FROM "Numerical Methods for Engineers"
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CHAPTER 29

Finite Difference: Elliptic Equations

Elliptic equations in engineering are typically used to characterize steady-state, boundary-value problems. Before demonstrating how they can be solved, we will illustrate how a simple case—the Laplace equation—is derived from a physical problem context.

29.1 THE LAPLACE EQUATION

As mentioned in the introduction to this part of the book, the Laplace equation can be used to model a variety of problems involving the potential of an unknown variable. Because of its simplicity and general relevance to most areas of engineering, we will use a heated plate as our fundamental context for deriving and solving this elliptic PDE. Homework problems and engineering applications (Chap. 32) will be employed to illustrate the applicability of the model to other engineering problem contexts.

Figure 29.1 shows an element on the face of a thin rectangular plate of thickness Δz. The plate is insulated everywhere but at its edges, where the temperature can be set at a prescribed level. The insulation and the thinness of the plate mean that heat transfer is limited to the x and y dimensions. At steady state, the flow of heat into the element over a unit time period Δt must equal the flow out, as in

$$q(x)\,\Delta y\,\Delta z\,\Delta t + q(y)\,\Delta x\,\Delta z\,\Delta t = q(x+\Delta x)\,\Delta y\,\Delta z\,\Delta t + q(y+\Delta y)\Delta x\,\Delta z\,\Delta t \tag{29.1}$$

where $q(x)$ and $q(y)$ = the heat fluxes at x and y, respectively [cal/(cm$^2\cdot$s)]. Dividing by Δz and Δt and collecting terms yields

$$[q(x) - q(x+\Delta x)]\Delta y + [q(y) - q(y+\Delta y)]\Delta x = 0$$

Multiplying the first term by $\Delta x/\Delta x$ and the second by $\Delta y/\Delta y$ gives

$$\frac{q(x) - q(x+\Delta x)}{\Delta x}\,\Delta x\,\Delta y + \frac{q(y) - q(y+\Delta y)}{\Delta y}\,\Delta y\,\Delta x = 0 \tag{29.2}$$

Dividing by $\Delta x\,\Delta y$ and taking the limit results in

$$-\frac{\partial q}{\partial x} - \frac{\partial q}{\partial y} = 0 \tag{29.3}$$

where the partial derivatives result from the definitions in Eqs. (PT7.1) and (PT7.2).

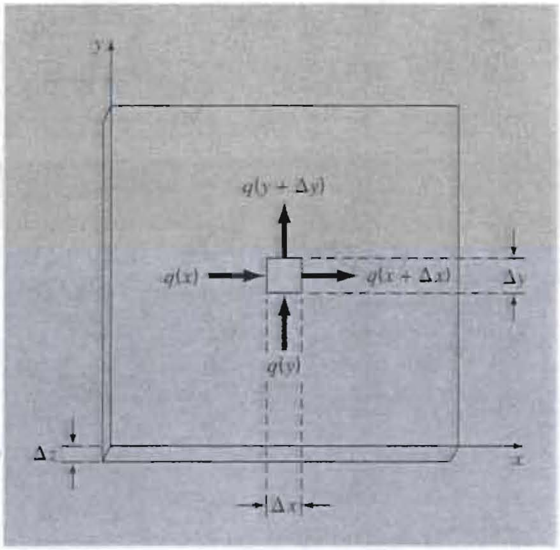

FIGURE 29.1
A thin plate of thickness Δz. An element is shown about which a heat balance is taken.

Equation (29.3) is a partial differential equation that is an expression of the conservation of energy for the plate. However, unless heat fluxes are specified at the plate's edges, it cannot be solved. Because temperature boundary conditions are given, Eq. (29.3) must be reformulated in terms of temperature. The link between flux and temperature is provided by *Fourier's law of heat conduction*, which can be represented as

$$q_i = -k\rho C \frac{\partial T}{\partial i} \tag{29.4}$$

where q_i = heat flux in the direction of the i dimension [cal/(cm$^2 \cdot$ s)], k = coefficient of *thermal diffusivity* (cm^2/s), ρ = density of the material (g/cm^3), C = heat capacity of the material [cal/(g · °C)], and T = temperature (°C), which is defined as

$$T = \frac{H}{\rho C V}$$

where H = heat (cal) and V = volume (cm^3). Sometimes the term in front of the differential in Eq. (29.3) is treated as a single term,

$$k' = k\rho C \tag{29.5}$$

where k' is referred to as the *coefficient of thermal conductivity* [cal/(s · cm · °C)]. In either case, both k and k' are parameters that reflect how well the material conducts heat.

