Chapter 1: Linear Systems and Matrices
Section: Matrix Operations
Lecture #2

Lebanese University

Prof. Ali WEHBE

Course Plan

Chapter 1 Matrices and Linear Systems

↓

Chapter 2 Determinant

↓

Chapter 3 Vector Spaces

↓

Chapter 5 Orthogonality

↓

Chapter 6 Eigenvalues

↓

Chapter 4 Linear Transformations

Table of contents

Matrix Definition

Definition (Matrix)

A matrix (plural matrices) is a rectangular array of numbers, or symbols, arranged in rows and columns.

Matrix Definition

Definition (Matrix)

A matrix (plural matrices) is a rectangular array of numbers, or symbols, arranged in rows and columns.

Notation:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} \text{ with a compact form: } A = [a_{ij}]$$

Matrix Definition

Definition (Matrix)

A matrix (plural matrices) is a rectangular array of numbers, or symbols, arranged in rows and columns.

Notation:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix}$$ with a compact form: $A = [a_{ij}]$

Example

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 4 & 8 & -6 \\ 7 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Definition (Size of a Matrix)

The size of a matrix is defined by the number of rows and columns that it contains. A matrix with m rows and n columns is called an $m \times n$ matrix or m-by-n matrix.

Definition (Size of a Matrix)

The size of a matrix is defined by the number of rows and columns that it contains. A matrix with m rows and n columns is called an $m \times n$ matrix or m-by-n matrix.

Example

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 4 & 8 & -6 \\ 7 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Definition (Size of a Matrix)

The size of a matrix is defined by the number of rows and columns that it contains. A matrix with m rows and n columns is called an $m \times n$ matrix or m-by-n matrix.

Example

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 4 & 8 & -6 \\ 7 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 5 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$size(A) = 3 \times 3, \quad size(B) = 4 \times 1, \quad size(X) = 3 \times 1$$

Operations with Matrices

Definition (Equal Matrices)

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if:

Operations with Matrices

Definition (Equal Matrices)

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if:
1. A and B have the same size

Operations with Matrices

Definition (Equal Matrices)

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if:
1. A and B have the same size
2. $a_{ij} = b_{ij}$ for all $i,\ j$

Operations with Matrices

Definition (Equal Matrices)

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if:
1. A and B have the same size
2. $a_{ij} = b_{ij}$ for all i, j

Definition (Matrix Addition)

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of same size, then their sum is the matrix given by

$$A + B = [a_{ij} + b_{ij}]$$

Operations with Matrices

Definition (Equal Matrices)

Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if:
1. A and B have the same size
2. $a_{ij} = b_{ij}$ for all i, j

Definition (Matrix Addition)

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of same size, then their sum is the matrix given by

$$A + B = [a_{ij} + b_{ij}]$$

Note: The sum of two matrices of different sizes is not defined.

Operations with Matrices

Example

Find $\boldsymbol{A}+\boldsymbol{B}$:

1. $\boldsymbol{A}=\begin{bmatrix} 1 & -2 & -3 \\ 4 & 8 & -6 \end{bmatrix}$, $\boldsymbol{B}=\begin{bmatrix} 7 & 0 & 1 \\ -4 & 0 & -6 \end{bmatrix}$

2. $\boldsymbol{A}=\begin{bmatrix} 3 & 2 \\ -7 & 0 \end{bmatrix}$, $\boldsymbol{B}=\begin{bmatrix} 8 & 0 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

Operations with Matrices

Definition (Scalar Multiplication)

If $A = [a_{ij}]$ is an $m \times n$ matrix and α is a scalar, then the scalar multiple of A by α is the matrix given by

$$\alpha A = [\alpha a_{ij}].$$

Operations with Matrices

Definition (Scalar Multiplication)

If $A = [a_{ij}]$ is an $m \times n$ matrix and α is a scalar, then the scalar multiple of A by α is the matrix given by

$$\alpha A = [\alpha a_{ij}].$$

Example

$$2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} =$$

Operations with Matrices

Definition (Scalar Multiplication)

If $A = [a_{ij}]$ is an $m \times n$ matrix and α is a scalar, then the scalar multiple of A by α is the matrix given by

$$\alpha A = [\alpha a_{ij}].$$

Example

$$2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}, \qquad -3 \begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 4 & 5 \end{bmatrix} =$$

Operations with Matrices

Definition (Scalar Multiplication)

