Calculus.

1.① Continuity - A function 'f' is said to be continuous at a point A. If

$$\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a).$$

In other words the graph without break is called continuous function.

2.① Differentiability - A function 'f' is said to be a differential at point 'a' if

$$f'(a) = \lim_{x\to 0} \frac{f(a+h) - f(a)}{h}$$ provide the limit should exist.

3① Every differentiable function is continuous but converse is not true. i.e. (continuity doesn't implies differentiability)
e.g. $f(x) = |x|$ which is continuous but not differentiable at $x=0$

4.① Every polynomial function is continuous

5.① Every constant function is continuous.

6.① Every Rational function $\left(\frac{f(x)}{g(x)}, g(x) \neq 0\right)$ is continuous. Then

$|x|$ has two values

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

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7 slope of x-axis = 0
if line $l_1 \parallel^{el}$ to line l_2 then slope of line l_1 = slope of line l_2

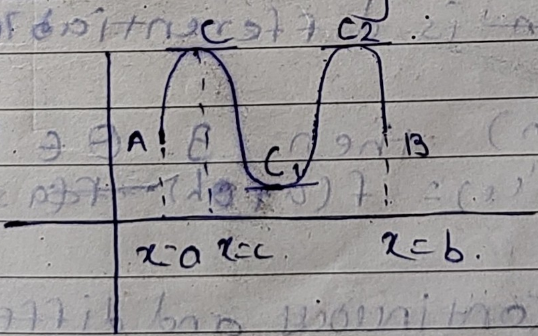
8 slope of line passing through points (x_1, y_1) (x_2, y_2) = $\frac{y_2 - y_1}{x_2 - x_1}$

9 If $y = f(x)$ be any curve then $\frac{dy}{dx}$ = slope of tangent to curve at any point.

10. $\left.\frac{dy}{dx}\right|_{\text{at } x=a} = f'(a)$ = slope of tangent to curve at $x = a$.

11 Rolle's theorem -
i) If $f(x)$ is continuous in $[a, b]$
ii) $f(x)$ is differentiable in (a, b) i.e. $f'(x)$ exist at every value of x in (a, b)
iii) $f(a) = f(b)$
then $\exists$ at least one point c ~~belongs to~~ $c \in (a, b)$ such that $f'(c) = 0$

12 Geometrical meaning of Rolle's th^m -



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consider the portion AB of a curve $y=f(x)$ lying betn $x=a$ and $x=b$ such that
i] It's go continuously from A to B.
ii] It has tangent at every point betn A and B.
iii] ordinate at A = ordinate at B.
iv] from Fig there is at least one point (may be more) of a curve at which tangent is parallel to x-axis
i.e. slope of tangent at $c=0$.
but the slope of tangent at c is the value of differential coefficient of $f(x)$ w.r.t. x.
i.e. $f'(c)=0$

13 Alternative form of Rolle's thm
Let $b=a+h$ i.e. $h=b-a$ for $a<c<b$
choose $\theta = \dfrac{c-a}{b-a}$

$c=a+\theta(b-a)$ = ~~c-a~~

$c=a+\theta h$ which lies betn a and $a+h$
provided $0<\theta<1$
Then Rolle's thm states that $f(x)$ is continuous in $[a,(a+h)]$
Rolle's thm ~~in~~ is differentiable is $(a,(a+h))$
$f(a)=f(a+h)$ then $\exists\ \theta \in (0,1)$.
Such that $f'(c)=f(a+\theta h)$ ~~-f(a)~~ $=0$

14 If $f(x)$ is continuous and differentiable and $f(a)=f(b)=0$ then by Rolle's thm There exist at least 1 point $c \in (a,b)$ such that $f'(c)=0$. i.e. the real roots c of the eqn

a & b which are two adjacent roots

$f'(x)=0$ lies betⁿ the real of the equation ~~a and b of the eq^n~~ $f(x)=0$

Q. verify Rolle's thᵐ for the following function $\frac{\sin x}{e^x}$ $(0, \pi)$.

Here Let $f(x) = \frac{\sin x}{e^x}$

Here $\sin x$ is continuous e^x is continuous

$\therefore$ $f(x) = \frac{\sin x}{e^x}$ is continuous function

$f'(x) = \frac{e^x \cos x - \sin x\, e^x}{e^{2x}}$

$f(x)$ is differentiable and $f(0) = \frac{\sin 0}{e^0} = 0$

$f(\pi) = \frac{\sin \pi}{e^\pi} = 0$

$f(0) = f(\pi)$

i.e. conditions of rolle's thᵐ are stasfied.

Now we have to find $c \in (0, \pi)$ such that

$f'(c) = 0$

$f'(c) = \frac{e^c \cos c - \sin c\, e^c}{e^{2c}} = 0$

$\therefore \cos c - \sin c = 0$

$\therefore \cos c = \sin c$

$c = \pi/4, 5\pi/4 \in (0, \pi)$

Q. $f(x) = x^3 - 4x$ $[-2, 2]$

$f(x)$ is continuous $[-2, 2]$

$f(x)$ is differentiable

$f(-2) =$

Here given function is polynomial hence it is continuous and differentiable

$f(-2) = 0$

$f(2) = 0$

i.e. $f(-2) = f(2)$

$\therefore$ condition for Rolle's th^m are satisfied

$\therefore$ $\exists$ $c \in (-2, 2)$ such that

$f'(c) = 0$

$3c^2 - 4 = 0$

$3c^2 = 4$

$c^2 = 4/3$

$c = \pm \frac{2}{\sqrt{3}} \in (-2, 2)$

Rolle's th^m is verified.

