• Vertices
— Edge

23/5/23
Thursday

GRAPH THEORY

GRAPHS & GRAPH MODELS:

• **GRAPH:**

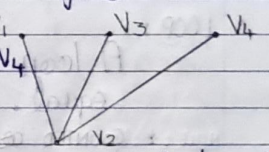
$G(V, E)$ — Vertex Set, Edge Set

A graph $G(V, E)$ is a non-empty set of vertices and edges.

• **ADJACENT VERTICES:**

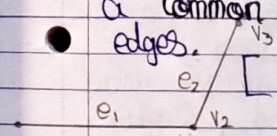
Any pair of vertices are incident with a common ~~vertex~~ edge are called adjacent vertices.

[Adjacent vertices: V_1V_2, V_2V_3, V_2V_4
Non-adjacent vertices: V_1V_3, V_3V_4, V_1V_4



• **ADJACENT EDGES:**

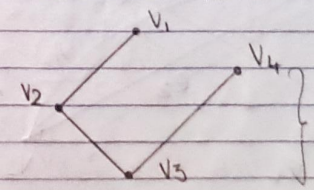
If two distinct edges are incident with a common vertex they are called adjacent edges.



[~~ADJACENT VERTICES~~ e_1 and e_2 are adjacent edge]
Since they have a common vertex V_1 in common

• **ISOLATED VERTEX:**

In any graph, a vertex which is not adjacent to any other vertex is called isolated vertex.



• V_5 DEG = 0 $\deg(V_5) = 0$

[V_5 is the isolated vertex]

disconnected graph have more than 1 compound.

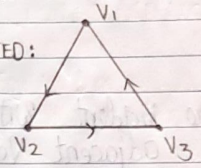
! Isolated Vertex exist only in disconnected graph ⇒ which has multiple COMPONENTS

- DIRECTED GRAPH AND UNDIRECTED GRAPH:

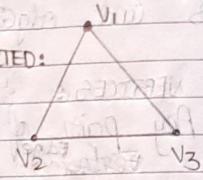
→ A graph in which every edge is directed is called directed graph.

→ A graph which is undirected is called undirected graph.

DIRECTED:



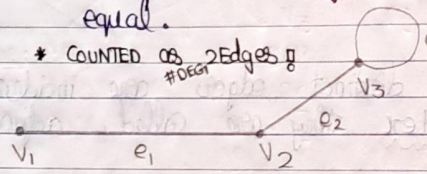
UNDIRECTED:



- LOOP:

A loop is an edge whose Vertices are equal.

* COUNTED as 2 Edges !! #DEGT

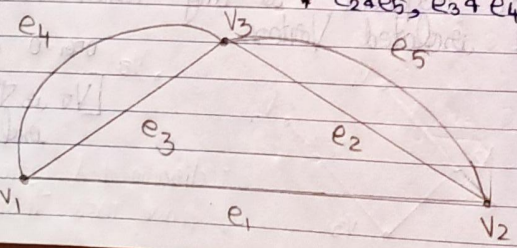


e_3 → is a loop

- PARALLEL EDGES: (MULTIPLE EDGES):

Edges having same pair of Vertices are called $\|^{el}$ edges.

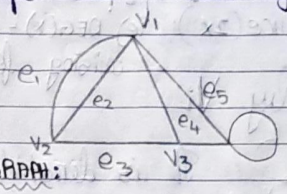
* e_2 & e_5, e_3 & e_4 are $\|^{el}$ Edges



- MULTI-GRAPH:

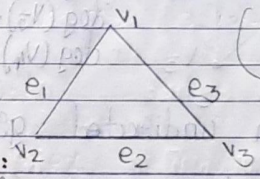
Any graph which contains some $\|^{el}$ edges (or) loop are multi graph.

ie) multiple EDGES are allowed between Vertices ⇒ They will have $\|^{el}$ EDGES



- SIMPLE GRAPH:

It is a graph having no loops or multiple edges. It's max is n-1.



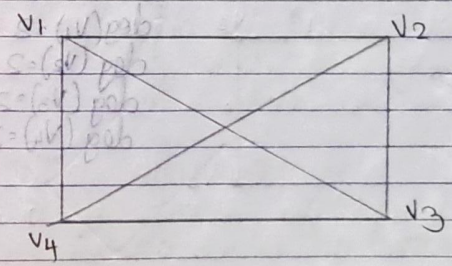
NO * LOOPS
* EDGES (multiple)

If n vertices. Max possible degree is

- PSEUDO GRAPH:

Any graph which contains some parallel edges and loops connecting the pair of vertices is called pseudo graph.

ie) MULTIPLE EDGES & LOOPS are NOT allowed



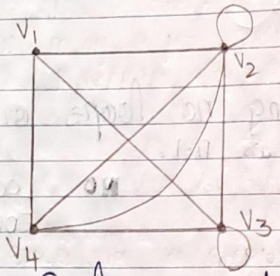
! CARDINALITY - no. of element in a set.

• DEGREE OF A VERTEX:

→ The degree of a vertex in an undirected graph is the no. of edges incident with it except the loop, which is counted as twice (2x) ie) DEG(x) = no. of edges arising from that vertex.

→ It is denoted by V_i

Eg:



It is denoted by (V_i)

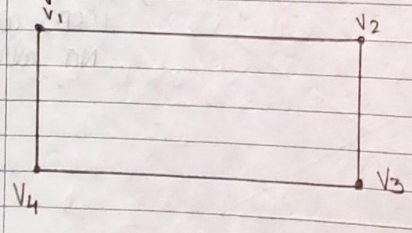
$deg(V_1) = 3$
$deg(V_2) = 6 = 4+2$
$deg(V_3) = 5 = 3+2$
$deg(V_4) = 4$

NOTE:
Let G be an undirected graph with |E| edges & |V| = n vertices then,

! $\sum_{i=1}^{n=|V|} deg(V_i) = 2|E|$

11- Cardinality

Eg:



$deg(V_1) = 2$
$deg(V_2) = 2$
$deg(V_3) = 2$
$deg(V_4) = 2$

$\sum_{i=1}^{n=|V|} deg(V_i) = 2|E|$

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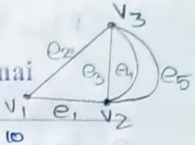
! Summation of all the Degree's in a GRAPH is equal to 2x the no. of EDGES.

Eg: $|V| = 3$ * $deg v_1 + deg v_2 + deg v_3 = 2|E| \Rightarrow 2+4+4 = 2|5| = 10$



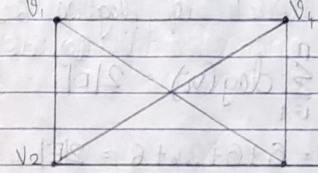
$\sum_{i=1}^{4} deg(V_i) = 2|E|$ $\quad |E| = 4$, $|V| = 4$ vertices

$2+2+2+2 = 2 \times 4$
$8 = 8$

PT-1

• * In any graph the no. of vertices of odd degree is always even
⇓
⊙ No of (odd DEGREE vertice) ! will be even.

ie) V_1, V_2, V_4, V_5

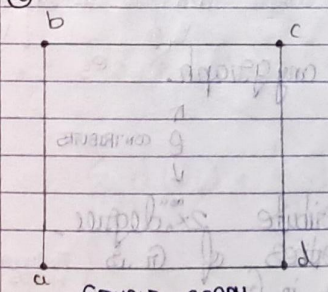


V_5 is a pendant vertex
ie) $deg(V_5) = 1$

• PENDANT - VERTEX:
→ A vertex of degree one is called pendant vertex.

