Gama function:

It is denoted by Γn and defined by,

1) $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \quad \ldots [n>0]$

For $n>0$, this integral is conversion [i.e. value exist]

2) $\Gamma n = \int_0^\infty e^{-t} t^{n-1} dt$ where "t" is a dummy variable

3) $\Gamma n = (n-1)!$ if n is positive integer

ex. $\Gamma 5 = 4! = 24$

$\Gamma 3 = 2! = 2$

4) $\Gamma \frac{1}{2} = \sqrt{\pi}$

5) $\Gamma \frac{1}{4} \, \Gamma \frac{3}{4} = \pi \sqrt{2}$

6) $\Gamma n+1 = n \Gamma n$

7) $\Gamma \frac{3}{2} = \frac{1}{2} \Gamma \frac{1}{2} = \frac{\sqrt{\pi}}{2}$

8) $\Gamma \frac{5}{2} = \frac{3}{2} \cdot \frac{1}{2} \Gamma \frac{1}{2}$

9) $\Gamma \frac{9}{2} = \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma \frac{1}{2}$

10) $\Gamma 0$ = not defined $\left[\because \Gamma n = \frac{\Gamma n+1}{n} \right] \left[\therefore \Gamma 0 = \frac{\Gamma 1}{0} = \infty \text{ (not defined)} \right]$

11) $\Gamma -1$ = not defined $\left[\because \Gamma n = \frac{\Gamma n+1}{n} \quad \because \Gamma -1 = \frac{\Gamma 0}{-1} = \text{not defined} \right]$

12) $\Gamma -\frac{1}{2} = \frac{\Gamma \frac{1}{2}}{-\frac{1}{2}} = -2\sqrt{\pi} \quad \left[\because \Gamma n = \frac{\Gamma n+1}{n} \right]$

13) $\Gamma -\frac{3}{2} = \frac{\Gamma -\frac{1}{2}}{-\frac{3}{2}} = \frac{-2\sqrt{\pi}}{-\frac{3}{2}} = \frac{4}{3}\sqrt{\pi}$

(15) $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma n}{k^n}$

14) Γn is defined for all n except $(n=0)$, $(n=-1,-2,\ldots)$

i.e. Γn is defined for +ve integer and -ve as well as +ve fraction

Q.1] Prove that $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma n}{k^n}$

We know that $\int_0^\infty e^{-t} t^{n-1} dt = \Gamma n$

Put, $t = kx$ $\quad$ Now, when $t=0$, $x=0$

$dt = k\,dx$ $\quad$ $t\to\infty$, $x\to\infty$

$$\therefore \Gamma n = \int_0^\infty e^{-kx} (kx)^{n-1} k\,dx$$

$$\Gamma n = (k^{n-1})(k) \int_0^\infty e^{-kx} (x)^{n-1} dx$$

$$\Gamma n = k^n \int_0^\infty e^{-kx} x^{n-1} dx$$

$$\therefore \frac{\Gamma n}{k^n} = \int_0^\infty e^{-kx} x^{n-1} dx$$

Q.2] Prove that $\Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1} dy$

We know that, $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

Put, $x = y^2$ $\quad$ When, $x=0$, $y=0$

$dx = 2y\,dy$ $\quad$ $x\to\infty$, $y\to\infty$

$$\therefore \Gamma n = \int_0^\infty e^{-y^2} (y^2)^{n-1} 2y\,dy$$

$$\Gamma n = 2\int_0^\infty e^{-y^2} (y^{2n-2}) y\,dy$$

$$\Gamma n = 2\int_0^\infty e^{-y^2} \cdot y^{2n-1} dy$$

Q.3] Evaluate. $\int_0^{\infty} e^{-x^2} dx$

let, $J = \int_0^{\infty} e^{-x^2} dx$

Put. $x^2 = t$ when $x=0$, $t=0$

$x = \sqrt{t}$ $x \to \infty$, $t \to \infty$

$\therefore dx = \frac{1}{2\sqrt{t}} dt$

$\therefore J = \int_0^{\infty} e^{-t} \frac{1}{2\sqrt{t}} dt = \frac{1}{2} \int_0^{\infty} e^{-t} . t^{-1/2} dt$

$= \frac{1}{2} \sqrt{1/2} = \frac{\sqrt{\pi}}{2}$ //

Q4] Evaluate. $\int_0^{\infty} \sqrt{x} . e^{-x^2} dx$

Put $x^2 = t$ when $x=0$, $t=0$

$x = \sqrt{t}$ $x \to \infty$, $t \to \infty$

$dx = \frac{1}{2\sqrt{t}} dt$

$\therefore J = \int_0^{\infty} t^{1/4} . e^{-t} \frac{1}{2\sqrt{t}} dt$

$= \frac{1}{2} \int_0^{\infty} e^{-t} . t^{-1/4} dt$

$= \frac{1}{2} \sqrt{3/4}$

Q.5] Evaluate. $\int_0^{\infty} \sqrt{x} e^{-x^3} dx$

Put $x^3 = t$ when $x=0$, $t=0$

$x = t^{1/3}$ $x \to \infty$, $t \to \infty$

$dx = \frac{1}{3} t^{-2/3} dt$

$$I = \int_0^\infty t^{1/6} \cdot e^{-t} \frac{1}{3} t^{-2/3} dt$$

$$= \frac{1}{3} \int_0^\infty e^{-t} \, t^{-1/2} \, dt$$

$$= \frac{1}{3} \sqrt{1/2} = \frac{\sqrt{\pi}}{3}$$

Q 6] Evaluate $\int_0^\infty x^7 \cdot e^{-2x^2} dx$

$$I = \int_0^\infty x^7 \cdot e^{-2x^2} dx$$

Put $2x^2 = t$ when $x=0, t=0$

$\therefore x = \sqrt{\frac{t}{2}} = \frac{\sqrt{t}}{\sqrt{2}}$ $\quad x \to \infty, t \to \infty$

$$dx = \frac{1}{2\sqrt{2t}} dt$$

$$\therefore I = \int_0^\infty \left(\frac{t}{2}\right)^{7/2} \cdot e^{-t} \frac{1}{2\sqrt{2}\sqrt{t}} dt$$

$$= \frac{1}{2^{7/2} \cdot 2^{3/2}} \int_0^\infty e^{-t} \, t^3 \, dt$$

$$= \frac{1}{2^5} \sqrt{4} = \frac{\cancel{24}\, 3!}{2^5} = \frac{6}{32} = \frac{3}{16}$$

Q.7] Evaluate. $\int_0^\infty x^n \cdot e^{-ax^n} dx$

$$I = \int_0^\infty x^n \cdot e^{-ax^n} dx$$

Put $ax^n = t$ when $x=0, t=0$

$x^n = \frac{t}{a}$ $\quad x \to \infty, t \to \infty$

$$x = \frac{t^{1/n}}{a^{1/n}}$$

$$dx = \frac{1}{na^{1/n}} t^{\frac{1-n}{n}} dt$$

$$J = \int_0^\infty \frac{t}{a} e^{-t} \cdot \frac{1}{n a^{1/n}} t^{1-n/n} \, dt$$

$$= \frac{1}{n a^{1/n+1}} \int_0^\infty e^{-t} \cdot t^{1/n} \, dt$$

$$= \frac{1}{n a^{\frac{1+n}{n}}} \Gamma(1/n + 1).$$

Q.8] Evaluate, $\int_0^\infty \sqrt{x}\, e^{-x^2} dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$

Let, $J = \int_0^\infty \sqrt{x}\, e^{-x^2} dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$

Put $x^2 = t$ when $x = 0$, $t = 0$

$x = \sqrt{t}$ $x \to \infty$, $t \to \infty$

$dx = \frac{1}{2\sqrt{t}}$

$$\therefore J = \int_0^\infty t^{1/4} e^{-t} \frac{1}{2\sqrt{t}} dt \times \int_0^\infty \frac{e^{-t}}{t^{1/4}} \frac{1}{2\sqrt{t}} dt$$

$$= \frac{1}{4} \int_0^\infty e^{-t} \cdot t^{-1/4} dt \times \int_0^\infty e^{-t} \cdot t^{-3/4} dt$$

$$= \frac{1}{4} \Gamma(3/4)\, \Gamma(1/4) = \frac{\pi\sqrt{2}}{4}$$

Q.9] Evaluate $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}} dx$

Put $x^{1/3} = t$ when $x = 0$, $t = 0$

$x = t^3$ $x \to \infty$, $t \to \infty$

$dx = 3t^2 dt$

$$\therefore J = \int_0^\infty t^{3/2} e^{-t} \, 3t^2 dt$$

$$= 3 \int_0^\infty e^{-t} \cdot t^{7/2} dt = 3\,\Gamma(9/2) = 3 \times \frac{\cancel{9}}{\cancel{2}} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \Gamma(1/2)$$

$$= \frac{315\sqrt{\pi}}{16}$$

Type : 2

1. If problem contains $\log x$, then put $\log x = -t$
2. If problem contains $\log(1/x)$, then put $\log(1/x) = t$

3. $e^{-\infty} = 0$ $\qquad \log(1) = 0$ $\qquad$ (5) $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$
4. $e^{\infty} = \infty$
7. $i = \cos\pi/2 + i\sin\pi/2$ $\qquad$ (6) $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