Q. discuss the applicability of rolle's th^m for the following function.

$f(x) = |x| \quad [-3, 3]$

Here, $f(x) = |x|$ is continuous everywhere.

Now $f'(0) = \lim\limits_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim\limits_{h \to 0} \frac{f(h) - f(0)}{h}$

$= \lim\limits_{h \to 0} \frac{|h|}{h}$

$f'(0^-) = \lim\limits_{h \to 0^-} \frac{-h}{h} = -1$

$f'(0^+) = \lim\limits_{h \to 0^+} \frac{h}{h} = 1$

lim doesn't exist

$\therefore$ $f(x)$ is not differentiable at $x = 0$

$\therefore$ rolle's th^m is not applicable.

$$f(x) = \begin{cases} 1+x & x \le 2 \\ 5-x & x > 2 \end{cases}$$

ii)

Now here $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 1+x = 3$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 5-x = 3$

$f(2) = 1+x = 3$

$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x) = f(2) = 3$

i.e. $f(x)$ is continuous.

Now we will check the differentiability at 2.

$f'(2) = \lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h}$ $\left(\begin{array}{l} \text{for } x \le 2 \\ f(x) = 1+x \\ f(h) = 3+h \end{array}\right)$

$= \lim_{h\to 0^-} \frac{(3+h) - 3}{h} = 1$

$\lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h}$ $\left(\begin{array}{l} \text{for } x > 2 \\ f(x) = 5-x \\ f(h) = 3-h \end{array}\right)$

$= \lim_{h\to 0^+} \frac{3-h-3}{h} = -1$

$\therefore \lim_{h\to 0^-} f(2) \neq \lim_{h\to 0^+} f(2)$

$\therefore$ $f(x)$ is not differential.

$\therefore$ Rolle's Th^m is not applicable.

Q. verify Rolle's th^m for the function $f(x) = (x-a)^m (x-b)^n$ $[a, b]$ m and n are +ve integer.

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Here, the given function is polynomial function so it is continuous and differentiable.

$f(a) = 0$

$f(b) = 0$

$\therefore f(a) = f(b).$

conditions of Rolle's th^m are satisfied

we have to find $c \in (a,b)$ such that

$f'(c) = 0$.

here $f(x) = (x-a)^m (x-b)^n$

$f'(x) = (x-a)^m n(x-b)^{n-1} + (x-b)^n m(x-a)^{m-1}$

as $f'(c) = 0$

~~$f'(c) = (c-a)^m n(c$~~

$(c-a)^m \cdot n(c-b)^{n-1} + (c-b)^n m(c-a)^{m-1} = 0$

$(c-a)^{m-1} (c-b)^{n-1} \{(c-a)n + (c-b)m\} = 0$

$n(c-a) + (c-b)m = 0.$

$nc - na + mc - mb = 0.$

$c = \frac{na+mb}{m+n}$

Q. ~~Prove that~~ discuss applicability of Rolle's th^m for function $f(x) = (x-2a)^{2/3}$ $[a, 3a]$

here,

$f(x)$ is continuous

$f'(x) = \frac{2}{3(x-2a)^{1/3}} = \frac{2}{3}(x-2a)^{-1/3}$

$f(x)$ is not differentiable at $x = 2a \in (a, 3a)$

$\therefore$ Rolle's th^m is not applicable.

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Q. Prove that $y = x^3 + x - 3 = 0$ has exactly one real root

$f(x) = x^3 + x - 3$

$f(x)$ is polynomial function it is continuous and differentiable.

let the eq^n have 2 real roots.

i.e. $x = a$ and $x = b$

here $f(x) = x^3 + x - 3 = 0$

Which is polynomial function and is always continuous and differentiable and as a and b are the roots

$f(a) = 0$, $f(b) = 0$

i.e. $f(a) = f(b)$.

i.e. conditions for Rolle's th^m are satisfied

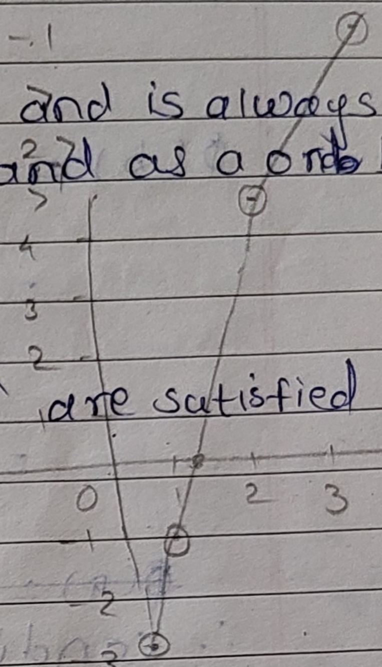
$\exists$ c such that $f'(c) = 0$

$f'(c) = 3c^2 + 1 \geq 1$

$\therefore$ since $f'(c) \neq 0$

$\therefore$ our assumption is wrong

$\therefore$ the given eq^n has exactly one real root.