If $A = [a_{ij}]$ is an $m \times n$ matrix and α is a scalar, then the scalar multiple of A by α is the matrix given by

$$\alpha A = [\alpha a_{ij}].$$

Example

$$2\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}, \qquad -3\begin{bmatrix} 2 & -1 \\ -1 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 3 & -6 \\ -12 & -15 \end{bmatrix}$$

Definition (Matrix Multiplication)

If $A = [a_{ij}]$ is an $\underline{m} \times n$ matrix and $B = [b_{ij}]$ is an $n \times \underline{r}$ matrix, then the product AB is an $\underline{m \times r}$ matrix $C = [c_{ij}]$ where

$$c_{ij} = a_{i1}b_{1j} + ai2b_{2j} + a_{i3}b_{3j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

for all $i = 1, \cdots, m$ and $j = 1, \cdots, r$

Example (Matrices Multiplication)

Find ***AB*** and ***BA***.

1. $\boldsymbol{A} = \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix}$

Solution:

Example (Matrices Multiplication)

Find $\boldsymbol{AB}$ and $\boldsymbol{BA}$.

1. $\boldsymbol{A} = \begin{bmatrix} 3 & -2 \\ 2 & 4 \\ 1 & -3 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} -2 & 1 & 3 \\ 4 & 1 & 6 \end{bmatrix}$

Solution:

$$\boldsymbol{AB} = \begin{bmatrix} -14 & 1 & -3 \\ 12 & 6 & 30 \\ -14 & -2 & -15 \end{bmatrix}, \quad \boldsymbol{BA} = \begin{bmatrix} -1 & -1 \\ 20 & -22 \end{bmatrix}$$

$$\boldsymbol{AB} \neq \boldsymbol{BA}$$

Definition (Matrix Multiplication)

If $A = [a_{ij}]$ is an $\underline{m} \times n$ matrix and $B = [b_{ij}]$ is an $n \times \underline{r}$ matrix, then the product AB is an $\underline{m \times r}$ matrix $C = [c_{ij}]$ where

$$c_{ij} = a_{i1}b_{1j} + ai2b_{2j} + a_{i3}b_{3j} + \ldots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

for all $i = 1, \cdots, m$ and $j = 1, \cdots, r$

Example (Matrices Multiplication)

1. $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{bmatrix}$

Solution:

Example (Matrices Multiplication)

1. $\boldsymbol{A} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 3 & 6 \end{bmatrix}$

Solution: The multiplication $\boldsymbol{AB}$ between the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is not defined, since the number of columns of the matrix$\boldsymbol{A} = 2 \neq 3 =$ number of rows of the matrix $\boldsymbol{B}$. However:

$$\boldsymbol{BA} = \begin{bmatrix} 5 & 8 \\ 17 & 26 \\ 15 & 24 \end{bmatrix}$$

Example (Matrices Multiplication)

2. $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Solution:

Example (Matrices Multiplication)

2. $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$

Solution:

$$AB = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

$$AB \neq BA$$

Example (Matrices Multiplication)

3. $A = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Solution:

Example (Matrices Multiplication)

3. $A = \begin{bmatrix} 1 & -2 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Solution:

$$AB = \begin{bmatrix} 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 2 & -4 & -6 \\ -1 & 2 & 3 \\ 1 & -2 & -3 \end{bmatrix}$$

$$AB \neq BA$$

Remark

In general, $AB \neq BA$. Matrix multiplication is not commutative.

Example (Matrices Multiplication)

Given $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 5 \\ -3 & 0 & 9 \end{bmatrix}$ and $\boldsymbol{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Find $\boldsymbol{AX}$.

Definition (Power of a matrix)

Let $\boldsymbol{A}$ be a square matrix of size $\boldsymbol{n} \times \boldsymbol{n}$ and let $\boldsymbol{k}$ be a strictly positive integer. Then

$$\boldsymbol{A}^{\boldsymbol{k}} =$$

Definition (Power of a matrix)

Let A be a square matrix of size $n \times n$ and let k be a strictly positive integer. Then

$$A^k = A.A.A...A, \quad k \quad \text{times}.$$

Definition (Power of a matrix)

Let A be a square matrix of size $n \times n$ and let k be a strictly positive integer. Then

$$A^k = A.A.A...A, \quad k \quad \text{times.}$$

Example

Find A^3 with $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Solution:

Theorem:

Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ be three matrices (such the following operations are defined), and let α and β be two scalars. Then the following statements are true:

