Chapter 1: Linear Systems and Matrices
Section: Introduction to Linear Systems
Lecture #1

Lebanese University

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Course Plan

- Chapter 1 Matrices and Linear Systems
- Chapter 2 Determinant
- Chapter 3 Vector Spaces
- Chapter 4 Orthogonality
- Chapter 5 Eigenvalues
- Chapter 6 Linear Transformations

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Linear Equation

Definition (Linear Equation)

A **linear equation** in n **unknowns** (or **variables**) is an equation of the form

$$a_1x_1 + a_2x_2 + \dots + a_nx_n = b,$$

where

$$\begin{cases} x_1, x_2, \dots, x_n : & \text{are the } \textbf{unknows} \text{ or the } \textbf{variables,} \\ a_1, a_2, \dots, a_n : & \text{The } \textbf{coefficients}\text{, are real numbers,} \\ b : & \text{The } \textbf{constant term}\text{, is a real number,} \\ a_1 : & \text{The } \textbf{leading coefficient,} \\ x_1 : & \text{The } \textbf{leading variable.} \end{cases}$$

Remark

Linear Equations have no products ($x_1 x_2$, x_1^2), no roots ($\sqrt{x}$, $\sqrt[3]{x}$), no trigonometric ($cos(x)$) , no exponential (e^x) , and no logarithmic functions ($ln(x)$) for the variables. Variables appear only in first power.

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Solve the following linear equations:

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2. $2x_1 + 3x_2 = 4$.
3. $2x - 3y + z = 5$.

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Solution:

1. We have $2x + 3 = 4 \quad \Rightarrow 2x = 4 - 3 = 1 \quad \Rightarrow x = 1/2$, so

$$S = \{\frac{1}{2}\}.$$

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If we choose the first one.We fix $x_2 = t$ where $t \in \mathbb{R}$.Then the solutions of the L.E is given by

$$\begin{cases} x_1 = \frac{4-3t}{2}, \\ x_2 = t. \end{cases}$$

thus the solution is: is given by

$$S = \left\{ x_1 = \frac{4-3t}{2},\ x_2 = t,\ \text{where } t \in \mathbb{R} \right\}.$$

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3. $2x - 3y + z = 5$.

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A **linear system** of m **equations** in n **unknowns (or variables)** is given as the following one:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2, \\ \quad\quad\quad\quad\quad\quad\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m, \end{cases}$$

where:

- a_{ij} (for $i = 1, .., m$ and $j = 1, .., n$): Is the **coefficient** of the **variable** x_j in the i^{th} **equation** number, is a real number,
- b_i (for $i = 1, .., m$): Is the **constant term** of the i^{th} **equation**.

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a) Which are the operations that we can applied to obtain the solution of the System of L. E.?

b) When can we say that two systems of Linear Equations are Equivalent?

An example to deduce the operations

Solve the following systems:

a)

$$(S_1)\begin{cases} 3x_1 & +2x_2 & -x_3 & = & -2, \\ & x_2 & & = & 3, \\ & & 2x_3 & = & 4. \end{cases}$$

b)

$$(S_2)\begin{cases} 3x_1 & +2x_2 & -x_3 & = & -2, \\ -3x_1 & -x_2 & +x_3 & = & 5, \\ 3x_1 & +2x_2 & +x_3 & = & 2. \end{cases}$$

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Equation (3) implies that $x_3 = 2$.

