What is Ramsey-equivalent to a clique?

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Abstract

A graph G is Ramsey for H if every two-colouring of the edges of G contains a monochromatic copy of H. Two graphs H and H' are Ramsey-equivalent if every graph G is Ramsey for H if and only if it is Ramsey for H'. In this paper, we study the problem of determining which graphs are Ramsey-equivalent to the complete graph K_k. A famous theorem of Nešetřil and Rödl implies that any graph H which is Ramsey-equivalent to K_k must contain K_k. We prove that the only connected graph which is Ramsey-equivalent to K_k is itself. This gives a negative answer to the question of Szabó, Zumstein, and Zürcher on whether K_k is Ramsey-equivalent to $K_k \cdot K_2$, the graph on $k+1$ vertices consisting of K_k with a pendent edge.

In fact, we prove a stronger result. A graph G is Ramsey minimal for a graph H if it is Ramsey for H but no proper subgraph of G is Ramsey for H. Let $s(H)$ be the smallest minimum degree over all Ramsey minimal graphs for H. The study of $s(H)$ was introduced by Burr, Erdős, and Lovász, where they show that $s(K_k) = (k-1)^2$. We prove that $s(K_k \cdot K_2) = k - 1$, and hence K_k and $K_k \cdot K_2$ are not Ramsey-equivalent.

We also address the question of which non-connected graphs are Ramsey-equivalent to K_k. Let $f(k,t)$ be the maximum f such that the graph $H = K_k + fK_t$, consisting of K_k and f disjoint copies of K_t, is Ramsey-equivalent to K_k. Szabó, Zumstein, and Zürcher gave a lower bound on $f(k,t)$. We prove an upper bound on $f(k,t)$ which is roughly within a factor 2 of the lower bound.

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1 Introduction

A graph G is *H-Ramsey* or *Ramsey for H*, denoted by $G \to H$, if any two-colouring of the edges of G contains a monochromatic copy of H. The fact that for every graph H there is a graph G such that G is H-Ramsey was first proved by Ramsey [10] in 1930 and rediscovered independently by Erdős and Szekeres a few years later [6]. Ramsey theory is currently one of the most active areas of combinatorics with connections to number theory, geometry, analysis, logic, and computer science.

A fundamental problem in graph Ramsey theory is to understand the graphs G that are K_k-Ramsey, where K_k denotes the complete graph on k vertices. The Ramsey number $r(H)$ is the minimum number of vertices of a graph G which is H-Ramsey. The most famous question in this area is that of estimating the Ramsey number $r(K_k)$. Classical results of Erdős [4] and Erdős and Szekeres [6] show that $2^{k/2} \leq r(K_k) \leq 2^{2k}$. While there have been several improvements on these bounds (see, for example, [3]), despite much attention, the constant factors in the above exponents remain the same. Given these difficulties, the field has naturally stretched in different directions. Many foundational results were proved in the 1970s which showed the depth and breadth of graph Ramsey theory. For instance, a famous theorem of Nešetřil and Rödl [9] states that for every graph H there is a graph G with the same clique number as H such that $G \to H$.

Szabó, Zumstein, and Zürcher [11] defined two graphs H and H' to be *Ramsey-equivalent* if for every graph G, G is H-Ramsey if and only if G is H'-Ramsey. The result of Nešetřil and Rödl [9] above implies that any graph H which is Ramsey-equivalent to the clique K_k must contain a copy of K_k. In this paper, we study the problem of determining which graphs are Ramsey-equivalent to K_k. In other words, knowing that G is Ramsey for K_k, what additional monochromatic subgraphs must occur in any two-colouring of the edges of G?

In [11] it was conjectured that, for large enough k, the clique K_k is Ramsey-equivalent to $K_k \cdot K_2$, the graph on $k+1$ vertices consisting of K_k with a pendent edge. We settle this conjecture in the negative, showing that, for all k, the graphs K_k and $K_k \cdot K_2$ are not Ramsey-equivalent. Together with the above discussion, this implies the following theorem.

Theorem 1.1. *Any graph which is Ramsey-equivalent to the clique K_k must be the disjoint union of K_k and a graph of smaller clique number.*

It is therefore natural to study the following function. Let $f(k,t)$ be the maximum f such that K_k and $K_k + f \cdot K_t$ are Ramsey-equivalent, where $K_k + f \cdot K_t$ denotes the disjoint union of a K_k and f copies of K_t. It is easy to see [11] that $f(k,k) = 0$ and $f(k,1) = R(K_k) - k$. For $t \leq k-2$, Szabó et al. [11] proved the lower bound

$$f(k,t) \geq \frac{R(k, k-t+1) - 2k}{2t}, \tag{1}$$

where $R(k,s)$ is the *Ramsey number* denoting the minimum n such that every red-blue edge-colouring of K_n contains a monochromatic red K_k or a monochromatic blue K_s.

We prove the following theorem which, together with (1), determines $f(k,t)$ up to roughly a factor 2.

Theorem 1.2. *For $k > t \geq 3$,*

$$f(k,t) \leq \frac{R(k, k-t+1) - 1}{t}.$$

While our proof does not apply for $t = 2$, we may get an upper bound on $f(k, 2)$ by taking a complete graph on $R(k, k)$ vertices. This is Ramsey for K_k by definition, but is not Ramsey for $K_k + fK_2$ for any f larger than $\frac{R(k,k)-k}{2}$, since for such an f the graph $K_k + fK_2$ has more than $R(k, k)$ vertices. This is within roughly a factor of 4 of the lower bound.

