# *Statistical Hotelling*

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**Abstract:** *This work aims at studying how privacy protection can have an impact on market welfare. It focuses on horizontal product differenciation through the Hotelling model, and investigates the market dynamics when a monopolist and a regulator face decisions under statistical uncertainty.*

## 1 Introduction

### 1.1 Privacy on markets

### 1.2 Horizontal product differenciation

### 1.3 Uncertainty

In this work, we study the dynamics of a monopolistic horizontally differentiated good market under uncertainty. That is, we study the incentives and possible regulations of a monopolist selling a good to consumers with heterogeneous preferences, when the distribution of these preferences is unknown but can be estimated through a finite number of samples.

**Objective.** *test*

**Idea.** *test*

## 2 Notations and Model

Let us consider the classical Hotelling model with a single firm. We consider a population of consumers with heterogeneous preferences [1] for a differentiated product $\theta \in [0,1]$ distributed according to measure $\mu \in \mathcal{P}([0,1])$ for a good. The consumers' valuation for the good is noted $v \in \mathbb{R}$. The products have a type on the same space as the consumers' preferences, so that the firms' product's type is also noted $\theta_f \in [0,1]$.

[1] We choose a quadratic cost for simplicity, as it makes variances appear in the latter calculations.

**Definition 2.1** (Utility). *The utility of a consumer of type $\theta$ for a product of type $\theta_f$ and price $p$ is given by :*

$$u_\theta(\theta_f, p) = \left(v - p - (\theta - \theta_f)^2\right)_+$$

**Remark 1** (Tiebreaks)**.** *The utility of not buying the product is normalized to* $0$*. We consider that tiebreaks are unfavorable, that is, a consumer with* $u_\theta(\theta_f, p) = 0$ *does not buy the product.*

**Definition 2.2** (Demand)**.** $D_\mu(\hat{\theta}, p) = \int_{\theta \in [0,1]} \mathbf{1}_{u_\theta(\hat{\theta}, p) \geq 0} \mu(d\theta)$

We assume a marginal cost of $c \in \mathbb{R}$. The profit of the firm is given by :

**Definition 2.3** (Profit)**.** $\pi_\mu(\hat{\theta}, p) = (p - c) \cdot D_\mu(\hat{\theta}, p)$

**Definition 2.4** (Consumer surplus)**.** *The consumer surplus is given by the net utility of the consumers :*

$$CS_\mu(\theta_f, p) = \int_{\theta \in [0,1]} u_\theta(\theta_f, p) \mu(d\theta)$$

**Definition 2.5** (Total welfare)**.** *The total welfare (with social preference parameter* $\alpha \in [0, 1]$*) is given by the weighted sum of the firm's profit and the consumer surplus :*

$$\mathcal{W}^\alpha_\mu(\theta_f, p) = CS_\mu(\theta_f, p) + \alpha \pi_\mu(\hat{\theta}, p) = \int_{\theta_f - \sqrt{v-p}}^{\theta_f + \sqrt{v-p}} (v - p - (\theta - \theta_f)^2 + \alpha(p - c)) \mu(d\theta)$$

Let us consider that the firm observes $N$ samples from the consumer distribution. We will try to derive how the samples can be used to derive a location and pricing strategy.

**Definition 2.6.** *Let* $\mathcal{F}(\overline{\theta}, \sigma^2)$ *be the set of all distributions on* $[0, 1]$ *with mean* $\overline{\theta}$ *and variance* $\sigma^2$*.*

# 3 Results

## 3.1 Known variety of tastes

We first consider that the firm knows the variance of the distribution of consumer tastes $\sigma^2$. As we chose a distribution-free approach, the metric for welfare will be the worst-case welfare over all distributions in $\mathcal{F}(\overline{\theta}, \sigma^2)$.

