Associativity and non-associativity of some hypergraph products

Richard H. Hammack, Marc Hellmuth, Lydia Ostermeier and Peter F. Stadler

Abstract. Several variants of hypergraph products have been introduced as generalizations of the strong and direct products of graphs. Here we show that only some of them are associative. In addition to the Cartesian product, these are the minimal rank preserving direct product, and the normal product. Counter-examples are given for the strong product as well as the non-rank-preserving and the maximal rank preserving direct product.

1. Introduction

Hypergraphs are natural generalizations of undirected graphs in which "edges" may consist of more than two vertices. Products of hypergraphs have been well-investigated since the 1960s, see e.g. [1, 2, 3, 4, 6, 7, 9, 10, 12, 13, 14, 15, 18].

The article [8] surveyed the literature on hypergraph products. In addition to well-known constructions such as the Cartesian product and the square product, it also considered various generalizations of graph products that had rarely been studied, if at all. In particular, it considered several variants of hypergraph products that generalize the direct and strong product of *graphs*, namely the direct products, $\overset{\frown}{\times}$ and $\widetilde{\times}$, and the strong product $\overset{\frown}{\boxtimes}$. In addition, it treated the normal product $\overset{\smile}{\boxtimes}$, a generalization of the strong graph product, and the direct product $\overset{\smile}{\times}$, which were introduced by Sonntag in the 1990's [16, 17].

Associativity is an important property of product operators. It is e.g. implicitly assumed in the standard definition of prime factors and thus for decompositions of a given hypergraph into prime factors w.r.t. a given product [5, 11].

The survey [8] mistakenly stated that the direct products $\overset{\frown}{\times}$ and $\widetilde{\times}$ and the strong product $\overset{\frown}{\boxtimes}$ are associative. Here we give a simple counterexample for these cases and prove associativity of the remaining hypergraph products. This contribution is an addendum to the results discussed in [8].

2. Preliminaries

We start our brief discussion with the formal definition of hypergraphs, and the hypergraph products in question.

A *(finite) hypergraph* $H = (V, E)$ consists of a (finite) vertex set V and a collection E of non-empty subsets of V. The *rank* of a hypergraph $H = (V, E)$ is $r(H) = \max_{e \in E} |e|$. A *homomorphism* from $H_1 = (V_1, E_1)$ into $H_2 = (V_2, E_2)$ is a map $\phi : V_1 \to V_2$ for which $\phi(e)$ is an edge in H_2 whenever e is an edge in H_1. A bijective homomorphism ϕ whose inverse is also a homomorphism is called an *isomorphism*. A hypergraph is *simple* if no edge is contained in any other edge and each edge contains two or more vertices.

In what follows, we consider six hypergraph products $\Box$, $\overset{\smile}{\times}$, $\overset{\frown}{\times}$, $\overset{\sim}{\times}$, $\overset{\smile}{\boxtimes}$, and $\overset{\frown}{\boxtimes}$. For each of these, the vertex set of the product is the Cartesian product of the vertex sets of its factors. To be more precise, given two hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ and some product $\circledast \in \{\Box, \overset{\smile}{\times}, \overset{\frown}{\times}, \overset{\sim}{\times}, \overset{\smile}{\boxtimes}, \overset{\frown}{\boxtimes}\}$, then $V(H_1 \circledast H_2) = V(H_1) \times V(H_2)$. The edge sets of the various products are defined as follows.

Cartesian Product $\Box$:
This is an immediate generalization of the standard Cartesian product of graphs. Its edges are

$$\begin{aligned} E(H_1 \Box H_2) = &\big\{\{x\} \times f \mid x \in V(H_1), f \in E(H_2)\big\} \\ &\cup \big\{e \times \{y\} \mid e \in E(H_1), y \in V(H_2)\big\}. \end{aligned}$$

There are several ways to generalize the direct product of graphs to a product of hypergraphs. Because we want such products to coincide with the usual direct product when the factors have rank 2 (and are therefore graphs) it is necessary to impose some *rank restricting* conditions on the edges. This can be accomplished in different ways and leads to different variants of the direct and strong graph products, respectively.

