Examples in Algebraic Geometry

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Contents

CHAPTER 1

Introduction

1. Motivation

Examples in algebraic geometry is an open source project whose content is succinctly described by its name.

There are many beautiful, instructional, and inspiring examples in algebraic geometry. On the other hand, the foundations of the subject are rather technical, which makes it more difficult for students to gain geometric intuition. Our aim of this project is to assemble a collection of "examples", short narratives of various phenomena, and to organize them thematically.

Learning algebraic geometry can be a lengthy and arduous process. Everyone has a story in which they are working hard trying grasp a complex idea and then they reach a moment of clarity. Afterwards, the concept seems natural and readily approachable. While these experiences can be very personal, they are often catalyzed by a good example. We would like to help future students by collecting and cataloging these nuggets of information.

2. Guidelines

Below is a set of guidelines which are used throughout of this document. They do not reflect any absolute practice in typesetting, but merely the combination of personal customs and some rational decisions aimed at standardizing various parts of the document.

- When using material from a book or article add a reference to it. If the discussion uses results or definitions which are not well-know, it could be beneficial to note these. Do not hesitate to add your own relevant observations as these could facilitate the learning process.
- When a source uses notation or language which is slightly old, it is better to update that. Best care should be taken to ensure notation is consistent throughout all sections.
- Use as many of the predefined customizations as possible.
- Referring to maps as “one-to-one” and “onto” can be very confusing. Replace these with “injective” and “surjective” respectively.
- Use $\setminus$ for the difference of sets instead of $-$.
- Use `\cn` (without the customizations `\colon`) for colons in functions, that is, visually $f\colon X \to Y$ looks better than $f : X \to Y$ (note the difference in spacing before the colon).
- Be consistent with wording and spelling. On a similar note, hyphenate “non” constructions such as “non-constant” and “non-vanishing”.

CHAPTER 2

Ideas

- Non-projective threefold (Hironaka's example)
- Non-projective surface (singular, see Vakil's Foundations of algebraic geometry)
- A curve with a non-smoothable singularity
- A curve with non-planar singularity (maybe give an example with arbitrary dimensional tangent space)
- Examples of plane singularities; it is not hard to write down specific examples with any torus knot; give an example with a cable of a torus knot
- Lines on a quadric in $\mathbb{P}^3$ and $\mathbb{P}^4$
- Variety whose effective cone is not polygonal
- Nef divisor which is not effective
- Several examples of semistable reduction for curves
- A complex manifold which is not algebraic (e.g., a torus) (see Shafarevich for more ideas)
- Monodromy of 27 lines on a cubic surface
- Monodromy of 2 rulings on smooth quadric surface degenerating to a cone
- Classify length 2 and 3 schemes (see Geometry of Schemes)
- Symmetric products versus Hilbert schemes of points
- Explain why symmetric products of curves are smooth, but this is not true for higher dimensions
- Examples which illustrate behavior specific to schemes
- Examples which illustrate behavior specific to stacks (Deligne-Mumford, Artin) (see Introduction in Knutson, Algebraic spaces)
- Interesting things about curves
- Interesting things about hypersurfaces
- Triple, quadruple covers, etc
- Cyclic covers
- Explicit examples of resolutions of singularities, especially emphasizing the differences between the classes of singularities coming from MMP
- Explain why hypersurfaces of small degree are uniruled
- Explain why locally free in the Zariski and étale topologies is equivalent (related to Hilbert theorem 90)
- Two pencils, everywhere tangent

1. Non-planar curve singularities

It is easy to construct a variety of dimension 1 whose tangent space at a given point has arbitrary dimension $n \geq 1$. For example, consider the union of n lines in $\mathbb{A}^n$, all passing through a given point $p \in \mathbb{A}^n$, whose tangent spaces span the ambient tangent space $T_p\mathbb{A}^n$. To be concrete, let $x_1, \ldots, x_n$ be global coordinates on $\mathbb{A}^n$; take $I_i = (x_1, \ldots, \widehat{x_i}, \ldots, x_n) \subset k[x_1, \ldots, x_n]$, $L_i = V(I_i)$, and $p = (0, \ldots, 0)$. At the origin p, the variety $C = \bigcup_{i=1}^n L_i = V(I_1 \cdots I_n)$ has tangent space

$$T_p C = \sum_{i=1}^n T_p L_i = \sum_{i=1}^n k\left\langle \frac{\partial}{\partial x_i} \right\rangle = k\left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle = T_p \mathbb{A}^n$$

of dimension n.

