Chapter 1

1 Functions and relations

Objectives

- To revise **set notation**, including the notation for **sets of numbers**.
- To understand the concepts of **relation** and **function**.
- To find the **domain** and **range** of a given relation.
- To find the **implied (maximal) domain** of a function.
- To work with **restrictions of a function**, **piecewise-defined functions**, **odd functions** and **even functions**.
- To decide whether or not a given function is **one-to-one**.
- To find the **inverse** of a one-to-one function.
- To understand **sums** and **products** of functions.
- To use **addition of ordinates** to help sketch the graph of a sum of two functions.
- To define and use **composite functions**.
- To understand the concepts of **strictly increasing** and **strictly decreasing**.
- To work with **power functions** and their graphs.
- To apply a knowledge of functions to **solving problems**.

In this chapter we introduce the notation that will be used throughout the rest of the book. You will have met much of it before and this will serve as revision. The language introduced in this chapter helps to express important mathematical ideas precisely. Initially they may seem unnecessarily abstract, but later in the book you will find them used more and more in practical situations.

In Chapters 2 to 8 we will study different families of functions. In Chapter 2 we will revise linear functions, in Chapter 4, polynomial functions in general and in Chapters 5 and 6, exponential, logarithmic and circular functions.

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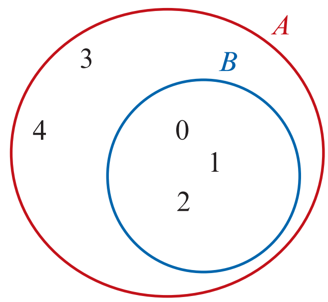
1A Set notation and sets of numbers

▶ Set notation

Set notation is used widely in mathematics and in this book where appropriate. This section summarises all of the set notation you will need.

- A **set** is a collection of objects. The objects that are in the set are known as **elements** or members of the set.
- If x is an element of a set A, we write $x \in A$. This can also be read as 'x is a member of the set A' or 'x belongs to A' or 'x is in A'.
- If x is **not an element** of A, we write $x \notin A$.
- A set B is called a **subset** of a set A if every element of B is also an element of A. We write $B \subseteq A$. This expression can also be read as 'B is contained in A' or 'A contains B'.



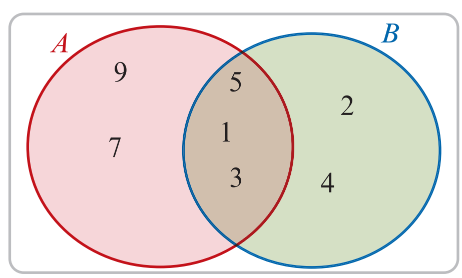
For example, let $B = \{0, 1, 2\}$ and $A = \{0, 1, 2, 3, 4\}$. Then

$$3 \in A, \quad 3 \notin B \quad \text{and} \quad B \subseteq A$$

as illustrated in the Venn diagram opposite.

- The set of elements common to two sets A and B is called the **intersection** of A and B, and is denoted by $A \cap B$. Thus $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- If the sets A and B have no elements in common, we say A and B are **disjoint**, and write $A \cap B = \emptyset$. The set $\emptyset$ is called the **empty set**.
- The set of elements that are in A or in B (or in both) is called the **union** of sets A and B, and is denoted by $A \cup B$.

For example, let $A = \{1, 3, 5, 7, 9\}$ and $B = \{1, 2, 3, 4, 5\}$. The intersection and union are illustrated by the Venn diagram shown opposite:



$$A \cap B = \{1, 3, 5\}$$
$$A \cup B = \{1, 2, 3, 4, 5, 7, 9\}$$

Example 1

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, find:

a $A \cap B$

b $A \cup B$

Solution

a $A \cap B = \{3, 7\}$

b $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

Explanation

The elements 3 and 7 are common to sets A and B.

The set $A \cup B$ contains all elements that belong to A or B (or both).

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The **set difference** of two sets A and B is given by

$$A \setminus B = \{ x : x \in A,\ x \notin B \}$$

The set $A \setminus B$ contains the elements of A that are not elements of B.

Example 2

For $A = \{1, 2, 3, 7\}$ and $B = \{3, 4, 5, 6, 7\}$, find:

a $A \setminus B$ **b** $B \setminus A$

Solution	*Explanation*
a $A \setminus B = \{1, 2, 3, 7\} \setminus \{3, 4, 5, 6, 7\} = \{1, 2\}$	The elements 1 and 2 are in A but not in B.
b $B \setminus A = \{3, 4, 5, 6, 7\} \setminus \{1, 2, 3, 7\} = \{4, 5, 6\}$	The elements 4, 5 and 6 are in B but not in A.

Sets of numbers

We begin by recalling that the elements of $\{1, 2, 3, 4, \dots\}$ are called the **natural numbers**, and the elements of $\{ \dots, -2, -1, 0, 1, 2, \dots \}$ are called **integers**.

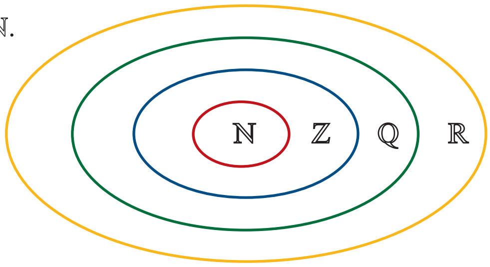
The numbers of the form $\frac{p}{q}$, with p and q integers, $q \neq 0$, are called **rational numbers**.

The real numbers which are not rational are called **irrational** (e.g. π and $\sqrt{2}$).

The rationals may be characterised as being those real numbers that can be written as a terminating or recurring decimal.

- The set of real numbers will be denoted by $\mathbb{R}$.
- The set of rational numbers will be denoted by $\mathbb{Q}$.
- The set of integers will be denoted by $\mathbb{Z}$.
- The set of natural numbers will be denoted by $\mathbb{N}$.

It is clear that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$, and this may be represented by the diagram on the right.



Describing a set

It is not always possible to list the elements of a set. There is an alternative way of describing sets that is especially useful for infinite sets.

The set of all x such that ___ is denoted by $\{ x : \_\_\_ \}$. Thus, for example:

- $\{ x : 0 < x < 1 \}$ is the set of all real numbers strictly between 0 and 1
- $\{ x : x \geq 3 \}$ is the set of all real numbers greater than or equal to 3
- $\{ x : x > 0,\ x \in \mathbb{Q} \}$ is the set of all positive rational numbers
- $\{ 2n : n = 0, 1, 2, \dots \}$ is the set of all non-negative even numbers
- $\{ 2n + 1 : n = 0, 1, 2, \dots \}$ is the set of all non-negative odd numbers.

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Interval notation

Among the most important subsets of $\mathbb{R}$ are the **intervals**. The following is an exhaustive list of the various types of intervals and the standard notation for them. We suppose that a and b are real numbers with $a < b$.

$$(a, b) = \{ x : a < x < b \} \qquad [a, b] = \{ x : a \le x \le b \}$$
$$(a, b] = \{ x : a < x \le b \} \qquad [a, b) = \{ x : a \le x < b \}$$
$$(a, \infty) = \{ x : a < x \} \qquad [a, \infty) = \{ x : a \le x \}$$
$$(-\infty, b) = \{ x : x < b \} \qquad (-\infty, b] = \{ x : x \le b \}$$

Intervals may be represented by diagrams as shown in Example 3.

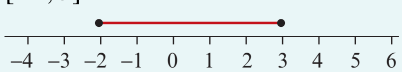
Example 3

Illustrate each of the following intervals of real numbers:

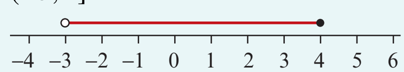
a $[-2, 3]$ **b** $(-3, 4]$ **c** $(-\infty, 5]$ **d** $(-2, 4)$ **e** $(-3, \infty)$

Solution

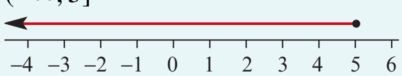
a $[-2, 3]$



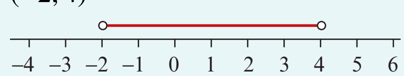
b $(-3, 4]$



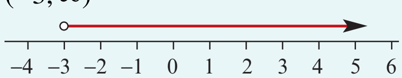
c $(-\infty, 5]$



d $(-2, 4)$



e $(-3, \infty)$



Notes:

- The 'closed' circle (●) indicates that the number is included.
- The 'open' circle (○) indicates that the number is not included.

The following are subsets of the real numbers for which we have special notation:

- Positive real numbers: $\mathbb{R}^+ = \{ x : x > 0 \}$
- Negative real numbers: $\mathbb{R}^- = \{ x : x < 0 \}$
- Real numbers excluding zero: $\mathbb{R} \setminus \{0\}$

Section summary

- If x is an element of a set A, we write $x \in A$.
- If x is not an element of a set A, we write $x \notin A$.
- If every element of B is an element of A, we say B is a **subset** of A and write $B \subseteq A$.
- The set $A \cap B$ is the **intersection** of A and B, where $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.
- The set $A \cup B$ is the **union** of A and B, where $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

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- The set $A \setminus B$ is the **set difference** of A and B, where $A \setminus B = \{\, x : x \in A,\ x \notin B \,\}$.
- If the sets A and B have no elements in common, we say A and B are **disjoint** and write $A \cap B = \emptyset$. The set $\emptyset$ is called the **empty set**.
- Sets of numbers:
  - Real numbers: $\mathbb{R}$
  - Rational numbers: $\mathbb{Q}$
  - Integers: $\mathbb{Z}$
  - Natural numbers: $\mathbb{N}$
- For real numbers a and b with $a < b$, we can consider the following **intervals**:

$$(a, b) = \{\, x : a < x < b \,\} \qquad [a, b] = \{\, x : a \le x \le b \,\}$$
$$(a, b] = \{\, x : a < x \le b \,\} \qquad [a, b) = \{\, x : a \le x < b \,\}$$
$$(a, \infty) = \{\, x : a < x \,\} \qquad [a, \infty) = \{\, x : a \le x \,\}$$
$$(-\infty, b) = \{\, x : x < b \,\} \qquad (-\infty, b] = \{\, x : x \le b \,\}$$

Exercise 1A

Example 1

1 For $A = \{3, 8, 11, 18, 22, 23, 24\}$, $B = \{8, 11, 25, 30, 32\}$ and $C = \{1, 8, 11, 25, 30\}$, find:

a $A \cap B$ **b** $A \cap B \cap C$ **c** $A \cup C$
d $A \cup B$ **e** $A \cup B \cup C$ **f** $(A \cap B) \cup C$

Example 2

2 For $A = \{3, 8, 11, 18, 22, 23, 24\}$, $B = \{8, 11, 25, 30, 32\}$ and $C = \{1, 8, 11, 25, 30\}$, find:

a $A \setminus B$ **b** $B \setminus A$ **c** $A \setminus C$ **d** $C \setminus A$

Example 3

3 Illustrate each of the following intervals on a number line:

a $[-2, 3)$ **b** $(-\infty, 4]$ **c** $[-3, -1]$ **d** $(-3, \infty)$ **e** $(-4, 3)$ **f** $(-1, 4]$

4 For $X = \{2, 3, 5, 7, 9, 11\}$, $Y = \{7, 9, 15, 19, 23\}$ and $Z = \{2, 7, 9, 15, 19\}$, find:

a $X \cap Y$ **b** $X \cap Y \cap Z$ **c** $X \cup Y$ **d** $X \setminus Y$
e $Z \setminus Y$ **f** $X \cap Z$ **g** $[-2, 8] \cap X$ **h** $(-3, 8] \cap Y$
i $(2, \infty) \cap Y$ **j** $(3, \infty) \cup Y$

5 For $X = \{a, b, c, d, e\}$ and $Y = \{a, e, i, o, u\}$, find:

a $X \cap Y$ **b** $X \cup Y$ **c** $X \setminus Y$ **d** $Y \setminus X$

6 For $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $B = \{2, 4, 6, 8, 10\}$ and $C = \{1, 3, 6, 9\}$, find:

a $B \cap C$ **b** $B \setminus C$ **c** $A \setminus B$ **d** $(A \setminus B) \cup (A \setminus C)$
e $A \setminus (B \cap C)$ **f** $(A \setminus B) \cap (A \setminus C)$ **g** $A \setminus (B \cup C)$ **h** $A \cap B \cap C$

7 Use the appropriate interval notation (i.e. $[a, b]$, (a, b), etc.) to describe each of the following sets:

a $\{\, x : -3 \le x < 1 \,\}$ **b** $\{\, x : -4 < x \le 5 \,\}$ **c** $\{\, y : -\sqrt{2} < y < 0 \,\}$
d $\left\{ x : -\dfrac{1}{\sqrt{2}} < x < \sqrt{3} \right\}$ **e** $\{\, x : x < -3 \,\}$ **f** $\mathbb{R}^+$
g $\mathbb{R}^-$ **h** $\{\, x : x \ge -2 \,\}$

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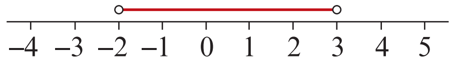
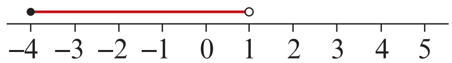
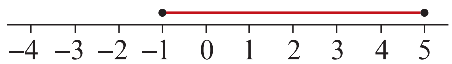
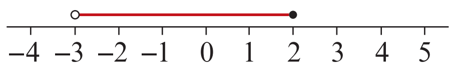
8 Describe each of the following subsets of the real number line using the interval notation $[a, b)$, (a, b), etc.:

a


b


c


d


9 Illustrate each of the following intervals on a number line:

a $(-3, 2]$ **b** $(-4, 3)$ **c** $(-\infty, 3)$ **d** $[-4, -1]$ **e** $[-4, \infty)$ **f** $[-2, 5)$

10 For each of the following, use one number line on which to represent the sets:

a $[-3, 6],\ [2, 4],\ [-3, 6] \cap [2, 4]$ **b** $[-3, 6],\ \mathbb{R} \setminus [-3, 6]$

c $[-2, \infty),\ (-\infty, 6],\ [-2, \infty) \cap (-\infty, 6]$ **d** $(-8, -2),\ \mathbb{R}^- \setminus (-8, -2)$

1B Identifying and describing relations and functions

▶ Relations, domain and range

An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.

A **relation** is a set of ordered pairs. The following are examples of relations:

a $S = \{(1, 1), (1, 2), (3, 4), (5, 6)\}$

b $T = \{(-3, 5), (4, 12), (5, 12), (7, -6)\}$

Every relation determines two sets:

- The set of all the first coordinates of the ordered pairs is called the **domain**.
- The set of all the second coordinates of the ordered pairs is called the **range**.

For the above examples:

a domain of $S = \{1, 3, 5\}$, range of $S = \{1, 2, 4, 6\}$

b domain of $T = \{-3, 4, 5, 7\}$, range of $T = \{5, 12, -6\}$

Some relations may be defined by a **rule** relating the elements in the domain to their corresponding elements in the range. In order to define the relation fully, we need to specify both the rule and the domain. For example, the set

$$\{\, (x, y) : y = x + 1,\ x \in \{1, 2, 3, 4\} \,\}$$

is the relation

$$\{(1, 2), (2, 3), (3, 4), (4, 5)\}$$

The **domain** is the set $X = \{1, 2, 3, 4\}$ and the **range** is the set $Y = \{2, 3, 4, 5\}$.

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When the domain of a relation is not explicitly stated, it is understood to consist of all real numbers for which the defining rule has meaning. For example:

- $S = \{ (x, y) : y = x^2 \}$ is assumed to have domain $\mathbb{R}$
- $T = \{ (x, y) : y = \sqrt{x} \}$ is assumed to have domain $[0, \infty)$.

Example 4

Sketch the graph of each of the following relations and state the domain and range of each:

a $\{ (x, y) : y = x^2 \}$

b $\{ (x, y) : y \leq x + 1 \}$

c $\{(-2, -1), (-1, -1), (-1, 1), (0, 1), (1, -1)\}$

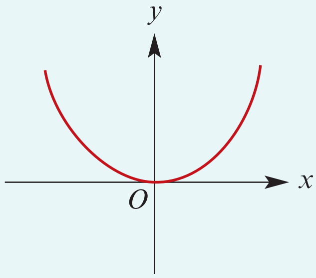
d $\{ (x, y) : x^2 + y^2 = 1 \}$

e $\{ (x, y) : 2x + 3y = 6,\ x \geq 0 \}$

f $\{ (x, y) : y = 2x - 1,\ x \in [-1, 2] \}$

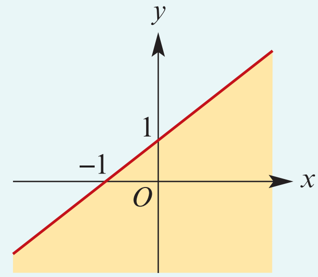
Solution

a



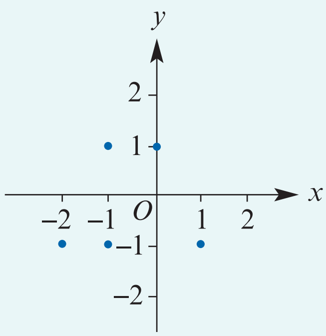
Domain = $\mathbb{R}$; Range = $\mathbb{R}^+ \cup \{0\}$

b



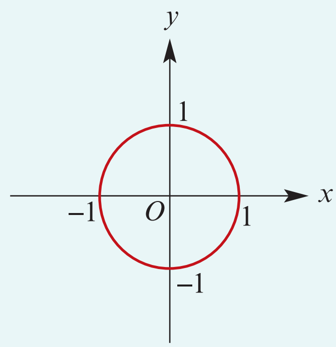
Domain = $\mathbb{R}$; Range = $\mathbb{R}$

c



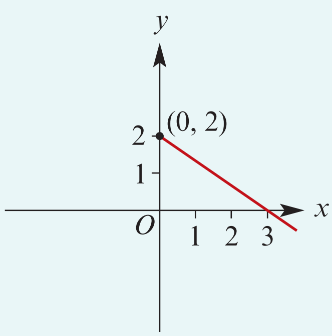
Domain = $\{-2, -1, 0, 1\}$
Range = $\{-1, 1\}$

d



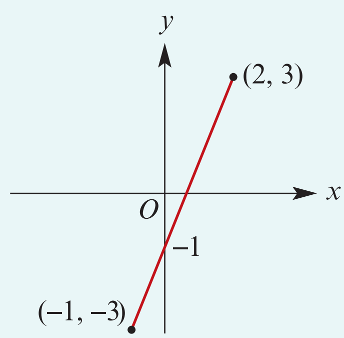
Domain = $[-1, 1]$; Range = $[-1, 1]$

e



Domain = $[0, \infty)$; Range = $(-\infty, 2]$

f



Domain = $[-1, 2]$; Range = $[-3, 3]$

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Sometimes set notation is not used in the specification of a relation.

For the previous example:

a is written as $y = x^2$

b is written as $y \leq x + 1$

e is written as $2x + 3y = 6$, $x \geq 0$

Functions

A **function** is a relation such that for each x-value there is only one corresponding y-value. This means that, if (a, b) and (a, c) are ordered pairs of a function, then $b = c$. In other words, a function cannot contain two different ordered pairs with the same first coordinate.

Example 5

Which of the following sets of ordered pairs defines a function?

a $S = \{(-3, -4), (-1, -1), (-6, 7), (1, 5)\}$

b $T = \{(-4, 1), (-4, -1), (-6, 7), (-6, 8)\}$

Solution

a S is a function because for each x-value there is only one y-value.

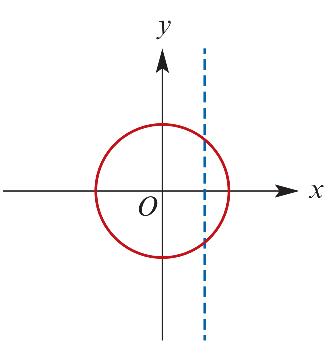
b T is not a function, because there is an x-value with two different y-values: the two ordered pairs $(-4, 1)$ and $(-4, -1)$ in T have the same first coordinate.

One way to identify whether a relation is a function is to draw a graph of the relation and then apply the following test.

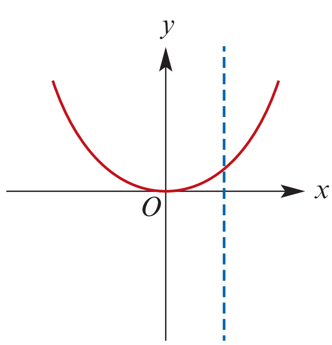
Vertical-line test

If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a **function**.

For example:



$x^2 + y^2 = 1$ is not a function



$y = x^2$ is a function

Functions are usually denoted by lowercase letters such as f, g, h.

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The definition of a function tells us that, for each x in the domain of f, there is a unique element y in the range such that $(x, y) \in f$. The element y is called 'the **image** of x under f' or 'the **value** of f at x', and the element x is called 'a **pre-image** of y'.

For $(x, y) \in f$, the element y is determined by x, and so we also use the notation $f(x)$, read as 'f of x', in place of y.

This gives an alternative way of writing functions:

- For the function $\{\,(x, y) : y = x^2\,\}$, write $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = x^2$.
- For the function $\{\,(x, y) : y = 2x - 1,\ x \in [0, 4]\,\}$, write $f\colon [0, 4] \to \mathbb{R},\ f(x) = 2x - 1$.
- For the function $\left\{(x, y) : y = \dfrac{1}{x}\right\}$, write $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = \dfrac{1}{x}$.

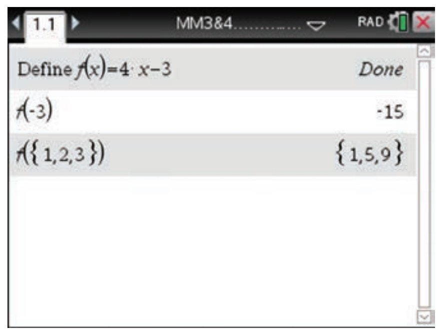
If the domain is $\mathbb{R}$, we often just write the rule: for example, $f(x) = x^2$.

Note that in using the notation $f\colon X \to Y$, the set X is the domain, but Y is not necessarily the range. It is a set that contains the range and is called the **codomain**. With this notation for functions, the domain of f is written as **dom** $\boldsymbol{f}$ and range of f as **ran** $\boldsymbol{f}$.

Using the TI-Nspire

Function notation can be used with a CAS calculator.