A graph G is H-*minimal* if G is H-Ramsey but no proper subgraph of G is H-Ramsey. We denote the class of all H-minimal graphs by $\mathcal{M}(H)$. Note that G is H-Ramsey if and only if G contains an H-minimal graph, so determining the H-Ramsey graphs reduces to determining the H-minimal graphs. Also, two graph H and H' are Ramsey-equivalent if and only if $\mathcal{M}(H) = \mathcal{M}(H')$.

A fundamental problem of graph Ramsey theory is to understand properties of graphs in $\mathcal{M}(H)$. For example, the minimum number of vertices of a graph in $\mathcal{M}(H)$ is precisely the Ramsey number $r(H)$. Another parameter of interest is $s(H)$, the smallest minimum degree of an H-minimal graph. That is,

$$s(H) := \min_{G \in \mathcal{M}(H)} \delta(G),$$

where $\delta(G)$ is the minimum degree of G.

It is a simple exercise to show [8] that for every graph H, we have

$$2\delta(H) - 1 \leq s(H) \leq r(H) - 1.$$

Somewhat surprisingly, the upper bound is far from optimal, at least for cliques. Indeed, Burr, Erdős, and Lovász [2] proved that $s(K_k) = (k-1)^2$. This is quite notable, as the simple upper bound mentioned above is exponential in k.

Szabó, Zumstein, and Zürcher [11] proved that $s(K_k \cdot K_2) \geq k - 1$, where $K_k \cdot K_2$ is the graph on $k+1$ vertices which contains a K_k and a vertex of degree 1. We prove the following theorem, showing that their lower bound is sharp.

Theorem 1.3. *For all $k \geq 2$,*

$$s(K_k \cdot K_2) = k - 1.$$

Note that Theorem 1.3 implies that K_k and $K_k \cdot K_2$ are not Ramsey-equivalent. Indeed, for $k = 2$ this is trivial, and for $k \geq 3$ we have $(k-1)^2 = s(K_k) > s(K_k \cdot K_2) = k - 1$. Hence, Theorem 1.1 is a corollary of Theorem 1.3.

Organization: In the next section, we prove Theorem 1.3, showing that $s(K_k \cdot K_2) = k - 1$; this implies Theorem 1.1. In Section 3, we prove Theorem 1.2 giving an upper bound on the maximum number $f = f(k,t)$ such that K_k is Ramsey-equivalent to $K_k + f \cdot K_t$. The final section contains relevant open problems of interest.

Conventions and Notation: All colourings are red-blue edge-colourings, unless otherwise specified. For a graph G, we write $V(G)$ for the vertex set of G and $v(G)$ for the number of vertices of G.

2 Hanging edges

In this section, we study the minimum degrees of graphs that are $K_k \cdot K_2$-minimal. Our plan is to construct a graph G that contains a vertex v of degree $k-1$ which is "crucial" for G to be $K_k \cdot K_2$-Ramsey. That is, $G \to K_k \cdot K_2$, but $G - v \nrightarrow K_k \cdot K_2$. Thus, any minimal $K_k \cdot K_2$-Ramsey subgraph $G' \subseteq G$ has to contain v and hence have minimum degree at most $k-1$. We therefore obtain the upper bound for Theorem 1.3.

We now proceed to develop tools useful for proving Theorem 1.3. The following theorem of Nešetřil and Rödl [9] states that there is a K_k-free graph F so that any two-colouring of the edges of F has a monochromatic K_{k-1}.

Theorem 2.1. *For every $k \geq 2$ there is some graph F so that F is K_k-free and $F \to K_{k-1}$.*

By a *circuit of length* s in a hypergraph $\mathcal{H} = (V, \mathcal{E})$ we mean a sequence $e_1, v_1, e_2, v_2, \ldots, e_s, v_s$ of distinct edges $e_1, \ldots, e_s \in \mathcal{E}$ and distinct vertices $v_1, \ldots, v_s \in V$ such that $v_j \in e_j \cap e_{j+1}$ for all $1 \leq j < s$, and $v_s \in e_s \cap e_1$. In particular, if two distinct hyperedges intersect in two or more vertices, we consider this as a circuit of length 2. By the *girth* of a hypergraph $\mathcal{H}$ we denote the length of the shortest circuit in $\mathcal{H}$. The following lemma is proved in [5] by a now standard application of the probabilistic method [1].

Lemma 2.2. *For all integers $k, m \geq 2$ and every $\epsilon > 0$ there is a k-uniform hypergraph of girth at least m and independence number at most ϵn, where n is the number of vertices in the hypergraph.*

We will need a strengthening of Theorem 2.1 which states that there is a K_k-free graph F so that any two-colouring of the edges of F has a monochromatic K_{k-1} inside of every ε fraction of the vertices.

Definition 2.3. We write $F \xrightarrow{\varepsilon} K_k$ to mean that for every $S \subseteq V(F)$, $|S| \geq \varepsilon v(F)$ implies $F[S] \to K_k$.