It is not much harder to come up with an irreducible curve whose tangent space has arbitrary dimension n. For example, we can crimp the affine line. Let k be a field and consider the subring

$$R = k\langle 1, t^n, t^{n+1}, \dots \rangle \subset k[t].$$

The ideal $m = k\langle t^n, t^{n+1}, \dots \rangle$ is maximal since $R/\mathfrak{m} \cong k$ is a field, and

$$t^n + \mathfrak{m}^2, \dots, t^{2n-1} + \mathfrak{m}^2$$

form a basis for the cotangent space $\mathfrak{m}/\mathfrak{m}^2$.

2. Vector bundle which does not extend

We will exhibit an example of a vector bundle E on an open subset U of a variety X which does not extend to all of X.

Let E' be a vector bundle on $\mathbb{P}^2$ which does not split into a direct sum of line bundles. To be more concrete, take the tangent bundle $E' = T_{\mathbb{P}^2}$. Consider $\mathbb{A}^3$ and the open subset $\mathbb{A}^3 \setminus \{0\}$. Let $\pi\colon \mathbb{A}^3 \setminus \{0\} \to \mathbb{P}^2$ be the natural projection map. We claim the vector bundle $E = \pi^* E'$ does not extend to $\mathbb{A}^3$. If it did, it has to be trivial (see 1). In particular, it would split which is not the case.

Since the projection $\pi\colon \mathbb{A}^3 \setminus \{0\} \to \mathbb{P}^2$ expends to a regular morphism $\widetilde{\pi}\colon \operatorname{Bl}_0 \mathbb{A}^3 \to \mathbb{P}^2$, it follows that E extends $\widetilde{\pi}^* E'$ over $\operatorname{Bl}_0 \mathbb{A}^3$.

Remark 1. Serre's problem (also known as Serre's conjecture) asks whether every (algebraic) vector bundle over an affine space is trivial. Phrased as a question in commutative algebra, the statement is that every projective module over a polynomial ring is free. Quillen [**4**] and Suslin [**7**] independently proved the conjecture is 1976. A simpler proof, due to Leonid Vaserusteuin, can be found in [**2**].

Remark 2. While we are providing a sort of counterexample, it would be interesting to investigate this problem more closely. For example, given a coherent sheaf E on X which is a vector bundle on U, there always exists a blowup X' of X with center in $X \setminus U$ such that $E|_U$ extends to a vector bundle over X' [**5**, Theorem 4.1]. Is there a way to carry this process in a canonical fashion?

3. An explanation why $T_{\mathbb{P}^2}$ does not split

There are (at least) two heavy-machinery reasons for this. One is that $T_{\mathbb{P}^2}$ is a stable vector bundle on $\mathbb{P}^2$, and stable bundles are simple (i.e. their endomorphisms are all homotheties) and therefore do not split. This last implication is easy to explain: if E splits as $E = E_1 \oplus E_2$, then it admits for instance the endomorphism $(x_1, x_2) \mapsto (x_1, 0)$, which is not a homothety.

The other is Horrocks criterion for splitting of vector bundles on the projective space [**1**]. A vector bundle E on $\mathbb{P}^n$ splits if and only if it has no intermediate cohomology, i.e. $\mathrm{H}^i\left(\mathbb{P}^n, E(a)\right) = 0$ for all $i \notin \{0, n\}$ and all $a \in \mathbb{Z}$. If we consider the dual of the Euler sequence and twist it by $\mathcal{O}_{\mathbb{P}^2}(-3)$ we obtain the sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-3) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(-2)^2 \longrightarrow T_{\mathbb{P}^2}(-3) \longrightarrow 0.$$

We can take cohomology and since

$$h^1\left(\mathcal{O}_{\mathbb{P}^2}(-2)^2\right) = h^2\left(\mathcal{O}_{\mathbb{P}^2}(-2)^2\right) = 0,$$

we get

$$\mathrm{H}^1\left(T_{\mathbb{P}^2}(-3)\right) \cong \mathrm{H}^2\left(\mathcal{O}_{\mathbb{P}^2}(-3)\right) \cong k.$$

For a definition of stable bundles and the result that they are simple, see chapter II, section 1 in [**3**]; for Horrocks criterion, see chapter I, section 2.3.