Q.1] Evaluate $\int_0^1 (x \boxed{\log x})^3 \, dx$

Put $\log x = -t$ $\qquad$ When, $x = 0$, $t \to \infty$

$\therefore x = e^{-t}$ $\qquad x = 1$, $t \to 0$

$dx = -e^{-t} dt$

$$\therefore I = \int_{\infty}^{0} e^{-3t} \cdot (-t)^3 \; -e^{-t} dt$$

$$= \int_0^{\infty} e^{-4t} (-t)^3 \, dt$$

$$= -\int_0^{\infty} e^{-4t} t^3 \, dt = \frac{-\Gamma 4}{4^4} = \frac{-3\cancel{6}}{256} = \frac{-3}{128}$$

Q.2] Evaluate $\int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{P-1} dx$

Put $\log\frac{1}{x} = t$ $\qquad$ when, $x = 0$, $t \to \infty$

$\frac{1}{x} = e^t$ $\qquad x = 1$, $t = 0$

$\therefore x = e^{-t}$

$dx = -e^{-t} dt$

$$\therefore I = \int_{\infty}^{0} (t)^{P-1} (-e^{-t}) \, dt = \int_0^{\infty} e^{-t} (t)^{P-1} \, dt$$

$$= \Gamma P$$

Q.3] Evaluate $\int_0^1 x^m (\log x)^n \, dx$

Put $\log x = -t$ When $x = 0$, $t \to \infty$

$x = e^{-t}$ $x = 1$, $t = 0$

$dx = -e^{-t} dt$

$$\therefore I = \int_\infty^0 e^{-mt} (-t)^n \cancel{dx} (-e^{-t}) dt$$

$$= (-1) \int_0^\infty e^{(-m-1)t} (t)^n dt = \frac{-\cancel{\Gamma n}}{-\cancel{(m+1)}^{(m+1)}} = \cancel{\Gamma n}$$

$$= (-1)^n \int_0^\infty e^{-(m+1)t} (t)^n dt$$

$$= \frac{(-1)^n \, \Gamma(n+1)}{(m+1)^{(m+1)}}$$

Q4] Evaluate $\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx$

Put $\log\left(\frac{1}{x}\right) = t$ when $x = 0$, $t \to \infty$

$x = 1$, $t = 0$

$\frac{1}{x} = e^t$

$x = e^{-t}$

$dx = -e^{-t} dt$

$$\therefore I = \int_\infty^0 e^{-mt} (t)^n (-e^{-t}) dt = \int_0^\infty e^{-(m+1)t} (t)^n dt$$

$$= \frac{\Gamma(n+1)}{(m+1)^{(n+1)}}$$

H.W. Q.5] Evaluate $\int_0^1 [x \log x]^4 dx$

H.W. Q.6] Evaluate $\int_0^1 \frac{1}{\sqrt{x \log x}} dx$

Q.7] Evaluate $\int_0^1 \sqrt{x \log(1/x)}\, dx$

Put $\log(1/x) = t$ when $x = 0,\ t \to \infty$

$\frac{1}{x} = e^t$ $x = 1,\ t \to 0$

$\therefore x = e^{-t}$

$dx = -e^{-t}\, dt$

$\therefore I = \int_\infty^0 e^{-t/2} \cdot t^{1/2} (-e^{-t}) dt$

$= \int_0^\infty e^{-3/2 t}\, t^{1/2}\, dt = \frac{\Gamma 3/2}{(3/2)^{3/2}} = \frac{1/2\, \Gamma 1/2}{(3/2)^{3/2}}$

$= \frac{2^{3/2}\ 1/2\, \sqrt{\pi}}{3^{3/2}} = \frac{\sqrt{2}\sqrt{\pi}}{3\sqrt{3}} = \frac{\sqrt{2\pi}}{3\sqrt{3}}$

→ Q.5] $I = \int_0^1 [x \log x]^4 dx$

Put $\log x = -t$ when $x = 0,\ t \to \infty$

$x = e^{-t}$ $x = 1,\ t = 0$

$dx = -e^{-t}\, dt$

$I = \int_\infty^0 e^{-4t} (-t)^4 \cdot -e^{-t} dt = \int_0^\infty e^{-5t}\, t^4\, dt$

$I = \frac{\Gamma 5}{5^5} =$

→ Q.6] $I = \int_0^1 \frac{1}{(x \log x)^{1/2}} dx = \int_0^1 (x \log x)^{-1/2} dx$

Put $\log x = t$

$x = e^t \quad \therefore dx = e^t\, dt$ when $x = 0,\ t \to$

$x = 1,\ t = 0$

Type : 3

Q.1] Evaluate $\int_0^\infty \frac{x^3}{3^x} dx$

Put $3^x = e^t$ when $x=0$, $t=0$

$\log 3^x = \log e^t$ $x \to \infty$, $t \to \infty$

$x \log 3 = t$

$\therefore x = \frac{t}{\log 3}$

$dx = \frac{dt}{\log 3}$

$\therefore I = \int_0^\infty \frac{t^3}{(\log 3)^3} \cdot e^{-t} \frac{dt}{\log 3} = \frac{1}{(\log 3)^4} \int_0^\infty e^{-t} t^3 dt$

$= \frac{1}{(\log 3)^4} \cdot \overline{|4} = \frac{6}{(\log 3)^4}$ //.

Q.2] Evaluate $\int_0^\infty 7^{-4x^2} dx$

Put $7^{-4x^2} = e^{-t}$ when $x=0$; $t=0$

$\log 7^{-4x^2} = \log e^{-t}$ $x \to \infty$, $t \Rightarrow \infty$

$-4x^2 \log 7 = -t$

$x^2 = \frac{t}{4 \log 7}$

$x = \frac{1}{2}\sqrt{\frac{t}{\log 7}}$ $\therefore dx = \frac{1}{4\sqrt{t}} \times \frac{1}{\sqrt{\log 7}} dt$

$\therefore I = \int_0^\infty e^{-t} \cdot \frac{1}{4} \cdot t^{-1/2} \frac{1}{\sqrt{\log 7}} dt$

$= \frac{1}{4\sqrt{\log 7}} \int_0^\infty e^{-t} t^{-1/2} dt = \frac{1}{4\sqrt{\log 7}} \overline{|1/2} = \frac{\sqrt{\pi}}{4\sqrt{\log 7}}$ //.

Q3] Evaluate $\int_0^\infty \frac{x^{(m-1)}}{(m-1)^x} dx$

when $x=0$, $t=0$
$x\to\infty$, $t\to\infty$

Put $(m-1)^x = e^t$

$x\log(m-1) = t$

$x = \frac{t}{\log(m-1)}$ $\therefore dx = \frac{dt}{\log(m-1)}$

$\therefore I = \int_0^\infty \frac{t^{(m-1)}}{(\log(m-1))^{(m-1)}} \cdot e^{-t} \frac{dt}{(\log(m-1))}$

$= \int_0^\infty \frac{e^{-t}\, t^{(m-1)}}{[\log(m-1)]^m} dt = \frac{\Gamma m}{[\log(m-1)]^m}$

HW Q4] Evaluate $\int_0^\infty \frac{x^a}{a^x} dx$

Put $a^x = e^t$

When $x=0$, $t=0$
$x\to\infty$, $t\to\infty$

$x\log a = t$

$x = \frac{t}{\log a}$ $\therefore dx = \frac{dt}{\log a}$

$\therefore I = \int_0^\infty \frac{t^a \cdot e^{-t}}{(\log a)^a} \cdot \frac{dt}{\log a}$

$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t}\, t^a\, dt$

$= \frac{1}{(\log a)^{a+1}} \cdot \Gamma{a+1}$

Q5] Evaluate ① $\int_0^\infty x^{m-1} \cos ax\, dx$ ② $\int_0^\infty x^{m-1} \sin ax\, dx$