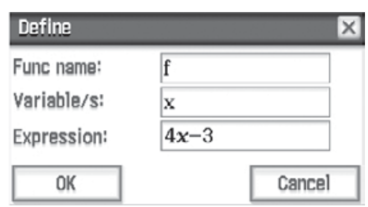
- Use (menu) > **Actions** > **Define** to define the function $f(x) = 4x - 3$.
- To find the value of $f(-3)$, type $f(-3)$ followed by (enter).
- To evaluate $f(1)$, $f(2)$ and $f(3)$, type $f(\{1, 2, 3\})$ followed by (enter).



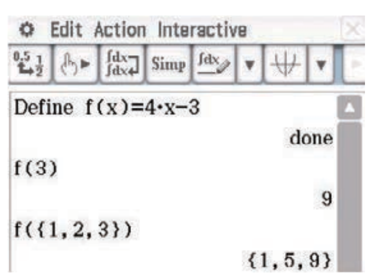
Using the Casio ClassPad

Function notation can be used with a CAS calculator.

- In $\overset{\text{Main}}{\sqrt{\alpha}}$, select **Interactive** > **Define**.
- Enter the expression $4x - 3$ as shown and tap OK.



- Enter $f(3)$ in the entry line and tap (EXE).
- Enter $f(\{1, 2, 3\})$ to obtain the values of $f(1)$, $f(2)$ and $f(3)$.



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Example 6

For $f(x) = 2x^2 + x$, find:

a $f(3)$ **b** $f(-2)$ **c** $f(x-1)$

Solution

a $f(3) = 2(3)^2 + 3$
$= 21$

b $f(-2) = 2(-2)^2 - 2$
$= 6$

c $f(x-1) = 2(x-1)^2 + (x-1)$
$= 2(x^2 - 2x + 1) + x - 1$
$= 2x^2 - 3x + 1$

Example 7

For $g(x) = 3x^2 + 1$:

a Find $g(-2)$ and $g(4)$.

b Express each the following in terms of x:

i $g(-2x)$ **ii** $g(x-2)$ **iii** $g(x+2)$ **iv** $g(x^2)$

Solution

a $g(-2) = 3(-2)^2 + 1 = 13$ and $g(4) = 3(4)^2 + 1 = 49$

b **i** $g(-2x) = 3(-2x)^2 + 1$
$= 3 \times 4x^2 + 1$
$= 12x^2 + 1$

ii $g(x-2) = 3(x-2)^2 + 1$
$= 3(x^2 - 4x + 4) + 1$
$= 3x^2 - 12x + 13$

iii $g(x+2) = 3(x+2)^2 + 1$
$= 3(x^2 + 4x + 4) + 1$
$= 3x^2 + 12x + 13$

iv $g(x^2) = 3(x^2)^2 + 1$
$= 3x^4 + 1$

Example 8

Consider the function defined by $f(x) = 2x - 4$ for all $x \in \mathbb{R}$.

a Find the value of $f(2)$, $f(-1)$ and $f(t)$.

b For what values of t is $f(t) = t$?

c For what values of x is $f(x) \geq x$?

d Find the pre-image of 6.

Solution

a $f(2) = 2(2) - 4 = 0$
$f(-1) = 2(-1) - 4 = -6$
$f(t) = 2t - 4$

b $f(t) = t$
$2t - 4 = t$
$t - 4 = 0$
$\therefore \quad t = 4$

c $f(x) \geq x$
$2x - 4 \geq x$
$x - 4 \geq 0$
$\therefore \quad x \geq 4$

d $f(x) = 6$
$2x - 4 = 6$
$x = 5$

Thus 5 is the pre-image of 6.

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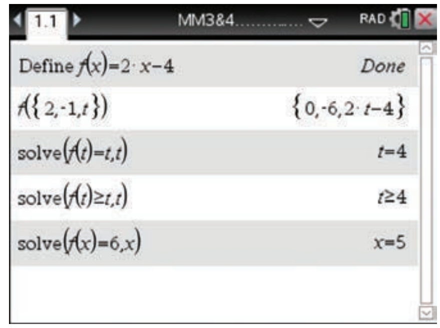
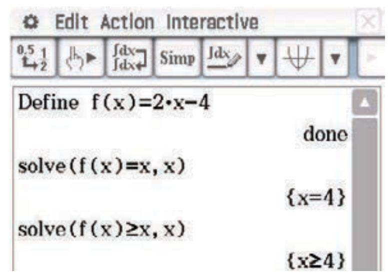
Using the TI-Nspire

- Use menu > **Actions** > **Define** to define the function and menu > **Algebra** > **Solve** to solve as shown.
- The symbol $\geq$ can be found using ctrl = or using ctrl menu > **Symbols**.



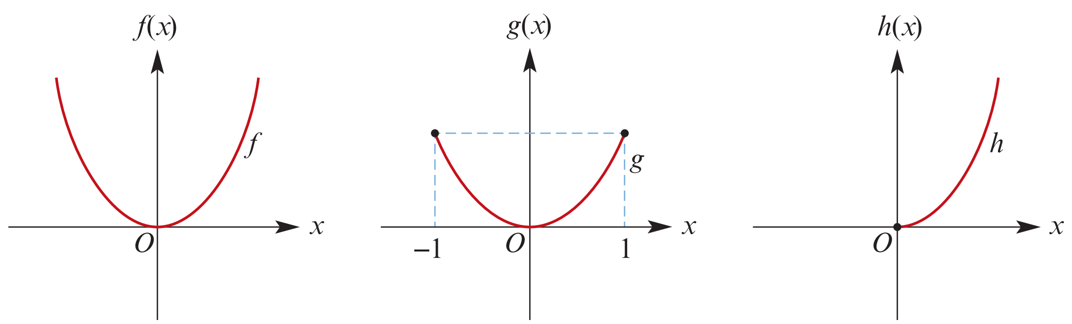
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, define the function $f(x) = 2x - 4$ using **Interactive** > **Define**.
- Now enter and highlight the equation $f(x) = x$.
- Select **Interactive** > **Equation/Inequality** > **solve**. Ensure the variable is set as x and tap OK.
- To enter the inequality, find the symbol $\geq$ in the Math3 keyboard.



▶ Restriction of a function

Consider the following functions:



$f(x) = x^2,\ x \in \mathbb{R}$ $\qquad$ $g(x) = x^2,\ -1 \leq x \leq 1$ $\qquad$ $h(x) = x^2,\ x \in \mathbb{R}^+ \cup \{0\}$

The different letters, f, g and h, used to name the functions emphasise the fact that there are three different functions, even though they each have the same rule. They are different because they are defined for different domains.

We call g and h **restrictions** of f, since their domains are subsets of the domain of f.

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Example 9

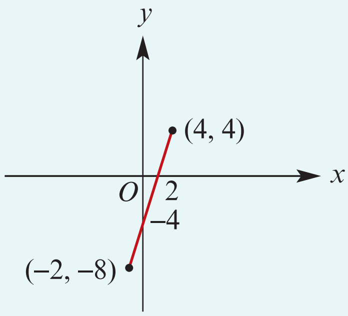
For each of the following, sketch the graph and state the range:

a $f\colon [-2, 4] \to \mathbb{R}$, $f(x) = 2x - 4$

b $g\colon (-1, 2] \to \mathbb{R}$, $g(x) = x^2$

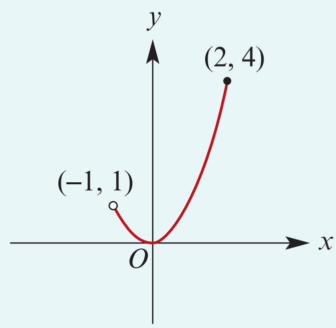
Solution

a



Range $= [-8, 4]$

b



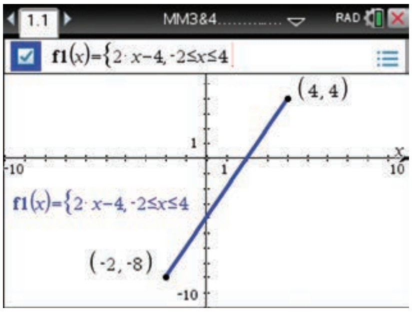
Range $= [0, 4]$

Using the TI-Nspire

Domain restrictions can be entered with the function if required.

For example: $f1(x) = 2x - 4 \mid -2 \le x \le 4$

Domain restrictions are entered using the 'with' symbol $\mid$, which is accessed using ctrl = or by using the **Symbols** palette ctrl 📖 and scrolling to the required symbol. The inequality symbols are also accessed from this palette.

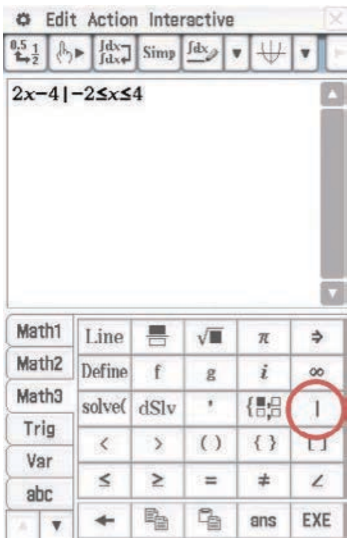


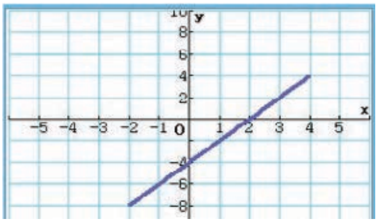
Using the Casio ClassPad

Domain restrictions can be entered with the function if required.

For example: $2x - 4 \mid -2 \le x \le 4$

Domain restrictions are entered using the 'with' symbol $\mid$, which is accessed from the Math3 palette in the soft keyboard. The inequality symbols are also accessed from Math3.





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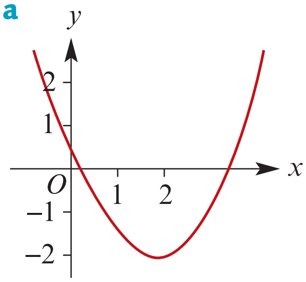
Section summary

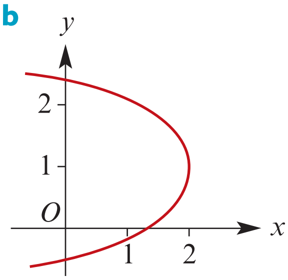
- An **ordered pair**, denoted (x, y), is a pair of elements x and y in which x is considered to be the first coordinate and y the second coordinate.
- A **relation** is a set of ordered pairs.
  - The set of all the first coordinates of the ordered pairs is called the **domain**.
  - The set of all the second coordinates of the ordered pairs is called the **range**.
- Some relations may be defined by a rule relating the elements in the domain to their corresponding elements in the range: for example, $\{ (x, y) : y = x + 1,\ x \in \mathbb{R}^+ \cup \{0\} \}$.
- A **function** is a relation such that for each x-value there is only one corresponding y-value.
- **Vertical-line test**: If a vertical line can be drawn anywhere on the graph and it only ever intersects the graph a maximum of once, then the relation is a function.
- For an ordered pair (x, y) of a function f, we say that y is the **image** of x under f or that y is the value of f at x, and we say that x is a **pre-image** of y. Since the y-value is determined by the x-value, we use the notation $f(x)$, read as 'f of x', in place of y.
- Notation for defining functions: For example, we write $f \colon [0, 4] \to \mathbb{R}$, $f(x) = 2x - 1$ to define a function f with domain $[0, 4]$ and rule $f(x) = 2x - 1$.
- A **restriction** of a function has the same rule but a 'smaller' domain.

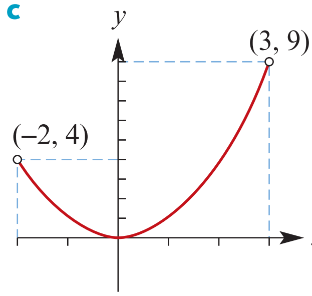
Exercise 1B

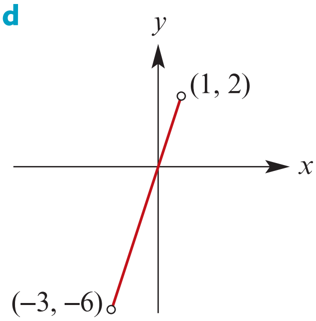
Skillsheet

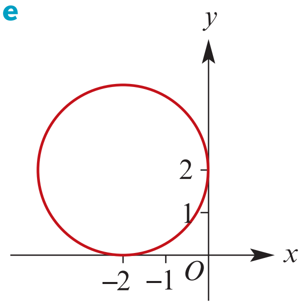
1 State the domain and range for the relations represented by each of the following graphs:

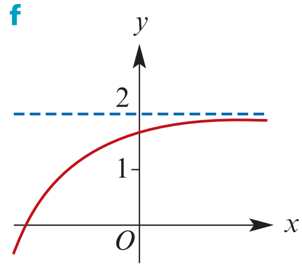
a


b


c


d


e


f


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Example 4 **2** Sketch a graph of each of the following relations and state its domain and range:

a $\{(x, y) : y = x^2 + 1\}$ **b** $\{(x, y) : x^2 + y^2 = 9\}$

c $\{(x, y) : 3x + 12y = 24, \ x \geq 0\}$ **d** $y = \sqrt{2x}$

e $\{(x, y) : y = 5 - x, \ x \in [0, 5]\}$ **f** $y = x^2 + 2, \ x \in [0, 4]$

g $y = 3x - 2, \ x \in [-1, 2]$ **h** $y = 4 - x^2$

i $\{(x, y) : y \leq 1 - x\}$

Example 5 **3** Which of the following relations are functions? State the domain and range for each.

a $\{(-1, 1), (-1, 2), (1, 2), (3, 4), (2, 3)\}$ **b** $\{(-2, 0), (-1, -1), (0, 3), (1, 5), (2, -4)\}$

c $\{(-1, 1), (-1, 2), (-2, -2), (2, 4), (4, 6)\}$ **d** $\{(-1, 4), (0, 4), (1, 4), (2, 4), (3, 4)\}$

4 Which of the following relations are functions? State the domain and range for each.

a $\{(x, 4) : x \in \mathbb{R}\}$ **b** $\{(2, y) : y \in \mathbb{Z}\}$ **c** $y = -2x + 4$

d $y \geq 3x + 2$ **e** $\{(x, y) : x^2 + y^2 = 16\}$

Example 6 **5** Let $f(x) = 2x^2 + 4x$ and $g(x) = 2x^3 + 2x - 6$.

a Evaluate $f(-1)$, $f(2)$, $f(-3)$ and $f(2a)$.

b Evaluate $g(-1)$, $g(2)$, $g(3)$ and $g(a - 1)$.

Example 7 **6** Consider the function $g(x) = 3x^2 - 2$.

a Find $g(-2)$ and $g(4)$.

b Express the following in terms of x:

i $g(-2x)$ **ii** $g(x - 2)$ **iii** $g(x + 2)$ **iv** $g(x^2)$

Example 8 **7** Consider the function $f(x) = 2x - 3$. Find:

a the image of 3 **b** the pre-image of 11

c $\{x : f(x) = 4x\}$ **d** $\{x : f(x) > x\}$

8 Consider the functions $g(x) = 6x + 7$ and $h(x) = 3x - 2$. Find:

a $\{x : g(x) = h(x)\}$

b $\{x : g(x) > h(x)\}$

c $\{x : h(x) = 0\}$

9 Rewrite each of the following using the $f : X \to Y$ notation:

a $\{(x, y) : y = 2x + 3\}$ **b** $\{(x, y) : 3y + 4x = 12\}$

c $\{(x, y) : y = 2x - 3, \ x \geq 0\}$ **d** $y = x^2 - 9, \ x \in \mathbb{R}$

e $y = 5x - 3, \ 0 \leq x \leq 2$

Example 9 **10** Sketch the graph of each of the following and state the range of each:

a $y = x + 1, \ x \in [2, \infty)$ **b** $y = -x + 1, \ x \in [2, \infty)$

c $y = 2x + 1, \ x \in [-4, \infty)$ **d** $y = 3x + 2, \ x \in (-\infty, 3)$

e $y = x + 1, \ x \in (-\infty, 3]$ **f** $y = 3x - 1, \ x \in [-2, 6]$

g $y = -3x - 1, \ x \in [-5, -1]$ **h** $y = 5x - 1, \ x \in (-2, 4)$

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11 For $f(x) = 2x^2 - 6x + 1$ and $g(x) = 3 - 2x$:

a Evaluate $f(2)$, $f(-3)$ and $f(-2)$. **b** Evaluate $g(-2)$, $g(1)$ and $g(-3)$.

c Express the following in terms of a:

i $f(a)$ **ii** $f(a+2)$ **iii** $g(-a)$ **iv** $g(2a)$

v $f(5-a)$ **vi** $f(2a)$ **vii** $g(a) + f(a)$ **viii** $g(a) - f(a)$

12 For $f(x) = 3x^2 + x - 2$, find:

a $\{ x : f(x) = 0 \}$ **b** $\{ x : f(x) = x \}$ **c** $\{ x : f(x) = -2 \}$

d $\{ x : f(x) > 0 \}$ **e** $\{ x : f(x) > x \}$ **f** $\{ x : f(x) \le -2 \}$

13 For $f(x) = x^2 + x$, find:

a $f(-2)$ **b** $f(2)$

c $f(-a)$ in terms of a **d** $f(a) + f(-a)$ in terms of a

e $f(a) - f(-a)$ in terms of a **f** $f(a^2)$ in terms of a

14 For $g(x) = 3x - 2$, find:

a $\{ x : g(x) = 4 \}$ **b** $\{ x : g(x) > 4 \}$ **c** $\{ x : g(x) = a \}$

d $\{ x : g(-x) = 6 \}$ **e** $\{ x : g(2x) = 4 \}$ **f** $\left\{ x : \dfrac{1}{g(x)} = 6 \right\}$

15 Find the value of k for each of the following if $f(3) = 3$, where:

a $f(x) = kx - 1$ **b** $f(x) = x^2 - k$ **c** $f(x) = x^2 + kx + 1$

d $f(x) = \dfrac{k}{x}$ **e** $f(x) = kx^2$ **f** $f(x) = 1 - kx^2$

16 Find the values of x for which the given functions have the given value:

a $f(x) = 5x - 4$, $f(x) = 2$ **b** $f(x) = \dfrac{1}{x}$, $f(x) = 5$ **c** $f(x) = \dfrac{1}{x^2}$, $f(x) = 9$

d $f(x) = x + \dfrac{1}{x}$, $f(x) = 2$ **e** $f(x) = (x+1)(x-2)$, $f(x) = 0$

1C Types of functions and implied domains

▶ One-to-one functions

A function is said to be **one-to-one** if different x-values map to different y-values. That is, a function f is one-to-one if $a \neq b$ implies $f(a) \neq f(b)$, for all $a, b \in \operatorname{dom} f$.

An equivalent way to say this is that a function f is one-to-one if $f(a) = f(b)$ implies $a = b$, for all $a, b \in \operatorname{dom} f$.

The function $f(x) = 2x + 1$ is one-to-one because

$$\begin{aligned} f(a) = f(b) &\Rightarrow 2a + 1 = 2b + 1 \\ &\Rightarrow 2a = 2b \\ &\Rightarrow a = b \end{aligned}$$

The function $f(x) = x^2$ is not one-to-one as, for example, $f(3) = 9 = f(-3)$.

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Example 10

Which of the following functions are one-to-one?

a $f = \{(2, -3), (4, 7), (6, 6), (8, 10)\}$

b $g = \{(1, 4), (2, 5), (3, 4), (4, 7)\}$

Solution

a The function f is one-to-one as the second coordinates of all of the ordered pairs are different.

b The function g is not one-to-one as the second coordinates of the ordered pairs are not all different: $g(1) = 4 = g(3)$.

The vertical-line test can be used to determine whether a relation is a function or not. Similarly, there is a geometric test that determines whether a function is one-to-one or not.

Horizontal-line test

If a horizontal line can be drawn anywhere on the graph of a function and it only ever intersects the graph a maximum of once, then the function is **one-to-one**.

Example 11

Which of the following functions are one-to-one?

a $y = x^2$

b $y = 2x + 1$

c $f(x) = 5$

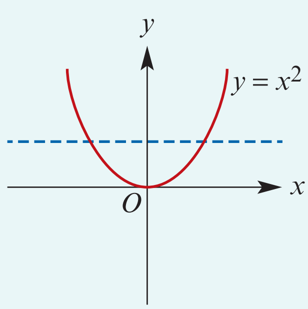
d $y = x^3$

e $y = \sqrt{9 - x^2}$

f $y = \dfrac{1}{x}$

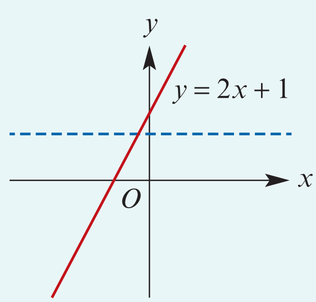
Solution

a



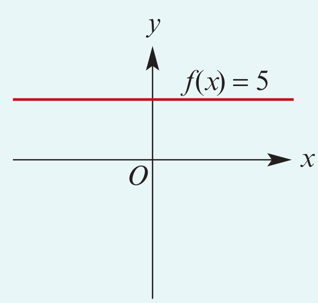
not one-to-one

b



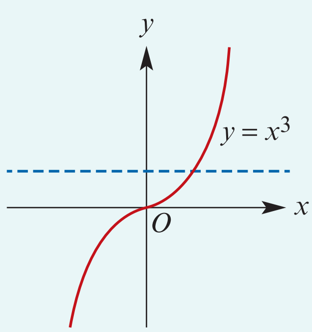
one-to-one

c



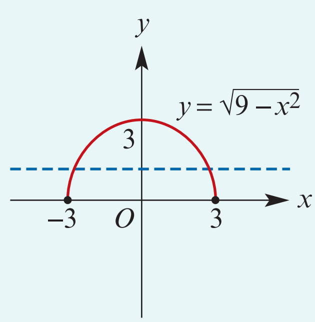
not one-to-one

d



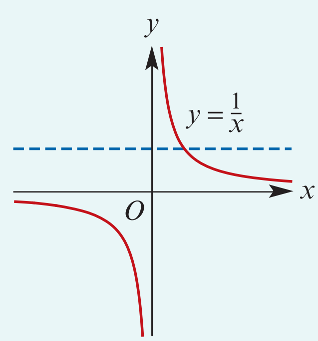
one-to-one

e



not one-to-one

f



one-to-one

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A function that is not one-to-one is **many-to-one**.

