Chapter 1: Linear Systems and Matrices
Section: Matrices and Gaussian Elimination
Lecture #3

Lebanese University

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Course Plan

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Definition

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a) The first nonzero entry in each nonzero row is **1** (called a leading **1**).
b) Every leading 1 is to the right of the one above it.
c) Any row that consists of zero entries must be at the bottom of the matrix.

Remark

All the entries below the leading 1 from each row are zeros.

Example (REF)

$$\begin{bmatrix} \mathbf{1} & \mathbf{4} & \mathbf{2} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix} \quad \begin{bmatrix} \mathbf{1} & \mathbf{4} & \mathbf{6} & \mathbf{4} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Example (Not REF)

$$\begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 6 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Definition

The **Elementary Row Operation** that we can apply on a matrix are:

a) Interchange two rows.
b) Multiply a row with a nonzero constant.
c) Add a multiple of a row to another row.

Example

Find the REF of the following Matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

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$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \quad R_1 + R_2 \longrightarrow R_2 \Longrightarrow A_1 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

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$$A_1 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix} R_3 - 2R_1 \longrightarrow R_3 \Longrightarrow$$

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$$A_1 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 2 & -5 & 5 & 17 \end{bmatrix} R_3 - 2R_1 \longrightarrow R_3 \Longrightarrow A_2 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

Example

$$A_2 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

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Example

$$A_2 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} R_3 + R_2 \longrightarrow R_3 \Longrightarrow A_3 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \frac{1}{2} R_2 \longrightarrow R_2 \Longrightarrow$$

Example

$$A_2 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} R_3 + R_2 \longrightarrow R_3 \Longrightarrow A_3 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \frac{1}{2} R_2 \longrightarrow R_2 \Longrightarrow A_4 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Example

$$A_2 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{bmatrix} R_3 + R_2 \longrightarrow R_3 \Longrightarrow A_3 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \frac{1}{2} R_2 \longrightarrow R_2 \Longrightarrow A_4 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

How can we get benefits from REF to solve systems of Linear Equations?

Definition (Coefficient Matrix)

Consider the system of linear equations

$$(S):\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2, \\ \quad\quad\quad\quad\quad\quad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m, \end{cases}$$

The **coefficient matrix** of S is:

$$C = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \vdots & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

Definition (Augmented Matrix)

Consider the system of equations

$$(S):\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2, \\ \qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m, \end{cases}$$

The **Augmented matrix** of S is:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & : & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & : & b_2 \\ & & \vdots & & : & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & : & b_m \end{bmatrix}$$

Strategy

To solve a system of Linear equation using **Gaussian Elimination with back substitution**, we apply the following steps:

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Strategy

To solve a system of Linear equation using **Gaussian Elimination with back substitution**, we apply the following steps:

a) Write the **Augmented Matrix** associated to the system.
b) Apply the **Elementary Row Operation** in order to transform the **Augmented Matrix in Row Echelon Form - REF**.

Strategy

To solve a system of Linear equation using **Gaussian Elimination with back substitution**, we apply the following steps:

a) Write the **Augmented Matrix** associated to the system.
b) Apply the **Elementary Row Operation** in order to transform the **Augmented Matrix in Row Echelon Form - REF**.
c) Write the **equivalent system** associated to the **REF Matrix**.

Strategy

To solve a system of Linear equation using **Gaussian Elimination with back substitution**, we apply the following steps:

a) Write the **Augmented Matrix** associated to the system.
b) Apply the **Elementary Row Operation** in order to transform the **Augmented Matrix in Row Echelon Form - REF**.
c) Write the **equivalent system** associated to the **REF Matrix**.
d) Use substitution method to solve the obtained system.