Fourier's law is sometimes referred to as a *constitutive equation*. It is given this label because it provides a mechanism that defines the system's internal interactions. Inspection

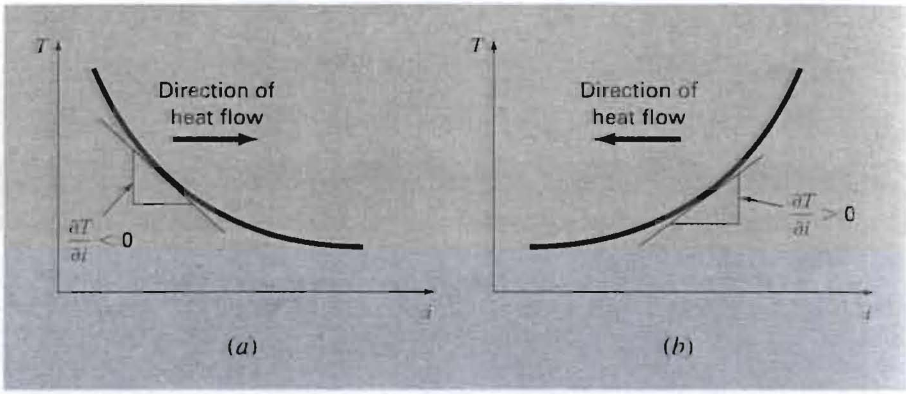

FIGURE 29.2
Graphical depiction of a temperature gradient. Because heat moves "downhill" from high to low temperature, the flow in (a) is from left to right in the positive i direction. However, due to the orientation of Cartesian coordinates, the slope is negative for this case. Thus, a negative gradient leads to a positive flow. This is the origin of the minus sign in Fourier's law of heat conduction. The reverse case is depicted in (b), where the positive gradient leads to a negative heat flow from right to left.

of Eq. (29.4) indicates that Fourier's law specifies that heat flux perpendicular to the i axis is proportional to the gradient or slope of temperature in the i direction. The negative sign ensures that a positive flux in the direction of i results from a negative slope from high to low temperature (Fig. 29.2). Substituting Eq. (29.4) into Eq. (29.3) results in

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{29.6}$$

which is the *Laplace equation.* Note that for the case where there are sources or sinks of heat within the two-dimensional domain, the equation can be represented as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y) \tag{29.7}$$

where $f(x, y)$ is a function describing the sources or sinks of heat. Equation (29.7) is referred to as the *Poisson equation.*

29.2 SOLUTION TECHNIQUE

The numerical solution of elliptic PDEs such as the Laplace equation proceeds in the reverse manner of the derivation of Eq. (29.6) from the preceding section. Recall that the derivation of Eq. (29.6) employed a balance around a discrete element to yield an algebraic difference equation characterizing heat flux for a plate. Taking the limit turned this difference equation into a differential equation [Eq. (29.3)].

For the numerical solution, finite-difference representations based on treating the plate as a grid of discrete points (Fig. 29.3) are substituted for the partial derivatives in Eq. (29.6). As described next, the PDE is transformed into an algebraic difference equation.

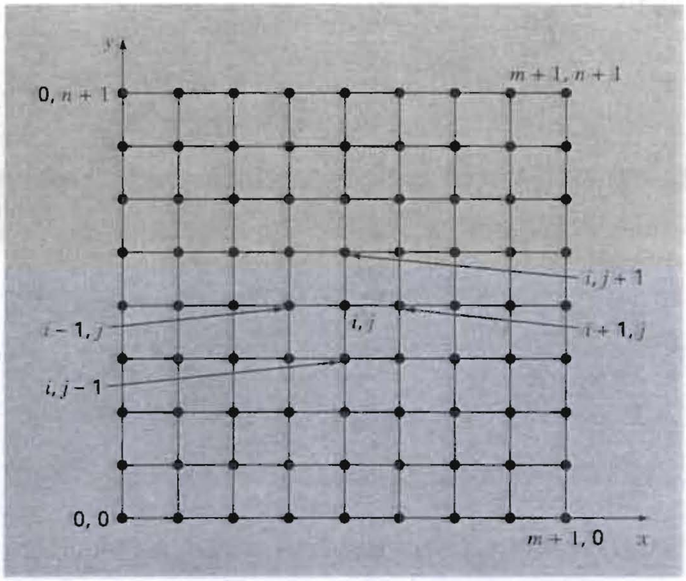

FIGURE 29.3
A grid used for the finite-difference solution of elliptic PDEs in two independent variables such as the Laplace equation.