**Proposition 3.1** (Worst case welfare for a given price)**.** *Let* $d = |\hat{\theta} - \overline{\theta}|$ *be the distance between the true mean and the empirical one. Let* $p \in \mathbb{R}$ *be so that* $2\sqrt{v - p} < 1$ *so that all consumers are not served in the worst case. Then, when the firm chooses* $\theta_f = \hat{\theta}$ *and a price* $p$*, its worst possible welfare is :*

$$\boxed{\mathcal{W}^{\alpha}_{min}(\sigma, d, p) = \min_{\substack{\overline{\theta}\in[0,1] \\ \mu\in\mathcal{F}(\overline{\theta},\sigma^2) \\ |\hat{\theta}-\overline{\theta}|\leq d}} \mathcal{W}^{\alpha}_{\mu}(\hat{\theta}, p) = \left(v - p - d^2 + \alpha(p-c)\right)\left(1 - \frac{\sigma^2}{v-p-d^2}\right)}$$

*Proof.* Deferred to Appendix A.1. □

**Corollary 3.1.** *The minimal welfare is positive if and only if* $p \leq v - \sigma^2 - d^2$.

Let us try to derive a model for the firm's price choice using its samples. We consider that the firm will maximise its profit in the worst case over the consumer distributions.

**Proposition 3.2** (Worst case demand for a given price). *Let* $p \in \mathbb{R}$ *be so that* $2\sqrt{v-p} < 1$ *so that all consumers are not served in the worst case. Let* $d = |\hat{\theta} - \overline{\theta}|$ *be the distance between the true mean and the empirical one. Then, when the firm chooses* $\theta_f = \hat{\theta}$ *and* $p$ *its worst possible demand is :*

$$\boxed{D_{min}(\sigma, d, p) = \min_{\substack{\overline{\theta}\in[0,1] \\ \mu\in\mathcal{F}(\overline{\theta},\sigma^2) \\ |\hat{\theta}-\overline{\theta}|\leq d}} D_{\mu}(\hat{\theta}, p) = 1 - \frac{\sigma^2 + d^2}{v-p}}$$

*Proof.* Deferred to Appendix A.2. □

**Remark 2.** *Although the expressions of the worst case demand and welfare are different, they are positive for the same parameters* $(v, c, \sigma, d)$*, which is very natural.*

**Proposition 3.3** (Worst case profit and associated optimal price). *Let* $d = |\hat{\theta} - \theta|$ *be the distance between the true mean and the empirical one. Then, when the firm chooses* $\theta_f = \hat{\theta}$*, its maximum profit in the worst case is :*

$$\boxed{\pi_{min}(\sigma, d) = \max_{p\in\mathbb{R}} \min_{\substack{\overline{\theta}\in[0,1] \\ \mu\in\mathcal{F}(\overline{\theta},\sigma^2) \\ |\hat{\theta}-\overline{\theta}|\leq d}} \pi_{\mu}(\hat{\theta}, p) = (p^{\star}(\sigma, d) - c)\left(1 - \frac{\sigma^2 + d^2}{v - p^{\star}(\sigma, d)}\right)}$$

*with :*

$$p^{\star}(\sigma, d) = v - \sqrt{(\sigma^2 + d^2)(v-c)}$$

*Proof.* Let us fix $\sigma$ and $d$. We can now study the variations of

$$\Pi : p \mapsto (p-c)D_{\min}(\sigma, d, p) = (p-c)\left(1 - \frac{\sigma^2 + d^2}{v-p}\right)$$

We compute $\frac{d\Pi}{dp} = 1 - \frac{(v-c)(\sigma^2+d^2)}{(v-p)^2}$. We see that $\Pi$ is increasing for $p < p^{\star}(\sigma, d)$ and decreasing for $p > p^\star(\sigma, d)$, thus $p^\star(\sigma, d)$ is the unique maximizer of $\Pi$, which concludes the proof. $\square$

**Corollary 3.2.** *The firm can ensure profit if and only if $c \leq v - \sigma^2 - d^2$.*

*Proof.* The proof follows directly from ensuring $p^\star(\sigma, d) \geq c$. $\square$

Now, we can analyse the welfare's trajectory as the firm gathers data and its error on the mean decreases.