Example

Solve the following System of Linear Equations:

$$(S_1)\begin{cases} x_1 & -2x_2 & +3x_3 & = & 9, \\ -x_1 & +3x_2 & & = & -4, \\ 2x_1 & -5x_2 & +5x_3 & = & 17. \end{cases}$$

The **Augmented matrix** of system (S_1) is given by

$$A = \left[\begin{array}{ccc:c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right]$$

Note: The **coefficient matrix** associated to system (S_1) is given by

$$B = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 3 & 0 \\ 2 & -5 & 5 \end{bmatrix}$$

Use the augmented matrix to solve system (S_1)

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix}$$

Use the augmented matrix to solve system (S_1)

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow{\text{REF}}$$

Use the augmented matrix to solve system (S_1)

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow{\text{REF}} A_4 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Use the augmented matrix to solve system (S_1)

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{bmatrix} \xrightarrow{\text{REF}} A_4 = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

So the following systems have the same solution:

$$(S_1) \begin{cases} x_1 & -2x_2 & +3x_3 & = & 9, \\ -x_1 & +3x_2 & & = & -4, \\ 2x_1 & -5x_2 & +5x_3 & = & 17. \end{cases} \quad (S_4) \begin{cases} x_1 & -2x_2 & +3x_3 & = & 9, \\ & x_2 & +3x_3 & & 5, \\ & & x_3 & = & 2. \end{cases}$$

Using back substitution the solution of system (S_4) is $x_1 = 1$, $x_2 = -1$ and $x_3 = 2$.
Thus the solution of (S_1) is $S = \{(1, -1, 2)\}$

Definition

A linear System is said to be **Overdetermined** if there are more equations than unknowns, $m > n$. Overdetermined systems are *usually* (but not always) inconsistent.

Example

Solve the following systems:

a)

$$\begin{cases} x_1 & +x_2 & = & 1, \\ x_1 & -x_2 & = & 3, \\ -x_1 & +2x_2 & = & -2. \end{cases}$$

Solution:

Example

Solve the following systems:

a)

$$\begin{cases} x_1 & +x_2 & = & 1, \\ x_1 & -x_2 & = & 3, \\ -x_1 & +2x_2 & = & -2. \end{cases}$$

Solution: Step 1 and Step 2:

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Solve the following systems:

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Solution: Step 1 and Step 2:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix}$$

Example

Solve the following systems:

a)

$$\begin{cases} x_1 & +x_2 & = & 1, \\ x_1 & -x_2 & = & 3, \\ -x_1 & +2x_2 & = & -2. \end{cases}$$

Solution: Step 1 and Step 2:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix} \quad R_2 - R_1 \longrightarrow R_2$$

Example

Solve the following systems:

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$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix} \quad R_2 - R_1 \longrightarrow R_2 \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

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Solve the following systems:

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<u>Solution:</u> Step 1 and Step 2:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix} \quad R_2 - R_1 \longrightarrow R_2 \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

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Solution: Step 1 and Step 2:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix} \quad R_2 - R_1 \longrightarrow R_2 \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \quad R_3 + R_1 \longrightarrow R_3$$

Example

Solve the following systems:

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$$\begin{cases} x_1 & +x_2 & = & 1, \\ x_1 & -x_2 & = & 3, \\ -x_1 & +2x_2 & = & -2. \end{cases}$$

Solution: Step 1 and Step 2:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{bmatrix} \quad R_2 - R_1 \longrightarrow R_2 \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ -1 & 2 & -2 \end{bmatrix} \quad R_3 + R_1 \longrightarrow R_3 \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 3 & -1 \end{bmatrix}$$

Example

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{-2} & \mathbf{2} \\ \mathbf{0} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 3 & -1 \end{bmatrix} \quad R_2 \times -\frac{1}{2}$$

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$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 3 & -1 \end{bmatrix} \quad R_2 \times -\frac{1}{2} \longrightarrow R_2 \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -1 \end{bmatrix} \quad R_3 - 3R_2 \longrightarrow R_3$$

Example

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 3 & -1 \end{bmatrix} \quad R_2 \times -\frac{1}{2} \longrightarrow R_2 \Longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -1 \end{bmatrix}$$

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Example

Step 3:

$$\begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Example

Step 3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{cases} x_1 & +x_2 & = & 1, \\ & x_2 & = & -1, \\ & 0 & = & 1. \end{cases}$$