Minimal Rank Preserving Direct Product $\overset{\smile}{\times}$:
Given $e_1 \in E_1$ and $e_2 \in E_2$, let $r^-_{e_1,e_2} = \min\{|e_1|, |e_2|\}$. The edge set of this product is defined as

$$E(H_1 \overset{\smile}{\times} H_2) := \left\{ e \in \binom{e_1 \times e_2}{r^-_{e_1,e_2}} \mid e_i \in E_i \text{ and } |p_i(e)| = r^-_{e_1,e_2},\ i = 1, 2 \right\}.$$

The edges are thus the subsets $e \subseteq e_1 \times e_2$ (with $e_i \in E_i$) for which both projections $p_i : e \to e_i$ are injective and at least one is surjective.

Maximal Rank Preserving Direct Product $\overset{\frown}{\times}$:
Given $e_1 \in E_1$ and $e_2 \in E_2$, let $r^+_{e_1,e_2} = \max\{|e_1|, |e_2|\}$. The edge set of this product is defined as

$$E(H_1 \overset{\frown}{\times} H_2) := \left\{ e \in \binom{e_1 \times e_2}{r^+_{e_1,e_2}} \mid e_i \in E_i \text{ and } p_i(e) = e_i,\ i = 1, 2 \right\}.$$

The edges are thus the subsets $e \subseteq e_1 \times e_2$ (with $e_i \in E_i$) for which both projections $p_i : e \to e_i$ are surjective and at least one is injective.

Non-rank-preserving Direct Product $\overset{\sim}{\times}$:

$$E(H_1 \overset{\sim}{\times} H_2) := \big\{\{(x, y)\} \cup \big((e \setminus \{x\}) \times (f \setminus \{y\})\big) \mid x \in e \in E_1;\ y \in f \in E_2\big\}.$$

The strong product of graphs is defined as $E(G_1 \boxtimes G_2) = E(G_1 \Box G_2) \cup E(G_1 \times G_2)$. This leads to the following generalizations to hypergraphs.

Normal Product $\overset{\smile}{\boxtimes}$:

$$E(H_1 \overset{\smile}{\boxtimes} H_2) = E(H_1 \Box H_2) \cup E(H_1 \overset{\smile}{\times} H_2).$$

Strong Product $\overset{\frown}{\boxtimes}$:

$$E(H_1 \overset{\frown}{\boxtimes} H_2) = E(H_1 \Box H_2) \cup E(H_1 \overset{\frown}{\times} H_2).$$

3. Associativity and Non-associativity of Hypergraph Products

It is well known that the Cartesian product is associative [9]. In contrast, we will show below that none of the products $\overset{\frown}{\times}$, $\overset{\sim}{\times}$ and $\overset{\frown}{\boxtimes}$ is associative. Our counterexamples require the following lemma.

Lemma 1. *If G and H are simple hypergraphs with $r(G) = 2$ and $r(H) \leq 3$, then $G \overset{\frown}{\times} H = G \overset{\sim}{\times} H$.*

Proof. By definition, $V(G \overset{\frown}{\times} H) = V(G \overset{\sim}{\times} H)$. We need to show that $E(G \overset{\frown}{\times} H) = E(G \overset{\sim}{\times} H)$. Given $* \in \{\sim, \frown\}$ and edges e_1, e_2, let $e_1 \overset{*}{\times} e_2$ denote the set $E\big((e_1, \{e_1\}) \overset{*}{\times} (e_2, \{e_2\})\big)$. Then

$$E(G \overset{*}{\times} H) = \bigcup_{e_1 \in E(G), e_2 \in E(H)} (e_1 \overset{*}{\times} e_2).$$

It suffices to show that $e_1 \overset{\frown}{\times} e_2 = e_1 \overset{\sim}{\times} e_2$ holds for all $e_1 \in E(G)$ and $e_2 \in E(H)$. Therefore, let $e_1 \in E(G)$ and $e_2 \in E(H)$ and assume first $|e_2| = 2$. Say $e_1 = \{x_1, y_1\}$ and $e_2 = \{x_2, y_2\}$. Then

$$e_1 \overset{\frown}{\times} e_2 = \big\{\{(x_1, x_2), (y_1, y_2)\}, \{(x_1, y_2), (y_1, x_2)\}\big\} = e_1 \overset{\sim}{\times} e_2.$$

