Problems and results on 1-cross intersecting set pair systems

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Abstract

The notion of cross intersecting set pair system of size m, $\Big(\{A_i\}_{i=1}^m, \{B_i\}_{i=1}^m\Big)$ with $A_i \cap B_i = \emptyset$ and $A_i \cap B_j \neq \emptyset$, was introduced by Bollobás and it became an important tool of extremal combinatorics. His classical result states that $m \leq \binom{a+b}{a}$ if $|A_i| \leq a$ and $|B_i| \leq b$ for each i.

Our central problem is to see how this bound changes with the additional condition $|A_i \cap B_j| = 1$ for $i \neq j$. Such a system is called 1-cross intersecting. We show that the maximum size of a 1-cross intersecting set pair system is

- at least $5^{n/2}$ for n even, $a = b = n$,
- equal to $\big(\lfloor \frac{n}{2} \rfloor + 1\big)\big(\lceil \frac{n}{2} \rceil + 1\big)$ if $a = 2$ and $b = n \geq 4$,
- at most $|\cup_{i=1}^m A_i|$,
- asymptotically n^2 if $\{A_i\}$ is a linear hypergraph ($|A_i \cap A_j| \leq 1$ for $i \neq j$),
- asymptotically $\frac{1}{2}n^2$ if $\{A_i\}$ and $\{B_i\}$ are both linear hypergraphs.

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1 Introduction, results

The notion of cross intersecting set pair systems was introduced by Bollobás [4] and it became a standard tool of extremal set theory. Because of its importance there are many proofs (e.g., Lovász [13], Kalai [11]) and generalizations (e.g., Alon [1], Füredi [6]). For applications and extensions of the concept the surveys of Füredi [7] and Tuza [14, 15] are recommended.

A *cross intersecting set pair system of size* $m \geq 2$ consists of finite sets $A_1, \dots, A_m$ and $B_1, \dots, B_m$ such that

$$A_i \cap B_i = \emptyset \text{ for every } 1 \leq i \leq m,$$

$$A_i \cap B_j \neq \emptyset \text{ for every } 1 \leq i \neq j \leq m.$$

We will consider further constrains but always keep these two basic properties.

Bollobás' theorem [4] states that

$$m \leq \binom{a+b}{a} \tag{1}$$

must hold for any cross intersecting set pair system if we have $|A_i| \leq a$ and $|B_i| \leq b$ for each i. This size can be achieved by the *standard example*, taking all a-element sets of an $(a+b)$-element set for the A_i-s and their complements as B_i-s.

Let $\mathcal{A} = \{A_i\}_{i=1}^m$ and $\mathcal{B} = \{B_i\}_{i=1}^m$. The set pair system (SPS for short) is denoted by $(\mathcal{A}, \mathcal{B}) = \{(A_i, B_i)\}_{i=1}^m$. The union of the two hypergraphs is denoted by $\mathcal{H} = \mathcal{A} \cup \mathcal{B}$. An SPS is (a, b)-*bounded* if $|A_i| \leq a$ and $|B_i| \leq b$ for each i.

For $v \in V$ we denote by $d_{\mathcal{A}}(v)$, $d_{\mathcal{B}}(v)$, $d_{\mathcal{H}}(v)$ the degree of v in the hypergraphs $\mathcal{A}, \mathcal{B}, \mathcal{H}$, respectively.

1.1 1-cross intersecting SPS of exponential sizes

An SPS $(\mathcal{A}, \mathcal{B})$ is 1-*cross intersecting* if $|A_i \cap B_j| = 1$ for each $i \neq j$. To find good estimates for the size under this condition leads to interesting but seemingly difficult problems. Somewhat surprisingly, 1-cross intersecting (n, n)-bounded SPS can have exponential size because of the following proposition.

Proposition 1.1. *If (a_1, b_1)-bounded and (a_2, b_2)-bounded 1-cross intersecting SPS exist with sizes m_1 and m_2, then (a_1+a_2, b_1+b_2)-bounded 1-cross intersecting SPS also exists with size $m_1 \cdot m_2$.*

The proof of this, and most other proofs, too, are postponed to later sections.

Starting from the standard example (with $a = b = 1$ and $m = 2$), Proposition 1.1 yields an (n, n)-bounded 1-cross intersecting SPS of size 2^n, exponential in n. Define the $(2, 2)$-bounded 1-cross intersecting SPS, called $\mathcal{H}(2, 2)$, of size five with the pairs $(\{i, i+1\}, \{i+2, i+4\})$ taken $\pmod 5$. Then Proposition 1.1 gives the following.

Corollary 1.2. *There exists an (n,n)-bounded 1-cross intersecting SPS of size $5^{n/2}$ if n is even and of size $2 \cdot 5^{(n-1)/2}$ if n is odd.* $\square$

This is the best lower bound known to us. It remains a challenge to decrease essentially the upper bound $\binom{2n}{n}$ in (1) for an (n,n)-bounded 1-cross intersecting SPS.

Corollary 1.2 gives a $(3,3)$-bounded 1-cross intersecting SPS of size 10, in fact two different ones, with 12 and with 15 vertices, depending on the order we apply Proposition 1.1. We have a third example, the pairs $(\{i, i+1, i+2\}, \{i+3, i+6, i+9\})$ taken $\pmod{10}$ has 10 vertices. It seems to be difficult to decide whether 10 is the largest size. The best upper bound we can prove is 12 (see Section 6).

One particular feature of a 1-cross intersecting SPS $(\mathcal{A}, \mathcal{B})$ is that its size is bounded by the sizes of the vertex sets of $\mathcal{A}$ (and $\mathcal{B}$). This can be considered as a variant of Fischer's inequality, and does not hold for general SPS.

Proposition 1.3. *Assume that $(\mathcal{A}, \mathcal{B})$ is 1-cross intersecting and $V := \cup\mathcal{A}$. Then the incidence vectors of the edges of $\mathcal{A}$ are linearly independent in $\mathbb{R}^V$.*

1.2 The case of $a = 2$, $b = n$

The main result of this subsection is the solution for $a = 2, b = n$, showing that the main term of the upper bound $\frac{1}{2}(n+2)(n+1)$ in (1) can be halved.

Theorem 1.4. *If $n \geq 4$, then a $(2,n)$-bounded 1-cross intersecting SPS has size at most*

$$\left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lceil \frac{n}{2} \right\rceil + 1\right).$$

This bound is the best possible. For $n = 2, 3$ the exact values are $m = 5, 7$.

The proof of Theorem 1.4 gives all *extremal* systems, i.e., systems with the maximum possible size for a given n. Since $\mathcal{A}$ is a graph, we only describe the graph part of the extremal systems, the corresponding $\mathcal{B}$ can be easily found.

