Multi-Variable Calculus

	Function	Derivative
1)	k	0
2)	x^n	$n \cdot x^{n-1}$
3)	$\sin x$	$\cos x$
4)	$\cos x$	$-\sin x$
5)	$\tan x$	$\sec^2 x$
6)	$\cot x$	$-\operatorname{cosec}^2 x$
7)	$\sec x$	$\sec x \cdot \tan x$
8)	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cdot \cot x$
9)	e^x	e^x
10)	a^x	$a^x \cdot \log a$
11)	$\log x$	$1/x$
12)	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
13)	$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
14)	$\tan^{-1} x$	$\frac{1}{1+x^2}$
15)	$\cot^{-1} x$	$\frac{-1}{1+x^2}$
16)	$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$

17)	$\operatorname{cosec}^{-1}x$	$\dfrac{-1}{x\sqrt{x^2-1}}$
18)	$\dfrac{1}{x}$	$\dfrac{-1}{x^2}$
19)	$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$
20)	kx	k

	function	Integration
1)	1	$x + C$
2)	x^n	$\dfrac{x^{n+1}}{n+1} + C$
3)	$(ax+b)^n$	$\dfrac{(ax+b)^{n+1}}{n+1} \times a + C$
4)	$\sin x$	$-\cos x + c$
5)	$\cos x$	$\sin x + C$
6)	$\tan x$	$\log(\sec x) + c$
7)	$\cot x$	$\log(\sin x) + c$
8)	$\sec x$	$\log(\sec x + \tan x) + c$
9)	$\operatorname{cosec} x$	$\log(\operatorname{cosec} x - \cot x) + c$
10)	e^x	$e^x + C$
11)	a^x	$\dfrac{a^x}{\log a} + C$
12)	$\dfrac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}(x/a) + c$

T

Y

13) $\dfrac{1}{\sqrt{x^2+a^2}}$	pe	$\log\left(x+\sqrt{x^2+a^2}\right)+c$
14) $\dfrac{1}{\sqrt{x^2-a^2}}$	1	$\log\left(x+\sqrt{x^2-a^2}\right)+c$
15) $\dfrac{1}{a^2-x^2}$	T	$\dfrac{1}{2a}\log\left(\dfrac{a+x}{a-x}\right)+c$
16) $\dfrac{1}{x^2+a^2}$	yp	$\dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right)+c$
17) $\dfrac{1}{x^2-a^2}$	e 2	$\dfrac{1}{2a}\log\left(\dfrac{x-a}{x+a}\right)+c$
18) $\sqrt{a^2-x^2}$	T	$\dfrac{x}{2}\sqrt{a^2-x^2}+\dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right)+c$
19) $\sqrt{x^2+a^2}$	yp	$\dfrac{x}{2}\sqrt{x^2+a^2}+\dfrac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)+c$
20) $\sqrt{x^2-a^2}$	e 3	$\dfrac{x}{2}\sqrt{x^2-a^2}-\dfrac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)+c$
21) $e^{ax}\cdot\sin bx$		$\dfrac{e^{ax}}{a^2+b^2}\left(a\cdot\sin bx-b\cdot\cos bx\right)+c$
22) $e^{ax}\cdot\cos bx$		$\dfrac{e^{ax}}{a^2+b^2}\left(a\cdot\cos bx+b\cdot\sin bx\right)+c$

$$\int_0^\infty e^{-x}\cdot x^{n-1}\,dx = \lvert\overline{n}$$

$$\int_0^\infty e^{-kx}\cdot x^{n-1}\,dx = \frac{\lvert\overline{n}}{k^n}$$

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$$

$$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{1}{2}\,\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\beta(m,n) = \frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}$$

$$\Gamma(1/2) = \sqrt{\pi}$$

$$\Gamma(1/4)\,\Gamma(3/4) = \pi\sqrt{2}$$

$$\Gamma(n) = (n-1)! \qquad n>0$$

$$1+\cos\theta = 2\cos^2\theta/2 \quad ; \quad 1-\cos\theta = 2\cdot\sin^2\theta/2$$

$$\sin 2\theta = 2\cdot\sin\theta\cdot\cos\theta \; ; \; \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$$= \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\sin 3\theta = 3\cdot\sin\theta - 4\sin^3\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cdot\cos\theta$$

$$\tan^{-1}(1) = \frac{\pi}{4} \; ; \; \tan^{-1}(\infty) = \frac{\pi}{2} \; ; \; \tan^{-1}(0) = 0$$

$$\sin^{-1}(1) = \pi/2 \; ; \; \sin^{-1}(0) = 0 \; ; \; \log 1 = 0$$

$$e^{\infty} = \infty \; ; \; e^{-\infty} = 0$$

$$\int \frac{f'(x)}{f(x)}\,dx = \log f(x) + c$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + c$$

$$\int \sec^2 x\,dx = \tan x \quad , \quad \int \operatorname{cosec}^2 x = -\cot x$$

$$\int \sec x \cdot \tan x\,dx = \sec x \quad , \quad \int \operatorname{cosec} x \cdot \cot x = -\operatorname{cosec} x$$

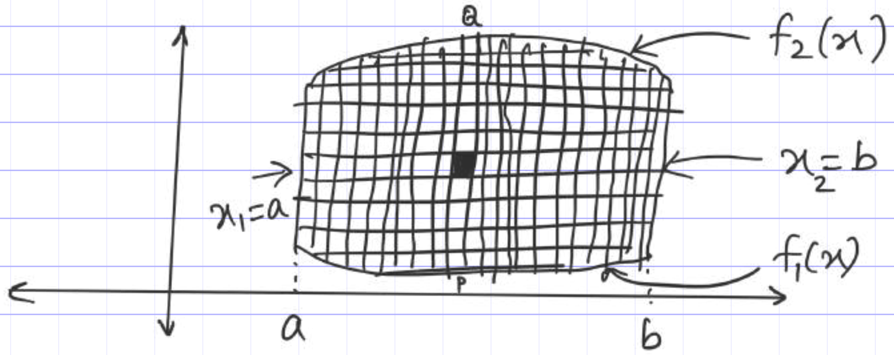
$$\int e^{-ax}\,dx = \frac{e^{-ax}}{-a}$$

☆ Multiple Integral :

◦ Evaluation :

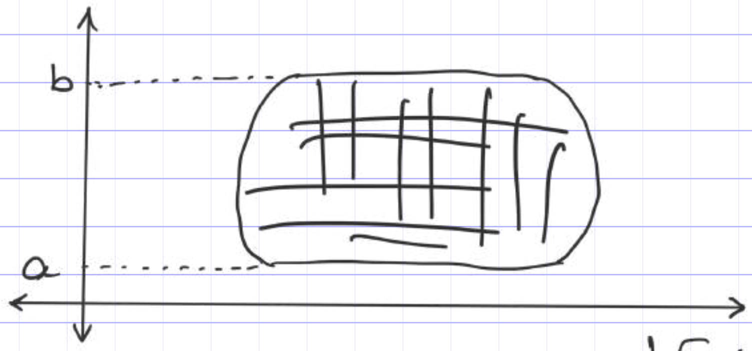
Let, A be region bounded by

$$y_1 = f_1(x) \quad , \quad y_2 = f_2(x) \quad ; \quad x_1 = a, \; x_2 = b$$



$$\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)\,dx\,dy = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy \right] dx$$

$x_1 = f_1(y)$, $x_2 = f_2(y)$, $y_1 = a$, $y_2 = b$



$$\int_a^b \int_{f_1(y)}^{f_2(y)} f(x,y)\, dx\, dy = \int_a^b \left[\int_{f_1(y)}^{f_2(y)} f(x,y)\, dx \right] dy$$

Interpretation

$\int f(x)\, dx \longrightarrow$ area under curve

if $f(x) = 1 \longrightarrow$ length of interval

$\iint f(x,y)\, dx\, dy \longrightarrow$ volume under surface

if $f(x,y) = 1 \longrightarrow$ area

$\iiint f(x,y,z)\, dx\, dy\, dz =$ Hypervolume

if $f(x,y,z) = 1 \longrightarrow$ volume in 3 dimension

Q 1 : Evaluate $\int_0^1\int_0^y xy \, dx \, dy$

$$I = \int_0^1\int_0^y xy \, dx \, dy$$

$$= \int_0^1 \left[y\int_0^y x \, dx \right] dy$$

$$= \int_0^1 y \left[x^2/2 \right]_0^y dy$$

$$= \frac{1}{2}\int_0^1 y \left(y^2 \right) dy$$

$$= \frac{1}{2} \left(y^4/4 \right)_0^1$$

$$\boxed{I = \frac{1}{8}}$$

Q 2 : Evaluate $\int_0^1\int_0^x e^{x+y} \, dx \, dy$

$$I = \int_0^1 \left[\int_0^x e^x \cdot e^y dy \right] dx$$

$$= \int_0^1 e^x \left(\int_0^x e^y \, dy \right) dx$$

$$= \int_0^1 e^x \left(e^x - 1 \right) dx$$

$$= \left(\frac{e^{2x}}{2} - e^x \right)_0^1$$

$$= \frac{e^2}{2} - e - \left(\frac{1}{2} - 1 \right)$$

$$= \frac{e^2}{2} - e + \frac{1}{2}$$

$$\boxed{I = \frac{(e-1)^2}{2}}$$

Q 3 : Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dx\,dy$

$$I = \int_0^1 \left[\int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2}\,dy \right] dx$$

$$= \int_0^1 \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}} \right) \right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left[\tan^{-1} 1 - \tan^{-1} 0 \right] dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \frac{\pi}{4}\,dx$$