Now suppose $|e_2| = 3$, say $e_2 = \{x_2, y_2, z_2\}$. Then

$$\begin{aligned} e_1 \overset{\frown}{\times} e_2 = &\big\{\{(x_1, x_2), (x_1, y_2), (y_1, z_2)\}, \{(x_1, x_2), (y_1, y_2), (y_1, z_2)\}, \\ &\{(x_1, x_2), (y_1, y_2), (x_1, z_2)\}, \{(y_1, x_2), (x_1, y_2), (x_1, z_2)\}, \\ &\{(y_1, x_2), (x_1, y_2), (y_1, z_2)\}, \{(y_1, x_2), (y_1, y_2), (x_1, z_2)\}\big\} = e_1 \overset{\sim}{\times} e_2. \end{aligned}$$

Thus the assertion follows. $\square$

Now we present a counterexample showing that none of the products $\overset{\frown}{\times}$, $\overset{\sim}{\times}$ and $\overset{\frown}{\boxtimes}$ is associative.

Counterexample. Consider the two hypergraphs $G = (\{a, b\}, \{\{a, b\}\})$ and $H = (\{x, y, z\}, \{\{x, y, z\}\})$. For $\circledast \in \{\overset{\frown}{\times}, \overset{\sim}{\times}, \overset{\frown}{\boxtimes}\}$, we claim $G \circledast (G \circledast H) \not\cong (G \circledast G) \circledast H$. Put $e = \{(a, (a, x)), (a, (b, y)), (b, (b, z))\}$. Note that e is an edge of $G \overset{\frown}{\times} (G \overset{\frown}{\times} H)$, and hence also of $G \overset{\frown}{\boxtimes} (G \overset{\frown}{\boxtimes} H)$. However, the set $\{((a, a), x), ((a, b), y), ((b, b), z)\}$ is not an edge in $(G \overset{\frown}{\boxtimes} G) \overset{\frown}{\boxtimes} H$, thus also not in $(G \overset{\frown}{\times} G) \overset{\frown}{\times} H$, because $\{(a, a), (a, b), (b, b)\}$ is neither an edge in $G \overset{\frown}{\boxtimes} G$ nor in $G \overset{\frown}{\times} G$. Thus the map $(g, (g', h)) \mapsto ((g, g'), h)$ is not an isomorphism $G \overset{\frown}{\times} (G \overset{\frown}{\times} H) \to (G \overset{\frown}{\times} G) \overset{\frown}{\times} H$, nor is it an isomorphism $G \overset{\frown}{\boxtimes} (G \overset{\frown}{\boxtimes} H) \to (G \overset{\frown}{\boxtimes} G) \overset{\frown}{\boxtimes} H$. Moreover, the following argument shows there is no isomorphism at all.

It is shown in [7] that the number of edges in $H_1 \overset{\frown}{\times} H_2$ is

$$|E(H_1 \overset{\frown}{\times} H_2)| = \sum_{e_1 \in E_1, e_2 \in E_2} (\min\{|e_1|, |e_2|\})! S_{\max\{|e_1|, |e_2|\}, \min\{|e_1|, |e_2|\}},$$

where $S_{n,k} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$ is a Stirling number of the second kind. Furthermore,

$$\begin{aligned} |E(H_1 \overset{\frown}{\boxtimes} H_2)| &= |E(H_1 \overset{\frown}{\times} H_2)| + |E(H_1 \Box H_2)| \\ &= |E(H_1 \overset{\frown}{\times} H_2)| + |V(H_1)||E(H_2)| + |E(H_1)||V(H_2)|. \end{aligned}$$

Using this, we see that $|E(G \overset{\frown}{\times} (G \overset{\frown}{\times} H))| = 36 \neq 12 = |E((G \overset{\frown}{\times} G) \overset{\frown}{\times} H)|$ and $|E(G \overset{\frown}{\boxtimes} (G \overset{\frown}{\boxtimes} H))| = 82 \neq 58 = |E((G \overset{\frown}{\boxtimes} G) \overset{\frown}{\boxtimes} H)|$. Thus $G \overset{\frown}{\times} (G \overset{\frown}{\times} H) \not\cong (G \overset{\frown}{\times} G) \overset{\frown}{\times} H$ and $G \overset{\frown}{\boxtimes} (G \overset{\frown}{\boxtimes} H) \not\cong (G \overset{\frown}{\boxtimes} G) \overset{\frown}{\boxtimes} H$.

