MLAI 504
NEURAL NETWORKS & DEEP LEARNING

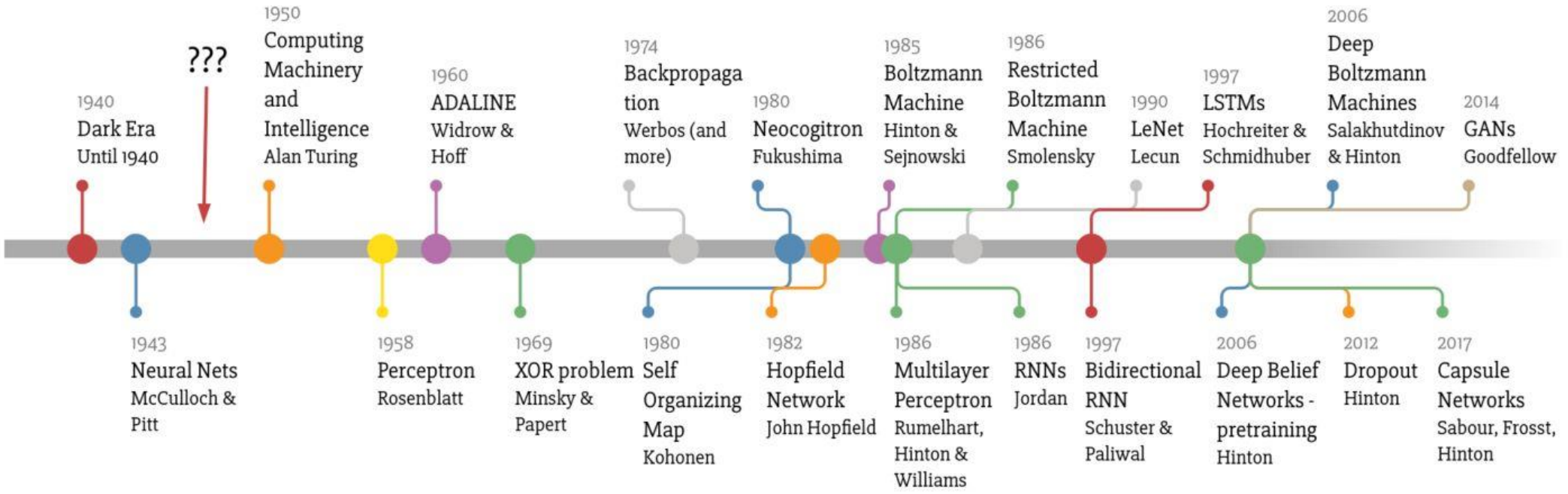
Dr. Zein Al Abidin IBRAHIM

zein.ibrahim@ul.edu.lb

Perceptron

NEURAL NETWORK

Deep Learning Timeline



Made by Favio Vázquez

- The perceptron is the simplest form of a neural network used for the classification of patterns said to be **linearly separable.**
  - (i.e. Patterns that lie on the opposite side of a hyperplane)
  - Basically, it consists of a single neuron with adjustable synaptic weight and bias.