1) $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$.
2) $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$.
3) $(\boldsymbol{AB})\boldsymbol{C}=\boldsymbol{A}(\boldsymbol{BC})$.
4) $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{AB}+\boldsymbol{AC}$.
5) $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C}=\boldsymbol{AC}+\boldsymbol{BC}$.
6) $(\alpha\beta)\boldsymbol{A}=\alpha(\beta\boldsymbol{A})=\beta(\alpha\boldsymbol{A})$.
7) $\alpha(\boldsymbol{AB})=(\alpha\boldsymbol{A})\boldsymbol{B}=\boldsymbol{A}(\alpha\boldsymbol{B})$.
8) $(\alpha+\beta)\boldsymbol{A}=\alpha\boldsymbol{A}+\beta\boldsymbol{A}$.
9) $\alpha(\boldsymbol{A}+\boldsymbol{B})=\alpha\boldsymbol{A}+\alpha\boldsymbol{B}$.

Example

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Verify that $A(BC) = (AB)C$ and $A(B + C) = AB + AC$.

Solution:

Example

Let

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}.$$

Verify that $A(BC) = (AB)C$ and $A(B+C) = AB + AC$.

Solution:

$$A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 16 & 11 \end{bmatrix}$$

$$(AB)C = \begin{bmatrix} -4 & 5 \\ -6 & 11 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 16 & 11 \end{bmatrix}$$

Thus

$$A(BC) = (AB)C.$$

Example

$$A(B+C) = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 5 & 15 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} -4 & 5 \\ -6 & 11 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 11 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 5 & 15 \end{bmatrix}$$

Hence, $A(B+C) = AB + AC$.

Definition

1) A **row** Matrix, is a matrix of size $\mathbf{1} \times n$ has the following form

$$A = [a_{11}, a_{12}, \cdots, a_{1n}].$$

Definition

1) A **row** Matrix, is a matrix of size $\mathbf{1} \times \boldsymbol{n}$ has the following form

$$A = [a_{11}, a_{12}, \cdots, a_{1n}].$$

2) A **Column** Matrix, is a matrix of size $\boldsymbol{m} \times \mathbf{1}$ has the following form

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}.$$

Definition

1) A **row** Matrix, is a matrix of size $\mathbf{1} \times n$ has the following form

$$A = [a_{11}, a_{12}, \cdots, a_{1n}].$$

2) A **Column** Matrix, is a matrix of size $m \times \mathbf{1}$ has the following form

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}.$$

3) A **square** matrix, is a matrix of size $n \times n$, equivalently, the number of rows is equal the number of columns.

Definition

1) A **row** Matrix, is a matrix of size $\mathbf{1} \times \boldsymbol{n}$ has the following form

$$A = [a_{11}, a_{12}, \cdots, a_{1n}].$$

2) A **Column** Matrix, is a matrix of size $\boldsymbol{m} \times \mathbf{1}$ has the following form

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}.$$

3) A **square** matrix, is a matrix of size $\boldsymbol{n} \times \boldsymbol{n}$, equivalently, the number of rows is equal the number of columns.

4) The **zero** matrix of size $\boldsymbol{m} \times \boldsymbol{n}$, denoted by O_{mn}, is the matrix with all the entries are equal to zero.

Special Matrices

Example (Square Matrix)

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 4 & 8 & -6 \\ 7 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & -2 & -3 & 2 \\ 4 & 8 & -6 & 3 \\ 7 & 0 & 1 & 2 \\ 9 & 3 & -5 & 0 \end{bmatrix}$$

Example (Zero Matrix)

$$\mathbf{0}_{1,1} = \begin{bmatrix} 0 \end{bmatrix} \qquad \mathbf{0}_{2,3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{0}_{4,4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem (Property of zero matrix)

Let A be a $m \times n$ matrix and let c be a scalar. Then, the following statements are true:

1) $A + O_{mn} = O_{mn} + A = A$.
2) $A + (-A) = O_{mn}$.
3) If $cA = O_{mn}$ then $c = 0$ or $A = O_{mn}$.