Example

Step 3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{cases} x_1 & +x_2 & = & 1, \\ & x_2 & = & -1, \\ & 0 & = & 1. \quad \text{✗} \end{cases}$$

Example

Step 3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{cases} x_1 & +x_2 & = & 1, \\ & x_2 & = & -1, \\ & 0 & = & 1. \end{cases} \quad \boldsymbol{X} \quad \text{False Statement}$$

Example

Step 3:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{cases} x_1 & +x_2 & = & 1, \\ & x_2 & = & -1, \\ & 0 & = & 1. \end{cases} \quad \boldsymbol{X} \quad \text{False Statement}$$

The system has no solution. The system is inconsistent.

b)

$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & -x_2 & +x_3 & = & 2, \\ 4x_1 & +3x_2 & +3x_3 & = & 4, \\ 2x_1 & -x_2 & +3x_3 & = & 5. \end{cases}$$

Solution:

Example

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{3} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} & \mathbf{5} \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{bmatrix} \textbf{\textit{REF}}$$

Example

$$\begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{1} & \mathbf{1} \\ \mathbf{2} & \mathbf{-1} & \mathbf{1} & \mathbf{2} \\ \mathbf{4} & \mathbf{3} & \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{-1} & \mathbf{3} & \mathbf{5} \end{bmatrix} \boldsymbol{REF} \Longrightarrow$$

Example

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{bmatrix} REF \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{bmatrix} \textbf{\textit{REF}} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Example

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$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ & x_2 & +\frac{1}{5}x_3 & = & 0, \\ & & x_3 & = & \frac{3}{2}, \\ & & 0 & = & 0. \end{cases}$$

Example

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{bmatrix} \textit{REF} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ & x_2 & +\frac{1}{5}x_3 & = & 0, \\ & & x_3 & = & \frac{3}{2}, \\ & & 0 & = & 0. \end{cases}$$

Using Back substitution, we deduce the system has a unique solution $x_1 = \frac{1}{10}$, $x_2 = -\frac{3}{10}$ and $x_3 = \frac{3}{2}$.

Example

c)

$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & -x_2 & +x_3 & = & 2, \\ 4x_1 & +3x_2 & +3x_3 & = & 4, \\ 3x_1 & +x_2 & +2x_3 & = & 3. \end{cases}$$

Solution:

Example

c)

$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & -x_2 & +x_3 & = & 2, \\ 4x_1 & +3x_2 & +3x_3 & = & 4, \\ 3x_1 & +x_2 & +2x_3 & = & 3. \end{cases}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

Example

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$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & -x_2 & +x_3 & = & 2, \\ 4x_1 & +3x_2 & +3x_3 & = & 4, \\ 3x_1 & +x_2 & +2x_3 & = & 3. \end{cases}$$

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$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{bmatrix} \textbf{\textit{REF}}$$

Example

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$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & -x_2 & +x_3 & = & 2, \\ 4x_1 & +3x_2 & +3x_3 & = & 4, \\ 3x_1 & +x_2 & +2x_3 & = & 3. \end{cases}$$

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Example

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$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & -x_2 & +x_3 & = & 2, \\ 4x_1 & +3x_2 & +3x_3 & = & 4, \\ 3x_1 & +x_2 & +2x_3 & = & 3. \end{cases}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{bmatrix} REF \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Example

c)

$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & -x_2 & +x_3 & = & 2, \\ 4x_1 & +3x_2 & +3x_3 & = & 4, \\ 3x_1 & +x_2 & +2x_3 & = & 3. \end{cases}$$

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$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{bmatrix} REF \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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Example

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$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & -x_2 & +x_3 & = & 2, \\ 4x_1 & +3x_2 & +3x_3 & = & 4, \\ 3x_1 & +x_2 & +2x_3 & = & 3. \end{cases}$$

Solution:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{bmatrix} REF \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow \begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ & x_2 & +\frac{1}{5}x_3 & = & 0, \\ & & 0 & = & 0, \\ & & 0 & = & 0. \end{cases}$$

Example

Setting $x_3 = t \in \mathbb{R}$. Then, the system has an infinite number of solutions given by

$$\begin{cases} x_1 &= 1 - \frac{6}{10}t, \\ x_2 &= -\frac{2}{10}t, \\ x_3 &= t. \end{cases}$$

Definition

A system of Linear Equation is said to be **under-determined** if the number of equations is less than the number of unknowns, i.e. if $m < n$.
Under-determined systems are usually (but not always) consistent with infinitely number of solutions.