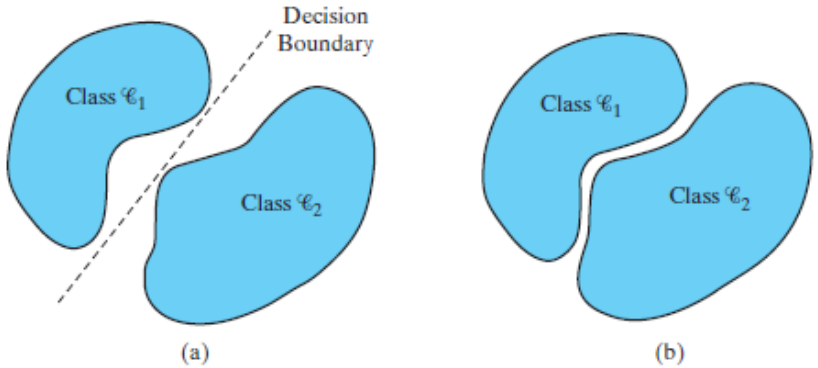


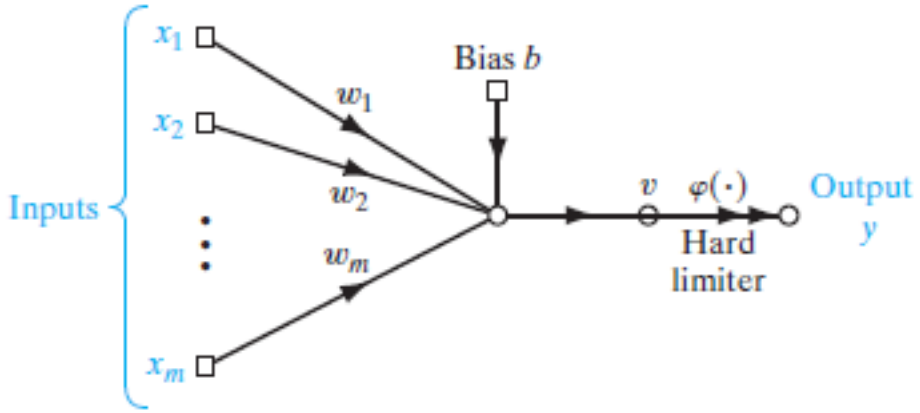
FIGURE 1.4 (a) A pair of linearly separable patterns. (b) A pair of non-linearly separable patterns.

  - The perceptron built around a single neuron is limited to performing pattern classification with only two classes(hypotheses).
  - By expanding the output (Computational) Layer we can perform classification on more than two classes .

Introduction

Perceptron

- Rosenblatt's perceptron is built around nonlinear neuron, called McCulloch-Pitts model of a neuron.
  - A neural model consisting of a linear combiner followed by a hard limiter (activation function ex, sigmoid function)
  - Accordingly, the neuron produce an output equal to +1 if the hard limiter input is positive or -1 if it is negative.
  - The *induced local field* or the hard limiter input is given by:
    - $v = \sum_{i=1}^{m} w_i x_i + b$
    - The goal of the perceptron is to correctly classify a set of externally applied stimuli $x_1, x_2, \ldots, x_m$ into one of two classes ζ_1 or ζ_2