Corollary 1.5. *The extremal systems for Theorem 1.4 are the following ones. $\mathcal{A} = C_5$ if $n = 2$, $\mathcal{A} = C_7$ if $n = 3$, $\mathcal{A} = C_9$ or $\mathcal{A} = 3 \cdot K_{1,3}$ if $n = 4$. For $n > 4$, $\mathcal{A} = k \cdot K_{1,k}$ if $n = 2k-2$; and either $\mathcal{A} = k \cdot K_{1,k+1}$ or $\mathcal{A} = (k+1) \cdot K_{1,k}$ if $n = 2k-1$.*

1.3 Intersection restrictions

A hypergraph $\mathcal{H}$ is called *linear* if the intersection of any two different edges has at most one vertex. $\mathcal{H}$ is called 1-*intersecting* if $|H \cap H'| = 1$ for all $H, H' \in \mathcal{H}$ whenever $H \neq H'$.

Although the main results of this article are about linear and 1-intersecting families we propose the problem in a very general setting. Let a, b positive integers and $I_A, I_B, I_{\text{cross}}$ three sets of non-negative integers. We denote by $m(a, b, I_A, I_B, I_{\text{cross}})$ the maximum size m of a cross intersecting SPS $(\mathcal{A}, \mathcal{B})$ with the following conditions.

i) $A_i \cap B_i = \emptyset$ for every $1 \le i \le m$,

ii) $|A_i| \le a$ for every $1 \le i \le m$,

iii) $|B_i| \le b$ for every $1 \le i \le m$,

iv) $|A_i \cap A_j| \in I_A$ for every $1 \le i \ne j \le m$,

v) $|B_i \cap B_j| \in I_B$ for every $1 \le i \ne j \le m$,

vi) $0 < |A_i \cap B_j| \in I_{\text{cross}}$ for every $1 \le i \ne j \le m$.

To avoid trivialities we always suppose that $0 \notin I_{\text{cross}}$, also that $m \ge 2$. If a constraint in iv)–vi) is vacuous (i.e., $\{0, 1, \dots, |X|\} \subseteq I_X$ or $\{1, \dots, \min\{a, b\}\} \subseteq I_{\text{cross}}$) then we use the symbol $*$ to indicate this. With this notation Bollobás' theorem [4] states

$$m(a, b, *, *, *) = \binom{a+b}{a},$$

and our Theorem 1.4 states (for $n \ge 4$)

$$m(2, n, *, *, 1) = \left(\left\lfloor \frac{n}{2} \right\rfloor + 1\right)\left(\left\lceil \frac{n}{2} \right\rceil + 1\right).$$

In the rest of the results we deal with the case $a = b = n$ and use the abbreviation of placing n as an index

$$m_n(I_A, I_B, I_{\text{cross}}) := m(n, n, I_A, I_B, I_{\text{cross}}).$$

Since in this paper the main results are about linear hypergraphs, we will have I_A (and also I_B) is either $\{0, 1\}$ ($\mathcal{A}$ is a linear hypergraph), or $\{1\}$ ($\mathcal{A}$ is a 1-intersecting hypergraph), or $*$. Instead of writing $I_A = \{1\}$ we write '1-int', instead of $I_A = \{0, 1\}$ we write '01-int', and for $I_{\text{cross}} = \{1\}$ we use just '1' (as we did above).

Adding more restrictions can only decrease the maximum size, so we have

$$m_n(\text{1-int}, \text{1-int}, 1) \le m_n(\text{1-int}, \text{01-int}, 1) \le m_n(\text{01-int}, \text{01-int}, 1). \tag{2}$$

Several problems under assumptions similar to 1-cross intersecting SPS and their refinements have been studied before, see, e.g., [3, 5, 8, 14].

1.4 Linear hypergraphs

Our first observation here is to show that if one of $(\mathcal{A}, \mathcal{B})$, say $\mathcal{A}$, in an SPS is linear, then the size of this SPS is bounded by $n^2 + O(n)$.

Proposition 1.6. $m_n(\text{01-int}, *, *) \le n^2 + n + 1$.

For a linear 1-cross intersecting SPS (that is, when $\mathcal{H}$ is linear) the bound of this Proposition can approximately be halved.

Theorem 1.7. *$m_n(\text{01-int}, \text{01-int}, 1) \le \frac{1}{2}n^2 + n + 1$.*

A further small decrement comes if in addition $\mathcal{A}$ and $\mathcal{B}$ are both 1-intersecting hypergraphs. For these hypergraphs their union $\mathcal{H} = \mathcal{A} \cup \mathcal{B}$ can be considered as a "geometry" where two lines intersect in at most one point, and every line has exactly one parallel line.

Theorem 1.8. *$m_n(\text{1-int}, \text{1-int}, 1) \le \binom{n}{2} + 1$ for $n > 2$.*

For $n \ge 4$, in the case of equality, $\mathcal{H}$ is n-uniform and n-regular, i.e., $d_H(v) = n$ for every $v \in V$. For small values we have $m_2(\text{1-int}, \text{1-int}, 1) = 3$, $m_3(\text{1-int}, \text{1-int}, 1) = 4$, $m_4(\text{1-int}, \text{1-int}, 1) = 7$, and $m_5(\text{1-int}, \text{1-int}, 1) = 10$.

In Section 4 we give constructive lower bounds, culminating in Constructions 4.1, 4.2 and 4.3 in subsection 4.4, showing that the last three results are asymptotically best possible. Constructions 4.1 and 4.2 show that

$$n^2 - o(n^2) \le m_n(\text{1-int}, \text{1-int}, *), \qquad n^2 - o(n^2) \le m_n(\text{1-int}, *, 1).$$

Since the right hand sides of these inequalities are bounded above by $m_n(\text{01-int}, *, *)$ (which is at most $n^2 + n + 1$), Proposition 1.6 is asymptotically the best possible.

Construction 4.3 shows that

$$\frac{1}{2}n^2 - o(n^2) \le m_n(\text{1-int}, \text{1-int}, 1),$$

Hence Theorems 1.7 and 1.8 are also asymptotically the best possible.

In fact, we examined all 18 cases for $m_n(I_A, I_B, I_{\text{cross}})$ where I_A and I_B are chosen from $\{1\}$, $\{0, 1\}$, or $*$ and I_{cross} is either $\{1\}$ or $*$. By symmetry they define twelve functions. Summarizing our results, $m_n(*, *, 1)$ and $m_n(*, *, *)$ are exponential as a function of n, the other cases are polynomial. Three of them, mentioned in (2), are asymptotically $\frac{1}{2}n^2$ while the other seven are asymptotically n^2.