① $I_1 = \int_0^\infty x^{m-1} \left(\frac{e^{iax} + e^{-iax}}{2}\right) dx$

$= \frac{1}{2}\left\{\int_0^\infty x^{m-1} \cdot e^{iax} \cdot dx + \int_0^\infty x^{m-1} \cdot e^{-iax} \cdot dx\right.$

$$I_1 = \frac{1}{2}\left\{\frac{\Gamma m}{(-ia)^m} + \frac{\Gamma m}{(ia)^m}\right\} = \frac{\Gamma m}{2a^m}\left\{\left(\frac{1}{-i}\right)^m + \left(\frac{1}{i}\right)^m\right\}$$

$$= \frac{\Gamma m}{2a^m}\left\{(i)^m + (-i)^m\right\}$$

$$= \frac{\Gamma m}{2a^m}\left\{(\cos \pi/2 + i\sin \pi/2)^m + (\cos \pi/2 - i\sin \pi/2)^m\right\}$$

$$= \frac{\Gamma m}{2a^m}\left\{\cos\frac{m\pi}{2} + i\sin\frac{m\pi}{2} + \cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right\}$$

$$= \frac{\Gamma m}{2a^m} \cdot 2\cos\frac{m\pi}{2}$$

$$= \frac{\Gamma m}{a^m}\cos\frac{m\pi}{2}$$

(2) $$I_2 = \int_0^\infty x^{m-1}\left[\frac{e^{iax} - e^{-iax}}{2i}\right]dx$$

$$= \frac{1}{i}\times\frac{1}{2}\left\{\int_0^\infty x^{m-1} e^{iax}\,dx - \int_0^\infty x^{m-1} e^{-iax}\,dx\right\}$$

$$= \frac{1}{2i}\left\{\frac{\Gamma m}{(-ia)^m} - \frac{\Gamma m}{(ia)^m}\right\} = \frac{\Gamma m}{2ia^m}\left\{\left(\frac{1}{-i}\right)^m - \left(\frac{1}{i}\right)^m\right\}$$

$$= \frac{\Gamma m}{2ia^m}\left\{(i)^m - (-i)^m\right\}$$

$$= \frac{\Gamma m}{2ia^m}\left\{\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)^m - \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)^m\right\}$$

$$= \frac{\Gamma m}{2ia^m} \cdot 2i\sin\frac{m\pi}{2} = \frac{\Gamma m}{a^m}\sin\left(\frac{m\pi}{2}\right).$$

Evaluate $\int_0^\infty (\cos ax^{1/n})\,dx$

Put $ax^{1/n} = t$ when $x = 0,\ t = 0$

$x^{1/n} = t/a$ $\quad x \to \infty,\ t \to \infty$

$$x = \frac{t^n}{a^n}$$

$$dx = \frac{n t^{n-1}}{a^n}\,dt$$

$$I = \int_0^\infty \cos(t) \cdot \frac{n}{a^n} t^{n-1} \, dt$$

$$= \frac{n}{a^n} \int_0^\infty t^{n-1} \cos(t) \, dt = \frac{n}{a^n} \int_0^\infty \left(\frac{e^{it} + e^{-it}}{2} \right) t^{n-1} \, dt$$

$$= \frac{n}{2a^n} \left[\int_0^\infty e^{-(-it)} \cdot t^{n-1} \, dt + \int_0^\infty e^{-it} \cdot t^{n-1} \, dt \right]$$

$$= \frac{n}{2a^n} \left[\frac{\overline{|n}}{(-i)^n} + \frac{\overline{|n}}{(i)^n} \right] = \frac{n\overline{|n}}{2a^n} \left[\left(\frac{1}{-i} \right)^n + \left(\frac{1}{i} \right)^n \right]$$

$$= \frac{n\overline{|n}}{2a^n} \left\{ (i)^n + (-i)^n \right\}$$

$$= \frac{n\overline{|n}}{2a^n} \left\{ \cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2} + \cos \frac{n\pi}{2} - i \sin \frac{n\pi}{2} \right\}$$

$$= \frac{n\overline{|n}}{2a^n} \left\{ 2 \cos \frac{n\pi}{2} \right\} = \frac{n\overline{|n}}{a^n} \cos \frac{n\pi}{2}$$

H.W. Q.7] Evaluate $\int_0^\infty \sin(ax^{1/n}) \, dx$

Put $ax^{1/n} = t$ when $x = 0, t = 0$

$x = \frac{t^n}{a^n}$ $x \to \infty, t \to \infty$

$\therefore dx = \frac{n t^{n-1}}{a^n} dt$

$$\therefore I = \int_0^\infty \sin(t) \cdot \frac{n}{a^n} t^{n-1} dt = \frac{n}{a^n} \int_0^\infty t^{n-1} \left(\frac{e^{it} - e^{-it}}{2i} \right) dt$$

$$= \frac{n}{a^n 2i} \left\{ \int_0^\infty e^{-(-it)} \cdot t^{n-1} \, dt - \int_0^\infty e^{-it} \cdot t^{n-1} \, dt \right\}$$

$$= \frac{n}{2i a^n} \left\{ \frac{\overline{|n}}{(-i)^n} - \frac{\overline{|n}}{(i)^n} \right\} = \frac{n\overline{|n}}{2i a^n} \left\{ \left(\frac{1}{-i} \right)^n - \left(\frac{1}{i} \right)^n \right\}$$

$$= \frac{n\overline{|n}}{2i a^n} \left\{ (i)^n - (-i)^n \right\} = \frac{n\overline{|n}}{2i a^n} \left\{ \cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2} - \cos \frac{n\pi}{2} + i \sin \frac{n\pi}{2} \right\}$$

$$= \frac{n\overline{|n}}{2i a^n} \left\{ 2i \sin \frac{n\pi}{2} \right\}$$

$$= \frac{n\overline{|n}}{a^n} \cdot \sin \frac{n\pi}{2}$$

Q.8] Evaluate $\int_0^\infty x e^{-ax} \sin bx \, dx$

let,

$$I = \int_0^\infty x e^{-ax} \left[\frac{e^{ibx} - e^{-ibx}}{2i} \right] dx$$

$$= \frac{1}{2i} \left\{ \int_0^\infty e^{ibx - ax} . x\,dx - \int_0^\infty e^{-(ibx+ax)} . x\,dx \right.$$

$$= \frac{1}{2i} \left\{ \frac{\Gamma 2}{(a-ib)^2} - \frac{\Gamma 2}{(a+ib)^2} \right\} = \frac{\Gamma 2}{2i} \left\{ \frac{(a+ib)^2 - (a-ib)^2}{[(a-ib)(a+ib)]^2} \right\}$$

$$= \frac{1}{2i} \left\{ \frac{a^2 + 2abi - b^2 - a^2 + 2abi + b^2}{[a^2+b^2]^2} \right\}$$

$$= \frac{1}{2i} \left\{ \frac{4iab}{(a^2+b^2)^2} \right\} = \frac{2ab}{(a^2+b^2)^2}$$

H.W. Q.9] Evaluate. $\int_0^\infty x e^{-ax} \cos bx \, dx$.

$$\text{let } I = \int_0^\infty x e^{-ax} \left[\frac{e^{ibx} + e^{-ibx}}{2} \right] dx$$

$$= \frac{1}{2} \left\{ \int_0^\infty e^{-(a-ib)x} . x\,dx + \int_0^\infty e^{-(a+ib)x} . x\,dx \right.$$

$$= \frac{1}{2} \left\{ \frac{\Gamma 2}{(a-ib)^2} + \frac{\Gamma 2}{(a+ib)^2} \right\} = \frac{\Gamma 2}{2} \left\{ \frac{(a+ib)^2 + (a-ib)^2}{[(a-ib)(a+ib)]^2} \right\}$$

$$= \frac{a^2 - b^2 + 2abi + a^2 - 2abi - b^2}{(a^2+b^2)^2}$$

$$= \frac{2(a^2-b^2)}{(a^2+b^2)^2}$$

Q.10] Prove that $\frac{2^n \Gamma n+1/2}{\sqrt{\pi}} = 1.3.5.7\ldots\ldots(2n-1)$

We know that, $\Gamma n+1 = n \Gamma n$

consider $\Gamma n+1/2$,

$$\Gamma\left(n+\tfrac{1}{2}\right) = \left(n-\tfrac{1}{2}\right)\Gamma\left(n-\tfrac{1}{2}\right)$$
$$= \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\Gamma\left(n-\tfrac{3}{2}\right)$$
$$= \left(n-\tfrac{1}{2}\right)\left(n-\tfrac{3}{2}\right)\left(n-\tfrac{5}{2}\right)\Gamma\left(n-\tfrac{5}{2}\right)$$

$$= \left[\left(n-\frac{1}{2}\right)\left(\left(n-\frac{1}{2}\right)-1\right)\left(\left(n-\frac{1}{2}\right)-2\right)\right]\Gamma\left(\left(n-\frac{1}{2}\right)-2\right)$$

$$= \left(n-\frac{1}{2}\right)\left(\left(n-\frac{1}{2}\right)-1\right)\left(\left(n-\frac{1}{2}\right)-2\right)\cdots\left(\left(n-\frac{1}{2}\right)-(n-1)\right)\Gamma\left(\left(n-\frac{1}{2}\right)-(n-1)\right)$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\cdots\frac{3}{2}\cdot\frac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right)$$

$$= \frac{(2n-1)}{2}\,\frac{(2n-3)}{2}\,\frac{(2n-5)}{2}\cdots\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$$

Now,

$$\text{L.H.S.} = \frac{2^n\,\Gamma\left(n+\tfrac{1}{2}\right)}{\sqrt{\pi}}$$
$$= \frac{2^n}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2^n}\cdot(2n-1)\cdot(2n-3)\cdots 3\cdot 1$$
$$= 1\cdot 3\cdot 5\cdot 7\cdots(2n-1)$$
$$= \text{R.H.S.}$$

Q 11] If $I_n = \dfrac{\frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}$ then show that $I_{n+2} = \dfrac{n+1}{n+2}\,I_n$ and hence find I_5.