Lemma 2.4. *For every $\varepsilon > 0$ and $k \geq 2$ there exists a graph F which is K_k-free and $F \xrightarrow{\varepsilon} K_{k-1}$.*

Proof. The case where $k = 2$ is trivial, so we will assume that $k \geq 3$. Take F_0 to be as in Theorem 2.1. By Lemma 2.2 there is some $v(F_0)$-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ of girth at least 4 and independence number less than $\varepsilon |V|$. We construct a graph F on vertex set V. The edges of F are created by placing a copy of F_0 inside of each hyperedge in $\mathcal{E}$.

Since $\mathcal{H}$ has girth at least 4, any triangle of F must be contained in a single hyperedge of $\mathcal{H}$. Therefore, the vertex set of any copy of K_k in F must be contained in a single hyperedge of $\mathcal{H}$ as well. However, a single hyperedge forms just a copy of F_0 in F and F_0 has no copy of K_k, so F has no copy of K_k.

Since $\mathcal{H}$ has independence number less than $\varepsilon |V|$, any set S of at least $\varepsilon |V|$ vertices must contain some hyperedge. Hence, $F[S]$ contains a copy of F_0. As $F_0 \to K_{k-1}$, we also have $F[S] \to K_{k-1}$. □

From this F we construct a gadget graph G_0 with a useful property, namely that a particular copy of K_k is forced to be monochromatic.

Lemma 2.5. *There exists a graph G_0 with a subgraph H isomorphic to K_k contained in G_0 such that*

1. there is a colouring of G_0 without a red $K_k \cdot K_2$ and without a blue K_k and

2. every colouring of G_0 without a monochromatic copy of $K_k \cdot K_2$ results in H being monochromatic.

In order to prove that H must be monochromatic in the above lemma, we will employ a technique we call colour focusing.

Lemma 2.6. *(Focusing Lemma) Let $G = (A \cup B, E)$ be a complete bipartite graph with a colouring $\chi : E \to \{red, blue\}$ of its edges. Then there exist subsets $A' \subseteq A$ and $B' \subseteq B$ with $|A'| \geq |A|/2$ and $|B'| \geq |B|/2^{|A|}$, such that*

(a) for every vertex $a \in A$, the set of edges from a to B' is monochromatic, and

(b) χ is constant on the edges between A' and B'.

Proof. Define, for some vertex $b \in B$, the colour pattern $\mathbf{c}_b$ of b to be the function with domain A that maps a vertex $a \in A$ to the colour of the edge $\{a, b\}$. Consider the most common of the $2^{|A|}$ possible colour patterns the vertices in B might have towards A and call it $\mathbf{c}$. We define $B' \subseteq B$ to be the set of vertices having colour pattern $\mathbf{c}$. By the pigeonhole principle $|B'| \geq |B|/2^{|A|}$. Now, each vertex a in A has only edges of colour $\mathbf{c}(a)$ to B', which proves part (a). Then, each vertex in A has either only red or only blue edges to B'. Therefore, for some colour $c \in \{red, blue\}$, at least half of the vertices in A have only edges of colour c towards B'. This is the set we choose to be A', concluding the proof of part (b). $\square$

We now use part (a) to prove Lemma 2.5.

Proof of Lemma 2.5. If $k = 2$ then taking G_0 to be a single edge suffices. We will henceforth assume $k \geq 3$. Take $\varepsilon = 2^{-k^2}$ and let $F_1, \ldots, F_{k-2}$ be copies of the graph F from Lemma 2.4. Add complete bipartite graphs between any two of these copies. Add a copy H of K_k and connect it to every vertex in every F_i. The resulting graph is G_0 (see Figure 1). To show $G_0 \nrightarrow K_k \cdot K_2$, colour all edges inside every F_i and inside H red, and all the remaining edges blue. The largest red clique is H, with only blue edges leaving H. The F_i are K_k-free, and any edge leaving F_i is blue as well. Since the graph of blue edges is $(k-1)$-chromatic ($F_1, \ldots, F_{k-2}, H$ is a partition into independent sets), the largest blue clique has order $k - 1$. This verifies (1).

For (2), assume χ is a red-blue colouring of the edges of G_0 without a monochromatic $K_k \cdot K_2$. We show that this forces H to be monochromatic. By taking $A = V(H)$ and $B = V(F_1)$ in part (a) of the Focusing Lemma, we find a subset $S_1 \subseteq V(F_1)$ such that $|S_1| \geq 2^{-k} v(F)$ and for each $a \in A$ the edges from a to S_1 are monochromatic (see Figure 2a). Then $|S_1| > \varepsilon v(F)$, hence $F_1[S_1] \to K_{k-1}$. Fix a monochromatic copy H_1 of K_{k-1} contained in S_1, and assume without loss of generality that H_1 is red. We claim that all edges between $V(H)$ and S_1 (and in particular to $V(H_1)$) are blue. Indeed, if one vertex i of H had red edges to S_1, then i along with H_1 and one (arbitrary) other vertex v of S_1 would form a red copy of $K_k \cdot K_2$, a contradiction to our assumption on the colouring χ.