1.5 Relation to clique and biclique partition problems

The notion of 1-cross intersecting SPS is closely related to the concept of clique and biclique partitions. A *clique partition* of a graph G is a partition of the edge set of G into complete graphs. Similarly, a *biclique partition* of a bipartite graph B is a partition of the edge set of B into complete bipartite graphs (bicliques). The minimum number of cliques (bicliques) needed for the clique (or biclique) partitions are well studied, see, for example [9]. Our problem relates to another parameter of clique (biclique) partitions. The *thickness* of a clique (biclique) partition of a graph (bipartite graph) is the maximum s such that every vertex of the graph (bipartite graph) is in at most s cliques (bicliques). Let T_{2m} be the cocktail party graph, i.e., the complete graph K_{2m} from which a perfect matching is removed. Let B_{2m} be the bipartite graph obtained from the complete bipartite graph $K_{m,m}$ by removing a perfect matching.

Theorem 1.9. *The maximum m such that B_{2m} has a biclique partition of thickness n is $m_n(*, *, 1)$. The maximum m such that T_{2m} has a clique partition of thickness n is $m_n(\text{1-int}, \text{1-int}, 1)$.*

2 1-cross intersecting SPS – proofs

Proposition 1.1. *If (a_1, b_1)-bounded and (a_2, b_2)-bounded 1-cross intersecting SPS exist with sizes m_1 and m_2, then (a_1+a_2, b_1+b_2)-bounded 1-cross intersecting SPS also exists with size $m_1 \cdot m_2$.*

Proof. We have to show that

$$m(a_1+a_2, b_1+b_2, *, *, 1) \geq m(a_1, b_1, *, *, 1) \cdot m(a_2, b_2, *, *, 1).$$

Consider $t = m(a_2, b_2, *, *, 1)$ pairwise disjoint ground sets $V_1, \ldots, V_t$ and for all $i \in [t]$ a copy $(\mathcal{A}_i, \mathcal{B}_i)$ of a construction giving an (a_1, b_1)-bounded 1-cross intersecting SPS of size s such that $\mathcal{A}_i = \{A_{i,1}, \ldots, A_{i,s}\}$, $\mathcal{B}_i = \{B_{i,1}, \ldots, B_{i,s}\}$, where $s = m(a_1, b_1, *, *, 1)$. Let $(\mathcal{A}, \mathcal{B})$ be a copy of an (a_2, b_2)-bounded 1-cross intersecting SPS of size t on the ground set V such that $\mathcal{A} = \{A_1, \ldots, A_t\}$, $\mathcal{B} = \{B_1, \ldots, B_t\}$, where V is disjoint from all V_i-s. For any $1 \leq i \leq t$, $1 \leq j \leq s$ define

$$A'_{i,j} = A_{i,j} \cup A_i, \; B'_{i,j} = B_{i,j} \cup B_i.$$

The pairs $(A'_{i,j}, B'_{i,j})$ form a 1-cross intersecting SPS such that $|A'_{i,j}| \leq a_1 + a_2$ and $|B'_{i,j}| \leq b_1 + b_2$. □

Proposition 1.3. *Assume that $(\mathcal{A}, \mathcal{B})$ is 1-cross intersecting and $V := \cup\mathcal{A}$. Then the incidence vectors of the edges of $\mathcal{A}$ are linearly independent in $\mathbb{R}^V$.*

Proof. Let $\mathbf{a}_i$ (resp. $\mathbf{b}_i$) denote the incidence vector of A_i (resp. B_i), such that $\mathbf{a}_i(v) = 1$ for $v \in V$ if and only if $v \in A_i$. Otherwise the coordinates are 0. Suppose that

$$\sum_{i=1}^{m} \lambda_i \mathbf{a}_i = \mathbf{0}.$$

Take the dot product of both sides of this equation with $\mathbf{b}_j$. Since $|A_i \cap B_j| = 1$ for $i \neq j$ and $|A_i \cap B_j| = 0$ for $i = j$, we get that

$$\sum_{i=1}^{m} \lambda_i = \lambda_j.$$

Adding these for all j shows that $m \sum_{i=1}^{m} \lambda_i = \sum_{j=1}^{m} \lambda_j$, consequently (using $m > 1$) $\sum_{i=1}^{m} \lambda_i = 0$ and thus $\lambda_j = 0$ for all j. □

Theorem 1.4. *If $n \geq 4$, then a $(2,n)$-bounded 1-cross intersecting SPS has size at most*

$$\left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)\left(\left\lceil\frac{n}{2}\right\rceil+1\right).$$

This bound is the best possible. For $n=2,3$ the exact values are $m=5,7$.

Proof. Let $(\mathcal{A},\mathcal{B})$ be a $(2,n)$-bounded 1-cross intersecting SPS. It is convenient to assume that $\mathcal{A}$ is two-uniform (a graph without multiple edges) and $\mathcal{B}$ is an n-uniform hypergraph. (For smaller sets dummy vertices can be added).

Consider the simple graph $\mathcal{A}$. The n-set B_i must be an independent transversal for all edges other than A_i and disjoint from the edge A_i. Consequently, the graph $\mathcal{A}$ cannot contain even cycles.

If there is an odd cycle C, it must contain all edges of $\mathcal{A}$ since any diagonal would create an even cycle and for any edge A_i with at most one vertex on C, B_i cannot be an independent transversal. Thus in this case $m \leq 2n+1$.

Assume next that $\mathcal{A}$ is an acyclic graph.

Lemma 2.1. *Assume that $T \subseteq \mathcal{A}$ is a non-star tree component with t edges. Then*

$$\max_{A_i \in T} |B_i \cap V(T)| \geq \left\lceil\frac{t}{2}\right\rceil.$$

Proof. Let $P = x, y, z, z_2, \ldots$ be a maximal path of T, set $A_1 = \{x,y\}, A_2 = \{y,z\}$. Let $S \subseteq V(T)$ the set of leaves connected to y. Note that $t \geq 3$, $|V(T)| = t+1$, $N_T(y) = S \cup \{z\}$ and $x \in S$. Then $B_1 \cap V(T)$ is the set X of vertices with odd distance from y in the tree $T-x$. On the other hand, $B_2 \cap V(T)$ is the set $X' = S \cup D$ where D is the set of vertices with odd distance from z in the tree $T-(S\cup\{y\})$. Then $|X|+|X'| = t+|S|-1 \geq t$. Therefore

$$\max_{A_i \in T} |B_i \cap V(T)| \geq \max(|B_1 \cap V(T)|, |B_2 \cap V(T)|) = \max\{|X|,|X'|\} \geq \left\lceil\frac{t}{2}\right\rceil. \quad \square$$

Using Lemma 2.1, if T is a non-star tree component, then we may replace T by two stars both with $\lceil\frac{t}{2}\rceil$ edges, the sizes of the corresponding new B_is will not exceed n. Thus we may assume that all components of $\mathcal{A}$ are stars, $S_1,\ldots,S_k$, where S_i has $t_i \geq 1$ edges. For any edge $A_j \in S_i$, $n \geq |B_j| = t_i - 1 + k - 1$. Adding these inequalities for $i = 1,\ldots,k$, we obtain that $kn \geq m - 2k + k^2$ which leads to $k(n+2-k) \geq m$. Hence

$$m \leq k(n+2-k) \leq \left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)\left(\left\lceil\frac{n}{2}\right\rceil+1\right).$$

Taking together the bounds for odd cycles and acyclic graphs, we get that

$$m \leq \max\left\{2n+1,\ \left(\left\lfloor\frac{n}{2}\right\rfloor+1\right)\left(\left\lceil\frac{n}{2}\right\rceil+1\right)\right\}.$$

For $n=2,3$ the first term is larger, for $n=4$ they are equal, and for $n \geq 5$ the second term takes over. This proves the upper bound for m.