LMVT

If a function $f(x)$ is continuous in $[a, b]$

$f(x)$ is differentiable in (a, b) i.e. $f'(x)$ is exist at every x in (a, b).

then there exist at least one value $c \in (a, b)$

such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Proof

choose $\phi(x) = f(x) - \dfrac{f(b) - f(a)}{b - a} x$.

since $f(x)$ is continuous $\therefore$ $\phi(x)$ is continuous in $[a, b]$.

and $f'(x)$ exist in (a,b) $\therefore$ $\phi(x)$ is differentiable (a,b)

$$\phi(a) = f(a) - \frac{f(b)-f(a)}{b-a} \times a.$$

$$= \frac{b f(a) - \cancel{a f(a)} - a f(b) + \cancel{a f(a)}}{b-a}$$

$$= \frac{b f(a) - a f(b)}{b-a}$$

$$\phi(b) = f(b) - \frac{f(b)-f(a)}{b-a} \times b$$

$$= \frac{\cancel{b f(b)} - a f(b) - \cancel{b f(b)} + b f(a)}{b-a}$$

$$= \frac{b f(a) - a f(b)}{b-a}$$

~~$\phi(a) = \phi(b)$~~ $\phi(a) = \phi(b)$

$\therefore$ conditions for Rolle's thm satisfied

$\therefore$ there exist at least $c \in (a,b)$ such that

$\phi'(c) = 0$

Now $\phi'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$

$$\therefore \phi'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

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Alternative form of LMVT –

let $b = a+h$ ie. $h = b-a$ and for $a<c<b$
$c = a+\theta h$ provided with $0<\theta<1$ (Refer type I)
then LMVT may be stated as
i) $f(x)$ is continuous in $[a,(a+h)]$.
ii) $f(x)$ is differentiable in $(a,(a+h))$.
then there exist at least one number θ
$\theta \in (0,1)$ such that

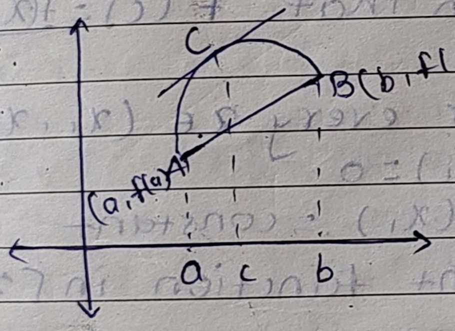
$$f'(a+\theta h) = \frac{f(a+h)-f(a)}{a+h-a}$$

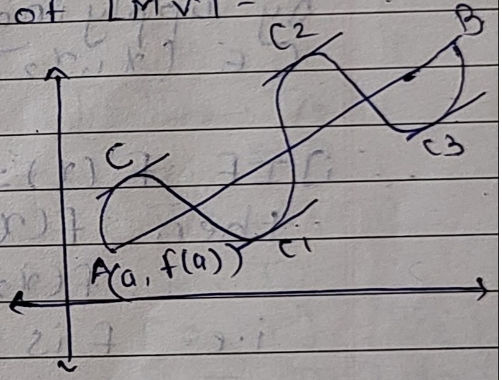
$$f'(a+\theta h) = \frac{f(a+h)-f(a)}{h}$$

$$h f'(a+\theta h) = f(a+h)-f(a)$$

$$f(a+h) = h f'(a+\theta h) + f(a)$$

Geometrical interpretation of LMVT –





consider A and B are the points on the curve $y=f(x)$ corresponding to $x=a$ and $x=b$ so that $A \equiv (a, f(a))$ and $B \equiv (b, f(b))$ here the slope of chord AB $= \frac{f(b)-f(a)}{b-a}$

by LMVT slope of chord AB $= f'(c)$..
slope of tangent to the curve at C at $(x=c)$ hence LMVT states that if the curve AB has a tangent at each of its point then there exist at least one point C on this curve such that the tangent at which is ||el to

chord AB.

- LMVT is generalisation of Rolle's th^m. in special case when $f(a) = f(b)$ then LMVT reduces to Rolle's th^m.

- The avarage rate of change of f over the interval (a, b) is given by $\frac{f(b) - f(a)}{b - a}$

Which is actual rate of change of f at some point of interval AB.

- If $f(x)$ ~~sta~~ satisfied the condition of LMVT in $[a, b]$ and x_1 and x_2 are any two points of the $[a, b]$ such that $x_1 < x_2$ then by applying LMVT in $[x_1, x_2]$ there exist $c \in (x_1, x_2)$ such that $f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ —①

i] If $f'(x) = 0$ for every $x \in (x_1, x_2)$.
then $f(x_2) - f(x_1) = 0$
$\therefore f(x_2) = f(x_1) = \text{constant}$
i.e. f is constant function in $[x_1, x_2]$

ii] If $f'(x) > 0$ for every $x \in (x_1, x_2)$
from ①
$f'(c)(x_2 - x_1) = f(x_2) - f(x_1)$
as $f'(c)(x_2 - x_1) > 0$ $\left(\begin{matrix} f'(x) > 0 \\ x_1 < x_2 \end{matrix} \right)$
since $f(x_2) - f(x_1) > 0$
$f(x_2) > f(x_1)$
$\therefore$ f is strictly increasing function.