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A system of Linear Equation is said to be **under-determined** if the number of equations is less than the number of unknowns, i.e. if $m < n$.
Under-determined systems are usually (but not always) consistent with infinitely number of solutions.

Remark:

It is not possible for an **under-determined** system to have a unique solution.

Example

a)

$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & +4x_2 & +2x_3 & = & 3. \end{cases}$$

solution:

Example

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$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & +4x_2 & +2x_3 & = & 3. \end{cases}$$

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$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{bmatrix}$$

Example

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$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & +4x_2 & +2x_3 & = & 3. \end{cases}$$

solution:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{bmatrix} REF$$

Example

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$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{bmatrix} REF \Longrightarrow$$

Example

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solution:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{bmatrix} \mathit{REF} \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

a)

$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & +4x_2 & +2x_3 & = & 3. \end{cases}$$

solution:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{bmatrix} REF \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example

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$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & +4x_2 & +2x_3 & = & 3. \end{cases}$$

solution:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{bmatrix} REF \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ & & 0 & = & 1. \end{cases}$$

Example

a)

$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & +4x_2 & +2x_3 & = & 3. \end{cases}$$

solution:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{bmatrix} REF \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ & & 0 & = & 1. \end{cases} \quad X$$

Example

a)

$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & +4x_2 & +2x_3 & = & 3. \end{cases}$$

solution:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{bmatrix} REF \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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alertFalse Statement

Example

a)

$$\begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ 2x_1 & +4x_2 & +2x_3 & = & 3. \end{cases}$$

solution:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{bmatrix} REF \Longrightarrow \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Longrightarrow \begin{cases} x_1 & +2x_2 & +x_3 & = & 1, \\ & & 0 & = & 1. \end{cases} \quad X \quad \text{alertFalse Statement}$$

The system is inconsistent. The system has **no solutions**.

Example

b)

$$\begin{cases} x_1 & +x_2 & +x_3 & +x_4 & +x_5 & = & 2, \\ x_1 & +x_2 & +x_3 & +2x_4 & +2x_5 & = & 3, \\ x_1 & +x_2 & +x_3 & +2x_4 & +3x_5 & = & 2. \end{cases}$$

Solution:

Example

b)

$$\begin{cases} x_1 & +x_2 & +x_3 & +x_4 & +x_5 & = & 2, \\ x_1 & +x_2 & +x_3 & +2x_4 & +2x_5 & = & 3, \\ x_1 & +x_2 & +x_3 & +2x_4 & +3x_5 & = & 2. \end{cases}$$

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{bmatrix}$$

Example

b)

$$\begin{cases} x_1 & +x_2 & +x_3 & +x_4 & +x_5 & = & 2, \\ x_1 & +x_2 & +x_3 & +2x_4 & +2x_5 & = & 3, \\ x_1 & +x_2 & +x_3 & +2x_4 & +3x_5 & = & 2. \end{cases}$$

Solution:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right] \textit{REF}$$

Example

b)

$$\begin{cases} x_1 & +x_2 & +x_3 & +x_4 & +x_5 & = & 2, \\ x_1 & +x_2 & +x_3 & +2x_4 & +2x_5 & = & 3, \\ x_1 & +x_2 & +x_3 & +2x_4 & +3x_5 & = & 2. \end{cases}$$

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{bmatrix} REF \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Example

b)

$$\begin{cases} x_1 & +x_2 & +x_3 & +x_4 & +x_5 & = & 2, \\ x_1 & +x_2 & +x_3 & +2x_4 & +2x_5 & = & 3, \\ x_1 & +x_2 & +x_3 & +2x_4 & +3x_5 & = & 2. \end{cases}$$

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Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{bmatrix} REF \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Longrightarrow$$