The matching lower bound for $n \geq 4$ comes from Proposition 1.1 applied to the standard construction with values $(1, \lceil \frac{n}{2} \rceil)$ and $(1, \lfloor \frac{n}{2} \rfloor)$. For $n = 2$ the hypergraph $\mathcal{H}(2,2)$ works (defined in the introduction). For $n = 3$ we can define $\mathcal{H}(2,3)$ as the pairs $\big(\{i, i{+}1\}, \{i{+}2, i{+}4, i{+}6\}\big)$ taken (mod 7). The proof of Corollary 1.5 is left to the reader. □

3 1-cross intersecting linear SPS – upper bounds

Proposition 1.6. $m_n(\text{01-int}, *, *) \leq n^2 + n + 1$.

Proof. Suppose that $(\mathcal{A}, \mathcal{B})$ is an (n,n)-bounded cross intersecting SPS of size $m \geq n^2 + n + 2$, where $\mathcal{A}$ is linear. We claim that $d_A(v) \leq n+1$ for each vertex v. Suppose $v \in A_1 \cap \ldots \cap A_{n+2}$. Then $v \notin B_i$ for $i \leq n+2$ and in $\bigcup_{i=1}^{n+2} A_i \setminus \{v\}$ the sets $A_i' = A_i \setminus \{v\}$ are pairwise disjoint. The set B_{n+2} must intersect each $A_1', \ldots, A_{n+1}'$ which is impossible.

Consider B_{n^2+n+2}. For $1 \leq i \leq n^2+n+1$ the set A_i intersects B_{n^2+n+2}, so there is vertex $v \in B_{n^2+n+2}$ with $d_A(v) > n+1$, a contradiction. □

Theorem 1.7. $m_n(\text{01-int}, \text{01-int}, 1) \leq \frac{1}{2}n^2 + n + 1$.

Proof. Suppose that $(\mathcal{A}, \mathcal{B})$ is an (n,n)-bounded 1-cross intersecting SPS of size m such that both $\mathcal{A}$ and $\mathcal{B}$ are linear hypergraphs. We have $m_2(\text{01-int}, \text{01-int}, 1) \leq 5$ by Theorem 1.4 so we may suppose that $n \geq 3$. If $m \leq 2n+2$ then there is nothing to prove, so from now on, we may suppose that $m \geq 2n+3$.

We claim that for every $v \in V$,

$$d_{\mathcal{A}}(v), d_{\mathcal{B}}(v) \leq n. \tag{3}$$

Indeed, $d_{\mathcal{A}}(v) \leq n+1$ (and in the same way $d_{\mathcal{B}}(v) \leq n+1$) is obvious. Otherwise, if, e.g., $v \in A_1 \cap \cdots \cap A_{n+2}$ then B_{n+2} cannot intersect some A_j with $j \in \{1, \ldots, n+1\}$. If $d_{\mathcal{A}}(v) = n+1$, say $v \in A_1 \cap \cdots \cap A_{n+1}$ then $m > 2n+2 \geq d_{\mathcal{A}}(v) + d_{\mathcal{B}}(v)$ so there is a pair A_i, B_i with $i > n+1$ such that $v \notin A_i \cup B_i$. Thus B_i cannot intersect all A_j-s containing v.

Since $(\mathcal{A}, \mathcal{B})$ is 1-cross intersecting we have $\sum_{v \in B_i} d_{\mathcal{A}}(v) = m - 1$ for each B_i. Adding up these m equations we get

$$\sum_v d_{\mathcal{A}}(v) d_{\mathcal{B}}(v) = m^2 - m. \tag{4}$$

Let $\mathcal{A}_i$ be the set of A_j-s that intersect A_i and different from A_i. Our crucial observation is that if A_i and A_j do not intersect then

$$|\mathcal{A}_i| + |\mathcal{A}_j| \leq n^2. \tag{5}$$

Indeed, the left hand side equals to $\sum_{\ell : \ell \neq i,j} |A_\ell \cap (A_i \cup A_j)|$. For $\ell \neq i, j$ we have $|A_\ell \cap (A_i \cup A_j)| \leq 2$. In case of $|A_\ell \cap (A_i \cup A_j)| = 2$ we select two pairs joining A_i

to A_j, namely $A_\ell \cap (A_i \cup A_j)$ and $B_\ell \cap (A_i \cup A_j)$. In case of $|A_\ell \cap (A_i \cup A_j)| = 1$ we select a pair joining A_i to A_j, namely $B_\ell \cap (A_i \cup A_j)$. These pairs are distinct, and there are n^2 pairs joining A_i to A_j so we obtain that $\sum_{\ell:\ell\neq i,j} |A_\ell \cap (A_i \cup A_j)| \leq n^2$.

If $A_i \cap A_j = \{v\}$ then a slightly more complicated argument gives

$$|\mathcal{A}_i| + |\mathcal{A}_j| \leq (n-1)^2 + d_{\mathcal{A}}(v) + d_{\mathcal{B}}(v) \leq n^2 + 1. \tag{6}$$

As before,

$$|\mathcal{A}_i| + |\mathcal{A}_j| = \sum_{\ell:\ell\neq i} |A_\ell \cap A_i| + \sum_{\ell:\ell\neq j} |A_\ell \cap A_j|.$$

For every $\ell \neq i, j$ we select (at most) two pairs joining $A_i \setminus \{v\}$ to $A_j \setminus \{v\}$, namely $A_\ell \cap ((A_i \setminus \{v\}) \cup (A_j \setminus \{v\}))$ and $B_\ell \cap ((A_i \setminus \{v\}) \cup (A_j \setminus \{v\}))$. In this way we selected at least $|A_\ell \cap A_i| + |A_\ell \cap A_j|$ distinct pairs except if $v \in A_\ell \cup B_\ell$. In the latter case we still have selected at least $|A_\ell \cap A_i| + |A_\ell \cap A_j| - 1$ pairs. So the left hand side of (6) is at most the number of pairs joining $A_i \setminus \{v\}$ to $A_j \setminus \{v\}$ plus $d_{\mathcal{A}}(v) + d_{\mathcal{B}}(v)$.