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iii] If $f'(x) < 0$ for every $x \in (x_1, x_2)$.
from ①

$f'(c)(x_2 - x_1) = f(x_2) - f(x_1)$.

$f'(c) < 0 \quad x_2 - x_1 > 0$

$f'(c)(x_2 - x_1) < 0$

$\therefore f(x_2) - f(x_1) < 0$

$f(x_2) < f(x_1)$

$\therefore$ f is strictly decreasing function.

⊙ If $a < b$ then $\frac{1}{a} > \frac{1}{b}$, $-a > -b$

⊙ the range of function is domain of f^{-1}

Inverse function	domain	Range
$\sin^{-1}x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}x$	$(-\infty, \infty) = R$	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	$(-\infty, \infty) - 0 = R - \{0\}$	$(0, \pi)$
$\sec^{-1}x$	$R - (-1, 1)$	$[0, \pi] - \{\pi/2$
$\operatorname{cosec}^{-1}x$	$R - (-1, 1)$	$[-\pi/2, \pi/2] \{0$

⊙ Inverse trignometric functions are continuous and differentiable in there domain.

⊙ continuity of some common function.

function	interval in which f is continuous.
$f(x) = k$	R
$f(x) = x$	R
$f(x) =$ polynomial $(a_0x^n + a_1x^{n-1} + \dots a_nx^0)$	R
$f(x) =$ modulus	R
$f(x) =$ Rational $\frac{p(x)}{q(x)}$	$R - \{x : (q(x) = 0\}$
if $q(x) \neq 0$	R
$f(x) = \sin x, \cos x$	R
$f(x) = e^x$	R
$f(x) = \log(x)$	$(0, \infty)$
$f(x) =$ inverse function $\sin^{-1}x, \cos^{-1}x$	in their respective domain
$f(x) = \tan x, \sec x$	$R - \{(2n+1)\pi/2\}$ $n \in z$
$f(x) = \cot, \operatorname{cosec} x$	$R - \{n\pi$ $n \in z\}$

Q. verify LMVT for $f(x) = 2x^2 - 7x - 10$ $[2, 5]$

$f(x) = 2x^2 - 7x - 10$

$f(x)$ is polynomial function so it is continuous and differentiable.

i.e. conditions for LMVT are satisfied

Now we have to find $c \in (2, 5)$

such that $f'(c) = \frac{f(b) - f(a)}{b - a}$ — (1)

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here, $a = 2$ $b = 5$

~~$f'(x)$~~ $f(a) = 2\times 4 - 14 - 10$
$= -16$

$f(b) = 5$

$f'(x) = 4x - 7$
$f'(c) = 4c - 7$
from eq^n ①
$4c - 7 = \frac{5+16}{5-2} = \frac{21}{3} = 7$

$4c = 14$
$c = 7/2 = 3.5 \in (2, 5)$.

Q. Find the point of curve $y = \log x$ at which the tangent to the curve is parallel to chord joining the points $(1,0)$ $(e,1)$

here $f(x) = \log x$ ~~and a = 1, b = e~~
and the points are $(1,0)$ $(e,1)$ $a = 1$ $b = e$
$f(a) = 0$ $f(b) = 1$
and function $f(x)$ is continuous in $[1, e]$
$f(x)$ is differentiable in $(1, e)$.
$\therefore$ by LMVT there exist at least one $c \in (1,e)$
such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

$= \frac{1-0}{e-1} = \frac{1}{e-1}$

$f'(c) = 1/c$ $\left(\begin{array}{l} f(x) = \log x \\ f'(x) = 1/x \end{array} \right)$

$\frac{1}{c} = \frac{1}{e-1}$

$c = e - 1 \in (1, e)$

$y = \log x$

Q. Apply LMVT to show that $\frac{h}{1+h^2} < \tan^{-1}h <$

where $h > 0$

Let $f(x) = \tan^{-1}x$ in $[0, h]$

$f(x)$ is continuous and differentiable

$\therefore$ by LMVT ~~at~~ there exist at least one point $c \in (0, h)$ such that $f'(c) = \frac{f(b) - f(a)}{b-a}$

$$\frac{1}{1+c^2} = \frac{f(h) - f(0)}{h} \qquad \left(\begin{array}{l} f(x) = \tan^{-1}x \\ f'(x) = \frac{1}{1+x^2} \end{array} \right)$$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}h}{h} \quad \text{—— (1)}$$

as $c \in (0, h)$ i.e. $0 < c < h$

$$0 < c^2 < h^2$$

$$1 < 1+c^2 < 1+h^2$$

$$1 > \frac{1}{1+c^2} > \frac{1}{1+h^2}$$

$$1 > \frac{\tan^{-1}h}{h} > \frac{1}{1+h^2} \quad \text{from (1)}$$

$$\frac{1}{1+h^2} < \frac{\tan^{-1}h}{h} < 1$$

$$\frac{h}{1+h^2} < \tan^{-1}h < h$$

Q. Apply LMVT to show that $0 < \frac{1}{x} \log\left(\frac{e^x - 1}{x}\right) < 1$ for $x > 0$.

$$\log\left(\frac{e^x-1}{x}\right)$$

$$\frac{1}{x} \log\left(\frac{e^x-1}{x}\right)$$

$$\frac{1}{x}\left(\log(e^x - 1) - \frac{1}{x}x \log x\right)$$

$$\frac{1}{x} \log\left(\frac{e^x-1}{x}\right) \frac{1}{x^2}$$

$$\frac{1}{x}$$

$$\frac{1}{x} - \frac{1}{x^2} \log\left(\frac{e^x-1}{x}\right).$$

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Let $f(x) = e^x$ in $[0, x]$