4. Double covers

Let X be a variety (smooth over an algebraically closed field k of characteristic 0). We would like to describe the set of (potentially ramified) double covers $\pi\colon Y \to X$.

Start by observing these correspond naturally to flat, locally free rank 2 $\mathcal{O}_X$-algebras $\mathcal{A}$. Given a cover $\pi\colon Y \to X$, we construct such a sheaf by $\mathcal{A} = \pi_*\mathcal{O}_Y$. Conversely, given a sheaf of algebras, taking Spec furnishes a cover. Therefore, it suffices to study such $\mathcal{O}_X$-algebras.

The unit in such an $\mathcal{O}_X$-algebra $\mathcal{A}$, gives an embedding $i\colon \mathcal{O}_X \to \mathcal{A}$. This map admits a splitting

$$s = \frac{1}{2}\operatorname{Tr}\colon \mathcal{A} \longrightarrow \mathcal{O}_X.$$

It is best to illustrate the trace map when $X = \operatorname{Spec} k$ and $\mathcal{A} = \widetilde{A}$ is a length 2 algebra over k. In simple terms, A is a dimension 2 vector space over k. Multiplication by an element $x \in A$ induces an endomorphism $x\cdot\colon A \to A$. The trace map $\operatorname{Tr}\colon A \to k$ is defined as the trace of this endomorphism, namely,

$$\operatorname{Tr}(x) = \operatorname{Tr}(x\cdot\colon A \longrightarrow A).$$

We are using the fact $\operatorname{Tr}(1_A) = \dim_k A = 2$, so $s = 1/2\operatorname{Tr}$ splits the inclusion $k \hookrightarrow A$. Note that $1/n\operatorname{Tr}$ defines a splitting for any length n algebra.

Back to a general base X, we have constructed a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_X \arrow[r, "i"] & \mathcal{A} \arrow[l, bend left, "s"] \arrow[r] & \widetilde{\mathcal{A}} \arrow[r] & 0. \end{tikzcd}$$

Since $\mathcal{A} = \mathcal{O}_X \oplus \widetilde{\mathcal{A}}$ is locally free of rank 2, it follows that $L = \widetilde{\mathcal{A}}$ is a line bundle. The multiplication map $\mu\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, can be written as a quadruple of morphisms

$$\mathcal{O}_X \otimes \mathcal{O}_X \longrightarrow \mathcal{A}, \qquad \mathcal{O}_X \otimes L \longrightarrow \mathcal{A}, \qquad L \otimes \mathcal{O}_X \longrightarrow \mathcal{A}, \qquad L \otimes L \longrightarrow \mathcal{A}.$$

The first three of these are uniquely determined by the fact $\mathcal{O}_X \subset \mathcal{A}$ is the subsheaf spanned by the unit. The last one can further be decomposed into a pair of morphisms

$$L \otimes L \longrightarrow \mathcal{O}_X, \qquad L \otimes L \longrightarrow L.$$

We claim that working with double covers forces $L \otimes L \to L$ to be the zero map. Again, it is best to see this by looking one point at a time.

There are two isomorphism classes of length 2 algebras, namely $A_1 = k \times k$ and $A_2 = k[\varepsilon]/(\varepsilon^2)$. The subspaces of trace-free elements (analogues of L above) in each are given by

$$L_1 = k \cdot (1, -1) \subset A_1, \qquad L_2 = k \cdot \varepsilon \subset A_2.$$

In the first case, the product $(1,-1)\cdot(1,-1) = (1,1)$ has projection 0 in the trace-free part. In the second case $\varepsilon \cdot \varepsilon = 0$, so there is no need to project. In each of the two cases, the map $L_i \otimes L_i \to L_i$ is zero. Note that the map $L_i \otimes L_i \to k$ is zero only in the second case. Geometrically, A_1 corresponds to reduced fiber of two points, while A_2 to a point of ramification. We will later use this observation to detect the ramification locus.

