Convex Optimization - HW1

Théotime de Charrin

12 octobre 2023

1 Exercise 1 : Which of the following sets are convex?

1) The rectangle set defined as $\left\{x \in \mathbb{R}^n \mid \forall i \in [\![1, n]\!], \alpha_i \leq x_i \leq \beta_i\right\}$
For a given i, a $\{x_i\}$ is the intersection of two halfspaces $\left\{x \in \mathbb{R}^n | x_i \leq \beta_i\right\}$ and $\left\{x \in \mathbb{R}^n | x_i \geq \alpha_i\right\}$
As i is finite, a rectangle is a finite intersection of halfspaces so it is a convex set.

2) The hyperbolic set defined as $\left\{H : x \in R_+^2 | x_1 x_2 \geq 1\right\}$.
We take $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in R^2$ such that $x, y \in H$ and $\theta \in [0;1]$ Hence, we have $x_1 x_2 \geq 1$ and the same for y.
Let's look at the convex combination of x and y :

$$\begin{aligned} \theta x + (1-\theta) y &= \begin{pmatrix} \theta x_1 + (1-\theta) y_1 \\ \theta x_2 + (1-\theta) y_2 \end{pmatrix} \\ &= \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \end{aligned}$$

$$z_1 z_2 = \theta^2 \overbrace{x_1 x_2}^{\geq 1} + (1-\theta)^2 \overbrace{y_1 y_2}^{\geq 1} + \theta(1-\theta)(x_1 y_2 + y_1 x_2)$$

Let's have a look at $x_1 y_2 + y_1 x_2$: we have $x_1 x_2 \geq 1 \Leftrightarrow x_1 y_2 \geq \frac{y_2}{x_2}$. In the same manner, $y_1 x_2 \geq \frac{x_2}{y_2}$.
We now have :

$$\begin{aligned} x_{1\,y_2} + y_1 x_2 &\geq \frac{y_2}{x_2} + \frac{x_2}{y_2} \\ &\geq \frac{y_2}{x_2} + \frac{1}{\frac{y_2}{x_2}} \\ \Leftrightarrow x_1 y_2 + y_1 x_2 - 2 &\geq \frac{y_2}{x_2} + \frac{1}{\frac{y_2}{x_2}} - 2 \\ &\geq \sqrt{\frac{y_2}{x_2}}^2 + \frac{1}{\sqrt{\frac{y_2}{x_2}}}^2 - 2\frac{\sqrt{\frac{y_2}{x_2}}}{\sqrt{\frac{y_2}{x_2}}} \\ &\geq \left(\sqrt{\frac{y_2}{x_2}} - \frac{1}{\sqrt{\frac{y_2}{x_2}}}\right)^2 \geq 0 \; \forall \left\{(x_1, x_2), (y_1, y_2)\right\} \end{aligned}$$

We can conclude that $x_1 y_2 + y_1 x_2 \geq 2$
$\theta(1-\theta) \geq 0 \ \forall \theta \in [0,1]$ so $\theta(1-\theta)(x_1 y_2 + y_1 x_2) \geq 2\theta(1-\theta)$ and :

$$\begin{aligned} z_1 z_2 &\geq \theta^2 + (1-\theta)^2 + 2\theta(1-\theta) \\ &\geq (\theta + 1 - \theta)^2 \geq 1 \end{aligned}$$

We have shown that $z_1 z_2 \geq 1$, i.e. $\theta x + (1-\theta) y \in H$
A convex combination of two elements in H is also in H, so the hyperbolic set is convex as well.