$$= \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}}\,dx$$

$$= \frac{\pi}{4} \left[\log\left(x + \sqrt{1+x^2} \right) \right]_0^1$$

$$\boxed{I = \frac{\pi}{4} \cdot \log(1+\sqrt{2})}$$

Q 4 : Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dy\,dx}{\sqrt{1-x^2-y^2}}$

$$\rightarrow \quad I = \int_0^1 \left[\int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx}{\sqrt{(1-y^2)-x^2}} \right] dy$$

$$= \int_0^1 \left[\sin^{-1}\left(\frac{x}{\sqrt{1-y^2}}\right) \right]_0^{\sqrt{\frac{1-y^2}{2}}} dy$$

$$= \int_0^1 \left[\sin^{-1}\left(1/\sqrt{2}\right) - \sin^{-1} 0 \right] dy$$

$$= \int_0^1 \frac{\pi}{4} \, dy$$

$$\boxed{I = \frac{\pi}{4}}$$

Q5 : Evaluate $\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2} \, dx \, dy$

$$\rightarrow \quad I = \int_0^a \left[\int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2} \, dx \right] dy$$

$$= \int_0^a \left[\frac{x}{2}\sqrt{a^2-y^2-x^2} + \frac{a^2-y^2}{2} \sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right) \right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left[\left(0 + \frac{a^2-y^2}{2} \frac{\pi}{2}\right) - (0+0) \right] dy$$

$$= \frac{\pi}{4} \int_0^a (a^2-y^2) \, dy$$

$$= \frac{\pi}{4} \left[a^2 y - \frac{y^3}{3} \right]_0^a$$

$$= \frac{\pi}{4} \times \frac{2a^3}{3}$$

$$\boxed{I = \frac{\pi a^3}{6}}$$

Q6: $\int_0^1 \int_0^y xy \cdot e^{-x^2} dx\, dy$

$$I = \int_0^1 \left[y \int_0^y x \cdot e^{-x^2} dx \right] dy \quad \cdots\cdots\cdots \; (*)$$

Let, $I_1 = \int_0^y x \cdot e^{-x^2} dx$

put, $x^2 = t$

$\therefore 2x\,dx = dt$

$\therefore x\,dx = \frac{dt}{2}$

limit $x = 0$, $t = 0$; $x = y$, $t = y^2$

$$I_1 = \frac{1}{2} \int_0^{y^2} e^{-t} dt$$

$$= \frac{1}{2} \left(\frac{e^{-t}}{-1} \right)_0^{y^2}$$

$$\therefore I_1 = \frac{-1}{2} \left(e^{-y^2} - 1 \right)$$

from $(*)$,

$$I = \int_0^1 y \times \frac{-1}{2} \times (e^{-y^2} - 1)\, dy$$

$$= \frac{-1}{2}\left[\int_0^1 y \cdot e^{-y^2} dy - \int_0^1 y \cdot dy\right]$$

$$= \frac{-1}{2}\left[\int_0^1 y \cdot e^{-y^2} dy - \frac{1}{2}\left(y^2\right)_0^1\right]$$

put, $y^2 = v$

$2y \cdot dy = v$

$y = 0,\ v = 0\ ;\ y = 1,\ v = 1$

$$I = \frac{-1}{2}\left[\frac{1}{2}\int_0^1 e^{-v} \cdot dv - \frac{1}{2}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{2}\left(e^{-1} - 1\right) - \frac{1}{2}\right]$$

$$= \frac{-1}{2}\left[\frac{-1}{2e} + \frac{1}{2} - \frac{1}{2}\right]$$

$$\boxed{I = \frac{1}{4 \cdot e}}$$

Q7 : Evaluate $\displaystyle\int_0^{\pi/2}\int_0^{3(1-\cos t)} x^2 \cdot \sin t \; dx \, dt$

$$I = \int_0^{\pi/2}\left[\sin t \int_0^{3(1-\cos t)} x^2 \, dx\right] dt$$

$$= \int_0^{\pi/2}\left[\frac{\sin t}{3}\left(x^3\right)_0^{3-3\cos t}\right] dt$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin t}{3}\left[3^3(1-\cos t)^3\right] dt$$

put, $1-\cos t = u$

$\therefore 0 - -\sin t\, dt = du$

$\sin t\, dt = du$

limits $\rightarrow$ $t=0,\ u=0$; $t=\frac{\pi}{2},\ u=1$

$$\therefore I = 9\int_0^1 u^3\, du$$

$$\boxed{\therefore I = \frac{9}{4}}$$

Q8: Evaluate $\int_0^{\pi/2}\int_{\pi/2}^{\pi} \cos(x+y)\, dx\, dy$

$$I = \int_0^{\pi/2}\left[\int_{\pi/2}^{\pi} \cos(x+y)\, dx\right] dy$$

$$= \int_0^{\pi/2}\left[\ \sin(x+y)\right]_{\pi/2}^{\pi} dy$$

$$= \int_0^{\pi/2}\left[\sin(\pi+y) - \sin\left(\frac{\pi}{2}+y\right)\right] dy$$

$$= \int_0^{\pi/2}\left[-\sin y - \cos y\right] dy$$

$$= (\cos y)_0^{\pi/2} - (\sin y)_0^{\pi/2}$$

$$= 0 - 1 - (1-0)$$

$$\boxed{I = -2}$$

Q 9: Evaluate $\int_0^\infty \int_0^\infty e^{-x^2(1+y^2)} \cdot x \cdot dx\, dy$

$\rightarrow$ $$I = \int_0^\infty \left[\int_0^\infty e^{-x^2(1+y^2)} \cdot x\, dx \right] dy$$

take $k = 1+y^2$

$$I = \int_0^\infty \left[\int_0^\infty e^{-kx^2} x\, dx \right] dy$$

put, $kx^2 = t$

$$k \cdot 2x \cdot dx = dt$$

$$x\, dx = \frac{dt}{2k}$$

$x = 0,\ t = 0$; $x \to \infty,\ t \to \infty$

$$\therefore\ I = \int_0^\infty \left[\int_0^\infty e^{-t} \cdot \frac{dt}{2k} \right] dy$$

$$= \int_0^\infty \frac{-1}{2k} \left(e^{-t} \right)_0^\infty dy$$

$$= \frac{-1}{2} \int_0^\infty \frac{1}{1+y^2} (0-1)\, dy$$

$$= \frac{1}{2} \int_0^\infty \frac{1}{1+y^2}\, dy$$

HW

$\int_0^2 \int_0^{\sqrt{2x-x^2}} xy\, dx\, dy$

Ans $= 2/3$

$$= \frac{1}{2}\left(\tan^{-1}y\right)_0^{\infty}$$

$$= \frac{1}{2}\left(\frac{\pi}{2}-0\right)$$

$$\boxed{I = \frac{\pi}{4}}$$

HW : $\int_0^{\pi/2}\int_{a(1-\cos\theta)}^{a} r^2 \cdot dr\, d\theta$

Ans :

HW : $\int_0^{1}\int_0^{\infty} e^{-x^2(1+y^2)}\, dx\, dy$

Ans : $\frac{\sqrt{\pi}}{2}\log(1+\sqrt{2})$

$\int_0^{\pi/2}\int_0^{a\cdot\cos\theta} r\sqrt{a^2-r^2}\, dr\, d\theta$

Ans: $(3\pi-4)\frac{a^3}{18}$

Q 10 : $\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\, d\theta$

$$I = \int_0^{\pi/4}\left[\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\, dr\right] d\theta$$

put, $1+r^2 = t$ $\qquad r = (t-1)^{1/2}$

$0 + 2r\,dr = dt$

$r = 0,\ t = 1 \ ;\ r = \sqrt{\cos 2\theta},\ t = 2\cdot\cos^2\theta$

$$I = \int_0^{\pi/4}\left[\int_1^{2\cos^2\theta} \frac{\sqrt{t-1}}{t^2}\,\frac{dt}{2\sqrt{(t-1)}}\right]$$

$$= \int_0^{\pi/4}\frac{1}{2}\left[\left(\frac{1}{-t}\right)_1^{2\cos^2\theta}\right]$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{-2\cdot\cos^2\theta} - \frac{1}{-1}$$

$$= \frac{1}{2}\int_0^{\pi/4}\left(\frac{1}{2}\sec^2\theta + 1\right) = \frac{1}{2}\left(\frac{-1}{2}(1-0) + \frac{\pi}{4} - 0\right)$$

$$\boxed{I = \frac{\pi}{8} - \frac{1}{4}}$$

☆ Triple Integral :

$$\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dx\, dy\, dz$$

$$= \int_a^b \left\{ \int_{\phi_1(x)}^{\phi_2(x)} \left[\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\, dz \right] dy \right\} dx$$

Type 2

Q 1 : Evaluate $\displaystyle\int_0^2 \int_0^y \int_{x-y}^{x+y} (x+y+z)\, dx\, dy\, dz$