→ Given that I_n, put $n = n+2$.

$$\therefore I_{n+2} = \frac{\frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{n+3}{2}\right)}{\Gamma\left(\frac{n+2}{2}+1\right)} = \frac{\frac{\sqrt{\pi}}{2}\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\left(\frac{n+2}{2}\right)\Gamma\left(\frac{n+2}{2}\right)}$$

$$= \frac{(n+1)}{(n+2)}\left[\frac{\frac{\sqrt{\pi}}{2}\,\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}\right] = \frac{(n+1)}{(n+2)}\,I_n$$

Put $n=3$

$$\therefore I_5 = \frac{4}{5}I_3 = \frac{4}{5}\left[\frac{\frac{\sqrt{\pi}}{2}\,\Gamma(2)}{\Gamma\left(\frac{5}{2}\right)}\right] = \frac{2\sqrt{\pi}}{3}\cdot\frac{2}{3}\cdot\frac{2}{\sqrt{\pi}}$$
$$= \frac{8}{15}$$

Type: 4 Beta Function

$$\Gamma n = \int_0^{\infty} e^{-x} x^{n-1} dx$$

Γn is defined all n except, $n = 0, -1, -2, \ldots$

Beta Function:

(1) $$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad , [m,n>0]$$

For $m,n>0$ the integral is convergent.

(2) $$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

(3) $$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

(4) $$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx$$

(5) $\beta(m,n) = \beta(n,m)$ (6) $\beta(m,n) = \dfrac{\Gamma m \cdot \Gamma n}{\Gamma m+n}$

(7) $1+\cos\theta = 2\cos^2\theta/2$

$1-\cos\theta = 2\sin^2\theta/2$

$1+\tan^2\theta = \sec^2\theta$

$1+\cot^2\theta = \mathrm{cosec}^2\theta$

(8) $\sin(\pi-\theta) = \sin\theta$

$\cos(\pi-\theta) = -\cos\theta$

(12) $$\int_a^b (uv)\,dx = \left[u\int v\,dx\right]_a^b - \int_a^b \left(\frac{du}{dx}\right)\left(\int v\,dx\right)_a^b dx$$

(9) $$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

(10) $$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

(11) $$2^{2m-1}\,\Gamma m\,\Gamma m+1/2 = \sqrt{\pi}\,\Gamma 2m$$

Q.1] Prove that $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Put, when $x=0$, $\theta=0$

$x=1$, $\theta=\pi/2$

$x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$$\therefore \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(\cos^2\theta)^{n-1}\, 2.\sin\theta.\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta.\cos^{2n-2}\theta.\sin\theta.\cos\theta.\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta.\cos^{2n-1}\theta.\,d\theta.$$

Q.2] Prove that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

We know that,

$$\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

Put $2m-1=p$ & $2n-1=q$

$m = (p+1)/2$ & $n=(q+1)/2$

$$\therefore \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta$$

$$\therefore \int_0^{\pi/2} \sin^p\theta.\cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Q.3] Prove that $\Gamma 1/2 = \sqrt{\pi}$

We know that,

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta.\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Put $p=q=0$,

$$\therefore \int_0^{\pi/2} d\theta = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{1}{2}\right) \Rightarrow [\theta]_0^{\pi/2} = \frac{1}{2}.\frac{\Gamma 1/2\,\Gamma 1/2}{\Gamma 1}$$

$\therefore \frac{\pi}{2} = \frac{1}{2}(\Gamma 1/2)^2 \quad \therefore (\Gamma 1/2)^2 = \pi \quad \therefore \Gamma 1/2 = \sqrt{\pi}$

Q.4] State and prove duplication formula. & hence prove that ~~g~~ $\Gamma 1/4 \, \Gamma 3/4 = \pi\sqrt{2}$

Duplication Formula is $2^{2m-1} \Gamma m \, \Gamma m+1/2 = \sqrt{\pi} \, \Gamma 2m$

We know that,

$$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Put $p = q$.

$$\therefore \int_0^{\pi/2} \sin^p\theta \cdot \cos^p\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right)$$

$$\therefore \frac{1}{2^p}\int_0^{\pi/2} 2^p \sin^p\theta \cdot \cos^p\theta \, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right)$$

$$\therefore \frac{1}{2^p}\int_0^{\pi/2} \sin^p 2\theta \, d\theta = \text{—}''\text{—}$$

Put $2\theta = t$ when $\theta = 0, t = 0$

$d\theta = \frac{dt}{2}$ $\theta = \pi/2, t = \pi$

$$\therefore \frac{1}{2^p}\int_0^{\pi} \sin^p t \, \frac{dt}{2} = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right)$$

$$\therefore \frac{1}{2^p}\left\{\int_0^{\pi/2} \sin^p t \, dt + \int_0^{\pi/2} \sin^p(\pi - t) \, dt\right\} = \beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right)$$

$$\therefore \frac{1}{2^p}\left\{\int_0^{\pi/2} \sin^p t \, dt + \int_0^{\pi/2} \sin^p t \, dt\right\}$$

$$\therefore \frac{2}{2^p}\int_0^{\pi/2} \sin^p t \cdot \cos^0 t \, dt = \beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right)$$

$$\therefore \frac{2}{2^p} \times \frac{1}{2} \cdot \beta\left(\frac{p+1}{2}, \frac{0+1}{2}\right) = \beta\left(\frac{p+1}{2}, \frac{p+1}{2}\right)$$

$$\therefore \frac{1}{2^p} \cdot \frac{\Gamma\frac{p+1}{2} \, \Gamma 1/2}{\Gamma\frac{p+2}{2}} = \frac{\Gamma\frac{p+1}{2} \cdot \Gamma\frac{p+1}{2}}{\Gamma\frac{2p+2}{2}}$$

$$\therefore \frac{1}{2^p} \frac{\sqrt{\pi}}{\Gamma\frac{p+2}{2}} = \frac{\Gamma p+1/2}{\Gamma p+1}$$

Put $p = 2m-1$

$$\therefore \frac{\sqrt{\pi}}{2^{2m-1}\,\Gamma\left(\frac{2m+1}{2}\right)} = \frac{\Gamma m}{\Gamma 2m}$$

$$\therefore \sqrt{\pi}\,\Gamma 2m = \Gamma m \cdot 2^{2m-1}\,\Gamma(m+1/2)$$

Now put $m = 1/4$

$$\therefore \sqrt{\pi}\,\Gamma 1/2 = \Gamma 1/4 \cdot 2^{-1/2}\,\Gamma 3/4$$

$$\sqrt{\pi}\,\sqrt{\pi} = \frac{1}{\sqrt{2}}\,\Gamma 1/4\,\Gamma 3/4$$

$$\therefore \Gamma 1/4 \cdot \Gamma 3/4 = \pi\sqrt{2}$$

Q.5) Evaluate. $\int_0^4 \sqrt{x}\,(4-x)^{3/2}\,dx$

Put $x = 4t$ when $x=0,\ t=0$

$dx = 4dt$ $x=4,\ t=1$

$$\therefore I = \int_0^1 2\sqrt{t}\,.\,(4)^{3/2}(1-t)^{3/2}\,4.dt$$

$$= 8.(4)^{3/2}\int_0^1 t^{1/2}(1-t)^{3/2}\,dt.$$

$$= 64\,\beta\left(\frac{3}{2},\frac{5}{2}\right) = 64\,\frac{\Gamma 3/2\,\Gamma 5/2}{\Gamma 4} = \frac{64.1/2\,\Gamma 1/2.\,3/2.\,1/2\,\Gamma 1/2}{6}$$

$$= 4\pi$$

Q.6] Evaluate $\int_0^2 x^3\sqrt{2-x}\,dx$

Put $x = 2t$ when $x=0,\ t=0$

$dx = 2dt$ $x=2,\ t=1$

$$\therefore I = \int_0^1 8t^3\,(2)^{1/2}(1-t)^{1/2}\,2dt$$

$$= 8\sqrt{2}.2\int_0^1 t^3\,(1-t)^{1/2}\,dt$$

$= 16\sqrt{2}\cdot\beta\left(4,\frac{3}{2}\right) = 16\sqrt{2}\,\frac{\Gamma 4\,\Gamma 3/2}{\Gamma 11/2} = \frac{16\sqrt{2}\cdot 6\cdot \frac{1}{2}\Gamma\frac{1}{2}}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}$

$= \frac{16\sqrt{2}\cdot 2^4\cdot 6}{9\times 7\times 5\times 3}$

$= \frac{2^9\cdot\sqrt{2}}{315} = \frac{512\sqrt{2}}{315}$ //.