We now iterate this argument. Assume we have found red cliques $H_1, \ldots, H_{t-1}$ in $F_1, \ldots, F_{t-1}$ with vertex sets $V_1, \ldots, V_{t-1}$, respectively, and that all the edges between these cliques as well as to H are blue. By part (a) of the Focusing Lemma, in F_t there is some subset $S_t \subseteq V(F_t)$ of the vertices of size at least $2^{-tk} v(F_t)$, so that each vertex $v \in V(H) \cup V_1 \cup V_2 \cup \cdots \cup V_{t-1}$ is

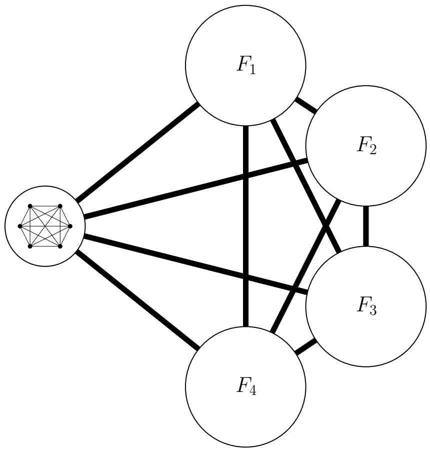


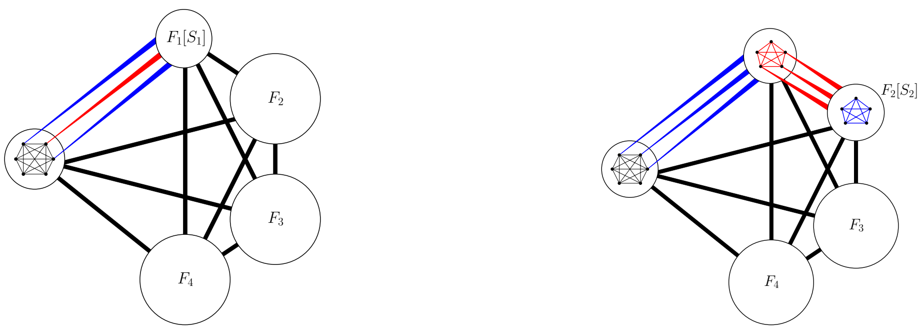
Figure 1: The gadget graph G_0 in Lemma 2.5 for $k = 6$. A thick line indicates that the vertices of the corresponding sets are pairwise connected.

monochromatic to S_t. Since $|S_t| > \varepsilon v(F_t)$, we have $F_t[S_t] \to K_{k-1}$. We find a monochromatic copy of K_{k-1} in S_t and call it H_t. Assume for contradiction that H_t is blue. In this case as before, all the edges between H_t and H as well as between H_t and $H_1, \ldots, H_{t-1}$ would have to be red, otherwise there would be a blue $K_k \cdot K_2$. But if all these edges are red, then any two vertices of H_t together with H_1 form a red $K_k \cdot K_2$ (see Figure 2b). Hence, H_t must be red, and as before all edges between H_t and H as well as between H_t and $H_1, \ldots, H_{t-1}$ must be blue.

After applying this argument to F_{k-2}, we have a collection $H_1, \ldots, H_{k-2}$ of red $(k-1)$-cliques and complete bipartite blue graphs between any two of $H, H_1, \ldots, H_{k-2}$. Now, if some edge in H were blue, this edge along with one vertex from each of $H_1, \ldots, H_{k-2}$ and any (arbitrary) other vertex from H_1 would create a blue $K_k \cdot K_2$. Therefore, every edge of H must be red, as desired. □

The following lemma completes the proof of Theorem 1.3.

Lemma 2.7. *For every $k \geq 3$ there is a graph G which contains a vertex v of degree $k-1$*



(a) Colour-focusing between $H = K_6$ and F_1.

(b) There cannot be a blue K_5 in $F_2[S_2]$.

Figure 2: Illustrating the proof of Lemma 2.5.

so that $G \to K_k \cdot K_2$ but $G - v \nrightarrow K_k \cdot K_2$.

Proof. Take $k-1$ copies $G_1, \dots, G_{k-1}$ of the gadget graph G_0 from Lemma 2.5, and let $H_1, \dots, H_{k-1}$ be the copies of K_k guaranteed to be monochromatic in any colouring without a monochromatic $K_k \cdot K_2$. Pick one vertex v_i in each H_i, and insert all edges between the v_i, so they form a K_{k-1}. In addition, pick an arbitrary vertex $v_k \neq v_2$ from $V(H_2)$ and insert an edge between it and v_1. Finally, add a vertex v to the graph, and connect it to $v_1, \dots, v_{k-1}$. This completes the construction of G (see Figure 3). Clearly, $\deg(v) = k-1$.

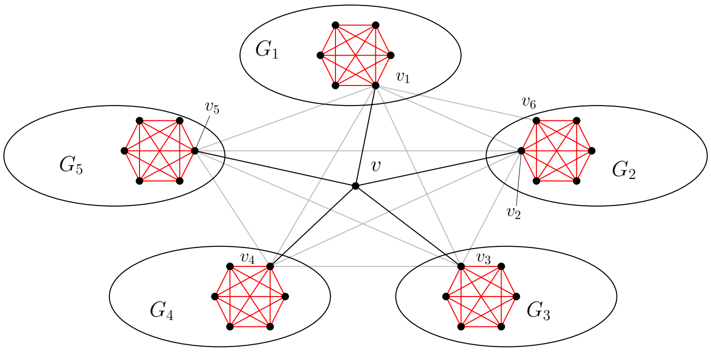


Figure 3: An example of the graph G in Lemma 2.7 for $k = 6$.