Example

b)

$$\begin{cases} x_1 & +x_2 & +x_3 & +x_4 & +x_5 & = & 2, \\ x_1 & +x_2 & +x_3 & +2x_4 & +2x_5 & = & 3, \\ x_1 & +x_2 & +x_3 & +2x_4 & +3x_5 & = & 2. \end{cases}$$

Solution:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{bmatrix} REF \Longrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \Longrightarrow \begin{cases} x_1 & +x_2 & +x_3 & +x_4 & +x_5 & = & 2, \\ & & & x_4 & +x_5 & = & 1, \\ & & & & x_5 & = & -1. \end{cases}$$

Example

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 2, \\ \qquad\qquad\qquad\quad x_4 + x_5 = 1, \\ \qquad\qquad\qquad\qquad\quad x_5 = -1. \end{cases}$$

Example

$$\begin{cases} x_1 & +x_2 & +x_3 & +x_4 & +x_5 & = & 2, \\ & & & x_4 & +x_5 & = & 1, \\ & & & & x_5 & = & -1. \end{cases}$$

Setting $x_2 = t \in \mathbb{R}$ and $x_3 = s \in \mathbb{R}$, then the system has an infinite number of solutions given by

Example

$$\begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 = 2, \\ x_4 + x_5 = 1, \\ x_5 = -1. \end{cases}$$

Setting $x_2 = t \in \mathbb{R}$ and $x_3 = s \in \mathbb{R}$, then the system has an infinite number of solutions given by

$$\begin{cases} x_1 = 1 - t - s, \\ x_2 = t, \\ x_3 = s, \\ x_4 = 2, \\ x_5 = -1. \end{cases}$$

Definition (RREF)

A matrix is said to be in **reduced row echelon form** if

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b) The first nonzero entry in each row (leading one) is the only nonzero entry in its columns.

Example

a) $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

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b) $\boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{1} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}$

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Example

a) $\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}$ is in **RREF**.

b) $\boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \mathbf{1} & \mathbf{3} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{2} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{bmatrix}$ is not in **RREF** ,

Definition (RREF)

A matrix is said to be in **reduced row echelon form** if

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Example

a) $\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is in **RREF**.

b) $\boldsymbol{B} = \begin{bmatrix} 1 & 3 & 1 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ is not in **RREF** , but it is in **REF**.

Strategy

To solve a system of Linear equation using **Gauss-Jordan Reduction - GJR** , we apply the following steps:

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a) Apply Gaussian Elimination method without back substitution.

Strategy

To solve a system of Linear equation using **Gauss-Jordan Reduction - GJR** , we apply the following steps:

a) Apply Gaussian Elimination method without back substitution.

b) Apply the **Elementary Row Operation** in order to transform the obtained matrix in **RREF**.

Example

Use **Gauss-Jordan Reduction** to solve the system

$$\begin{cases} -x_1 & +x_2 & -x_3 & +3x_4 & = & 0, \\ 3x_1 & +x_2 & -x_3 & -x_4 & = & 0, \\ 2x_1 & -x_2 & -2x_3 & -x_4 & = & 0. \end{cases}$$

Solution:

Example

Use **Gauss-Jordan Reduction** to solve the system

$$\begin{cases} -x_1 & +x_2 & -x_3 & +3x_4 & = & 0, \\ 3x_1 & +x_2 & -x_3 & -x_4 & = & 0, \\ 2x_1 & -x_2 & -2x_3 & -x_4 & = & 0. \end{cases}$$

Solution:

$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{bmatrix}$$

Example

Use **Gauss-Jordan Reduction** to solve the system

$$\begin{cases} -x_1 & +x_2 & -x_3 & +3x_4 & = & 0, \\ 3x_1 & +x_2 & -x_3 & -x_4 & = & 0, \\ 2x_1 & -x_2 & -2x_3 & -x_4 & = & 0. \end{cases}$$

Solution:

$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{bmatrix} \begin{array}{l} R_2 + 3R_1 \longrightarrow R_2 \\ R_3 + 2R_1 \longrightarrow R_3 \end{array}$$