Example

Solve for $\boldsymbol{X}$ in the equation $\boldsymbol{3X + A = B}$ with

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{0} & \mathbf{3} \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} \mathbf{-3} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$$

Example

Solve for $\boldsymbol{X}$ in the equation $\mathbf{3}\boldsymbol{X} + \boldsymbol{A} = \boldsymbol{B}$ with

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-2} \\ \mathbf{0} & \mathbf{3} \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} \mathbf{-3} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} \end{bmatrix}$$

Solution: $\mathbf{3}\boldsymbol{X} + \boldsymbol{A} = \boldsymbol{B} \Longrightarrow \mathbf{3}\boldsymbol{X} = \boldsymbol{B} - \boldsymbol{A} \Longrightarrow \boldsymbol{X} = \frac{\mathbf{1}}{\mathbf{3}}(\boldsymbol{B} - \boldsymbol{A})$. Hence

$$\boldsymbol{B} - \boldsymbol{A} = \begin{bmatrix} \mathbf{-4} & \mathbf{6} \\ \mathbf{2} & \mathbf{-2} \end{bmatrix} \Longrightarrow \boldsymbol{X} = \begin{bmatrix} -\frac{\mathbf{4}}{\mathbf{3}} & \mathbf{2} \\ \frac{\mathbf{2}}{\mathbf{3}} & -\frac{\mathbf{2}}{\mathbf{3}} \end{bmatrix}$$

Definition (Diagonal of a Matrix)

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **Diagonal** of A is the set that contains the entries a_{ij} such that $i = j$

Definition (Diagonal of a Matrix)

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The **Diagonal** of A is the set that contains the entries a_{ij} such that $i = j$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

Diagonal of a matrix

Definition

Let $A = [a_{ij}]$ be a square matrix. The **Diagonal** of A is the set that contains the entries a_{ij} such that $i = j$.

Example

$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$. The diagonal of A is $S = \{1, 6, 7, 16\}$.

Identity Matrix

Definition

The $n \times n$ **identity matrix** is the matrix $I_n = [\delta_{ij}]$ with

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Identity Matrix

Definition

The $n \times n$ **identity matrix** is the matrix $I_n = [\delta_{ij}]$ with

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j, \\ 0 & \text{if} \quad i \neq j. \end{cases}$$

Example

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Identity Matrix

Theorem

Let A be an $m \times n$ matrix. Then, the following statements are true:

1. $AI_n = A$.
2. $I_mA = A$.

Identity Matrix

Theorem

Let A be an $m \times n$ matrix. Then, the following statements are true:

1. $AI_n = A$.
2. $I_mA = A$.

Example

$A = \begin{bmatrix} 3 & 4 & 1 \\ 2 & 6 & 3 \\ 0 & 1 & 8 \end{bmatrix}$. Show that $I_3A = AI_3 = A$.

Solution: Homework.

Definition

Let $\boldsymbol{A} = [\boldsymbol{a_{ij}}] \in \boldsymbol{M_{n,n}}(\mathbb{R})$.

Definition

Let $A = [a_{ij}] \in M_{n,n}(\mathbb{R})$.

1- The matrix A is said to be an upper triangular matrix if $a_{ij} = 0$ for all $i > j$ and there exists $a_{ij} \neq 0$ for $i < j$.

Definition

Let $A = [a_{ij}] \in M_{n,n}(\mathbb{R})$.

1- The matrix A is said to be an upper triangular matrix if $a_{ij} = 0$ for all $i > j$ and there exists $a_{ij} \neq 0$ for $i < j$.

2- The matrix A is said to be an lower triangular matrix if $a_{ij} = 0$ for all $i < j$ and there exists $a_{ij} \neq 0$ for $i > j$.

Definition

Let $A = [a_{ij}] \in M_{n,n}(\mathbb{R})$.

1- The matrix A is said to be an upper triangular matrix if $a_{ij} = 0$ for all $i > j$ and there exists $a_{ij} \neq 0$ for $i < j$.

2- The matrix A is said to be an lower triangular matrix if $a_{ij} = 0$ for all $i < j$ and there exists $a_{ij} \neq 0$ for $i > j$.

3- The matrix A is said to be a diagonal matrix if $a_{ij} = 0$ for all $i > j$, $i < j$, and there exists $a_{ii} \neq 0$.

Example

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 4 & -5 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 4 & -5 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

- A is an upper triangular matrix

Example

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 7 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 4 & -5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

- A is an upper triangular matrix
- B i a lower triangular matrix

Example

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix} \qquad B = \begin{bmatrix} 7 & 0 & 0 \\ -1 & 3 & 0 \\ 1 & 4 & -5 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

- A is an upper triangular matrix
- B i a lower triangular matrix
- C is a diagonal matrix.

Definition (Matrix Transpose)

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The **transpose** of A is formed by writing its columns as rows and it is denoted by A^T.
Accordingly, the size of A^T is $n \times m$ and if $A^T = [b_{ij}]$ then

$$b_{ij} = a_{ji}$$

for $i = 1, \cdots, m$ and $j = 1, \cdots, n$.

