

Mathematical Analysis of 2048, The Game

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Abstract

The game 2048 took the internet by storm, spawning countless rip-offs. People around the world have poured millions of hours trying to create the 2048 tile. Besides the addictive game play, the game also presents an interesting opportunity to explore mathematics. The paper attempts to analyse the game mathematically, through Mathematical Induction, Number Theory, Fuzzy Theory and Topology, in the process also attempting to find the optimal strategy to ensure victory.

Keywords: Mathematical Analysis, 2048, Pascals Triangle, Optimization

INTRODUCTION

2048 is sliding block puzzle game developed by Gabriele Cirulli. It's a game played on a 4×4 grid with tiles numbered 2^n where 'n' represents a natural number. The objective of the game is to combine tiles of the same number to eventually form the number 2048. The user can move in the four cardinal directions and after every move a new tile is generated randomly in the grid which is either numbered 2 or 4 with a probability of about .10. A move is legal if at least one tile can be slid into an empty spot or if the tiles can be combined in the chosen direction. The game ends when the user does not have any legal moves left.

One cannot help but ask the question, since the game is based on mathematics, whether moves can be optimised to improve our score by applying different concepts of mathematics.

Mathematical Induction

At the start of the game, we are given two tiles in random locations. The tiles could be either numbered 2 or 4 with the probability as discussed earlier. By observation, one can see that the grid can only contain numbers which are of the form 2^n . We can also prove this mathematically through the use of mathematical induction.

Step 1: The tiles given in the beginning are both 2s, one 2 one 4 or both 4s. In all, there are $2!2! - 1 = 3$ cases. The numbers here can be represented in the form 2^n .

Step 2: Let us assume that after 'k' steps, the numbers in the grid can be represented in the form of 2^n .

Step 3: In $(k+1)^{th}$ step, the following cases arise-

- I. We slid the tile into an empty spot, no tiles combined with one another
- II. We combined two or more existing tiles with one another
- III. A combination of cases 1 and 2

In all the three cases, a new tile will be randomly generated, which will be numbered 2 or 4 and by default can be represented as a power of 2.

In the first case, we have the only one new number in addition to the ones at k^{th} step which could all be expressed in powers of 2. In the other case, since the tiles numbered 2^r would combine to make tile numbered 2^{r+1} , all the tiles could still be represented in powers of 2.

So we can say that at any point of time the grid will only have numbers which can be represented in the form of 2^n .

Maximal tile

From the previous discussion, we know that all the tiles are of the form 2^n , so the maximal tile will also be of the same form.

Let us assume 2^r is the maximal tile where $r \in \mathbb{N}$. To create this tile, 2 tiles of 2^{r-1} are required, to create tile of 2^{r-1} , two tiles of 2^{r-2} are required and so on. The number of tiles is limited by the space on the grid which is 16 spaces, which means $r = 16$. So $2^{16} = 65536$ is the maximal tile if we get 2 at the end.

Assuming that we get lucky and get tiles numbered 4 towards the end; 2^{r+1} will be the maximal tile.

So $2^{17} = 131072$ will be the maximal tile.

Maximum possible sum (S_n)

All the tiles are of the form 2^n . Assuming the best case scenario that there are no two tiles of the same number and that we have the highest numbered tiles in the grid that is possible, one can easily see that they will form a Geometric Progression (GP). Also, since we are considering best case scenario, we can take the lowest number to be 4 instead of 2. The sum, S_n will be given by-

$$\begin{aligned} S_n &= 2^2 + 2^3 + 2^4 + \dots + 2^{17} \\ &= 2(2^{17}-1)/(2-1) - 2^1 \\ &= 262142 - 2 \\ &= 262140 \end{aligned}$$

If 2 is the last tile instead of 4, then the sum would be

$$\begin{aligned} S_n &= 2^1 + 2^3 + 2^4 + \dots + 2^{17} \\ &= 2(2^{17}-1)/(2-1) - 2^2 \\ &= 262142 - 4 \\ &= 262138 \end{aligned}$$

Or we can just do-

$$\begin{aligned} S_{n'} &= 262140 - 2 \\ &= 262138 \end{aligned}$$

MINIMUM NUMBER OF TURNS TO WIN (T_{\min})

The game is won by creating the 2048 tile (although one can choose to keep playing till no more legal moves are left).

