# Chapter 1: Linear Systems and Matrices
## Section: Determinants
## Lecture #6

Lebanese University

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# Course Plan

Chapter 1 Matrices and Linear Systems

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Chapter 2 Determinant

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3) The **determinant** can be applied only on **square** matrices.
4) Why we study the determinant of a matrix ?
   The determinant give us an idea for the existence of its inverse. Then, we can know if a square system has an unique solution or not.

## General Strategy

1) We start by the determinant of $\mathbf{2} \times \mathbf{2}$ matrices.
2) To determine the determinant of an $\boldsymbol{n} \times \boldsymbol{n}$ matrix, we transform the problem to $(\boldsymbol{n}-\mathbf{1}) \times (\boldsymbol{n}-\mathbf{1})$ matrix. Then to $(\boldsymbol{n}-\mathbf{2}) \times (\boldsymbol{n}-\mathbf{2})$ matrix,..., to $\mathbf{2} \times \mathbf{2}$ matrix.

# Determinant of $2 \times 2$ Matrices

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an $2 \times 2$ matrix. Then, its determinant is given by

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**Remark:**

Remember that in the previous lecture, we put an essential condition for the existence of the inverse of a square matrix like $A$. Indeed, the condition is $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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## Example (Finding the determinant)

a) $\boldsymbol{A} = \begin{bmatrix} \mathbf{2} & \mathbf{4} \\ \mathbf{1} & \mathbf{-3} \end{bmatrix}$.

$$\boldsymbol{ad - bc = -10.}$$

b) $\boldsymbol{B} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{4} \end{bmatrix}$.

$$\boldsymbol{ad - bc = 0.}$$

# Determinant of $3 \times 3$ matrices

## Strategy

1. Choose an arbitrary column or row.
2. Assign a sign to each entry in the selected column/row.
3. For each entry, multiply the entry with the determinant of the $2 \times 2$ matrix obtained by eliminating the row and column containing this entry.
4. Add the values obtained for all the three entries, and this is the determinant.

## Example

Find the determinant of the following $3 \times 3$ matrix: $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$

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$$A = \begin{vmatrix} 2^{+} & 5 & 4 \\ 3^{-} & -1 & 2 \\ -5^{+} & 4 & 6 \end{vmatrix}$$

$$= +2 \begin{vmatrix} -1 & 2 \\ 4 & 6 \end{vmatrix} - 3 \begin{vmatrix} 5 & 4 \\ 4 & 6 \end{vmatrix} + (-5) \begin{vmatrix} 5 & 4 \\ -1 & 2 \end{vmatrix}$$

$$= +2\,(-6-8) - 3(30-16) + (-5)(10-(-4))$$

$$= +2\,(-14) - 3(14) + (-5)(14)$$

$$= -28 - 42 - 70 = -130$$

# Minors and Cofactors

## Definition (Minors and Cofactors)

Let $A = [a_{ij}]$ be an $n \times n$ matrix.
Let $M_{ij}$ denote the $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting the $i^{\text{th}}$ row and $j^{\text{th}}$ column.
The Minor of $a_{ij}$ is the determinant $|M_{ij}|$ of the matrix $M_{ij}$.
The Cofactor $A_{ij}$ of $a_{ij}$ is defined by:

$$A_{ij} = (-1)^{i+j}|M_{ij}|.$$

## Example

Consider the matrix:

$$A = \begin{bmatrix} 2 & 5 & 4 & -2 \\ 0 & 1 & 2 & 1 \\ 5 & -1 & 0 & -2 \\ 3 & 7 & 4 & 3 \end{bmatrix}$$

# Determinant of $n \times n$ Matrix

**Definition (Expanding through Rows)**

The Determinant of an $n \times n$ matrix $A$, denoted by $det(A)$, is a **scalar** given by

$$\begin{aligned}
\det(A) &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + \ldots + a_{1n}A_{1n} = \sum_{j=1}^{n} a_{1j}A_{1j} \\
&= a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} + \ldots + a_{2n}A_{2n} = \sum_{j=1}^{n} a_{2j}A_{2j} \\
&. \\
&. \\
&= a_{i1}A_{i1} + a_{i2}A_{i2} + a_{i3}A_{i3} + \ldots + a_{in}A_{in} = \sum_{j=1}^{n} a_{ij}A_{ij}
\end{aligned}$$

# Determinant of $n \times n$ Matrices

## Definition (Expanding through Columns)

The Determinant of an $n \times n$ matrix $A$, denoted by $det(A)$, is a **scalar** given by

$$\begin{aligned}
\det(A) =& a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} \ldots + a_{n1}A_{n1} = \sum_{i=1}^{n} a_{i1}A_{i1} \\
=& a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} + \ldots + a_{n2}A_{n2} = \sum_{i=1}^{n} a_{i2}A_{i2} \\
&. \\
&. \\
=& a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j} + \ldots + A_{nj} = \sum_{i=1}^{n} a_{ij}A_{ij}
\end{aligned}$$

**Remark:**

Since the determinant of a matrix depends of the entries of the chosen row or column, then logically we choose the row or the column that contains more zero entries.