To see that $G - v \nrightarrow K_k \cdot K_2$, colour each G_i so it has no red $K_k \cdot K_2$ and no blue K_k. By property (2) of the gadget G_0 this also means that every H_i is monochromatic red. Colour the edges between $\{v_1, \dots, v_{k-1}\}$ and the additional edge $\{v_1, v_k\}$ blue. Since none of the G_i had a red $K_k \cdot K_2$ and we did not add any red edges, this colouring has no red $K_k \cdot K_2$. The G_i have no blue K_k, and for $i = 1, \dots, k-1$ the vertex v_i has no blue edges leaving G_i except those to the other v_j. But the edge $\{v_2, v_k\}$ is red, therefore there is no blue K_k and in particular no blue $K_k \cdot K_2$.

Finally, we show that $G \to K_k \cdot K_2$. Let any colouring of G be given, and suppose none of the copies of G_0 contains a monochromatic copy of $K_k \cdot K_2$. Then all of $H_1, \dots, H_{k-1}$ are monochromatic. We claim they have the same colour. Indeed, if H_i and H_j had different colours, then the edge $v_i v_j$ would induce a monochromatic $K_k \cdot K_2$ with whichever copy of K_k had the same colour as its own.

So all of the H_i have the same colour; without loss of generality, let this colour be red. If any of the edges $v_i v_j$, for $1 \leq i < j \leq k-1$ or for $i = 1$, $j = k$ were red, then along with H_i it would form a red $K_k \cdot K_2$. Similarly, if any of the edges vv_i were red, then along with H_i it would induce a red $K_k \cdot K_2$. Otherwise, all these edges are blue and then $v, v_1, \dots, v_{k-1}$ and v_k form a blue $K_k \cdot K_2$, as desired. $\square$

3 Clique with some disjoint smaller cliques

Recall that $K_k + f \cdot K_t$ denotes the disjoint union of a K_k and f copies of K_t. Also, $f(k,t)$ is the largest number f so that K_k and $K_k + f \cdot K_t$ are Ramsey-equivalent. In this section,

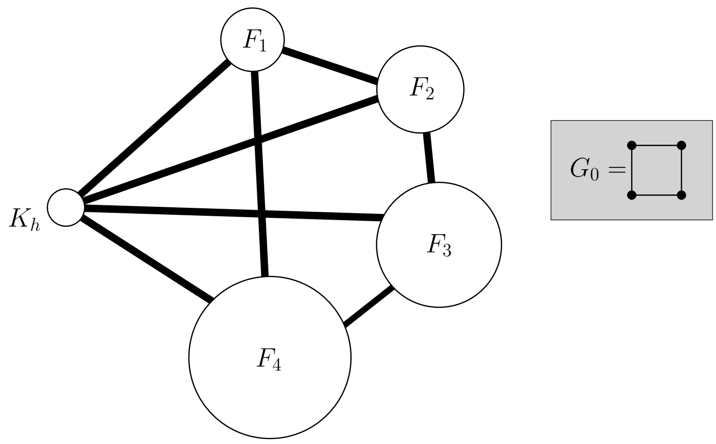


Figure 4: An illustration of $G(h, G_0, F_1, F_2, F_3, F_4)$ when $G_0 = C_4$. A thick line indicates that the vertices of the corresponding sets are pairwise connected.

we prove Theorem 1.2, which gives an upper bound on $f(k,t)$ for $t \geq 3$ and determines it up to roughly a factor of 2.

Proof of Theorem 1.2. Let $f = \left\lfloor \frac{R(k,k-t+1)-1}{t} \right\rfloor + 1$. We will construct a graph G with the following two properties.

$(G1)$ $G \to K_k$ and

$(G2)$ $G \nrightarrow K_k + f \cdot K_t$.

Construction of G.
G will be constructed by combining a number of smaller graphs. For a positive integer h and graphs $G_0, F_1, \dots, F_{n_0}$ where n_0 is the number of vertices of G_0, define $G = G(h, G_0, F_1, \dots, F_{n_0})$ as follows.

Take pairwise disjoint sets V_H and V_i, $1 \leq j \leq n_0$, such that $|V_H| = h$ and $|V_j| = |V(F_j)|$. Set $V := V_H \cup \bigcup_{j=1}^{n_0} V_j$. Label the vertices of H as $v_1, \dots, v_h$. The edge set E is defined as follows.

- $G[V_H] \cong K_h$,
- $G[V_j] \cong F_j$ for all $1 \leq j \leq n_0$,
- $v_i w \in E(G)$ for all $1 \leq i \leq h$, and all $w \in \bigcup_{j=1}^{n_0} V_j$,
- for all $u \in V_i$, $w \in V_j$, $uw \in E(G)$ if and only if $ij \in E(G_0)$.

That is, our gadget graph consists of one copy of each F_j together with a copy of a complete graph on h vertices. Furthermore, we place a complete bipartite graph between F_i and F_j whenever ij is an edge in G_0, and a complete bipartite graph between V_H and $\bigcup_{j=1}^{n_0} F_j$ (see Figure 4).

