Symmetric Stress Tensor:

To proceed further, into symmetric S.T, first lets see the divergence of the Canonical Stress Tensor.

The canonical stress tensor from ④ is,

$$T^{\alpha\beta} = \sum_K \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_K)} \partial^\beta \phi_K - g^{\alpha\beta} \mathcal{L}$$

Taking divergence, in 4-dimension of the Canonical Stress Tensor.

$$\partial_\alpha T^{\alpha\beta} = \sum_K \partial_\alpha \left[\left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_K)} \right) \partial^\beta \phi_K \right] - \partial_\alpha g^{\alpha\beta} \mathcal{L}.$$

$$= \sum_K \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_K)} \right) \partial^\beta \phi_K + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_K)} \partial_\alpha \partial^\beta \phi_K - \partial^\beta \mathcal{L} \quad \text{— (A)}$$

But from the Euler-Lagrange Equation,

$$\partial_\alpha \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_K)} - \frac{\partial \mathcal{L}}{\partial \phi_K} = 0 \quad \text{— (B)}$$

$$\text{So, } \partial_\alpha T^{\alpha\beta} = \sum_K \frac{\partial \mathcal{L}}{\partial \phi_K} \partial^\beta \phi_K + \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_K)}}_{} \partial^\beta (\partial_\alpha \phi_K) - \partial^\beta \mathcal{L} \quad \text{— (C).}$$

Derivative of the lagrangian density with respect to the derivatives of the field.

Now let us consider any lagrangian which depends on fields & the derivative of the field. i.e.

$$\mathcal{L} = \mathcal{L}(\phi_k, \partial_\alpha \phi_k)$$

If we take the derivative of above,

$$\partial^\beta \mathcal{L} = \sum_k \frac{\partial \mathcal{L}}{\partial \phi_k} \partial^\beta \phi_k + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi_k)} \partial^\beta(\partial_\alpha \phi_k) \quad \text{(D)}$$

If we compare (C) and (D)

Then,

$$\partial_\alpha T^{\alpha\beta} = \partial^\beta \mathcal{L} - \partial^\beta \mathcal{L} = 0.$$

$$\Rightarrow \boxed{\partial_\alpha T^{\alpha\beta} = 0} \quad \text{—— (E)}.$$

This means that divergence of Canonical stress Tensor is zero. (E) is important because it gives us conservation laws.

$\Rightarrow$ We can obtain the conservation of energy & momentum by integrating over the complete 3D space,

So,

$$\int \partial_\alpha T^{\alpha\beta} \, d^3x = 0$$

$$\Rightarrow \partial_0 \underbrace{\int T^{0\beta} \, d^3x}_{\text{Time part}} + \int_V \partial_i \underbrace{T^{i\beta} \, d^3x}_{\text{Space part}} = 0.$$

If we turn this volume integral into surface integral by divergence theorem then we get $\int_S da \, n_i T^{i\beta}$, when the surface goes to infinity all the field die, so it goes to zero.

Then,

$$\Rightarrow \int \partial_0 T^{0\beta} \, d^3x = 0$$

$\Rightarrow$ If we take the volume integral then every thing depends on time, we do not have to write partial derivative we can write it as exact derivative. $\frac{\partial_0}{\partial_0} \rightarrow \frac{d}{\partial t} \rightarrow \frac{d}{dt}$

~~For~~ $\frac{d}{dt}\int T^{00} d^3x + \frac{d}{dt}\int T^{0i} d^3x = 0$

$\Rightarrow \frac{d}{dt} E_{field} = 0$, $\Rightarrow \frac{d}{dt} P_{field} = 0$

$\therefore E_{field} = \text{constant}$ $\therefore P_{field} = \text{constant}$.

Problems with Canonical Stress Tensor ?

Symmetrical Stress Tensor :

To get the symmetrical stress Tensor we manipulate the canonical stress Tensor. The canonical stress Tensor is given by

$$T^{\alpha\beta} = -\frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \underbrace{\partial^{\beta} A^{\lambda}} - g^{\alpha\beta} \mathcal{L}_{em} \quad \text{--- (1)}$$

We can manipulate this as,

$$F^{\lambda\beta} = \partial^{\lambda} A^{\beta} - \partial^{\beta} A^{\lambda}$$

$$\text{So, } \partial^{\beta} A^{\lambda} = -F^{\lambda\beta} + \partial^{\lambda} A^{\beta}$$

So (1) becomes,

$$T^{\alpha\beta} = -\frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \{-F^{\lambda\beta} + \partial^{\lambda} A^{\beta}\} + \frac{g^{\alpha\beta}}{16\pi} F_{\mu\nu} F^{\mu\nu}$$

$$\left[\because \mathcal{L}_{em} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}\right]$$

$$= \underbrace{\frac{1}{4\pi}\left(g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{g^{\alpha\beta}}{4} F_{\mu\nu} F^{\mu\nu}\right)} - \underbrace{\frac{1}{4\pi} g^{\alpha\mu} F_{\mu\lambda} \partial^{\lambda} A^{\beta}}_{T_D^{\alpha\beta}}$$

This expression is free of potential so it is gauge invariant.