Implied domains

If the domain of a function is not specified, then the domain is the largest subset of $\mathbb{R}$ for which the rule is defined; this is called the **implied domain** or the **maximal domain**.

Thus, for the function $f(x) = \sqrt{x}$, the implied domain is $[0, \infty)$. We write:

$$f\colon [0, \infty) \to \mathbb{R},\ f(x) = \sqrt{x}$$

Example 12

Find the implied domain and the corresponding range for the functions with rules:

a $f(x) = 2x - 3$ **b** $f(x) = \dfrac{1}{(x-2)^2}$ **c** $f(x) = \sqrt{x+6}$ **d** $f(x) = \sqrt{4 - x^2}$

Solution

a $f(x) = 2x - 3$ is defined for all x. The implied domain is $\mathbb{R}$. The range is $\mathbb{R}$.

b $f(x) = \dfrac{1}{(x-2)^2}$ is defined for $x \neq 2$. The implied domain is $\mathbb{R} \setminus \{2\}$. The range is $\mathbb{R}^+$.

c $f(x) = \sqrt{x+6}$ is defined for $x + 6 \geq 0$, i.e. for $x \geq -6$.
Thus the implied domain is $[-6, \infty)$. The range is $\mathbb{R}^+ \cup \{0\}$.

d $f(x) = \sqrt{4 - x^2}$ is defined for $4 - x^2 \geq 0$, i.e. for $x^2 \leq 4$.
Thus the implied domain is $[-2, 2]$. The range is $[0, 2]$.

Example 13

Find the implied domain of the functions with the following rules:

a $f(x) = \dfrac{2}{2x-3}$ **b** $g(x) = \sqrt{5 - x}$

c $h(x) = \sqrt{x-5} + \sqrt{8-x}$ **d** $f(x) = \sqrt{x^2 - 7x + 12}$

Solution

a $f(x)$ is defined when $2x - 3 \neq 0$, i.e. when $x \neq \frac{3}{2}$. Thus the implied domain is $\mathbb{R} \setminus \{\frac{3}{2}\}$.

b $g(x)$ is defined when $5 - x \geq 0$, i.e. when $x \leq 5$. Thus the implied domain is $(-\infty, 5]$.

c $h(x)$ is defined when $x - 5 \geq 0$ and $8 - x \geq 0$, i.e. when $x \geq 5$ and $x \leq 8$. Thus the implied domain is $[5, 8]$.

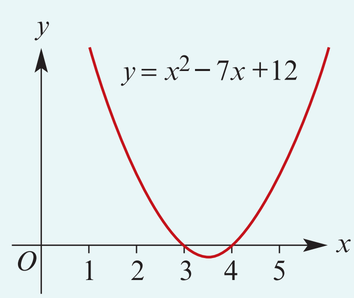
d $f(x)$ is defined when

$$x^2 - 7x + 12 \geq 0$$

which is equivalent to

$$(x-3)(x-4) \geq 0$$

Thus $f(x)$ is defined when $x \geq 4$ or $x \leq 3$.
The implied domain is $(-\infty, 3] \cup [4, \infty)$.



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▶ Piecewise-defined functions

Functions which have different rules for different subsets of their domain are called **piecewise-defined functions**. They are also known as **hybrid functions**.

Example 14

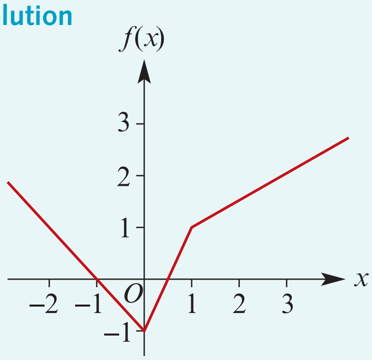
a Sketch the graph of the function f given by:

$$f(x) = \begin{cases} -x-1 & \text{for } x < 0 \\ 2x-1 & \text{for } 0 \le x \le 1 \\ \frac{1}{2}x + \frac{1}{2} & \text{for } x > 1 \end{cases}$$

b State the range of f.

Solution

a



b The range is $[-1, \infty)$.

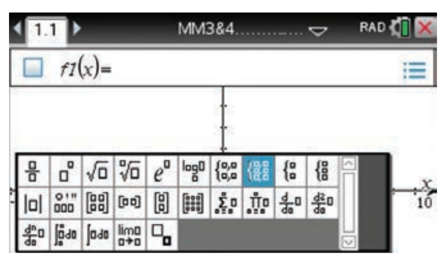
Explanation

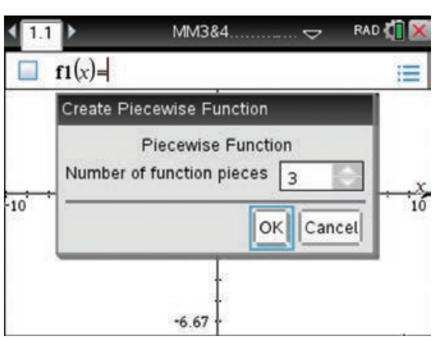
- The graph of $y = -x - 1$ is sketched for $x < 0$. Note that when $x = 0$, $y = -1$ for this rule.
- The graph of $y = 2x - 1$ is sketched for $0 \le x \le 1$. Note that when $x = 0$, $y = -1$ and when $x = 1$, $y = 1$ for this rule.
- The graph of $y = \frac{1}{2}x + \frac{1}{2}$ is sketched for $x > 1$. Note that when $x = 1$, $y = 1$ for this rule.

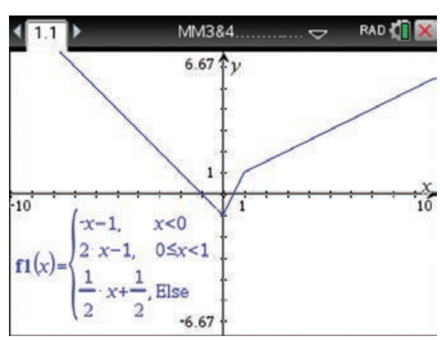
Note: For this function, the sections of the graph 'join up'. This is not always the case.

Using the TI-Nspire

- In a **Graphs** application with the cursor in the entry line, select the piecewise function template as shown. (Access the templates using (template key) or (ctrl) (menu) > **Math Templates**.)
- If the domain of the last function piece is the remaining subset of $\mathbb{R}$, then leave the final condition blank and it will autofill as 'Else' when you press (enter).







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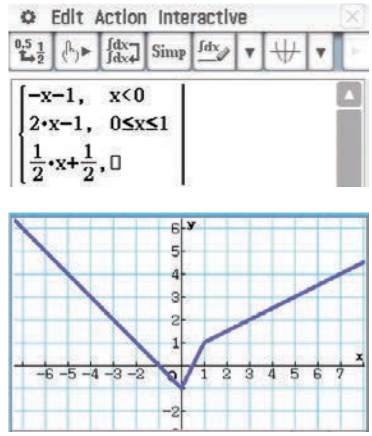
Using the Casio ClassPad

- In Main $\sqrt{\alpha}$, open the keyboard and select the Math3 palette.
- Tap the piecewise template twice.
- Enter the function as shown.

Note: If the domain of the last function piece is the remaining subset of $\mathbb{R}$, then the last domain box can be left empty.



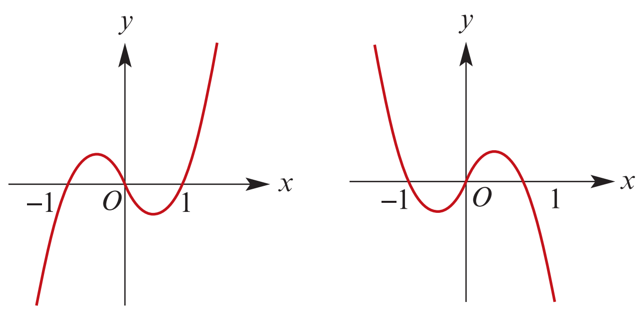
Odd and even functions

Odd functions

An **odd** function has the property that $f(-x) = -f(x)$. The graph of an odd function has rotational symmetry with respect to the origin: the graph remains unchanged after rotation of 180° about the origin.

For example, $f(x) = x^3 - x$ is an odd function, since

$$\begin{aligned} f(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \\ &= -f(x) \end{aligned}$$



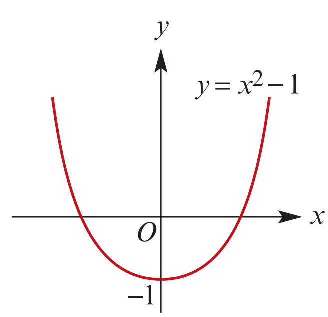
$y = f(x)$ $\quad$ $y = f(-x)$

Even functions

An **even** function has the property that $f(-x) = f(x)$. The graph of an even function is symmetrical about the y-axis.

For example, $f(x) = x^2 - 1$ is an even function, since

$$\begin{aligned} f(-x) &= (-x)^2 - 1 \\ &= x^2 - 1 \\ &= f(x) \end{aligned}$$



The properties of odd and even functions often facilitate the sketching of graphs.

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Example 15

State whether each function is odd or even or neither:

a $f(x) = x^2 + 7$

b $f(x) = x^4 + x^2$

c $f(x) = -2x^3 + 7$

d $f(x) = \dfrac{1}{x}$

e $f(x) = \dfrac{1}{x - 3}$

f $f(x) = x^5 + x^3 + x$

Solution

a $f(-a) = (-a)^2 + 7$
$= a^2 + 7$
$= f(a)$

The function is even.

b $f(-a) = (-a)^4 + (-a)^2$
$= a^4 + a^2$
$= f(a)$

The function is even.

c $f(-1) = -2(-1)^3 + 7 = 9$
but $f(1) = -2 + 7 = 5$
and $-f(1) = -5$

The function is neither even nor odd.

d $f(-a) = \dfrac{1}{-a}$
$= -\dfrac{1}{a}$
$= -f(a)$

The function is odd.

e $f(-1) = -\frac{1}{4}$
but $f(1) = -\frac{1}{2}$
and $-f(1) = \frac{1}{2}$

The function is neither even nor odd.

f $f(-a)$
$= (-a)^5 + (-a)^3 + (-a)$
$= -a^5 - a^3 - a$
$= -f(a)$

The function is odd.

Section summary

- A function f is **one-to-one** if different x-values map to different y-values. Equivalently, a function f is one-to-one if $f(a) = f(b)$ implies $a = b$, for all $a, b \in \text{dom } f$.
- **Horizontal-line test**: If a horizontal line can be drawn anywhere on the graph of a function and it only ever intersects the graph a maximum of once, then the function is one-to-one.
- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning; this is called the **implied domain** or the **maximal domain** of the function.
- Functions which have different rules for different subsets of their domain are called **piecewise-defined functions**.
- A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
- A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f.

Exercise 1C

Skillsheet

Example 10

1 State which of the following functions are one-to-one:

a $\{(2, 3), (3, 4), (5, 4), (4, 6)\}$

b $\{(1, 2), (2, 3), (3, 4), (4, 6)\}$

c $\{(7, -3), (11, 5), (6, 4), (17, -6), (12, -4)\}$

d $\{(-1, -2), (-2, -2), (-3, 4), (-6, 7)\}$

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Example 11 **2** State which of the following functions are one-to-one:

a $\{(x, y) : y = x^2 + 2\}$ **b** $\{(x, y) : y = 2x + 4\}$ **c** $f(x) = 2 - x^2$

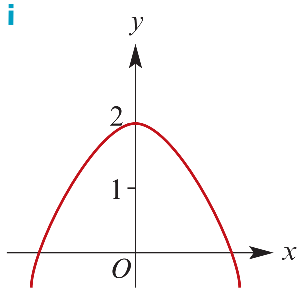
d $y = x^2, \ x \geq 1$ **e** $y = \dfrac{1}{x^2}, \ x \neq 0$ **f** $y = (x - 1)^3$

3 Each of the following is the graph of a relation.

a State which are the graph of a function.

b State which are the graph of a one-to-one function.

i



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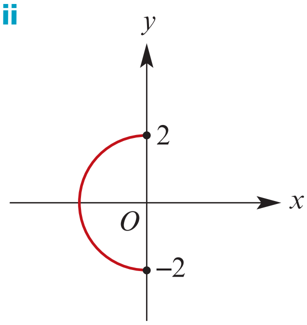

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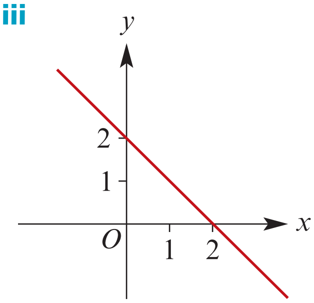

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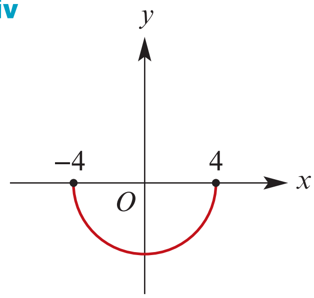

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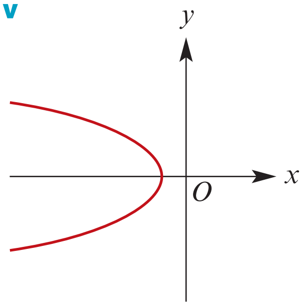

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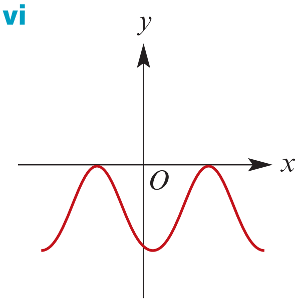

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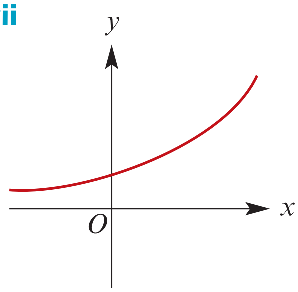

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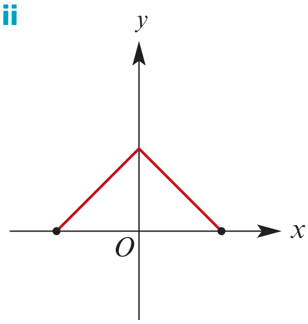
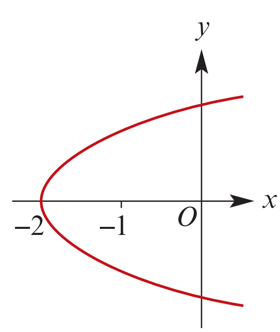


4 The graph of the relation $\{(x, y) : y^2 = x + 2, \ x \geq -2\}$ is shown on the right. From this relation, form two functions and specify the range of each.



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5 **a** Draw the graph of $g: \mathbb{R} \to \mathbb{R}, g(x) = x^2 + 2$.

b By restricting the domain of g, form two one-to-one functions that have the same rule as g.

Example 12

6 State the largest possible domain and range for the functions defined by each of the following rules:

a $y = 4 - x$ **b** $y = \sqrt{x}$ **c** $y = x^2 - 2$ **d** $y = \sqrt{16 - x^2}$

e $y = \dfrac{1}{x}$ **f** $y = 4 - 3x^2$ **g** $y = \sqrt{x - 3}$

7 Each of the following is the rule of a function. In each case, write down the implied domain and the range.

a $y = 3x + 2$ **b** $y = x^2 - 2$

c $f(x) = \sqrt{9 - x^2}$ **d** $g(x) = \dfrac{1}{x - 1}$

Example 13

8 Find the implied domain for each of the following rules:

a $f(x) = \dfrac{1}{x - 3}$ **b** $f(x) = \sqrt{x^2 - 3}$

c $g(x) = \sqrt{x^2 + 3}$ **d** $h(x) = \sqrt{x - 4} + \sqrt{11 - x}$

e $f(x) = \dfrac{x^2 - 1}{x + 1}$ **f** $h(x) = \sqrt{x^2 - x - 2}$

g $f(x) = \dfrac{1}{(x + 1)(x - 2)}$ **h** $h(x) = \sqrt{\dfrac{x - 1}{x + 2}}$

i $f(x) = \sqrt{x - 3x^2}$ **j** $h(x) = \sqrt{25 - x^2}$

k $f(x) = \sqrt{x - 3} + \sqrt{12 - x}$

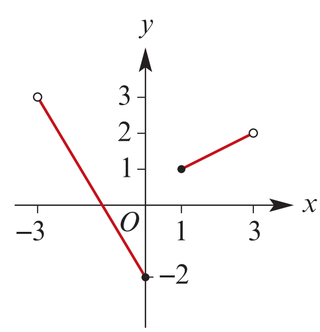
Example 14

9 **a** Sketch the graph of the function

$$f(x) = \begin{cases} -2x - 2, & x < 0 \\ x - 2, & 0 \le x < 2 \\ 3x - 6, & x \ge 2 \end{cases}$$

b What is the range of f?

10 State the domain and range of the function for which the graph is shown.



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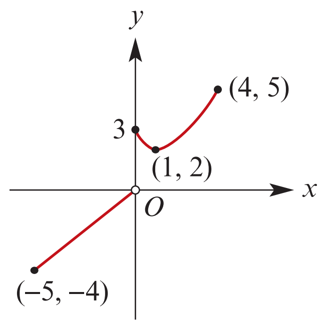
11 State the domain and range of the function for which the graph is shown.



12 **a** Sketch the graph of the function with rule

$$f(x) = \begin{cases} 2x + 6, & 0 < x \le 2 \\ -x + 5, & -4 \le x \le 0 \\ -4, & x < -4 \end{cases}$$

b State the domain and range of the function.

13 **a** Sketch the graph of the function with rule

$$g(x) = \begin{cases} x^2 + 5, & x > 0 \\ 5 - x, & -3 \le x \le 0 \\ 8, & x < -3 \end{cases}$$

b State the range of the function.

14 Given that

$$f(x) = \begin{cases} \dfrac{1}{x}, & x > 3 \\ 2x, & x \le 3 \end{cases}$$

find:

a $f(-4)$ **b** $f(0)$ **c** $f(4)$
d $f(a+3)$ in terms of a **e** $f(2a)$ in terms of a **f** $f(a-3)$ in terms of a

15 Given that

$$f(x) = \begin{cases} \sqrt{x - 1}, & x \ge 1 \\ 4, & x < 1 \end{cases}$$

find:

a $f(0)$ **b** $f(3)$ **c** $f(8)$
d $f(a+1)$ in terms of a **e** $f(a-1)$ in terms of a

16 Sketch the graph of the function

$$g(x) = \begin{cases} -x - 2, & x < -1 \\ \dfrac{x - 1}{2}, & -1 \le x < 1 \\ 3x - 3, & x \ge 1 \end{cases}$$

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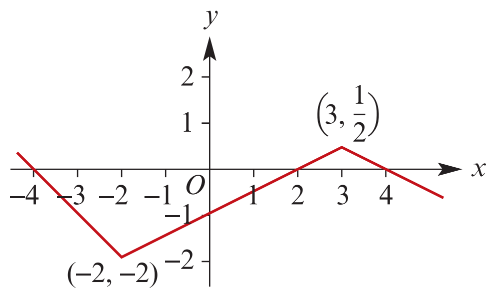
17 Specify the function illustrated by the graph.



Example 15 **18** State whether each of the following functions is odd, even or neither:

a $f(x) = x^4$ **b** $f(x) = x^5$ **c** $f(x) = x^4 - 3x$

d $f(x) = x^4 - 3x^2$ **e** $f(x) = x^5 - 2x^3$ **f** $f(x) = x^4 - 2x^5$

19 State whether each of the following functions is odd, even or neither:

a $f(x) = x^2 - 4$ **b** $f(x) = 2x^4 - x^2$ **c** $f(x) = -4x^3 + 7x$

d $f(x) = \dfrac{1}{2x}$ **e** $f(x) = \dfrac{1}{x+5}$ **f** $f(x) = 3 + 2x^2$

g $f(x) = x^2 - 5x$ **h** $f(x) = 3^x$ **i** $f(x) = x^4 + x^2 + 2$

1D Sums and products of functions

The domain of f is denoted by dom f and the domain of g by dom g. Let f and g be functions such that dom $f \cap$ dom $g \neq \emptyset$. The **sum**, $f + g$, and the **product**, fg, as functions on dom $f \cap$ dom g are defined by

$$(f + g)(x) = f(x) + g(x) \qquad \text{and} \qquad (fg)(x) = f(x)\,g(x)$$

The domain of both $f + g$ and fg is the intersection of the domains of f and g, i.e. the values of x for which both f and g are defined.

Example 16

If $f(x) = \sqrt{x-2}$ for all $x \geq 2$ and $g(x) = \sqrt{4-x}$ for all $x \leq 4$, find:

a $f + g$ **b** $(f + g)(3)$ **c** fg **d** $(fg)(3)$

Solution

Note that dom $f \cap$ dom g $= [2, 4]$.

a
$$(f + g)(x) = f(x) + g(x) = \sqrt{x-2} + \sqrt{4-x}$$

dom$(f + g) = [2, 4]$

b
$$(f + g)(3) = \sqrt{3-2} + \sqrt{4-3} = 2$$

c
$$(fg)(x) = f(x)\,g(x) = \sqrt{(x-2)(4-x)}$$

dom$(fg) = [2, 4]$

d
$$(fg)(3) = \sqrt{(3-2)(4-3)} = 1$$

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Addition of ordinates

We have seen that, for two functions f and g, a new function $f + g$ can be defined by

$$(f+g)(x) = f(x) + g(x)$$
$$\text{dom}(f+g) = \text{dom}\, f \cap \text{dom}\, g$$

We now look at how to graph the new function $f + g$. This is a useful graphing technique and can be combined with other techniques such as finding axis intercepts, stationary points and asymptotes.

Example 17

Sketch the graphs of $f(x) = x + 1$ and $g(x) = 3 - 2x$ and hence the graph of $(f + g)(x)$.