$f(x)$ is continuous and differentiable

by LMVT $\exists$ at least one point $c \in (0, x)$

such that $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

$$f'(c) = \frac{f(x) - f(0)}{x}$$

$$e^c = \frac{e^x - 1}{x} \qquad \left(\begin{array}{l} f(x) = e^x \\ f'(x) = e^x \end{array} \right)$$

$$e^c = \frac{e^x - 1}{x} \quad \text{—— (1)}$$

as ~~$c \in (0, x)$~~ $0 < c < x$.

$$c = \log\left(\frac{e^x - 1}{x}\right) \quad \text{—— (1)}$$

as $c \in (0, x)$ $0 < c < x$

$$0 < \log\left(\frac{e^x - 1}{x}\right) < x \qquad \text{- from (1)}$$

$$0 < \frac{1}{x} \log\left(\frac{e^x - 1}{x}\right) < 1$$

Q. If A and B are any two real numbers ~~then $\exists$~~ show that there $\exists$ at least one real number $c \in (a, b)$ such that $b^2 + ab + a^2 = 3c^2$

Let

$f(x) = x^3$ which is continuous and differentiable in (a, b)

by LMVT $\exists$ c ~~such th~~ $\in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$3c^2 = \frac{b^3 - a^3}{b - a}$$

$$3c^2 = \frac{(b-a)(b^2 + ab + a^2)}{(b-a)}$$

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$3c^2 = b^2 + ab + a^2$

Q. show that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$

$a < b$ hence deduce that $\pi/4 + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \pi/4 + \frac{1}{6}$

here $f(x) = \tan^{-1}x$ which is continuous and differentiable in (a, b)

by LMVT $\exists\ c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b-a}$

here $f(x) = \tan^{-1}x$

$f'(x) = \frac{1}{1+x^2}$

$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$ — (1)

as $a < c < b$.

$a < c < b$

$a^2 < c^2 < b^2$

$1+a^2 < 1+c^2 < 1+b^2$

$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$

i.e. $\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$

$\frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$ from (1)

$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$

take $b = 4/3 \quad a = 1$

$\frac{4/3 - 1}{1 + 16/9} < \tan^{-1}4/3 - \tan^{-1}1 < \frac{4/3 - 1}{1+1}$

$\frac{1/3}{25/9} < \tan^{-1}4/3 - \pi/4 < \frac{1/3}{2}$

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$\frac{3}{25}+\frac{\pi}{4} < \tan^{-1}4/3 < \frac{1}{6}+\pi/4$

Q. Apply LMVT to show that $\frac{a-b}{a} < \log\left(\frac{a}{b}\right) < \frac{a-b}{b}$

if $a > b > 0$

Let $f(x) = \log x$ which is continuous and differentiable in (b, a).

by LMVT $\exists$ $c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

here, $f(x) = \log x$

$f'(x) = 1/x$

$\frac{1}{c} = \frac{\log a - \log b}{a-b}$

$\frac{1}{c} = \frac{\log(b/a)}{a-b}$ — (1)

$c \in (b, a)$

$b < c < a$

$\frac{1}{b} > \frac{1}{c} > \frac{1}{a}$

$\frac{1}{a} < \frac{1}{c} < \frac{1}{b}$

from (1)

$\frac{1}{a} < \frac{\log(b/a)}{a-b} < \frac{1}{b}$

$\frac{a-b}{a} < \log\left(\frac{a}{b}\right) < \frac{a-b}{b}$

Q. Apply LMVT to show that $1 < \frac{\sin^{-1}x}{x} < \frac{1}{\sqrt{1-x^2}}$

$0 < x < 1$

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here $f(x) = \sin^{-1}x$ is continuous and differentiable in $(0, x)$

by LMVT $c \in (0, x)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

$f(x) = \sin^{-1}x$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$

$\frac{1}{\sqrt{1-c^2}} = \frac{\cancel{\sin^{-1}b - \sin^{-1}a} + \sin^{-1}x}{(0, x)}$ — ①

$c \in (0, x)$

$0 < c < x$

$0 < c^2 < x^2$

$1-0 > 1-c^2 > 1-x^2$

$1-0 > \sqrt{1-c^2} > \sqrt{1-x^2}$

$1\ \cancel{1-0} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-x^2}}$

from ①

$1 < \frac{\sin^{-1}x}{x} < \frac{1}{\sqrt{1-x^2}}$

Q. Show that $0 < \frac{1}{\log(1+x)} - \frac{1}{x} < 1$, $x > 0$

here $f(x) = \log(1+x)$ which continuous and differentiable in $(0, x)$.