Moreover, Lemma 1 implies $G \overset{\sim}{\times} (G \overset{\sim}{\times} H) = G \overset{\frown}{\times} (G \overset{\frown}{\times} H) \neq (G \overset{\frown}{\times} G) \overset{\frown}{\times} H = (G \overset{\sim}{\times} G) \overset{\sim}{\times} H$.

The remainder of this contribution proves that the direct product $\overset{\smile}{\times}$ and the normal product $\overset{\smile}{\boxtimes}$ are associative. To our knowledge, these results have not yet appeared in the literature.

Proposition 2. *The direct product $\overset{\smile}{\times}$ is associative.*

Proof. Let $H_1 = (V_1, E_1)$, $H_2 = (V_2, E_2)$, and $H_3 = (V_3, E_3)$ be hypergraphs and consider the map $\psi : V\big(H_1 \overset{\smile}{\times} (H_2 \overset{\smile}{\times} H_3)\big) \to V\big((H_1 \overset{\smile}{\times} H_2) \overset{\smile}{\times} H_3\big)$ defined as $(x, (y, z)) \mapsto ((x, y), z)$. We will show that ψ is an isomorphism. Clearly ψ is bijective. Hence it remains to show the isomorphism property, that is, e is an edge in $H_1 \overset{\smile}{\times} (H_2 \overset{\smile}{\times} H_3)$ if and only if $\psi(e)$ is an edge in $(H_1 \overset{\smile}{\times} H_2) \overset{\smile}{\times} H_3$. Let $e = \{((x_1, y_1), z_1), \ldots, ((x_r, y_r), z_r)\}$ be an edge in $(H_1 \overset{\smile}{\times} H_2) \overset{\smile}{\times} H_3$. There are two cases that can occur.

First, $\{z_1, \dots, z_r\}$ is an edge in H_3 and $\{(x_1, y_1), \dots, (x_r, y_r)\}$ is therefore a subset of an edge in $H_1 \breve{\times} H_2$. Hence $\{x_1, \dots, x_r\}$ and $\{y_1 \dots, y_r\}$ must be subsets of edges in H_1 and H_2, respectively. But then $\{(y_1, z_1), \dots, (y_r, z_r)\}$ is an edge in $H_2 \breve{\times} H_3$, which implies that $\psi(e) = \{(x_1, (y_1, z_1)), \dots, (x_r, (y_r, z_r))\}$ is an edge in $H_1 \breve{\times} (H_2 \breve{\times} H_3)$.

Second, $\{(x_1, y_1), \dots, (x_r, y_r)\}$ is an edge in $H_1 \breve{\times} H_2$ and $\{z_1, \dots, z_r\}$ is a subset of an edge in H_3. Then $\{x_1, \dots, x_r\}$ is an edge in H_1 and $\{y_1 \dots, y_r\}$ is a subset of an edge in H_2, or vice versa. In the first case $\{(y_1, z_1), \dots, (y_r, z_r)\}$ is a subset of an edge in $H_2 \breve{\times} H_3$, hence $\psi(e)$ is an edge in $H_1 \breve{\times} (H_2 \breve{\times} H_3)$, and in the second case $\{(y_1, z_1), \dots, (y_r, z_r)\}$ is an edge in $H_2 \breve{\times} H_3$ and thus $\psi(e)$ is an edge in $H_1 \breve{\times} (H_2 \breve{\times} H_3)$.

This implies that if e is an edge in $(H_1 \breve{\times} H_2) \breve{\times} H_3$, then $\psi(e)$ is an edge in $H_1 \breve{\times} (H_2 \breve{\times} H_3)$. The converse follows analogously. Thus $(H_1 \breve{\times} H_2) \breve{\times} H_3 \cong H_1 \breve{\times} (H_2 \breve{\times} H_3)$. □

Proposition 3. *The normal product $\breve{\boxtimes}$ is associative.*

Proof. As in the previous proof, consider the bijection $\psi : V\big((H_1 \breve{\boxtimes} H_2) \breve{\boxtimes} H_3\big) \to V\big(H_1 \breve{\boxtimes} (H_2 \breve{\boxtimes} H_3)\big)$ defined as $((x, y), z) \mapsto (x, (y, z))$. We claim this is an isomorphism.