Q.7] Evaluate. $\int_0^1 x^6(1-x^2)^{1/2}\,dx$

Put $x^2 = t$ when $x=0,\ t=0$

$x = \sqrt{t}$ $x=1,\ t=1$

$dx = \frac{1}{2\sqrt{t}}\,dt$

$I = \int_0^1 t^3(1-t)^{1/2}\, \cancel{dx}\, \frac{1}{2\sqrt{t}}\,dt = \frac{1}{2}\int_0^1 t^{5/2}(1-t)^{1/2}\,dt$

$= \frac{1}{2}\beta\left[\frac{7}{2},\frac{3}{2}\right] = \frac{1}{2}\,\frac{\Gamma 7/2\cdot\Gamma 3/2}{\Gamma 5} = \frac{1}{2}\cdot\frac{\frac{5}{2}\,\frac{3}{2}\,\frac{1}{2}\Gamma\frac{1}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{4!}$

$= \frac{15\pi}{24\times 2^5} = \frac{5\pi}{256}$ //.

Q.8] Evaluate. $\int_0^{2a} x\sqrt{2ax-x^2}\,dx$.

$I = \int_0^{2a} x\sqrt{x(2a-x)}\,dx = \int_0^{2a} x^{3/2}(2a-x)^{1/2}\,dx$

Put $x = 2at$ when $x=0,\ t=0$

$dx = 2a\,dt$ $x=2a,\ t=1$

$I = \int_0^1 (2at)^{3/2}(2a-2at)^{1/2}(2a)\,dt$.

$= \int_0^1 (2a)^{3/2}(2a)^{1/2}(\cancel{a}t)^{3/2}(1-t)^{1/2}(2a)\,dt$

$= (2a)^3\int_0^1 t^{3/2}(1-t)^{1/2}\,dt = (2a)^3\beta[5/2, 3/2]$

$= 8a^3\cdot\frac{\Gamma 5/2\ \Gamma 3/2}{\Gamma 4} = \frac{8a^3\ \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\frac{1}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{3!} = \frac{8a^3\pi\times 3}{2^3\cdot 6}$

$= \frac{a^3\pi}{2}$ //.

Q.9] Evaluate $\int_0^1 (1-\sqrt[3]{x})^{1/2}\, dx$

Put $x^{1/3} = t$ when $x=0,\ t=0$

$x = t^3$ $\quad x=1,\ t=1$

$dx = 3t^2 dt$

$\therefore I = \int_0^1 (1-t)^{1/2}\cdot 3t^2 dt = 3\int_0^1 t^2(1-t)^{1/2} dt$

$= 3\beta(3, 3/2)$

$= 3\dfrac{\Gamma 3\ \Gamma 3/2}{\Gamma 9/2} = \dfrac{3.2.\ 1/2\ \Gamma 1/2}{7/2\ 5/2\ 3/2\ 1/2\ \Gamma 1/2}$

$= \dfrac{6\times 2^3}{7\times 5\times 3} = \dfrac{16}{35}$ //

Q.10] Evaluate. $\int_0^2 x^4 (8-x^3)^{-1/3} dx$

Put $x^3 = 8t$ when $x=0,\ t=0$

$x = 2t^{1/3}$ $\quad x=2,\ t=1$

$dx = \frac{2}{3} t^{-2/3} dt$

$I = \int_0^1 (2)^4 t^{4/3} \cdot (8)^{-1/3} (1-t)^{-1/3} \cdot \frac{2}{3} t^{-2/3} dt$

$= (2)^4 \cdot \frac{1}{2}\cdot\frac{2}{3} \int_0^1 t^{2/3} (1-t)^{-1/3} dt$

$= \frac{16}{3}\beta\left(\frac{5}{3}, \frac{2}{3}\right) = \frac{16}{3}\dfrac{\Gamma 5/3 \cdot \Gamma 2/3}{\Gamma 7/3}$

$= \frac{16}{3}\dfrac{2/3\ \Gamma 2/3\ \Gamma 2/3}{4/3\ 1/3\ \Gamma 1/3} = \dfrac{8(\Gamma 2/3)^2}{(\Gamma 1/3)}$ //

Q.11] Evaluate $\int_0^2 x^7 (16-x^4)^{10} dx$

Put $x^4 = 16t$ when $x=0,\ t=0$

$x = 2t^{1/4}$ $\quad x=2,\ t=1$

$dx = \frac{1}{2} t^{-3/4} dt$

$I = \int_0^1 (2)^7 t^{7/4} (16)^{10} (1-t)^{10} \frac{1}{2} t^{-3/4} dt$

$$= (2)^7 (16)^{10} \cdot \frac{1}{2} \int_0^1 t^{1} (1-t)^{10}\, dt$$

$$= 2^{46} \beta[2,11] = (2)^{46} \frac{\Gamma 2 \cdot \Gamma 11}{\Gamma 13} = \frac{(2)^{46} \cdot 10!}{12!}$$

$$= \frac{(2)^{46}}{12 \times 11} = \frac{(2)^{44}}{33}$$

H.W.Q.12] Evaluate $\int_0^2 x^2 (2-x)^{-1/2} dx$ $\quad 64\sqrt{2}/15$

Put $x = 2t$ $\quad$ when $x=0,\ t=0$

$dx = 2dt$ $\quad$ $x=2,\ t=1$

$$I = \int_0^1 4t^2 (2)^{-1/2} (1-t)^{-1/2}\, 2\, dt$$

$$= 4\sqrt{2} \int_0^1 t^2 (1-t)^{-1/2} dt = 4\sqrt{2}\, \beta(3, \tfrac{1}{2})$$

$$= 4\sqrt{2} \frac{\Gamma 3\, \Gamma 1/2}{\Gamma 7/2} = \frac{4\sqrt{2} \cdot 2 \cdot \Gamma 1/2}{5/2 \cdot 3/2 \cdot 1/2 \cdot \Gamma 1/2} = \frac{8\sqrt{2}}{15/8}$$

$$= \frac{64\sqrt{2}}{15}$$

H.W.Q.13] Evaluate $\int_0^3 x^{5/2} (3-x)^{1/2} dx$ $\quad 405\pi/128$

Put $x = 3t$ $\quad$ when $x=0,\ t=0$

$dx = 3dt$ $\quad$ $x=3,\ t=1$

$$I = \int_0^1 3^{5/2} \cdot t^{5/2} (3)^{1/2} (1-t)^{1/2}\, 3\, dt$$

$$= 3^4 \int_0^1 t^{5/2} (1-t)^{1/2} dt = 81 \int_0^1 t^{5/2} (1-t)^{1/2} dt$$

$$= 81\, \beta(\tfrac{7}{2}, \tfrac{3}{2}) = \frac{81\, \Gamma 7/2 \cdot \Gamma 3/2}{\Gamma 5} = \frac{81 \cdot 5/2 \cdot 3/2 \cdot 1/2 \Gamma 1/2 \cdot 1/2 \Gamma 1/2}{4!}$$

$$= \frac{81 \times 5 \times 3 \times \pi}{24 \times 2^4} = \frac{1215\pi}{384} = \frac{405\pi}{128}$$

H.W. Q.14) Evaluate $\int_0^2 x^2 (2-x)^{13} dx$ $\quad 2^{12}/105$

Put $x = 2t$ $\quad$ when $x=0,\ t=0$

$dx = 2dt$ $\quad x=2,\ t=1$

$I = \int_0^1 4t^2 \, (2)^{13} (1-t)^{13} \, 2dt$

$= 2^{16} \int_0^1 t^2 (1-t)^{13} dt$

$= 2^{16} \beta(3,14) = 2^{16} \dfrac{\Gamma 3 \, \Gamma 14}{\Gamma 17} = \dfrac{2^{16} . 2. 13!}{\cancel{17} \times 16 \times 15 \times 14 \times 13!}$

$= \dfrac{2^{17}}{\cancel{17} \times 16 \times 15 \times 14}$

$= \dfrac{2^{13}}{15 \times 14} = \dfrac{2^{12}}{15 \times 7} = \dfrac{2^{12}}{105}$ //.