by LMVT $c \in (0, x)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

$f'(c) = \frac{\log(1+x)}{x}$

$f(x) = \log(1+x)$

$f'(x) = \frac{\cancel{1}}{1+x}$ $\log x$

$\frac{1}{1+c} = \frac{\log(1+x)}{x}$

$f(x) = \frac{x}{1+\theta x}$

$\log(1+x) = \frac{x}{1+\theta x}$

$0 < c < x$

$0 < \theta x < x$

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$1 < 1+c < 1+x$

$1 > \frac{1}{1+c} > \frac{1}{1+x}$

from ①

$1 > \frac{\log(1+x)}{x} > \frac{1}{1+x}$

$x > \log(1+x) > \frac{x}{1+x}$

$\frac{1}{x} < \frac{1}{\log(1+x)} < \frac{1+x}{x}$

$0 < \frac{1}{\log(1+x)} - \frac{1}{x} < \frac{1+x}{x} - \frac{1}{x}$

Q. prove that $\log(1+x) = \frac{x}{1+\theta x}$ where $0<\theta$

hence deduce that $\frac{x}{x+1} < \log(1+x) < x$

let $f(x) = \log(1+x)$ which is continuous an differentiable in $(0,x)$

by 2^{nd} form of LMVT in $(0,x)$

there exist $\theta \in (0,1)$ such that

$f(a+h) = f(a) + h \cdot f'(a+\theta h)$

here we consider interval (a, x)

$a=0 \quad h = x = b-a$

$f(x) = f(0) + x \cdot f'(\theta x)$ — ①

Now $f(x) = \log(1+x)$

$f(0) = 0$

$f'(x) = \frac{1}{1+x}$

$f'(\theta x) = \frac{1}{1+\theta x}$

from eq^n ①

$f(x) = f(0) + h \cdot f'(\theta x)$

$$\log(1+x) = \frac{x}{1+\theta x}$$

as $\theta \in (0,1)$ i.e. $0 < \theta < 1$

$$0 < \theta x < x$$

$$1 < 1+\theta x < 1+x$$

$$1 > \frac{1}{1+\theta x} > \frac{1}{1+x}$$

$$x > \frac{x}{1+\theta x} > \frac{x}{1+x}$$

$$x > \log(1+x) > \frac{x}{1+x}$$

$$\frac{x}{1+x} < \log(1+x) < x$$

$0 < \theta < 1$

$0 < \theta x < x$

$(0,x)$

$f(x) = e^x$

$f'(x) = e^x$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$e^c = \frac{e^x - 1}{x}$

$c = \log\left(\frac{e^x-1}{x}\right)$

Q. show that for any $x \geq 0$ $1+x < e^x < 1+xe^x$

here $f(x) = e^x$ which continuous and differential $(0,x)$

by LMVT $\exists$ $c \in (0,x)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$e^c = \frac{e^x - 1}{x}$$

$$c = \log\left(\frac{e^x-1}{x}\right)$$

$$0 < c < x$$

$$0 < \log\left(\frac{e^x-1}{x}\right) < x$$

$$1 < \frac{e^x-1}{x} < e^x$$

$0 < \log\left(\frac{e^x-1}{x}\right) < x$

$1 < \frac{e^x-1}{x} < e^x$

$x < e^x - 1 < xe^x$

$1+x < e^x < 1+xe^x$

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$x < e^x - 1 < xe^x$

$1+x < e^x < 1+xe^x$

OR

Let $f(x) = \cancel{1+x}\ e^x - (1+x)$ $\quad x \geq 0$

$f'(x) = e^x - 1 \quad x \geq 0$

and as $f'(x) > 0 \quad f(0) = 0$

$\Rightarrow$ f is increasing function.

$f > 0$

$e^x - (1+x) > 0$

$e^x > 1+x$

$1+x < e^x$ — ①

similarly consider $g(x) = 1+xe^x - e^x$

$g'(x) = \cancel{1+}\ xe^x + e^x - e^x$

$g'(x) = xe^x$

$g(0) = 0$

g is increasing

$g > 0$

$1+xe^x - e^x > 0$

$1+xe^x > e^x$

$1+x < e^x < 1+xe^x$

Q. Show that for $x \in (0, 1)$ $x < -\log(1-x) < \frac{x}{1-x}$

here ~~$f(x) = \log(1-x)$~~

We have to prove that $x < -\log(1-x) < \frac{x}{1-x}$

$-x > \log(1-x) > \frac{-x}{1-x} = \frac{x}{x-1}$

here $f(x) = \log(1-x)$

continuous and differentiable in $(0, x)$

by LMVT $c \in (0, x)$ s.t. $f'(c) = \frac{f(b) - f(a)}{b-a}$

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$$\frac{-1}{1-c} = \frac{\log(1-x)}{x} \quad \text{--- (1)}$$

$$0 < c < x$$

$$1-0 > 1-c > 1-x$$

$$-1 > \frac{-1}{1-c} > \frac{-1}{1-x}$$

$$-1 > \log\left\{\frac{(1-x)}{(x)}\right\} > \frac{-1}{1-x} \quad \text{--- from (1)}$$

$$-x > \log(1-x) > \frac{-x}{1-x}$$

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Type-3
Cauchy's mean value theorem-

If i] $f(x)$ and $g(x)$ is continuous function in $[a, b]$

ii] $f(x)$ & $g(x)$ be differentiable in (a, b).
and $g'(x) \neq 0$ for any value of $x \in (a, b)$
then $\exists$ at least one value c which belongs to (a, b) such that $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$