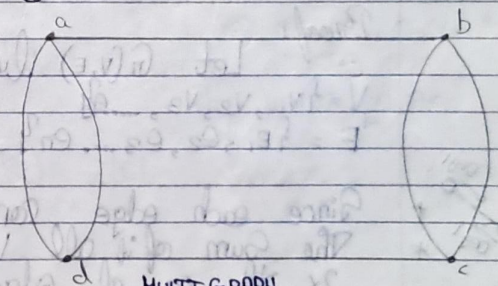
• PROBLEM:
① Mention the type of the following graphs:

ⓐ

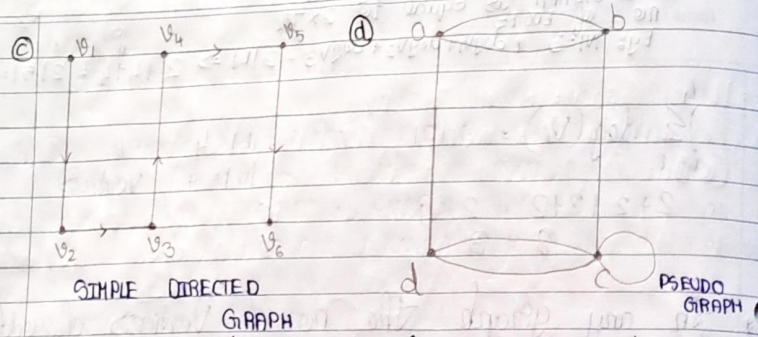


SIMPLE GRAPH

ⓑ



MULTI GRAPH



② How many edges are there in a graph with 10 vertices of each of degree 6.

$V = 10 = |V| = n$ $\quad \sum_{i=1}^{n} \deg(v_i) = 2|E|$

$\deg = 6$

$(6)(10) = 6+6+\dots+6 = 2|E|$

$60 = 2E$

$E = 30$

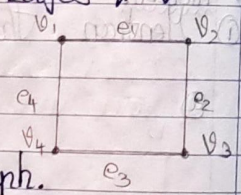
GRAPH TERMINOLOGY & SPECIAL TYPE OF GRAPHS:

IMPORTANT 2 MARK (%) THEOREM-1:

[HAND SHAKING THEOREM]

for any graph G with E Edges & V Vertices

$$\sum_{i=1}^{n} d(v_i) = 2|E|$$

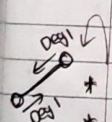


Proof:

Let $G(V,E)$ be any graph.

$V = \{v_1, v_2, v_3, \dots v_n\}$

$E = \{e_1, e_2, e_3 \dots e_n\}$



e CONTRIBUTES

Since each edge contribute 2x degree.

The sum of all vertices of G is 2x the no. of edges in G. } FORMULA IN WORDS !

$$\sum_{i=1}^{n} \deg(v_i) = 2|E|$$

APPLICABLE → Multiple Edges (2deg.)

→ Loops

NOTE: This theorem is applicable even if it has multiple edges and loops are present.

THEOREM-2: PT-1

The no. of odd deg Vertices in a graph is always even.

Proof:

Let $G = \{V, E\}$ graph with in number of Vertices and E Edges.

Let $V_1, V_2, V_3 \dots V_m$ be the Vertices of odd degree and $V_1', V_2', V_3' \dots V_n'$ be the Vertices of even degree.

WKT from HAND SHAKING THEOREM:

$$\sum_{i=1}^{n} d(v_i) = 2|E|$$

$$\underbrace{\sum_{i=1}^{m} d(v_i)}_{\text{odd}} + \underbrace{\sum_{i=1}^{n} d(v_i')}_{\text{even}} = \underbrace{2|E|}_{\text{even}}$$

The sum of n no. of even deg Vertices is always EVEN !

$$\sum_{i=1}^{m} d(v_i) + \text{Even} = \text{Even}$$

$$\sum_{i=1}^{m} d(v_i) = \text{Even}$$

$\Rightarrow m$ is even

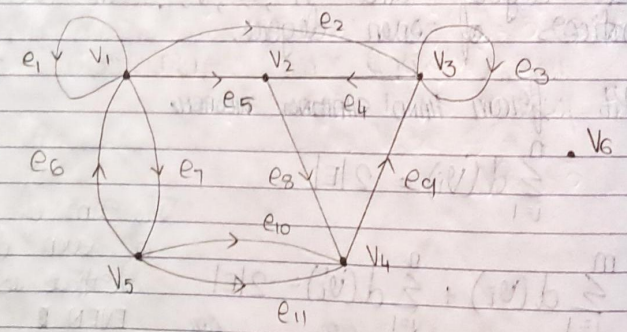
Hence proved //-

- **IN-DEGREE:**

In a directed graph G. The in-degree of V is denoted by $\deg^+ G(V)$ is the no. of edges ending with V. ie) $\deg^+(v_i)$

- **OUT-DEGREE:**

In a directed graph G, the out-degree of a vertex V of G is denoted by $\deg^- G(V)$ is no. of edges beginning with V. ie) $\deg^-(v_i)$



$\deg(v_i)$	IN-DEGREE $\deg^+(v_i)$	OUT-DEGREE $\deg^-(v_i)$
V_1	2	4
V_2	2	1
V_3	3	2
V_4	2	2
V_5	2	2
V_6	0	0

Here, $\sum_{i=1}^{n} \deg^+(v_i) = \sum_{i=1}^{n} \deg^-(v_i) = |E|$

$2+2+3+2+2+0 = 4+1+2+2+2+0 = 11$

- **DEGREE SEQUENCE:**

* A degree of each vertices arranged in ascending (or) descending order is called degree sequence.

③ Is there a simple graph corresponding to the following deg. sequence.

a) (1, 1, 2, 3)

By Theorem 2, the no. of odd degree vertices is always even.

* Here no. of odd degree vertices is → 3 which is not even. [#THEOREM-2]
* Hence no simple graph corresponds to the given degree sequence.

DELTA:

δ Δ

Small min deg. Big max deg.

NOTE: In a "Simple graph" with n vertices, each vertex is of max deg n-1.

$n = |V|$

b) Deg Sequence (6,3,2,1) Determine whether a simple graph is Possible or NOT.

* No. of Vertices, in this graphical Seq is 4. And the max degree of a vertex is 6 which is not possible as the max deg cannot exceed n-1

$n-1 \Rightarrow 4-1=3$

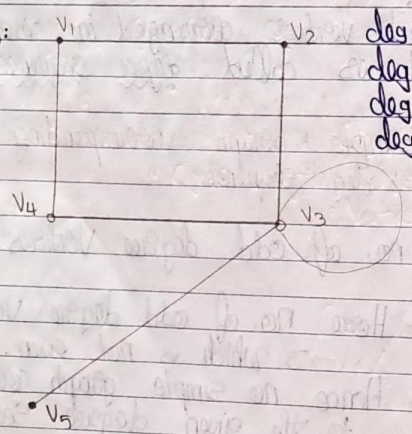
∴ It is not simple.

• MAXIMUM AND MINIMUM DEGREE OF A GRAPH:

$\delta(G) = \min\{\deg(v) / v \in V(G)\}$

$\Delta(G) = \max\{\deg(v) / v \in V(G)\}$

Eg:



$\deg(V_1) = 2$ $\deg(V_5) = 1$

$\deg(V_2) = 2$

$\deg(V_3) = 5 = 3+2$

$\deg(V_4) = 2$

clearly $\delta(G) \le \Delta(G)$

$\delta(G) = 1$

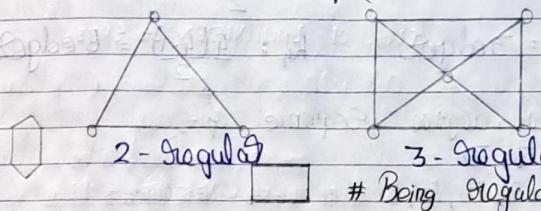
$\Delta(G) = 5$

K → DEG → K+1 (EDGES)

! K - regular graph

• REGULAR GRAPH:

* If $\delta(G) = \Delta(G) = K$ if each vertex of a graph G has degree 'K', Then G is a K-regular graph.



2 - regular 3 - regular

They have degree 'K' for each vertex "SAME DEGREE"

Being regular in every deg.

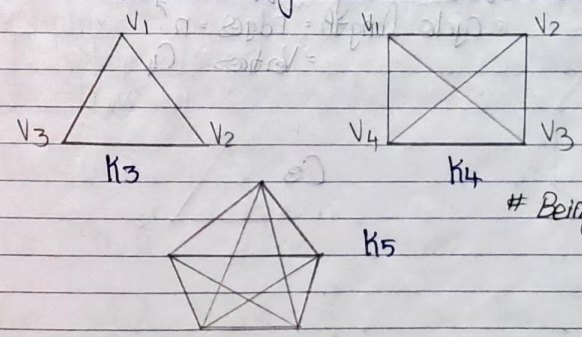
• COMPLETE GRAPH:

* All vertices must be connected to one another

* A simple graph in which every pair of distinct vertices are adjacent is called a complete graph.