Set $h := R(k, k-t+1) + k - 1$ and $\varepsilon_0 := 2^{-h-1}$. Let G_0 be a graph (given by Lemma 2.4) such that

$$K_{k-1} \not\subseteq G_0 \quad \text{and} \quad G_0 \xrightarrow{\varepsilon_0} K_{k-2}. \tag{2}$$

Now, set $n_0 := v(G_0)$ and assume without loss of generality that $V(G_0) = [n_0]$. For every $1 \leq j \leq n_0$, we define F_j iteratively. First, let $\varepsilon_1 := 2^{-(h+n_0)}$ and let F_1 be a graph (given by Lemma 2.4) such that $K_t \not\subseteq F_1$ and $F_1 \xrightarrow{\varepsilon_1} K_{t-1}$. For $2 \leq j \leq n_0$, assume we have defined $\varepsilon_1, \dots, \varepsilon_{j-1}$ and $F_1, \dots, F_{j-1}$. We then set

$$\varepsilon_j := 2^{-\left(h+n_0-j+\sum_{i=1}^{j-1} v(F_i)\right)} \tag{3}$$

and let F_j be a graph (given by Lemma 2.4) such that

$$K_t \not\subseteq F_j \quad \text{and} \quad F_j \xrightarrow{\varepsilon_j} K_{t-1}. \tag{4}$$

Define the graph $G := G(h, G_0, F_1, \dots, F_{n_0})$, and take $V = V(G), E = E(G)$. Take H to be the copy of K_h.

We now show that G fulfills the two conditions $(G1)$ and $(G2)$ above.

The graph G has property $(G2)$.
To see that $G \nrightarrow K_k + f \cdot K_t$, colour all edges inside H and inside the copy of each F_j red, and all edges between H and F_j's blue. Then the largest blue clique has size $k-1$ (since G_0 is K_{k-1}-free). So any monochromatic copy of $K_k + f \cdot K_t$ would need to be red. Since all the F_j's are K_t-free, the red copy of $K_k + f \cdot K_t$ needs to lie inside H. However, $v(K_k + f \cdot K_t) = k + ft \geq k + R(k, k-t+1) > v(H)$. So H cannot host a copy of $K_k + f \cdot K_t$.

The graph G has property $(G1)$.
Let $\chi : E \to \{\text{red, blue}\}$ be a 2-colouring of G. We apply a similar "colour-focusing" procedure as in the proof of Lemma 2.5. This technique is used to obtain Lemma 3.1, which shows that there is a vertex subset for which the colouring is highly structured. From this lemma, it is not difficult to prove that there must be a monochromatic K_k.

Lemma 3.1. *There exist a subset $J \subseteq [n_0]$ and subsets $W_j \subseteq V_j$ for each $j \in J$ such that the following holds.*

- *(a)* $|J| \geq n_0/2^h = 2\varepsilon_0 n_0$,
- *(b) for all $j \in J$, W_j is the vertex set of a monochromatic K_{t-1} under χ,*
- *(c) for all $i, j \in J$ with $ij \in E(G_0)$, there exists $c_{ij} \in \{$red, blue$\}$ such that for all $u \in W_i$, $w \in W_j$, $\chi(uw) = c_{ij}$.*
- *(d) for all $v_i \in V_H$, there exists $c_i \in \{$red, blue$\}$ such that for all $u \in \bigcup_{j \in J} W_j$, $\chi(v_i u) = c_i$.*

The structure of the sets J and W_j in Lemma 3.1 is depicted in Figure 5. Before proving the lemma, we first show how it implies that there is a monochromatic K_k in G, which implies $(G1)$.

Proof of $(G1)$ assuming Lemma 3.1.
Let $J' \subseteq J$ with $|J'| \geq |J|/2$ be such that all W_j with $j \in J'$ are monochromatic of the *same* colour. Consider the induced subgraph $G_0' := G_0[J']$ of G_0. Let χ' be the edge-colouring of G_0'

where each edge $ij \in E(G_0')$ has colour $\chi'(ij) := c_{ij}$. Since $|J'| \geq |J|/2 \geq \varepsilon_0 n_0$ by property (a), and since $G_0 \xrightarrow{\varepsilon_0} K_{k-2}$ by definition of G_0, there exists a monochromatic copy of K_{k-2} in G_0' under χ'. Let $I \subseteq J'$ denote the vertex set of this monochromatic copy, and assume without loss of generality that it is blue. Then, for all $i, j \in I$, $i \neq j$, the sets W_i and W_j are connected by complete bipartite graphs, all edges being blue under χ. The monochromatic W_j with $j \in I \subseteq J$ are all the same colour. If they were all blue, the union of the W_j with $j \in I$, each of which is of order $t-1$ by property (b), form a monochromatic blue clique of order $(k-2)(t-1) \geq k$ (since $k > t \geq 3$), and thus there is a monochromatic K_k. Therefore, we may assume from now on that each W_j, $j \in I$, is a red K_{t-1}.