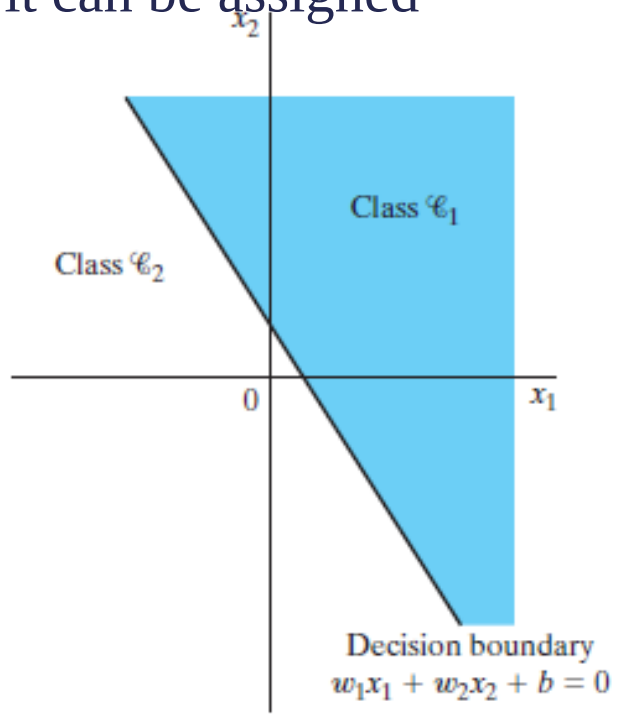


McCulloch-Pitts Vs Rosenblatt perceptron

Feature	McCulloch-Pitts Model	Rosenblatt Perceptron
Year	1943	1958
Purpose	Theoretical model of biological neurons	Practical implementation of a single artificial neuron
Inputs	Binary (0 or 1)	Real-valued
Weights	All identical	Can be different and positive or negative
Threshold	Single, fixed	Single, adjustable during learning
Activation function	Step function	Step function
Learning rule	None	Perceptron learning rule
Applications	Theoretical foundation for artificial neural networks	Simple pattern recognition, linear binary classification
Strengths	Laid the groundwork for artificial neurons, provided a simple model to understand	Introduced the concept of learning in neural networks, enabled practical applications
Weaknesses	Limited to binary inputs and outputs, couldn't solve non-linearly separable problems, no learning mechanism	Limited to linearly separable problems, slow convergence in some cases

Perceptron

- There are two decision regions separated by a hyperplane. Which is defined by:
  - $\sum_{i=1}^{m} w_i x_i + b = 0$
  - For the case of two input variables x_1 and x_2, the decision boundary takes the form of a straight line.
    - Depending on the location of point (x_1, x_2) with respect top the line, it can be assigned whether to Class 1 or Class2.
    - The synaptic weights $w_1, w_2, \ldots w_m$ of the perceptron can be adapted on an iteration-by-iteration basis.
    - For the adaptation, we may use an error-correction rule known as the perceptron conversion algorithm. (Discussed next)

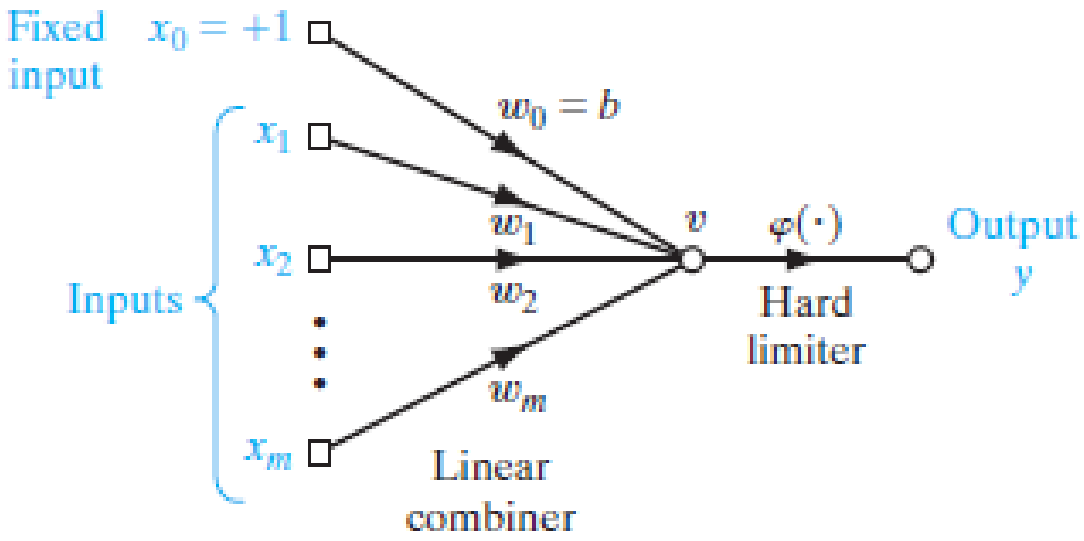


- Consider the below signal flow diagram where bias $b(n)$ is treated as a synaptic weight driven by a fixed input of +1.
  - We can thus define the $(m+1)-by-1$ input vector
    - $x(n) = [+1, x_{1(n)}, x_2(n), \dots, x_m(n)]^T$
    - Where n denotes the time step in applying the algorithm.
  - Correspondingly we can define the $(m+1)-by-1$ weight vector
    - $w(n) = [b, w_{1(n)}, w_2(n), \dots, w_m(n)]^T$
  - And the linear combiner output is written in the compact form as:
    - $v(n) = \sum_{i=0}^{m} w_i(n)x_i(n)$
    - $= w^T(n)x(n)$
  - The equation $w^T x = 0$ plotted in

m- dimensional space represent with coordinates

$x_1, x_2, \dots x_m$ defines a hyperplane as the

decision surface consists of a hyperplane.



