BIAN THEOPHILE

Neural Networks & Biological Modelling: Graded Exercise

1. Phase plane analysis & epidemics

$$\frac{dx}{dt} = f(x,y) \qquad \frac{dy}{dt} = g(x,y)$$

a) From the course: $N = S+I+R$ (with N constant) and $\Delta I = \left(\frac{\beta}{N} \Delta t\, S\right) I(t) - \gamma \Delta t\, I(t)$

therefore $\frac{dI}{dt} = \frac{\beta}{N} SI - \gamma I$; $\frac{dS}{dt} = \frac{-\beta}{N} SI$

if we set $R_0 = \beta/\gamma$; $x = S/N \Leftrightarrow S = Nx$; $y = I/N \Rightarrow I = Ny$ we get

$$\frac{dNy}{dt} = \frac{\beta}{N} Nx\, Ny - \beta \frac{\gamma}{\beta} Ny \Leftrightarrow \frac{dy}{dt} = \beta\left(xy - \frac{1}{R_0} y\right) = \beta y\left(x - \frac{1}{R_0}\right)$$

$$\frac{d\,Nx}{dt} = \frac{-\beta}{N} Nx\, Ny \Leftrightarrow \frac{dx}{dt} = -\beta\, xy$$

we can also write these as (as in the video)
$$\begin{cases} \frac{1}{\beta}\frac{dx}{dt} = -xy \\ \frac{1}{\beta}\frac{dy}{dt} = y\left(x - \frac{1}{R_0}\right) \end{cases}$$

$\boxed{f(x,y) = -\beta xy}$ and $\boxed{g(x,y) = \beta y\left(x - \frac{1}{R_0}\right)}$

we will use these forms as the exercise specifies $f(x,y) = \frac{dx}{dt}$ and not $f(x,y) = \frac{1}{\beta}\frac{dx}{dt}$ But we will disregard the parameter during analysis

b) $R_0 = 2.0$

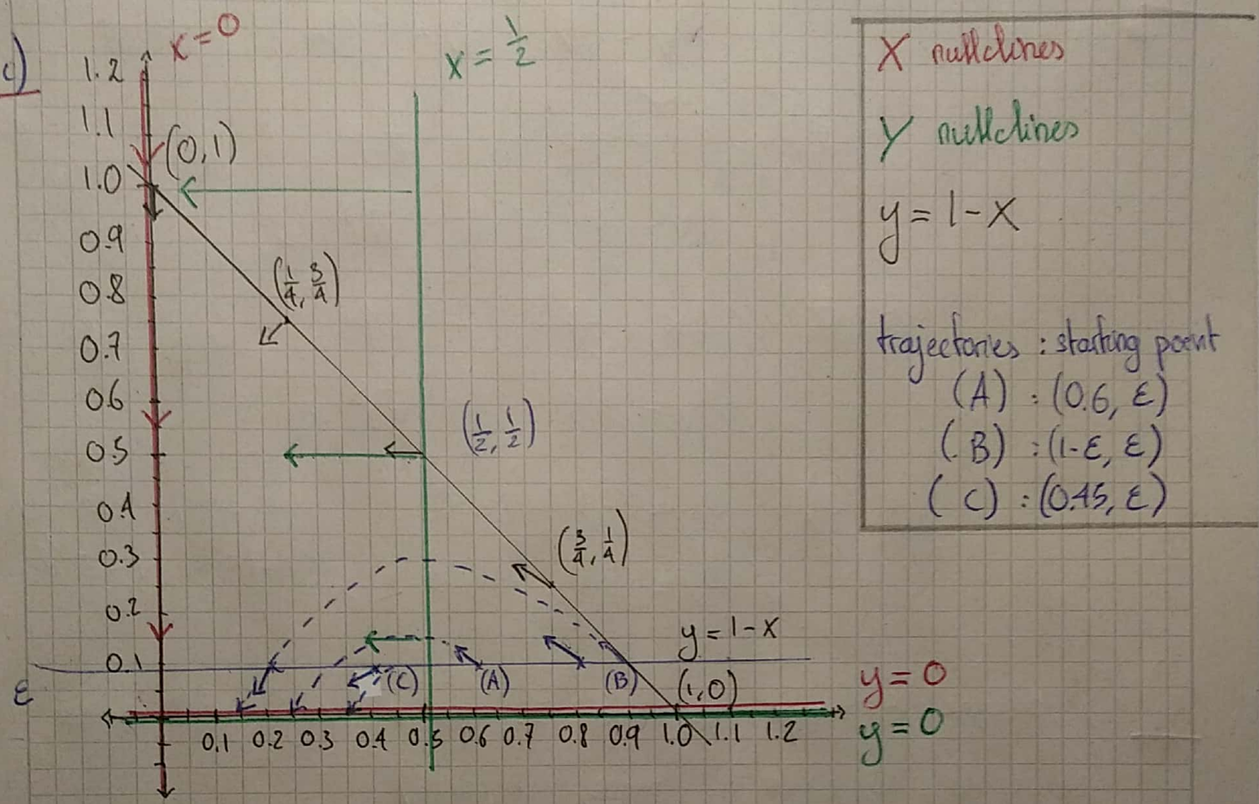
X-nullclines: $\frac{dx}{dt} = 0 = -\beta xy \Rightarrow x = 0$ or $y = 0$