**Definition 3.1.** *We consider the surplus in the worst case when the firm maximises its profit in the worst case, that is :*

$$\begin{aligned}
\mathcal{S}^\alpha : d^2 &\mapsto \mathcal{W}^\alpha_{min}(\sigma, d, p^\star(\sigma, d)) \\
&\mapsto \Big(v - p^\star(\sigma, d) - d^2 + \alpha(p^\star(\sigma, d) - c)\Big)\left(1 - \frac{\sigma^2}{v - p^\star(\sigma, d) - d^2}\right) \\
&\mapsto \left(\sqrt{(\sigma^2 + d^2)(v - c)} - d^2 + \alpha\big(v - c - \sqrt{(\sigma^2 + d^2)(v - c)}\big)\right)\left(1 - \frac{\sigma^2}{\sqrt{(\sigma^2 + d^2)(v - c)} - d^2}\right)
\end{aligned}$$

### *Local behavior of $S^\alpha(d^2)$ around $d^2 = 0$*

We investigate whether the welfare function $S^\alpha(d^2)$ increases or decreases around $d^2 = 0$, given parameters $(v, c, \sigma, \alpha)$, where $\sigma$ is the belief variance, $v - c > 0$ the net valuation, and $\alpha \in [0, 1]$ the interpolation parameter.

Expanding $S^\alpha(d^2)$ for small $d^2$ yields: $\frac{\partial S^\alpha}{\partial(d^2)}(0) = \frac{\frac{v-c}{2} - (1+\alpha)\sigma\sqrt{v-c} + \alpha\sigma^2}{\sigma\sqrt{v-c}}$, whose denominator is positive. Hence the sign of the derivative is governed by $N(v, c, \sigma, \alpha) = \frac{v-c}{2} - (1+\alpha)\sigma\sqrt{v-c} + \alpha\sigma^2$.
Setting $x = \sqrt{v-c}$, the numerator is quadratic: $N = \frac{1}{2}x^2 - (1+\alpha)\sigma x + \alpha\sigma^2$. The two positive roots of $N = 0$ are $x_\pm = \sigma\big((1+\alpha) \pm \sqrt{1+\alpha^2}\big)$, so that $S^\alpha$ locally *increases* when $\sqrt{v-c} \leq \sigma\big((1+\alpha) - \sqrt{1+\alpha^2}\big)$ or $\sqrt{v-c} \geq \sigma\big((1+\alpha) + \sqrt{1+\alpha^2}\big)$, and *decreases* otherwise.

In normalized form, letting $t = \sigma/\sqrt{v-c} \in [0, 1]$, this condition is equivalent to the sign of $P(t) = \frac{1}{2} - (1+\alpha)t + \alpha t^2$, where $S^\alpha$ increases near $d^2 = 0$ if $P(t) \geq 0$ and decreases otherwise. The frontier $P(t) = 0$ defines the curve $t_*(\alpha) = \begin{cases} \dfrac{(1+\alpha) - \sqrt{1+\alpha^2}}{2\alpha}, & \alpha > 0, \\ \dfrac{1}{2}, & \alpha = 0, \end{cases}$ $t \in [0, 1]$.

*Entry constraint.* The firm enters only if $\sigma^2 + d^2 < v - c$, i.e. $t < 1$ at $d^2 = 0$. Hence the upper boundary $t = 1$ corresponds to a knife-edge case: for any $d^2 > 0$, the profit (and thus welfare) collapses to zero. Consequently, although the formula above predicts an "increasing" region for $t \to +\infty$ the domain effectively stops at $t = 1$, beyond which the firm does not enter.

*Interpretation.* Two opposing forces determine the sign of $\partial_{d^2} S^\alpha(0)$: (i) mislocalization, which lowers welfare, and (ii) price softening due to uncertainty, which raises it. The second effect dominates when $v - c$ is either very small or very large relative to $\sigma^2$; otherwise, the welfare locally decreases with $d^2$.

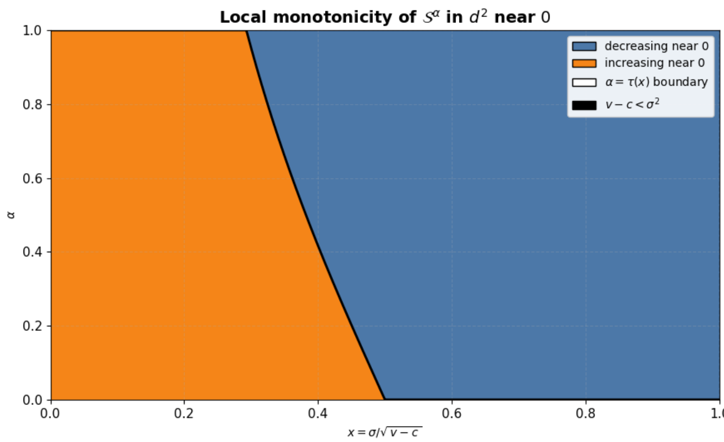


Figure 1: Diagram of when the surplus function $\mathcal{S}(d)$ can be increasing in $d$ near 0.