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**Example**

Calculate the determinants of the following matrices:

a) $\boldsymbol{A} = \begin{bmatrix} \mathbf{4} & \mathbf{3} & \mathbf{0} \\ \mathbf{3} & \mathbf{1} & \mathbf{2} \\ \mathbf{5} & \mathbf{-1} & \mathbf{-4} \end{bmatrix}.$

**Remark:**

Since the determinant of a matrix depends of the entries of the chosen row or column, then logically we choose the row or the column that contains more zero entries.

**Example**

Calculate the determinants of the following matrices:

a) $\boldsymbol{A} = \begin{bmatrix} 4 & 3 & 0 \\ 3 & 1 & 2 \\ 5 & -1 & -4 \end{bmatrix}.$

b) $\boldsymbol{B} = \begin{bmatrix} 1 & 2 & 4 \\ -3 & 3 & 5 \\ 7 & 0 & 6 \end{bmatrix}.$

## Example

b) $C = \begin{bmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{bmatrix}$

**Theorem:**

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## Example

Find the determinant of

$$A = \begin{bmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}$$

Solution: Using row operations:

$$|A| = \begin{vmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -2 \\ 2 & -3 & 10 \\ 0 & 1 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -2 \\ 0 & -7 & 14 \\ 0 & 1 & -3 \end{vmatrix}$$

$$= -(-7)\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 1 & -3 \end{vmatrix} = 7\begin{vmatrix} 1 & 2 & -2 \\ 0 & 1 & -2 \\ 0 & 0 & -1 \end{vmatrix} = 7(1)(1)(-1) = -7$$

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The operations:

$$R_1 \longleftrightarrow R_2$$
$$R_2 - 2R_1 \longrightarrow R_2$$
$$\frac{-1}{7} R_2 \longrightarrow R_2$$
$$R_3 - R_2 \longrightarrow R_3$$

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Let $A$ and $B$ be two $n \times n$ matrices and let $c$ be a nonzero scalar. Then the following statements are true:

**Theorem:**

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two $\boldsymbol{n} \times \boldsymbol{n}$ matrices and let $\boldsymbol{c}$ be a nonzero scalar. Then the following statements are true:

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**Theorem:**

Let $A$ and $B$ be two $n \times n$ matrices and let $c$ be a nonzero scalar. Then the following statements are true:

1) $det(A^T) = det(A)$.
2) If $A$ is invertible, then $det(A^{-1}) = \dfrac{1}{det(A)}$.

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4) If $A$ has a row or column consisting entirely of zeros, then $det(A) = 0$.

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5) $det(AB) = det(A)det(B)$.

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5) $det(AB) = det(A)det(B)$.
6) If $A$ is a triangular matrix, then $det(A) = \prod_{i=1}^{n} a_{ii}$.

## Supplementary Exercise 1

solve the following independent parts:

1. Find all values of $\lambda$ for which $det(A) = 0$, where

$$A = \begin{bmatrix} \lambda - 4 & 3 & 0 \\ 0 & \lambda & 3 \\ 0 & 0 & \lambda - 4 \end{bmatrix}.$$

2. Let $B \in \mathbb{M}_2(\mathbb{R})$. Show that

$$det(B) = \frac{1}{2} \begin{vmatrix} tr(B) & 1 \\ tr(B^2) & tr(B) \end{vmatrix},$$

where $tr$ is the trace function.

3. Verify that $|-3C| = (-3)^2|C|$ where $C = \begin{bmatrix} 2 & -2 \\ -3 & 1 \end{bmatrix}$.

## Solution

1. Since $A$ is a triangular matrix, then

$$det(A) = (\lambda - 4)(\lambda)(\lambda - 4)$$

$$\text{So } det(A) = 0 \leftrightarrow (\lambda - 4)(\lambda)(\lambda - 4) = 0 \leftrightarrow \lambda = 0 \text{ or } \lambda = 4.$$

2. Let $B \in \mathbb{M}_2(\mathbb{R})$, then

$$B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and } B^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} \text{ and } tr(B) = a + d$$

Thus

$$\frac{1}{2}\begin{vmatrix} tr(B) & 1 \\ tr(B^2) & tr(B) \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a + d & 1 \\ a^2 + 2bc + d^2 & a + d \end{vmatrix}$$

$$= \frac{1}{2}\left((a + d)^2 - \left(a^2 + 2bc + d^2\right)\right) = \frac{1}{2}(2ad - 2bc) = ad - bc$$

and this final answer is equal to the determinant of $B$.

## Solution

3. We have:

$$\boldsymbol{det(C)} = \begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} = 2 - 6 = -4$$

so

$$(-3)^2|C| == 9(-4) = -36$$

and

$$-3C = \begin{bmatrix} -6 & 6 \\ 9 & -3 \end{bmatrix} \text{ so } |-3C| = \begin{vmatrix} -6 & 6 \\ 9 & -3 \end{vmatrix} = 18 - 54 = -36$$

so they are equal.