Proof -
1^{st} of all we note that denominator of RHS is not zero. i.e. $g(b) - g(a) \neq 0$.
[If $g(b) - g(a) = 0$ then $g(b) = g(a)$ here
g is continuous and differentiable so by Rolle's th^m there exist $d \in (a, b)$ such that $g'(d) =$
which is contradiction to hypothesis $g'(x) \neq 0$]

Now consider $\phi(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)} g(x)$.

here $f(x)$, $g(x)$ are continuous
$\therefore$ $\phi(x)$ is also continuous
$f(x)$ and $g(x)$ are differentiable in (a, b)
$\therefore$ $\phi(x)$ is also differentiable.

$$\phi(a) = f(a) - \frac{f(b) - f(a)}{g(b) - g(a)} g(a)$$

$$= \frac{f(a)g(b) - f(a)g(a) - g(a)f(b) + f(a)g(a)}{g(b) - g(a)}$$

$$= \frac{f(a)g(b) - g(a)f(b)}{g(b) - g(a)}$$

$\frac{1}{v^2} = x^3$.

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$$\phi(b) = f(b) - \frac{f(b) - f(a)}{g(b) - g(a)} g(b).$$

$$= \frac{f(b)g(b) - f(b)g(a) - g(b).f(b) + f(a)g(b)}{g(b) - g(a)}$$

$$= \frac{f(a).g(b) - g(a).f(b)}{g(b) - g(a)}.$$

i.e. $f(a) = f(b)$.

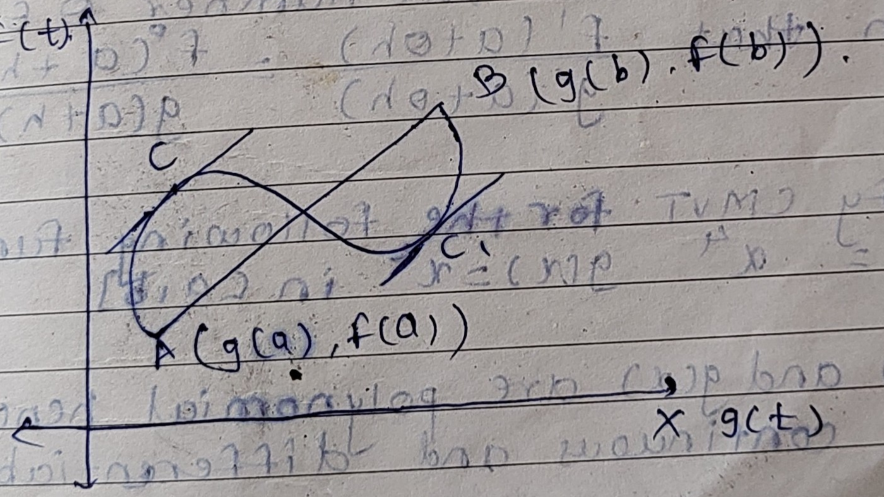
conditions of Rolle's th^m are satisfied.

$\therefore \exists$ at least one point $c \in (a,b)$ such that

$$\phi'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) = 0$$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

⊙ Geometrical interpretation Cauchy's MVT -



suppose that a curve γ is discribed by parametric eq^n $x = g(t)$ $y = f(t)$. Where the parameter ranges in the interval $[a,b]$ i.e. $t \in [a,b]$.

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When changing the parameter t the point of curve runs from $A \equiv (g(a), f(a))$ $B \equiv (g(b), f(b))$ according to th^m there is at least one point c on the curve & where tangent is $\parallel^{el}$ to chord joining the ends points A and B of curve

⊙ If $g(x) = x$ then CMVT reduced to LMVT
⊙ If $g(x) = x$ and $f(b) = f(a)$ then CMVT reduces to Rolle's th^m.

⊙ Alternative form of CMVT -
Take $b = a+h$ then for $a < c < b$ &
$c = a + \theta h$ provided with $0 < \theta < 1$
then CMVT may be stated as
i] $f(x)$ & $g(x)$ are continuous in $[a, a+h]$
ii] $f(x)$ & $g(x)$ are differentiable in $(a, a+h)$
then ∃ at least one number $\theta \in (0,1)$
such that $\dfrac{f'(a+\theta h)}{g'(a+\theta h)} = \dfrac{f(a+h) - f(a)}{g(a+h) - g(a)}$

Q. verify CMVT for the following function.
i] $f(x) = x^4$ $g(x) = x^2$ in $[a, b]$

$f(x)$ and $g(x)$ are polynomial hence both are continuous and differentiable.

~~$f(x) = x^4$, $g(x) = x^2$~~

i.e. conditions of CMVT are satisfied so
∃ at least one point $c \in (a, b)$.
Such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b) - f(a)}{g(b) - g(a)}$

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$$\frac{4c^3}{2c} = \frac{b^4 - a^4}{b^2 - a^2}$$

$$2c^2 = \frac{(b^2-a^2)(b^2+a^2)}{b^2-a^2} = b^2+a^2$$

$$c^2 = \frac{b^2+a^2}{2}$$

$$c = \pm\sqrt{\frac{a^2+b^2}{2}} \in (a,b)$$

Q. $f(x) = \log(x)$ $g(x) = 1/x$. in $[1, e]$.

$f(x)$ ~~is to~~ $g(x)$ is continuous and differentiab[le]
i.e. conditions of CMVT are satisfied s[o]
$\exists$ at least one point $c \in (1, e)$.
such that $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$.