**Remark 3** (Conclusion for known variety of tastes)**.** *For the case where the firm know the variance of the distribution of consumer tastes $\sigma^2$, we see that two effects oppose. On the one hand, as the firm gathers more data, it can better estimate the mean of the distribution of consumer tastes, which tends to increase welfare. On the other hand, as the firm gathers more data, it can better optimise its price in the worst case, which tends to decrease welfare as the price gets closer to the monopoly price. When the variance of consumer tastes is large enough compared to the gap between value and cost, the second effect dominates the first one for small d, leading to a decrease in welfare as the firm gathers more data.*

## 4 Page Layout

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[1] Footnotes will appear on the margins

**Definition 4.1.** *Here's is the beautiful Schrödinger equation*

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\Psi(x,t)$$

## 4.1 *Headings*

*Proof (Theorem 1.1).* Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. □

**Lemma 4.1.** *Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris.*

*Proof.* Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. □

**Corollary 4.1.** *Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris.*

**Proposition 4.1.** *Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris.*

**Problem 1.** *Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis.*

*Proof.* Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.[2] □

[2] Harold Hotelling. Stability in Competition. 39(153):41–57

# A Proofs

## A.1 Proof of Proposition 3.1

*Proof of Proposition 3.1.* First, we argue that, in our worst case analysis, we can consider $\overline{\theta}$ to be so that $[\hat{\theta} - \sqrt{v-p}, \hat{\theta} + \sqrt{v-p}] \subset [0,1]$. This is because this case leads us to consider adversarial consumer distributions with mass on both sides of the served consumers interval, which are the one that achieve the least welfare. Also, as we will see more precisely, there is no point in putting mass far from the served consumers interval, as it does not change directly the welfare but only increases the variance.

We decompose any measure $\mu$ as $\lambda\mu_{\text{in}} + (1-\lambda)\mu_{\text{out}}$ where $\mu_{\text{in}}$ is supported on $J = (\theta_f - \sqrt{v-c}, \theta_f + \sqrt{v-c})$, and $\mu_{\text{out}}$ is supported on $[0,1] \setminus J$. We note $\theta_{\text{in}} = \mathbb{E}_{\theta\sim\mu_{\text{in}}}[\theta]$, $\theta_{\text{out}} = \mathbb{E}_{\theta\sim\mu_{\text{out}}}[\theta]$, $\sigma^2_{\text{in}} = \text{Var}_{\theta\sim\mu_{\text{in}}}(\theta)$ and $\sigma^2_{\text{out}} = \text{Var}_{\theta\sim\mu_{\text{out}}}(\theta)$.
We have the following constraints in order for $\mu$ to be in $\mathcal{F}(\theta, \sigma^2)$:

$$\lambda\theta_{\text{in}} + (1-\lambda)\theta_{\text{out}} = \overline{\theta} \tag{1}$$

$$\lambda\sigma^2_{\text{in}} + (1-\lambda)\sigma^2_{\text{out}} + \lambda(1-\lambda)(\theta_{\text{in}} - \theta_{\text{out}})^2 = \sigma^2 \tag{2}$$

We can then express the welfare as a function of these parameters. By noticing that $\mathbb{E}_{\theta\sim\mu_{\text{in}}}[(\theta-\theta_f)^2] = \mathbb{E}_{\theta\sim\mu_{\text{in}}}[(\theta-\theta_{\text{in}})^2] + (\theta_{\text{in}} - \theta_f)^2 - 2(\theta_{\text{in}} - \theta_f)\mathbb{E}_{\theta\sim\mu_{\text{in}}}[\theta - \theta_{\text{in}}] = \sigma^2_{\text{in}} + (\theta_{\text{in}} - \theta_f)^2$, we can write :

$$\mathcal{W}^{\alpha}_{\mu}(p,d) = \int_J (v - p - (\theta-\theta_f)^2 + \alpha(p-c))d\mu(\theta) = \lambda\left(v - p - (\theta_{\text{in}} - \theta_f)^2 - \sigma^2_{\text{in}} + \alpha(p-c)\right) \tag{3}$$