Let $p_{1,2}$ be the projection from $(H_1 \boxtimes H_2) \boxtimes H_3$ onto $H_1 \boxtimes H_2$, defined by $p_{1,2}(((x, y), z)) = (x, y)$. Let $p_{2,3}$ be projection from $H_1 \boxtimes (H_2 \boxtimes H_3)$ to $H_2 \boxtimes H_3$, whereas p_j is the usual projection to H_j. By definition, $e = \{((x_1, y_1), z_1), \dots, ((x_r, y_r), z_r)\}$ is an edge in $(H_1 \breve{\boxtimes} H_2) \breve{\boxtimes} H_3$ if and only if one of the following conditions is satisfied:

(i) $p_{1,2}(e) = e_{1,2} \in E(H_1 \breve{\boxtimes} H_2)$ and $|p_3(e)| = 1$,
(ii) $p_3(e) = e_3 \in E(H_3)$ and $|p_{1,2}(e)| = 1$,
(iii) $p_{1,2}(e) = e_{1,2} \in E(H_1 \breve{\boxtimes} H_2)$ and $p_3(e) \subseteq e_3 \in E(H_3)$ and $|e| = |e_{1,2}| = |p_{1,2}(e)| = |p_3(e)| \leq |e_3|$,
(iv) $p_3(e) = e_3 \in E(H_3)$ and $p_{1,2}(e) \subseteq e_{1,2} \in E(H_1 \breve{\boxtimes} H_2)$ and $|e| = |e_3| = |p_3(e)| = |p_{1,2}(e)| \leq |e_{1,2}|$.

Condition (i) is equivalent to one of the following conditions holding:

(i a) $p_1(e) = p_1(e_{1,2}) = e_1 \in E(H_1)$ and $|p_2(e)| = |p_2(e_{1,2})| = |p_3(e)| = 1$, or
(i b) $p_2(e) = p_2(e_{1,2}) = e_2 \in E(H_2)$ and $|p_1(e)| = |p_1(e_{1,2})| = |p_3(e)| = 1$, or
(i c) $p_1(e) = p_1(e_{1,2}) = e_1 \in E(H_1)$ and $p_2(e) = p_2(e_{1,2}) \subseteq e_2 \in E(H_2)$ and $|e| = |e_1| = |p_1(e)| = |p_2(e)| \leq |e_2|$ and $|p_3(e)| = 1$, or
(i d) $p_2(e) = p_2(e_{1,2}) = e_2 \in E(H_2)$ and $p_1(e) = p_1(e_{1,2}) \subseteq e_1 \in E(H_1)$ and $|e| = |e_2| = |p_2(e)| = |p_1(e)| \leq |e_1|$ and $|p_3(e)| = 1$.

Condition (iii) is equivalent to one of the following conditions holding:

(iii a) $p_1(e) = p_1(e_{1,2}) = e_1 \in E(H_1)$ and $|p_2(e)| = |p_2(e_{1,2})| = 1$ and $p_3(e) \subseteq e_3 \in E(H_3)$ and $|e| = |p_3(e)| \leq |e_3|$, or
(iii b) $p_2(e) = p_2(e_{1,2}) = e_2 \in E(H_2)$ and $|p_1(e)| = |p_1(e_{1,2})| = 1$ and $p_3(e) \subseteq e_3 \in E(H_3)$ and $|e| = |p_3(e)| \leq |e_3|$, or
(iii c) $p_1(e) = p_1(e_{1,2}) = e_1 \in E(H_1)$ and $p_2(e) = p_2(e_{1,2}) \subseteq e_2 \in E(H_2)$ and $|e| = |e_1| = |p_1(e)| = |p_2(e)| \leq |e_2|$ and $p_3(e) \subseteq e_3 \in E(H_3)$ and $|e| = |p_3(e)| \leq |e_3|$, or
(iii d) $p_2(e) = p_2(e_{1,2}) = e_2 \in E(H_2)$ and $p_1(e) = p_1(e_{1,2}) \subseteq e_1 \in E(H_1)$ and $|e| = |e_2| = |p_2(e)| = |p_1(e)| \leq |e_1|$ and $p_3(e) \subseteq e_3 \in E(H_3)$ and $|e| = |p_3(e)| \leq |e_3|$.