Example

Determine the Transpose of the following Matrices:

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Solution:

Example

Determine the Transpose of the following Matrices:

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Solution: A is an 2×3 matrix with entries a_{ij} given by:

$$\begin{cases} a_{11} = 1, & a_{12} = 2, & a_{13} = 3, \\ a_{21} = 4, & a_{22} = 5, & a_{23} = 6. \end{cases}$$

Example

Determine the Transpose of the following Matrices:

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Solution: A is an 2×3 matrix with entries a_{ij} given by:

$$\begin{cases} a_{11} = 1, & a_{12} = 2, & a_{13} = 3, \\ a_{21} = 4, & a_{22} = 5, & a_{23} = 6. \end{cases}$$

Then, A^T is an 3×2 matrix with entries

Example

Determine the Transpose of the following Matrices:

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Solution: A is an 2×3 matrix with entries a_{ij} given by:

$$\begin{cases} a_{11} = 1, & a_{12} = 2, & a_{13} = 3, \\ a_{21} = 4, & a_{22} = 5, & a_{23} = 6. \end{cases}$$

Then, A^T is an 3×2 matrix with entries

$$\begin{matrix} b_{11} = 1, & b_{12} = 4, \\ b_{21} = 2, & b_{22} = 5, \\ b_{31} = 3, & b_{32} = 6. \end{matrix} \quad \Rightarrow$$

Example

Determine the Transpose of the following Matrices:

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$.

Solution: A is an 2×3 matrix with entries a_{ij} given by:

$$\begin{cases} a_{11} = 1, & a_{12} = 2, & a_{13} = 3, \\ a_{21} = 4, & a_{22} = 5, & a_{23} = 6. \end{cases}$$

Then, A^T is an 3×2 matrix with entries

$$\begin{array}{ll} b_{11} = 1, & b_{12} = 4, \\ b_{21} = 2, & b_{22} = 5, \\ b_{31} = 3, & b_{32} = 6. \end{array} \quad \Rightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Example

2. $\boldsymbol{B} = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$

Example

2. $B = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix} \Longrightarrow B^T = \begin{bmatrix} -3 & 4 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}.$

Example

2. $B = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix} \Longrightarrow B^T = \begin{bmatrix} -3 & 4 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.
3. $C = \begin{bmatrix} 1 \\ 2 \\ -10 \end{bmatrix}$

Example

2. $B = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix} \Longrightarrow B^T = \begin{bmatrix} -3 & 4 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.
3. $C = \begin{bmatrix} 1 \\ 2 \\ -10 \end{bmatrix} \Longrightarrow C^T = \begin{bmatrix} 1 & 2 & -10 \end{bmatrix}$.

Example

2. $B = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix} \implies B^T = \begin{bmatrix} -3 & 4 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.
3. $C = \begin{bmatrix} 1 \\ 2 \\ -10 \end{bmatrix} \implies C^T = \begin{bmatrix} 1 & 2 & -10 \end{bmatrix}$.

Definition (Symmetric Matrix)

An $n \times n$ matrix A is said to be **symmetric** if $A^T = A$, equivalently, when $a_{ij} = a_{ji}$.

Example

2. $B = \begin{bmatrix} -3 & 2 & 1 \\ 4 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix} \Longrightarrow B^T = \begin{bmatrix} -3 & 4 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 5 \end{bmatrix}$.
3. $C = \begin{bmatrix} 1 \\ 2 \\ -10 \end{bmatrix} \Longrightarrow C^T = \begin{bmatrix} 1 & 2 & -10 \end{bmatrix}$.

Definition (Symmetric Matrix)

An $n \times n$ matrix A is said to be **symmetric** if $A^T = A$, equivalently, when $a_{ij} = a_{ji}$.
An $n \times n$ matrix A is said to be **skew-symmetric** if $A^T = -A$, equivalently, when $a_{ij} = -a_{ji}$.

Example

1. $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix}$

Example

1. $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \Longrightarrow A = A^T$

Example

1. $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \Longrightarrow A = A^T$

   $\Longrightarrow$ A is symmetric.

2. $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix}$

Example

1. $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \Longrightarrow A = A^T$

   $\Longrightarrow$ A is symmetric.

2. $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \Longrightarrow B^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} \Longrightarrow B^T = -B$

Example

1. $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \Longrightarrow A = A^T$

   $\Longrightarrow$ A is symmetric.

2. $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \Longrightarrow B^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} \Longrightarrow B^T = -B$

   $\Longrightarrow$ B is skew-symmetric.