We see that the the welfare only depends on the dispersion of $\mu_{\text{in}}$ and not on the one of $\mu_{\text{out}}$. It is decreasing in $\sigma^2_{\text{in}}$, thus we want to minimize the dispersion outside of the support of $\mu_{\text{in}}$ so that we can use all of the variance budget to decrease the inside mass $\lambda$. In the worst case, we then have that $\mu_{\text{out}}$ is supported on $\{\theta_f - \sqrt{v-p}, \theta_f + \sqrt{v-p}\}$ (see remark 1).
We start by considering a fix error $|\theta_f - \overline{\theta}| = d$, specifically the case where $\theta_f = \overline{\theta} + d$ with $d > 0$ (without loss of generality by symmetry).
We also note $t = \theta_{\text{in}} - \overline{\theta}$ so that $\theta_{\text{out}} = \overline{\theta} - \frac{\lambda}{1-\lambda}t$.
Thus, we can compute $\sigma^2_{\text{out}}$ :

$$\begin{aligned}
\sigma^2_{\text{out}} &= \left(\theta_{\text{out}} - (\theta_f - \sqrt{v-p})\right)\left(\theta_f + \sqrt{v-p} - \theta_{\text{out}}\right) \\
&= \left(\theta_{\text{out}} - ((\overline{\theta}+d) - \sqrt{v-p})\right)\left((\overline{\theta}+d) + \sqrt{v-p} - \theta_{\text{out}}\right) \\
&= \left(-d - \frac{\lambda t}{1-\lambda} + \sqrt{v-p}\right)\left(d + \frac{\lambda t}{1-\lambda} + \sqrt{v-p}\right) \\
&= v - p - \left(d + \frac{\lambda t}{1-\lambda}\right)^2 \\
&= v - p - d^2 - 2\frac{\lambda d t}{1-\lambda} - \left(\frac{\lambda t}{1-\lambda}\right)^2
\end{aligned}$$

We also have using 1 that :

$$\sigma_{\text{out}} = \frac{\overline{\theta} - \lambda\theta_{\text{in}}}{1-\lambda} = \overline{\theta} - \frac{\lambda}{1-\lambda}t$$

Combining it with 2, we get :

$$\begin{aligned}\sigma^2 &= \lambda\sigma_{\text{in}}^2 + (1-\lambda)\left(v-p-d^2\right) - 2\lambda dt - \frac{\lambda^2 t^2}{1-\lambda} + \lambda(1-\lambda)\left(t + \overline{\theta} - \overline{\theta} + \frac{\lambda t}{1-\lambda}\right) \\ &= (1-\lambda)\left(v-p-d^2\right) - \lambda\left(t^2 - 2dt + \sigma_{\text{in}}^2\right) \qquad \text{(4a)}\end{aligned}$$

Combining 3 and 4a, we get :

$$\mathcal{W}_\mu^\alpha(p,d) = \lambda(v-p-d^2+\alpha(p-c)) - \sigma^2 + (1-\lambda)(v-p-d^2)$$

$$\mathcal{W}_\mu^\alpha(p,d) = v-p-d^2+\lambda(p-c)-\sigma^2$$

We notice that the welfare does not depend on t. This is because adding internal dispersion decreases the welfare but it is compensated by its cost in the total variance budget. We choose to fix $t=0$ without loss of generality for the rest of the calculations.

We have constrained the problem so that the only remaining parameteter is $\lambda$ and the welfare is decreasing in $\lambda$ so we are looking for its lowest achievable value.
$\sigma_{\text{in}}^2 > 0$ in 4a yields :

$$\lambda \geq 1 - \frac{\sigma^2}{v-p-d^2}$$

By construction, this value is achieved for the three point distribution charging $\overline{\theta}$, $\theta_f - \sqrt{v-p}$ and $\theta_f + \sqrt{v-p}$ with weights satisfying 1 and 2.

