Chapter 1: Linear Systems and Matrices
Section: Determinants
Lecture #7

Lebanese University

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Course Plan

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Example

Find the determinant of the following $\mathbf{3} \times \mathbf{3}$ matrix:

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Solution:

Example

Find the determinant of the following $\mathbf{3} \times \mathbf{3}$ matrix:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

Solution:

$$|A| = \begin{vmatrix} 0 & 2 & 1 \\ 3 & 1 & 2 \\ 4 & 0 & 1 \end{vmatrix} = \begin{vmatrix} +0 & -2 & +1 \\ 3 & 1 & 2 \\ 4 & 0 & 1 \end{vmatrix}$$

$$= +0 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} -2 \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} +1 \begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix}$$

$$= +0(1-0) - 2(3-8) + 1(0-4)$$

$$= 0 + 10 - 4 = 6$$

Definition (Cofactor Matrix)

Let $A = [a_{ij}]$ be an $n \times n$ matrix. We define the **Cofactor** matrix of A by:

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$$Cof(A) = [A_{ij}] = \begin{bmatrix} +|M_{11}| & -|M_{12}| & \cdots & \cdots \\ -|M_{21}| & +|M_{22}| & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}|M_{n1}| & \cdots & \cdots & \cdots \end{bmatrix}$$

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where $A_{ij} = (-1)^{i+j}|M_{ij}|$ is the Cofactor of A associated to the entry a_{ij} and M_{ij} is the Minor of A associated to the entry a_{ij} .

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where $A_{ij} = (-1)^{i+j}|M_{ij}|$ is the Cofactor of A associated to the entry a_{ij} and M_{ij} is the Minor of A associated to the entry a_{ij} .

Definition (Adjoint Matrix)

Let $A = [a_{ij}]$ be an $n \times n$ matrix. We define the **Adjoint** matrix of A the new matrix given by:

$$Adj(A) = Cof(A)^T$$

Example

Find the Adjoint of the following matrix:

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{2} & \mathbf{2} \\ \mathbf{1} & \mathbf{2} & \mathbf{3} \end{bmatrix}$$

Solution:

Example

Find the Adjoint of the following matrix:

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Solution: The matrix of cofactors of A is:

$$Cof(A) = \begin{bmatrix} +\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} & +\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} +2 & -7 & +4 \\ -(-1) & +4 & -3 \\ +-2 & -(-2) & +1 \end{bmatrix}$$

Example

Thus

$$Cof(A) = \begin{bmatrix} 2 & -7 & 4 \\ 1 & 4 & -3 \\ -2 & 2 & 1 \end{bmatrix}$$

So the adjoint of A is:

$$Adj(A) = Cof(A)^T = \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

Theorem

Let A be an $n \times n$ matrix. If A is invertible ($|A| \neq 0$), then the inverse matrix of A is given by

$$A^{-1} = \frac{1}{det(A)} adj(A).$$

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$$A^{-1} = \frac{1}{det(A)} adj(A).$$

Example

The inverse of $A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ is

$$A^{-1} = \frac{1}{det(A)} \cdot Adj(A) = \frac{1}{6} \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

Steps

So to find the inverse of a matrix $\boldsymbol{A}$:

Step 1. Compute the $\boldsymbol{Cof(A)}$.

Step 2. Find $\boldsymbol{Adj(A) = Cof(A)^T}$.

Step 3. Set $\boldsymbol{A^{-1} = \frac{1}{det(A)} Adj(A)}$.

Example

Find the inverse of the following $\mathbf{3} \times \mathbf{3}$ matrix:

$$A = \begin{bmatrix} \mathbf{0} & \mathbf{2} & \mathbf{1} \\ \mathbf{3} & \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Solution:

Example

Find the inverse of the following $\mathbf{3} \times \mathbf{3}$ matrix:

$$A = \begin{bmatrix} 0 & 2 & 1 \\ 3 & 1 & 2 \\ 4 & 0 & 1 \end{bmatrix}$$

Solution:

$$Cof(A) = \begin{bmatrix} +\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ 4 & 0 \end{vmatrix} \\ -\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 4 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 4 & 0 \end{vmatrix} \\ +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 1 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} +(1) & -(-5) & +(-4) \\ -(2) & +(-4) & -(-8) \\ +(3) & -(-3) & +(-6) \end{bmatrix}$$