* It is denoted by K_n.

EACH VERTEX IS ADJACENT TO ONE ANOTHER



K_3 K_4 K_5

Being completed

REGULAR GRAPH: SAME DEGREE FOR ALL VERTICES
ie) Said as K-regular graph! (deg)

COMPLETE GRAPH: All VERTICES ARE ADJACENT
ie) Said as K_n (n = NO OF VERTICES)

* K_n: is a (n-1)-regular graph.
→ Since each vertex is of degree (n-1)

K_n

∴ Degree sequence of K_n,
(n-1), (n-1), (n-1) ..... (n-1)

VERTICES & EDGES:
K_n has n-vertices and $\frac{n(n-1)}{2}$ edges

K_3: $\frac{3(3-1)}{2} = 3$ edges K_4: $\frac{4(4-1)}{2} = 6$ edges

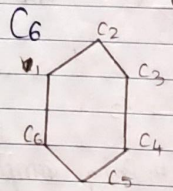
⑤ What is the degree sequence of

a) K_4 - 3, 3, 3, 3
b) K_6 - 5, 5, 5, 5, 5, 5
c) K_2 - 1, 1

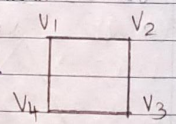
• CYCLE GRAPH:

* A Cycle graph of order 'n' is a Connected graph whose edge form a cycle of length 'n'
* It is denoted by C_n.
⊙ Cycle length = Edges = n (ORDER)
= Vertices

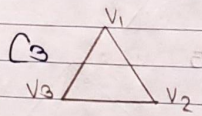
C_6



C_4



C_3



VERTICES, EDGES: In C_n: n-vertices, n-edges
All cycles are 2 regular graph.

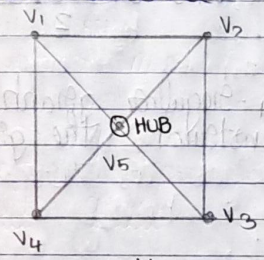
② Write the sequence of C_5. 2, 2, 2, 2, 2

• WHEEL GRAPH:

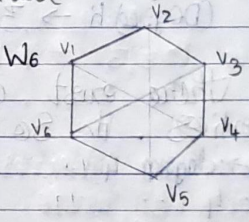
* A wheel graph of order 'n' is obtained by joining a new vertex called hub of the wheel and connect the all other vertices to the hub.
* It is denoted by W_n.

DEG: 3, n (HUB)
VER: n+1
EDG: 2n

W_4



W_6



⑥ How many vertices and edges in the following graph.

SUMMARY

	VERTICES	EDGES	
K_n	n	$\frac{n(n-1)}{2}$	"COMPLETE GRAPH"
W_n	(n+1)	2n	"WHEEL GRAPH"
C_n	n	n	"CYCLE GRAPH"

⑦ For what values of n are these graph are regular?

RG - Regular graph

* COMPLETE GRAPH is a K-regular graph deg n-1 ie) (n-1) - reg graph.
DEG

a) $K_n \rightarrow n \geq 1$ (COMPLETE GRAPH), bcoz, all VERTICES are adj, have same deg n-1, n-1-RG
= ORDER OF OUTER EDGES

b) $C_n \rightarrow n \geq 3$, bcoz min ORDER for cycle is 3, deg 2, 2-RG

c) $W_n \rightarrow n = 3$, bcoz this is the ONLY case where the hub deg. = deg of Vertex.

NOTE:

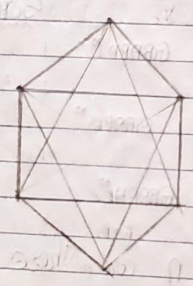
In a K-regular graph (DEG. OF EACH VERTEX IS SAME.) $q = \frac{PK}{2}$

(Vertices) $P \rightarrow 3$
(Edges) $q \rightarrow PK/2$
(Degree) $K \rightarrow 3$

$$\text{EDGES} = \frac{\text{VERTICES} \times \text{DEG}}{2}$$

⑧ Does there exist a 4-regular graph on 6 vertices. if So construct the graph.

REGUL → degree 4
DEG: $K = 4$
VER: $P = 6$
EDGE: $q = \frac{PK}{2} = \frac{6 \times 4}{2} = 12$



$\Rightarrow 2|E| = \sum_{i=1}^{n=|V|} \deg(v_i)$ } $2|E| = |V| \times \deg$

⑨ How many Vertices does a regular graph of degree 4 with 10 edges

q = No of edges = $\frac{\text{Deg} \times \text{No. Vertices}}{2}$ $\left[q = \frac{KP}{2} \right]$

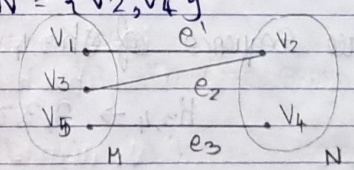
$10 = \frac{4 \times V}{2} \Rightarrow V = 5$

• Bi-Partite graph:
In an undirected graph, Vertex Set can be partitioned into two sets M and N in such a way that each edge joins a vertex in M to a Vertex N and no edge joins either two vertices in M or N.

Set of Vertices

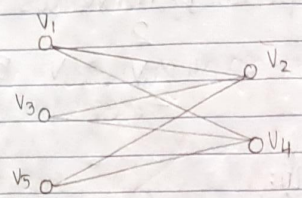
disjoint / distinct Vertex ⇒ They don't have any common vertex.

$V(G) = \{V_1, V_2, V_3, V_4, V_5\}$
$M = \{V_1, V_3, V_5\}$
$N = \{V_2, V_4\}$



- COMPLETE BIPARTITE GRAPH:

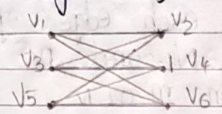
A Complete bipartite graph is a bipartite graph in which each Vertex of m is adjacent to n



$K_{3,2}$

* It is denoted by $K_{m,n}$ → m,n - denotes no. of Vertices in the set m,n

1+ (9) Draw $K_{3,3}$

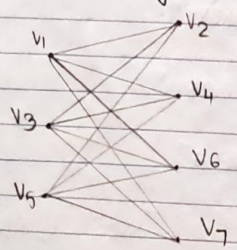


1+ (10) How many Vertices & Edges are there in a Complete bi-partite graph $K_{m,n}$

Vertices ⇒ m+n

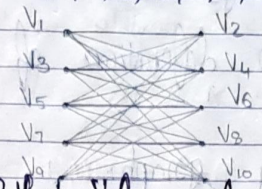
Edges ⇒ mn

1+ (11) Find the degree sequence of a) $K_{3,4}$



$K_{3,4}$ → 3,3,3,3,4,4,4

b) $K_{5,5}$ → 5,5,5,5,5,5,5,5,5,5



1+ (12) For What Values of m and n $K_{m,n}$ is a regular graph.

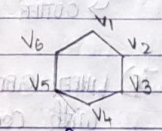
REGULAR GRAP ⇒ K-regular (degree - K) → SAME DEG.

ie) m = n

1+ (13) Check whether C_6 is ~~bipartite~~? bi-partite?

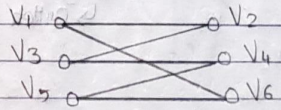
⇓

ie) C_6 CYCLE GRAPH WITH 6 EDGES / ORDER = 6



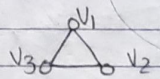
"Colour fill" Separate as "ODD & EVEN"

$m = \{V_1, V_3, V_5\}$ $n = \{V_2, V_4, V_6\}$



Yes, it is a bipartite graph

1+ (14) Is K_3 Bipartite?



$m = \{V_1, V_3\}$

$n = \{V_2\}$

It is not bipartite graph.

→ COMPLE GRAPH OF OUTER EDGE (3)

If we divide the Vertex set of K_3 into 2 disjoint set. One must contain 2 vertices that must be always connected.

This is against the definition of bipartite graph.