Since the tiles are randomly generated; 2 with a probability of 0.9 ($P_2 = 0.9$) and 4 with a probability of 0.1 ($P_4 = 0.1$); we cannot give an exact number for T_{\min} , however we can give a range for it.

- I. Assuming the best-case scenario that all the tiles we get are of number 4 and that every move we make results in the merging of 2 tiles into 1.

Now, to make 2048 we need at least 512 tiles ($512 \times 4 = 2048$). But we start with 2 tiles so that means we need $512 - 2 = 510$ turns so that we get all the required tiles. The last tile we get will have to be combined to make 8, then 8s into 16, 16s into 32 and so on until we combine 1024s into 2048

So we get,

$$\begin{aligned} T_{\min} &= 512 - 2 + 9 \\ &= 519 \text{ turns} \end{aligned}$$

The probability of getting all 4 numbered tiles is

$$0.1^{519} = 10^{-519}$$

- II. Consider a case in which we get all tiles numbered 2 rather than 4. Proceeding as above, we need 1024 tiles to make 2048 and then combine the last tile we get with another 2 numbered tile to eventually get 2048.

In this case

$$\begin{aligned} T_{\min} &= 1024 - 2 + 10 \\ &= 1032 \text{ turns} \end{aligned}$$

The probability of getting all 2 numbered tiles is $0.9^{1032} \sim 6.0 \times 10^{-48}$

So the minimum number of turns required to win the game, assuming we play a perfect game is in the range of [519, 1032].

Fuzzy Theory

If a player plays the game, will he be able to confidently say that he can win the game? Furthermore if "n" number of players start playing at the same time, can any or all say that they can win the game?

Now if we consider the same with "n" number of players, "m" times, then again the answer on victory over the game before finishing it is not sure.

Consider $S = \{P_1, P_2, P_3, \dots, P_{11}\}$ is a set of eleven player who played the game.

P1 fails to win but he reached 32

P2 fails to win but he reached 64

P3 fails to win but he reached 32

P4 fails to win but he reached 128

P5 fails to win but he reached 512

P6 fails to win but he reached 256

P7 fails to win but he reached 128
 P8 fails to win but he reached 1024
 P9 fails to win but he reached 512
 P10 fails to win but he reached 256
 P11 fails to win but he reached 64

Now consider P8, who was closer to victory compared to rest except P9. Similarly, P5 was closer to victory compared to rest except P9 and P8. Now between P1 and P2, P1 is less close to victory than P2.

From the above discussion, we find that the victory over the game is not a sure event but we can talk about the degree of win for each player, giving us a glimpse of 'Fuzzy Theory' which helps us to know to which extent player was close to win.

Basically, Fuzzy theory enables us to know the degree of truth of an event (whether precise or imprecise) at any defined instant.

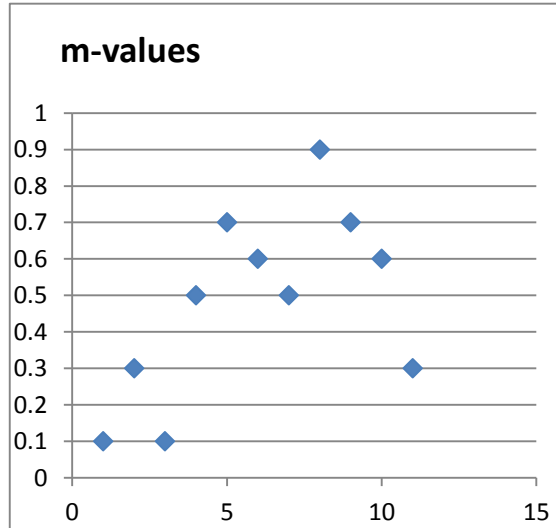
Define Membership function $m: S \rightarrow [0, 1]$ as given in Table 1.