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$$\begin{cases} x_1 = -2, \\ x_2 = 3, \\ x_3 = 2. \end{cases}$$

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Equation (2) implies that $x_3 = 2$. Then $\boxed{\text{substituting}}$ equation (2) and the result of equation (3) in equation (1), we get $x_1 = -2$. Hence, the solution of system (S_1) is given by

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Operation is **Back Substition**

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$$(S_2)\begin{cases} 3x_1 & +2x_2 & -x_3 & = & -2, \\ -3x_1 & -x_2 & +x_3 & = & 5, \\ 3x_1 & +2x_2 & +x_3 & = & 2. \end{cases}$$

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b)

$$(S_2)\begin{cases} 3x_1 & +2x_2 & -x_3 & = & -2, & (1)\\ -3x_1 & -x_2 & +x_3 & = & 5, & (2)\\ 3x_1 & +2x_2 & +x_3 & = & 2. & (3)\end{cases}$$

Eliminating x_1 from the second equation: $(2) + (1) \Longrightarrow (2')$

$$(S_2')\begin{cases} 3x_1 & +2x_2 & -x_3 & = & -2, & (1)\\ & x_2 & & = & 3, & (2')\\ 3x_1 & +2x_2 & +x_3 & = & 2. & (3)\end{cases}$$

Eliminating x_1 and x_2 from the third equation:

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$(3) - (1) \Longrightarrow (3')$

$$
(S_2') \begin{cases} 3x_1 & +2x_2 & -x_3 & = & -2, & (1) \\ & x_2 & & = & 3, & (2') \\ & & 2x_3 & = & 4. & (3') \end{cases}
$$

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$(3) - (1) \Longrightarrow (3')$

$$(S_2') \begin{cases} 3x_1 & +2x_2 & -x_3 & = & -2, & (1) \\ & x_2 & & = & 3, & (2) \\ & & 2x_3 & = & 4. & (3') \end{cases}$$

We remark that the system (S_2') is the system (S_1). Thus, system (S_2') has the same solution of (S_1). Equivalently, system (S_2) has the same solution.

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Definition (Equivalent Systems)

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There are three operations that can be used on a system to obtain an **equivalent** system.These operations are named by **Elementary operations**:

- I- Interchange two equations.
- II- Multiply an equation by a nonzero real number.
- III- Replace an equation by its sum with a multiple of another equation.

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x_1 & & +x_3 & +42x_4 & +8x_5 & +4x_6 & -23x_7 & +7x_8 = & 2, \\
 & -2x_2 & +4x_3 & +4x_4 & +5x_5 & +6x_6 & -13x_7 & -x_8 = & 3, \\
-x_1 & -3x_2 & & +4x_4 & +x_5 & -9x_6 & & -x_8 = & 4, \\
2x_1 & +9x_2 & +2x_3 & -x_4 & & -7x_6 & -2x_7 & -x_8 = & 5, \\
3x_1 & +8x_2 & +2x_3 & +4x_4 & +5x_5 & +6x_6 & +11x_7 & = & 6, \\
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8x_1 & +3x_2 & +2x_3 & +4x_4 & +5x_5 & +6x_6 & +x_7 & -x_8 = & 9, \\
-11x_1 & -3x_2 & -x_3 & & +5x_5 & & +11x_7 & -x_8 = & -9.
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Logically, it is hard to apply the Elementary operations in order to solve the previous problem since it is contains **10** equations and **8** variables. We must use another tools. The most important tools to solve these type of systems are the

Matrices

From a system of Linear equations to a matrix

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From a system of Linear equations to a matrix

We see the coefficients of every **variable** in each **equation**

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-x_1 & -3x_2 & +2x_3 & +4x_4 & +5x_5 & +6x_6 & +11x_7 & -x_8 = & 1, \\
1x_1 & & +x_3 & +42x_4 & +8x_5 & +4x_6 & -23x_7 & +7x_8 = & 2, \\
0x_1 & -2x_2 & +4x_3 & +4x_4 & +5x_5 & +6x_6 & -13x_7 & -x_8 = & 3, \\
-x_1 & -3x_2 & & +4x_4 & +x_5 & -9x_6 & & -x_8 = & 4, \\
2x_1 & +9x_2 & +2x_3 & -x_4 & & -7x_6 & -2x_7 & -x_8 = & 5, \\
3x_1 & +8x_2 & +2x_3 & +4x_4 & +5x_5 & +6x_6 & +11x_7 & = & 6, \\
-2x_1 & +7x_2 & +2x_3 & +4x_4 & +5x_5 & +6x_6 & -9x_7 & -x_8 = & 7, \\
0x_1 & -3x_2 & +2x_3 & +4x_4 & +5x_5 & +6x_6 & +11x_7 & -x_8 = & 8, \\
8x_1 & +3x_2 & +2x_3 & +4x_4 & +5x_5 & +6x_6 & +x_7 & -x_8 = & 9, \\
-11x_1 & -3x_2 & -x_3 & & +5x_5 & & +11x_7 & -x_8 = & -9,
\end{array}\right.$$