Next we prove that

$$\sum_{v\in V} d_{\mathcal{A}}(v)^2 \leq m\left(\frac{1}{2}n^2 + n + \frac{1}{2}\right). \tag{7}$$

Add up inequalities (5) and (6) for all $1 \leq i < j \leq m$

$$\frac{1}{m-1}\sum_{1\leq i<j\leq m} |\mathcal{A}_i| + |\mathcal{A}_j| \leq \frac{1}{m-1}\binom{m}{2}(n^2+1) = m\left(\frac{1}{2}n^2 + \frac{1}{2}\right).$$

Here the left hand side is

$$\sum_{1\leq i\leq m} |\mathcal{A}_i| = \sum_{1\leq i\leq m}\left(\sum_{v\in A_i}(d_{\mathcal{A}}(v)-1)\right) = \sum_{v\in V}\left(d_{\mathcal{A}}(v)^2 - d_{\mathcal{A}}(v)\right) = \left(\sum_{v\in V} d_{\mathcal{A}}(v)^2\right) - mn.$$

The last two displayed formulas yield (7) and equality can hold only if (5) was not used. Note that similar upper bound must hold for $\sum_{v\in V} d_{\mathcal{B}}(v)^2$, too.

Apply (7) to $\mathcal{A}$ and to $\mathcal{B}$ and subtract the double of (4). We obtain

$$\begin{aligned}0 \leq \sum_{v\in V}(d_{\mathcal{A}}(v) - d_{\mathcal{B}}(v))^2 &= \sum_v d_{\mathcal{A}}(v)^2 + \sum_v d_{\mathcal{A}}(v)^2 - 2\sum_v d_{\mathcal{A}}(v)d_{\mathcal{B}}(v)\\ &\leq 2m\left(\frac{1}{2}n^2 + n + \frac{1}{2}\right) - 2m(m-1) = 2m\left(\frac{1}{2}n^2 + n + \frac{3}{2} - m\right).\end{aligned}$$

This implies $m \leq \frac{1}{2}n^2 + n + \frac{3}{2}$. However, here equality can hold only if (5) was never used to $\mathcal{A}$ or to $\mathcal{B}$. This implies that $\mathcal{A}$ and $\mathcal{B}$ are 1-intersecting and there exists a v with $d_{\mathcal{A}}(v) = d_{\mathcal{B}}(v) = n$. But this easily leads to a contradiction. This completes the proof of the upper bound on m. $\square$

For the case $n = 3$ we have $7 \leq m_3(\text{01-int}, \text{01-int}, 1) \leq 8$. The lower bound 7 is provided by the following set pairs:

$$A_i := \{2i, 2i+1, 2i+2\},\ B_i := \{2i+4, 2i+8, 2i+12\} \pmod{14}.$$

Theorem 1.8. $m_n(\text{1-int}, \text{1-int}, 1) \leq \binom{n}{2} + 1$ *for* $n > 2$.

For $n \geq 4$*, in the case of equality,* $\mathcal{H}$ *is* n*-uniform and* n*-regular, i.e.,* $d_H(v) = n$ *for every* $v \in V$*. For small values we have* $m_2(\text{1-int}, \text{1-int}, 1) = 3$*,* $m_3(\text{1-int}, \text{1-int}, 1) = 4$*,* $m_4(\text{1-int}, \text{1-int}, 1) = 7$*, and* $m_5(\text{1-int}, \text{1-int}, 1) = 10$*.*

Proof. Suppose that $(\mathcal{A}, \mathcal{B})$ is an (n, n)-bounded 1-cross intersecting SPS of size m such that both $\mathcal{A}$ and $\mathcal{B}$ are 1-intersecting, and recall that $\mathcal{H} = \mathcal{A} \cup \mathcal{B}$. First, consider the case when there exists a vertex v with $d_{\mathcal{H}}(v) \geq n+1$, say $v \in A_i \cup B_i$ for $i \in \{1, 2, \ldots, n+1\}$. Then one of the members of $\{A_{n+2}, B_{n+2}\}$ does not cover v, say, $v \notin A_{n+2}$. Then A_{n+2} cannot intersect all members of $\{A_i, B_i\}_{1 \leq i \leq n+1}$ containing v, a contradiction. So in this case $m = n + 1$ and we are done.

From now on, we may suppose that $m > n + 1$, and $d_{\mathcal{H}}(v) \leq n$ for all $v \in V$. Since only B_1 is disjoint from A_1 we get

$$2m = |\mathcal{H}| = 2 + \sum_{v \in A_1} (d_{\mathcal{H}}(v) - 1) \leq 2 + n(n-1).$$

and we conclude that $m \leq \binom{n}{2} + 1$. For $n \geq 4$ in the case of equality all vertices of A_1 (and of all other hyperedges) must have degree n.

For $n = 3$ and 4 the upper bounds 4 and 7 are sharp. For $n = 3$ one can consider the pairs of triples $(\{i, i+1, i+3\}, \{i+4, i+5, i+7\})$ taken $\pmod 8$ for $i = 1, 2, 3, 4$. For $n = 4$ there are several ways to describe a construction showing $m_4(\text{1-int}, \text{1-int}, 1) \geq 7$. For example the difference set $D = \{1, 2, 5, 7\}$ that generates $PG(2, 3) \pmod{13}$ can be considered $\pmod{14}$, then $A_i := D + i$, $B_i := D + 7 + i$ is a 1-cross intersecting SPS for $n = 4$ such that both $\mathcal{A}$ and $\mathcal{B}$ are 1-intersecting families (see also in [16]). Another representation is the following. Fix a Baer subplane $F = PG(2, 2)$ in $G = PG(2, 4)$. Each point $p_i \in F$ determines two lines through p that intersect $V(G) \setminus V(F)$ in four element sets A_i, B_i for $i = 1, \ldots, 7$.