Condition (iv) is equivalent to one of the following conditions holding:

(iv a) $p_3(e) = e_3 \in E(H_3)$ and $p_1(e) = p_1(p_{1,2}(e)) \subseteq e_1 \in E(H_1)$ and $|p_2(e)| = |p_2(p_{1,2})(e)| = 1$ and $|e| = |p_3(e)| = |e_3| = |p_1(e)| \leq |e_1|$, or
(iv b) $p_3(e) = e_3 \in E(H_3)$ and $p_2(e) = p_2(p_{1,2}(e)) \subseteq e_2 \in E(H_2)$ and $|p_1(e)| = |p_1(p_{1,2})(e)| = 1$ and $|e| = |p_3(e)| = |e_3| = |p_2(e)| \leq |e_2|$, or

(iv c) $p_3(e) = e_3 \in E(H_3)$ and $p_1(e) = p_1(p_{1,2}(e)) \subseteq e_1 \in E(H_1)$ and $p_2(e) = p_2(p_{1,2}(e)) \subseteq e_2 \in E(H_2)$ and $|e| = |p_3(e)| = |e_3| = |p_1(e)| = |p_2(e)| \leq \min_{i=1,2} |e_i|$.

Then Condition (i a) implies the following condition:

(I) $p_1(e) = e_1 \in E(H_1)$ and $|p_{2,3}(e)| = 1$.

Conditions (i b), (ii), (iii b) and (iv b) each imply the following condition:

(II) $|p_1(e)| = 1$ and $p_{2,3}(e) = e_{2,3} \in E(H_2 \breve{\boxtimes} H_3)$.

Conditions (i c), (iii a) and (iii c) each imply the following condition:

(III) $p_1(e) = e_1 \in E(H_1)$ and $p_{2,3}(e) \subseteq e_{2,3} \in E(H_2 \breve{\boxtimes} H_3)$ and $|e| = |e_1| = |p_1(e)| = |p_{2,3}(e)| \leq |e_{2,3}|$.

Conditions (i d), (iii d), (iv a) and (iv c) each imply the following condition:

(IV) $p_1(e) \subseteq e_1 \in E(H_1)$ and $p_{2,3}(e) = e_{2,3} \in E(H_2 \breve{\boxtimes} H_3)$ and $|e| = |e_{2,3}| = |p_{2,3}(e)| = |p_1(e)| \leq |e_1|$.

By definition of the normal product, if any of the Conditions (I)–(IV) are satisfied, then $\psi(e) = \{(x_1, (y_1, z_1)), \ldots, (x_r, (y_r, z_r))\}$ is an edge in $H_1 \breve{\boxtimes} (H_2 \breve{\boxtimes} H_3)$. It follows that ψ is a homomorphism. In the same way, the inverse $(x, (y, z)) \mapsto ((x, y), z)$ is also a homomorphism. □

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Richard H. Hammack
Department of Mathematics and Applied Mathematics
Virginia Commonwealth University
Richmond, VA 23284-2014, USA
e-mail: rhammack@vcu.edu

Marc Hellmuth
Department of Mathematics and Computer Science
University of Greifswald
Walther-Rathenau-Straße 47, D-17487 Greifswald, Germany

Center for Bioinformatics
Saarland University
Building E 2.1, Room 413, P.O. Box 15 11 50, D-66041 Saarbrücken, Germany
e-mail: mhellmuth@mailbox.org

Lydia Ostermeier
Bioinformatics Group,
Department of Computer Science and Interdisciplinary Center for Bioinformatics
University of Leipzig,
Härtelstraße 16-18, D-04107 Leipzig, Germany
e-mail: glydia@bioinf.uni-leipzig.de

Peter F. Stadler
Bioinformatics Group,
Department of Computer Science; and Interdisciplinary Center for Bioinformatics,
University of Leipzig,
Härtelstraße 16-18, D-04107 Leipzig, Germany

Max Planck Institute for Mathematics in the Sciences
Inselstrasse 22, D-04103 Leipzig, Germany

RNomics Group, Fraunhofer Institut für Zelltherapie und Immunologie, Deutscher Platz 5e, D-04103 Leipzig, Germany

Department of Theoretical Chemistry, University of Vienna, Währingerstraße 17, A-1090 Wien, Austria

Santa Fe Institute, 1399 Hyde Park Rd., Santa Fe, NM87501, USA
e-mail: studla@bioinf.uni-leipzig.de