H.W. Q.15] Evaluate $\int_0^9 x^{3/2} (9-x)^{1/2} dx$ $\quad 729\pi/16$

Put $x = 9t$ $\quad$ when $x=0,\ t=0$

$dx = 9dt$ $\quad x=9,\ t=1$

$I = \int_0^1 (9)^{3/2} t^{3/2} (1-t)^{1/2} 9 . (9)^{1/2} dt$

$= 27.3.9. \int_0^1 t^{3/2} (1-t)^{1/2} dt$

$= 729 \beta\left(\frac{5}{2}, \frac{3}{2}\right) = \dfrac{729 \, \Gamma 5/2 . \Gamma 3/2}{\Gamma 4}$

$= 729. \dfrac{3/2 . 1/2 \, \Gamma 1/2 . 1/2 \, \Gamma 1/2}{3!} = \dfrac{729 \pi}{16}$ //.

H.W. Q.16] Evaluate $\int_0^1 x^3 (1-\sqrt{x})^5 dx$ $\quad 2\beta(8,6)$

Put $x = t^2$ $\quad$ when $x=0,\ t=0$

$dx = t2dt$ $\quad x=1,\ t=1$

$$I = \int_0^1 t^6 (1-t)^5 2.t.dt$$

$$= 2\int_0^1 t^7 (1-t)^5 dt$$

$$= 2\beta(8,6)$$

$$= 2\frac{\Gamma 8 . \Gamma 6}{\Gamma 14}$$

$$= \frac{2. 7! . 5!}{13!} = \frac{\cancel{241560}\ 2 \times 120 \times \cancel{5040}\ 7!}{13 \times 12 \times 11 \times 10 \times 9 \times 8 \times 7!}$$

$$= \frac{2 \times 120}{1235520} = \frac{24}{123552} = \frac{12}{61776} = \frac{6}{30888} = \frac{3}{15444} = \frac{1}{5148}$$

Type : 5

Q.1] Evaluate $\int_0^{\pi/6} \sin^2 6\theta \cos^6 3\theta \, d\theta$

Put $3\theta = t$ when $\theta = 0$, $t = 0$

$\therefore d\theta = \frac{dt}{3}$ $\theta = \pi/6$, $t = \pi/2$

$$\therefore I = \int_0^{\pi/2} \sin^2 2t \cos^6 t . \frac{dt}{3} = \frac{1}{3}\int_0^{\pi/2} (2)^2 \sin^2 t . \cos^2 t . \cos^6 t \, dt$$

$$= \frac{4}{3}\int_0^{\pi/2} \sin^2 t . \cos^8 t . dt$$ [Here, $p = 2$ & $q = 8$]

$$= \frac{4}{3} . \frac{1}{2} \beta\left(\frac{3}{2}, \frac{9}{2}\right)$$

$$= \frac{2}{3} \frac{\Gamma 3/2 \, \Gamma 9/2}{\Gamma 6} = \frac{2}{3} \frac{1/2 \Gamma 1/2 . 7/2 . 5/2 . 3/2 . 1/2 \Gamma 1/2}{120}$$

$$= \frac{35\pi}{120 \times 2^4} = \frac{7\pi}{384}$$

Q.2] Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta} \, d\theta$

$$\therefore I = \int_0^{\pi/2} \sin^{1/2}\theta \cos^{-1/2}\theta \, d\theta$$ [$p = 1/2$ & $q = -1/2$]

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2} \frac{\Gamma 3/4 \, \Gamma 1/4}{\Gamma 1} = \frac{\pi\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}}$$

H.W. Q.3] Evaluate $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

~~Proof~~ Now, $I = \int_0^{\pi/2} \sin^{-1/2}\theta . \cos^{1/2}\theta . d\theta$

$= \frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$

$= \frac{1}{2} \frac{\Gamma 1/4 . \Gamma 3/4}{\Gamma 1} = \frac{\pi\sqrt{2}}{2} = \frac{\pi}{\sqrt{2}}$ //.

Q.4) Evaluate $\int_0^{\pi} (1-\cos\theta)^3 d\theta$.

$I = \int_0^{\pi} (2\sin^2\theta/2)^3 d\theta = 8 \int_0^{\pi/2} \sin^6\theta/2 \; d\theta$.

Put $\theta/2 = t$ when $\theta = 0, t = 0$

$d\theta = 2dt$ $\theta = \pi, t = \pi/2$

$I = 8\int_0^{\pi/2} \sin^6 t \; 2\, dt$

$= 16 \int_0^{\pi/2} \sin^6 t . \cos^0 t . dt$

$= 16 . \frac{1}{2} \beta\left(\frac{7}{2}, \frac{1}{2}\right) = 8 . \frac{\Gamma 7/2 \Gamma 1/2}{\Gamma 4}$ ~~$= \frac{8 \times 2}{5/2 . 3/2 . 1/2}$~~

$= \frac{\cancel{27}}{\cancel{15}} = \frac{\cancel{128}}{\cancel{15}}$

$= \frac{8 \Gamma 7/2 \Gamma 1/2}{\Gamma 4}$

$= \frac{8 . 5/2 . 3/2 . 1/2 \Gamma 1/2 . \Gamma 1/2}{6} = \frac{8 \times 5 \times 3 \pi}{2^3 \times 6} = \frac{5\pi}{2}$ //

Q.5) Evaluate. $\int_0^{a} (a^2 - x^2)^{5/2} dx$

Put $x^2 = a^2 t$ when $x = 0, t = 0$

$x = a t^{1/2}$ $x = a, t = 1$

$dx = \frac{a}{2} t^{-1/2} dt$

$$I = \int_0^1 (a^2)^{5/2} (1-t)^{5/2} \frac{a}{2} t^{-1/2} dt$$

$$= \frac{a^6}{2} \int_0^1 t^{-1/2} (1-t)^{5/2} dt$$

$$= \frac{a^6}{2} \beta\left(\frac{1}{2}, \frac{7}{2}\right) = \frac{a^6}{2} \frac{\Gamma 1/2 \, \Gamma 7/2}{\Gamma 4} = \frac{a^6}{2\times 6} \pi . 7/2 . 5/2 . 3/2 . 1/2$$

$$= \frac{5a^6 \pi}{32}$$

H.W.Q.6] Evaluate, $\int_0^a x^4 \sqrt{a^2 - x^2}\, dx$ $\quad a^6\pi/32$

Put $x^2 = a^2 t$ $\quad$ when $x = 0, t = 0$

$x = a t^{1/2}$ $\quad$ $x = a, t = 1$

$dx = \frac{a}{2} t^{-1/2} dt$

$$I = \int_0^1 a^4 t^2 (a^2)^{1/2} (1-t)^{1/2} \frac{a}{2} t^{-1/2} dt$$

$$= \frac{a^6}{2} \int_0^1 t^{3/2} (1-t)^{1/2} dt$$

$$= \frac{a^6}{2} \beta\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{a^6}{2} . \frac{\Gamma 5/2 . \Gamma 3/2}{\Gamma 4} = \frac{a^6}{2} . \frac{3/2 . 1/2 \Gamma 1/2 . 1/2 \Gamma 1/2}{3!}$$

$$= \frac{a^6 \pi}{25} = \frac{a^6 \pi}{32}$$

Q.7] Evaluate $\int_0^{\pi/4} \cos^3(2x) \sin^4(4x)\, dx$

Put $2x = t$ $\quad$ when $x = 0, t = 0$

$dx = \frac{dt}{2}$ $\quad$ $x = \pi/4, t = \pi/2$

$$I = \int_0^{\pi/2} \cos^3(t) \sin^4(2t) \frac{dt}{2} = \frac{1}{2} \int_0^{\pi/2} \cos^3 t . (2)^4 \sin^4 t \cos^4 t\, dt$$

$$= 8 \int_0^{\pi/2} \sin^4 t \cos^7 t\, dt = 8 . \frac{1}{2} \beta\left(\frac{5}{2}, 4\right)$$

$$= 4 \frac{\Gamma 5/2 \, \Gamma 4}{\Gamma 13/2} = \frac{4\times 6 \times 3/2 . 1/2 \Gamma 1/2}{11/2 . 9/2 . 7/2 . 5/2 . 3/2 . 1/2 \Gamma 1/2} = \frac{24 \times 2^4}{11 \times 9 \times 7 \times 5}$$

$$= 384/3465 = 128/1155$$

Type : 6

Q.1] Evaluate. $\int_0^\infty \frac{1}{1+x^4} dx$

Put $x^2 = \tan\theta$ when $x=0$, $\theta=0$

$x = \sqrt{\tan\theta}$ $x=\infty$, $\theta=\pi/2$

$dx = \frac{1}{2\sqrt{\tan\theta}} \sec^2\theta . d\theta$

$I = \int_0^{\pi/2} \frac{1}{1+\tan^2\theta} . \frac{\sec^2\theta \, d\theta}{2\sqrt{\tan\theta}} = \frac{1}{2}\int_0^{\pi/2} (\tan\theta)^{-1/2} d\theta$