Solution

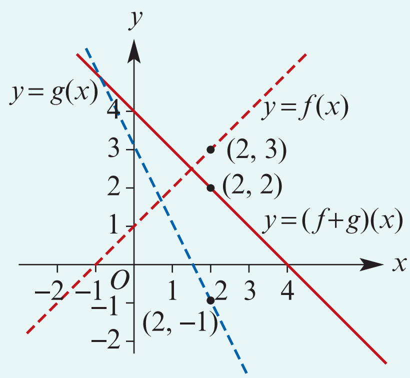
For $f(x) = x + 1$ and $g(x) = 3 - 2x$, we have

$$\begin{aligned}(f+g)(x) &= f(x) + g(x)\\ &= (x+1) + (3-2x)\\ &= 4 - x\end{aligned}$$

For example:

$$\begin{aligned}(f+g)(2) &= f(2) + g(2)\\ &= 3 + (-1) = 2\end{aligned}$$

i.e. the ordinates are added.



Now check that the same principle applies for other points on the graphs. A table of values can be a useful aid to find points that lie on the graph of $y = (f + g)(x)$.

The table shows that $(-1, 5)$, $(0, 4)$, $(\frac{3}{2}, \frac{5}{2})$ and $(2, 2)$ lie on the graph of $y = (f + g)(x)$.

x	$f(x)$	$g(x)$	$(f+g)(x)$
-1	0	5	5
0	1	3	4
$\frac{3}{2}$	$\frac{5}{2}$	0	$\frac{5}{2}$
2	3	-1	2

Example 18

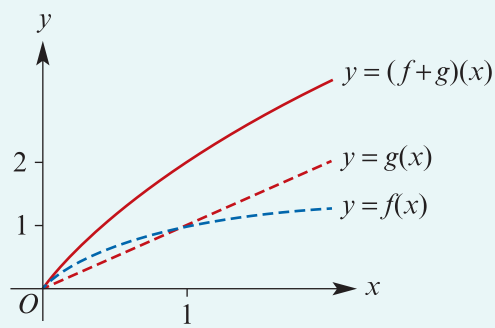
Sketch the graph of $y = (f + g)(x)$, where $f(x) = \sqrt{x}$ and $g(x) = x$.

Solution

The function with rule

$$(f+g)(x) = \sqrt{x} + x$$

is defined by the addition of the two functions f and g.



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Example 19

Sketch the graph of $y = (f - g)(x)$, where $f(x) = x^2$ and $g(x) = \sqrt{x}$.

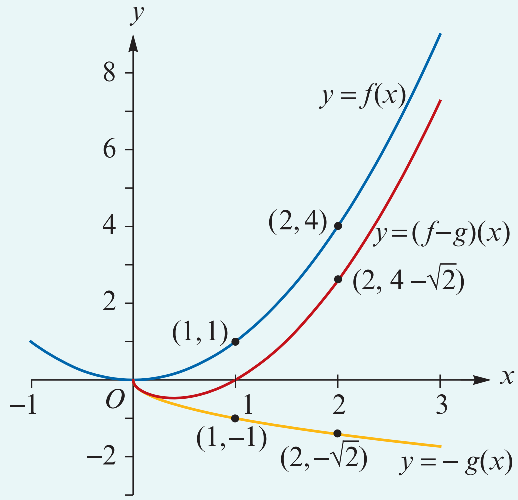
Solution

The function with rule

$$(f - g)(x) = x^2 - \sqrt{x}$$

is defined by the addition of the two functions f and $-g$.

The implied domain of $f - g$ is $[0, \infty)$.



Section summary

- Sum of functions $(f + g)(x) = f(x) + g(x)$, where $\operatorname{dom}(f + g) = \operatorname{dom} f \cap \operatorname{dom} g$
- Difference of functions $(f - g)(x) = f(x) - g(x)$, where $\operatorname{dom}(f - g) = \operatorname{dom} f \cap \operatorname{dom} g$
- Product of functions $(f \cdot g)(x) = f(x) \cdot g(x)$, where $\operatorname{dom}(f \cdot g) = \operatorname{dom} f \cap \operatorname{dom} g$
- Addition of ordinates This technique can be used to help sketch the graph of the sum of two functions. Key points to consider when sketching $y = (f + g)(x)$:
  - When $f(x) = 0$, $(f + g)(x) = g(x)$.
  - When $g(x) = 0$, $(f + g)(x) = f(x)$.
  - If $f(x)$ and $g(x)$ are positive, then $(f + g)(x) > g(x)$ and $(f + g)(x) > f(x)$.
  - If $f(x)$ and $g(x)$ are negative, then $(f + g)(x) < g(x)$ and $(f + g)(x) < f(x)$.
  - If $f(x)$ is positive and $g(x)$ is negative, then $g(x) < (f + g)(x) < f(x)$.
  - Look for values of x for which $f(x) + g(x) = 0$.

Exercise 1D

Example 16

1 For each of the following, find $(f + g)(x)$ and $(fg)(x)$ and state the domain for both $f + g$ and fg:

a $f(x) = 3x$ and $g(x) = x + 2$

b $f(x) = 1 - x^2$ for all $x \in [-2, 2]$ and $g(x) = x^2$ for all $x \in \mathbb{R}^+$

c $f(x) = \sqrt{x}$ and $g(x) = \dfrac{1}{\sqrt{x}}$ for $x \in [1, \infty)$

d $f(x) = x^2$, $x \geq 0$ and $g(x) = \sqrt{4 - x}$, $0 \leq x \leq 4$

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2 Functions f, g, h and k are defined by:

i $f(x) = x^2 + 1, \ x \in \mathbb{R}$ **ii** $g(x) = x, \ x \in \mathbb{R}$

iii $h(x) = \dfrac{1}{x^2}, \ x \neq 0$ **iv** $k(x) = \dfrac{1}{x}, \ x \neq 0$

a State which of the above functions are odd and which are even.

b Give the rules for the functions $f + h$, fh, $g + k$, gk, $f + g$ and fg, stating which are odd and which are even.

Example 17 **3** Sketch the graphs of $f(x) = x + 2$ and $g(x) = 4 - 3x$ and hence the graph of $(f + g)(x)$.

Example 18 **4** Sketch the graph of $f\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $f(x) = \sqrt{x} + 2x$ using addition of ordinates.

5 Sketch the graph of $f\colon [-2, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x + 2} + x$ using addition of ordinates.

6 Sketch the graph of $f\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $f(x) = -\sqrt{x} + x$ using addition of ordinates.

7 Sketch the graph of $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x} + \dfrac{1}{x^2}$ using addition of ordinates.

8 For each of the following, sketch the graph of $f + g$:

a $f\colon [-2, \infty) \to \mathbb{R}$, $f(x) = \sqrt{2 + x}$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = -2x$

b $f\colon (-\infty, 2] \to \mathbb{R}$, $f(x) = \sqrt{2 - x}$ and $g\colon [-2, \infty) \to \mathbb{R}$, $g(x) = \sqrt{x + 2}$

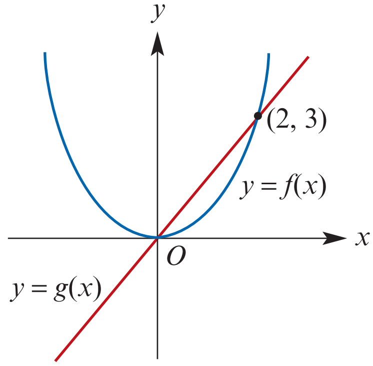
Example 19 **9** Sketch the graph of $y = (f - g)(x)$, where $f(x) = x^3$ and $g(x) = \sqrt{x}$.

10 Sketch the graph of $y = (f - g)(x)$, where $f(x) = 2x^2$ and $g(x) = 3\sqrt{x}$.

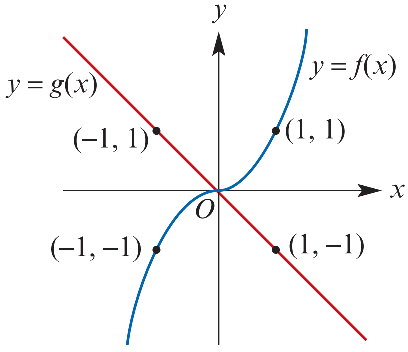
11 Sketch the graph of $f(x) = x^2$ and $g(x) = 3x + 2$ on the one set of axes and hence, using addition of ordinates, sketch the graph of $y = x^2 + 3x + 2$.

12 Copy and add the graph of $y = (f + g)(x)$ using addition of ordinates:

a



b



13 For each of the following, sketch the graph of $f + g$:

a $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 3$

b $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 2x$ and $g\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $g(x) = \sqrt{x}$

c $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = -x^2$ and $g\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $g(x) = \sqrt{x}$

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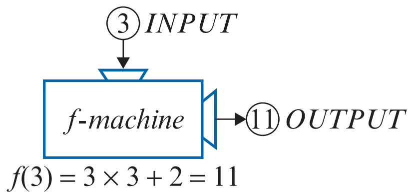
1E Composite functions

A function may be considered to be similar to a machine for which the input (domain) is processed to produce an output (range). For example, the diagram on the right represents an 'f-machine' where $f(x) = 3x + 2$.



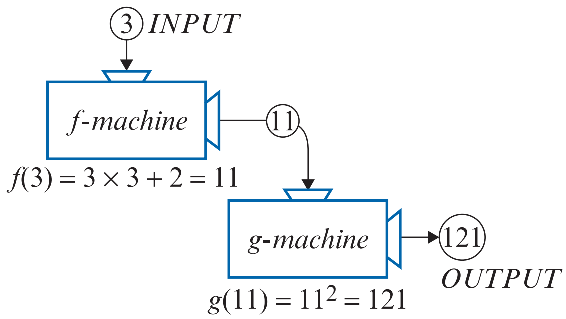
With many processes, more than one machine operation is required to produce an output.

Suppose an output is the result of one function being applied after another.

For example: $f(x) = 3x + 2$

followed by $g(x) = x^2$

This is illustrated on the right.



A new function h is formed. The rule for h is $h(x) = (3x + 2)^2$.

The diagram shows $f(3) = 11$ and then $g(11) = 121$. This may be written:

$$h(3) = g(f(3)) = g(11) = 121$$

The new function h is said to be the **composition** of g with f. This is written $h = g \circ f$ (read 'composition of f followed by g') and the rule for h is given by $h(x) = g(f(x))$.

In the example we have considered:

$$\begin{aligned} h(x) &= g(f(x)) \\ &= g(3x + 2) \\ &= (3x + 2)^2 \end{aligned}$$

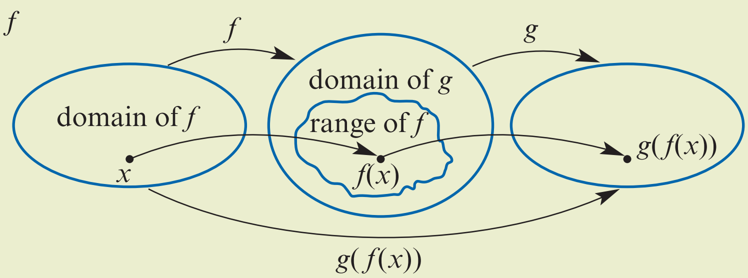
In general, for functions f and g such that

$$\operatorname{ran} f \subseteq \operatorname{dom} g$$

we define the **composite function** of g with f by

$$g \circ f(x) = g(f(x))$$

$$\operatorname{dom}(g \circ f) = \operatorname{dom} f$$



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Example 20

Find both $f \circ g$ and $g \circ f$, stating the domain and range of each, where:

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = 2x - 1 \quad \text{and} \quad g\colon \mathbb{R} \to \mathbb{R},\ g(x) = 3x^2$$

Solution

To determine the existence of a composite function, it is useful to form a table of domains and ranges.

We see that $f \circ g$ is defined since $\operatorname{ran} g \subseteq \operatorname{dom} f$, and that $g \circ f$ is defined since $\operatorname{ran} f \subseteq \operatorname{dom} g$.

	Domain	Range
g	$\mathbb{R}$	$\mathbb{R}^+ \cup \{0\}$
f	$\mathbb{R}$	$\mathbb{R}$

$$\begin{aligned} f \circ g(x) &= f(g(x)) \\ &= f(3x^2) \\ &= 2(3x^2) - 1 \\ &= 6x^2 - 1 \end{aligned}$$

$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(2x - 1) \\ &= 3(2x - 1)^2 \\ &= 12x^2 - 12x + 3 \end{aligned}$$

$\operatorname{dom}(f \circ g) = \operatorname{dom} g = \mathbb{R}$

$\operatorname{ran}(f \circ g) = [-1, \infty)$

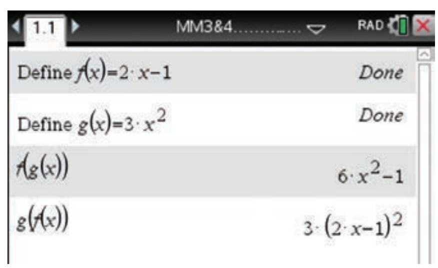
$\operatorname{dom}(g \circ f) = \operatorname{dom} f = \mathbb{R}$

$\operatorname{ran}(g \circ f) = [0, \infty)$

Note: It can be seen from this example that in general $f \circ g \neq g \circ f$.

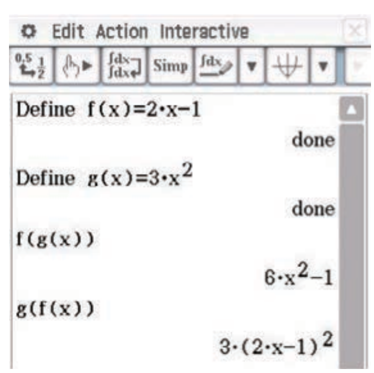
Using the TI-Nspire

- Define $f(x) = 2x - 1$ and $g(x) = 3x^2$.
- The rules for $f \circ g$ and $g \circ f$ can now be found using $f(g(x))$ and $g(f(x))$.



Using the Casio ClassPad

- Define $f(x) = 2x - 1$ and $g(x) = 3x^2$.
- The rules for $f \circ g$ and $g \circ f$ can now be found using $f(g(x))$ and $g(f(x))$.



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Example 21

For the functions $g(x) = 2x - 1$, $x \in \mathbb{R}$, and $f(x) = \sqrt{x}$, $x \geq 0$:

a State which of $f \circ g$ and $g \circ f$ is defined.

b For the composite function that is defined, state the domain and rule.

Solution

a Range of $f \subseteq$ domain of g

Range of $g \nsubseteq$ domain of f

Thus $g \circ f$ is defined, but $f \circ g$ is not defined.

	Domain	Range
g	$\mathbb{R}$	$\mathbb{R}$
f	$\mathbb{R}^+ \cup \{0\}$	$\mathbb{R}^+ \cup \{0\}$

b
$$\begin{aligned} g \circ f(x) &= g(f(x)) \\ &= g(\sqrt{x}) \\ &= 2\sqrt{x} - 1 \end{aligned}$$

$\text{dom}(g \circ f) = \text{dom}\, f = \mathbb{R}^+ \cup \{0\}$

Example 22

For the functions $f(x) = x^2 - 1$, $x \in \mathbb{R}$, and $g(x) = \sqrt{x}$, $x \geq 0$:

a State why $g \circ f$ is not defined.

b Define a restriction f^* of f such that $g \circ f^*$ is defined, and find $g \circ f^*$.

Solution

a Range of $f \nsubseteq$ domain of g

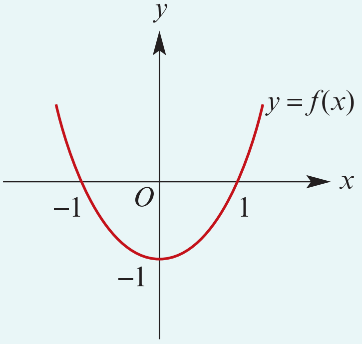
Thus $g \circ f$ is not defined.

	Domain	Range
f	$\mathbb{R}$	$[-1, \infty)$
g	$\mathbb{R}^+ \cup \{0\}$	$\mathbb{R}^+ \cup \{0\}$

b For $g \circ f^*$ to be defined, we need range of $f^* \subseteq$ domain of g, i.e. range of $f^* \subseteq \mathbb{R}^+ \cup \{0\}$.

For the range of f^* to be a subset of $\mathbb{R}^+ \cup \{0\}$, the domain of f must be restricted to a subset of

$$\{\, x : x \leq -1 \,\} \cup \{\, x : x \geq 1 \,\} = \mathbb{R} \setminus (-1, 1)$$



So we define f^* by

$$f^* : \mathbb{R} \setminus (-1, 1) \to \mathbb{R},\ f^*(x) = x^2 - 1$$

Then
$$\begin{aligned} g \circ f^*(x) &= g(f^*(x)) \\ &= g(x^2 - 1) \\ &= \sqrt{x^2 - 1} \end{aligned}$$

$$\text{dom}(g \circ f^*) = \text{dom}\, f^* = \mathbb{R} \setminus (-1, 1)$$

The composite function is $g \circ f^* : \mathbb{R} \setminus (-1, 1) \to \mathbb{R}$, $g \circ f^*(x) = \sqrt{x^2 - 1}$

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Section summary

- If range of $f \subseteq$ domain of g, the composition $g \circ f$ is defined and

$$g \circ f(x) = g(f(x)) \quad \text{with } \operatorname{dom}(g \circ f) = \operatorname{dom} f$$

- If range of $g \subseteq$ domain of f, the composition $f \circ g$ is defined and

$$f \circ g(x) = f(g(x)) \quad \text{with } \operatorname{dom}(f \circ g) = \operatorname{dom} g$$

- In general, $f \circ g \neq g \circ f$.

Exercise 1E

Skillsheet

Example 20

1 For each of the following, find $f(g(x))$ and $g(f(x))$:

a $f(x) = 2x - 1,\ g(x) = 2x$ **b** $f(x) = 4x + 1,\ g(x) = 2x + 1$

c $f(x) = 2x - 1,\ g(x) = 2x - 3$ **d** $f(x) = 2x - 1,\ g(x) = x^2$

e $f(x) = 2x^2 + 1,\ g(x) = x - 5$ **f** $f(x) = 2x + 1,\ g(x) = x^2$

2 For the functions $f(x) = 2x - 1$ and $h(x) = 3x + 2$, find:

a $f \circ h(x)$ **b** $h(f(x))$ **c** $f \circ h(2)$ **d** $h \circ f(2)$

e $f(h(3))$ **f** $h(f(-1))$ **g** $f \circ h(0)$

3 For the functions $f(x) = x^2 + 2x$ and $h(x) = 3x + 1$, find:

a $f \circ h(x)$ **b** $h \circ f(x)$ **c** $f \circ h(3)$

d $h \circ f(3)$ **e** $f \circ h(0)$ **f** $h \circ f(0)$

4 For the functions $h: \mathbb{R} \setminus \{0\} \to \mathbb{R},\ h(x) = \dfrac{1}{x^2}$ and $g: \mathbb{R}^+ \to \mathbb{R},\ g(x) = 3x + 2$, find:

a $h \circ g$ (state rule and domain) **b** $g \circ h$ (state rule and domain)

c $h \circ g(1)$ **d** $g \circ h(1)$

Example 21

5 Consider the functions $f: \mathbb{R} \to \mathbb{R},\ f(x) = x^2 - 4$ and $g: \mathbb{R}^+ \cup \{0\} \to \mathbb{R},\ g(x) = \sqrt{x}$.

a State the ranges of f and g. **b** Find $f \circ g$, stating its range.

c Explain why $g \circ f$ does not exist.

6 Let f and g be functions given by

$$f: \mathbb{R} \setminus \{0\} \to \mathbb{R},\ f(x) = \frac{1}{2}\left(\frac{1}{x} + 1\right) \qquad g: \mathbb{R} \setminus \{\tfrac{1}{2}\} \to \mathbb{R},\ g(x) = \frac{1}{2x - 1}$$

Find:

a $f \circ g$ **b** $g \circ f$

and state the range in each case.

7 The functions f and g are defined by $f: \mathbb{R} \to \mathbb{R},\ f(x) = x^2 - 2$ and $g: [0, \infty) \to \mathbb{R}$, $g(x) = \sqrt{x}$.

a Explain why $g \circ f$ does not exist. **b** Find $f \circ g$ and sketch its graph.

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Example 22 **8** $f: (-\infty, 3] \to \mathbb{R}$, $f(x) = 3 - x$ and $g: \mathbb{R} \to \mathbb{R}$, $g(x) = x^2 - 1$

a Show that $f \circ g$ is not defined.

b Define a restriction g^* of g such that $f \circ g^*$ is defined and find $f \circ g^*$.

9 $f: \mathbb{R}^+ \to \mathbb{R}$, $f(x) = x^{-\frac{1}{2}}$ and $g: \mathbb{R} \to \mathbb{R}$, $g(x) = 3 - x$

a Show that $f \circ g$ is not defined.

b By suitably restricting the domain of g, obtain a function g_1 such that $f \circ g_1$ is defined.

10 Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ and let $g: (-\infty, 3] \to \mathbb{R}$, $g(x) = \sqrt{3 - x}$. State with reasons whether:

a $f \circ g$ exists **b** $g \circ f$ exists.

11 Let $f: S \to \mathbb{R}$, $f(x) = \sqrt{4 - x^2}$, where S is the set of all real values of x for which $f(x)$ is defined. Let $g: \mathbb{R} \to \mathbb{R}$, where $g(x) = x^2 + 1$.

a Find S.

b Find the range of f and the range of g.

c State whether or not $f \circ g$ and $g \circ f$ are defined and give a reason for each assertion.

12 Let a be a positive number, let $f: [2, \infty) \to \mathbb{R}$, $f(x) = a - x$ and let $g: (-\infty, 1] \to \mathbb{R}$, $g(x) = x^2 + a$. Find all values of a for which both $f \circ g$ and $g \circ f$ exist.

1F Inverse functions

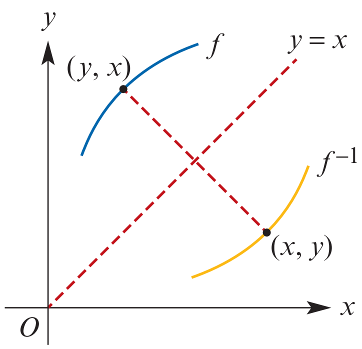
If f is a one-to-one function, then for each number y in the range of f there is exactly one number x in the domain of f such that $f(x) = y$.

Thus if f is a one-to-one function, a new function f^{-1}, called the **inverse** of f, may be defined by:

$$f^{-1}(x) = y \text{ if } f(y) = x, \quad \text{for } x \in \operatorname{ran} f \text{ and } y \in \operatorname{dom} f$$

Note: The function f^{-1} is also a one-to one function, and f is the inverse of f^{-1}.