For example, the **variable** x_1 has the following coefficients: -1, 1, 0, -1, $+2$, $+3$, -2, 0, $+8$ and -11.

From a system of Linear equations to a matrix

We use the coefficients of the variables in each equation and we construct the following object:

$$\begin{bmatrix} -1 & -3 & 2 & 4 & 5 & 6 & 11 & -1 \\ 1 & 0 & 1 & 42 & 8 & 4 & -23 & 7 \\ 0 & -2 & 4 & 4 & 5 & 6 & -13 & -1 \\ -1 & -3 & 0 & 4 & 1 & -9 & 0 & -1 \\ 2 & 9 & 2 & -1 & 0 & -7 & -2 & -1 \\ 3 & 8 & 2 & 4 & 5 & 6 & 11 & 0 \\ -2 & 7 & 2 & 4 & 5 & 6 & -9 & -1 \\ 0 & -3 & 2 & 4 & 5 & 6 & 11 & -1 \\ 8 & 3 & 2 & 4 & 5 & 6 & 1 & -1 \\ -11 & -3 & -1 & 0 & 5 & 0 & 11 & -1 \end{bmatrix}$$

From a system of Linear equations to a matrix

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How can we characterize a Matrix?

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$$\begin{bmatrix} -1 & -3 & 2 & 4 & 5 & 6 & 11 & -1 \\ 1 & 0 & 1 & 42 & 8 & 4 & -23 & 7 \\ 0 & -2 & 4 & 4 & 5 & 6 & -13 & -1 \\ -1 & -3 & 0 & 4 & 1 & -9 & 0 & -1 \\ 2 & 9 & 2 & -1 & 0 & -7 & -2 & -1 \\ 3 & 8 & 2 & 4 & 5 & 6 & 11 & 0 \\ -2 & 7 & 2 & 4 & 5 & 6 & -9 & -1 \\ 0 & -3 & 2 & 4 & 5 & 6 & 11 & -1 \\ 8 & 3 & 2 & 4 & 5 & 6 & 1 & -1 \\ -11 & -3 & -1 & 0 & 5 & 0 & 11 & -1 \end{bmatrix}$$

Number of row is **10**=number of equations of the system, and number of columns is **8**=number of variables of the system.

Definition (Matrix Notation)

A matrix A of m rows and n columns is given as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ . & . & . & \dots & . \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

where a_{ij} is a real number (constant) that represent the entry of the matrix A, for $i = 1, .., m$ and $j = 1, .., n$. In this case, we say that A is an $m \times n$ matrix and we denote $A := [a_{ij}]$.

Example

Give all the characterization of the following matrix

$$A = \begin{bmatrix} 1 & 2 & -7 \\ 3 & 0 & 6 \end{bmatrix}$$

Example

Give all the characterization of the following matrix

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{-7} \\ \mathbf{3} & \mathbf{0} & \mathbf{6} \end{bmatrix}$$

Solution:

$\boldsymbol{A}$ is $\mathbf{2} \times \mathbf{3}$ matrix. $\boldsymbol{A} = [\boldsymbol{a_{ij}}]$, $\boldsymbol{i} = \mathbf{1, 2}$ number of rows, $\boldsymbol{j} = \mathbf{1, 2, 3}$ number of columns.