The Perceptron Conversion Theorem

- For the perceptron to function properly, the two classes must be linearly separable.
  - If the classes are allowed to be close as shown in Figure below (b) then it becomes beyond the capability of perceptron.

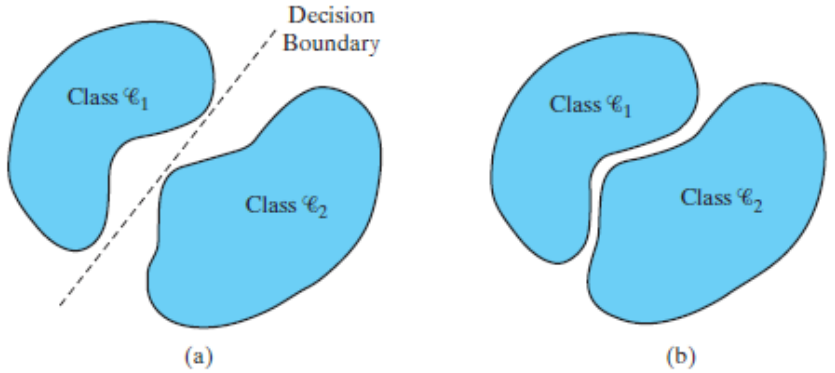
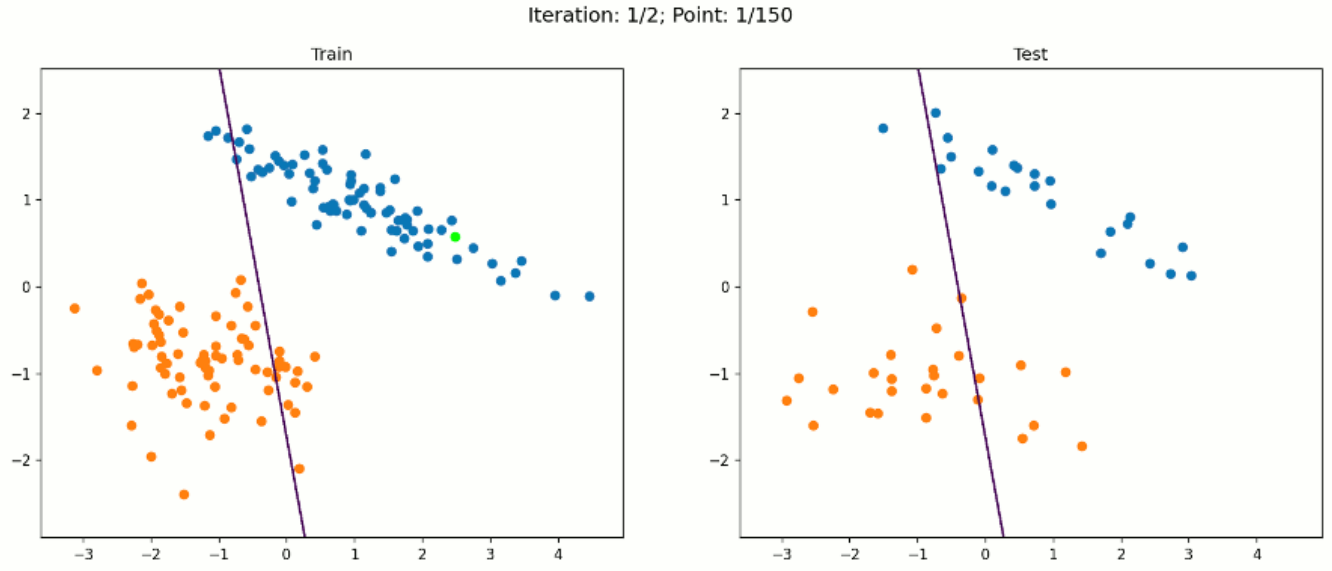


FIGURE 1.4 (a) A pair of linearly separable patterns. (b) A pair of non-linearly separable patterns.