It is not difficult to see what the relation between f and f^{-1} means geometrically. The point (x, y) is on the graph of f^{-1} if the point (y, x) is on the graph of f. Therefore to get the graph of f^{-1} from the graph of f, the graph of f is to be reflected in the line $y = x$.



From this the following is evident:

$$\operatorname{dom} f^{-1} = \operatorname{ran} f$$
$$\operatorname{ran} f^{-1} = \operatorname{dom} f$$

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A function has an inverse function if and only if it is one-to-one. Using the notation for composition we can write:

$$f \circ f^{-1}(x) = x, \quad \text{for all } x \in \operatorname{dom} f^{-1}$$
$$f^{-1} \circ f(x) = x, \quad \text{for all } x \in \operatorname{dom} f$$

Example 23

Find the inverse function f^{-1} of the function $f(x) = 2x - 3$.

Solution

Method 1

The graph of f has equation $y = 2x - 3$ and the graph of f^{-1} has equation $x = 2y - 3$, that is, x and y are interchanged.

Solve for y:

$$\begin{aligned} x &= 2y - 3 \\ x + 3 &= 2y \\ \therefore \quad y &= \tfrac{1}{2}(x + 3) \end{aligned}$$

Thus $f^{-1}(x) = \frac{1}{2}(x + 3)$ and $\operatorname{dom} f^{-1} = \operatorname{ran} f = \mathbb{R}$.

Method 2

We require f^{-1} such that

$$\begin{aligned} f(f^{-1}(x)) &= x \\ 2f^{-1}(x) - 3 &= x \\ \therefore \quad f^{-1}(x) &= \tfrac{1}{2}(x + 3) \end{aligned}$$

Thus $f^{-1}(x) = \frac{1}{2}(x + 3)$ and $\operatorname{dom} f^{-1} = \operatorname{ran} f = \mathbb{R}$.

Example 24

Find the inverse of each of the following functions, stating the domain and range for each:

a $f\colon [-2, 1] \to \mathbb{R}, f(x) = 2x + 3$

b $g(x) = \dfrac{1}{5 - x}, \ x > 5$

c $h(x) = x^2 - 2, \ x \geq 1$

Solution

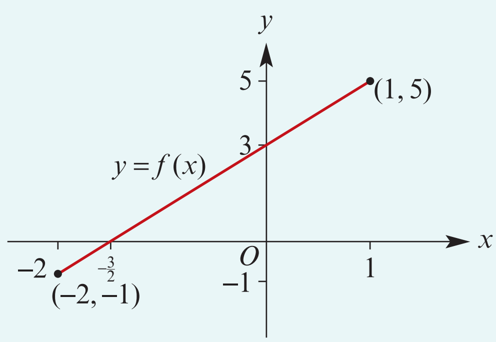
a $f\colon [-2, 1] \to \mathbb{R}, f(x) = 2x + 3$

$\operatorname{ran} f^{-1} = \operatorname{dom} f = [-2, 1]$
$\operatorname{dom} f^{-1} = \operatorname{ran} f = [-1, 5]$

Let $y = 2x + 3$. Interchange x and y:

$$\begin{aligned} x &= 2y + 3 \\ x - 3 &= 2y \\ y &= \frac{x - 3}{2} \end{aligned}$$

$$\therefore f^{-1}\colon [-1, 5] \to \mathbb{R}, f^{-1}(x) = \frac{x - 3}{2}$$



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b $g(x) = \dfrac{1}{5-x}, \quad x > 5$

ran g^{-1} = dom $g = (5, \infty)$
dom g^{-1} = ran $g = (-\infty, 0)$

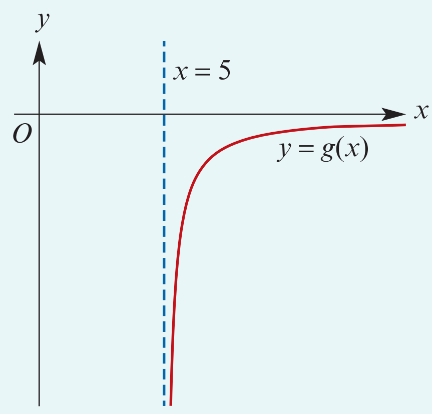
Let $y = \dfrac{1}{5-x}$. Interchange x and y:

$$x = \frac{1}{5-y}$$

$$5 - y = \frac{1}{x}$$

$$y = 5 - \frac{1}{x}$$

$\therefore\ g^{-1}\colon (-\infty, 0) \to \mathbb{R},\ g^{-1}(x) = 5 - \dfrac{1}{x}$



c $h(x) = x^2 - 2, \quad x \ge 1$

ran h^{-1} = dom $h = [1, \infty)$
dom h^{-1} = ran $h = [-1, \infty)$

Let $y = x^2 - 2$. Interchange x and y:

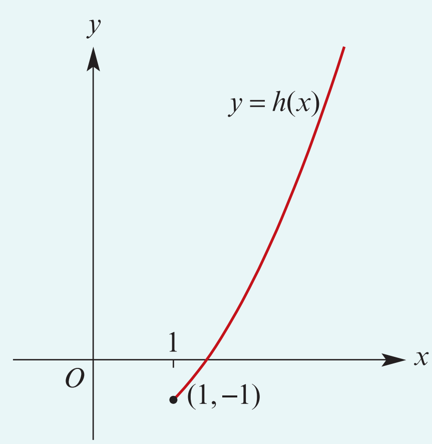
$$x = y^2 - 2$$

$$y^2 = x + 2$$

$$y = \pm\sqrt{x+2}$$

$\therefore\ h^{-1}\colon [-1, \infty) \to \mathbb{R},\ h^{-1}(x) = \sqrt{x+2}$

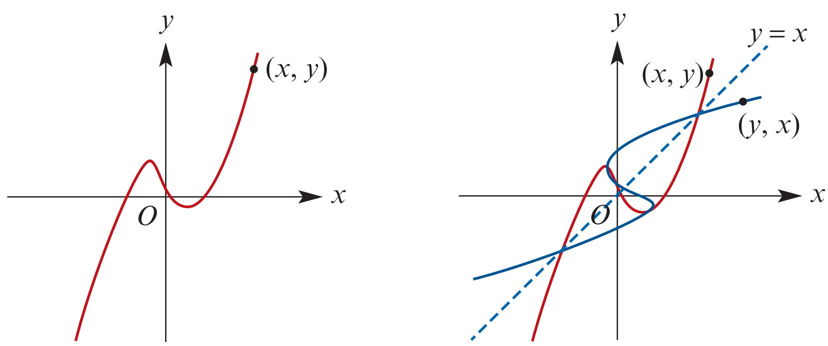
The positive square root is taken because of the known range.



▶ Graphing inverse functions

The transformation which reflects each point in the plane in the line $y = x$ can be described as 'interchanging the x- and y-coordinates of each point in the plane' and can be written as $(x, y) \to (y, x)$. This is read as 'the ordered pair (x, y) is mapped to the ordered pair (y, x)'.

Reflecting the graph of a function in the line $y = x$ produces the graph of its **inverse relation**. Note that the image in the graph below is not a function.



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If the function is one-to-one, then the image is the graph of a function. (This is because, if the function satisfies the horizontal-line test, then its reflection will satisfy the vertical-line test.)

Example 25

Find the inverse of the function $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \dfrac{1}{x} + 3$ and sketch both functions on one set of axes, showing the points of intersection of the graphs.

Solution

We use method 2.

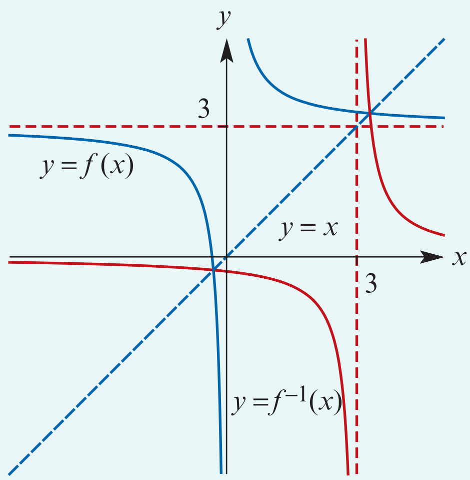
Let $x \in \operatorname{dom} f^{-1} = \operatorname{ran} f$. Then

$$\begin{aligned} f(f^{-1}(x)) &= x \\ \frac{1}{f^{-1}(x)} + 3 &= x \\ \frac{1}{f^{-1}(x)} &= x - 3 \\ \therefore \quad f^{-1}(x) &= \frac{1}{x-3} \end{aligned}$$

The inverse function is

$$f^{-1}\colon \mathbb{R} \setminus \{3\} \to \mathbb{R}, \quad f^{-1}(x) = \frac{1}{x-3}$$

The graphs of f and f^{-1} are shown opposite.



The two graphs intersect when

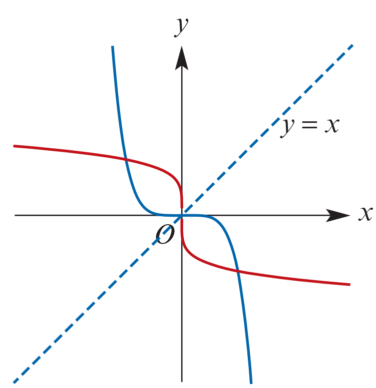
$$\begin{aligned} f(x) &= f^{-1}(x) \\ \frac{1}{x} + 3 &= \frac{1}{x-3} \\ 3x^2 - 9x - 3 &= 0 \\ x^2 - 3x - 1 &= 0 \end{aligned}$$

$$\therefore \quad x = \tfrac{1}{2}(3 - \sqrt{13}) \text{ or } x = \tfrac{1}{2}(3 + \sqrt{13})$$

The points of intersection are

$$\left(\tfrac{1}{2}(3 - \sqrt{13}), \tfrac{1}{2}(3 - \sqrt{13})\right) \quad \text{and} \quad \left(\tfrac{1}{2}(3 + \sqrt{13}), \tfrac{1}{2}(3 + \sqrt{13})\right)$$

Note: In this example, the points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ can also be found by solving either $f(x) = x$ or $f^{-1}(x) = x$, rather than the more complicated equation $f(x) = f^{-1}(x)$.
However, there can be points of intersection of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ that *do not* lie on the line $y = x$, as shown in the diagram opposite.



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Example 26

Find the inverse of the function with rule $f(x) = 3\sqrt{x+2} + 4$ and sketch both functions on one set of axes.

Solution

Consider $x = 3\sqrt{y+2} + 4$ and solve for y:

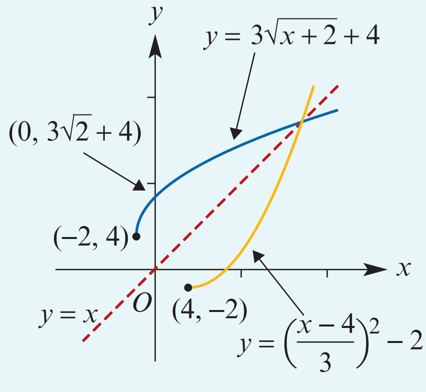
$$\frac{x-4}{3} = \sqrt{y+2}$$

$$y = \left(\frac{x-4}{3}\right)^2 - 2$$

$$\therefore \quad f^{-1}(x) = \left(\frac{x-4}{3}\right)^2 - 2$$

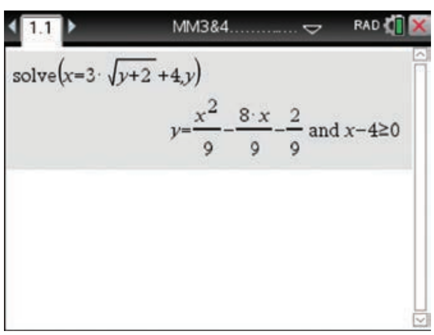
The domain of f^{-1} equals the range of f. Thus

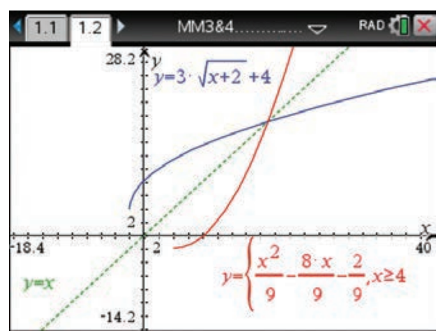
$$f^{-1}\colon [4, \infty) \to \mathbb{R},\ f^{-1}(x) = \left(\frac{x-4}{3}\right)^2 - 2$$



Using the TI-Nspire

- First find the rule for the inverse of $y = 3\sqrt{x+2} + 4$ by solving the equation $x = 3\sqrt{y+2} + 4$ for y.
- Insert a **Graphs** page and enter $f1(x) = 3\sqrt{x+2} + 4$, $f2(x) = \frac{x^2}{9} - \frac{8x}{9} - \frac{2}{9} \mid x \geq 4$ and $f3(x) = x$.



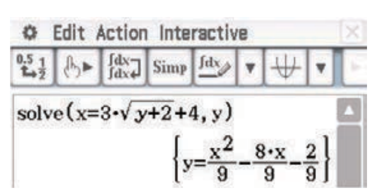


Note: To change the graph label to $y =$, place the cursor on the plot, press ctrl menu > **Attributes**, arrow down to the **Label Style** and select the desired style using the arrow keys. The **Attributes** menu can also be used to change the **Line Style**.

Using the Casio ClassPad

To find the rule for the inverse of $f(x) = 3\sqrt{x+2} + 4$:

- In Main $\sqrt{\alpha}$, enter and highlight $x = 3\sqrt{y+2} + 4$.
- Select **Interactive** > **Equation/Inequality** > **solve** and set the variable as y. Then tap OK.



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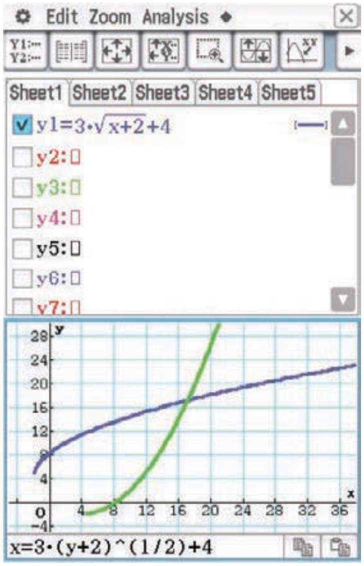
To graph the inverse of $f(x) = 3\sqrt{x+2} + 4$:

- In Graph & Table, enter the rule for the function f in $y1$.
- Tick the box and tap $\boxed{\Psi}$.
- Use $\boxed{\Leftrightarrow}$ to adjust the window view.
- To graph the inverse function f^{-1}, select **Analysis** > **Sketch** > **Inverse**.



Example 27

Express $\frac{x+4}{x+1}$ in the form $\frac{a}{x+b} + c$. Hence find the inverse of the function $f(x) = \frac{x+4}{x+1}$.
Sketch both functions on the one set of axes.

Solution

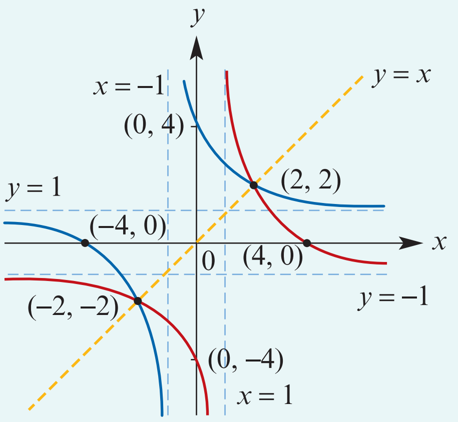
$$\frac{x+4}{x+1} = \frac{3+x+1}{x+1} = \frac{3}{x+1} + \frac{x+1}{x+1} = \frac{3}{x+1} + 1$$

Consider $x = \frac{3}{y+1} + 1$ and solve for y:

$$x - 1 = \frac{3}{y+1}$$

$$y + 1 = \frac{3}{x-1}$$

$$\therefore \quad y = \frac{3}{x-1} - 1$$



The range of f is $\mathbb{R} \setminus \{1\}$ and thus the inverse function is

$$f^{-1}: \mathbb{R} \setminus \{1\} \to \mathbb{R}, \ f^{-1}(x) = \frac{3}{x-1} - 1$$

Note: The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.
The two graphs meet where

$$\frac{3}{x+1} + 1 = x, \quad x \neq -1$$

i.e. where $x = \pm 2$. Thus the two graphs meet at the points $(2, 2)$ and $(-2, -2)$.

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Example 28

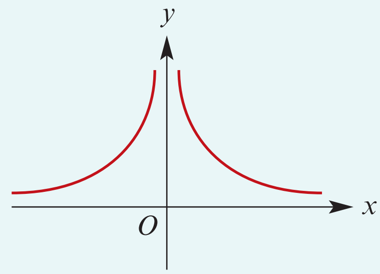
Let f be the function given by $f(x) = \dfrac{1}{x^2}$ for $x \in \mathbb{R} \setminus \{0\}$. Define a suitable restriction g of f such that g^{-1} exists, and find g^{-1}.

Solution

The function f is not one-to-one. Therefore the inverse function f^{-1} is not defined. The following restrictions of f are one-to-one:

$f_1\colon (0, \infty) \to \mathbb{R},\quad f_1(x) = \dfrac{1}{x^2}\quad$ Range of $f_1 = (0, \infty)$

$f_2\colon (-\infty, 0) \to \mathbb{R},\ f_2(x) = \dfrac{1}{x^2}\quad$ Range of $f_2 = (0, \infty)$



Let g be f_1 and determine f_1^{-1}.

Using method 2, we require f_1^{-1} such that

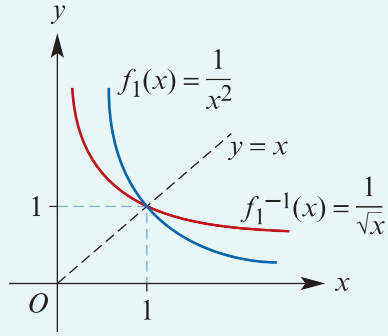
$$f_1(f_1^{-1}(x)) = x$$

$$\frac{1}{(f_1^{-1}(x))^2} = x$$

$$f_1^{-1}(x) = \pm\frac{1}{\sqrt{x}}$$

But ran f_1^{-1} = dom $f_1 = (0, \infty)$ and so

$$f_1^{-1}(x) = \frac{1}{\sqrt{x}}$$



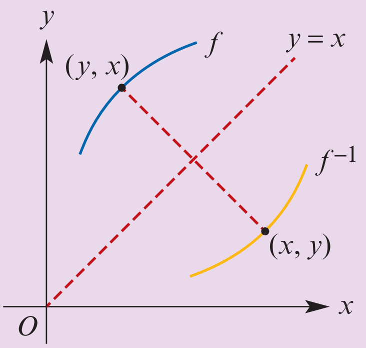
As dom f_1^{-1} = ran $f_1 = (0, \infty)$, the inverse function is $f_1^{-1}\colon (0, \infty) \to \mathbb{R},\ f_1^{-1}(x) = \dfrac{1}{\sqrt{x}}$

Section summary

- If f is a one-to-one function, then a new function f^{-1}, called the **inverse** of f, may be defined by

$$f^{-1}(x) = y \text{ if } f(y) = x, \quad \text{for } x \in \operatorname{ran} f,\ y \in \operatorname{dom} f$$

- dom f^{-1} = ran f
- ran f^{-1} = dom f
- $f \circ f^{-1}(x) = x$, for all $x \in \operatorname{dom} f^{-1}$
- $f^{-1} \circ f(x) = x$, for all $x \in \operatorname{dom} f$
- The point (x, y) is on the graph of f^{-1} if and only if the point (y, x) is on the graph of f. Thus the graph of f^{-1} is the reflection of the graph of f in the line $y = x$.



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Exercise 1F

Skillsheet
Example 23

1 Find the inverse function f^{-1} of the function:

a $f(x) = 2x + 3$
b $f(x) = 4 - 3x$
c $f(x) = 4x + 3$

2 For each of the following, find the rule for the inverse:

a $f: \mathbb{R} \to \mathbb{R}, f(x) = x - 4$
b $f: \mathbb{R} \to \mathbb{R}, f(x) = 2x$
c $f: \mathbb{R} \to \mathbb{R}, f(x) = \dfrac{3x}{4}$
d $f: \mathbb{R} \to \mathbb{R}, f(x) = \dfrac{3x - 2}{4}$

Example 24

3 For each of the following functions, find the inverse and state its domain and range:

a $f: [-2, 6] \to \mathbb{R}, f(x) = 2x - 4$
b $g(x) = \dfrac{1}{9 - x}, \ x > 9$
c $h(x) = x^2 + 2, \ x \geq 0$
d $f: [-3, 6] \to \mathbb{R}, f(x) = 5x - 2$
e $g: (1, \infty) \to \mathbb{R}, g(x) = x^2 - 1$
f $h: \mathbb{R}^+ \to \mathbb{R}, h(x) = \sqrt{x}$

4 Consider the function $g: [-1, \infty) \to \mathbb{R}, g(x) = x^2 + 2x$.

a Find g^{-1}, stating the domain and range.
b Sketch the graph of g^{-1}.

Example 25

5 Find the inverse of the function $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}, f(x) = \dfrac{1}{x} - 3$. Sketch both functions on one set of axes, showing the points of intersection of the graphs.