$$\therefore \frac{1/c}{-1/c^2} = \frac{1}{1/e - 1}$$

$$\frac{1}{c} \times \frac{c^2}{-1} = \frac{1}{1-e} \times e$$

$$-c = \frac{e}{1-e}$$

$$c = \frac{e}{e-1} \in (1, e).$$

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Q. If $f(x) = x^2$, $g(x) = x$ $x \in [a, b]$. Verify CMVT and show that the value of c is the Airthematic mean of a and b.

here, f(x) & g(x) are continuous polynomial function hence they are continuous & differentiable $g'(x) = 1 \neq 0$

by CMVT there exist at least one point $c \in (a, b)$ s.t. $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

$$\frac{2c}{1} = \frac{b^2 - a^2}{b - a} = \frac{(b-a)(b+a)}{(b-a)}$$

$$2c = b + a$$

$$c = \frac{b+a}{2}$$

Which is Airthmatic mean of a & b.

Q. $f(x) = e^x$, $g(x) = e^{-x}$ $x \in [a, b]$ verify CMVT.

here f(x) & g(x) are exponential function hence continuous and differentiable.

$g'(x) = -e^{-x} \neq 0$

by CMVT $\exists$ at least one point $c \in (a, b)$

s.t. $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

$$\frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$\cancel{2c\, e^{2c} = \frac{b-a}{b+a}}$$

$$\cancel{c = \frac{b-a}{}}$$

$$e^{2c} = \frac{e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}}$$

$$e^{2c} = \frac{(e^b - e^a)(e^b \cdot e^a)}{(e^a - e^b)}$$

$$e^{2c} = e^{a+b}$$

$$c = \frac{a+b}{2}$$

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$\tan^{-1}x$

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Q. Apply CMVT to prove that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ $\quad 0 < a < b$.

Let,
$f(x) = \tan^{-1}x \qquad g(x) = x$ in $[a,b]$ here $f(x)$ & $g(x)$ continuous & differentiable

~~here~~ by CMVT $\exists$ at least one point $c \in (a,b)$

s.t. $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$ —— ①

$f(x) = \tan^{-1}x$

$f'(x) = \frac{1}{1+x^2} \qquad g(x) = x$

$\qquad\qquad\qquad\quad g'(x) = 1$

from ①

$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$ —— ②

as $c \in [a,b]$

$a < c < b$

$a^2 < c^2 < b^2$

$1+a^2 < 1+c^2 < 1+b^2$

$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$

from ②.

$\frac{1}{1+a^2} > \frac{\tan^{-1}b - \tan^{-1}a}{b-a} > \frac{1}{1+b^2}$

$\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$

It is proved.

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verify CMVT

Q. $f(x) = \sin x$ $g(x) = \cos x$ $[-\pi/2, 0]$

$f(x)$ & $g(x)$ are continuous & differentiable.
$g'(x) = -\sin x \neq 0$
by CMVT there exist at least one pt $c \in$
s.t. $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

$$\frac{\cos c}{-\sin c} = \frac{1}{1}$$

$$-\cot c = 1$$

$$\cot c = -1$$

$$c = -\pi/4 \in (-\pi/2, 0)$$

Q. verify CMVT for the function $f(x) = 2x+1$ $g(x) = 3x-4$ in $(1, 3)$
$f(x)$ & $g(x)$ are continuous & diffrentiable
$g'(x) = 3 \neq 0$ $\therefore \exists\, c \in (1, 3)$ $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

$\therefore$ CMVT is satisfied for every point in the interval $(1, 3)$.

Q. By CMVT show that $\frac{\sin b - \sin a}{e^b - e^a} = \frac{\cos c}{e^c}$ a<c<b
hence deduce that $e^c \sin x = \cos c\,(e^x - 1)$.

$f(x) = \sin x$ $g(x) = e^x$ both are continuous and differentiable
$g'(x) = e^x \neq 0$
$\therefore$ by CMVT $\exists$ at least one point $c \in (a, b)$
s.t. $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

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$$\frac{\cos c}{e^c} = \frac{\sin b - \sin a}{e^b - e^a}$$

It is proved

put $a = 0$ $b = x$.

$$\frac{\cos c}{e^c} = \frac{\sin x}{e^x - 1}$$

$\sin x \cdot e^c = \cos c(e^x - 1)$

It is proved.

Q. Prove that $\frac{\sin b - \sin a}{\cos a - \cos b} = \cot c$ hence deduce that $c = x/2$ $(a < c < b)$

$f(x) = \sin x$ $g(x) = \cos x$ are continuous & differentiable $g'(x) = -\sin x \neq 0$

$\therefore$ by CMVT $\exists$ at least one pt $c \in (a, b)$ s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\frac{\cos c}{-\sin c} = \frac{\sin b - \sin a}{\cos b - \cos a}$$

$$\cot c = \frac{\sin b - \sin a}{\cos a - \cos b}$$

It is proved.

$a = 0$ $b = x$

$$\cot c = \frac{\sin x}{1 - \cos x}$$

$$\cot c = \frac{\sin x}{2\sin^2 x/2} = \frac{2\sin x/2 \cdot \cos x/2}{2\sin^2 x/2}$$

$\cot c = \cot x/2$ $\boxed{c = x/2}$ It is proved.

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