(16) Draw the following graph:

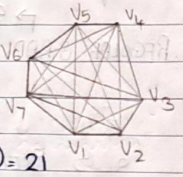
a) K_7

$n=7$

$n-1 = 7-1 = 6$ degree

7 vertices $V: n=7$

$E: \frac{n(n-1)}{2} = 21$



ie) COMPLETE GRAPH

⇓

All the VERTICES are adjacent to each other.

K_n

↳ OUTTER EDGES

b) W_8

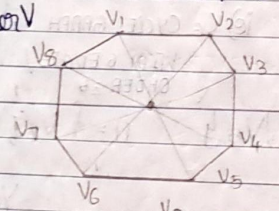
8 outter V, 1 inner V

Deg 3 (OUTTER), $n=8$

$V: n+1 = 9$

$E: 2n = 16$

DEG: $n-1 = 6$

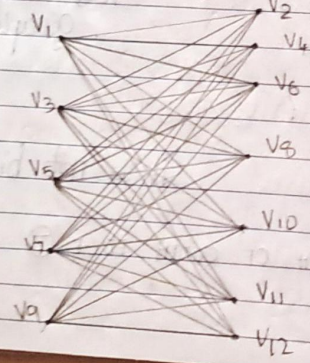


ie) WHEEL GRAPH ⇒ HUB with center

W_n

↳ OUTTER EDGE/VERTICES

c) $K_{5,7}$



ie) COMPLETE BIPARTITE G

$K_{m,n}$

vertices.

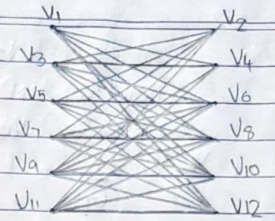
$V: m+n = 12$

$E: m \times n = 35$

DS: $\underbrace{n, n, \ldots}_{m}, \underbrace{m, m, m \ldots}_{n}$

7,7,7,7,7,5,5,5,5,5,5,5

d) $K_{6,6}$



ie) COMPLETE GRAPH

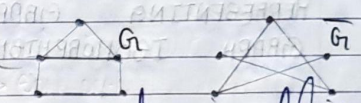
$K_{m,n}$

$V: 6+6 = 12$

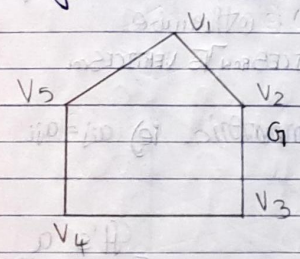
$E: 6 \times 6 = 36$

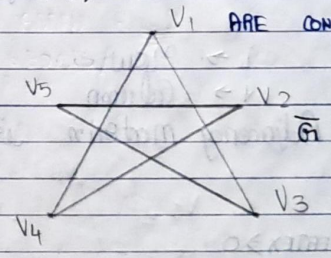
DS: 6, 6 6

- COMPLEMENT OF A GRAPH:



The complement $\overline{G}$ of a graph is defined as the simple graph with same vertex set as G and two vertices and two vertices u and v are adjacent only in $\overline{G}$ when they are not adjacent. ie) NON-ADJACENT VERTEX ARE CONNECTED.





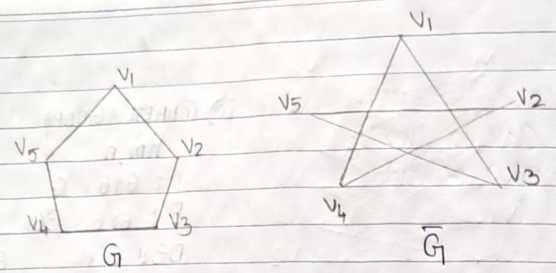
- REPRESENTING GRAPH AS MATERIALS & GRAPH ISOMORPHISM

Adjacency matrix:

The adjacency matrix $A = [a_{ij}]_{m \times n}$ of a $n \times n$ symmetric matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{where } V_i \text{ is adjacent to } V_j \\ 0 & \text{otherwise} \end{cases}$$

SYMMETRIC MATRIX : $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$



REPRESENTING GRAPHS AS MATRICES AND GRAPH ISOMORPHISM:

- ADJACENT MATRIX: The adjacency matrix $A = [a_{ij}]_{n \times n}$ of a graph of $n \times n$ symmetric matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{where } V_i \text{ is adjacent to } V_j \\ 0 & \text{otherwise} \end{cases}$$

1.6.23

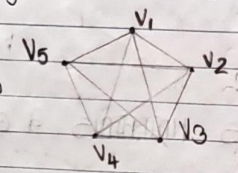
$i \rightarrow$ row ie) VERTICES ROW TO VERTICES COL

$j \rightarrow$ column

* Adjacency matrix is symmetric. ie) $a_{ij} = a_{ji}$

ADJ ⇒ 1

NOT-ADJ VERTEX ⇒ 0



It's a symmetric matrix

$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

m Km = $\frac{1000}{3600}$

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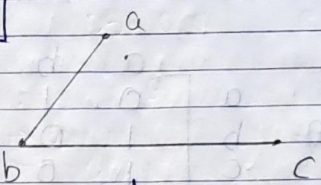
① Draw the graph for $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

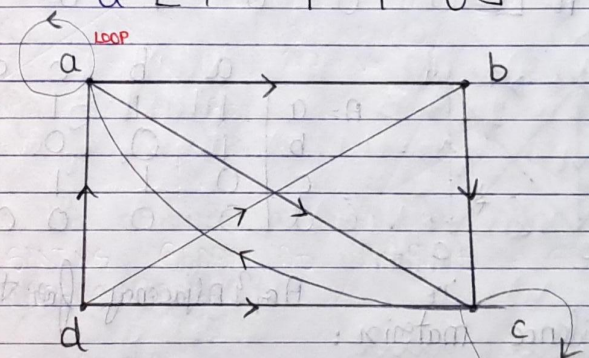
$$A = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$



② Draw the graph for

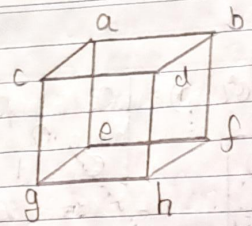
$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(x,y)



NOTE: Labelling should be either CLOCKWISE/ ANTICLOCKWISE.

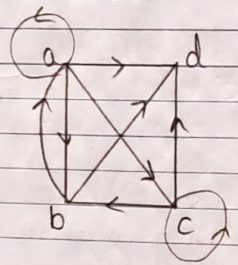
(A)



(ROW, COLUMN)
ROW → COLUMN

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \\ h \end{array} \begin{array}{c} \begin{array}{cccccccc} a & b & c & d & e & f & g & h \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(2)



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

A_{G_1} } Adjacency for the matrix

- Incidence matrix:

$A = [a_{ij}] = I_{G_1}$ { 1 where V_i is adjacency with
0 otherwise.

IF: VER ADJ EDGE = 1
ELSE = 0
NOT ADJ

! EDGE COL TO VERTEX ROWS
VERTEX TO COLUMN (EDGE)

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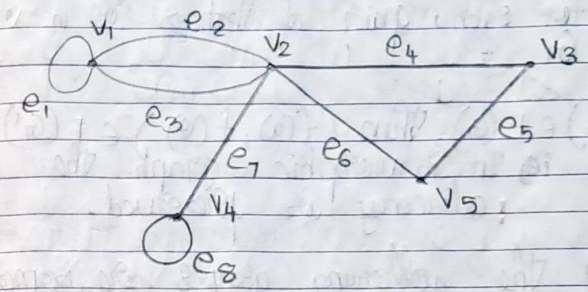
Eg:



e_1 e_2 e_3

$I_{G_1} = e_1$
e_2
e_3
e_4
e_5
e_6
e_7
e_8

$$I_{G_1} = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

HW:

① Write the incidence matrix for a) K_n b) C_n c) W_n

- ISOMORPHIC GRAPH:

The graphs G and G' are isomorphic if there is a function $f: V(G) \to V(G')$ from the vertices of G to the vertices of G' such that,

PART-B (2q → theorem + isomorphism)

! $V(G) \to V(G')$ ⇒ VERTICE SET OF G to VERTICE SET OF G'

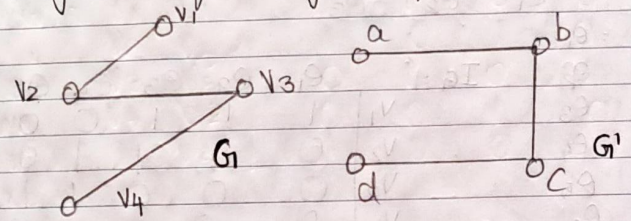
i) f is one to one
ii) f is onto Codomain = Range (B has all pre-image)
iii) for each pair of vertices 'u' in 'v' in G

$(u, v) \in E(G)$, then $(f(u), f(v)) \in E(G')$

ie) In isomorphic graph the adjacency is preserved.