Example

Then the cofactor matrix

$$\boldsymbol{Cof(A)} = \begin{bmatrix} 1 & 5 & -4 \\ -2 & -4 & 8 \\ 3 & 3 & -6 \end{bmatrix}$$

So

$$\boldsymbol{Adj(A)} = \begin{bmatrix} 1 & -2 & 3 \\ 5 & -4 & 3 \\ -4 & 8 & -6 \end{bmatrix}$$

Thus

$$\boldsymbol{A^{-1}} = \frac{1}{6}\begin{bmatrix} 1 & -2 & 3 \\ 5 & -4 & 3 \\ -4 & 8 & -6 \end{bmatrix}$$

Theorem:

Let A be an invertible $n \times n$ matrix and let $b = (b_1, b_2, .., b_n)^T \in \mathbb{R}^n$. Let A_i be the matrix obtained by replacing the ith column of A by b. If $X = (x_1, x_2, .., x_n)^T$ is the unique solution of $Ax = b$, then

$$x_i = \frac{det(A_i)}{det(A)}, \quad , i = 1, .., n.$$

Example (Find the solution a system using Cramer's Rule)

$$(S) : \begin{cases} -x & -2y & +3z & = & 9, \\ -x & +3y & & = & -4, \\ 2x & -5y & +5z & = & 17. \end{cases}$$

Solution:

Example (Find the solution a system using Cramer's Rule)

$$(S):\begin{cases} -x & -2y & +3z & = & 9, \\ -x & +3y & & = & -4, \\ 2x & -5y & +5z & = & 17. \end{cases}$$

Solution: The coefficient matrix of this system is:

$$A = \begin{bmatrix} -1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{bmatrix} \text{ and } |A| = \begin{vmatrix} -1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{vmatrix} = -28 \neq 0$$

So the system has a unique solution:

$$x = \frac{|A_1|}{|A|}, \qquad y = \frac{|A_2|}{|A|}, \qquad z = \frac{|A_3|}{|A|}$$

where:

$$|A_1| = \begin{vmatrix} 9 & -2 & 3 \\ -4 & 3 & 0 \\ 17 & -5 & 5 \end{vmatrix}, \quad |A_2| = \begin{vmatrix} -1 & 9 & 3 \\ -1 & -4 & 0 \\ 2 & 17 & 5 \end{vmatrix}, \quad |A_3| = \begin{vmatrix} -1 & -2 & 9 \\ -1 & 3 & -4 \\ 2 & -5 & 17 \end{vmatrix}$$

Example

so the solution is:

$$x = \frac{2}{-28}, \qquad y = \frac{38}{-28}, \qquad z = \frac{-58}{-28}$$

Example (1)

Find all possible choices of c that would make the following matrix singular: $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{bmatrix}$

Solution

We have

$$det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 9 & c \\ 1 & c & 3 \end{vmatrix} = +(27 - c^2) - (3 - c) + (c - 9)$$

$$= -c^2 + 2c + 15 = (c - 5)(c + 3).$$

Thus

$$det(A) = 0 \leftrightarrow (c - 5)(c + 3) = 0 \leftrightarrow c = 5 \text{ or } c = -3.$$

So A is singular if $det(A) = 0$, so if $c = 5$ or $c = -3$ the matrix A is singular.

Example

Use row operations to find the determinant of this matrix:

1. $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{bmatrix}$

2. $B = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{bmatrix}$

Solution

1. We have

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix} = - \begin{vmatrix} 2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5 \end{vmatrix}$$
$$= -(2 \times -6 \times -5) = -60.$$

Solution

1. We have

$$|A| = \begin{vmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 6 & -3 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 0 & -5 \\ 0 & -6 & -5 \end{vmatrix} = - \begin{vmatrix} 2 & 1 & 3 \\ 0 & -6 & -5 \\ 0 & 0 & -5 \end{vmatrix}$$
$$= -(2 \times -6 \times -5) = -60.$$

The operations:

$$R_2 - 2R_1 \longleftrightarrow R_2$$
$$R_3 - 3R_1 \longrightarrow R_3$$
$$R_2 \leftrightarrow R_3$$