29.2.1 The Laplacian Difference Equation

Central differences based on the grid scheme from Fig. 29.3 are (recall Fig. 23.3)

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2}$$

and

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2}$$

which have errors of $O[\Delta(x)^2]$ and $O[\Delta(y)^2]$, respectively. Substituting these expressions into Eq. (29.6) gives

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} = 0$$

For the square grid in Fig. 29.3, $\Delta x = \Delta y$, and by collection of terms, the equation becomes

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0 \tag{29.8}$$

This relationship, which holds for all interior points on the plate, is referred to as the *Laplacian difference equation.*

In addition, boundary conditions along the edges of the plate must be specified to obtain a unique solution. The simplest case is where the temperature at the boundary is set at a fixed value. This is called a *Dirichlet boundary condition.* Such is the case for Fig. 29.4,

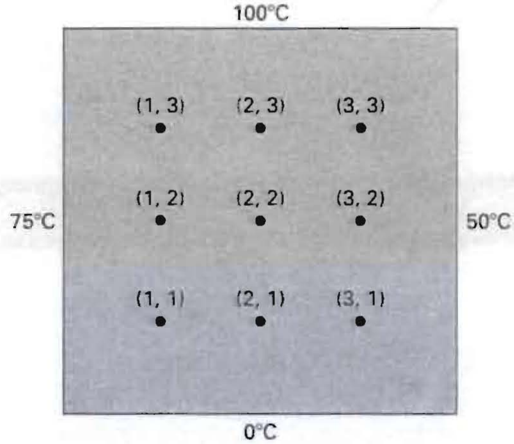

FIGURE 29.4
A heated plate where boundary temperatures are held at constant levels. This case is called a Dirichlet boundary condition.

where the edges are held at constant temperatures. For the case illustrated in Fig. 29.4, a balance for node (1, 1) is, according to Eq. (29.8),

$$T_{21} + T_{01} + T_{12} + T_{10} - 4T_{11} = 0 \tag{29.9}$$

However, $T_{01} = 75$ and $T_{10} = 0$, and therefore, Eq. (29.9) can be expressed as

$$-4T_{11} + T_{12} + T_{21} = -75$$

Similar equations can be developed for the other interior points. The result is the following set of nine simultaneous equations with nine unknowns:

$$\begin{array}{cccccccccl}
4T_{11} & -T_{21} & & -T_{12} & & & & & & = 75 \\
-T_{11} & +4T_{21} & -T_{31} & & -T_{22} & & & & & = 0 \\
& -T_{21} & +4T_{31} & & & -T_{32} & & & & = 50 \\
-T_{11} & & & +4T_{12} & -T_{22} & & -T_{13} & & & = 75 \\
& -T_{21} & & -T_{12} & +4T_{22} & -T_{32} & & -T_{23} & & = 0 \\
& & -T_{31} & & -T_{22} & +4T_{32} & & & -T_{33} & = 50 \\
& & & -T_{12} & & & +4T_{13} & -T_{23} & & = 175 \\
& & & & -T_{22} & & -T_{13} & +4T_{23} & -T_{33} & = 100 \\
& & & & & -T_{32} & & -T_{23} & +4T_{33} & = 150
\end{array} \tag{29.10}$$

29.2.2 The Liebmann Method

Most numerical solutions of the Laplace equation involve systems that are much larger than Eq. (29.10). For example, a 10-by-10 grid involves 100 linear algebraic equations. Solution techniques for these types of equations were discussed in Part Three.