(I) Verify the following graphs are isomorphic

a)



Sol:

$|V(G)| = 4$ $|V(G')| = 4$
$|E(G)| = 3$ $|E(G')| = 3$

Deg. Sequence of G = (1,1,2,2) Deg. seq. of G' = (1,1,2,2)

$f: V(G) \to V(G')$

$f(v_1) = a$ $f(v_2) = b$
$f(v_3) = c$ $f(v_4) = d$

'f' is clearly 1-1 and onto.

$f(x)$: one - one : SET-A (every element) have UNIQUE IMAGE.
on - to : SET-B (every element) have ATLEAST (1) PRE-IMAGE.

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$(v_1, v_2) \in E(G)$ and $(f(v_1), f(v_2)) = (a, b) \in E(G')$
$(v_2, v_3) \in E(G)$ and $(f(v_2), f(v_3)) = (b, c) \in E(G')$
$(v_3, v_4) \in E(G)$ and $(f(v_3), f(v_4)) = (c, d) \in E(G')$

⇒ G and G' are Isomorphic

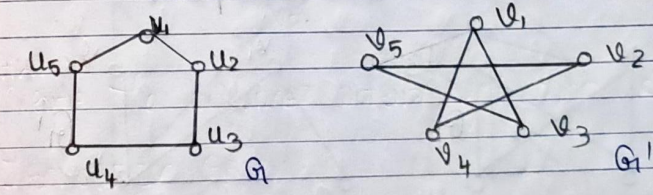
NOTE: In Isomorphism, the adjacency is preserved.

$$A(G) = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 0 \\ v_2 & 1 & 0 & 1 & 0 \\ v_3 & 0 & 1 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 0 \end{array}$$

$$A(G') = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 0 & 0 \\ b & 1 & 0 & 1 & 0 \\ c & 0 & 1 & 0 & 1 \\ d & 0 & 0 & 1 & 0 \end{array}$$

Deg wise the v should be plotted. Here we can change the ORDER OF VERTICES.

b)



$|V(G)| = 5$ $|V(G')| = 5$
$|E(G)| = 5$ $|E(G')| = 5$
DSG = (2,2,2,2,2) DSG' = (2,2,2,2,2)

$f: V(G) \to V(G')$

OUTLOOK: order of mapping

$f(u_1) = v_1$ $\quad f(u_2) = v_3$ $\quad f(u_3) = v_5$

$f(u_4) = v_2$ $\quad f(u_5) = u_4$

$$A(G) = \begin{array}{c|ccccc} & u_1 & u_2 & u_3 & u_4 & u_5 \\ \hline u_1 & 0 & 1 & 0 & 0 & 1 \\ u_2 & 1 & 0 & 1 & 0 & 0 \\ u_3 & 0 & 1 & 0 & 1 & 0 \\ u_4 & 0 & 0 & 1 & 0 & 1 \\ u_5 & 1 & 0 & 0 & 1 & 0 \end{array}$$

$G \to G'$ $\quad A(G') =$

$\therefore u \to v$

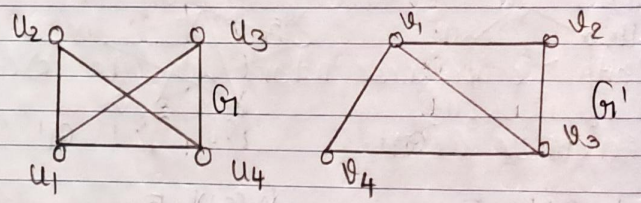
$$\begin{array}{c|ccccc} & v_1 & v_3 & v_5 & v_2 & v_4 \\ \hline v_1 & 0 & 1 & 0 & 0 & 1 \\ v_3 & 1 & 0 & 1 & 0 & 0 \\ v_5 & 0 & 1 & 0 & 1 & 0 \\ v_2 & 0 & 0 & 1 & 0 & 1 \\ v_4 & 1 & 0 & 0 & 1 & 0 \end{array}$$

NOTE: Row × column should be mapped as it is ordered

$\Rightarrow A(G) = A(G')$

$\Rightarrow G$ & G' are isomorphic.

(c)



$|V(G)| = 4$
$|E(G)| = 5$
$DSG = (2, 2, 3, 3)$

$|V(G')| = 4$
$|E(G')| = 5$
$DSG' = (2, 2, 3, 3)$

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$f : V(G) \to V(G')$

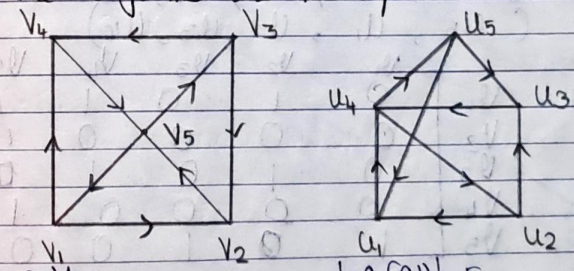
$f(u_1) = v_1$ $\quad f(u_2) = v_2$ $\quad f(u_3) = v_3$

$f(u_3) = v_4$

$$A(G) = \begin{array}{c|cccc} & u_1 & u_2 & u_3 & u_4 \\ \hline u_1 & 0 & 1 & 1 & 1 \\ u_2 & 1 & 0 & 0 & 1 \\ u_3 & 1 & 0 & 0 & 1 \\ u_4 & 1 & 1 & 1 & 0 \end{array}$$

$$A'(G) = \begin{array}{c|cccc} & v_1 & v_2 & v_4 & v_3 \\ \hline v_1 & 0 & 1 & 1 & 1 \\ v_2 & 1 & 0 & 0 & 1 \\ v_4 & 1 & 0 & 0 & 1 \\ v_3 & 1 & 1 & 1 & 0 \end{array}$$

(d) ST: The diagrams are isomorphic.



$|V(G)| = 5$
$|E(G)| = 8$

$|V(G')| = 5$
$|E(G')| = 8$

Consider the indegree & outdegree of vertices of G & G'

$\deg^+ v_i \Rightarrow$ IN-DEGREE
$\deg^- v_i \Rightarrow$ OUT - DEGREE

G	$\deg^+$ (in degree)	$\deg^-$ (out degree)
v_1	1	2
v_2	2	1
v_3	1	2
v_4	2	1
v_5	2	2

G'	$\deg^+$ (in degree)	$\deg^-$ (out degree)
u_1	2	1
u_2	1	2
u_3	2	1
u_4	2	2
u_5	1	2

$f(v_1) = u_5$ $\quad f(v_2) = u_1$ $\quad f(v_3) = u_2$
$f(v_4) = u_3$ $\quad f(v_5) = u_4$

v_i $(v_1, v_2, v_3, v_4, v_5)$
u_i $(u_5, u_1, u_2, u_3, u_4)$

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$A(G') = \begin{array}{c} \\ u_5 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} \begin{array}{c} \begin{array}{ccccc} u_5 & u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

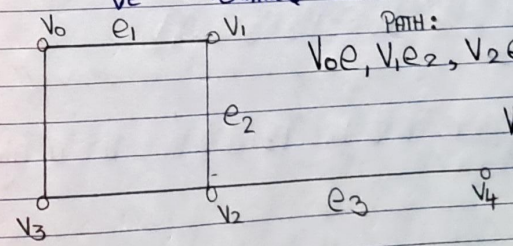
$\Rightarrow A(G) = A(G')$
$\Rightarrow G$ & G' are isomorphic.