→

$$I = \int_0^2 \int_0^y \left[\int_{x-y}^{x+y} (x+y+z)\, dz \right] dx\, dy$$

$$= \int_0^2 \int_0^y \left[\frac{(x+y+z)^2}{2} \right]_{x-y}^{x+y} dx\, dy$$

$$= \frac{1}{2} \int_0^2 \int_0^y \left\{ (2x+2y)^2 - (2x)^2 \right\} dx\, dy$$

$$= \frac{4}{2} \int_0^2 \left\{ \int_0^y \left[(x+y)^2 - x^2 \right] dx \right\} dy$$

$$= 2\int_0^2 \left[\frac{(x+y)^3}{3} - \frac{x^3}{3}\right]_0^y dy$$

$$= 2\int_0^2 \left\{\left[(2y)^3 - y^3\right] - \left(y^3 - 0\right)\right\} dy$$

$$= \frac{2}{3}\int_0^2 6y^3\, dy$$

$$= \frac{2}{3}\times 6 \left(y^4\right)_0^2$$

$$= \frac{4}{4}\left(16-0\right)$$

$$\boxed{I = 16}$$

Q2: Evaluate $\displaystyle\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\, dx\, dy\, dz$

$$\longrightarrow \quad I = \int_0^{\log 2}\int_0^{x}\left[e^{x+y}\cdot\left(e^{z}\right)_0^{x+y}\right] dx\, dy$$

$$= \int_0^{\log 2}\int_0^{x} e^{x+y}\left(e^{x+y}-1\right) dx\, dy$$

$$= \int_0^{\log 2}\left[\int_0^{x}\left(e^{2(x+y)} - e^{x+y}\right) dy\right] dx$$

$$= \int_0^{\log 2} \left[\frac{e^{2x+2y}}{2} - \frac{e^{x+y}}{1} \right]_0^x dx$$

$$= \int_0^{\log 2} \left[\left(\frac{e^{4x}}{2} - e^{2x} \right) - \left(\frac{e^{2x}}{2} - e^{x} \right) \right] dx$$

$$= \int_0^{\log 2} \left(\frac{e^{4x}}{2} - \frac{3}{2} e^{2x} + e^{x} \right) dx$$

$$= \frac{1}{2} \int_0^{\log 2} \left(e^{4x} - 3 e^{2x} + 2 e^{x} \right) dx$$

$$= \frac{1}{2} \left[\frac{e^{4x}}{4} - 3 \cdot \frac{e^{2x}}{2} + 2 \cdot e^{x} \right]_0^{\log 2}$$

$$= \frac{1}{2} \left\{ \left(\frac{e^{4 \log 2}}{4} - 3 \cdot \frac{e^{2 \log 2}}{2} + 2 \cdot e^{\log 2} \right) - \left(\frac{1}{4} - \frac{3}{2} + 2 \right) \right\}$$

$$= \frac{1}{2} \left(2 - \frac{3}{4} \right)$$

$$\boxed{I = \frac{5}{8}}$$

Q 3: Evaluate $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\, dy\, dz}{\sqrt{1-x^2-y^2-z^2}}$

$$\rightarrow I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left\{ \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{(1-x^2-y^2)-z^2}}\, dz \right\} dy\, dx$$

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \left[\sin^{-1}\left(\frac{z}{\sqrt{1-x^2-y^2}} \right) \right]_0^{\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^1 \left[\int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy \right] dx$$

$$= \frac{\pi}{2} \int_0^1 \left[y \right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2} \left\{ \int_0^1 \sqrt{1-x^2}\, dx \right\}$$

$$= \frac{\pi}{2} \left\{ \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x}{1}\right) \right]_0^1 \right\}$$

$$= \frac{\pi}{2} \left\{ \left(0 + \frac{1}{2} \times \frac{\pi}{2} \right) - (0+0) \right\}$$

$$\boxed{I = \frac{\pi^2}{8}}$$

Q 4: $\displaystyle\int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$

$$I = \int_{-2}^{2}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \Big[z \Big]_{x^3+3y^2}^{8-x^2-y^2} dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{\dots}}^{\sqrt{\dots}} (8-x^2-y^2)-(x^2+3y^2)\, dy\, dx$$

$$= \int_{-2}^{2}\int_{-\sqrt{\dots}}^{\sqrt{\dots}} (8-2x^2-4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} 2 \times \int_{0}^{\text{upper}} (8-2x^2-4y^2)\, dx \quad \dots\dots \text{even function} \quad \int_{-a}^{a}\text{even} = 2\int_{0}^{a}\text{even}$$

$$= 2\int_{-2}^{2} \left\{ \left[(8-2x^2)y - \frac{4y^3}{3} \right]_{0}^{\sqrt{\frac{4-x^2}{2}}} \right.$$

$$= 2\int_{-2}^{2} \left\{ \left[(8-2x^2)\sqrt{\frac{4-x^2}{2}} - \frac{4}{3}\left(\frac{4-x^2}{2}\right)\sqrt{\frac{4-x^2}{2}} \right] - \Big[0 - 0 \Big] \right\} dx$$

$$= 2\int_{-2}^{2}\sqrt{\frac{4-x^2}{2}}\left\{8-2x^2-\frac{8}{3}+\frac{2x^2}{3}\right\}dx$$

$$= 2\int_{-2}^{2}\sqrt{\frac{4-x^2}{2}}\left(16/3-\frac{4}{3}x^2\right)dx$$

$$= 2\times\frac{4}{3}\int_{-2}^{2}\sqrt{\frac{4-x^2}{2}}\left(4-x^2\right)$$

$$= \frac{8}{3}\int_{-2}^{2}\frac{1}{\sqrt{2}}\left(4-x^2\right)^{3/2}dx$$

$$= \frac{8}{3\sqrt{2}}\int_{-2}^{2}\left(4-x^2\right)^{3/2}dx$$

$$= \frac{8}{3\sqrt{2}}\times 2\int_{0}^{2}\left(4-x^2\right)^{3/2}dx \qquad \left(\text{even function}\right)$$

$$= \frac{16}{3\sqrt{2}}\int_{0}^{2}\left(4-x^2\right)^{3/2}dx$$

put, $x^2 = 4t$

$$x = 2\sqrt{t}$$

$$dx = 2\frac{1}{2\sqrt{t}}\,dt$$

$x=0,\ t=0\ ;\ x=2,\ t=1$

$$I=\frac{16}{3\sqrt{2}}\int_0^1 (4-4t)^{3/2}\, t^{-1/2}\, dt$$

$$=\frac{16\times 8}{3\sqrt{2}}\int_0^1 (1-t)^{3/2}\, t^{-1/2}\, dt$$

$$=\frac{16\times 8}{3\sqrt{2}}\ \beta\left(\frac{1}{2},\frac{5}{2}\right)$$

$$=\frac{16\times 8}{3\sqrt{2}}\ \frac{\Gamma(1/2)\ \Gamma(5/2)}{\Gamma(3)}$$

$$=\frac{16\times 8}{3\sqrt{2}}\times\frac{\Gamma(1/2)\times 3/2\times 1/2\times\Gamma(1/2)}{2\times 1}$$

$$=\frac{\overset{\cancel{32}\,\cancel{16}\ 8\times 2}{\cancel{64}}}{\cancel{3}\sqrt{2}}\ \frac{\times\pi\times\cancel{3}\times 1}{\cancel{2}\times\cancel{2}\times\cancel{2}}$$

$$\boxed{I=\frac{16\pi}{\sqrt{2}}=8\pi\sqrt{2}}$$

HW : $$\int_0^a\int_0^{a-x}\int_0^{a-x-y} x^2\, dz\, dy\, dx=\frac{a^5}{60}$$

Answer

HW : $\displaystyle\int_0^2\int_x^{4-x}\int_{\frac{3x}{2}-y}^{3} dz\, dy\, dx = 16$

Answer

Q5 : $\displaystyle\int_1^e\int_1^{\log y}\int_1^{e^x} \log z \cdot dz\, dy\, dx$

$$\rightarrow \quad I = \int_1^e\int_1^{\log y}\left[\int_1^{e^x} \log z \cdot dz\right] dy\, dx$$

$$= \int_1^e\int_1^{\log y}\left\{\left[\log z \times z\right]_1^{e^x} - \int_1^{e^x} z \cdot \frac{1}{z} \cdot dz\right\} dy\, dx$$

$$= \int_1^e\int_1^{\log y}\left\{(x \cdot e^x - 0) - (e^x - 1)\right\} dy\, dx$$

$$= \int_1^e\left[\int_1^{\log y}\left[(x-1)e^x + 1\right] dx\right] dy$$

$$= \int_1^e\left\{\left[(x-1)e^x - 1 \cdot e^x\right]_1^{\log y} + (\log y - 1)\right\} dy$$

$$= \int_1^e\left[(\log y - 1)y - y - (0 - e) + \log y - 1\right] dy$$

$$= \int_1^e\left[y \log y - 2y + e + \log y - 1\right] dy$$

Ans : $\dfrac{e^2-8e+13}{4}$

$$= \int_1^e \left[(y+1)\log y - 2y + e - 1\right] dy$$

$$= \quad \vdots$$

$$= \quad \vdots$$

$$= \quad \vdots$$

$$\boxed{I = \frac{e^2 - e + 13}{4}}$$

Type 3

☆ Change the Order of Integration (For Cartesian Co-ord.) :

① When evaluation of integration with given order is dificult, then in that case, order of integration is to be changed.

② $x = 0 \Rightarrow$ Y axis ; $y = 0 \Rightarrow$ X axis

$x = h \Rightarrow$ line ||el to Y axis ; $y = k \Rightarrow$ line ||el to X axis

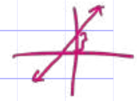
$x^2 + y^2 = a^2 \Rightarrow$ circle with centre $(0,0)$ & radius $= a$

$(x-h)^2 + (y-k)^2 = a^2 \Rightarrow$ Circle with centre (h,k) & radius $= a$

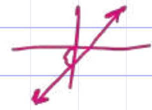
$y = mx \quad (m>0) \Rightarrow$ line passing through $(0,0)$ from 1st Quad. to 3rd Quad.

$y = mx \quad (m<0) \Rightarrow$ line passing through $(0,0)$ from 2nd Quad. to 4th Quad.

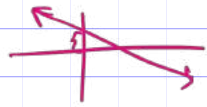
$y = mx + c$ $\quad (m>0, c>0) \Rightarrow$ line passing from 1st to 3rd Quad. with +ve intercept.