Y-nullclines: $\frac{dy}{dt} = 0 = y\left(x - \frac{1}{R_0}\right)\beta \Rightarrow x = \frac{1}{R_0} = 0.5$ or $y = 0$

the two branches for $\frac{dx}{dt} = 0$ (i) $x = 0$ (ii) $y = 0$

the two branches for $\frac{dy}{dt} = 0$: (i) $x = \frac{1}{R_0} = 0.5$ (ii) $y = 0$

c)



d) /

e) $\Delta x = \Delta t \frac{dx}{dt}$, $\Delta y = \Delta t \frac{dy}{dt}$

at $(x,y) = (\frac{1}{2}, \frac{1}{2})$ we get $\Delta x = \Delta t(-\beta(\frac{1}{2})(\frac{1}{2})) = -\beta \Delta t/4$

and $\Delta y = \Delta t((\frac{1}{2})(\frac{1}{2} - \frac{1}{2})) = 0$

As we have seen in a), we can use β on the other side of the equation since for now we are not interested in this parameter

we get $\boxed{\Delta x = -\frac{1}{4} \; ; \Delta y = 0}$ by choosing $\Delta t = 1$

f) $\frac{\Delta y}{\Delta x} = dir(x)$; and we are on $y = 1-x$

so $\Delta x = -\beta \Delta t \, x(1-x) = \beta \Delta t (-1) x (1-x)$

$\Delta y = \beta \Delta t (x - \frac{1}{R_0})(1-x) = \beta \Delta t (x - \frac{1}{2})(1-x)$

$$\frac{\Delta y}{\Delta x} = \frac{(\beta \Delta t)(1-x)(x - 1/2)}{(\beta \Delta t)(1-x)(-1)(x)} = \boxed{\frac{1}{x}\left(\frac{1}{2} - x\right)}$$

g) let's look at $(0,1) \Rightarrow \Delta x = -\beta \Delta t\, xy = 0$
with $\Delta t = 1$
$\Delta y = \beta \Delta t\, y\left(x - \frac{1}{R_0}\right) = \left(-\frac{1}{2}\right)\beta$ } $\lim_{x \to 0^+} dir(x) = -\infty$

$\left(\frac{1}{4}, \frac{3}{4}\right) \Rightarrow \Delta x = -\beta \frac{1}{4} \cdot \frac{3}{4} = -\frac{3}{16}\beta$
$\Delta y = \beta\left(\frac{3}{4}\right)\left(\frac{1}{4} - \frac{1}{2}\right) = \frac{-3}{16}\beta$ } $dir(x) = 1$

$\left(\frac{3}{4}, \frac{1}{4}\right) \Rightarrow \Delta x = -\beta \frac{3}{4} \cdot \frac{1}{4} = -\frac{3}{16}\beta$
$\Delta y = \beta\left(\frac{1}{4}\right)\left(\frac{3}{4} - \frac{1}{2}\right) = \frac{3}{16}\beta$ } $dir(x) = -1$

$(1, 0) \Rightarrow \Delta x = -\beta(1)(0) = 0$
$\Delta y = \beta(0)\left(1 - \frac{1}{2}\right) = 0$ } using $dir(x) = \frac{1}{x}\left(\frac{1}{2} - x\right)$ we get $dir(x) = -1/2$

h) we assume ε to be close to 0.1 (not a "very" small number but it helps to see the trajectories on the graph)

$R_0 = 2$;
$(0.2, \varepsilon) \rightarrow (\Delta x, \Delta y) = (-0.2\beta\varepsilon, -0.3\beta\varepsilon)$
$(0.4, \varepsilon) \rightarrow (\Delta x, \Delta y) = (-0.4\beta\varepsilon, -0.1\beta\varepsilon)$
$(0.6, \varepsilon) \rightarrow (\Delta x, \Delta y) = (-0.6\beta\varepsilon, 0.1\beta\varepsilon)$
$(0.8, \varepsilon) \rightarrow (\Delta x, \Delta y) = (-0.8\beta\varepsilon, 0.3\beta\varepsilon)$

i) /

j) We can use the previous questions to get the direction of the flow at the nullclines.
Also: at $x = 0$, the x nullcline is s.t. $\frac{dx}{dt} = 0$; $\frac{dy}{dt} = \beta\left(0 - \frac{1}{2}\right)y = -\frac{\beta}{2}y$
therefore $\frac{dx}{dt} < 0$ if $y > 0$;
and at $x = \frac{1}{2}$, the y-nullcline is s.t. $\frac{dy}{dt} = 0$; $\frac{dx}{dt} = \beta\left(\frac{1}{2}\right)y = \frac{\beta}{2}y$
therefore $\frac{dy}{dt} < 0$ if $y < 0$;

k)

l) $(1-\varepsilon, \varepsilon) \rightarrow (\Delta_x, \Delta_y) = \left(-\beta\varepsilon(1-\varepsilon),\ \beta(1-\varepsilon-\tfrac{1}{2})\varepsilon\right)$