! METHOD FOR ISOMORPHISM:
STEP-1: $|V(G)|, |E(G)|; |V(G')|, |E(G')|$
STEP-2: Degree Sequence G $\deg^+(G)$ $\deg^-(G)$
STEP-3: $f: V(G) \to V(G')$, 1-1, onto
STEP-4: $f(v_G) = v_{G'}$ .....
STEP-5: $A(G)$ & $A(G')$

6.6.23 ! PATHS, CIRCUITS AND CONNECTDNESS:

- PATH:
  A path of length 'n' from the vertex V_0 to V_n is sequence of the form
  * $V_0 e_1, V_1 e_2, V_2 e_3, V_3 e_4 \ldots e_n V_n$!

* Where the V_0 & V_n are called end points of the path.
  $V_0 \to$ Initial pt
  $V_e \to$ Terminal pt



PATH:
$V_0 e_1, V_1 e_2, V_2 e_3 V_4$

V_4 - PENDANT VERTEX

NOTE:

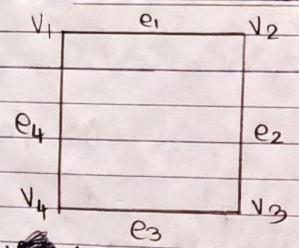
* A path from v_0 to v_n which "does not" contain repeated vertices is called a SIMPLE PATH ie) NO REPEATED VERTICES

* The no. of edges in a path is called the LENGTH of the path. ie) NO. OF EDGES = LENGTH → PATH

* A path of length ZERO(0) is called trivial path. ie) LENGTH = 0

• CIRCUIT:

* A non-trivial path is called a circuit if it is started & ends with same vertex.

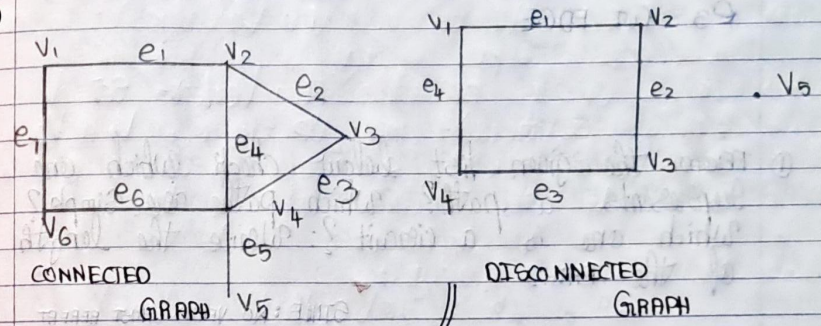


$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$

Q NOTE:!
A loop is not pmu. considered as a cycle of length, one.

• CONNECTED GRAPH:

A graph is connected if there is a path between every pair of distinct vertices otherwise the graph is said to be a disconnected. ie) PATH btw all vertices.

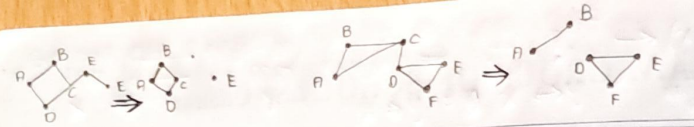


* In disconnected graph we have more than 1 component.

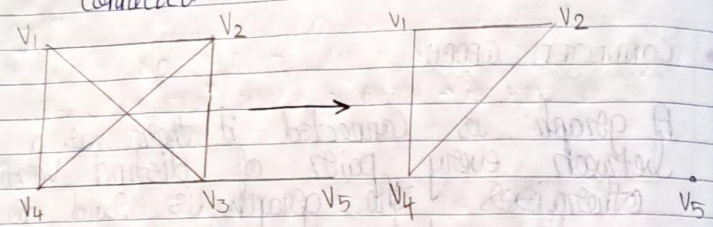
* Any graph with ISOLATED VERTEX (DEG = 0) is called a DISCONNECTED GRAPH.

• Cut Vertex & cut edge: (Bridges)

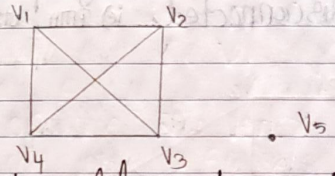
* A vertex (Edge) whose removal increases the no. of components.
ie) Removal makes the GRAPH disconnected.



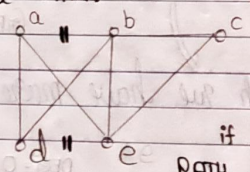
Connected — Disconnected



V_3 - CUT VERTEX
e_3 - CUT EDGE



① From the given list below check which one represents a path? Which path are simple? Which one is a circuit? Write the length of the PATH.



SIMLE : NO VERTEX MOST REPEAT
CIRCUIT : START, END @ SAME VERTEX
LENGTH: NO OF EDGES.

PATH	SIMPLE	CIRCUIT	LENGTH
a) a, e, b, c, b -	✓	$a^{b \text{ is REPEATED}}$	4
b) e, b, a, d, b, c -	$a^{b \to a}$ NO PATH		
c) a, e, a, d, b, c, a -	$a^{c \to a}$ NO PATH		
d) c, b, d, a, e, c -	✓	$a^{c \text{ is REPEATED}}$	5

Sol: a) Not a simple path, not a circuit, length is 4
b) Not a path
c) Not a path.
d) Circuit, length is 5.

SIMPLE : NO REPEATED VERTICES
CIRCUIT : STARTS + ENDS @ SAME VERTEX + SIMPLE BTW

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SIMPLE PATH ⇔ BE A CIRCUIT

In circuits no Vertic

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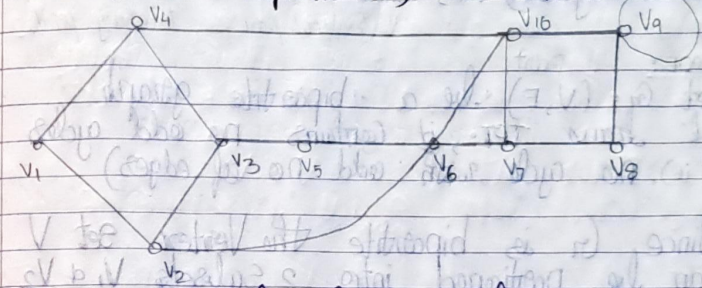
② For the following graph find length of path, circuit, closed path (or) SIMLE PATH.



In circuits only the 1st + last pt/vertice must repeat no other pt in between should not repeat.

a) $V_1 - \underline{\underline{V_4}} - V_3 - V_5 - V_6 - V_{10} - \underline{\underline{V_4}} - V_1$

7 / Not simple / Not a circuit / closed path.
Since a circuit + V_4 repeats — V_4 repeats

b) $V_5 - \underline{\underline{V_6}} - V_7 - V_{10} - \underline{\underline{V_6}} - V_2$

5 / Path / Not simple / Not circuit / Not closed.
V_6 repeats — V_6 repeats. START ≠ END

c) $V_2 - V_3 - V_5 - \underline{\underline{V_6}} - V_7 - V_{10} - V_6 - \underline{\underline{V_6}} - V_2$

8 / Not simple / Not a circuit / closed path
V_6 repeat + Not circuit — V_6 repeat

SUFFICENT PART $\rightleftarrows$ NECESSARY PART iff $\Leftrightarrow$ ($\rightleftarrows$) !

x iff y

$x \Rightarrow y = x \leftrightarrow y$

$(x \rightarrow y) \wedge (y \rightarrow x)$

PREMISE NECESSARY PART — HYPOTH — PREMISE SUFFICENT PART

8 month ⇒ THEOREM-3 ⊛

ST "a simple graph G is BIPARTITE iff it has NO CIRCUIT (cycle) of odd length (odd cycles)". ie) SIMPLE GRAPH → BIPARTITE iff (No circuits → odd)

Sufficent part — Necessary part.

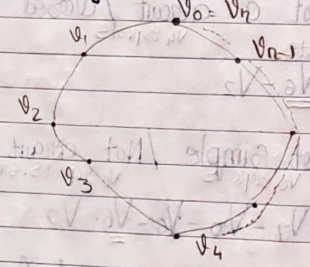
NECESSARY PART: PROOF: if part

→ Let $G = (V, E)$ be a bipartite graph.

→ We have TPT:, it contains no odd cycles (ie). no cycle with odd no. of edges)

→ Since G is bipartite the Vertex set V can be partioned into 2 subsets V_1 & V_2 such that every edge of G has one of its set in V_1 and other end in V_2.