Consider now the vertices in V_H. Any such vertex has either only red edges or only blue edges to $\bigcup_{j \in J'} W_j$, by property (d). We call $v_i \in V_H$ *red* if $c_i = red$, and *blue* otherwise. Suppose there exist two vertices, $v_i, v_j \in V_H$ which are both *blue*, such that $\chi(v_i v_j) = \text{blue}$. Then they form a blue K_k with one vertex from each $W_j, j \in I$. So we can assume that for two *blue* vertices $v_i, v_j \in V_H$ we have $\chi(v_i v_j) = \text{red}$. But then we can also assume that there are at most $k-1$ *blue* vertices inside H, since otherwise they form a red K_k inside H. So, there are at least $v(H) - (k-1) = R(k, k-t+1)$ *red* vertices $V_{red} \subseteq V_H$ in H. By definition of $R(k, k-t+1)$, V_{red} contains either a red K_{k-t+1} or a blue K_k. In the second case, we are done. In the first case, the vertex set $V_{red} \cup W_j$ contains a red K_k for any $j \in I$, so we are done as well. $\square$

Proof of Lemma 3.1.
We prove the lemma in two steps. First, we apply part (a) of the Focusing Lemma with $A = V_H$ and each $V(F_j)$ as B in order to ensure property (d). Then, in order to ensure property (c), we restrict to smaller and smaller sets inside $V(F_j)$ by repeatedly applying part (b) of the Focusing Lemma. These two steps are illustrated in Figure 6.

Recall that we are given a 2-colouring $\chi : E \to \{red, blue\}$ of the edge set of G. First we show that there exists an index set $J \subseteq [n_0]$ and subsets $V_j' \subseteq V_j$ for each $j \in J$ such that the following properties hold.

(a) $|J| \geq n_0/2^h$,

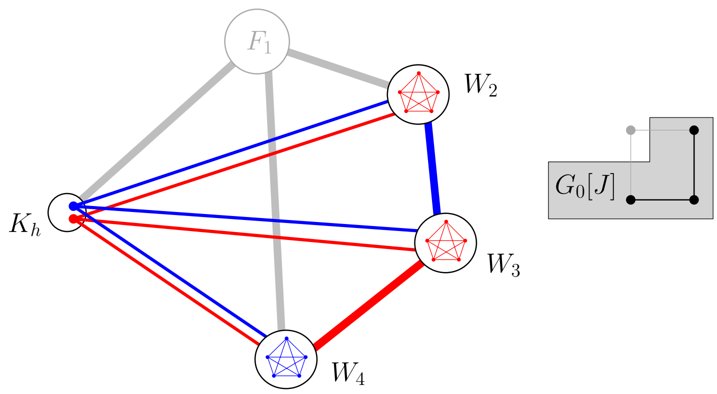
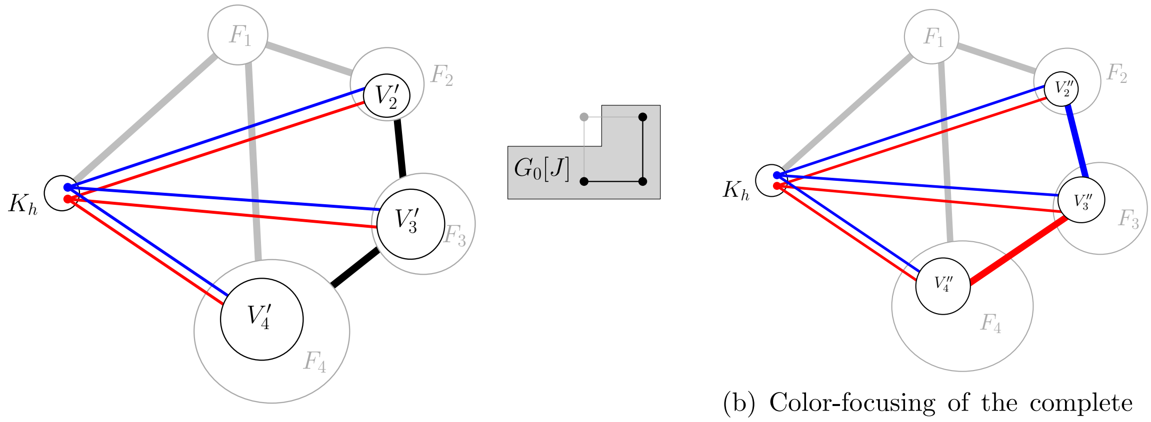


Figure 5: The colour patterns we find with Lemma 3.1.

(b') for all $j \in J$, $|V'_j| \geq v(F_j)/2^h$, and

(d') for all $v_i \in V_H$, there exists $c_i \in \{red, blue\}$ such that for all $u \in \bigcup_{j \in J} V'_j$, $\chi(v_i u) = c_i$.

To see this, for each $j \in [n_0]$ apply part (a) of the Focusing Lemma to the complete bipartite graphs between V_H and V_j to obtain subsets $V'_j \subseteq V_j$ of size at least $v(F_j)/2^h$ such that for each vertex $v \in V_H$ and $j \in [n_0]$, the set of edges between v and V'_j is monochromatic. In other words, for each index $j \in [n_0]$ there is a function $\mathbf{c}_j : V_h \to \{red, blue\}$ where $\mathbf{c}_j(v_i)$ is the colour of the edges from v_i to V'_j. There are 2^h possible functions, so there must be a set $J \subseteq [n_0]$ of at least $n_0/2^h$ indices with a function $\mathbf{c}$ such that for any $j \in J$ we have $\mathbf{c}_j = \mathbf{c}$. Choosing $c_i := \mathbf{c}(v_i)$ guarantees property (d').