3. $C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & -2 & -3 \end{bmatrix}$

Example

1. $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \Longrightarrow A = A^T$

   $\Longrightarrow$ A is symmetric.

2. $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \Longrightarrow B^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} \Longrightarrow B^T = -B$

   $\Longrightarrow$ B is skew-symmetric.

3. $C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & -2 & -3 \end{bmatrix} \Longrightarrow C^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & -2 \\ 2 & 2 & -3 \end{bmatrix}$

Example

1. $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \Longrightarrow A^T = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 1 & 5 \\ 4 & 5 & 3 \end{bmatrix} \Longrightarrow A = A^T$

   $\Longrightarrow$ A is symmetric.

2. $B = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \Longrightarrow B^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} \Longrightarrow B^T = -B$

   $\Longrightarrow$ B is skew-symmetric.

3. $C = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & -2 & -3 \end{bmatrix} \Longrightarrow C^T = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & -2 \\ 2 & 2 & -3 \end{bmatrix}$

   $\Longrightarrow$ C is neither symmetric and nor skew-symmetric.

Theorem (Properties of the Transpose)

Let A and B be two $m \times n$ matrices and let α be a scalar.

Theorem (Properties of the Transpose)

Let A and B be two $m \times n$ matrices and let α be a scalar. Then, the following properties are true:

1. $(A^T)^T = A$.

Theorem (Properties of the Transpose)

Let A and B be two $m \times n$ matrices and let α be a scalar. Then, the following properties are true:

1. $(A^T)^T = A$.
2. $(\alpha A)^T = \alpha A^T$.

Theorem (Properties of the Transpose)

Let A and B be two $m \times n$ matrices and let α be a scalar. Then, the following properties are true:

1. $(A^T)^T = A$.
2. $(\alpha A)^T = \alpha A^T$.
3. $(A + B)^T = A^T + B^T$.

Theorem (Properties of the Transpose)

Let A and B be two $m \times n$ matrices and let α be a scalar. Then, the following properties are true:

1. $(A^T)^T = A$.
2. $(\alpha A)^T = \alpha A^T$.
3. $(A + B)^T = A^T + B^T$.
4. $(AB)^T = B^T A^T$.

Example

Calculate $(AB)^T$ in two methods, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix}$.

Solution:

Example

Calculate $(AB)^T$ in two methods, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix}$.

Solution:
First method:

Example

Calculate $(AB)^T$ in two methods, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix}$.

Solution:

First method:

$$AB = \begin{bmatrix} 10 & 6 & 5 \\ 34 & 23 & 14 \\ 15 & 8 & 9 \end{bmatrix}$$

Example

Calculate $(AB)^T$ in two methods, where $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix}$.

Solution:

First method:

$$AB = \begin{bmatrix} 10 & 6 & 5 \\ 34 & 23 & 14 \\ 15 & 8 & 9 \end{bmatrix} \Longrightarrow (AB)^T = \begin{bmatrix} 10 & 34 & 15 \\ 6 & 23 & 8 \\ 5 & 14 & 9 \end{bmatrix}.$$

Example

Second method:

Example

Second method:

$$A^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 4 \\ 1 & 5 & 1 \end{bmatrix}$$

Example

Second method:

$A^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 4 \\ 1 & 5 & 1 \end{bmatrix}$ and $B^T = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 2 & 1 & 1 \end{bmatrix}$. Then

$$(AB)^T =$$

Example

Second method:

$$A^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 4 \\ 1 & 5 & 1 \end{bmatrix}$$ and $$B^T = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 2 & 1 & 1 \end{bmatrix}.$$ Then

$$(AB)^T = B^T A^T =$$

Example

Second method:

$\boldsymbol{A^T} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{5} & \mathbf{1} \end{bmatrix}$ and $\boldsymbol{B^T} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{5} \\ \mathbf{0} & \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} \end{bmatrix}$. Then

$$\boldsymbol{(AB)^T = B^T A^T} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{5} \\ \mathbf{0} & \mathbf{1} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} & \mathbf{1} \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{5} & \mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{10} & \mathbf{34} & \mathbf{15} \\ \mathbf{6} & \mathbf{23} & \mathbf{8} \\ \mathbf{5} & \mathbf{14} & \mathbf{9} \end{bmatrix}.$$

Example

Given $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{2} \\ \mathbf{2} & \mathbf{3} & \mathbf{4} \end{bmatrix}$, Find $\boldsymbol{AA^T}$ and $\boldsymbol{(AA^T)^T}$. What can you conclude?