$y = mx + c$ $\quad (m>0, c<0) \Rightarrow$ line passing from 1st to 3rd Quad. with −ve intercept

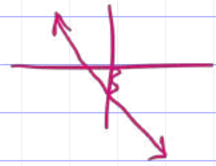


$y = mx + c$ $\quad (m<0, c>0) \Rightarrow$ line passing from 2nd to 4th Quad. ; + inter.



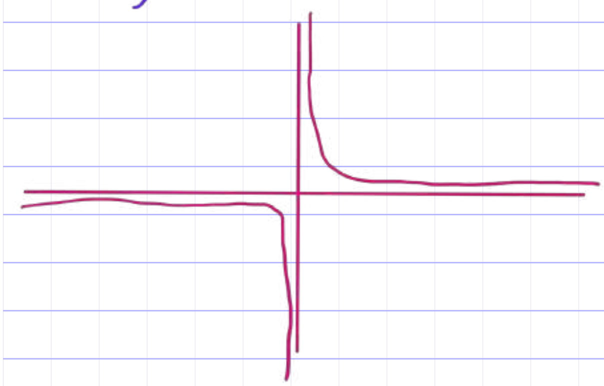
line

$y = mx + c$ $\quad (m<0, c<0) \Rightarrow$ same but −ve intercept.



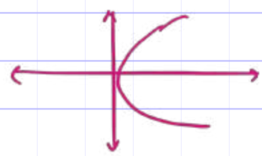
- Rectangular Hyperbola ⤵

$xy = k$

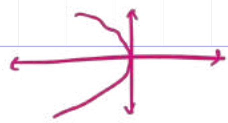


$xy = -k$

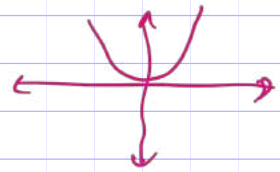
$y^2 = 4ax \Rightarrow$ parabola with vertex (0, 0) ; open towards right side



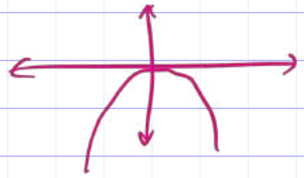
$y^2 = -4ax \Rightarrow$ same ; open towards left



$x^2 = 4by \Rightarrow$ parabola with vertex $(0,0)$; open upwards



$x^2 = -4by \Rightarrow$ same ; open downwards



$(y-k)^2 = 4a(x-h) \Rightarrow$ Parabola with vertex (h,k) & open rightwards

$(y-k)^2 = -4a(x-h) \Rightarrow$ Parabola with vertex (h,k) & open leftwards

$(x-h)^2 = 4a(y-k) \Rightarrow$ parabola " " (h,k) & open upwards

$(x-h)^2 = -4a(y-k) \Rightarrow$ parabola " " (h,k) & open downwards

$$\int \frac{f'(x)}{f(x)}\,dx = \log f(x)$$

$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}$$

$$\int e^x\left[f(x) + f'(x)\right]dx = e^x \cdot f(x)$$

$\sin^{-1}(1) = \pi/2$, $\sin^{-1}(0) = 0$, $\tan^{-1}(1) = \frac{\pi}{4}$

$\tan^{-1}(\infty) = \pi/2$, $\tan^{-1}(0) = 0$

☆ for change the order of int^n –

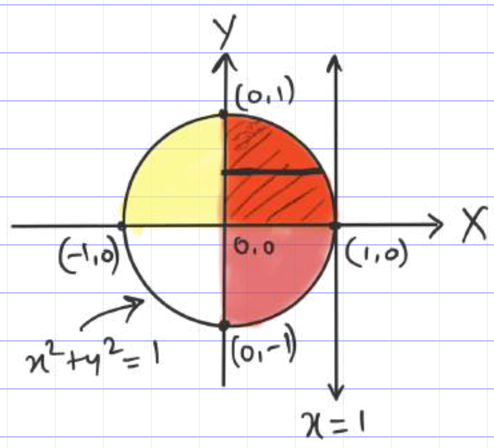
1) If limits are functions of x , consider strip parallel to X axis.

2) If limits are functions of y , consider strip parallel to Y axis.

Q : Change order of integration & hence Evaluate.

1) $$\int_0^1 \int_0^{\sqrt{1-x^2}} \frac{1}{(1+e^y)\sqrt{1-x^2-y^2}}\, dy\, dx$$

→ Given region is $y = 0$ to $y = \sqrt{1-x^2}$ ⇒ $\boxed{x^2+y^2 = 1}$ circle with centre $(0,0)$ & rad. = 1
& $x = 0$ to $x = 1$



In given prob., strip is //el to Y axis.

Consider, strip parallel to X axis.

Move the strip from bottom to top, it remains in same region.

$\therefore$ limits are $x=0$ to $x=\sqrt{1-y^2}$

$y=0$ to $y=1$

$$\therefore \quad I = \int_0^1 \left(\int_0^{\sqrt{1-y^2}} \frac{1}{(1+e^y)\sqrt{1-x^2-y^2}}\, dx \right.$$

$$= \int_0^1 \frac{1}{1+e^y} \left[\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{(1-y^2)-x^2}}\, dx \right] dy$$

$$= \int_0^1 \frac{1}{1+e^y} \sin^{-1}\left(\frac{x}{\sqrt{1-y^2}}\right)_0^{\sqrt{1-y^2}} dy$$

$$= \frac{\pi}{2} \int_0^1 \frac{1}{1+e^y}\, dy$$

$$= \frac{\pi}{2} \int_0^1 \left[\frac{1+e^y}{1+e^y} - \frac{e^y}{1+e^y}\right] dy$$

$$= \frac{\pi}{2} \left\{ \int_0^1 dy - \int_0^1 \frac{e^y}{1+e^y}\, dy \right\}$$

$$= \pi/2 \left\{ [y]_0^1 - \left[\log(1+e^y)\right]_0^1 \right\}$$

$$= \frac{\pi}{2} \left\{ 1 - 0 - \left[\log(1+e) - \log 2\right] \right\}$$

$$= \frac{\pi}{2} \left\{ \log e - \log(1+e) + \log 2 \right\}$$

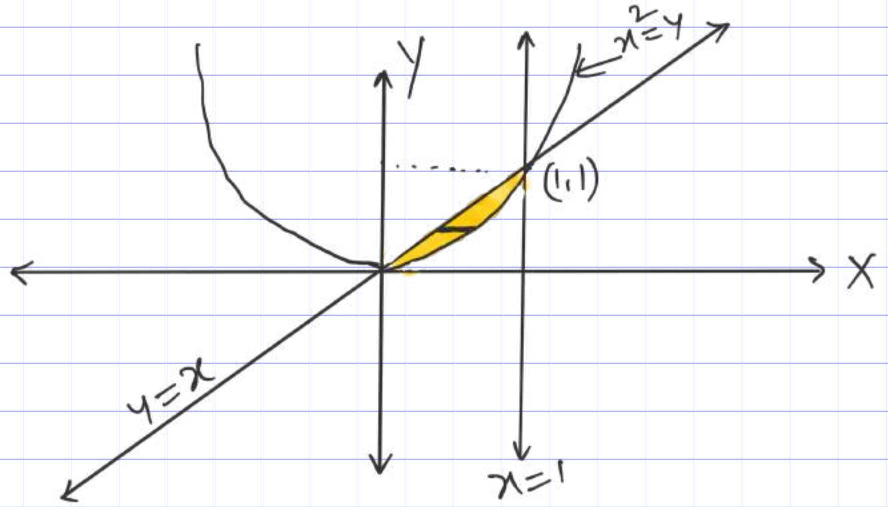
$$\boxed{I = \frac{\pi}{2} \log\left(\frac{2e}{1+e}\right)}$$

Q 2) $\displaystyle\int_0^1 \int_{x^2}^{x} xy \, dy \, dx$

$\rightarrow$ Given region is $y = x^2$ to $y = x$
& $x = 0$ to $x = 1$.

here, $y = x^2 \Rightarrow$ parabola with Ver $(0,0)$ & open upwards

$y = x \Rightarrow$ line passing thr. $(0,0)$ from 1^{st} to 3^{rd} Quad.



In given prob., strip is //el to Y axis. hence, to change the order, Consider, strip parallel to X axis. & move it from bottom to top. So. that, entire region is covered.

As we move the strip, it remains in same region.