3) The set of points closer to a given point than a given set, *i.e.*

$$A = \left\{ x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \ \forall y \in S \right\} \text{where} S \subseteq \mathbb{R}^n.$$

$$Let$$

y $\in S$ and $x \in A$. Then we must have :

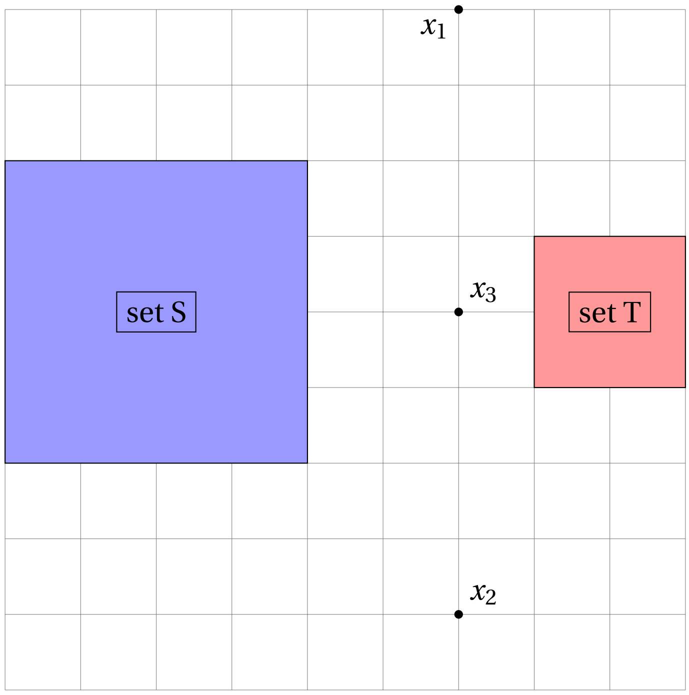
$$\begin{aligned} \|x - x_0\|_2 &\leq \|x - y\|_2 \\ (x - x_0)^T (x - x_0) &\leq (x - y)^T (x - y) \\ (x^T - {x_0}^T)(x - x_0) &\leq (x^T - y^T)(x - y) \\ x^T x - x^T x_0 - {x_0}^T x + {x_0}^T x_0 &\leq x^T x - x^T y - y^T x + y^T y \\ -2x^T x_0 + \|x_0\|_2^2 &\leq -2x^T y + \|y\|_2^2 \\ 2x^T (y - x_0) &\leq \|x_0\|_2^2 + \|y\|_2^2 \\ (y - x_0)^T x &\leq \frac{\|x_0\|_2^2 + \|y\|_2^2}{2} \end{aligned}$$

This is the equation of an halfspace ($a^T x \leq b$), which is a convex set. S is the intersection over $y \in S$ of these halfspaces, hence it's also a convex set.

4) The set of points closer to one set than another, *i.e.*

$$A = \{x \mid \textbf{dist}(x, S) \leq \textbf{dist}(x, T)\}, \text{ where } S, T \subseteq \mathbb{R}^n, \text{ and } \textbf{dist}(x, S) = \inf\{ \|x - z\|_2 \mid z \in S\}$$

A visual example should better explain why this is not always a convex set :



Here, we see that x_1 and x_2 are closer to the set S rather than the set T. Therefore, they would be part of the set A. However, x_3, a convex combination of $x_1,\ x_2$, *i.e.* on the same segment, is closer to the set T than the set S.
Therefore, A is not a convex set.

5) The set

$$A = \{x \mid x + S_2 \subseteq S_1\}$$

where $S_1,\ S_2\ \in \mathbb{R}^n$, S_1 convex.

$$x + S_2 \subseteq S_1 \Leftrightarrow \forall y\ \in S_2,\ x + y \in S_1$$

Let's look at the convex combination of $u,\ v \in A$, $\theta \in [0,1]$, and see if it's still in A :

$$\theta u + (1-\theta)v + y = \theta(u+y) + (1-\theta)(v+y) \in S_1$$

By definition, $u + y,\ v + y \in S_1$ because $u,\ v \in A$ and $y \in S_2$.
As S_1 is convex, any convex combination of $z \in S_1$ is also in S_1. We conclude that A is a convex set.

2 Exercise 2 : For each of the following functions determine whether it is convex or concave or not.

1) $f(x_1, x_2) = x_1 x_2$ on $\mathbb{R}^2_{++}$