(a) Color-focusing of vertices of K_h.

(b) Color-focusing of the complete bipartite graphs

Figure 6: The colour patterns we find in G.

In the remainder of the proof we consider only the vertices in the sets V'_j we have just defined. We will maintain subsets $V''_j \subseteq V'_j$, starting with $V''_j = V'_j$, and keep reducing their size until the edges between them are monochromatically coloured for each pair.

For ease of notation we assume $J = [\ell]$. For each pair $i, j \in J$ with $ij \in E(G_0)$, we apply part (b) of the Focusing Lemma for the complete bipartite graph between the sets V''_i and V''_j, where V''_i plays the role of A and V''_j plays the role of B if $i < j$. These applicatons are done one after another, in an arbitrary order, and after each of them the participating subsets V''_i and V''_j are redefined to be the subsets $A' \subseteq A = V''_i$ and $B' \subseteq B = V''_j$ given by the Focusing Lemma. Hence, after an application for the pair i, j, the edges between the sets V''_i and V''_j are monochromatic.

Let $i \in J$ be an arbitrary index. The set V''_i participates in an application of the Focusing Lemma $(i-1)$-times as the set B and $(\ell - i)$-times as the set A. The size of V''_i might be reduced with each application, but the Focusing Lemma gives us a lower bound on the new size: it is at least half of the old size if V''_i participated as A and it is at least the $2^{-|V''_j|}$-fraction if V''_i participated as B together with some other set V''_j as A (with $j < i$). Since we know how many times V''_i participated as the set A and how many times as the set B, we have a bound on its order at the end:

$$|V_i''| \geq \frac{v(F_i)}{2^h} \cdot \frac{1}{2^{\ell-i}} \cdot \frac{1}{2^{\sum_{j=1}^{i-1} v(F_j)}} \geq v(F_i) \cdot 2^{-\left(h+n_0-i+\sum_{j<i} v(F_j)\right)} = \varepsilon_i \cdot v(F_i),$$

where we used property (b') to estimate the size of V_i' at the beginning.

Since we applied the Focusing Lemma for every pair $i, j \in J$, $ij \in E(G_0)$, there exist $c_{ij} \in \{red, blue\}$ such that the edges between V_i'' and V_j'' are monochromatic of colour c_{ij}, for every such pair.

It is now straight-forward to see that Lemma 3.1 follows. Since each $F_i \xrightarrow{\varepsilon_i} K_{t-1}$ and by the above V_i'' at the end has size at least $\varepsilon_i v(F_i)$, V_i'' does host a monochromatic K_{t-1}. Let W_i be the vertex set of this K_{t-1}. Now, since $W_i \subseteq V_i'' \subseteq V_i'$, (c) and (d) follow.

As we saw earlier, the proof of Lemma 3.1 completes the proof of Theorem 1.2. $\square$

4 Open problems

Despite the progress made in this paper, we note the following interesting problems that remain open.

Recall that $f(k,t)$ is the maximum f such that K_k and $K_k + f \cdot K_t$ are Ramsey-equivalent. We determined $f(k,t)$ up to roughly a factor 2 for $k-1 > t > 2$. It would be of interest to close the gap between the lower and upper bounds.

Problem 4.1. *Determine $f(k,t)$.*

A special case of this problem already asked in [11] is the following. Note that we have shown that $f(k,k-1) \leq 1$. That is, if K_k and $K_k + K_{k-1}$ are Ramsey-equivalent, then $f(k,k-1) = 1$ and otherwise $f(k,k-1) = 0$. It is easy to see that $f(2,1)$ and $f(3,2)$ are 0. We conjecture that for larger k we have $f(k,k-1) = 1$.

Conjecture 4.2. *For k at least 4, K_k and $K_k + K_{k-1}$ are Ramsey-equivalent.*

We proved that every graph (other than K_k) that is Ramsey-equivalent to K_k is not connected. This naturally leads to the following question.

Question 4.3. *Is there a pair of non-isomorphic connected graphs H_1, H_2 that are Ramsey-equivalent?*

An interesting special case of this question is about pairs of graphs such that one contains the other. This motivates the following question.

Question 4.4. *Is there a connected graph H which is Ramsey-equivalent to a graph formed by adding a pendent edge to H?*

We have recently shown [7] that $K_{t,t}$ and $K_{t,t} \cdot K_2$, the graph formed by adding a pendent edge to $K_{t,t}$, are *not* Ramsey-equivalent. Furthermore, we proved $s(K_{t,t} \cdot K_2) = 1$ while it was shown in [8] that $s(K_{t,t}) = 2t-1$.

We do not have a good understanding of how large of a connected subgraph can be added to K_k and still be Ramsey-equivalent to K_k. For example, we have the following problem.

Problem 4.5. *Let $g(k)$ be the maximum g such that K_k is Ramsey-equivalent to $K_k + K_{1,g}$, the disjoint union of K_k and the star $K_{1,g}$ with g leaves. Determine $g(k)$.*

We only know that $g(k)$ is at least linear in k and at most exponential in k.

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