$= \frac{1}{2}\int_0^{\pi/2} \sin^{-1/2}\theta . \cos^{1/2}\theta . d\theta$

$= \frac{1}{2} . \frac{1}{2} \beta\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{4} . \frac{\Gamma 1/4 . \Gamma 3/4}{\Gamma 1} = \frac{\pi\sqrt{2}}{4} = \frac{\pi}{2\sqrt{2}}$ //

Q.2] Evaluate $\int_0^\infty \left(\frac{t}{1+t^2}\right)^4 dt$

Put $t = \tan\theta$ when $t=0$, $\theta=0$

$dt = \sec^2\theta . d\theta$ $t\to\infty$, $\theta = \pi/2$

$I = \int_0^{\pi/2} \frac{\tan^4\theta}{(\sec^2\theta)^4} . \sec^2\theta . d\theta$

$= \int_0^{\pi/2} \frac{\tan^4\theta}{\sec^6\theta} d\theta = \int_0^{\pi/2} \sin^4\theta . \cos^2\theta . d\theta$

$= \frac{1}{2}\beta\left(\frac{5}{2}, \frac{3}{2}\right) = \frac{1}{2} \frac{\Gamma 5/2 \, \Gamma 3/2}{\Gamma 4} = \frac{1}{2} \times \frac{3/2 . 1/2 \Gamma 1/2 . 1/2 \Gamma 1/2}{6}$

$= \frac{\pi}{32}$ //

Q.3] Evaluate. $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx$

Put $a+bx = \frac{a}{1-t}$, $x = \frac{at}{b(1-t)}$ when $x=0$, $t=0$

$x\to\infty$, $t=1$

$b\,dx = \frac{a}{(1-t)^2} dt$

$$J = \int_0^1 \frac{a^{m-1} t^{m-1}}{b^{m-1} (1-t)^{m-1}} \cdot \frac{(1-t)^{m+n}}{a^{m+n}} \cdot \frac{a}{b} \cdot \frac{a}{(1-t)^2} dt$$

$$= \frac{a^{m-1} \cdot a}{b^{m-1} \cdot a^{m+n} \cdot b} \int_0^1 \frac{t^{m-1}}{(1-t)^{m-1}} \frac{(1-t)^{m+n}}{(1-t)^2} dt$$

$$= \frac{1}{a^n b^m} \int_0^1 t^{m-1} \cdot (1-t)^{n-1} dt$$

$$= \frac{1}{a^n b^m} \cdot \beta(m, n).$$

Q4] Express $\int_a^b (x-a)^m (b-x)^n dx$ as a β-function & hence find $\int_0^1 x^4 (1-x)^{3/2} dx$

(upper-lower)

Put $x - a = (b-a)t$ when $x = a$, $t = 0$

$dx = (b-a) dt$ $x = b$, $t = 1$

Now, $b - x = b - [a + (b-a)t]$

$= (b-a) - (b-a)t$

$= (b-a) \cdot (1-t)$

$$J = \int_0^1 (b-a)^m t^m (b-a)^n (1-t)^n (b-a) dt$$

$$= (b-a)^{m+n+1} \int_0^1 t^m (1-t)^n dt = (b-a)^{m+n+1} \beta(m+1, n+1)$$

Now, we have to evaluate the integral,

$$J = \int_0^1 x^4 (1-x)^{3/2} dx$$

Here, $a = 0$, $b = 1$, $m = 4$, $n = 3/2$

$$\therefore \int_0^1 x^4 (1-x)^{3/2} dx = (1-0)^{4+3/2+1} \cdot \beta\left(5, \frac{5}{2}\right)$$

$$= \frac{\Gamma 5 \, \Gamma 5/2}{\Gamma 15/2} = \frac{24 \times 3/2 \times 1/2 \, \Gamma 1/2}{13/2 \cdot 11/2 \cdot 9/2 \cdot 7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2 \, \Gamma 1/2}$$

$$= \frac{24 \times 2^5}{13 \times 11 \times 9 \times 7 \times 5} = \frac{768}{45045} = \frac{256}{15015}$$

Q.5) Evaluate $\int_0^1 x^5 \sin^{-1}x \, dx$

$$I = \left[\sin^{-1}x \cdot \frac{x^6}{6}\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-x^2}} \frac{x^6 \, dx}{6}$$

$$= \left(\frac{\pi}{2} \cdot \frac{1}{6} - 0\right) - \frac{1}{6}\int_0^1 \frac{x^6}{(1-x^2)^{1/2}} dx$$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^1 x^6 (1-x^2)^{-1/2} dx$$

$= \frac{\pi}{12} -$ Put $x^2 = t$ when $x=0, \ t=0$

$x = \sqrt{t}$ $\quad x=1, \ t=1$

$dx = \frac{1}{2\sqrt{t}} dt$

$$= \frac{\pi}{12} - \frac{1}{6}\int_0^1 t^3 (1-t)^{-1/2} \cdot \frac{1}{2\sqrt{t}} dt = \frac{\pi}{12} - \frac{1}{12}\int_0^1 t^{5/2}(1-t)^{-1/2} dt$$

$$= \frac{\pi}{12} - \frac{1}{12} \beta\left(\frac{7}{2}, \frac{1}{2}\right)$$

$$= \frac{\pi}{12} - \frac{1}{12} \frac{\Gamma 7/2 \cdot \Gamma 1/2}{\Gamma 4} = \frac{\pi}{12} - \frac{1}{12} \cdot \frac{5/2 \cdot 3/2 \cdot 1/2 \, \pi}{6}$$

$$= \frac{\pi}{12} - \frac{5\pi}{192} = \frac{11\pi}{192}$$

Q.6) Evaluate $\int_0^\pi x \sin^5 x \cdot \cos^4 x \, dx$

Here, $\int_0^\pi (\pi - x) \sin^5(\pi - x) \cos^4(\pi - x) \, dx$

$$= \int_0^\pi (\pi - x) \sin^5 x \cos^4 x \, dx \quad - \left[\begin{array}{l} \sin(\pi-x) = \sin x \\ \cos(\pi - x) = -\cos x \end{array}\right]$$

$$I = \int_0^\pi \pi \sin^5 x \cdot \cos^4 x \cdot dx - \int_0^\pi x \sin^5 x \cos^4 x \, dx$$

$$2I = \pi \int_0^\pi \sin^5 x \cdot \cos^4 x \, dx$$

$$I = \frac{\pi}{2} \cdot \left\{ \int_0^{\pi/2} \sin^5 x \cdot \cos^4 x \, dx + \int_0^{\pi/2} \sin^5(\pi - x) \cdot \cos^4(\pi - x) \, dx \right\}$$

$$I = \frac{\pi}{2}\left\{2\int_0^{\pi/2} \sin^5 x \cdot \cos^4 x\, dx\right\}$$

$$I = \pi \cdot \frac{1}{2}\beta\left[3, \frac{5}{2}\right] = \frac{\pi}{2}\,\frac{\Gamma 3 \cdot \Gamma 5/2}{\Gamma 11/2} = \frac{\pi \cdot 2 \cdot 3/2 \cdot 1/2\, \Gamma 1/2}{2\ \ 9/2\ 7/2\ 5/2\ 3/2\ 1/2\ \Gamma 1/2}$$

$$= \frac{8\pi}{9\times7\times5} = \frac{8\pi}{315}$$

Q7) Evaluate $\int_0 x^5 \sqrt{\frac{1+x^2}{1-x^2}}\, dx$

Put $x^2 = \cos\theta$ When $x=0,\ \theta = \pi/2$

$x = \sqrt{\cos\theta}$ $x = 1,\ \theta = 0$

$dx = \frac{1}{2\sqrt{\cos\theta}}(-\sin\theta)\, d\theta$

$$\therefore J = \int_{\pi/2}^{0} \cos^{5/2}\theta \cdot \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\ \frac{-\sin\theta}{2\sqrt{\cos\theta}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^2\theta \cdot \sin\theta \sqrt{\frac{2\cos^2\theta/2}{2\sin^2\theta/2}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^2\theta \cdot 2 \cdot \sin\theta/2 \cdot \cos\theta/2 \cdot \frac{\cos\theta/2}{\sin\theta/2}\, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta \cdot \cos^2\theta/2\, d\theta = \int_0^{\pi/2} \cos^2\theta \left(\frac{1+\cos\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^2\theta\, d\theta + \frac{1}{2}\int_0^{\pi/2} \cos^3\theta\, d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\beta(1/2, 3/2) + \frac{1}{2}\beta\left(\frac{1}{2}, 2\right)\right] = \frac{1}{4}\left[\frac{\Gamma 1/2\, \Gamma 3/2}{\Gamma 2} + \frac{\Gamma 1/2\, \Gamma 2}{\Gamma 5/2}\right]$$

$$= \frac{1}{4}\left[\frac{\sqrt{\pi} \cdot 1/2 \cdot \sqrt{\pi}}{1} + \frac{\sqrt{\pi}}{3/4\sqrt{\pi}}\right]$$

$$= \frac{1}{4}\left[\frac{\pi}{2} + \frac{4}{3}\right] = \frac{\pi}{8} + \frac{1}{3} = \frac{3\pi + 8}{24}$$

