weak magnetic fields in accretion disk flow. For a Keplerian disk, $|\omega_{\max}| = \frac{3}{4}\Omega$ occurs at $kv_A = \frac{\sqrt{15}}{4}\Omega$. So timescale (e-folding time) of the instability is of order inverse angular velocity:

$$|\omega_{\max}|^{-1} \approx \Omega^{-1} = 10^{-6}\mathrm{s}\left(\frac{R}{10^{10}\mathrm{cm}}\right)^{1/2}\left(\frac{M}{M_\odot}\right)^{-1/2}. \tag{9}$$

The reason for the instability to require a magnetic field may be understood as follows. In a non-magnetised fluid, a perturbed fluid element tends to conserve its specific angular momentum. So when a fluid element is displaced radially outward, it has too little specific angular momentum for its new position, and so relaxes back towards its initial position. However, in MHD, fluid flow entrains magnetic field lines, which enforce rigid rotation. So a perturbed fluid element tends to conserve its angular velocity. So when a fluid element is displaced radially outward, it has too much angular velocity for its new position, and the excess centrifugal force drives it further out. Indeed, the instability criterion (7) shows that an instability will set in if the magnetic tension, whose effect is to resist compression or rarefaction of field lines, is not strong enough to counteract the net tidal force (centrifugal minus gravitational force) acting on it.

2.1 Caveats and Limitations

The claim that there always exist a small enough k for Inequality 7 to be satisfied for positive RHS is not strictly true, since the perturbation lengthscale is obviously limited by the disk scale height H. So for a thin disk, even if Inequality 7 is satisfied, the disk is stable if $v_A/\Omega > H$. For a thin disk, $\Omega H \sim c_s$, and so the disk is stable if $v_A > c_s$.

Another assumption is that for MRI to be applied to accretion disks, matter must be coupled to the magnetic field, i.e. there must be some degree of ionisation. This is, for example, not satisfied in protoplanetary disks where there is a very low ionisation fraction. In fact, the gravitational instability is the main source of viscosity in protoplanetary disks.

The other caveat stems from the assumption of ideal MHD, which neglects magnetic diffusivity η_B in the induction equation. However, it is possible for magnetic diffusivity to suppress an instability if they act on a similar lengthscale. Recalling that η_B is the diffusion coefficient for $\mathbf{B}$, the timescale for magnetic diffusion over a lengthscale $1/k$ is $\tau_{\text{diffusion}} \sim 1/(\eta_B k^2)$. On the other hand, the timescale of the MRI was previously shown to be $\tau_{\text{growth}} \sim 1/\Omega$. So we expect magnetic diffusivity to stabilise the disk against the MRI if $\tau_{\text{growth}} \gtrsim \tau_{\text{diffusion}} \implies \eta_B \gtrsim \Omega/k^2$, i.e. the magnetic Reynolds number is not large, $Re_m \lesssim 1$. This makes sense intuitively: large magnetic diffusivity and small perturbations favour the suppression of the MRI. But even then, the instability is present at large lengthscales. In fact, linear stability analyses including the effects of resistivity, ambipolar diffusion, and the Hall effect show the instability to be present for many non-ideal astrophysical plasmas.

Numerical simulations of magnetised accretion disks confirm the presence of the MRI, which is found to transport angular momentum outward in the disk. The dynamo action of the accretion disk is also found to regenerate the magnetic field to sustain this instability.

2.2 Application to the Shakura-Sunyaev Disk Model

Continuing from the discussion of turbulent viscosity, a turbulent disk may be described by a Reynolds stress $-\Sigma\langle\tilde{u_r}\tilde{u_\phi}\rangle$. We associate this with a shear viscosity ν by

$$-\Sigma\langle\tilde{u_r}\tilde{u_\phi}\rangle = \Sigma\nu r\frac{d\Omega}{dr}. \tag{10}$$

For a Keplerian disk, this gives $\langle\tilde{u_r}\tilde{u_\phi}\rangle \sim \nu\Omega$. Using $\nu = \alpha c_s H$ in the Shakura-Sunyaev prescription and $H \sim c_s/\Omega$ in a thin disk,

$$\alpha \sim \frac{\langle\tilde{u_r}\tilde{u_\phi}\rangle}{c_s^2}, \tag{11}$$

which gives the expression for α in disk accretion driven by shear turbulence. Balbus & Papaloizou (1999) point out that as long as the velocity correlation $\langle\tilde{u_r}\tilde{u_\phi}\rangle$ is positive, the magnetic turbulence may indeed be treated as an effective viscosity. If it also gives rise to a constant α, Equation 11 formally closes the Shakura-Sunyaev disk equations and solves the viscosity problem.