→ Now consider any cycle $v_0, v_1, v_2 \ldots v_n = v_0$ of the length n.



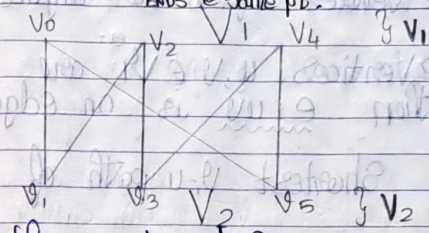
$V_1^{\#V_1}$	$V_2^{\#V_2}$
v_0	v_1
v_2	v_3
v_4	v_5
v_6	v_7
v_8	v_9
⋮	⋮

→ Suppose $v_0 \in V_1$, $v_1 \in V_2$, $v_2 \in V_1$, $v_3 \in V_2 \ldots$ } BIPARTING

* Thus $v_0, v_2, v_4, v_6 \ldots \in V_1$ and $v_1, v_3, v_5 \ldots \in V_2$

→ also $v_0 = v_n \in V_1$ and so 'n' is even,

ie) STARTS & ENDS @ same pt.



* So the cycle has even no. of edges. Hence the graph has no odd cycles (NO)

SUFFICENT NECESSARY PART: PROOF: only if

Conversely, let the graph G has no odd cycles, we have TP its bipartite.

→ Without loss of generality we may assume that G is connected.

Let v_1 be any vertex of G $v_1 \in V$

Define

$$V_1^{set} = \{ v \in V_1 / d(v, v_1) \text{ is even} \}$$

$$V_2^{set} = \{ v \in V_2 / d(v, v_1) \text{ is odd} \}$$

→ clearly V_1, V_2 are non-empty dis-joint subsets of V.

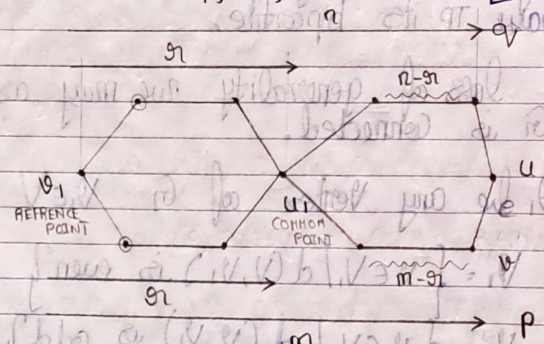
* We have to prove every edge of G has one end in V_1 SET and other end in V_2 SET
ie)
TPT no 2 Vertices of V_1 are adjacent.

→ Suppose 2 Vertices $u, v \in V_1$ SET are adjacent then $e = uv$ is an edge.

* Let p the shortest v_1-u path of length m.

* Let q be the shortest v_1-v of length n.

→ Since $u, v \in V_1$, m, n is even. [∵ define]



→ Now let u_1 be the last common point of the path p and q.

* Then the v_1-u_1 path along p and the v_1-u_1 path along q are both shortest path.
→ And Hence have the same length r.

→ Now the path u_1-u along p the line uv and the path v-u_1 along q from a cycle of length $(m-r) + 1 + (n-r)$ which is ODD

m, n → even ODD if in V_2 A contradition
r → odd

$(m-r)$ } odd
$(n-r)$ }
$(m-r) + (n-r)$ = Even
Even + 1 = odd ☹

∴ u and v in V_1 are NOT adjacent

⇒ No 2 Vertices of V_1 are adjacent.

Similarly we can prove that no 2 Vertices in adjacent in V_2.
Hence (V_1, V_2) is Bipartition of G
⇒ G is Bipartite.

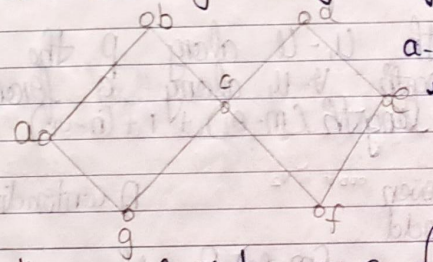
* CIRCUIT: STARTS & ENDS AT SAME POINT
* (+) NO

07.06.2023 EULER AND HAMILTON GRAPH:

Defn: Euler → Edge (No-Repeat)

• EULER'S CIRCUIT:
An Euler circuit (cycle) in a graph G is a simple circuit containing every edge of G. ie) ALL EDGE ARE IN A CIRCUIT

* An Euler path in a graph G is a simple path containing every edge of G.



a-b-c-d-e-f-g-a is an Eulerian circuit

R: Vertice can repeat any no. of times but no. edges should be repeated while completing the graph.

NOTE:
* If all the vertices in a graph has even degree then it has both Euler path & Euler circuit ie) K-regular graph, K ∈ N. Then EULER'S GRAPH EXIST.

* In a Eulerian graph, vertices may be repeated but no edge is repeated.
! RULE: All edges must be in a circuit. Vertices can be repeated, edges can't be

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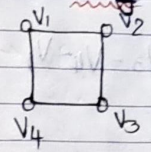
• DEFN: HAMILTONIAN:

A simple path in a connected graph G is called a Hamiltonian path if it contains every vertex exactly by one.

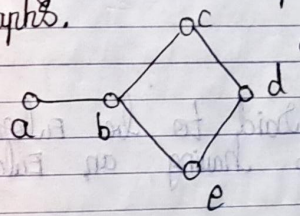
Eg:



$V_1 - V_2 - V_3 - V_4 - V_1$

* A graph is said to be hamiltonian if it has a hamilton circuit.

① Identify the Euler path in the following graphs.

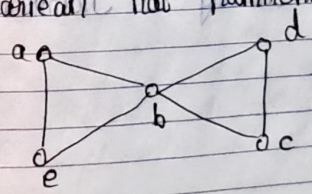


Sol. There is a circuit b-c-d-e-b But Eulerian graph doesn't exist.

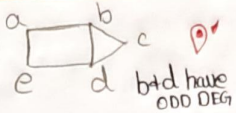
b-c-d-e-b-a
a-b-c-d-e-b

b has ODD DEG FROM NOTE

② Give an example for which is
i) Eulerian not hamiltonian



a-b-d-c-b-e-a

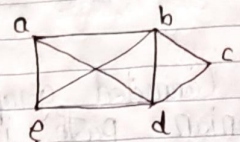
ii) 

b & d have ODD DEG

- EULER: All E
- HAMILTONIAN: All V

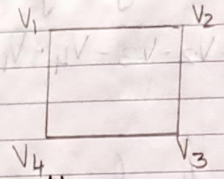
V
E

ii) Hamiltonian but NOT Eulerian:



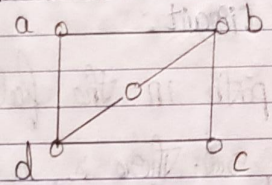
a-b-c-d-e-a

iii) Both hamiltonian & Euler:



$V_1 - V_2 - V_3 - V_4 - V_1$

iv) Neither Hamiltonian nor Eulerian:



NOTE:

* A graph is said to be Eulerian graph if it is having an Eulerian circuit.

* A graph is said to be hamiltonian graph if it is having a hamiltonian circuit.

	EDGE	VERTEX
EULER	All EDGE (NR)	R
HAMIL	R	ALL VERTEX (NR)

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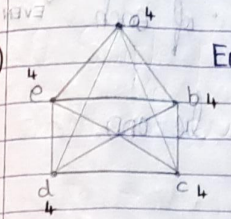
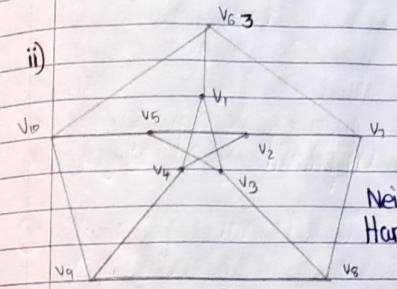
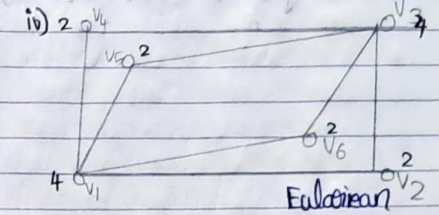
③ check whether the graph is Euler / Hamiltonian.

i) 

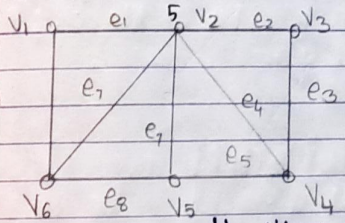
Eulerian and Hamiltonian

ii) 

Neither Hamitonian nor Eulerian

iv) 

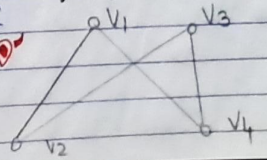
Eulerian not hamiltonian

iii) 

Hamiltonian not Eulerian

④ For what values of m & n, $K_{m,n}$ is Eulerian?