$\therefore$ limits are $x = y$ to $x = \sqrt{y}$

& $y = 0$ to $y = 1$

$$\therefore \quad I = \int_0^1 y \left[\int_y^{\sqrt{y}} x \, dx \right] dy$$

$$= \int_0^1 y \left[\frac{x^2}{2}\right]_{\sqrt{y}}^{y} dy$$

$$= \int_0^1 y \left(y/2 - y^2/2 \right) dy$$

$$= \frac{1}{2} \int_0^1 \left(y^2 - y^3\right) dy$$

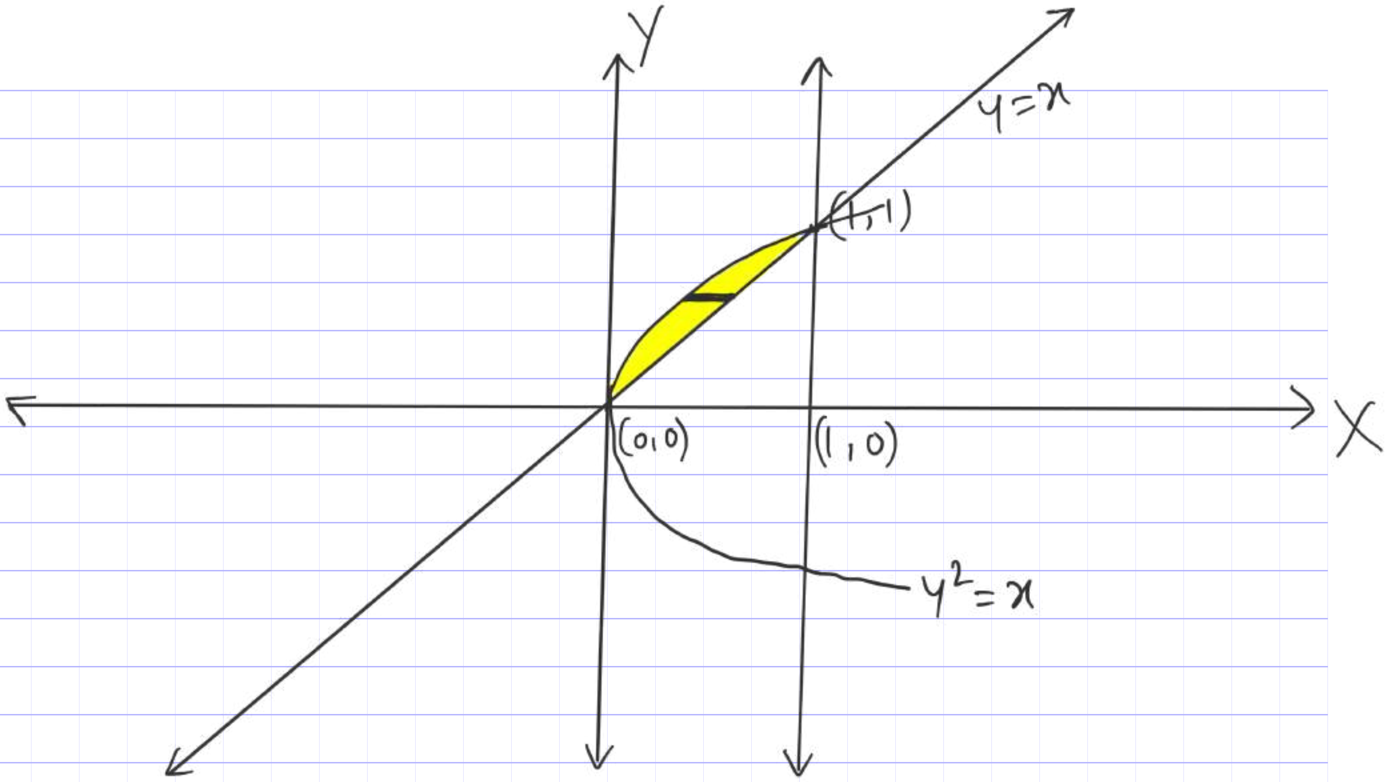
$$\boxed{I = \frac{1}{24}}$$

Q3: $\displaystyle\int_0^1 \int_x^{\sqrt{x}} (x^2 + y^2) \, dy \, dx$

$\rightarrow$ Given region is $y = x$ to $y = \sqrt{x}$

& $x = 0$ to $x = 1$

here, $y = \sqrt{x} \Rightarrow y^2 = x$ parabola with vertex $(0,0)$ & open rightwards



In given prob., strip is ||el to Y axis.
To change order, consider strip ||el to X axis. Move the strip from bottom to top so that entire region will get covered.

$x = y^2$ to $x = y$

& $y = 0$ to $y = 1$

$$I = \int_0^1 \left[\int_{y^2}^{y} (x^2+y^2)\, dx \right] dy$$

$$I = \int_0^1 \left[\frac{x^3}{3} + y^2 x \right]_{y^2}^{y} dy$$

$$I = \int_0^1 \left[\left(\frac{y^3}{3} + y^3 \right) - \left(\frac{y^6}{3} + y^4 \right) \right] dy$$

$$I = \int_0^1 \left(\frac{4y^3}{3} - \frac{y^6}{3} - y^4 \right) dy$$

$$I = \left[\frac{\cancel{4}}{3}\frac{y^4}{\cancel{4}} - \frac{y^7}{21} - \frac{y^5}{5}\right]_0^1$$

$$I = \frac{1}{3} - \frac{1}{21} - \frac{1}{5}$$

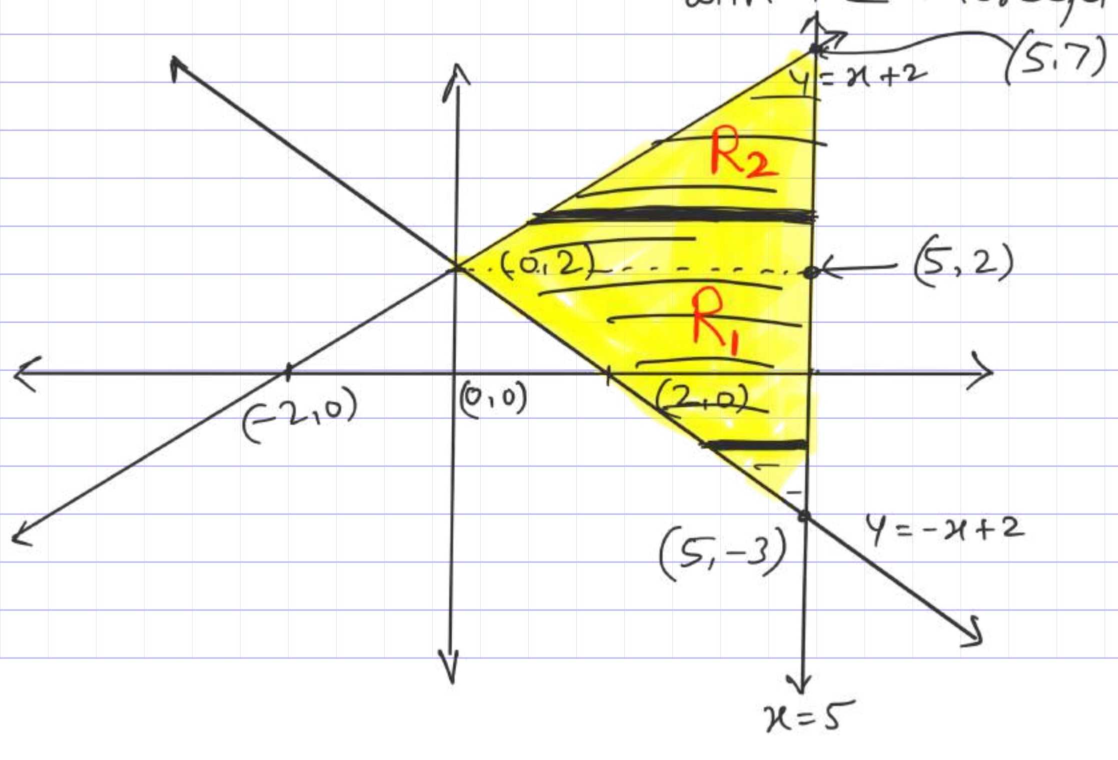
$$\boxed{\therefore\ I = \frac{3}{35}}$$

Q4: $\displaystyle\int_0^5\int_{2-x}^{2+x} f(x,y)\,dy\,dx$

→ Given region is $y = 2-x$ to $y = 2+x$
$x = 0$ to $x = 5$.

$y = -x + 2 \Rightarrow$ line passing from 2^{nd} to 4^{th} Quad
with $+2$ intercept

& $y = x + 2 \Rightarrow$ line passing from 1^{st} to 3^{rd} Quad.
with $+2$ intercept



In given problem, strip is //el to Y axis. To change order of integration, consider strip parallel to X axis. & move the strip from bottom to top so that entire region will get covered.

As we are moving the strip from bottom to top, it changes the position i.e. it takes jump & enter into another region.

here, region of int. is divided into 2 parts. i.e. R_1 & R_2.

$\therefore$ limits are, In R_1 : $x = 2-y$ to $x = 5$
& $y = -3$ to $y = 2$

, In R_2 : $x = y-2$ to $x = 5$
$y = 2$ to $y = 7$

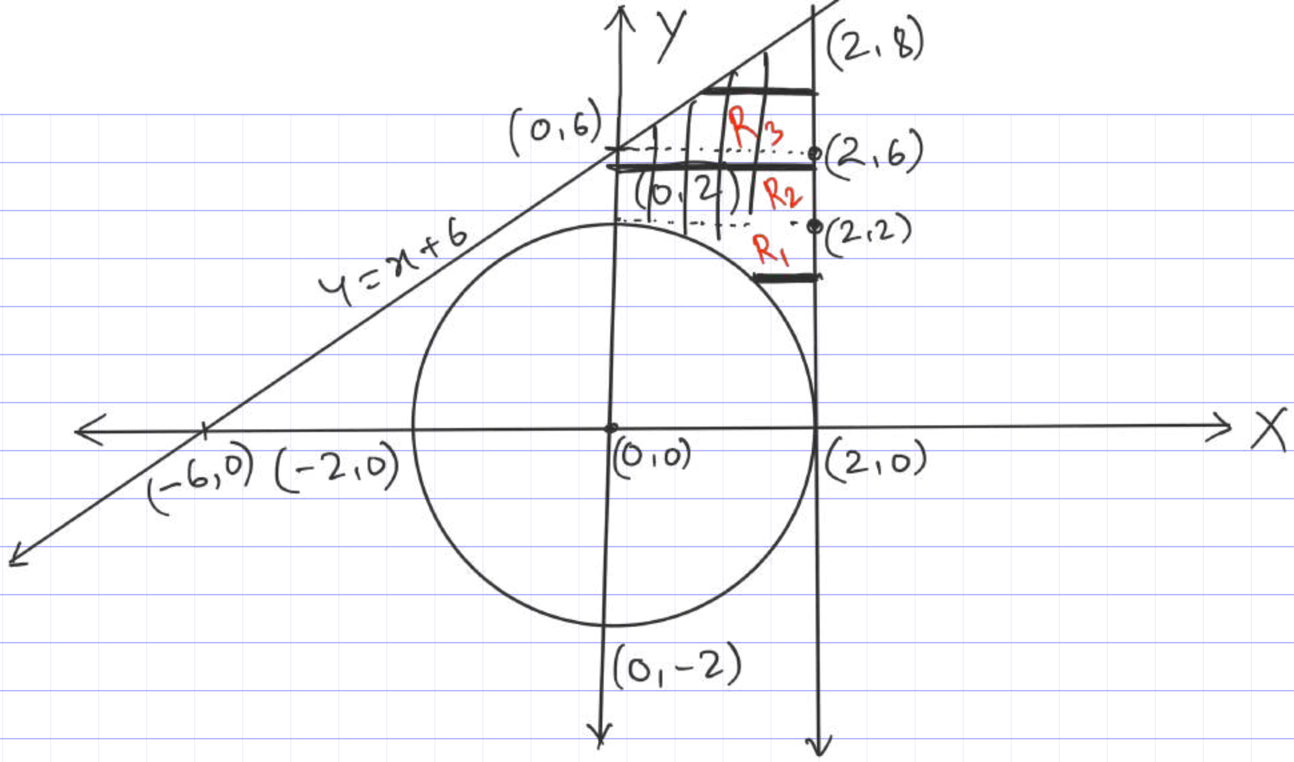
$$\boxed{I = \int_{-3}^{2}\int_{2-y}^{5} f(x,y)\,dy\,dx + \int_{2}^{7}\int_{y-2}^{5} f(x,y)\cdot dx\,dy}$$