6 Let $f: [0, 3] \to \mathbb{R}, f(x) = 3 - 2x$. Find $f^{-1}(2)$ and the domain of f^{-1}.

7 For each of the following functions, find the inverse and state its domain and range:

a $f: [-1, 3] \to \mathbb{R}, f(x) = 2x$
b $f: [0, \infty) \to \mathbb{R}, f(x) = 2x^2 - 4$
c $\{(1, 6), (2, 4), (3, 8), (5, 11)\}$
d $h: \mathbb{R}^- \to \mathbb{R}, h(x) = \sqrt{-x}$
e $f: \mathbb{R} \to \mathbb{R}, f(x) = x^3 + 1$
f $g: (-1, 3) \to \mathbb{R}, g(x) = (x + 1)^2$
g $g: [1, \infty) \to \mathbb{R}, g(x) = \sqrt{x - 1}$
h $h: [0, 2] \to \mathbb{R}, h(x) = \sqrt{4 - x^2}$

Example 26

8 For each of the following functions, sketch the graph of the function and on the same set of axes sketch the graph of the inverse function. For each of the functions, state the rule, domain and range of the inverse. It is advisable to draw in the line with equation $y = x$ for each set of axes.

a $y = 2x + 4$
b $f(x) = \dfrac{3 - x}{2}$
c $f: [2, \infty) \to \mathbb{R}, f(x) = (x - 2)^2$
d $f: [1, \infty) \to \mathbb{R}, f(x) = (x - 1)^2$
e $f: (-\infty, 2] \to \mathbb{R}, f(x) = (x - 2)^2$
f $f: \mathbb{R}^+ \to \mathbb{R}, f(x) = \dfrac{1}{x}$
g $f: \mathbb{R}^+ \to \mathbb{R}, f(x) = \dfrac{1}{x^2}$
h $h(x) = \dfrac{1}{2}(x - 4)$

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9 Find the inverse function of each of the following, and sketch the graph of the inverse function:

a $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}, f(x) = \sqrt{x} + 2$ **b** $f: \mathbb{R} \setminus \{3\} \to \mathbb{R}, f(x) = \dfrac{1}{x-3}$

c $f: [2, \infty) \to \mathbb{R}, f(x) = \sqrt{x-2} + 4$ **d** $f: \mathbb{R} \setminus \{3\} \to \mathbb{R}, f(x) = \dfrac{1}{x-3} + 1$

e $f: \mathbb{R} \setminus \{1\} \to \mathbb{R}, f(x) = \dfrac{5}{x-1} - 1$ **f** $f: (-\infty, 2] \to \mathbb{R}, f(x) = \sqrt{2-x} + 1$

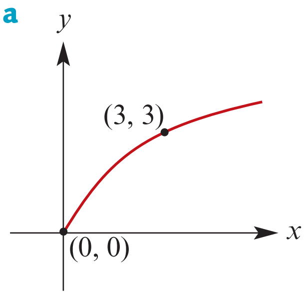
Example 27 **10** Find the rule for the inverse of each of the following functions:

a $f: \mathbb{R} \setminus \{1\} \to \mathbb{R}, f(x) = \dfrac{x+1}{x-1}$ **b** $f: [2, \infty) \to \mathbb{R}, f(x) = \sqrt{x-2}$

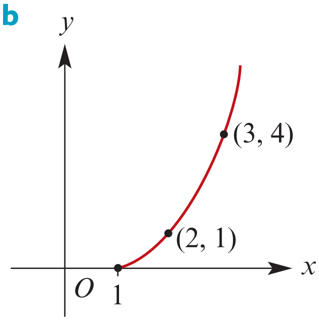
c $f: \mathbb{R} \setminus \{\frac{2}{3}\} \to \mathbb{R}, f(x) = \dfrac{2x+3}{3x-2}$

11 Copy each of the following graphs and on the same set of axes draw the inverse of each of the corresponding functions:

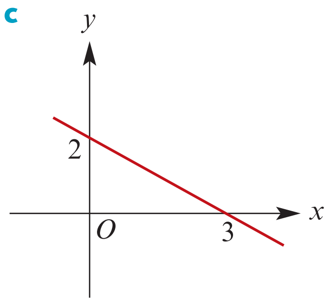
a



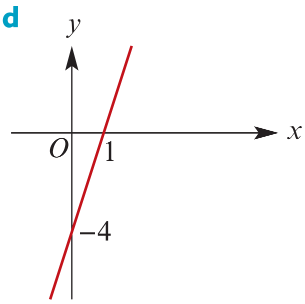
b



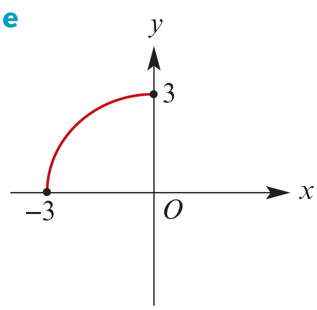
c



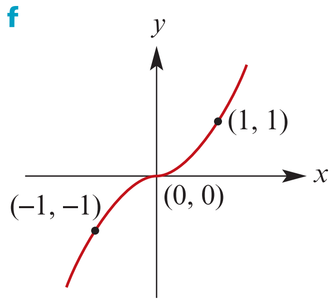
d



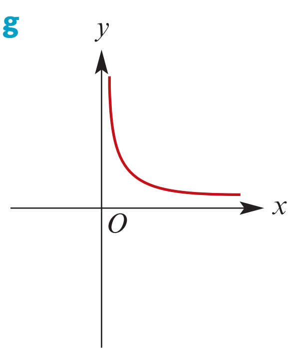
e



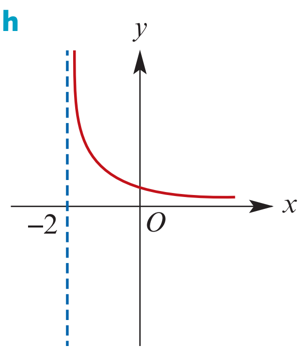
f



g



h



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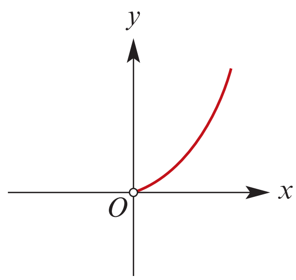
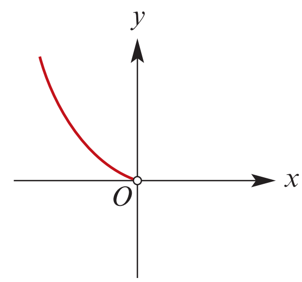
12 Match each of the graphs of **a**, **b**, **c** and **d** with its inverse.

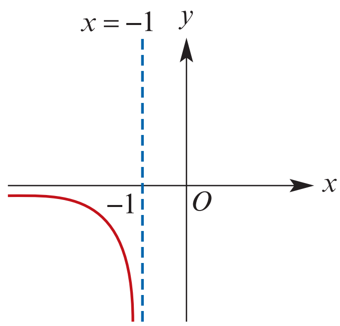
a



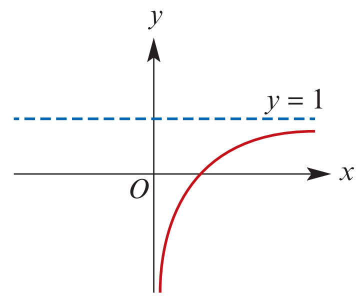
b



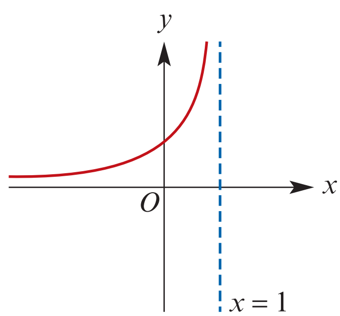
c



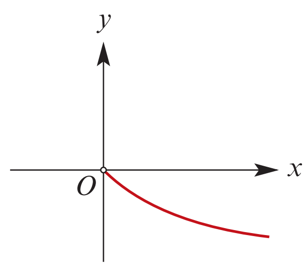
d



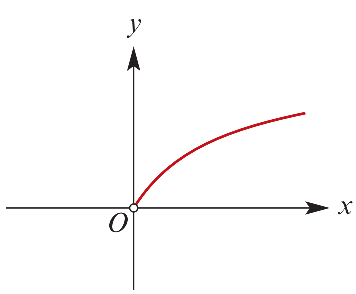
A



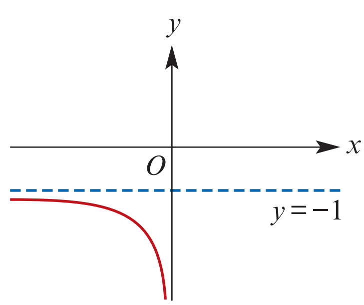
B



C



D



13 **a** Let $f\colon A \to \mathbb{R}$, $f(x) = \sqrt{3 - x}$. If A is the set of all real values of x for which $f(x)$ is defined, find A.

Example 28

b Let $g\colon [b, 2] \to \mathbb{R}$, $g(x) = 1 - x^2$. If b is the smallest real number such that g has an inverse function, find b and $g^{-1}(x)$.

14 Let $g\colon [b, \infty) \to \mathbb{R}$, where $g(x) = x^2 + 4x$. If b is the smallest real number such that g has an inverse function, find b and $g^{-1}(x)$.

15 Let $f\colon (-\infty, a) \to \mathbb{R}$, where $f(x) = x^2 - 6x$. If a is the largest real number such that f has an inverse function, find a and $f^{-1}(x)$.

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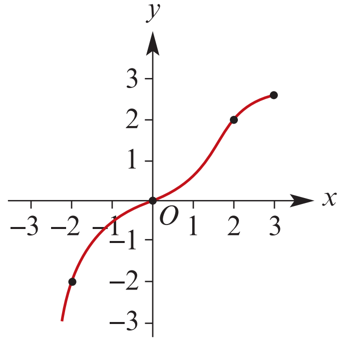
16 For each of the following functions, find the inverse function and state its domain:

a $g(x) = \dfrac{3}{x}$ **b** $g(x) = \sqrt[3]{x+2} - 4$ **c** $h(x) = 2 - \sqrt{x}$

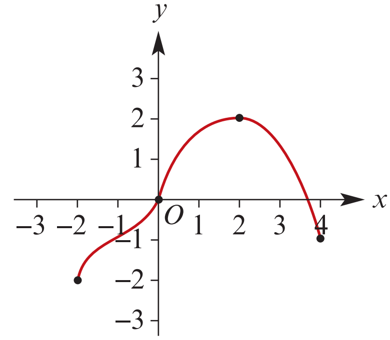
d $f(x) = \dfrac{3}{x} + 1$ **e** $h(x) = 5 - \dfrac{2}{(x-6)^3}$ **f** $g(x) = \dfrac{1}{(x-1)^{\frac{3}{4}}} + 2$

17 For each of the following, copy the graph onto a grid and sketch the graph of the inverse on the same set of axes. In each case, state whether the inverse is or is not a function.

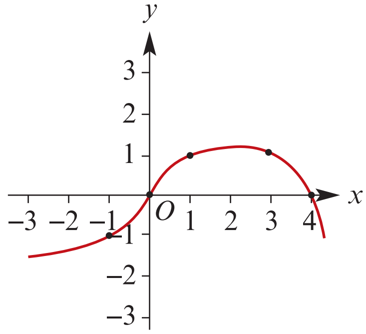
a



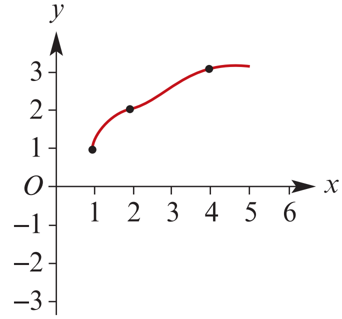
b



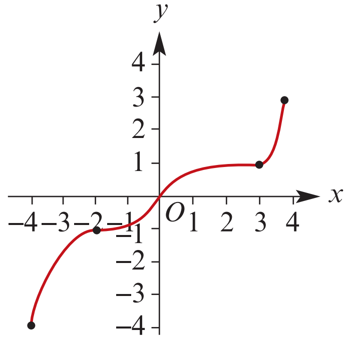
c



d



e



18 Let $f: S \to \mathbb{R}$ be given by $f(x) = \dfrac{x+3}{2x-1}$, where $S = \mathbb{R} \setminus \{\frac{1}{2}\}$.

a Show that $f \circ f$ is defined.

b Find $f \circ f(x)$ and sketch the graph of $f \circ f$.

c Write down the inverse of f.

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1G Power functions

In this section we look at functions of the form $f(x) = x^r$, where r is a rational number. These functions are called **power functions**.

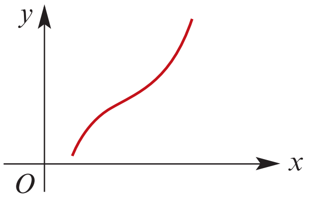
In particular, we look at functions with rules such as

$$f(x) = x^4, \quad f(x) = x^{-4}, \quad f(x) = x^{\frac{1}{4}}, \quad f(x) = x^5, \quad f(x) = x^{-5}, \quad f(x) = x^{\frac{1}{3}}$$

We will not concern ourselves with functions such as $f(x) = x^{\frac{2}{3}}$ at this stage, but return to consider these functions in Chapter 7.

▶ Increasing and decreasing functions

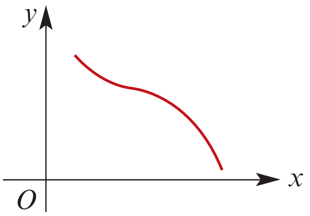
We say a function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.



For example:

- The graph opposite shows a strictly increasing function.
- A straight line with positive gradient is strictly increasing.
- The function $f: (0, \infty) \to \mathbb{R}$, $f(x) = x^2$ is strictly increasing.

We say a function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.

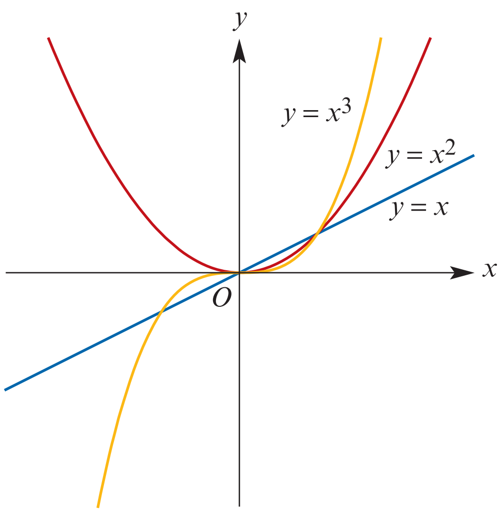


For example:

- The graph opposite shows a strictly decreasing function.
- A straight line with negative gradient is strictly decreasing.
- The function $f: (-\infty, 0) \to \mathbb{R}$, $f(x) = x^2$ is strictly decreasing.

▶ Power functions with positive integer index

We start by considering power functions $f(x) = x^n$ where n is a positive integer.



Taking $n = 1, 2, 3$, we obtain the linear function $f(x) = x$, the quadratic function $f(x) = x^2$ and the cubic function $f(x) = x^3$.

We have studied these functions in Mathematical Methods Units 1 & 2 and have referred to them in the earlier sections of this chapter.

The general shape of the graph of $f(x) = x^n$ depends on whether the index n is odd or even.

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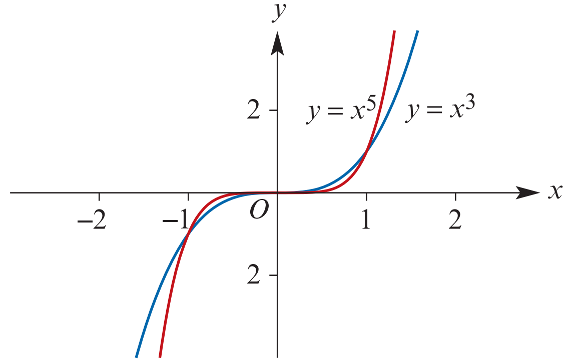
The function $f(x) = x^n$ where n is an odd positive integer

The graph has a similar shape to those shown below. The maximal domain is $\mathbb{R}$ and the range is $\mathbb{R}$.

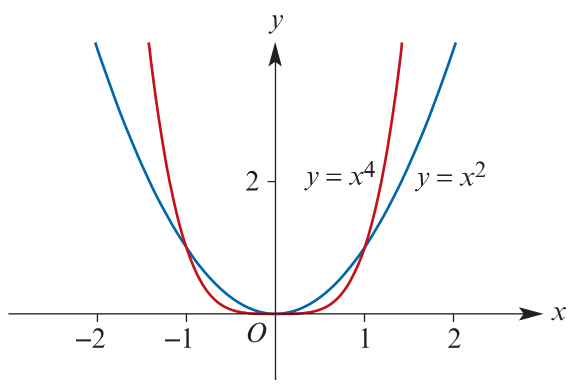


Some properties of $f(x) = x^n$ where n is an odd positive integer:

- f is an odd function
- f is strictly increasing
- f is one-to-one
- $f(0) = 0$, $f(1) = 1$ and $f(-1) = -1$
- as $x \to \infty$, $f(x) \to \infty$ and as $x \to -\infty$, $f(x) \to -\infty$.

The function $f(x) = x^n$ where n is an even positive integer

The graph has a similar shape to those shown below. The maximal domain is $\mathbb{R}$ and the range is $\mathbb{R}^+ \cup \{0\}$.



Some properties of $f(x) = x^n$ where n is an even positive integer:

- f is an even function
- f strictly increasing for $x > 0$
- f is strictly decreasing for $x < 0$
- $f(0) = 0$, $f(1) = 1$ and $f(-1) = 1$
- as $x \to \pm\infty$, $f(x) \to \infty$.

Power functions with negative integer index

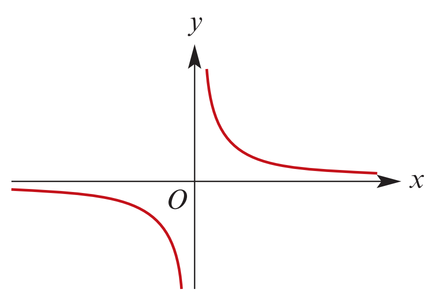
Again, the general shape of the graph depends on whether the index n is odd or even.

The function $f(x) = x^n$ where n is an odd negative integer

Taking $n = -1$, we obtain

$$f(x) = x^{-1} = \frac{1}{x}$$

The graph of this function is shown on the right. The graphs of functions of this type are all similar to this one.



In general, we consider the functions $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-k}$ for $k = 1, 3, 5, \ldots$

- the maximal domain is $\mathbb{R} \setminus \{0\}$ and the range is $\mathbb{R} \setminus \{0\}$
- f is an odd function
- there is a horizontal asymptote with equation $y = 0$
- there is a vertical asymptote with equation $x = 0$.

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Example 29

For the function f with rule $f(x) = \frac{1}{x^5}$:

a State the maximal domain and the corresponding range.

b Evaluate each of the following:

i $f(2)$ **ii** $f(-2)$ **iii** $f(\frac{1}{2})$ **iv** $f(-\frac{1}{2})$

c Sketch the graph without using your calculator.

Solution

a The maximal domain is $\mathbb{R} \setminus \{0\}$ and the range is $\mathbb{R} \setminus \{0\}$.

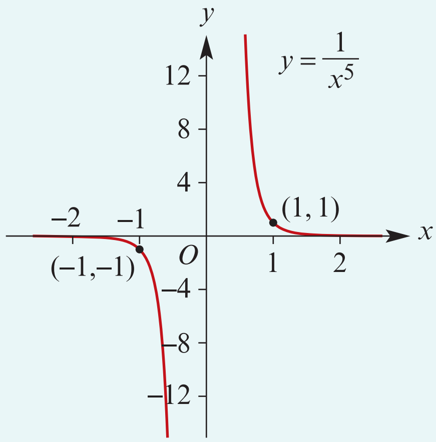
b **i** $f(2) = \frac{1}{2^5} = \frac{1}{32}$

ii $f(-2) = \frac{1}{(-2)^5} = -\frac{1}{32}$

iii $f(\frac{1}{2}) = \frac{1}{(\frac{1}{2})^5} = 32$

iv $f(-\frac{1}{2}) = \frac{1}{(-\frac{1}{2})^5} = -32$

c



Example 30

Let $f: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-1}$ and $g: \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $g(x) = x^{-3}$.

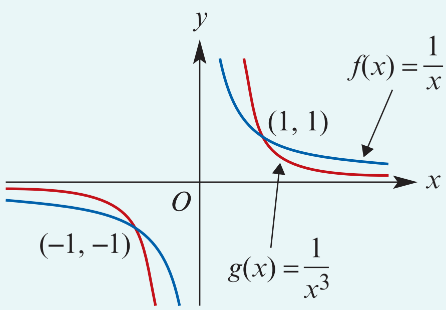
a Find the values of x for which $f(x) = g(x)$.

b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Solution

a
$$f(x) = g(x)$$
$$x^{-1} = x^{-3}$$
$$\frac{1}{x} = \frac{1}{x^3}$$
$$x^2 = 1$$
$$\therefore \quad x = 1 \text{ or } x = -1$$

b



Note:

If $x > 1$, then $x^3 > x$ and so $\frac{1}{x} > \frac{1}{x^3}$.

If $0 < x < 1$, then $x^3 < x$ and so $\frac{1}{x} < \frac{1}{x^3}$.

If $x < -1$, then $x^3 < x$ and so $\frac{1}{x} < \frac{1}{x^3}$.

If $-1 < x < 0$, then $x^3 > x$ and so $\frac{1}{x} > \frac{1}{x^3}$.

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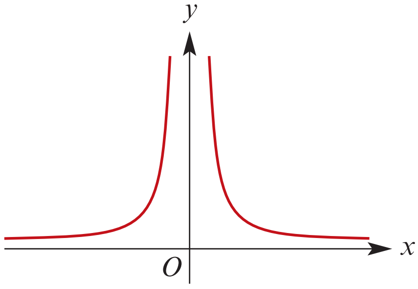
The function $f(x) = x^n$ where n is an even negative integer

Taking $n = -2$, we obtain

$$f(x) = x^{-2} = \frac{1}{x^2}$$

The graph of this function is shown on the right. The graphs of functions of this type are all similar to this one.



In general, we consider the functions $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-k}$ for $k = 2, 4, 6, \dots$

- the maximal domain $\mathbb{R} \setminus \{0\}$ and the range is $\mathbb{R}^+$
- f is an even function
- there is a horizontal asymptote with equation $y = 0$
- there is a vertical asymptote with equation $x = 0$.

The function $f(x) = x^{\frac{1}{n}}$ where n is a positive integer

Let a be a positive real number and let $n \in \mathbb{N}$. Then $a^{\frac{1}{n}}$ is defined to be the nth root of a. That is, $a^{\frac{1}{n}}$ is the positive number whose nth power is a. We can also write this as $a^{\frac{1}{n}} = \sqrt[n]{a}$. For example: $9^{\frac{1}{2}} = 3$, since $3^2 = 9$.

We define $0^{\frac{1}{n}} = 0$, for each natural number n, since $0^n = 0$.

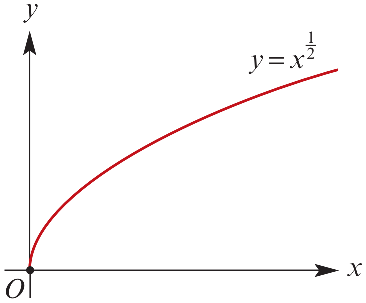
If n is odd, then we can also define $a^{\frac{1}{n}}$ when a is negative. If a is negative and n is odd, define $a^{\frac{1}{n}}$ to be the number whose nth power is a. For example: $(-8)^{\frac{1}{3}} = -2$, as $(-2)^3 = -8$.