Hence, we can call it a symmetric stress tensor $\Theta^{\alpha\beta}$.

So,

$$\Theta^{\alpha\beta} = T^{\alpha\beta} - T_D^{\alpha\beta}$$

$$\therefore \boxed{\Theta^{\alpha\beta} = \frac{1}{4\pi}\left[g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{g^{\alpha\beta}}{4} F_{\mu\nu} F^{\mu\nu}\right]}$$

Different Components of the symmetric - Stress Tensor :

1. Time - Time Component $(\alpha = \beta = 0)$

We have,

$$\theta^{\alpha\beta} = \frac{1}{4\pi}\left(g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{g^{\alpha\beta}}{4} F_{\mu\nu} F^{\mu\nu}\right)$$

$$\theta^{00} = \frac{1}{4\pi}\left(g^{0\mu} F_{\mu\lambda} F^{\lambda 0} + \frac{g^{00}}{4} F_{\mu\nu} F^{\mu\nu}\right)$$

$$= \frac{1}{4\pi}\left(g^{0\mu} F_{\mu\lambda} F^{\lambda 0} + \frac{g^{00}}{4}\left\{-2(E^2-B^2)\right\}\right) \qquad \left[\because \frac{E^2-B^2}{8\pi} = \frac{-1}{16\pi} F_{\mu\nu} F^{\mu\nu}\right]$$

$$= \frac{1}{4\pi}\left[F_{0\lambda} F^{\lambda 0} - \frac{(E^2-B^2)}{2}\right]$$

$$= \frac{1}{4\pi}\left[F_{00} F^{00} + F_{01} F^{10} + F_{02} F^{20} + F_{03} F^{30} - \frac{(E^2-B^2)}{2}\right]$$

$$= \frac{1}{4\pi}\left[0 + E_x^2 + E_y^2 + E_z^2 - \frac{E^2}{2} + \frac{B^2}{2}\right]$$

$$= \frac{1}{4\pi}\frac{(E^2+B^2)}{2} = \frac{1}{8\pi}(E^2+B^2)$$

$$\therefore \theta^{00} = \frac{E^2+B^2}{8\pi}$$

This shows that the time-time component of the symmetric stress tensor represents the energy density of the EM field.
The total energy of the EM field.

$$E_{field} = \int \theta^{00} d^3x .$$

2. Time space Component: $\alpha = 0$, $\beta = i$

We have, $$\theta^{\alpha\beta} = \frac{1}{4\pi}\left[g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{g^{\alpha\beta}}{4} F_{\mu\nu} F^{\mu\nu}\right].$$

$$\theta^{0i} = \frac{1}{4\pi}\left[g^{0\mu} F_{\mu\lambda} F^{\lambda i} + \frac{g^{0i}}{4} F_{\mu\nu} F^{\mu\nu}\right]$$

$$= \frac{1}{4\pi}\left[g^{00} F_{0\lambda} F^{\lambda i}\right].$$

$$= \frac{1}{4\pi}\left[F_{01}F^{1i} + F_{02}F^{2i} + F_{03}F^{3i} + \cancel{F_{00}F^{0i}}^{\,0}\right]$$

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$$\theta^{0i} = \frac{1}{4\pi}\left[F_{01}F^{1i} + F_{02}F^{2i} + F_{03}F^{3i}\right].$$

let $i = 1$

$$\theta^{01} = \frac{1}{4\pi}\left(\cancel{F_{01}F^{10}}^{\,0} + F_{02}F^{21} + F_{03}F^{31}\right)$$

$$= \frac{1}{4\pi}\left[E_y B_z - E_z B_y\right]$$

$$\theta^{01} = \frac{1}{4\pi}\left[\vec{E}\times\vec{B}\right]_x.$$

Similarly $\theta^{02} = \frac{1}{4\pi}\left[\vec{E}\times\vec{B}\right]_y$ & $\theta^{03} = \frac{1}{4\pi}(\vec{E}\times\vec{B})_z$

So,

$$\therefore \theta^{0i} = \frac{1}{4\pi}(\vec{E}\times\vec{B}).$$

The i^{th} time space component of the symmetric stress tensor represents the i^{th} component of the momentum density.

The total momentum can be determined by taking volume integration of θ^{0i} i.e. $cp^i_{field} = \int \theta^{0i} d^3x$.