Q.8] Evaluate $\int_1^{\infty} \frac{(x-1)^7}{x^{12}} dx$ Put $x = 1/t$

$\therefore dx = \frac{-1}{t^2} dt$ when $x=1$, $t=1$

$x \to \infty$, $t=0$

$I = \int_1^0 \frac{t^{12}}{t^7} (1-t)^7 \cdot \frac{-1}{t^2} dt$

$= \int_0^1 t^3 (1-t)^7 dt = \beta(4,8) = \frac{\Gamma 4 \, \Gamma 8}{\Gamma 12}$

$= \frac{3! \times 7!}{11!} = \frac{6}{11 \times 10 \times 9 \times 8} = \frac{1}{1320}$ //

Q.9] ~~Evaluate~~ Prove that $\int_0^{\infty} \frac{x^6 (1-x^{10})}{(1+x)^{24}} dx = 0$

let, $I = \int_0^{\infty} \frac{x^6(1-x^{10}) dx}{(1+x)^{24}}$

$= \int_0^{\infty} \frac{x^6}{(1+x)^{24}} dx - \int_0^{\infty} \frac{x^{16}}{(1+x)^{24}} dx$

Put $1+x = \frac{1}{1-t}$ when $x=0$, $t=0$

$x \to \infty$, $t=1$

$dx = \frac{-1(-dt)}{(1-t)^2}$ Now, $x = \frac{1}{1-t} - 1 = \frac{t}{1-t}$

$\therefore I = \int_0^1 \frac{t^6}{(1-t)^6} \cdot (1-t)^{24} \frac{dt}{(1-t)^2} - \int_0^1 \frac{t^{16}}{(1-t)^{16}} \cdot (1-t)^{24} \frac{dt}{(1-t)^2}$

$= \int_0^1 t^6 (1-t)^{16} dt - \int_0^1 t^{16} \cdot (1-t)^6 dt$

$= \beta(7,17) - \beta(17,7)$

$= 0$ —— $[\because \beta(m,n) = \beta(n,m)]$

Q.10] Prove that $\beta(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

Put $(1+x) =$

$I = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$

Put $1+x=\frac{1}{1-t}$ (*) also $x=\frac{t}{1-t}$

$$dx=\frac{dt}{(1-t)^2}$$

When $x=0$, From (*) $t=0$

$x=1$, From (*) $t=1/2$

$$I=\int_0^{1/2}\frac{t^{m-1}}{(1-t)^{m-1}}\cdot(1-t)^{m+n}\frac{dt}{(1-t)^2}+\int_0^{1/2}\frac{t^{n-1}}{(1-t)^{n-1}}\cdot(1-t)^{m+n}\frac{dt}{(1-t)^2}$$

$$=\int_0^{1/2}t^{m-1}(1-t)^{n-1}dt+\int_0^{1/2}t^{n-1}\cdot(1-t)^{m-1}dt$$

$=$ R.H.S — eq^n ①

Now,

$$\text{L.H.S.}=\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx=\int_0^1 t^{m-1}(1-t)^{n-1}dt$$

We know that, $\int_0^{2a}F(x)dx=\int_0^a F(x)dx+\int_0^a F(2a-x)dx$

$$\therefore=\int_0^{1/2}t^{m-1}(1-t)^{n-1}dt+\int_0^{1/2}(1-t)^{m-1}t^{n-1}dt \quad \text{— eq}^n \text{ ②}$$

$\therefore$ from ① and ②

$$\beta(m,n)=\int_0^1\frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx$$

Q.11] Prove that $\int_0^1\frac{x^{2n}}{\sqrt{1-x^2}}dx=\frac{(2n)!}{2^{2n}(n!)^2}\cdot\frac{\pi}{2}$, where $n\in I^+$

let

$$I=\int_0^1\frac{x^{2n}}{(1-x^2)^{1/2}}dx$$

Put $x^2=t$ when $x=0$, $t=0$

$x=\sqrt{t}$ $x=1$, $t=1$

$dx=\frac{1}{2}t^{-1/2}dt$

$$I=\int_0^1 t^n(1-t)^{-1/2}\cdot\frac{1}{2}t^{-1/2}dt=\frac{1}{2}\int_0^1 t^{n-1/2}(1-t)^{-1/2}dt$$

$$=\frac{1}{2}\beta\left(n+\frac{1}{2},\frac{1}{2}\right)$$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

$$= \frac{1}{2} \frac{\Gamma(n+1/2)\,\Gamma(1/2)}{\Gamma(n+1)} = \frac{\sqrt{\pi}}{2} \frac{\Gamma(n+1/2)}{\Gamma(n+1)} \quad \text{——— (1)}$$

We know that,

duplication formula, $2^{2n-1}\,\Gamma(n)\,\Gamma(n+1/2) = \sqrt{\pi}\,\Gamma(2n)$

$$\therefore \Gamma(n+1/2) = \frac{\sqrt{\pi}\,\Gamma(2n)}{\Gamma(n)\,2^{2n-1}}$$

From (1).

$$\therefore = \frac{\sqrt{\pi}}{2} \cdot \frac{\sqrt{\pi}\,\Gamma(2n)}{\Gamma(n).\,2^{2n-1}} \cdot \frac{1}{\Gamma(n+1)}$$

$$= \frac{\pi}{2^{2n}} \cdot \frac{(2n-1)!}{n!\,(n-1)!}$$

multiplying & dividing by $(2n)$.

$$= \frac{\pi}{2^{2n}} \cdot \frac{2n\,(2n-1)!}{\underline{2n}.\,(n)!\,\underline{(n-1)!}} = \frac{\pi}{2^{2n}} \cdot \frac{(2n)!}{(n!)\,2\,(n!)}$$

$$= \frac{(2n)!}{2^{2n}\,(n!)^2} \cdot \frac{\pi}{2}$$

Hence proved.

Q.12] Prove that $\int_{-\infty}^{\infty} \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{2}\beta\left(\frac{n}{2}, \frac{n}{2}\right)$ and hence evaluate $\int_{-\infty}^{\infty} \operatorname{sech}^8 x\, dx$

Put $e^x = \tan\theta$ when $x \to -\infty,\ t = 0$

$x = \log(\tan\theta)$ $\quad x \to \infty,\ t = \pi/2$

$dx = \frac{\sec^2\theta}{\tan\theta}\, d\theta$

$$\therefore I = \int_0^{\pi/2} \frac{\sec^2\theta}{\tan\theta}\, d\theta \cdot \frac{1}{(\sec^2\theta/\tan\theta)^n}$$

$$= \int_0^{\pi/2} \frac{\sec^2\theta}{\tan\theta} \cdot \frac{\tan^n\theta}{\sec^{2n}\theta}\, d\theta = \int_0^{\pi/2} \frac{1}{\cos^2\theta} \cdot \frac{\cos\theta}{\sin\theta} \cdot \frac{\sin^n\theta}{\cos^n\theta} \cdot \cos^{2n}\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^{n-1}\theta \cdot \cos^{2n+1-2-n}\theta\, d\theta$$

$$= \frac{1}{2}\beta\left(\frac{n}{2}, \frac{n}{2}\right) \quad \text{——— (*)}$$

Now, we have to evaluate

$$I = \int_{-\infty}^{\infty} \operatorname{sech}^8 x \, dx = \int_{-\infty}^{\infty} \frac{1}{\cosh^8 x} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\left(\frac{e^x + e^{-x}}{2}\right)^8} dx = 2^8 \int_{-\infty}^{\infty} \frac{1}{(e^x + e^{-x})^8} dx$$

comparing with ⊛, $n = 8$

$$= 2^8 \cdot \frac{1}{2} \beta(4,4) = 2^7 \frac{\Gamma 4 \, \Gamma 4}{\Gamma 8} = \frac{2^7 \cdot (3!)^2}{7!} = \frac{32}{35}$$