Linear Algebra in Image Compression.

* Apply SVD to an image matrix to decompose it into singular vectors and values. By retaining only the most significant components, you can represent the image using fewer parameters, achieving compression while preserving essential features.

1. Overview:

- SVD is a factorization method that decomposes a matrix into three other matrices, representing the singular vectors and values
- For an image matrix A, the SVD is written as $A = U \Sigma V^T$, where U and V are orthogonal matrices, and Σ is a diagonal matrix with singular values

2. Compression Steps:

- Decomposition: given an image matrix A, perform SVD to obtain U, Σ, and V
- Rank Reduction: retain only the first K singular values and their corresponding columns in U and V. This effectively reduces the rank of the original matrix.

3. Compression Ratio:

- The compression ratio is determined by the choice of K. Smaller K values result in higher compression but may lead to a loss of image quality
- Compression ratio is calculated as : $\frac{\text{original size}}{\text{compressed size}}$

4. Image Reconstruction:

- The compressed image can be reconstructed using the truncated matrices U_K, Σ_K, and V_K^T where K is the nb of retained singular values
- The reconstructed image $A_K = U_K \Sigma_K V_K^T$

Scanned with CamScanner

STEPS

→ • A: m×n

• U: m×m orthogonal matrix containing the left singular vectors

• Σ: m×n diagonal matrix containing the singular values

• V^T: n×n orthogonal matrix containing the right singular vecto

→ Truncation:

• to compress the image, you truncate these matrices by keepin only the first K columns in U and V, and first K×K subma in Σ.

• the truncation effectively reduces the dimensions of the matri and K determines the level of compression. Smaller K values in higher compression but may lead to loss of informatio

→ Truncated Matrices:

The truncated matrices are denoted as U_K, Σ_K, V_K^T where

• U_K contains the 1st K columns of U.

• Σ_K contains the upper left K×K submatrix of Σ.

• V_K^T " the first K rows of V^T.

Exercise 1.

$$A = \begin{pmatrix} 4 & 11 \\ 7 & 2 \end{pmatrix}$$

1) Compute A^TA

$$A^TA = \begin{pmatrix} 4 & 7 \\ 11 & 2 \end{pmatrix}\begin{pmatrix} 4 & 11 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} 65 & 46 \\ 46 & 125 \end{pmatrix}$$

2) Eigenvalues of A^TA:

$$|A^TA - \lambda I| = 0 \Rightarrow \begin{vmatrix} 65-\lambda & 46 \\ 46 & 125-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 190\lambda + 2109 = 0$$

$$\Rightarrow \lambda_1 = 178.16 \quad \lambda_2 = 11.84$$

3) Eigenvectors of A^TA:

$(A^TA - \lambda I)v = 0$

For λ_1,
$$\begin{pmatrix} 65-178.16 & 46 \\ 46 & 125-178.16 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For λ_2,
$$\begin{pmatrix} 65-11.84 & 46 \\ 46 & 125-11.84 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow v_1 = \begin{pmatrix} -0.879 \\ 0.476 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -0.476 \\ -0.879 \end{pmatrix}.$$

These vectors form the colomns of V.

4) Compute Σ:

the singular values σ are the square roots of the eigenval
of A^TA. We arrange them in decreasing order along the di
of Σ.

$\sigma_1 = \sqrt{\lambda_1}$; $\sigma_2 = \sqrt{\lambda_2}$ $\rightarrow$ $\Sigma = \begin{pmatrix} 13.35 & 0 \\ 0 & 3.44 \end{pmatrix}$

$\sigma_1 = 13.35$; $\sigma_2 = 3.44$

5) Compute U:

To compute U, we use $u_i = \frac{1}{\sigma_i} A v_i$ for each column u_i
of U, where v_i are the columns of V.

$$U = \begin{pmatrix} 0.314 & -0.915 \\ -0.949 & -0.403 \end{pmatrix}$$

$\rightarrow$ For $K=1$,

U_1 = column 1 of U

$\Sigma_1 = \sigma_1 = 13.35$

V_1^T = row 1 of V^T

– E.V unit vectors

– $U^TU = I$, $V^TV = I$

$$A_1 = \begin{pmatrix} -3.69 & 1.998 \\ 11.14 & -6.04 \end{pmatrix}.$$

Exercise 2.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 0 \end{pmatrix}$$

- $A^T A = \begin{pmatrix} 10 & 0 \\ 0 & 4 \end{pmatrix}$

- $\lambda_1 = 10 \quad \lambda_2 = 4 \rightarrow \sigma_1 = \sqrt{10} \quad \sigma_2 = 2 \rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

- $\Sigma = \begin{pmatrix} \sqrt{10} & 0 \\ 0 & 2 \\ \cancel{0} & \cancel{0} \end{pmatrix} \quad V = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

- For $K=1$, $A_1 = u_1 \sigma_1 v_1^T$
- For $K=2$, $A_2 = U \Sigma V^T = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 3 & 0 \end{pmatrix}$