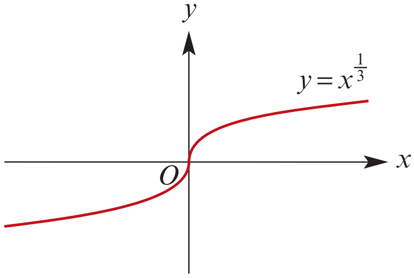
In all three cases we can write:

$$a^{\frac{1}{n}} = \sqrt[n]{a} \quad \text{with} \quad \left(a^{\frac{1}{n}}\right)^n = a$$

In particular, $x^{\frac{1}{2}} = \sqrt{x}$.

Let $f(x) = x^{\frac{1}{n}}$. When n is even the maximal domain is $\mathbb{R}^+ \cup \{0\}$ and when n is odd the maximal domain is $\mathbb{R}$. The graphs of $f(x) = \sqrt{x} = x^{\frac{1}{2}}$ and $f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$ are as shown.





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Example 31

Let $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^{\frac{1}{3}}$ and $g: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $g(x) = x^{\frac{1}{2}}$.

a Find the values of x for which $f(x) = g(x)$.

b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Solution

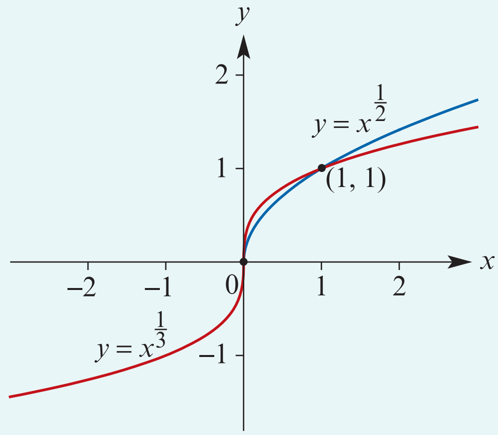
a

$$\begin{aligned} f(x) &= g(x) \\ x^{\frac{1}{3}} &= x^{\frac{1}{2}} \\ x^{\frac{1}{3}} - x^{\frac{1}{2}} &= 0 \\ x^{\frac{1}{3}}\left(1 - x^{\frac{1}{6}}\right) &= 0 \end{aligned}$$

$\therefore \quad x = 0 \text{ or } 1 - x^{\frac{1}{6}} = 0$

$\therefore \quad x = 0 \text{ or } x = 1$

b



Inverses of power functions

We prove the following result in the special case when $n = 5$. The general proof is similar.

> If n is an odd positive integer, then $f(x) = x^n$ is strictly increasing for $\mathbb{R}$.

Proof Let $f(x) = x^5$ and let $a > b$. To show that $f(a) > f(b)$, we consider five cases.

Case 1: $a > b > 0$ We have

$$\begin{aligned} f(a) - f(b) &= a^5 - b^5 \\ &= (a-b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4) \qquad \text{(Show by expanding.)} \end{aligned}$$

Since $a > b$, we have $a - b > 0$. Since we are assuming that a and b are positive in this case, all the terms of $a^4 + a^3b + a^2b^2 + ab^3 + b^4$ are positive. Therefore $f(a) - f(b) > 0$ and so $f(a) > f(b)$.

Case 2: $a > 0$ and $b < 0$ In this case, we have $f(a) = a^5 > 0$ and $f(b) = b^5 < 0$ (an odd power of a negative number). Thus $f(a) > f(b)$.

Case 3: $a = 0$ and $b < 0$ We have $f(a) = 0$ and $f(b) < 0$. Thus $f(a) > f(b)$.

Case 4: $b = 0$ and $a > 0$ We have $f(a) > 0$ and $f(b) = 0$. Thus $f(a) > f(b)$.

Case 5: $0 > a > b$ Let $a = -c$ and $b = -d$, where c and d are positive. Then $a > b$ implies $-c > -d$ and so $c < d$. Hence $f(c) < f(d)$ by Case 1 and thus $f(-a) < f(-b)$. But f is an odd function and so $-f(a) < -f(b)$. Finally, we have $f(a) > f(b)$.

Note: For the general proof, use the identity

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + a^2b^{n-3} + ab^{n-2} + b^{n-1})$$

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If f is a strictly increasing function on $\mathbb{R}$, then it is a one-to-one function and so has an inverse. Thus $f(x) = x^n$ has an inverse function, where n is an odd positive integer.

Similar results can be achieved for restrictions of functions with rules $f(x) = x^n$, where n is an even positive integer. For example, $g\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $g(x) = x^6$ is a strictly increasing function and $h\colon \mathbb{R}^- \cup \{0\} \to \mathbb{R}$, $h(x) = x^6$ is a strictly decreasing function. In both cases, these restricted functions are one-to-one.

If f is an odd one-to-one function, then f^{-1} is also an odd function.

Proof Let $x \in \operatorname{dom} f^{-1}$ and let $y = f^{-1}(x)$. Then $f(y) = x$. Since f is an odd function, we have $f(-y) = -x$, which implies that $f^{-1}(-x) = -y$. Hence $f^{-1}(-x) = -f^{-1}(x)$.

By this result we see that, if n is odd, then $f(x) = x^{\frac{1}{n}}$ is an odd function. It can also be shown that, if f is a strictly increasing function, then f^{-1} is strictly increasing.

Example 32

Find the inverse of each of the following functions:

a $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^5$

b $f\colon (-\infty, 0] \to \mathbb{R}$, $f(x) = x^4$

c $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 8x^3$

d $f\colon (1, \infty) \to \mathbb{R}$, $f(x) = 64x^6$

Solution

a $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^5$

Write $y = x^5$. Interchange x and y and then solve for y:

$$x = y^5$$
$$\therefore \quad y = x^{\frac{1}{5}}$$

Thus $f^{-1}\colon \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = x^{\frac{1}{5}}$

b $f\colon (-\infty, 0] \to \mathbb{R}$, $f(x) = x^4$

Note that f has range $[0, \infty)$. Therefore f^{-1} has domain $[0, \infty)$ and range $(-\infty, 0]$.

Write $y = x^4$. Interchange x and y and then solve for y:

$$x = y^4$$
$$\therefore \quad y = \pm x^{\frac{1}{4}}$$

Thus $f^{-1}\colon [0, \infty) \to \mathbb{R}$, $f^{-1}(x) = -x^{\frac{1}{4}}$

c $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 8x^3$

Write $y = 8x^3$. Interchange x and y and then solve for y:

$$x = 8y^3$$
$$y^3 = \frac{x}{8}$$
$$\therefore \quad y = \frac{1}{2}x^{\frac{1}{3}}$$

Thus $f^{-1}\colon \mathbb{R} \to \mathbb{R}$, $f^{-1}(x) = \frac{1}{2}x^{\frac{1}{3}}$

d $f\colon (1, \infty) \to \mathbb{R}$, $f(x) = 64x^6$

Note that f has range $(64, \infty)$. Therefore f^{-1} has domain $(64, \infty)$ and range $(1, \infty)$.

Write $y = 64x^6$. Interchange x and y and then solve for y:

$$x = 64y^6$$
$$y^6 = \frac{x}{64}$$
$$\therefore \quad y = \pm\frac{1}{2}x^{\frac{1}{6}}$$

Thus $f^{-1}\colon (64, \infty) \to \mathbb{R}$, $f^{-1}(x) = \frac{1}{2}x^{\frac{1}{6}}$

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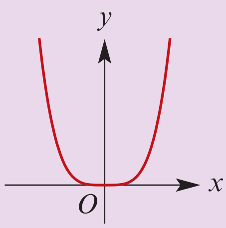
Section summary

- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
- A **power function** is a function f with rule $f(x) = x^r$, where r is a rational number.
- For a power function $f(x) = x^n$, where n is a non-zero integer, the general shape of the graph depends on whether n is positive or negative and whether n is even or odd:

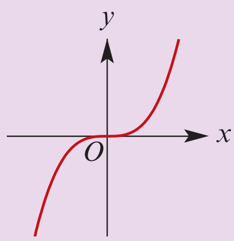
Even positive

$f(x) = x^4$



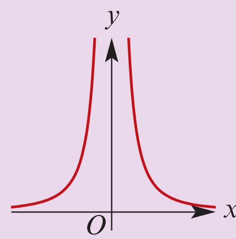
Odd positive

$f(x) = x^3$



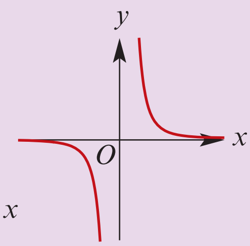
Even negative

$f(x) = x^{-2}$



Odd negative

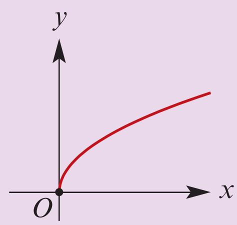
$f(x) = x^{-3}$



- For a power function $f(x) = x^{\frac{1}{n}}$, where n is a positive integer, the general shape of the graph depends on whether n is even or odd:

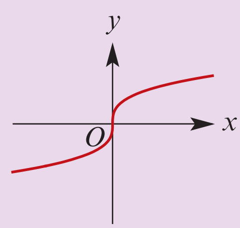
Even

$f(x) = x^{\frac{1}{2}}$



Odd

$f(x) = x^{\frac{1}{3}}$



Exercise 1G

Example 29

1 For the function f with rule $f(x) = \dfrac{1}{x^4}$:

a State the maximal domain and the corresponding range.

b Evaluate each of the following:

i $f(2)$ **ii** $f(-2)$ **iii** $f(\frac{1}{2})$ **iv** $f(-\frac{1}{2})$

c Sketch the graph without using your calculator.

2 For each of the following, state whether the function is odd, even or neither:

a $f(x) = 2x^5$ **b** $f(x) = x^2 + 3$ **c** $f(x) = x^{\frac{1}{5}}$

d $f(x) = \dfrac{1}{x}$ **e** $f(x) = \dfrac{1}{x^2}$ **f** $f(x) = \sqrt[3]{x}$

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Example 30 **3** Let $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = x^{-2}$ and $g\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $g(x) = x^{-4}$.

a Find the values of x for which $f(x) = g(x)$.

b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Example 31 **4** Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^{\frac{1}{3}}$ and $g\colon \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$, $g(x) = x^{\frac{1}{4}}$.

a Find the values of x for which $f(x) = g(x)$.

b Sketch the graphs of $y = f(x)$ and $y = g(x)$ on the one set of axes.

Example 32 **5** Find the inverse of each of the following functions:

a $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^7$

b $f\colon (-\infty, 0] \to \mathbb{R}$, $f(x) = x^6$

c $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 27x^3$

d $f\colon (1, \infty) \to \mathbb{R}$, $f(x) = 16x^4$

1H Applications of functions

In this section we use function notation in the solution of some problems.

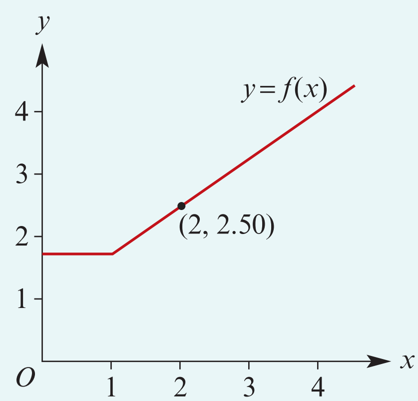
Example 33

The cost of a taxi trip in a particular city is \$1.75 up to and including 1 km. After 1 km the passenger pays an additional 75 cents per kilometre. Find the function f which describes this method of payment and sketch the graph of $y = f(x)$.

Solution

Let x denote the length of the trip in kilometres.
Then the cost in dollars is given by

$$f(x) = \begin{cases} 1.75 & \text{for } 0 \le x \le 1 \\ 1.75 + 0.75(x - 1) & \text{for } x > 1 \end{cases}$$



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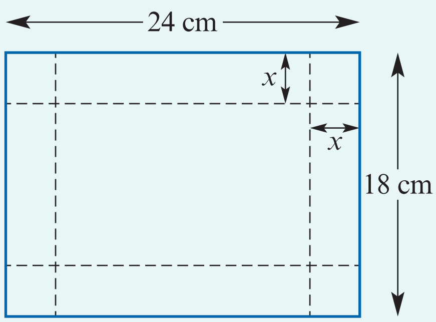
Example 34

A rectangular piece of cardboard has dimensions 18 cm by 24 cm. Four squares each x cm by x cm are cut from the corners. An open box is formed by folding up the flaps.

Find a function V which gives the volume of the box in terms of x, and state the domain of the function.



Solution

The dimensions of the box will be $24 - 2x$, $18 - 2x$ and x.

Thus the volume of the box is determined by the function

$$V(x) = (24 - 2x)(18 - 2x)x$$

For the box to be formed:

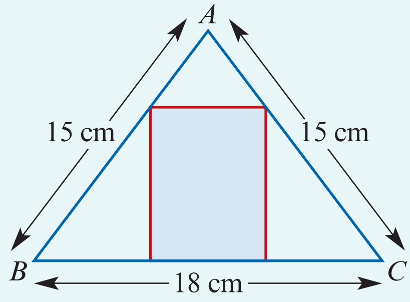
$$24 - 2x \geq 0 \quad \text{and} \quad 18 - 2x \geq 0 \quad \text{and} \quad x \geq 0$$

Therefore $x \leq 12$ and $x \leq 9$ and $x \geq 0$. The domain of V is $[0, 9]$.

Example 35

A rectangle is inscribed in an isosceles triangle with the dimensions as shown.

Find an area-of-the-rectangle function and state the domain.

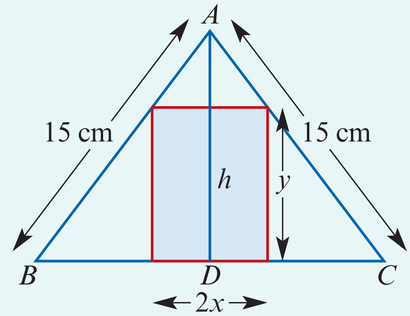


Solution

Let the height of the rectangle be y cm and the width $2x$ cm.

The height (h cm) of the triangle can be determined by Pythagoras' theorem:

$$h = \sqrt{15^2 - 9^2} = 12$$



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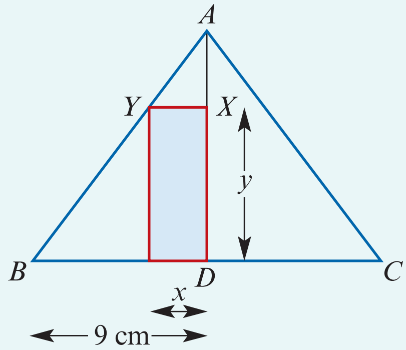
In the diagram opposite, the triangle AYX is similar to the triangle ABD. Therefore

$$\frac{x}{9} = \frac{12-y}{12}$$

$$\frac{12x}{9} = 12 - y$$

$$\therefore \quad y = 12 - \frac{12x}{9}$$



The area of the rectangle is $A = 2xy$, and so

$$A(x) = 2x\left(12 - \frac{12x}{9}\right) = \frac{24x}{9}(9 - x)$$

For the rectangle to be formed, we need

$$x \geq 0 \quad \text{and} \quad 12 - \frac{12x}{9} \geq 0$$

$$\therefore \quad x \geq 0 \quad \text{and} \quad x \leq 9$$

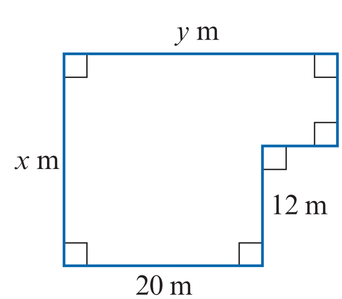
The domain is $[0, 9]$, and so the function is $A\colon [0, 9] \to \mathbb{R}$, $A(x) = \dfrac{24x}{9}(9 - x)$

Exercise 1H

Example 33 **1** The cost of a taxi trip in a particular city is \$4.00 up to and including 2 km. After 2 km the passenger pays an additional \$2.00 per kilometre. Find the function f which describes this method of payment and sketch the graph of $y = f(x)$, where x is the number of kilometres travelled. (Use a continuous model.)

Example 34 **2** A rectangular piece of cardboard has dimensions 20 cm by 36 cm. Four squares each x cm by x cm are cut from the corners. An open box is formed by folding up the flaps. Find a function V which gives the volume of the box in terms of x, and state the domain for the function.

3 The dimensions of an enclosure are shown. The perimeter of the enclosure is 160 m.



a Find a rule for the area, A m^2, of the enclosure in terms of x.

b State a suitable domain of the function $A(x)$.

c Sketch the graph of A against x.

d Find the maximum possible area of the enclosure and state the corresponding values of x and y.

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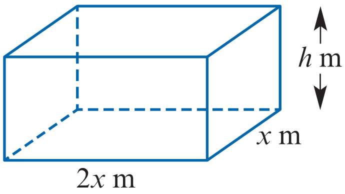
4 A cuboid tank is open at the top and the internal dimensions of its base are x m and $2x$ m. The height is h m. The volume of the tank is V m^3 and the volume is fixed. Let S m^2 denote the internal surface area of the tank.



a Find S in terms of:

i x and h

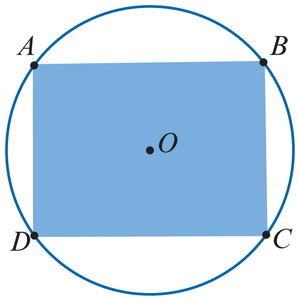
ii V and x

b State the maximal domain for the function defined by the rule in part a ii.

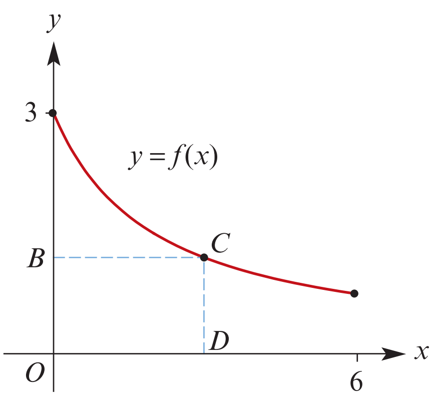
c If $2 \le x \le 15$, find the maximum value of S if $V = 1000$ m^3.

Example 35

5 A rectangle $ABCD$ is inscribed in a circle of radius a. Find an area-of-the-rectangle function and state the domain.



6 Let $f : [0, 6] \to \mathbb{R}$, $f(x) = \dfrac{6}{x+2}$.
Rectangle $OBCD$ is formed so that the coordinates of C are $(a, f(a))$.



a Find an expression for the area-of-rectangle function A.

b State the implied domain and range of A.

c State the maximum value of $A(x)$ for $x \in [0, 6]$.

d Sketch the graph of $y = A(x)$ for $x \in [0, 6]$.

7 A man walks at a speed of 2 km/h for 45 minutes and then runs at 4 km/h for 30 minutes. Let S km be the distance the man has travelled after t minutes. The distance travelled can be described by

$$S(t) = \begin{cases} at & \text{if } 0 \le t \le c \\ bt + d & \text{if } c < t \le e \end{cases}$$

a Find the values a, b, c, d, e.

b Sketch the graph of $S(t)$ against t.

c State the range of the function.

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Review

Chapter summary

Spreadsheet

AS

Nrich

Relations

- A **relation** is a set of ordered pairs.
- The **domain** is the set of all the first coordinates of the ordered pairs in the relation.
- The **range** is the set of all the second coordinates of the ordered pairs in the relation.

Functions

- A **function** is a relation such that no two ordered pairs in the relation have the same first coordinate.
- For each x in the domain of a function f, there is a unique element y in the range such that $(x, y) \in f$. The element y is called the **image** of x under f or the **value** of f at x and is denoted by $f(x)$.
- When the domain of a function is not explicitly stated, it is assumed to consist of all real numbers for which the rule has meaning; this is called the **implied domain** or the **maximal domain** of the function.
- For a function f, the domain is denoted by **dom f** and the range by **ran f**.
- Let f and g be functions such that $\operatorname{dom} f \cap \operatorname{dom} g \neq \varnothing$. Then the **sum**, $f + g$, and the **product**, fg, as functions on $\operatorname{dom} f \cap \operatorname{dom} g$ are defined by

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (fg)(x) = f(x) \cdot g(x)$$

- The **composition** of functions f and g is denoted by $f \circ g$. The rule is given by

$$f \circ g(x) = f(g(x))$$

The domain of $f \circ g$ is the domain of g. The composition $f \circ g$ is defined only if the range of g is a subset of the domain of f.

One-to-one functions and inverses

- A function f is said to be **one-to-one** if $a \neq b$ implies $f(a) \neq f(b)$, for all $a, b \in \operatorname{dom} f$.
- If f is a one-to-one function, then a new function f^{-1}, called the **inverse** of f, may be defined by

$$f^{-1}(x) = y \text{ if } f(y) = x, \quad \text{for } x \in \operatorname{ran} f,\ y \in \operatorname{dom} f$$

- For a one-to-one function f and its inverse f^{-1}:

$$\operatorname{dom} f^{-1} = \operatorname{ran} f$$
$$\operatorname{ran} f^{-1} = \operatorname{dom} f$$

Types of functions

- A function f is **odd** if $f(-x) = -f(x)$ for all x in the domain of f.
- A function f is **even** if $f(-x) = f(x)$ for all x in the domain of f.
- A function f is **strictly increasing** on an interval if $x_2 > x_1$ implies $f(x_2) > f(x_1)$.
- A function f is **strictly decreasing** on an interval if $x_2 > x_1$ implies $f(x_2) < f(x_1)$.
- A **power function** is a function f with rule $f(x) = x^r$, where r is a rational number.