Q5: $\displaystyle\int_{0}^{2}\int_{\sqrt{4-x^2}}^{x+6} f(x,y)\,dy\,dx$

$\rightarrow$ Given region is $y = \sqrt{4-x^2}$ to $y = x+6$
& $x = 0$ to $x = 2$

here, $y = \sqrt{4-x^2}$ i.e. $x^2+y^2 = 4 \Rightarrow$ (0,0) 2 with centre $(0,0)$
& $r = 2$

$y = x+6$ $\Rightarrow$ line passing from 1st to 3rd Quad. with +6 y intercept.



In given prob., strip parallel to Y axis.
To change the order, consider strip //el to X axis.

Move the strip from bottom to top so that entire region is covered.

for R_1, $x = \sqrt{4-y^2}$ to $x = 2$
& $y = 0$ to $y = 2$

for R_2, $x = 0$ to $x = 2$
& $y = 2$ to $y = 6$

for R_3, $x = y-6$ to $x = 2$
& $y = 6$ to $y = 8$

$$\therefore I = \int_0^2 \int_{\sqrt{4-y^2}}^2 f(x,y)\,dy\,dx + \int_2^6 \int_0^2 f(x,y)\,dy\,dx + \int_6^8 \int_{y-6}^2 f(x,y)\,dy\,dx$$

0

$$Q6: \int_0^2 \int_{\frac{x^2+4}{4}}^{\frac{6-x}{2}} f(x,y)\, dy\, dx$$

$\rightarrow$ region is $y = \dfrac{x^2+4}{4}$ to $y = \dfrac{6-x}{2}$

$x = 0$ to $x = 2$

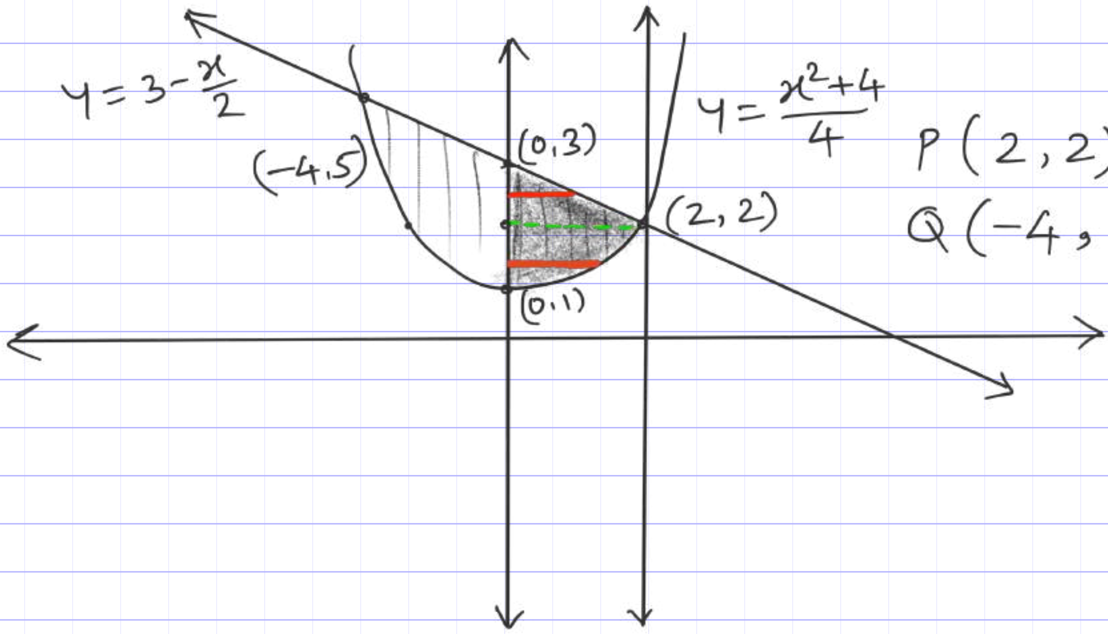
$$4y = x^2 + 4$$

$$x^2 = 4(y-1) \Rightarrow \text{Parabola with vertex } (0,1) \text{ open upwards}$$

$$y = \frac{6-x}{2}$$

$$y = 3 - \frac{x}{2}$$

$$y = \frac{-1}{2}x + 3 \Rightarrow \text{line passing from } 2^{th} \text{ to } 4^{th} \text{ Quad. with } +3 \text{ y intercept}$$



$P(2,2)$

$Q(-4,$

here, P & Q are pt. of int. of line & parabola

$$3 - \frac{x}{2} = \frac{x^2+4}{4}$$

$$12 - 2x = x^2 + 4$$

$$(x+4)(x-2) = 0$$

$$\boxed{x = 2 \quad \& \quad x = -4}$$

here, Region divided into 2 parts.

In R_1, $x = 0$ to $x = \sqrt{4y-4}$

$y = 1$ to $y = 2$

In R_2, $x = 0$ to $x = 6 - 2y$

$y = 2$ to $y = 3$

$$\therefore I = \iint f(x,y)\,dy\,dx + \iint f(x,y)\,dy\,dx$$

Q7: $$\int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} f(x,y)\,dy\,dx$$

→ region is $x = \sqrt{a^2-y^2}$ to $x = y + a$

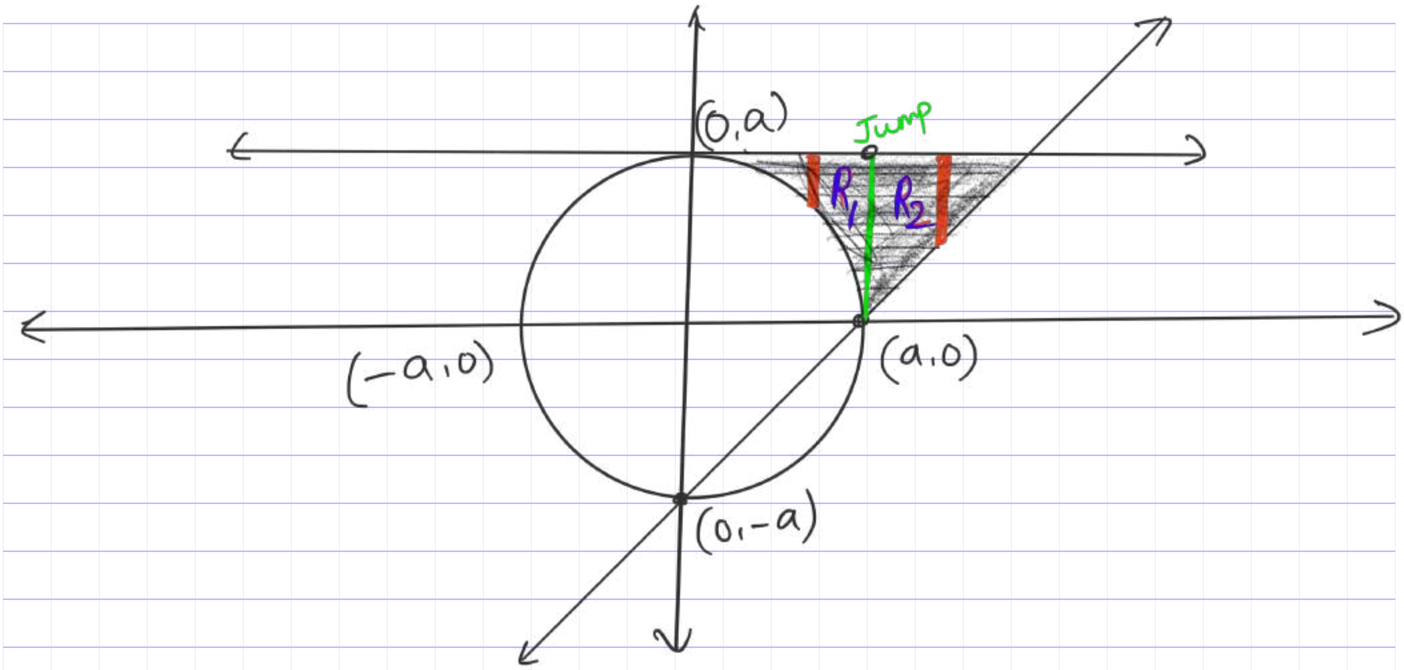
& $y = 0$ to $y = a$

here, $$x = \sqrt{a^2 - y^2}$$

$\therefore x^2 + y^2 = a^2 \Rightarrow$ ⊙ with centre $(0,0)$ & rad $= a$

$$x = y + a$$

$\therefore y = x - a \Rightarrow$ line passing from 1st to 3rd Qud with −ve intercept



In given prob., strip is //el to X axis.
So, to change order, consider strip //el to Y axis & move it from left to right. as we move the strip, it changes pos^n i.e. takes jump.
∴ Region divided into 2 parts.

