Chapter 1: Linear Systems and Matrices
Section: LU - Factorization
Lecture #5

Lebanese University

Prof Ali WEHBE

Course Plan

Chapter 1 Matrices and Linear Systems

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Chapter 2 Determinant

↓

Chapter 3 Vector Spaces

↓

Chapter 5 Orthogonality

↓

Chapter 6 Eigenvalues

↓

Chapter 4 Linear Transformations

Table of contents

1. Some Additional Definitions
   - Trace of a matrix
   - LU - Factorization
2. Exercises on RREF - GJE (From Lecture 3)
3. Exercises on Inverse of Matrices (From Lecture 4)

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Definition

Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$. The trace function tr defined by $tr: \mathbb{M}_n(\mathbb{C}) \longrightarrow \mathbb{C}$, is given by

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3- $tr(AB) = tr(BA)$.

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Introduction

Our aim is to write a matrix $\boldsymbol{A} = [\boldsymbol{a_{ij}}] \in \mathbb{M}_{\boldsymbol{n}}(\mathbb{C})$ as follows:

$$\boldsymbol{A} = \boldsymbol{LU},$$

where $\boldsymbol{L} = [\boldsymbol{l_{ij}}] \in \mathbb{M}_{\boldsymbol{n}}(\mathbb{C})$ is a Lower Triangular matrix and $\boldsymbol{U} = [\boldsymbol{u_{ij}}] \in \mathbb{M}_{\boldsymbol{n}}(\mathbb{C})$ is an Upper Triangular one.

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Equivalently $\boldsymbol{l_{ij}} = \boldsymbol{0}$ for all $\boldsymbol{i} < \boldsymbol{j}$ and $\boldsymbol{u_{ij}} = \boldsymbol{0}$ for all $\boldsymbol{i} > \boldsymbol{j}$. We will assume in addition, that $\boldsymbol{l_{ii}} = \boldsymbol{1}$.
Thus

$$\boldsymbol{L} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{l_{21}} & \boldsymbol{1} & \boldsymbol{0} \\ \boldsymbol{l_{31}} & \boldsymbol{l_{32}} & \boldsymbol{1} \end{bmatrix},$$

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$$\boldsymbol{L} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \qquad \boldsymbol{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

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By applying the matrix multiplication between L and U and by comparing with A, we can find the entries of the matrices L and U.

Example

Find the ***LU*** factorization of the following matrices:

1- $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{4} \\ \mathbf{3} & \mathbf{8} & \mathbf{14} \\ \mathbf{2} & \mathbf{6} & \mathbf{13} \end{bmatrix}$.

2- $\boldsymbol{B} = \begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{6} \\ \mathbf{-6} & \mathbf{0} & \mathbf{-16} \\ \mathbf{0} & \mathbf{8} & \mathbf{-17} \end{bmatrix}$.

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Example (Supplementary 1)

Discuss according the value of ***a*** for which the consistence of the following system:

$$(S)\begin{cases} x & +2y & +z & = & 2, \\ 2x & -2y & +3z & = & 1, \\ x & +2y & -az & = & a. \end{cases}$$

Solution:

Example (Supplementary 2)

Discuss according the value of a for which the consistence of the following system:

$$(S)\begin{cases} x & +y & +7z & = & -7, \\ 2x & +3y & +17z & = & 11, \\ x & +2y & +(a^2+1)z & = & 6a. \end{cases}$$

Solution:

Example (Supplementary 3)

Discuss according the values of a and b for which the consistence of the following system:

$$(S)\begin{cases} ax & & +bz & = & 2, \\ ax & +ay & +4z & = & 4, \\ & ay & +2z & = & b. \end{cases}$$

Solution:

Example (Supplementary 4)

Solve the following system of non-linear equations for the unknowns angles α, β and γ, where $0 \leq \alpha \leq \frac{\pi}{2}$, $0 \leq \beta < 2\pi$ and $0 \leq \gamma \leq \pi$.

$$(S)\begin{cases} 2\sin\alpha & +\cos\beta & -\tan\gamma & = & 1, \\ -4\sin\alpha & +\cos\beta & +\tan\gamma & = & 0, \\ -2\sin\alpha & +3\cos\beta & +2\tan\gamma & = & 4. \end{cases}$$

Solution:

Example (Supplementary 5)

Find the values of a, b and c such that the graph of the polynomial $p(x) = ax^2 + bx + c$ passes through the points $(1, 2)$, $(-1, 6)$ and $(2, 3)$.

Solution:

Example (Solve the system using the inverse of Matrices)

a)

$$(S)\begin{cases} 2x & +3y & +z & = & -1, \\ 3x & +3y & +z & = & 1, \\ 2x & +4y & +z & = & -2. \end{cases}$$

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b)

$$(S)\begin{cases} 2x & +3y & +z & = & 4, \\ 3x & +3y & +z & = & 8, \\ 2x & +4y & +z & = & 5. \end{cases}$$