A tale of position and momenta

What we know from my thesis

In my thesis we take a soliton coherent state centered in zero $|\varphi(0)\rangle$ and we let it evolve. Suppose that the evolution of this coherent state is given by a rigid diffusion onto its Goldstone's modes:

$$|\Psi(t=0)\rangle = |\varphi(0)\rangle$$
$$|\Psi(t)\rangle = \int \frac{dz}{L} \exp(iKz)\exp(-iE_0 t)|\varphi(z)\rangle$$

where the momentum of the state is $K = \langle\varphi(0)|P|\varphi(0)\rangle = \langle\varphi(z)|P|\varphi(z)\rangle$ and $E_0 = \langle\varphi(0)|H|\varphi(0)\rangle = \langle\varphi(z)|H|\varphi(z)\rangle$ is it's energy. We introduce the energy averages

$$\left\langle\varphi\left(\frac{r}{2}\right)|H|\varphi\left(-\frac{r}{2}\right)\right\rangle \equiv E(r)\left\langle\varphi\left(\frac{r}{2}\right)|\varphi\left(-\frac{r}{2}\right)\right\rangle$$

For small enough times δt, we can approximate

$$|\Psi(\delta t)\rangle = (1 - i\delta t H)|\varphi(0)\rangle$$

Introducing the phase corrected overlap $\mathcal{O}(r) = e^{-iKr}\langle\varphi(r/2)|\varphi(-r/2)\rangle$, one can show that the Fourier transform $\tilde{f}(k,t) \equiv \frac{1}{\sqrt{L}}\int dr f(r,t)e^{ikr}$ must satisfy

$$\frac{1}{\sqrt{L}}\tilde{f}(k,\delta t) = 1 + iE_0\delta t - i\delta t\frac{\widetilde{\mathcal{O}E}(k)}{\tilde{\mathcal{O}}(k)}$$

Let α be the width of the packet. In a semi-classical fashion, we Taylor expand

$$E(r) = E_0\left(1 + ie_1\alpha r + e_2(\alpha r)^2\right)$$

$$\mathcal{O}(r) = \left(1 - i\mathcal{N}f_3\frac{(\alpha r)^3}{6}\right)\exp\left(-f_2\mathcal{N}\frac{(\alpha r)^2}{2}\right)$$

The Fourier transforms are given by

$$\alpha\frac{\sqrt{L}}{\sqrt{2}\pi}\tilde{\mathcal{O}}(k) = \frac{1}{\sqrt{f_2\mathcal{N}}}\exp\left(-\frac{1}{2f_2\mathcal{N}}\left(\frac{k}{\alpha}\right)^2\right)\left(1 + \frac{f_3}{2f_2^2\mathcal{N}}\frac{k}{\alpha} - \frac{f_3}{6f_2^3\mathcal{N}^2}\left(\frac{k}{\alpha}\right)^3\right)$$

Thanks to the exponential prefactor, the relevant momenta are of the order $k \approx \alpha\sqrt{\mathcal{N}f_2}$, this means that in the previous expansion, both terms are $O\left(1/\sqrt{\mathcal{N}}\right)$ and thus relevant. The product is given by

$$\alpha\frac{\sqrt{L}}{\sqrt{2}\pi}\widetilde{\mathcal{O}E}(k) = \frac{E_0}{\sqrt{f_2\mathcal{N}}}\exp\left(-\frac{1}{2f_2\mathcal{N}}\left(\frac{k}{\alpha}\right)^2\right)\left[\left(1 + e_1\frac{f_3}{2f_2^2\mathcal{N}} + \frac{e_2}{f_2\mathcal{N}}\right)\right.$$
$$\left. + \left(\frac{f_3}{2f_2^2\mathcal{N}} - \frac{e_1}{f_2\mathcal{N}} - \frac{5e_2f_3}{2f_2^3\mathcal{N}^2}\right)\left(\frac{k}{\alpha}\right) + \left(\frac{e_1f_3}{f_2^3\mathcal{N}^2} + \frac{e_2}{f_2^2\mathcal{N}^2}\right)\left(\frac{k}{\alpha}\right)^2 + \#\left(\frac{k}{\alpha}\right)^3 + \#\left(\frac{k}{\alpha}\right)^4\right]$$

Keeping the leading terms in $O(\mathcal{N})$ and up to k^2:

$$\alpha\frac{\sqrt{L}}{\sqrt{2}\pi}\widetilde{\mathcal{O}E}(k) = \frac{E_0}{\sqrt{f_2\mathcal{N}}}\exp\left(-\frac{1}{2f_2\mathcal{N}}\left(\frac{k}{\alpha}\right)^2\right)\left[1 + \left(\frac{f_3}{2f_2^2\mathcal{N}} - \frac{e_1}{f_2\mathcal{N}}\right)\left(\frac{k}{\alpha}\right) + \left(\frac{e_1f_3}{f_2^3\mathcal{N}^2} + \frac{e_2}{f_2^2\mathcal{N}^2}\right)\left(\frac{k}{\alpha}\right)^2\right]$$