In R_1 ; $y = \sqrt{a^2-x^2}$ to $y = a$

$x = 0$ to $x = a$

In R_2 ; $y = x-a$ to $y = a$

$x = a$ to $x = 2a$

$$\therefore \; I = \int_{\sqrt{a^2-x^2}}^{a}\int_{0}^{a} f(x,y)\,dy\,dx + \int_{x-a}^{a}\int_{a}^{2a} f(x,y)\,dy\,dx$$

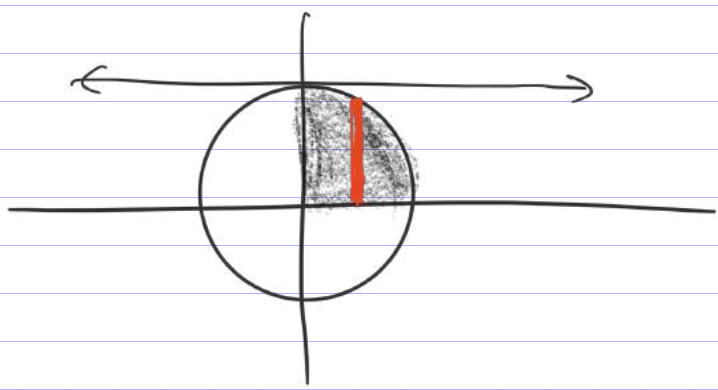
Q8 : $$\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{\cos^{-1}x}{\sqrt{1-x^2}\,\sqrt{1-x^2-y^2}}\,dy\,dx$$

here, region is $x = 0$ to $x = \sqrt{1-y^2}$

& $y = 0$ to $y = 1$

$x = \sqrt{1-y^2}$

$x^2 = 1 - y^2$ $\therefore$ $\odot$ with rad 1



strip is ||el X axis to change order, consider it ||el to Y axis.

$\therefore$ $y = 0$ to $y = \sqrt{1-x^2}$

& $x = 0$ to $x = 1$

$$\therefore \quad I = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\cos^{-1} x}{\sqrt{1-x^2}\,\sqrt{1-x^2-y^2}}\, dy\, dx$$

$$= \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}} \times \left[\sin^{-1}\left(\frac{y}{\sqrt{1-x^2}}\right) \right]_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}} \times \frac{\pi}{2} \times dx$$

put, $\cos^{-1}x = t$

$$\frac{-1}{\sqrt{1-x^2}}\,dx = dt$$

$x=0,\ t=\pi/2$; $x=1,\ t=0$

$$I = \frac{\pi}{2}\int_{\pi/2}^{0} -t\,.\,dt$$

$$= \frac{\pi}{2}\int_{0}^{\pi/2} t\,dt$$

$$= \frac{\pi}{4}\left(t^2\right)_0^{\pi/2}$$

$$= \frac{\pi}{4}\times\left(\frac{\pi^2}{4}-0\right)$$

$$\boxed{I = \frac{\pi^3}{16}}$$

HW

$$\int_0^3\int_{y^2/9}^{\sqrt{10-y^2}} dx\,dy$$

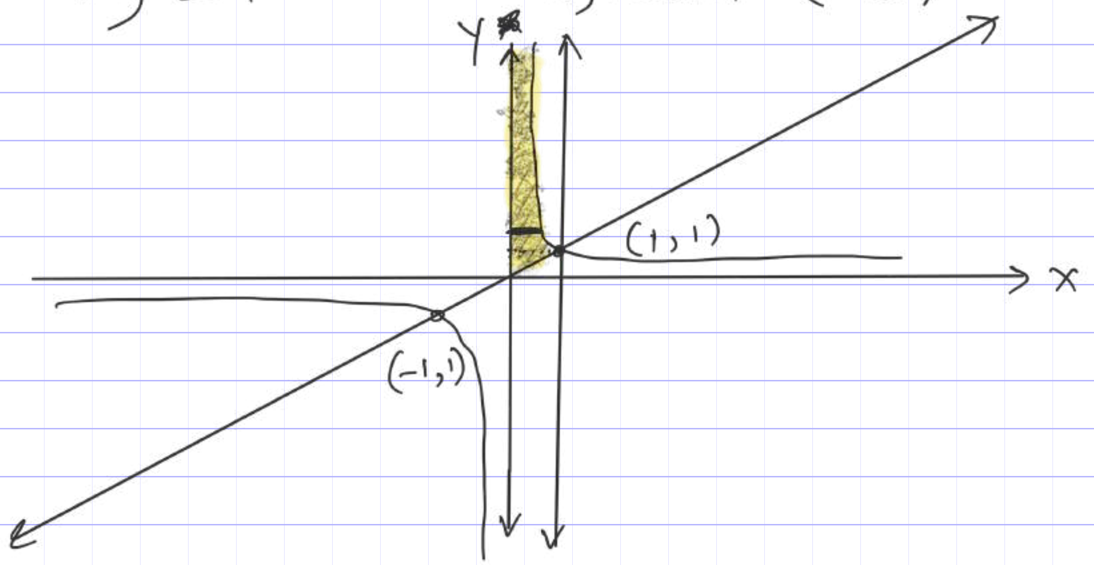
$$\int_0^2\int_{y}^{2+\sqrt{4-2y}} f(x,y)\,dx\,dy$$

$$\int_0^1\int_x^{1/x}\frac{y}{(1+y^2)(1+xy)^2}\,dx\,dy$$

$\rightarrow$ $y=x$ to $y=1/x$

$x=0$ to $x=1$

$xy = 1$ hyberbola (rect.)



strip ||el to Y, consider ||el to X axis
move ~~bt~~ bottom to top, jump

in R_1 ; $x=0$, $x = y$

$y=0$, $y=1$

in R_2 ; $x=0$, $x = 1/y$

$y=1$, $y=\infty$

$$\therefore I = \underbrace{\int_0^1\int_0^y \frac{y}{(1+y^2)(1+xy)^2}\,dx\,dy}_{1} + \underbrace{\int_1^\infty\int_0^{1/y} \frac{y}{(1+y^2)(1+xy)^2}\,dx\,dy}_{2}$$

(1) $$I_1 = \int_0^1 \frac{y}{1+y^2} \times \left[\frac{(1+xy)^{-1}}{-1\cdot y}\right]_0^y dy$$

$$= \int_0^1 \frac{-1}{1+y^2}\left[(1+y^2)^{-1} - (1)^{-1}\right] dy$$

$$I_1 = -\int_0^1 \frac{1}{(1+y^2)^2} + \int_0^1 \frac{1}{1+y^2}\,dy$$

put, $y = \tan\theta$

$\therefore dy = \sec^2\theta\,d\theta$

$y=0,\ \theta=0\ ;\ y=1,\ \theta=\frac{\pi}{4}$

$$I_1 = -\int_0^{\pi/4} \cos^2\theta\,d\theta + \left(\tan^{-1} y\right)_0^1$$

$$= -\int_0^{\pi/4} \cos^2\theta\,d\theta + \frac{\pi}{4}$$

$$= -\int_0^{\pi/4} \left(\frac{1+\cos 2\theta}{2}\right) d\theta + \frac{\pi}{4}$$

$$= -\frac{1}{2}\left\{\theta + \frac{\sin 2\theta}{2}\right\}_0^{\pi/4} + \frac{\pi}{4}$$

$$= \frac{-1}{2}\left[\frac{\pi}{4} - 0 + \frac{1}{2} - 0\right] + \frac{\pi}{4}$$

$$= \frac{-1}{2}\left[\frac{\pi}{4} + \frac{2}{4}\right] + \frac{\pi}{4}$$

$$I_1 = \frac{\pi}{8} - \frac{1}{4}$$

$$I_2 = \int_1^\infty \left[\int_0^{1/y} \frac{y}{(1+y^2)(1+xy)^2}\,dx\right] dy$$

$$= \int_1^\infty \frac{y}{1+y^2}\left[\frac{(1+xy)^{-1}}{(-1)y}\right]_0^{1/y} dy$$

$$= -\int_1^\infty \frac{1}{1+y^2}\left(\frac{1}{2}-1\right) dy$$

$$= \frac{1}{2}\int_1^\infty \frac{1}{1+y^2}\,dy$$

$$= \frac{1}{2}\left[\tan^{-1} y\right]_1^\infty$$

$$= \frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right)$$

$$I_2 = \frac{\pi}{8}$$

$$\therefore\ I = I_1 + I_2$$

$$= \frac{\pi}{8} - \frac{1}{4} + \frac{\pi}{8}$$

$$\boxed{I = \frac{\pi}{4} - \frac{1}{4}}$$

Type 4

☆ Evaluation of double integral over the region of cartesian co-ordinate :