The Perceptron Conversion Theorem

The Perceptron Conversion Theorem

- Suppose that two classes to be classified are linearly separable then:
  - **$w^T x > 0$ for each input vector x belonging to class 1**
  - **$w^T x \leq 0$ for each input vector x belonging to class 2**
  - **For the case $w^T x = 0$ we arbitrary said that it belongs to class 2**
- The algorithm for adapting weight vector of the elementary perceptron may be formulated as follows:
  1. If the nth member of the training set, $x(n)$, is **correctly** classified by the weight vector $w(n)$ computed at the nth iteration of the algorithm, no changes are done to the previous weights: **(no change)**
     1. $\mathbf{w(n+1) = w(n)}$ **if $\mathbf{w^T(n)x(n) > 0}$ and $\mathbf{x(n)}$ belongs to class 1**
     2. $\mathbf{w(n+1) = w(n)}$ **if $\mathbf{w^T(n)x(n) \leq 0}$ and $\mathbf{x(n)}$ belongs to class 2**
  2. Otherwise, the weight vectors are updated in accordance to the below:
     1. $\mathbf{w(n+1) = w(n) - \eta(n)x(n)}$ **if $\mathbf{w^T(n)x(n) > 0}$ and $\mathbf{x(n)}$ belongs to class 2**
     2. $\mathbf{w(n+1) = w(n) + \eta(n)x(n)}$ **if $\mathbf{w^T(n)x(n) \leq 0}$ and $\mathbf{x(n)}$ belongs to class 1**

     Where $\eta(n)$ is the learning rate parameter that controls the adjustment applied to the weight vector at iteration n

Learning rate

- If $\eta(n) = \eta > 0$ where η is a constant independent of the iteration number n then we have **a fixed-increment adaption rule** for the perceptron.
- Types of learning rates used by other algorithms:
  - Constant learning rate → Easy to implement & can be effective for simple problems
  - Decreasing learning rate → Helps model to converge smoothly and avoid oscillations
  - Adaptive learning rate → Adjusts itself dynamically based on parameters and loss function
  - Others → Cyclic learning rate, Warmup learning rate, ….

TABLE 1.1 Summary of the Perceptron Convergence Algorithm

Variables and Parameters:

$\mathbf{x}(n)$ = $(m + 1)$-by-1 input vector
$= [+1, x_1(n), x_2(n), \ldots, x_m(n)]^T$
$\mathbf{w}(n)$ = $(m + 1)$-by-1 weight vector
$= [b, w_1(n), w_2(n), \ldots, w_m(n)]^T$
b = bias
$y(n)$ = actual response (quantized)
$d(n)$ = desired response
η = learning-rate parameter, a positive constant less than unity

1. *Initialization.* Set $\mathbf{w}(0) = \mathbf{0}$. Then perform the following computations for time-step $n = 1, 2, \ldots$.
2. *Activation.* At time-step n, activate the perceptron by applying continuous-valued input vector $\mathbf{x}(n)$ and desired response $d(n)$.
3. *Computation of Actual Response.* Compute the actual response of the perceptron as

$$y(n) = \text{sgn}[\mathbf{w}^T(n)\mathbf{x}(n)]$$

where sgn(·) is the signum function.