3. Space-Space Component $(\alpha = i, \beta = j)$

We have,

$$\theta^{\alpha\beta} = \frac{1}{4\pi}\left(g^{\alpha\mu} F_{\mu\lambda} F^{\lambda\beta} + \frac{1}{4} g^{\alpha\beta} F_{\mu\nu}F^{\mu\nu}\right)$$

Then
$$\theta^{ij} = \frac{1}{4\pi}\left(g^{i\mu} F_{\mu\lambda}F^{\lambda j} + \cancel{\frac{1}{4} g^{ij} F_{\mu\nu}F^{\mu\nu}}^{\,0}\right)$$

$$= \frac{1}{4\pi}\left(g^{i\mu} F_{\mu\lambda}F^{\lambda j}\right)$$

let $i = 1, j = 2$

$$\theta^{12} = \frac{1}{4\pi}\left(g^{1\mu} F_{\mu\lambda}F^{\lambda 2}\right)$$

$$= \frac{1}{4\pi} g^{11} F_{1\lambda} F^{\lambda 2}$$

$$= -\frac{1}{4\pi} [F_{1\lambda} F^{\lambda 2}] \quad \because g^{11} = -1$$

$$= -\frac{1}{4\pi} [F_{10} F^{02} + \cancel{F_{11} F^{12}} + F_{13} F^{32} + \cancel{F_{12} F^{22}})$$

$$= -\frac{1}{4\pi} (F_{10} F^{02} + F_{13} F^{32})$$

$F_{10} = -E_x$

$F^{02} = -E_y$.

$$\theta^{12} = -\frac{1}{4\pi} (E_x E_y + B_x B_y)$$

$$\therefore \theta^{ij} = -\frac{1}{4\pi} (E_i E_j + B_i B_j) \qquad \text{For } i \neq j$$

Also, $$\theta^{ii} = -\frac{1}{4\pi} [E_i^2 + B_i^2 - \frac{1}{2} (E^2 + B^2)].$$

In the combined form,

$$\theta^{ij} = -\frac{1}{4\pi} (E_i E_j + B_i B_j - \frac{\delta_{ij}}{2} (E^2 + B^2))$$

Conservation laws of em field by using the Symmetric Stress Tensor:

The differential Conservation law of symmetric stress tensor states that,

$$\partial_\alpha \theta^{\alpha\beta} = 0 \qquad \text{—} \quad (1)$$

$\boxed{\text{Case I: Taking } \beta = 0 \text{ from } (1)}$

$$\partial_\alpha \theta^{\alpha 0} = 0$$

Then, $$\partial_0 \theta^{00} + \partial_1 \theta^{10} + \partial_2 \theta^{20} + \partial_3 \theta^{30} = 0.$$

Since the time-time component θ^{00} = energy density U & the Space time component $\theta^{i0} = c g_i$

We have, $$\partial_0 U + \partial_1 (c g_x) + \partial_2 (c g_y) + \partial_3 (c g_z) = 0$$

$$\Rightarrow \frac{\partial U}{\partial x^0} + \frac{\partial}{\partial x^1}(c g_x) + \frac{\partial}{\partial x^2}(c g_y) + \frac{\partial}{\partial x^3} c(g_z) = 0.$$

$$\Rightarrow \frac{1}{c}\frac{\partial U}{\partial t} + \frac{\partial}{\partial x^1}(c g_x) + \frac{\partial}{\partial x^2}(c g_y) + \frac{\partial}{\partial x^3}(c g_z) = 0$$

or, $\frac{1}{c}\frac{\partial U}{\partial t} + c\,\nabla\cdot g = 0$

$$\Rightarrow \frac{1}{c}\left[\frac{\partial U}{\partial t} + \nabla\cdot c^2 g\right] = 0.$$

$\therefore \frac{1}{c}\left(\frac{\partial U}{\partial t} + \nabla\cdot\vec{S}\right) = 0$, where, $S = c^2\vec{g}$ is the poynting vector which is defined as the energy flow per unit area per unit time. Eq^n (2) represents the continuity equation & hence represents the conservation laws of energy for free-em fields.

Case II : Taking $\beta = i$ then eq^n ① gives,

$$\partial_\alpha \theta^{\alpha i} = 0$$

$$\Rightarrow \partial_0\theta^{0i} + \partial_1\theta^{1i} + \partial_2\theta^{2i} + \partial_3\theta^{3i} = 0$$

Since the space-time component is the c-times the momentum density (g) above becomes,

$$\theta^{0i} = c g_i$$

$$\therefore \frac{1}{c}\frac{\partial (c g_i)}{\partial t} + \sum_{j=1}^{3}\partial_j\,\theta^{ji} = 0$$

Since $\theta^{ji} = -T_M^{ji}$ i.e. maxwell-stress Tensor.

$$\Rightarrow \boxed{\frac{\partial g_i}{\partial t} - \sum_{j=1}^{3}\frac{\partial T}{\partial x^j} = 0.}$$

This Eq^n ③ is the continuity equation & hence represents the conservation law of momentum for the source free Em field.

Case III : Conservation of Angular momentum for the Em-field.