We finally get

$$\frac{\widetilde{\mathcal{O}E}(k)}{\tilde{\mathcal{O}}(k)} = E_0\left[1 - \frac{e_1}{f_2\mathcal{N}}\left(\frac{k}{\alpha}\right) + \left(\frac{e_1 f_3}{f_2^3\mathcal{N}^2} + \frac{e_2}{f_2^2\mathcal{N}^2}\right)\left(\frac{k}{\alpha}\right)^2\right]$$

In my thesis we showed that the velocity of the soliton is indeed given by

$$v_{sol} = \frac{e_1}{f_2\mathcal{N}\alpha}$$

From the previous expression we deduce the effective mass

$$\frac{1}{2m} = -E_0\left[\frac{e_1 f_3}{f_2^3\mathcal{N}^2} + \frac{e_2}{f_2^2}\mathcal{N}^2\right]$$

To sum up we discovered that we performed a semiclassical expansion of the following averages

$$\langle\varphi(r/2)|H|\varphi(-r/2)\rangle \equiv (E_0 + iE_1 r + E_2 r^2 + ...)\langle\phi(r/2)|\phi(-r/2)\rangle$$

$$\langle\phi(r/2)|\phi(-r/2)\rangle = e^{iKr}\left(1 - i\mathcal{N}f_3\frac{(\alpha r)^3}{6} + ...\right)\exp\left(-f_2\mathcal{N}\frac{(\alpha r)^2}{2}\right)$$

and from this expansion we discovered that the effective mass is given by

$$\frac{1}{2m} = -\left(\frac{E_1 f_3}{\alpha f_2^3\mathcal{N}^2} + \frac{E_2}{\alpha^2}f_2^2\mathcal{N}^2\right)$$

Band dispersion

We consider a soliton coherent state with width α and center in r:

$$|\varphi_\alpha(r)\rangle$$

Alberto discovered numerically that these states live on the upper branch of the theory. The eigenstates of the upper branch $H|k\rangle = \omega(k)|k\rangle$ are normalized such that $\langle k|k'|k|k'\rangle = \delta(k - k')$. The energy dispersion of the upper branch is given by $\omega(k)$. We start by consider the soliton coherent state center in $r = 0$:

$$|\phi_\alpha(0)\rangle = \int dk u_\alpha(k)|k\rangle$$

The displaced soliton is given by

$$|\phi_\alpha(r)\rangle = e^{-iPr}|\phi_\alpha(0)\rangle = \int dk u_\alpha(k)e^{-ikr}|k\rangle$$

This band expression is clearly related to the symmetry breaking ansatz:

$$\int dr e^{iKr}|\phi_\alpha(r)\rangle = \int dk u_\alpha(k)|k\rangle \int dr e^{-i(k-K)r} = 2\pi u_\alpha(K)|K\rangle \propto |K\rangle$$

Our goal is to better understand the properties of this function $u_{\alpha(k)}$ and relate it to what I did in my thesis. Since $\langle\varphi_\alpha(0)|\ \varphi_\alpha(0)\rangle = 1$, we know that

$$\int dk\ |u_\alpha(k)|^2 = 1$$

The overlap between two displaced solitons is given by

$$O(r) = \langle(\phi_\alpha(r/2)|\phi_\alpha(-r/2))\rangle = \int dk \int dk' u_\alpha^*(k') u_\alpha(k) e^{ik'r/2+ikr/2} \langle\rangle(k|k') = \int dk |u_\alpha(k)|^2 \ e^{ikr}$$

This means that we can find the function $|u_\alpha(k)|^2$ by Fourier transforming $O(r)$:

$$|u_\alpha(k)|^2 = \frac{1}{2\pi} \int dr O(r) e^{-ikr}$$

Let $K = \langle\varphi_\alpha(r)|P|\varphi_\alpha(r)\rangle$ the momentum of the soliton. For the KdV soliton the overlap $O(r)$ is given by a Gaussian + a small quantum correction:

$$O(r) = e^{iKr} \left(1 - i\mathcal{N} f_3 \frac{(\alpha r)^3}{6}\right) \exp\left(-f_2 \mathcal{N} \frac{(\alpha r)^2}{2}\right)$$

Thus

$$|u_\alpha(k)|^2 = d_\alpha(k - K)$$

$$d_\alpha(k) = \frac{1}{\sqrt{2\pi\mathcal{N} f_2}} \exp\left(-\frac{1}{2 f_2 \mathcal{N}} \left(\frac{k}{\alpha}\right)^2\right) \left(1 - \frac{f_3}{2 f_2^2 \mathcal{N}} \frac{k}{\alpha} + \frac{f_3}{6 f_2^3 \mathcal{N}^2} \left(\frac{k}{\alpha}\right)^3\right)$$