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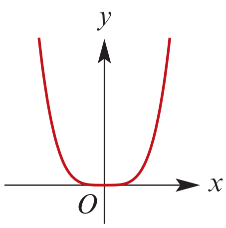
- For a power function $f(x) = x^n$, where n is a non-zero integer, the general shape of the graph depends on whether n is positive or negative and whether n is even or odd:

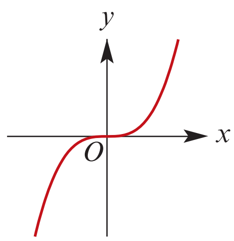
Even positive

$f(x) = x^4$



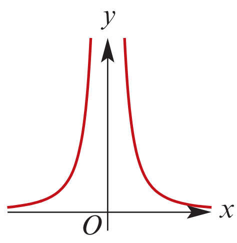
Odd positive

$f(x) = x^3$



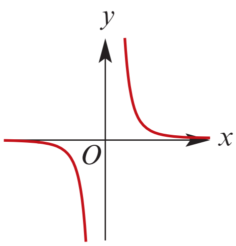
Even negative

$f(x) = x^{-2}$



Odd negative

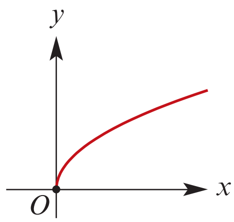
$f(x) = x^{-3}$



- For a power function $f(x) = x^{\frac{1}{n}}$, where n is a positive integer, the general shape of the graph depends on whether n is even or odd:

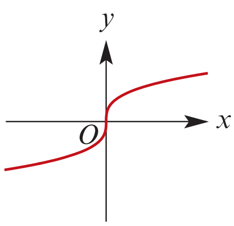
Even

$f(x) = x^{\frac{1}{2}}$



Odd

$f(x) = x^{\frac{1}{3}}$



Technology-free questions

1 Sketch the graph of each of the following relations and state the implied domain and range:

a $f(x) = x^2 + 1$ **b** $f(x) = 2x - 6$ **c** $\{(x, y) : x^2 + y^2 = 25\}$

d $\{(x, y) : y \geq 2x + 1\}$ **e** $\{(x, y) : y < x - 3\}$

2 For the function $g\colon [0, 5] \to \mathbb{R}$, $g(x) = \dfrac{x+3}{2}$:

a Sketch the graph of $y = g(x)$. **b** State the range of g.

c Find g^{-1}, stating the domain and range of g^{-1}.

d Find $\{x : g(x) = 4\}$. **e** Find $\{x : g^{-1}(x) = 4\}$.

3 For $g(x) = 5x + 1$, find:

a $\{x : g(x) = 2\}$ **b** $\{x : g^{-1}(x) = 2\}$ **c** $\left\{x : \dfrac{1}{g(x)} = 2\right\}$

4 Sketch the graph of the function f for which

$$f(x) = \begin{cases} x + 1 & \text{for } x > 2 \\ x^2 - 1 & \text{for } 0 \leq x \leq 2 \\ -x^2 & \text{for } x < 0 \end{cases}$$

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5 Find the implied domain for each of the following:

a $f(x) = \dfrac{1}{2x - 6}$ **b** $g(x) = \dfrac{1}{\sqrt{x^2 - 5}}$ **c** $h(x) = \dfrac{1}{(x-1)(x+2)}$

d $h(x) = \sqrt{25 - x^2}$ **e** $f(x) = \sqrt{x - 5} + \sqrt{15 - x}$ **f** $h(x) = \dfrac{1}{3x - 6}$

6 For $f(x) = (x + 2)^2$ and $g(x) = x - 3$, find $(f + g)(x)$ and $(fg)(x)$.

7 For $f\colon [1, 5] \to \mathbb{R}$, $f(x) = (x - 1)^2$ and $g\colon \mathbb{R} \to \mathbb{R}$, $g(x) = 2x$, find $f + g$ and fg.

8 For $f\colon [3, \infty) \to \mathbb{R}$, $f(x) = x^2 - 1$, find f^{-1}.

9 For $f(x) = 2x + 3$ and $g(x) = -x^2$, find:

a $(f + g)(x)$ **b** $(fg)(x)$ **c** $\{\, x : (f + g)(x) = 0 \,\}$

10 Let $f\colon (-\infty, 2] \to \mathbb{R}$, $f(x) = 3x - 4$. On the one set of axes, sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$.

11 Find the inverse of each of the following functions:

a $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 8x^3$ **b** $f\colon (-\infty, 0] \to \mathbb{R}$, $f(x) = 32x^5$

c $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = 64x^6$ **d** $f\colon (1, \infty) \to \mathbb{R}$, $f(x) = 10\,000x^4$

12 For $f(x) = 2x + 3$ and $g(x) = -x^3$, find:

a $f \circ g(x)$ **b** $g \circ f(x)$ **c** $g \circ g(x)$ **d** $f \circ f(x)$

e $f \circ (f + g)(x)$ **f** $f \circ (f - g)(x)$ **g** $f \circ (f \cdot g)(x)$

13 If the function f has the rule $f(x) = \sqrt{x^2 - 16}$ and the function g has rule $g(x) = x + 5$, find the largest domain for g such that $f \circ g$ is defined.

14 For the function h with rule $h(x) = 2x^5 + 64$, find the rule for the inverse function h^{-1}.

Multiple-choice questions

1 For the function with rule $f(x) = \sqrt{6 - 2x}$, which of the following is the maximal domain?

A $(-\infty, 6]$ **B** $[3, \infty)$ **C** $(-\infty, 6]$ **D** $(3, \infty)$ **E** $(-\infty, 3]$

2 For $f\colon [-1, 3) \to \mathbb{R}$, $f(x) = -x^2$, the range is

A $\mathbb{R}$ **B** $(-9, 0]$ **C** $(-\infty, 0]$ **D** $(-9, -1]$ **E** $[-9, 0]$

3 For $f(x) = 3x^2 + 2x$, $f(2a) =$

A $20a^2 + 4a$ **B** $6a^2 + 2a$ **C** $6a^2 + 4a$ **D** $36a^2 + 4a$ **E** $12a^2 + 4a$

4 For $f(x) = 2x - 3$, $f^{-1}(x) =$

A $2x + 3$ **B** $\dfrac{1}{2}x + 3$ **C** $\dfrac{1}{2}x + \dfrac{3}{2}$ **D** $\dfrac{1}{2x - 3}$ **E** $\dfrac{1}{2}x - 3$

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5 For $f\colon (a, b] \to \mathbb{R}$, $f(x) = 10 - x$ where $a < b$, the range is

A $(10 - a, 10 - b)$ **B** $(10 - a, 10 - b]$ **C** $(10 - b, 10 - a)$

D $(10 - b, 10 - a]$ **E** $[10 - b, 10 - a)$

6 For the function with rule

$$f(x) = \begin{cases} x^2 + 5 & x \geq 3 \\ -x + 6 & x < 3 \end{cases}$$

the value of $f(a + 3)$, where a is a negative real number, is

A $a^2 + 6a + 14$ **B** $-a + 9$ **C** $-a + 3$ **D** $a^2 + 14$ **E** $a^2 + 8a + 8$

7 Which one of the following sets is a possible domain for the function with rule $f(x) = (x + 3)^2 - 6$ if the inverse function is to exist?

A $\mathbb{R}$ **B** $[-6, \infty)$ **C** $(-\infty, 3]$ **D** $[6, \infty)$ **E** $(-\infty, 0)$

8 For which one of the following functions does an inverse function not exist?

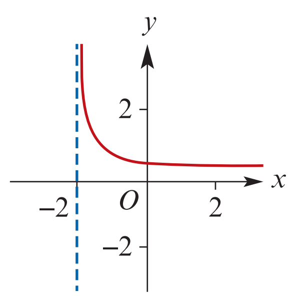
A $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2x - 4$ **B** $g\colon [-4, 4] \to \mathbb{R}$, $g(x) = \sqrt{16 - x^2}$

C $h\colon [0, \infty) \to \mathbb{R}$, $h(x) = -\frac{1}{5}x^2$ **D** $p\colon \mathbb{R}^+ \to \mathbb{R}$, $p(x) = \frac{1}{x^2}$

E $q\colon \mathbb{R} \to \mathbb{R}$, $q(x) = 2x^3 - 5$

9 The graph of the function f is shown on the right.

Which one of the following is most likely to be the graph of the inverse function of f?



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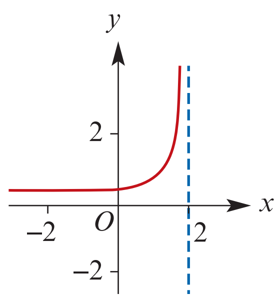

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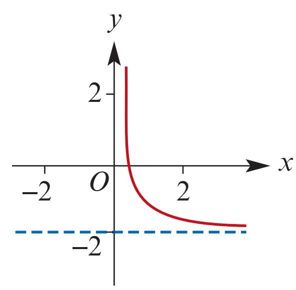

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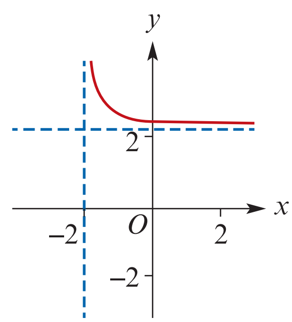

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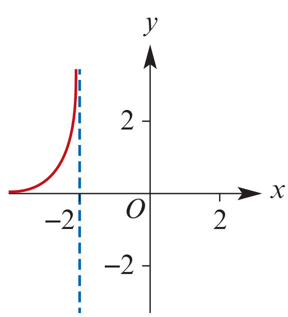

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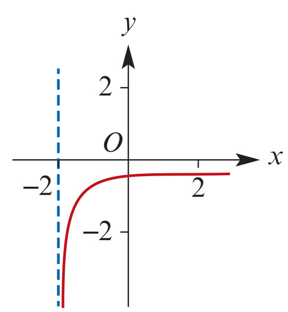

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Review

10 The maximal domain and range of $f(x) = \dfrac{2x+1}{x-1}$ are

A $\mathbb{R} \setminus \{0\}$, $\mathbb{R} \setminus \{2\}$ **B** $\mathbb{R} \setminus \{1\}$, $\mathbb{R} \setminus \{-2\}$ **C** $\mathbb{R} \setminus \{1\}$, $\mathbb{R} \setminus \{2\}$
D $\mathbb{R} \setminus \{2\}$, $\mathbb{R} \setminus \{1\}$ **E** $\mathbb{R} \setminus \{-2\}$, $\mathbb{R} \setminus \{-1\}$

11 If $f(x) = 3x^2$ and $g(x) = 2x + 1$, then $f(g(a))$ is equal to

A $12a^2 + 3$ **B** $12a^2 + 12a + 3$ **C** $6a^2 + 1$
D $6a^2 + 4$ **E** $4a^2 + 4a + 1$

12 The range of the function $f : [-2, 4) \to \mathbb{R}$, $f(x) = x^2 + 2x - 6$ is

A $\mathbb{R}$ **B** $(-3, 18]$ **C** $(-6, 18)$ **D** $[0, 6]$ **E** $[-7, 18)$

13 Which of the following functions is strictly increasing on the interval $(-\infty, -1]$?

A $f(x) = x^2$ **B** $f(x) = x^4$ **C** $f(x) = x^{\frac{1}{5}}$
D $f(x) = \sqrt{4 - x}$ **E** $f(x) = -x^3$

14 If $f : (-1, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x+1}$ and $g : (-\infty, 4] \to \mathbb{R}$, $g(x) = \sqrt{4-x}$, then the maximal domain of the function $f + g$ is

A $\mathbb{R}$ **B** $(-\infty, -1)$ **C** $(-1, 4]$ **D** $(-1, \infty)$ **E** $[-4, 1)$

15 If $f : (2, \infty) \to \mathbb{R}$, $f(x) = \sqrt{2x+3}$, then the inverse function is

A $f^{-1} : (\sqrt{7}, \infty) \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x^2 - 3}{2}$ **B** $f^{-1} : (7, \infty) \to \mathbb{R}$, $f^{-1}(x) = \sqrt{\dfrac{x}{2} - 3}$

C $f^{-1} : (\sqrt{7}, \infty) \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x^2 + 3}{2}$ **D** $f^{-1} : (7, \infty) \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x^2 - 3}{2}$

E $f^{-1} : (2, \infty) \to \mathbb{R}$, $f^{-1}(x) = \dfrac{x^2 - 2}{3}$

16 The linear function $f : D \to \mathbb{R}$, $f(x) = 5 - x$ has range $[-2, 3)$. The domain D is

A $[-7, 2)$ **B** $(2, 7]$ **C** $\mathbb{R}$ **D** $[-2, 7)$ **E** $[2, 7)$

17 The function $g : \mathbb{R} \setminus \{3\} \to \mathbb{R}$, where $g(x) = \dfrac{1}{x-3} + 2$, has an inverse g^{-1}. The rule and domain of g^{-1} are

A $g^{-1}(x) = \dfrac{1}{x-2} + 3$, $\operatorname{dom} g^{-1} = \mathbb{R} \setminus \{2\}$

B $g^{-1}(x) = \dfrac{1}{x-2} + 3$, $\operatorname{dom} g^{-1} = \mathbb{R} \setminus \{3\}$

C $g^{-1}(x) = \dfrac{1}{x+2} - 3$, $\operatorname{dom} g^{-1} = \mathbb{R} \setminus \{2\}$

D $g^{-1}(x) = \dfrac{-1}{x+2} - 3$, $\operatorname{dom} g^{-1} = \mathbb{R} \setminus \{3\}$

E $g^{-1}(x) = \dfrac{1}{x-2} + 3$, $\operatorname{dom} g^{-1} = \mathbb{R} \setminus \{-3\}$

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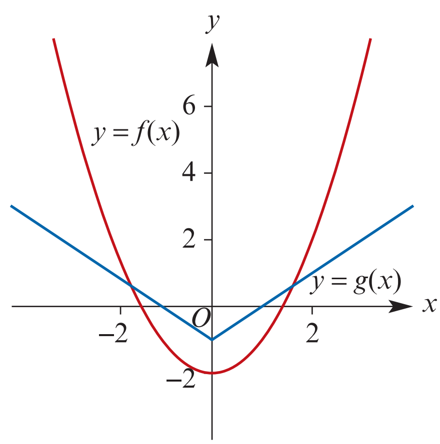
18 The graphs of $y = f(x)$ and $y = g(x)$ are as shown on the right.

Which one of the following best represents the graph of $y = f(g(x))$?



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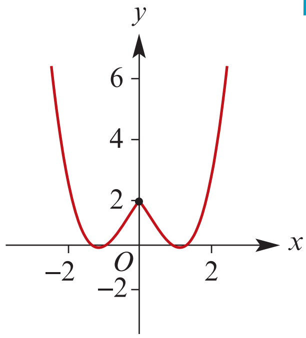

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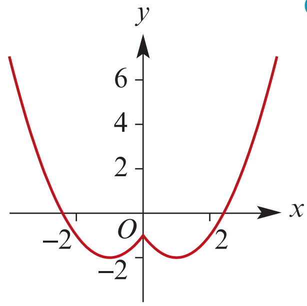

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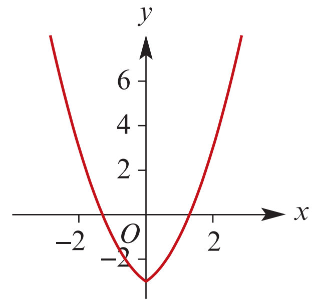

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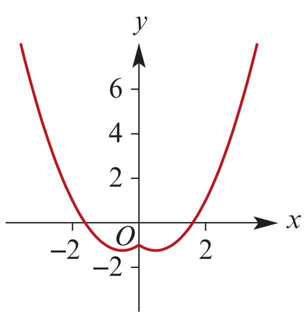

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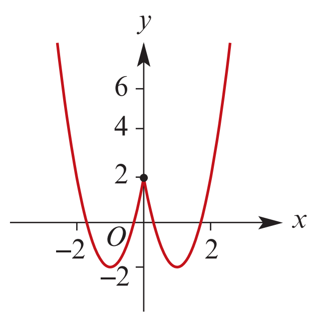

19 Let $g(x) = \dfrac{3}{(x+1)^3} - 2$. The equations of the asymptotes of the inverse function g^{-1} are

A $x = -2,\ y = 1$ **B** $x = -2,\ y = -1$ **C** $x = 1,\ y = -2$
D $x = -1,\ y = -2$ **E** $x = 2,\ y = -1$

20 The equations of the vertical and horizontal asymptotes of the graph with equation $y = \dfrac{-2}{(x+3)^4} - 5$ are

A $x = 3,\ y = -5$ **B** $x = -5,\ y = -3$ **C** $x = -3,\ y = -5$
D $x = -2,\ y = -5$ **E** $x = -3,\ y = 5$

21 Which one of the following functions does not have an inverse function?

A $f\colon [0, \infty) \to \mathbb{R},\ f(x) = (x-2)^2$ **B** $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = x^3$
C $f\colon [-3, 3] \to \mathbb{R},\ f(x) = \sqrt{9 - x}$ **D** $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = x^{\frac{1}{3}} + 4$
E $f\colon \mathbb{R} \to \mathbb{R},\ f(x) = 3x + 7$

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Review

22 A function with rule $f(x) = \frac{1}{x^4}$ can be defined on different domains. Which one of the following does not give the correct range for the given domain?

A $\text{dom } f = [-1, -0.5], \ \text{ran } f = [1, 16]$
B $\text{dom } f = [-0.5, 0.5] \setminus \{0\}, \ \text{ran } f = [16, \infty)$
C $\text{dom } f = (-0.5, 0.5) \setminus \{0\}, \ \text{ran } f = (16, \infty)$
D $\text{dom } f = [-0.5, 1] \setminus \{0\}, \ \text{ran } f = [1, 16]$
E $\text{dom } f = [0.5, 1), \ \text{ran } f = (1, 16]$

Extended-response questions

1 Self-Travel, a car rental firm, has two methods of charging for car rental:

Method 1 \$64 per day + 25 cents per kilometre
Method 2 \$89 per day with unlimited travel.

a Write a rule for each method if x kilometres per day are travelled and the cost in dollars is C_1 using method 1 and C_2 using method 2.
b Draw the graph of each, using the same axes.
c Determine, from the graph, the distance that must be travelled per day if method 2 is cheaper than method 1.

2 Express the total surface area, S, of a cube as a function of:

a the length x of an edge
b the volume V of the cube.

3 Express the area, A, of an equilateral triangle as a function of:

a the length s of each side
b the altitude h.

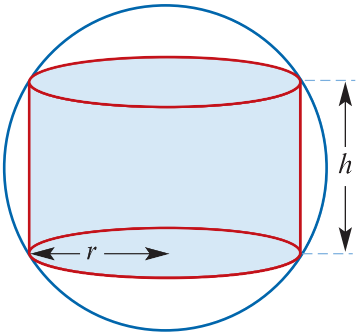
4 The base of a 3 m ladder leaning against a wall is x metres from the wall.

a Express the distance, d, from the top of the ladder to the ground as a function of x and sketch the graph of the function.
b State the domain and range of the function.

5 A car travels half the distance of a journey at an average speed of 80 km/h and half at an average speed of x km/h. Define a function, S, which gives the average speed for the total journey as a function of x.

6 A cylinder is inscribed in a sphere with a radius of length 6 cm.

a Define a function, V_1, which gives the volume of the cylinder as a function of its height, h. (State the rule and domain.)
b Define a function, V_2, which gives the volume of the cylinder as a function of the radius of the cylinder, r. (State the rule and domain.)



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7 Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, where $f(x) = x + 1$ and $g(x) = 2 + x^3$.

a State why $g \circ f$ exists and find $g \circ f(x)$.

b State why $(g \circ f)^{-1}$ exists and find $(g \circ f)^{-1}(10)$.

8 A function f is defined as follows:

$$f(x) = \begin{cases} x^2 - 4 & \text{for } x \in (-\infty, 2) \\ x & \text{for } x \in [2, \infty) \end{cases}$$

a Sketch the graph of f.

b Find the value of:

i $f(-1)$ **ii** $f(3)$

c Given $g : S \to \mathbb{R}$ where $g(x) = f(x)$, find the largest set S such that the inverse of g exists and $-1 \in S$.

d If $h(x) = 2x$, find $f(h(x))$ and $h(f(x))$.

9 Find the rule for the area, $A(t)$, enclosed by the graph of the function

$$f(x) = \begin{cases} 3x, & 0 \le x \le 1 \\ 3, & x > 1 \end{cases}$$

the x-axis, the y-axis and the vertical line $x = t$ (for $t > 0$). State the domain and range of the function A.

10 Let $f : \mathbb{R} \setminus \left\{\dfrac{-d}{c}\right\} \to \mathbb{R}$, $f(x) = \dfrac{ax + b}{cx + d}$.

a Find the inverse function f^{-1}.

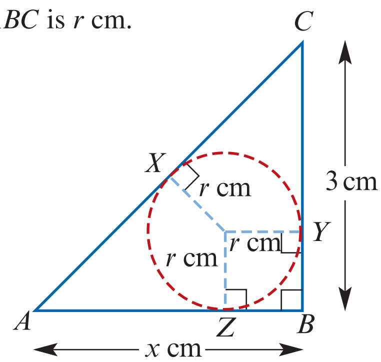
b Find the inverse function when:

i $a = 3,\ b = 2,\ c = 3,\ d = 1$ **ii** $a = 3,\ b = 2,\ c = 2,\ d = -3$

iii $a = 1,\ b = -1,\ c = -1,\ d = -1$ **iv** $a = -1,\ b = 1,\ c = 1,\ d = 1$

c Determine the possible values of a, b, c and d if $f = f^{-1}$.

11 The radius of the incircle of the right-angled triangle ABC is r cm.



a Find:

i YB in terms of r

ii ZB in terms of r

iii AZ in terms of r and x

iv CY

b Use the geometric results $CY = CX$ and $AX = AZ$ to find an expression for r in terms of x.

c **i** Find r when $x = 4$.

ii Find x when $r = 0.5$.

d Use a CAS calculator to investigate the possible values r can take.

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Review

12 Let $f(x) = \dfrac{px+q}{x+r}$ where $x \in \mathbb{R} \setminus \{-r, r\}$.

a If $f(x) = f(-x)$ for all x, show that $f(x) = p$ for $x \in \mathbb{R} \setminus \{-r, r\}$.

b If $f(-x) = -f(x)$ for $x \neq 0$, find the rule for $f(x)$ in terms of q.

c If $p = 3$, $q = 8$ and $r = -3$:

i find the inverse function of f

ii find the values of x for which $f(x) = x$.

13 **a** Let $f(x) = \dfrac{x+1}{x-1}$.

i Find $f(2)$, $f(f(2))$ and $f(f(f(2)))$.

ii Find $f(f(x))$.

b Let $f(x) = \dfrac{x-3}{x+1}$. Find $f(f(x))$ and $f(f(f(x)))$.

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