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If the determinant of the coefficient square matrix is equal to zero, then the system has an **infinite number of solutions** or has **no solution**.

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**Remark:**

If the determinant of the coefficient square matrix is equal to zero, then the system has an **infinite number of solutions** or has **no solution**.

**Example**

Find the values of $\boldsymbol{k}$ for which the following matrix $\boldsymbol{A}$ is invertible

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{1} & \mathbf{3} & \boldsymbol{k} \\ \mathbf{2} & \mathbf{1} & \mathbf{3} \\ \mathbf{4} & \mathbf{6} & \mathbf{2} \end{bmatrix}.$$

## Supplementary Exercise 2

Let $A = \begin{bmatrix} -1 & -3 & 1 \\ 3 & 6 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

1) Verify that $A$ is invertible.
2) Find $A^{-1}$.
3) a) Write the system associated to the problem $AX - 2b = c$, where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$ and $c = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$.

   b) Deduce its solution.

## Solution

1. $A = \begin{pmatrix} -1 & -3 & 1 \\ 3 & 6 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Then the determinate of $A$:

$$\det(A) = -3.$$

2. $A = \begin{pmatrix} -1 & -3 & 1 \\ 3 & 6 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. Let us calculate $A^{-1}$:

$$(A|I_3) = \left(\begin{array}{ccc|ccc} -1 & -3 & 1 & 1 & 0 & 0 \\ 3 & 6 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2' = R_2 + 3R_1} \xrightarrow{R_3' = R_3 + R_1}$$

$$\left(\begin{array}{ccc|ccc} -1 & -3 & 1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 3 & 1 & 0 \\ 0 & -3 & 2 & 1 & 0 & 1 \end{array}\right) \xrightarrow{R_3' = R_3 - R_2}$$

## Solution

$$\left(\begin{array}{ccc|ccc} -1 & -3 & 1 & 1 & 0 & 0 \\ 0 & -3 & 3 & 3 & 1 & 0 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right) \xrightarrow{R_2'=R_2+3R_3} \xrightarrow{R_1'=R_1+R_3}$$

$$\left(\begin{array}{ccc|ccc} -1 & -3 & 0 & -1 & -1 & 1 \\ 0 & -3 & 0 & -3 & -2 & 3 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right) \xrightarrow{R_1'=R_1-R_2}$$

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 0 & 2 & 1 & -2 \\ 0 & -3 & 0 & -3 & -2 & 3 \\ 0 & 0 & -1 & -2 & -1 & 1 \end{array}\right) \xrightarrow{R_1'=-R_1} \xrightarrow{R_2'=\frac{-R_2}{3}} \xrightarrow{R_3'=-R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 2 \\ 0 & 1 & 0 & 1 & \frac{2}{3} & -1 \\ 0 & 0 & 1 & 2 & 1 & -1 \end{array}\right) = \left(I_3 | A^{-1}\right).$$

## Solution

2. Then $A^{-1} = \begin{pmatrix} -2 & -1 & 2 \\ 1 & \frac{2}{3} & -1 \\ 2 & 1 & -1 \end{pmatrix}$.

3.
$$AX - 2b = c \implies AX = c + 2b \implies X = A^{-1}(c + 2b).$$

Then

$$X = \begin{pmatrix} -2 & -1 & 2 \\ 1 & \frac{2}{3} & -1 \\ 2 & 1 & -1 \end{pmatrix} \left[ \begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \right]$$

$$= \begin{pmatrix} -2 & -1 & 2 \\ 1 & \frac{2}{3} & -1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 2 \\ -\frac{1}{3} \\ 7 \end{pmatrix}.$$

## Supplementary Exercise 3

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 0 \end{bmatrix}$, $P = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} -2 & 4 \\ 0 & 3 \end{bmatrix}$.

1) Find $(AB)^T$.
2) Calculate $A^2$ and $B^2$.
3) Find $X$, if $XP + 4I_2 = 0$.
4) Find $det(3D^{-1}) + det(AB)$.

## Solution

1.

$$AB = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 0 \end{pmatrix}$$

$$\Longrightarrow (AB)^\top = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

2.

$$A^2 = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{pmatrix}, \quad B^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 0 \end{pmatrix}.$$

## Solution

3. We have $\boldsymbol{det(P) = -2}$, then $\boldsymbol{P}$ is an invertible matrix, so

$$P^{-1} = \frac{1}{-2}\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

We have $PX + 4I_2 = 0 \implies PX = -4I_2$

$$\implies X = -4P^{-1}I_2 = -4P^{-1} = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}.$$

4. We have
   - $det(3D^{-1}) = 3^2 det(D^{-1}) = 3^2 \cdot \dfrac{1}{det(D)} = \dfrac{9}{-6} = -\dfrac{3}{2}$.
   - $det(AB) = 0$, since $AB$ a upper triangular matrix, so the determinant is the product of the diagonal entries.

so $det(3D^{-1}) + det(AB) = -\dfrac{3}{2}$.