Since the exponential kills momenta bigger than $\alpha\sqrt{f_2\mathcal{N}}$, both quantum corrections are relevant and of order $O\left(\sqrt{N}\right)$. This means that $|u_\alpha(k)|^2$ is also a gaussian, centered in K, with some small quantum correction. Notice that the broadening of this gaussian is due to the fact that the soliton coherent state is not an exact eigenstate of the momenta. The momenta of this distribution are related to the following averages:

$$\langle\phi_\alpha(r)|(P - K)^n \ |\phi_\alpha(r)\rangle = \int dk (k - K)^n \ |u_\alpha(k)| = \int dk k^n d_\alpha(k)$$

Also notice that the Fourier transform

$$\int dk d_\alpha(k) e^{ikr} = \int dk |u_\alpha(k + K)|^2 \ e^{ikr} = e^{-iKr} \int dk |u_\alpha(k)|^2 \ e^{ikr} = e^{-iKr} O(r) = \mathcal{O}(r)$$

corresponds to the phase corrected overlap $\mathcal{O}(r)$.

We now consider the energy overlap

$$\langle\phi_\alpha(r/2)|H|\phi_\alpha(-r/2)\rangle = \int dk \omega(k) |u_\alpha(k)|^2 \ e^{ikr}$$

The idea is now to Taylor the upper band energy dispersion to second order. Since the width of the $|u_\alpha(k)|^2$ is $\alpha\sqrt{f_2\mathcal{N}}$, this Taylor approximation is valid if

$$\omega'''(K)\left(\alpha\sqrt{f_2\mathcal{N}}\right)^3 \ll \omega''(K)\left(\alpha\sqrt{f_2\mathcal{N}}\right)^2$$

Since $\omega(K) = cK^{\frac{5}{3}}$, the last inequality translates to

$$\frac{K}{\alpha\sqrt{\mathcal{N}}} \propto \frac{\alpha}{\gamma} \gg 1$$

which coincides with the semi-classical limit. Expanding the band dispersion around K we get

$$\omega(k) = \omega(K) + \omega'(K)(k - K) + \omega''(K)\frac{(k - K)^2}{2}$$

This means that

$$\langle\phi_\alpha(r/2)|H|\phi_\alpha(-r/2)\rangle = \int dk\left[\omega(K)+\omega'(K)(k-K)+\omega''(K)\frac{(k-K)^2}{2}\right]|u_\alpha(k)|^2\ e^{ikr}$$

$$= e^{iKr}\int dk\left[\omega(K)+\omega'(K)k+\omega''(K)\frac{k^2}{2}\right]d_\alpha(k)e^{ikr}$$

$$= O(r) - ie^{iKr}\mathcal{O}'(r) - \frac{e^{iKr}}{2}\mathcal{O}''(r)$$

Up to leading order

$$E(r) \equiv \frac{\langle\phi_\alpha(r/2)|H|\phi_\alpha(-r/2)\rangle}{O(r)} = \omega(K) + if_2\mathcal{N}(\omega'(K)\alpha)(\alpha r) - \left[\frac{f_3\mathcal{N}(\omega'(K)\alpha) + f_2^2\mathcal{N}^2(\omega''(K)\alpha^2)}{2}\right](\alpha r)^2$$

In my work we used the following expansion for the energy averages

$$E(r) = E_0 + iE_1 r + E_2 r^2 + \ldots$$

this means that for a coherent state living on the upper branch (with a dispersion $\omega(K)$) the coefficients are given by

$$E_0 = \omega(K)$$
$$E_1 = f_2\mathcal{N}\omega'(K)\alpha$$
$$E_2 = -\left[\frac{f_3\mathcal{N}(\omega'(K)\alpha) + f_2^2\mathcal{N}^2(\omega''(K)\alpha^2)}{2}\right]$$

Evolving the state, we also found its effective mass

$$\frac{1}{2m} = -\left(\frac{E_1 f_3}{\alpha f_2^3\mathcal{N}^2} + \frac{E_2}{\alpha^2}f_2^2\mathcal{N}^2\right)$$

Substituting the previous expression we get that

$$\frac{1}{m} = \omega''(K)$$

which is indeed consistent with the fact that the status lives on the upper branch. Keeping just the classical part (a simple power law) in the band dispersion:

$$\omega(K) \approx E_0(K) \equiv cK^{5/3}$$
$$\omega'(K) = \frac{5}{3}\frac{E_0(K)}{K}$$
$$\omega''(K) = \frac{10}{9}\frac{E_0(K)}{K^2}$$

Finally, using that $K = \mathcal{N}f_1\alpha$ we get

$$E(r) = E_0(K)\left[1 + i\frac{5f_2}{3f_1}(\alpha r) - \left(\frac{5f_3}{6f_1} + \frac{5f_2^2}{9f_1^2}\right)(\alpha r)^2\right]$$

In my thesis we expanded the energy average as

$$E(r) = E_0(K)[1 + ie_1(\alpha r) + e_2(\alpha r)^2]$$

We already knew that $e_1 = \frac{5f_2}{3f_1}$, but we were able to find e_2 only numerically. Now we showed that:

$$e_2 = -\left(\frac{5f_3}{6f_1} + \frac{5f_2^2}{9f_1^2}\right)$$