$K_{m,n}$ is Eulerian iff both m & n are even



⑤ For what Values of n, K_n is Eulerian graph?

K_n - COMPLETE GRAPH, all Vertices are adjacent to each other. → DEG = $n-1$ ⇓ EVEN

* Since K_n is a complete graph · of each Vertex is $n-1$.

* To get $n-1$ as even, n must be ODD.

GRAPH THEORY pts to be NOTED:

① Loop is counted as 2× EDGE

② $\sum_{i=1}^{n=|V|} \deg(v_i) = 2|E|$

③ HANDSHAKE THEOREM:-1 INTRO ; V, E Sets ; ; ENG TRANSLATION

④ "NO OF ODD VERTICES IN A GRAPH IS ALWAYS EVEN"

⑤ THEOREM - 2: INTRO ; ODD - DEG, EVEN - DEG ; HST (THEOREM-1) ; MATHEMATICS.

⑥ "IN A SIMPLE GRAPH (n VERTICES) EACH VERTEX OF MAX DEG $n-1$"

⑦ K- REGULAR : Same degree , K - DEGREE

⑧ COMPLETE GRAPH: K_n , n denotes SIDE

⑨

			DEGREE	EDGES	VERTICES	REGULAR?
COMPLETE GRAPH:	K_n	n-sides	$n-1$	$n(n-1)/2$	n	$n \geq 1$
CYCLE GRAPH:	C_n	n-sides	2	n	n	$n \geq 3$
WHEEL GRAPH:	W_n	n-sides	3, n	$2n$	$n+1$	$n=3$
COMPLETE BIPARTITE:	$K_{m,n}$	m,n vertices	$\frac{m}{n}$, $\frac{n}{m}$	$m \times n$	$m+n$	$m=n$

⑩ # K - REGULAR GRAPH: $q = kp/2$ ⇒ $q \to |E|$, $k \to$ DEG, $p \to |V|$

⑪ COMPLETE - BIPARTITE: $K_{m,n}$

Walk ≡ Path

TP: Eulerian G
WSP: E Circuit.

THEOREM - 4

" A given connected graph is an Eulerian graph if all vertices of G is of even degree" $P \Leftrightarrow q \Rightarrow \underbrace{(P \rightarrow q)}_{\text{NECESSARY PART}} \wedge \underbrace{(q \rightarrow P)}_{\text{SUFFICENT PART}}$

PROOF:

* Suppose G is an Euler graph.
* It contains an Euler line (Path)
* It contains a closed path ~~of even~~ containing ~~degree~~. all the edges.

TP:

All vertices in G is of even degree.

Proof:

NECESSARY PART:

→ In traversing the closed walk everytime.
→ The walk meets a vertex V it goes through 2 new edges incident on v with one we entered and other exited.
→ This is TRUE for all verticesos, because it is a closed walk.

* The degree of every vertex is even.

SUFFICENT PART:

Conversely, Suppose all vertices of G is even degree.

TP:

G is an Euler graph

Construct

Proof:

* Construct a closed path /walk starting at an arbitrary vertex V and going through the edges of G such that no edge is repeated.
* Because, each vertex is of even degree, we can exit and enter to each and every vertex, the tracing only stops at vertex V.

Name the closed path 'h'.

CASE-1:

If walk 'h' converses all edges of G, then h becomes the Eulerian graph. The G is Euler's graph.

CASE-2:

* If h does not cover all the edges of G then remove all the edges of the path h and obtain the remaining graph G' because both the G + G' have all the vertices of even degree.
* Every vertex of G' is also even degree.

* Since G is connected, h will touch G' in atleast one vertex V, start from V.
* We again construct a new walk in G'. This will terminate only at a V because every vertex of G' is of even degree.
* Now h walk combined with G' closed walk start and ends at V.
* And have more edges than V.
* This procedure is repeated until we obtain a normal closed walk covering all edges of G.
* Thus G is Euler graph.

NOTE:

- MAXIMAL NON-HAMILTONIAN GRAPH:

A maximal-non Hamiltonian is an graph such that for every pair of non-adjacent vertices u and v, adding the edges uv makes the graph hamiltonian

ie) deg of each vertex is $\frac{n}{2}$, then prove that G is hamiltonian

! THEOREM-5:

DIRAC'S THEOREM

"If G is a simple connected graph with n vertices where $n \geq 3$. Then G is hamiltonian of degree of each vertex is atleast $\frac{n}{2}$"

IF $\frac{n}{2}$ THEN Hamiltonian

PROOF:

→ Consider the contraposition if a simple connected graph G.

→ With 'n' vertices is non-homogeneous Hamiltonian, it has atleast one vertex with degree less than n/2.

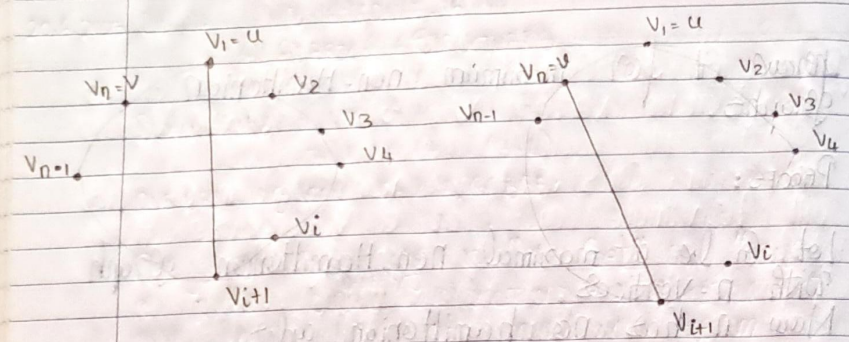
* Clearly the theorem is true for all non-hamiltonian graph if we can

prove it for maximum non-Hamiltonian graphs.

PROOFS:

* Let G be a maximal non-Hamiltonian graph with n-vertices.
* Now n has no hamiltonian cycles.
* Now, Let's consider a pair of non-adjacent vertices u and v.
* By definition of maximal non-hamiltonian graph there exist one hamiltonian path if we are able to connect non-adjacent vertices.
- Let's define,

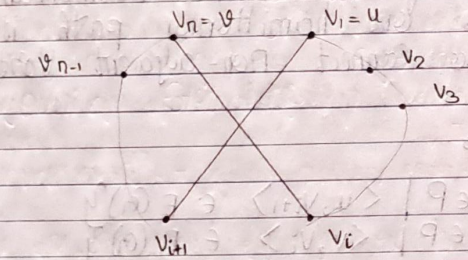
$$S = \{v_i \in P \mid \langle u, v_{i+1} \rangle \in E(G)\}$$
$$T = \{v_i \in P \mid \langle v, v_i \rangle \in E(G)\}$$



S T

$|S| = \deg(u)$ $|T| = \deg(v)$

With, $|S \cup T| = |S| + |T| - |S \cap T|$ — ①



$|S \cap T| = 0$, it is a non-Hamiltonian graph, so there are no vertices common in the set of vertices S and T. ie) $|S \cap T| = 0$ — ②

Also, Both S and T does not contain $v_n = v$,

So the total union of S and T

has vertices less than total no of vertices.

$|S \cup T| < n$ — ③

So $\Rightarrow |S \cup T| = |S| + |T|$ — ④ [from ②]

$|S| + |T| < n$ [from ③ & ④]

$\Rightarrow \deg(u) + \deg(v) < n$

Thus, atleast one of $\deg(u)$ and $\deg(v)$ must be less than $\frac{n}{2}$

Thus the proof is complete.