Given below is a graph (Graph 1), in which X-axis represents player and Y-axis represents their corresponding membership values

Table 1: Membership values of Players

Player	Membership value (m-value)
1	0.1
2	0.3
3	0.1
4	0.5
5	0.7
6	0.6
7	0.5
8	0.9
9	0.7
10	0.6
11	0.3

Graph 1: m Values of corresponding players



From above fuzzy graph we can see that no payer has m-value 1, that is, no one wins but we can easily examine how close the players are to winning and we can even compare their closeness of victory.

Topology for optimality

To win the game, we must combine similar numbered tiles which combine to form a bigger numbered tile for example, two tiles numbered 2^n can be combined to form a tile numbered 2^{n+1} . Since we can only move in the four cardinal directions, the two tiles

must be adjacent to each other, either horizontally or vertically. This connects the game with topology.

In an ideal configuration, the tiles must not only be adjacent to similar numbered tile but the higher numbered tile so formed must be adjacent to another tile of the same number and so on with the highest numbered tile being left in the end. It is preferable to have monotonicity in the grid along a line. The position of head should be at a place where it can combine with other tiles easily without the new randomly generated tile breaking the chain midway. In other words, it would be better that the highest numbered tile is at the corner of the grid. The resulting configuration would be a snake line (Image 1).

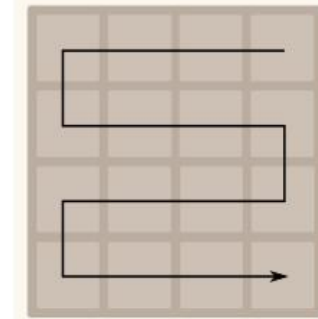


Image 1: Snake line through the grid ^[1]

However, practically such a configuration is hard to achieve due to non-deterministic nature of the game. The tiles will not land always exactly where we need them to be. So this can be classified as a case for Best-Case Optimality. Since we are actually combining two adjacent tiles to form a new higher number, which is actually the sum of the two tiles, it is similar to Pascal Triangle (Image 2).

The snake line (Hoang, 2014) in the above discussion, sounds good but is hard to replicate since it restricts the movement to only a single direction. But in Pascal's triangle, we need only restrict two directions and can even use third one although sparingly.

Experimenting with this, we find the gameplay not only smoother but are also able to create the 2048 tile with ease and win the game (Image 3). This approach is closer to the stochastic optimality for the original game.

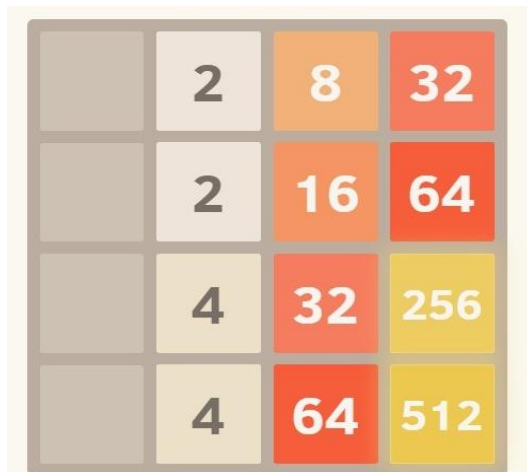


Image 2: Pascal triangle overlaid on grid



Image 3: Gameplay using optimality condition

For Robust optimisation however, this approach is also impractical since the tile is no longer randomly generated. It will pop up in places which are least optimal according to minimax tree. In this case though the concept remains same and following the Pascal triangle as closely as possible, the best score that I got was 1176 (Image 4).

CONCLUSION

The paper has successfully shown that behind the simple game, there lies a world of mathematics to explore. Mathematical concepts helped us to analyse the game and understand how to form an optimum strategy and maximise our score. The above analysis can be taken forwarded to develop an Artificial Intelligence Agent to maximise the score.

This paper is an attempt at showing the game in different light and help students to learn mathematics in an interesting way by applying the concepts learned in classroom. At the same time, it is hoped that it will inspire researchers to explore the relatively newer and emerging fields for applied mathematics.



Image 4: Gameplay for Robust Optimisation

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