Example

Use **Gauss-Jordan Reduction** to solve the system

$$\begin{cases} -x_1 & +x_2 & -x_3 & +3x_4 & = & 0, \\ 3x_1 & +x_2 & -x_3 & -x_4 & = & 0, \\ 2x_1 & -x_2 & -2x_3 & -x_4 & = & 0. \end{cases}$$

Solution:

$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{bmatrix} \begin{matrix} R_2 + 3R_1 \longrightarrow R_2 \\ R_3 + 2R_1 \longrightarrow R_3 \end{matrix} \Longrightarrow$$

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Solution:

$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{bmatrix} \begin{matrix} R_2 + 3R_1 \longrightarrow R_2 \\ R_3 + 2R_1 \longrightarrow R_3 \end{matrix} \Longrightarrow \begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$$

Example

Use **Gauss-Jordan Reduction** to solve the system

$$\begin{cases} -x_1 & +x_2 & -x_3 & +3x_4 & = & 0, \\ 3x_1 & +x_2 & -x_3 & -x_4 & = & 0, \\ 2x_1 & -x_2 & -2x_3 & -x_4 & = & 0. \end{cases}$$

Solution:

$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{bmatrix} \begin{matrix} R_2 + 3R_1 \longrightarrow R_2 \\ R_3 + 2R_1 \longrightarrow R_3 \end{matrix} \Longrightarrow \begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$$

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Solution:

$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{bmatrix} \begin{matrix} R_2 + 3R_1 \longrightarrow R_2 \\ R_3 + 2R_1 \longrightarrow R_3 \end{matrix} \Longrightarrow \begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} R_2 \times \frac{1}{4} \longrightarrow R_4$$

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Use **Gauss-Jordan Reduction** to solve the system

$$\begin{cases} -x_1 & +x_2 & -x_3 & +3x_4 & = & 0, \\ 3x_1 & +x_2 & -x_3 & -x_4 & = & 0, \\ 2x_1 & -x_2 & -2x_3 & -x_4 & = & 0. \end{cases}$$

Solution:

$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{bmatrix} \begin{matrix} R_2 + 3R_1 \longrightarrow R_2 \\ R_3 + 2R_1 \longrightarrow R_3 \end{matrix} \Longrightarrow \begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$$

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Example

Use **Gauss-Jordan Reduction** to solve the system

$$\begin{cases} -x_1 & +x_2 & -x_3 & +3x_4 & = & 0, \\ 3x_1 & +x_2 & -x_3 & -x_4 & = & 0, \\ 2x_1 & -x_2 & -2x_3 & -x_4 & = & 0. \end{cases}$$

Solution:

$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{bmatrix} \begin{matrix} R_2 + 3R_1 \longrightarrow R_2 \\ R_3 + 2R_1 \longrightarrow R_3 \end{matrix} \Longrightarrow \begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} R_2 \times \frac{1}{4} \longrightarrow R_4 \Longrightarrow \begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix}$$

Example

$$\begin{bmatrix} \mathbf{-1} & \mathbf{1} & \mathbf{-1} & \mathbf{3} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{2} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{-4} & \mathbf{5} & \mathbf{0} \end{bmatrix}$$

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$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{bmatrix} R_3 - R_2 \longrightarrow R_3$$

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$$\begin{bmatrix} -1 & 1 & -1 & 3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{bmatrix} \begin{array}{c} R_3 \times -\frac{1}{3} \longrightarrow R_3 \\ R_1 \times (-1) \longrightarrow R_1 \end{array}$$

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$$\begin{bmatrix} 1 & -1 & 1 & -3 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} \begin{matrix} R_2 + R_3 \longrightarrow R_2 \\ R_1 - R_3 \longrightarrow R_1 \end{matrix}$$

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Example

$$\begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{-2} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{-1} & \mathbf{0} \end{bmatrix}$$

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Setting $x_4 = t \in \mathbb{R}$, then the system has an infinite number of solutions given by

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$$\begin{cases} x_1 & = & t, \\ x_2 & = & -t, \\ x_3 & = & t, \\ x_4 & = & t. \end{cases}$$