For $n = 5$ the upper bound is 11. However, from Proposition 5.1 this can be sharp only when T_{22} can be partitioned into 22 cliques. This is impossible, as derived in [9] from earlier results. Thus the following construction implies that $m_5(\text{1-int}, \text{1-int}, 1) = 10$. Take AG(2, 4), the affine plane of order four, and add ten distinct new points $w_1, \ldots, w_{10}$ to it. Take two pairs of lines (A_i^*, B_i^*), (A_{i+1}^*, B_{i+1}^*) in each parallel class, $i = 1, 3, 5, 7, 9$. Then set

$$A_i := A_i^* \cup \{w_i\},\ A_{i+1} := A_{i+1}^* \cup \{w_{i+1}\},\ B_i := B_i^* \cup \{w_{i+1}\},\ B_{i+1} := B_{i+1}^* \cup \{w_i\}$$

for $i = 1, 3, 5, 7, 9$. □

4 Constructing cross-intersecting linear hypergraphs

Here we give constructions of large cross intersecting SPS such that $\mathcal{A}$ is an intersecting linear hypergraph. These constructions show that the upper bounds in Theorems 1.6, 1.7, 1.8 are asymptotically sharp. The outline of the constructions is the following. We use that the function $m_n(I_A, I_B, I_{\text{cross}})$ is monotone increasing in n so we have to make constructions only for a dense set of special values of n. We prove the asymptotic for the lower bound for m_n in three steps. In subsection 4.1 we show that $m_n \geq n$ for all functions and for all n. In subsection 4.2 we consider the affine plane $\mathrm{AG}(2,q)$ where q is an odd prime and define an SPS (A_i, B_i) such that $A_i = A'_i \cup A''_i$ and $B_i = B'_i \cup B''_i$ where A'_i and B'_i, are parallel line pairs of this affine plane. Some pairs A'_i and B'_j will not intersect, so to ensure the cross intersecting property we extend them with members of another (smaller) construction consisting of pairs (A''_i, B''_i). We define three different extensions $C_i(q)$ $(i = 1, 2, 3)$ of the affine plane and in subsection 4.3 we use these to show that $m_n \geq \Omega(n^2)$. Then in subsection 4.4 we use the same kind of extensions again to extend the affine plane $AG(2,q)$ with $C_i(p)$ where now q is a little bit smaller than n and $p \approx \sqrt{q}$ to get the final constructions.

4.1 Double stars

The vertex set of a *double star of size* s consist of $\{v_{i,j} \mid 1 \leq i,\ j \leq s,\ i \neq j\}$ and two additional special vertices w_a and w_b. Define for $i = 1, \dots, s$ sets $A_i = \{w_a\} \cup \{v_{i,j} \mid 1 \leq j \leq s,\ j \neq i\}$ and $B_i = \{w_b\} \cup \{v_{j,i} \mid 1 \leq j \leq s,\ j \neq i\}$. It is easy to check that $(\mathcal{A}, \mathcal{B})$ is a 1-cross intersecting SPS of size s containing s-element sets such that both $\mathcal{A}$ and $\mathcal{B}$ are 1-intersecting.

4.2 Extending parallel line pairs of affine planes

The affine plane $\mathrm{AG}(2,q)$ has lines with q points, has $q+1$ directions (parallel classes of lines) and each parallel class contains q lines. Let δ be a direction and $A'_{1,\delta}, \dots, A'_{q,\delta}$ be the elements of the parallel class determined by δ. We give three types of parallel line pairs of $AG(2,q)$ and their extensions to obtain a cross intersecting SPS.

Extension I. Let $B'_{i+1,\delta} := A'_{i,\delta}$ for $i = 1, \dots, q-1$ and $B'_{1,\delta} := A'_{q,\delta}$. For each δ we take a copy of a cross-intersecting SPS $(A''_{i,\delta}, B''_{i,\delta})$ for $i = 1, \dots, q$, so that the ground sets of the copies are pairwise disjoint and also disjoint from $AG(2,q)$. Then define $A_{i,\delta} := A'_{i,\delta} \cup A''_{i,\delta}$ and $B_{i,\delta} := B'_{i,\delta} \cup B''_{i,\delta}$.

Extension II. Let $B'_{1,\delta} = \ldots = B'_{q-1,\delta} = A'_{q,\delta}$. Again, for each δ we take a copy of a cross-intersecting SPS $(A''_{i,\delta}, B''_{i,\delta})$ for $i = 1, \dots, q-1$ so that their ground sets are pairwise disjoint and also disjoint from $AG(2,q)$. Define $A_{i,\delta} := A'_{i,\delta} \cup A''_{i,\delta}$ and $B_{i,\delta} := B'_{i,\delta} \cup B''_{i,\delta}$ for each δ and $1 \leq i \leq q-1$.

Extension III. Set $h := (q-1)/2$ and $B'_{i+h,\delta} := A'_{i,\delta}$ for $i = 1, \ldots, h$. For each δ we take a copy of a cross-intersecting SPS $(A''_{i,\delta}, B''_{i,\delta})$ for $i = 1, \ldots, h$, so that their ground sets are pairwise disjoint and also disjoint from $AG(2,q)$. Set $A_{i,\delta} := A'_{i,\delta} \cup A''_{i,\delta}$ and $B_{i,\delta} := B'_{i,\delta} \cup B''_{i,\delta}$ for each δ and $1 \leq i \leq h$.

4.3 Extensions with double stars

Let $C_1(q)$ be the SPS obtained with Extension I by selecting $(A''_{i,\delta}, B''_{i,\delta})$ for $i = 1, \ldots, q$ as a double star with $|A''_{i,\delta}| = |B''_{i,\delta}| = q$.

Claim 4.1. *$C_1(q)$ is a cross intersecting SPS $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}, \mathcal{B}$ are both intersecting linear hypergraphs. Therefore $m_{2q}(\text{1-int}, \text{1-int}, *) \geq q^2 + q$.*

Proof. Indeed, $A_{i,\delta} \cap B_{i,\delta} = \emptyset$ because $A'_{i,\delta} \cap B'_{i,\delta} = \emptyset$ and $A''_{i,\delta} \cap B''_{i,\delta} = \emptyset$. For $i \neq j$ $A_{i,\delta} \cap B_{j,\delta} \neq \emptyset$ because $A''_{i,\delta} \cap B''_{j,\delta} \neq \emptyset$. For $\delta \neq \delta'$ $A_{i,\delta} \cap B_{j,\delta'} \neq \emptyset$ because the lines $A'_{i,\delta}$ and $B'_{j,\delta'}$ are not parallel, thus they intersect in one point and $A''_{i,\delta} \cap B''_{j,\delta'} = \emptyset$ (they are subsets of two disjoint ground sets).

We claim that $\mathcal{A}$ (and similarly $\mathcal{B}$) is an intersecting linear hypergraph. For $\delta \neq \delta'$ the lines $A'_{i,\delta}$ and $A'_{j,\delta'}$ intersect in one point, and $A''_{i,\delta}$ and $A''_{j,\delta'}$ are disjoint (they are subsets of two disjoint ground sets). For $i \neq j$ $A'_{i,\delta}$ and $A'_{j,\delta}$ are disjoint (parallel) but $|A''_{i,\delta} \cap A''_{j,\delta}| = 1$. $\square$

Let $C_2(q)$ be the SPS obtained with Extension II by selecting $(A''_{i,\delta}, B''_{i,\delta})$ for $i = 1, \ldots, q-1$ as a double star with $|A''_{i,\delta}| = |B''_{i,\delta}| = q-1$.

