Effective Domination

Daniel Filan, `daniel.filan@anu.edu.au`

Here, we define effective domination, uniform effective domination, motivate these definitions, and show that S, the speed prior, effectively dominates the class of lower semicomputable semimeasures.

First, assume μ and ν are semimeasures on $\{0,1\}^\omega$. Then, we say that μ *effectively dominates* ν if there exists some $f_\nu : \mathbb{N} \to \mathbb{R}_{>0}$ and some $c \in \mathbb{R}$ such that for all $x \in \{0,1\}^*$,

$$\mu(x) \geq c f_\nu(|x|)\nu(x) \tag{1}$$

Thus, μ dominates ν except for some function of the length of the string. Similarly, we say that μ effectively dominates a class of semimeasures if it dominates each one of them.

Next, we say that μ *uniformly effectively dominates* a class $\mathcal{M}$ of semimeasures if there exists some $f : \mathbb{N} \to \mathbb{R}_{>0}$ such that for all $\nu \in \mathcal{M}$, there exists some $c \in \mathbb{R}_{>0}$ such that for all $x \in \{0,1\}^*$,

$$\mu(x) \geq c f(|x|)\nu(x) \tag{2}$$

Note the difference between (1) and (2): in (1), f_ν depends on the semimeasure ν, while in (2), it does not.

Uniform effective dominance might seem to be useful: if we are in a sequence prediction setting, then the Λ_μ predictor acts just like the $\Lambda_{\mu/f}$ predictor, and μ/f dominates the class $\mathcal{M}$, meaning that good bounds could apply.

However, we show that any non-vanishing semimeasure μ effectively dominates any semimeasure class of semimeasures $\mathcal{M}$: let $f(n) = \min_{|x|=n} \mu(x)$. Then, letting $\nu \in \mathcal{M}$,

$$\begin{aligned} \mu(x) &\geq f(|x|) \\ &\geq f(|x|)\nu(x) \end{aligned} \tag{3}$$

As (3) shows, uniform effective dominance is in itself nothing special, making it therefore somewhat unwisely named. Despite this, we will show that S effectively dominates the class of lower semicomputable semimeasures, and hope that this is worth something.

We use the fact that for all lower semicomputable semi-measures μ, there is some Turing machine T such that $\mu(x) = \sum_{p\colon T(p)=x} 2^{-|p|}$. Now, if $\mu(x) < 2^{-|x|}$, then μ is clearly dominated by the Lebesgue measure, which in turn is clearly effectively dominated by S. On the other hand, suppose that $\mu(x) \geq 2^{-|x|}$. Then, let $f_\nu(n)$ be the maximum time it takes for T to output a string of length n PROBLEM - THIS MIGHT NOT EXIST!!! There must be some program of length $\leq |x|$ that outputs x: call the shortest such program p^x. Then,

$$\begin{aligned}
S(x) &= \sum_{i=1}^{\infty} 2^{-i} \sum_{p \to_i x} 2^{-|p|} \\
&\geq \sum_{i=|p^x|+c+\log t(p^x,x)} 2^{-i-|p^x|-c} \\
&\geq 2^{-c} \sum_{i=|p^x|+c+\log f_\nu(|x|)} 2^{-i-|x|} \\
&\geq 2^{-c} 2^{-|x|} 2^{-|p^x|-\log f_\nu(|x|)-c+1} \\
&\geq c' \frac{2^{-2|x|}}{f_\nu(|x|)} \nu(x)
\end{aligned}$$

meaning that S effectively dominates ν.

If this result is to be useful, it will be due to the form of f. Alternatively, if it turns out that S uniformly effectively dominates the class of lower semi-computable semimeasures with the function $f(|x|)$, and that S/f is still a semimeasure, that would mean that S would perform well in a sequence prediction setting.