**Remark 4.** *The three point distribution is only one of the minimising distributions, as others can be found with other values of $t$. However, they will reach the same level of welfare.*

Finally, as we see that this expression is decreasing in $d$, the worst case over all possible $d'$ satisfying $|\theta_f - \overline{\theta}| \leq d'$ is obtained for $d' = d$. This concludes the proof. □

*Back to Proposition 3.1, p. 2.*

## A.2 Proof of Proposition 3.2

*Proof of Proposition 3.2.* In the same fashion as the proof of 3.1, we decompose any measure $\mu$ as $\lambda\delta_{\theta_{\text{in}}} + w_L\delta_{\theta_f - \sqrt{v-p}} + w_R\delta_{\theta_f+\sqrt{v-p}}$ where $\mu_{\text{in}}$ is supported on $J = (\theta_f - \sqrt{v-c}, \theta_f + \sqrt{v-c})$, and $\delta_x$ is the Dirac mass at point $x$, so that $D_\mu(\hat{\theta}, p) = \lambda$.

We note $\theta_{\text{in}} = \mathbb{E}_{\theta\sim\mu_{\text{in}}}[\theta]$, $\sigma_{\text{in}}^2 = \text{Var}_{\theta\sim\mu_{\text{in}}}(\theta)$.

We then have the following equations for $\mu$ to be in $\mathcal{F}(\overline{\theta}, \sigma^2)$:

$$w_L + w_R = 1 - \lambda \qquad (5)$$

$$\lambda\theta_{\text{in}} + w_L(\theta_f - \sqrt{v-p}) + w_R(\theta_f + \sqrt{v-p}) = \overline{\theta} \tag{6}$$

$$\lambda\left(\theta_{\text{in}} - \theta_f\right)^2 + w_L\left(\theta_f - \sqrt{v-p} - \overline{\theta}\right)^2 + w_R\left(\theta_f + \sqrt{v-p} - \overline{\theta}\right)^2 = \sigma^2 \tag{7}$$

We start by considering a fix error $|\theta_f - \overline{\theta}| = d$, specifically the case where $\overline{\theta} = \theta_f + d$ with $d > 0$ (without loss of generality by symmetry). Building on 5 and 6, we obtain :

$$w_L = \frac{1}{2}\left((1-\lambda) - \frac{\lambda(\overline{\theta} - \theta_{\text{in}}) + d(1-\lambda)}{\sqrt{v-p}}\right)$$

Building on 5 and 7, we obtain :

$$\sigma^2 = \lambda(\theta_{\text{in}} - \overline{\theta})^2 + 4dw_L\sqrt{v-p} + (1-\lambda)(\sqrt{v-p} - d)^2$$

Combining these two equations, we get :

$$\begin{aligned}
\sigma^2 &= \lambda(\theta_{\text{in}} - \overline{\theta})^2 + (1-\lambda)(2d\sqrt{v-p} - 2d^2) - 2\lambda d(\overline{\theta} - \theta_{\text{in}}) + (1-\lambda)(\sqrt{v-p} - d)^2 \\
&= \lambda(\theta_{\text{in}} - \overline{\theta})^2 + (1-\lambda)(v - p - d^2) + 2d\lambda(\theta_{\text{in}} - \overline{\theta}) \\
&= (1-\lambda)(v - p - d^2) + \lambda((\theta_{\text{in}} - \theta_f)^2 - d^2)
\end{aligned}$$

We can thus express :

$$\lambda = \frac{v - p - (\sigma^2 + d^2)}{v - p - (\theta_{\text{in}} - \theta_f)^2}$$

All parameters have been expressed without loss of generality in the worst case. We see that the minimum is achieved for $\theta_{\text{in}} = \theta_f$ and is

$$\lambda^\star = 1 - \frac{\sigma^2 + d^2}{v - p}$$

Finally, as we see that this expression is decreasing in $d$, the worst case over all possible $d'$ satisfying $|\theta_f - \overline{\theta}| \leq d'$ is obtained for $d' = d$. This concludes the proof.

□

## *References*

[1] Harold Hotelling. Stability in Competition. 39(153):41–57.