Claim 4.2. *$C_2(q)$ is a 1-cross intersecting SPS $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}$ is an intersecting linear hypergraph. Therefore $m_{2q-1}(\text{1-int}, *, 1) \geq q^2 - 1$.*

Proof. Since the proof is straightforward and very similar to the proof of Claim 4.1 it is omitted. Note that $\mathcal{B}$ is *not* linear. $\square$

Let $C_3(q)$ be the SPS obtained with Extension III by selecting $(A''_{i,\delta}, B''_{i,\delta})$ for $i = 1, \ldots, h$ as a double star with $|A''_{i,\delta}| = |B''_{i,\delta}| = h$. The following statement is also an easy consequence of the definitions.

Claim 4.3. *$C_3(q)$ is a 1-cross intersecting SPS $(\mathcal{A}, \mathcal{B})$ such that $\mathcal{A}$ and $\mathcal{B}$ are both intersecting linear hypergraphs. Therefore $m_r(\text{1-int}, \text{1-int}, 1) \geq (q^2 - 1)/2$ where $r = \lfloor \frac{3}{2} q \rfloor$.* $\square$

4.4 Final constructions

For the final constructions we need results about the density of primes.

Theorem 4.4 (Hoheisel [10])**.** *There are constants x_0 and $0.5 \leq \alpha < 1$ such that for all $x \geq x_0$ the interval $[x - x^\alpha, x]$ contains a prime number.*

The currently known best α is 0.525 by Baker, Harman and Pintz [2].

Lemma 4.5. *There is a constant n_0 such that for $n > n_0$ we may choose an odd prime q between $n - 5n^\alpha$ and $n - 4n^\alpha$ by Theorem 4.4. Then $n - q \geq 4\sqrt{q+1}$ and $q^2 - 1 \geq n^2 - 10n^{1+\alpha} = n^2 - O(^{1+\alpha}) = n^2 - o(n^2)$.*

Proof. $(n-q) \geq 4n^\alpha \geq 4(q+1)^\alpha \geq 4\sqrt{q+1}$. $\square$

Corollary 4.6. *For q chosen in Lemma 4.5, by Bertrand's postulate, we can choose another odd prime p from the interval $[\sqrt{q+1},\ 2\sqrt{q+1}]$. Then $2p \leq n - q$ and $p^2 - 1 \geq q$.* $\square$

Remark. Actually with use of the result in [2] we can prove stronger statements, namely $o(n^2)$ can be replaced by $O(n^{1.525})$. Define q and p as in Lemma 4.5 and Corollary 4.6, and let $h = (q-1)/2$ and $r = \lfloor \frac{3}{2}p \rfloor$.

Construction 4.1, showing $m_n(\text{1-int}, \text{1-int}, *) \geq n^2 - o(n^2)$.
Starting from $AG(2,q)$, apply Extension I so that SPS $C_1(p)$ is selected for the extension. The size of $C_1(p)$ is $p^2 + p \geq q$ thus we need only the first q set pairs from it. Now $|A_i| = |B_i| = q + 2p \leq n$ and $(\mathcal{A}, \mathcal{B})$ satisfies the properties required.

Construction 4.2, showing $m_n(\text{1-int}, *, 1) \geq n^2 - o(n^2)$.
Starting from $AG(2,q)$, apply Extension II so that SPS $C_2(p)$ is selected for the extension. Again, we need only its first q set pairs ($p^2 - 1 \geq q$). Now $|A_i| = |B_i| = q + 2p - 1 \leq n$ and $(\mathcal{A}, \mathcal{B})$ satisfies the properties required.

Construction 4.3, showing $m_n(\text{1-int}, \text{1-int}, 1) \geq \frac{1}{2}n^2 - o(n^2)$.
Starting from $AG(2,q)$, apply Extension III so that SPS $C_3(p)$ is selected for the extension. We need only its first h set pairs ($\frac{1}{2}(p^2 - 1) \geq h$). Now $|A_i| = |B_i| = q + r \leq q + 2p \leq n$ and $(\mathcal{A}, \mathcal{B})$ satisfies the properties required.

5 Connection with clique and biclique partitions

Theorem 1.9. *The maximum m such that B_{2m} has a biclique partition of thickness n is $m_n(*, *, 1)$. The maximum m such that T_{2m} has a clique partition of thickness n is $m_n(\text{1-int}, \text{1-int}, 1)$.*

Proof. Assume that $(\mathcal{A}, \mathcal{B})$ is an (n,n)-bounded 1-cross intersecting SPS of size m, and $\mathcal{H} = \mathcal{A} \cup \mathcal{B}$. The dual of this hypergraph, $\mathcal{H}^*$, has vertex set

$$V^* = \{x_1, \ldots, x_m, y_1, \ldots, y_m\}$$

where x_i, y_i correspond to A_i, B_i. The hyperedges of $\mathcal{H}^*$ correspond to vertices of $\mathcal{H}$. Since $|A_i \cap B_j| = 1$ for $i \neq j$, every pair x_i, y_j for $i \neq j$ is covered exactly once by the hyperedges of $\mathcal{H}^*$. On the other hand, $|A_i \cap B_i| = 0$ for every i thus the pairs x_i, y_i are not covered by any hyperedge of $\mathcal{H}^*$. Thus the complete graphs induced by the hyperedges of $\mathcal{H}^*$ form a biclique partition of thickness n of the bipartite graph B_{2m}.

The second statement follows by the same argument, but in this case the pairs x_i, x_j and the pairs y_i, y_j are also covered exactly once by the hyperedges of $\mathcal{H}^*$. Thus in this case the subgraphs of T_{2m} induced by the hyperedges of $\mathcal{H}^*$ form a clique partition of thickness n of the cocktail party graph T_{2m}. □

Proposition 5.1. *For $n \geq 4$, $m_n(\text{1-int}, \text{1-int}, 1) = \binom{n}{2} + 1$ if and only if $T_{n(n-1)+2}$ has a clique partition into $n(n-1)+2$ cliques.*

Proof. By Theorem 1.8 for $n \geq 4$ (see in Section 3) $|\mathcal{H}| = m_n(\text{1-int}, \text{1-int}, 1) = \binom{n}{2}+1$ implies that $\mathcal{H}$ is n-uniform and n-regular. So the corresponding clique partition of the cocktail party graph $T_{n(n-1)+2}$ consists of cliques of size n. For the number of these edge disjoint cliques we have

$$\frac{e(T)}{\binom{n}{2}} = \frac{\binom{n(n-1)+2}{2} - \left(\binom{n}{2}+1\right)}{\binom{n}{2}} = n(n-1)+2.$$

The other direction follows (using Theorem 1.9) from a result in [17] (via [9]) stating that a clique partition of T into $n(n-1)+2$ cliques must have thickness n. □

Note that such perfect partitions of $T_{n(n-1)+2}$ (and rather the non-existence of those for infinitely many values) were also investigated by Lamken, Mullin, and Vanstone [12] (under the name of 'twisted projective planes').