Notice that there are a maximum of five unknown terms per line in Eq. (29.10). For larger-sized grids, this means that a significant number of the terms will be zero. When applied to such sparse systems, full-matrix elimination methods waste great amounts of computer memory storing these zeros. For this reason, approximate methods provide a viable approach for obtaining solutions for elliptical equations. The most commonly employed approach is *Gauss-Seidel,* which when applied to PDEs is also referred to as *Liebmann's method.* In this technique, Eq. (29.8) is expressed as

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4} \tag{29.11}$$

and solved iteratively for $j = 1$ to n and $i = 1$ to m. Because Eq. (29.8) is diagonally dominant, this procedure will eventually converge on a stable solution (recall Sec. 11.2.1). Overrelaxation is sometimes employed to accelerate the rate of convergence by applying the following formula after each iteration:

$$T_{i,j}^{\text{new}} = \lambda T_{i,j}^{\text{new}} + (1 - \lambda) T_{i,j}^{\text{old}} \tag{29.12}$$

where $T_{i,j}^{\text{new}}$ and $T_{i,j}^{\text{old}}$ are the values of $T_{i,j}$ from the present and the previous iteration, respectively, and λ is a weighting factor that is set between 1 and 2.

As with the conventional Gauss-Seidel method, the iterations are repeated until the absolute values of all the percent relative errors $(\varepsilon_a)_{i,j}$ fall below a prespecified stopping criterion ε_s. These percent relative errors are estimated by

$$|(\varepsilon_a)_{i,j}| = \left| \frac{T_{i,j}^{\text{new}} - T_{i,j}^{\text{old}}}{T_{i,j}^{\text{new}}} \right| 100\% \tag{29.13}$$

EXAMPLE 29.1 Temperature of a Heated Plate with Fixed Boundary Conditions

Problem Statement. Use Liebmann's method (Gauss-Seidel) to solve for the temperature of the heated plate in Fig. 29.4. Employ overrelaxation with a value of 1.5 for the weighting factor and iterate to $\varepsilon_s = 1\%$.

Solution. Equation (29.11) at $i = 1, j = 1$ is

$$T_{11} = \frac{0 + 75 + 0 + 0}{4} = 18.75$$

and applying overrelaxation yields

$$T_{11} = 1.5(18.75) + (1 - 1.5)0 = 28.125$$

For $i = 2, j = 1$,

$$T_{21} = \frac{0 + 28.125 + 0 + 0}{4} = 7.03125$$

$$T_{21} = 1.5(7.03125) + (1 - 1.5)0 = 10.54688$$

For $i = 3, j = 1$,

$$T_{31} = \frac{50 + 10.54688 + 0 + 0}{4} = 15.13672$$

$$T_{31} = 1.5(15.13672) + (1 - 1.5)0 = 22.70508$$

The computation is repeated for the other rows to give

$$T_{12} = 38.67188 \qquad T_{22} = 18.45703 \qquad T_{32} = 34.18579$$
$$T_{13} = 80.12696 \qquad T_{23} = 74.46900 \qquad T_{33} = 96.99554$$

Because all the $T_{i,j}$'s are initially zero, all ε_a's for the first iteration will be 100%.

For the second iteration the results are

$$T_{11} = 32.51953 \qquad T_{21} = 22.35718 \qquad T_{31} = 28.60108$$
$$T_{12} = 57.95288 \qquad T_{22} = 61.63333 \qquad T_{32} = 71.86833$$
$$T_{13} = 75.21973 \qquad T_{23} = 87.95872 \qquad T_{33} = 67.68736$$

The error for $T_{1,1}$ can be estimated as [Eq. (29.13)]

$$|(\varepsilon_a)_{1,1}| = \left|\frac{32.51953 - 28.12500}{32.51953}\right| 100\% = 13.5\%$$

Because this value is above the stopping criterion of 1%, the computation is continued. The ninth iteration gives the result

$$T_{11} = 43.00061 \qquad T_{21} = 33.29755 \qquad T_{31} = 33.88506$$
$$T_{12} = 63.21152 \qquad T_{22} = 56.11238 \qquad T_{32} = 52.33999$$
$$T_{13} = 78.58718 \qquad T_{23} = 76.06402 \qquad T_{33} = 69.71050$$

where the maximum error is 0.71%.

Figure 29.5 shows the results. As expected, a gradient is established as heat flows from high to low temperatures.

FIGURE 29.5
Temperature distribution for a heated plate subject to fixed boundary conditions.