Returning to a general base, the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is determined by a map $\sigma\colon L \otimes L \to \mathcal{O}_X$. Conversely, it is not hard to check that any map $\sigma\colon L \otimes L \to \mathcal{O}_X$ can be augmented to give a commutative, associative $\mathcal{O}_X$-algebra structure on $\mathcal{A} = \mathcal{O}_X \oplus L$, the isomorphism type of the algebra depends only on σ up to scaling. Finally, we can take duals to get

$$\sigma^\vee\colon \mathcal{O}_X \longrightarrow L^{-2}.$$

Up to scaling this is the same as a divisor D in the linear system $|L^{-2}|$. Note that D is the vanishing locus of $\sigma^\vee$ and σ, hence also the ramification locus of the corresponding double cover $\mathrm{Spec}(\mathcal{O}_X \otimes L) \to X$. We can replace L with L^{-1} for convenience to obtain the result as it is commonly stated.

Proposition 3. *A pair (L, D) consisting of a line bundle $L \in \mathrm{Pic}(X)$ together with a divisor $D \in |L^2|$ determine a double cover, and conversely. The ramification divisor of the cover corresponding to a pair (L, D) is D.*

We can use this result to derive a useful corollary.

Corollary 4. *Unramified double covers of X are in a correspondence with 2-torsion elements of $\mathrm{Pic}(X)$.*

Given a pair (L, D) as above, there is an alternative geometric construction of the associated double cover. Start by using D to produce a map $f\colon X \to \mathbb{P}^1$ such that $D = [f^{-1}(0)] - [f^{-1}(\infty)]$. It is easy to construct a double cover $C \to \mathbb{P}^1$ with unique ramification over 0 and ∞. The total space of double cover corresponding to the pair (L, D) is the fiber product $Y = X \times_{\mathbb{P}^2} C \to X$. We realize Y as a two-sheeted cover by projecting to the first factor.

$$\begin{tikzcd} Y = X \times_{\mathbb{P}^1} C \arrow[r] \arrow[d] & C \arrow[d] \\ X \arrow[r] & \mathbb{P}^1 \end{tikzcd}$$

5. The scroll $\mathbb{F}(a_1, \ldots, a_n)$ as a quotient

It is customary to construct the scroll $\mathbb{F}(a_1, \ldots, a_n)$ as the projectivization $\mathbb{P}E$ of the vector bundle

$$E = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$$

over $\mathbb{P}^1$. Alternatively, it is also possible to construct this space as the quotient of $(\mathbb{A}^2 \setminus 0) \times (\mathbb{A}^n \setminus 0)$ by an action of the algebraic group $\mathbb{G}_m \times \mathbb{G}_m$. In coordinates, the action is given by

$$(\lambda, \mu) \cdot (t_1, t_2, x_1, \ldots, x_n) = (\lambda t_1, \lambda t_2, \mu\lambda^{-a_1} x_1, \ldots, \mu\lambda^{-a_n} x_n).$$

For more details on both points of view, see [**6**, Chapter 2].

Bibliography

[1] G. Horrocks, *Vector bundles on the punctured spectrum of a local ring*, Proc. London Math. Soc. (3) **14** (1964), 689–713. MR 0169877 (30 #120)

[2] Serge Lang, *Algebra*, third ed., Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002. MR 1878556 (2003e:00003)

[3] Christian Okonek, Michael Schneider, and Heinz Spindler, *Vector bundles on complex projective spaces*, Modern Birkhäuser Classics, Birkhäuser/Springer Basel AG, Basel, 2011, Corrected reprint of the 1988 edition, With an appendix by S. I. Gelfand. MR 2815674 (2012d:14073)

[4] Daniel Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171. MR 0427303 (55 #337)

[5] M. Raynaud, *Flat modules in algebraic geometry*, Compositio Math. **24** (1972), 11–31. MR 0302645 (46 #1789)

[6] Miles Reid, *Chapters on algebraic surfaces*, City **3** (1996), 147.

[7] A. A. Suslin, *Projective modules over polynomial rings are free*, Dokl. Akad. Nauk SSSR **229** (1976), no. 5, 1063–1066. MR 0469905 (57 #9685)