6 $m_3(*, *, 1) \leq 12$

The proof of Theorem 1.4 with some efforts leads to the following technical statements.

Corollary 6.1. *Suppose that $(\mathcal{A}, \mathcal{B})$ is a $(2,3)$-bounded 1-cross intersecting SPS of size at least 6. Then $\mathcal{A}$ is 2-uniform, $\mathcal{B}$ is 3-uniform and $\cup\mathcal{A} = \cup\mathcal{B}$.*

Moreover $\mathcal{B}$ has transversal number $\tau(\mathcal{B}) = 2$ or $\tau(\mathcal{B}) = 3$. If $\tau(\mathcal{B}) = 2$, then $|\mathcal{B}| = 6$ and either the optimal transversal is unique or the union of optimal transversals is $\{x, y, z\}$ with the property $d_{\mathcal{A}}(x) = d_{\mathcal{A}}(y) = d_{\mathcal{A}}(z) = 2$, moreover every 3-element transversal intersects $\{x, y, z\}$. (In this last case $\mathcal{A} = 3 \cdot K_{1,2}$ and $\{x, y, z\}$ are the centers.)

Proposition 6.2. $m_3(*, *, 1) \leq 12$.

Proof. Assume that a $(3,3)$-bounded 1-cross intersecting SPS has size 13. We may assume that every vertex is incident to at least one hyperedge from both $\mathcal{A}, \mathcal{B}$. By Proposition 1.3, $|V| \geq |\mathcal{A}| = 13$. This implies that the average of $d_{\mathcal{A}}(v)$ is at most three.

We first claim that $d_{\mathcal{A}}(v) \le 6$ for each $v \in V$. Suppose that $v \in A_1 \cap \ldots \cap A_s$ for $s \ge 7$, we may also assume that $v \in B_{13}$. Let $A'_i = A_i \setminus \{v\}$, $U = \cup_{i=1}^{s} A'_i$, $\mathcal{A}' = \{A'_i\}_{i=1}^{s}$ and $\mathcal{B}' = \{B_i\}_{i=1}^{s}$. Consider $(\mathcal{A}', \mathcal{B}')$, it is $(2,3)$-bounded 1-cross intersecting SPS because $v \notin B_i$ for $i \le s$. By Corollary 6.1 $s = 7$ and $B_i \subseteq U \subseteq V \setminus B_{13}$ for all $i \le s$, and $\tau(\mathcal{B}') = 3$. $v \notin A_8$ and A_8 intersects B_{13}, so $|A_8 \cap U| \le 2$. However A_8 intersects each set in $\mathcal{B}'$, this contradicts to $\tau(\mathcal{B}') = 3$.

Next we claim that $d_{\mathcal{A}}(v) \le 5$ for each $v \in V$. Suppose $d_{\mathcal{A}}(v) = 6$. We use the same notation but now $s = 6$. We use Corollary 6.1 again, if $\tau(\mathcal{B}') = 3$, then we have the same contradiction. If $\tau(\mathcal{B}') = 2$ and there is only one 2-element transversal $\{x, y\}$ then $x \in A_j$ for $7 \le j \le 12$ and as $x \in U$ also $x \in A'_i$ for an $i \le 6$. This means $d_{\mathcal{A}}(x) \ge 7$, contradicting to the previous claim. The last case is when the 2-element transversals are $\{x, y\}$ and $\{y, z\}$ and $\{z, x\}$. If one of x, y, z is in at least five A_j for $7 \le j \le 12$, then its $\mathcal{A}$-degree is at least 7 again. Otherwise all three are in four of these A_j sets and in two of A'_i sets ($i \le 6$), however in this case at least one of them is contained also in A_{13}, leading again to a contradiction.

As a consequence we also proved $d_{\mathcal{A}}(v) \ge 2$ because $\sum_{u \in B_j} d_{\mathcal{A}}(u) = 12$. By symmetry this is also true for the $\mathcal{B}$-degrees, so for each v we have $2 \le d_{\mathcal{A}}(v), d_{\mathcal{B}}(v) \le 5$.

Let $a_2 = |\{v \mid d_{\mathcal{A}}(v) = 2\}|$ and $a_5 = |\{v \mid d_{\mathcal{A}}(v) = 5\}|$. The average $\mathcal{A}$-degree is at most three, so $a_2 \ge 2a_5$. If $a_2 = a_5 = 0$, then $\mathcal{A}$ is 3-regular contradicting to $\sum_{u \in B_j} d_{\mathcal{A}}(u) = 12$. The number of indices i for which B_i contains a vertex of $\mathcal{A}$-degree 2 is at least $2a_2$ because if $d_{\mathcal{A}}(v) = 2$, then v is in two different B_j (as $d_{\mathcal{B}}(v) \ge 2$), and one B_j can contain only one vertex of $\mathcal{A}$-degree 2. All of these $2a_2$ B_j contains two vertices of $\mathcal{A}$-degree 5, counting with multiplicity this sums up to $4a_2 \ge 8a_5$.

Thus there is a vertex w with $d_{\mathcal{A}}(w) = 5$ and $d_{\mathcal{B}}(w) \ge 8$, a contradiction. $\square$

7 Concluding conjectures

We strongly believe that the following holds.

Conjecture 1. There exist a positive ε such that $m_n(*, *, 1) \le (1-\varepsilon)\binom{2n}{n}$ for every $n \ge 2$.

Although Constructions 4.1, and 4.3 together with Proposition 1.6 and Theorem 1.7 show that

$$\lim_{n\to\infty} \frac{m_n(\text{1-int}, \text{1-int}, 1)}{m_n(\text{1-int}, \text{1-int}, *)} = \lim_{n\to\infty} \frac{m_n(\text{01-int}, \text{01-int}, 1)}{m_n(\text{01-int}, \text{01-int}, *)} = \frac{1}{2},$$

we think that the following is also true.

Conjecture 2.

$$\lim_{n\to\infty} \frac{m_n(*, *, 1)}{m_n(*, *, *)} = 0.$$

The proof of Proposition 1.6 gives $m(a, b, 01\text{-int}, *, *) \le b^2 + b + 1$. Since the function m is monotone in a and b (one can add a different new vertex to each member of $\mathcal{A}$), we get the asymptotic $m = b^2 + O(b)$ for many of the cases we have considered. We also have a lower bound for all $a < b$ from Theorem 1.4 since $m(2, b, *, *, 1) \approx b^2/4$. It would be interesting to investigate all cases and the case $a < b$ as well.

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