

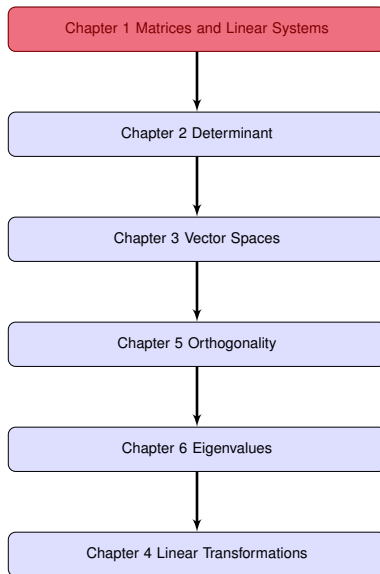
# Chapter 1: Linear Systems and Matrices

## Section: LU - Factorization

### Lecture #5

Lebanese University

Prof Ali WEHBE



- 1 Some Additional Definitions
  - Trace of a matrix
  - LU - Factorization
  
- 2 Exercises on RREF - GJE (From Lecture 3)
  
- 3 Exercises on Inverse of Matrices (From Lecture 4)

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- Trace of a matrix
- LU - Factorization

## 2 Exercises on RREF - GJE (From Lecture 3)

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## Definition

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Our aim is to write a matrix  $\mathbf{A} = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$  as follows:

$$\mathbf{A} = \mathbf{L}\mathbf{U},$$

where  $\mathbf{L} = [l_{ij}] \in \mathbb{M}_n(\mathbb{C})$  is a Lower Triangular matrix and  $\mathbf{U} = [u_{ij}] \in \mathbb{M}_n(\mathbb{C})$  is an Upper Triangular one.

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By applying the matrix multiplication between  $\mathbf{L}$  and  $\mathbf{U}$  and by comparing with  $\mathbf{A}$ , we can find the entries of the matrices  $\mathbf{L}$  and  $\mathbf{U}$ .



### Example

Find the **LU** factorization of the following matrices:

1-  $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}.$

2-  $B = \begin{bmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{bmatrix}.$

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### Example (Supplementary 1)

Discuss according the value of  $a$  for which the consistence of the following system:

$$(S) \begin{cases} x & +2y & +z & = & 2, \\ 2x & -2y & +3z & = & 1, \\ x & +2y & -az & = & a. \end{cases}$$

**Solution:**

### Example (Supplementary 2)

Discuss according the value of  $a$  for which the consistence of the following system:

$$(S) \begin{cases} x & +y & +7z & = & -7, \\ 2x & +3y & +17z & = & 11, \\ x & +2y & +(a^2 + 1)z & = & 6a. \end{cases}$$

Solution:

### Example (Supplementary 3)

Discuss according the values of  $a$  and  $b$  for which the consistence of the following system:

$$(S) \begin{cases} ax & & +bz & = & 2, \\ ax & +ay & +4z & = & 4, \\ & ay & +2z & = & b. \end{cases}$$

Solution:

### Example (Supplementary 4)

Solve the following system of non-linear equations for the unknowns angles  $\alpha$ ,  $\beta$  and  $\gamma$ , where  $0 \leq \alpha \leq \frac{\pi}{2}$ ,  $0 \leq \beta < 2\pi$  and  $0 \leq \gamma \leq \pi$ .

$$(S) \begin{cases} 2 \sin \alpha & + \cos \beta & - \tan \gamma & = & 1, \\ -4 \sin \alpha & + \cos \beta & + \tan \gamma & = & 0, \\ -2 \sin \alpha & + 3 \cos \beta & + 2 \tan \gamma & = & 4. \end{cases}$$

**Solution:**

**Example (Supplementary 5)**

Find the values of  **$a$** ,  **$b$**  and  **$c$**  such that the graph of the polynomial  **$p(x) = ax^2 + bx + c$**  passes through the points  **$(1, 2)$** ,  **$(-1, 6)$**  and  **$(2, 3)$** .

**Solution:**

## Example (Solve the system using the inverse of Matrices )

a)

$$(S) \begin{cases} 2x + 3y + z = -1, \\ 3x + 3y + z = 1, \\ 2x + 4y + z = -2. \end{cases}$$



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b)

$$(S) \begin{cases} 2x & +3y & +z & = & 4, \\ 3x & +3y & +z & = & 8, \\ 2x & +4y & +z & = & 5. \end{cases}$$