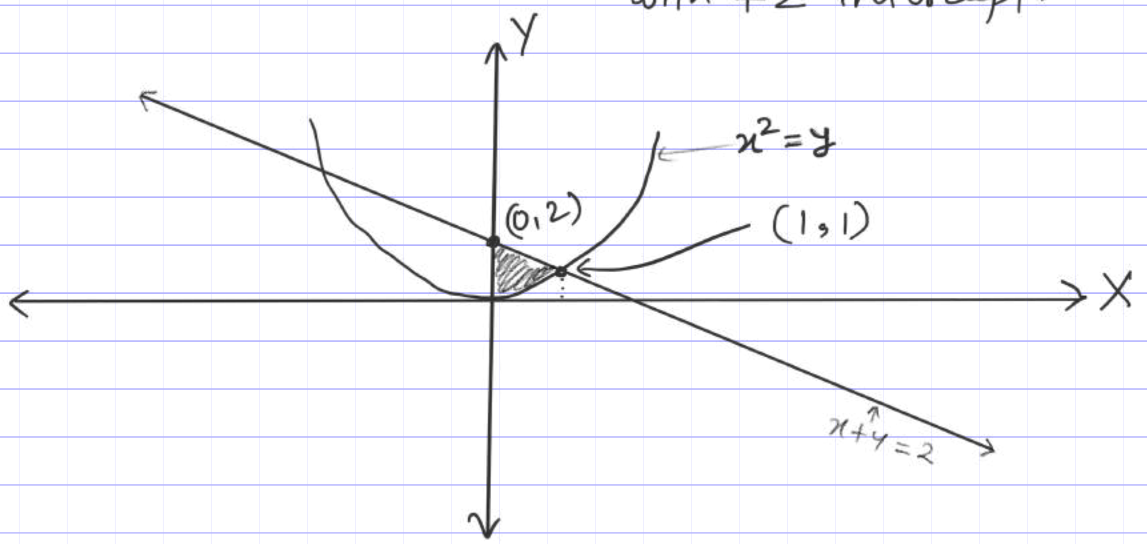
In this type, we have a choice that either we can take a strip //el to X axis or Y axis.

Q1: Evaluate $\iint y \, dx \, dy$ over Region R.

where R is the region of integration such that, it is the area bounded by $x=0$, $y=x^2$, $x+y=2$ in first quadrant.

→ here, Region of intⁿ is $x=0$, $y=x^2$ ... parabola, $(0,0)$ Vert., open ↑

& $y = -x + 2$ ... line with -ve slope i.e 2nd to 4th with +2 intercept.



here, consider strip //el to Y axis, move it from left to right. As we move the strip, it remains in same region.

$\therefore$ limits are $y = x^2$ to $y = 2-x$

$x = 0$ to $x = 1$

$$\therefore \quad I = \int_0^1 \left[\int_{x^2}^{2-x} y \, dy \right] dx$$

$$= \int_0^1 \left(\frac{y^2}{2} \right)_{x^2}^{2-x} dx$$

$$= \frac{1}{2} \int_0^1 \left((2-x)^2 - x^4 \right) dx$$

$$= \frac{1}{2} \left\{ \left[\frac{(2-x)^3}{3(-1)} - \frac{x^5}{5} \right]_0^1 \right\}$$

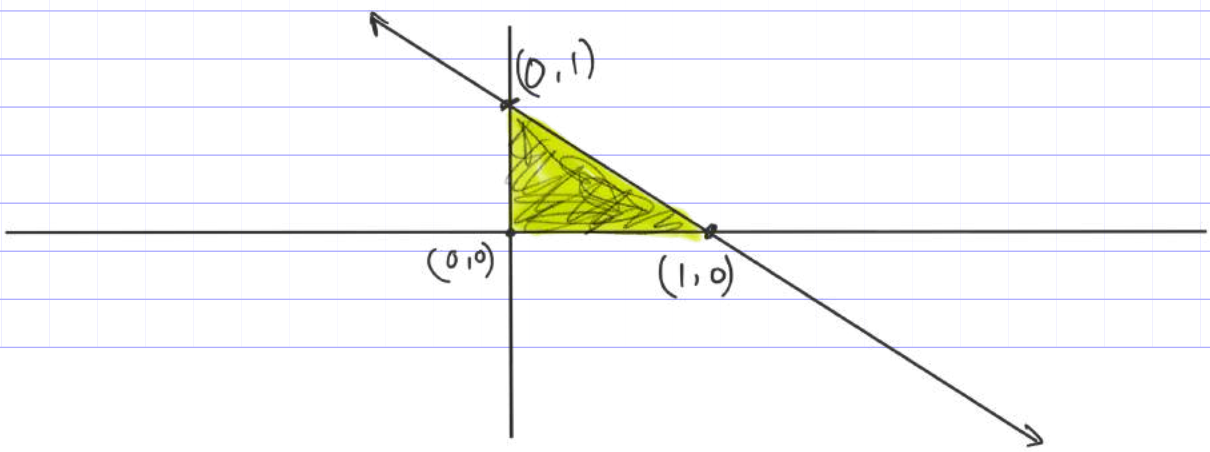
$$= \frac{1}{2} \left\{ \frac{-1}{3}(1-8) - \frac{1}{5}(1-0) \right\}$$

$$= \frac{1}{2} \left\{ \frac{7}{3} - \frac{1}{5} \right\}$$

$$\boxed{I = \frac{16}{15}}$$

Q2 : Evaluate $\iint \sqrt{xy(1-x-y)} \; dy \, dx$

$x \geqslant 0 \,, \quad y \geqslant 0 \,, \quad x + y < 1$



Consider, Strip ||el to X axis. move it from bottom to top. remains in same region

$x = 0$ to $x = 1-y$

$y = 0$ to $y = 1$

$$I = \int_0^1 \left[\int_0^{1-y} \sqrt{xy(1-x-y)}\, dx \right] dy$$

$$= \int_0^1 \sqrt{y} \left[\int_0^{1-y} \sqrt{x}\, \sqrt{(1-y)-x}\, dx \right] dy$$

take, $a = 1-y$

$$\int_0^1 \sqrt{y} \left[\int_0^a \sqrt{x}\, \sqrt{a-x}\, dx \right] dy$$

$x = at$

$\therefore\ dx = a \cdot dt$ limit change

$$\therefore\ I = \int_0^1 \sqrt{y} \left(\int_0^1 \sqrt{at}\, \sqrt{a-at}\; a\, dt \right) dy$$

$$= \int_0^1 \sqrt{y}\, a^2 \left[\int_0^1 t^{1/2} (1-t)^{1/2}\, dt \right] dy$$

$$= \int_0^1 \sqrt{y}\, (1-y)^2 \times \beta\left(3/2, 3/2\right) dy$$

$$= \beta\left(\tfrac{3}{2}, \tfrac{3}{2}\right) \times \beta\left(\tfrac{3}{2}, 3\right)$$

$$= \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)} \times \frac{\Gamma\left(\frac{3}{2}\right)\Gamma(3)}{\Gamma\left(\frac{9}{2}\right)}$$

$$= \frac{\frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right)}{2 \times 1} \times \frac{\frac{1}{\cancel{2}} \times \cancel{\Gamma\left(\frac{1}{2}\right)} \times \cancel{2}}{\frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \cancel{\Gamma\left(\frac{1}{2}\right)}}$$

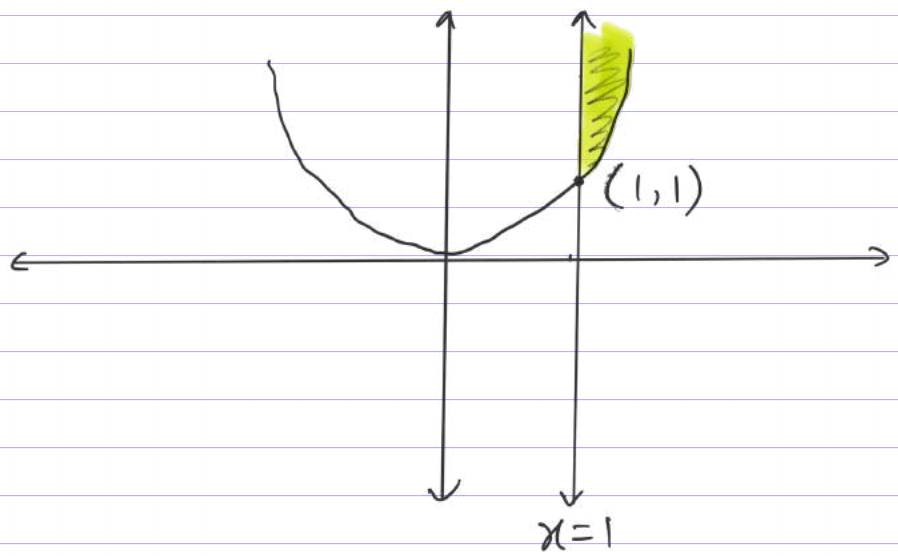
$$= \frac{1}{8}\sqrt{\pi}\sqrt{\pi} \times \frac{\cancel{16}\,2}{7 \times 5 \times 3}$$

$$= \frac{2\pi}{105}$$

Q3 : Evaluate $\displaystyle\iint \frac{1}{x^4 + y^2}\, dx\, dy$

$x \geq 1 \;,\; y \geq x^2$

→



Consider, strip ||el to Y axis. Move it from left to right.

$y = x^2$ to $y = \infty$ & $x = 1$ to $x = \infty$

$$I = \int_1^\infty \left[\int_{x^2}^{\infty} \frac{1}{y^2 + (x^2)^2}\, dy\right] dx$$

$$= \int_1^\infty \frac{1}{x^2}\left[\tan^{-1}\left(y/x^2\right)\right]_{x^2}^{\infty} dx$$

$$= \int_1^\infty \frac{1}{x^2}\left(\pi/2 - \pi/4\right) dx$$

$$= \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\left[\frac{-1}{x}\right]_1^\infty$$

$$= \frac{-\pi}{4}\,(0 - 1)$$

$$\boxed{I = \frac{\pi}{4}}$$