$R_0=2;$

$$\lim_{\varepsilon \to 0} (\Delta_x, \Delta_y)\Big|_{\substack{x=1-\varepsilon \\ y=\varepsilon}} = (0,0)$$

m) $(0.45, \varepsilon) \rightarrow (\Delta_x, \Delta_y) = (-0.45\beta\varepsilon, -0.05\beta\varepsilon)$

$R_0=2;$

n)

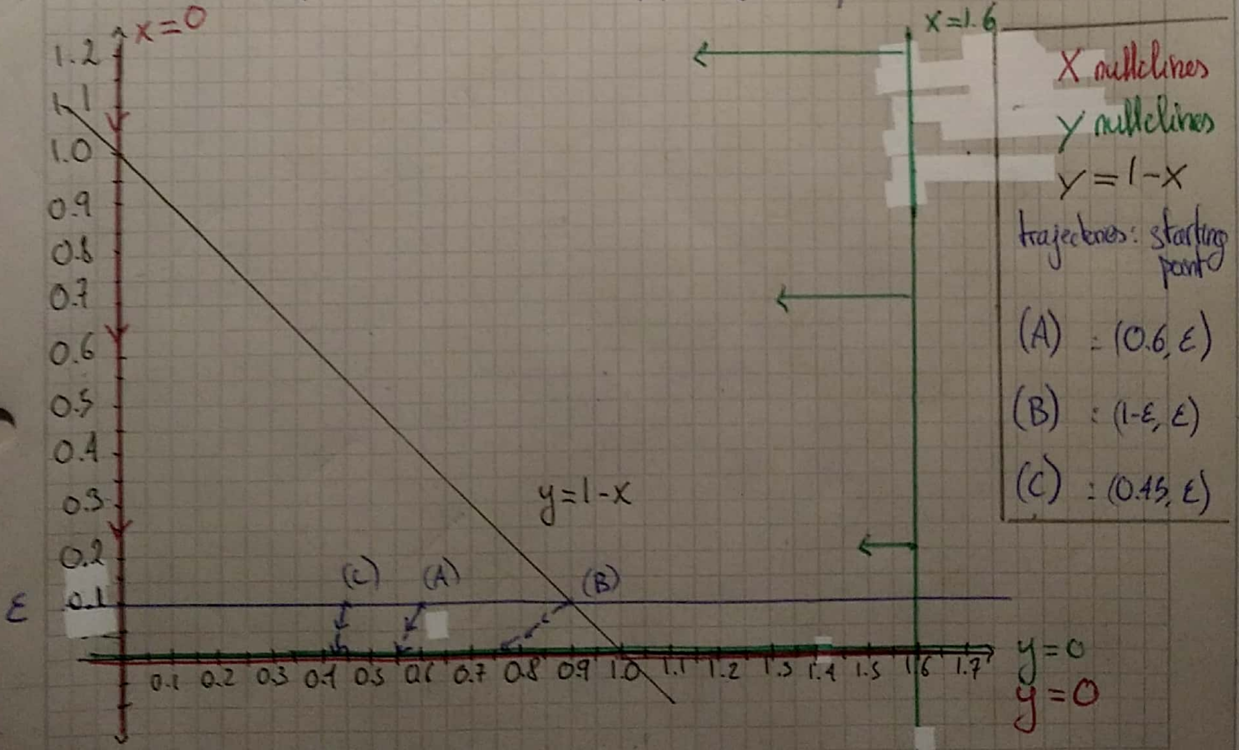
$R_0 = 0.8$

$\Rightarrow \frac{1}{R_0} = 1.6$

$(0.6, \varepsilon) \rightarrow (\Delta_x, \Delta_y) = (-0.6\beta\varepsilon, -\beta\varepsilon)$

$(1-\varepsilon, \varepsilon) \rightarrow (\Delta_x, \Delta_y) = (-\beta\varepsilon(1-\varepsilon), \beta(1-\varepsilon-1.6)\varepsilon)$

$(0.45, \varepsilon) \rightarrow (\Delta_x, \Delta_y) = (-0.45\beta\varepsilon, -1.15\beta\varepsilon)$



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