Solution

2. We have

$$|B| = \begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 \end{vmatrix} =$$

$$= (1 \times 3 \times 2 \times 5) = 30.$$

Solution

2. We have

$$|B| = \begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ -1 & -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 3 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 5 \end{vmatrix} =$$

$$= (1 \times 3 \times 2 \times 5) = 30.$$

The operations:

$$R_4 + R_1 \longleftrightarrow R_4$$

Example

Use Cramer's rule to find the solution of the following systems

1. $(S_1):\begin{cases} x + 2y = 3, \\ 3x - y = 1. \end{cases}$

2. $(S_2):\begin{cases} -x + 2y - 3z = 1, \\ 2x + z = 0, \\ 3x - 4y + 4z = 2. \end{cases}$

Solution

1. We have: The coefficient matrix of this system is:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \text{ and } |A| = -1 - 6 = -7 \neq 0$$

So the system has a unique solution:

$$x = \frac{|A_1|}{|A|}, \qquad y = \frac{|A_2|}{|A|}$$

where:

$$|A_1| = \begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} = -5, \quad |A_2| = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -8$$

so

$$x = \frac{-5}{-7}, \qquad y = \frac{-8}{-7} \qquad \Longrightarrow S = \left\{\left(\frac{5}{7}, \frac{8}{7}\right)\right\}$$

Solution

2. We have: The coefficient matrix of this system is:

$$A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{bmatrix} \text{ and } |A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10 \neq 0$$

So the system has a unique solution:

$$x = \frac{|A_1|}{|A|}, \qquad y = \frac{|A_2|}{|A|}, \qquad z = \frac{|A_3|}{|A|}$$

where:

$$|A_1| = \begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}, \quad |A_2| = \begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix}, \quad |A_3| = \begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix}$$

solution

So the solution is:

$$x = \frac{8}{10}, \qquad y = \frac{-15}{10}, \qquad z = \frac{-16}{10}$$

$$\Longrightarrow \; S = \left\{ \left(\frac{8}{10}, \frac{-15}{10}, \frac{-16}{10} \right) \right\}$$

Example

Find the inverse of the following matrices using Adjoint and Cofactors:

1. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -1 \\ 2 & 2 & 2 \end{bmatrix}$

2. $A = \begin{bmatrix} -3 & -5 & -7 \\ 2 & 4 & 3 \\ 0 & 1 & -1 \end{bmatrix}$

Solution

1. We have

$$Cof(A) = \begin{bmatrix} +\begin{vmatrix} 1 & -1 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 0 & 1 \\ 2 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(4) & -(2) & +(-2) \\ -(-2) & +(-4) & -(-2) \\ +(-5) & -(-1) & +(1) \end{bmatrix}$$

block

Then the cofactor matrix

$$Cof(A) = \begin{bmatrix} 4 & -2 & -2 \\ -2 & -4 & 2 \\ -5 & 1 & 1 \end{bmatrix}$$

So

$$Adj(A) = \begin{bmatrix} 4 & -2 & -5 \\ -2 & -4 & 1 \\ -2 & 2 & 1 \end{bmatrix}$$

Thus

$$A^{-1} = \frac{-1}{6}\begin{bmatrix} 4 & -2 & -5 \\ -2 & -4 & 1 \\ -2 & 2 & 1 \end{bmatrix}$$

Solution

2. We have

$$Cof(A) = \begin{bmatrix} +\begin{vmatrix} 4 & 3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix} & +\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} \\ -\begin{vmatrix} -5 & -7 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} -3 & -7 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} -3 & -5 \\ 0 & 1 \end{vmatrix} \\ +\begin{vmatrix} -5 & -7 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} -3 & -7 \\ 2 & 3 \end{vmatrix} & +\begin{vmatrix} -3 & -5 \\ 2 & 4 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} +(-7) & -(-2) & +(2) \\ -(12) & +(3) & -(-3) \\ +(13) & -(5) & +(-2) \end{bmatrix}$$

block

Then the cofactor matrix

$$Cof(A) = \begin{bmatrix} -7 & 2 & 2 \\ -12 & 3 & 3 \\ 13 & -5 & -2 \end{bmatrix}$$

So

$$Adj(A) = \begin{bmatrix} -7 & -12 & 13 \\ 2 & 3 & -5 \\ 2 & 3 & -2 \end{bmatrix}$$

Thus

$$A^{-1} = \frac{-1}{3} \begin{bmatrix} -7 & -12 & 13 \\ 2 & 3 & -5 \\ 2 & 3 & -2 \end{bmatrix}$$