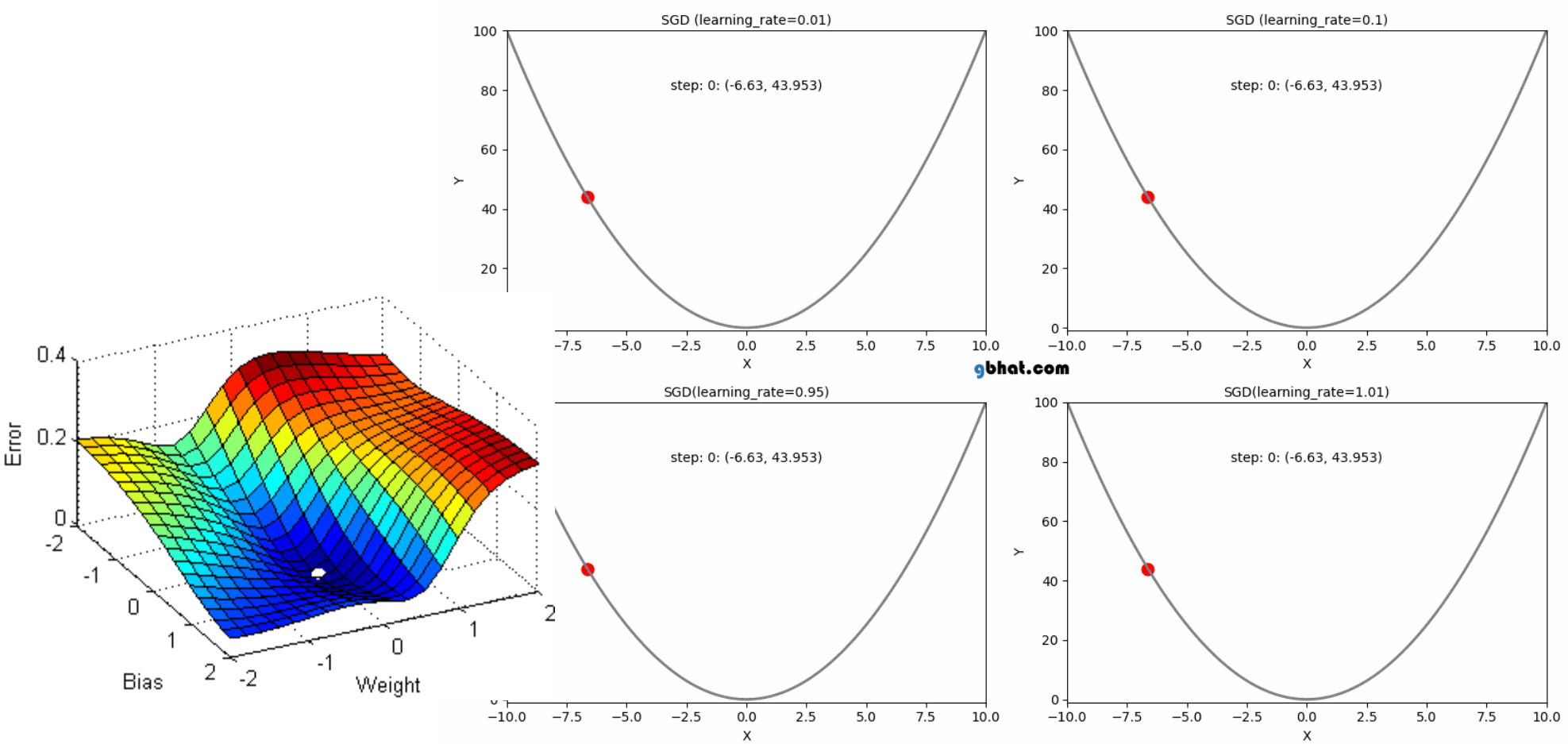
4. *Adaptation of Weight Vector.* Update the weight vector of the perceptron to obtain

$$\mathbf{w}(n + 1) = \mathbf{w}(n) + \eta[d(n) - y(n)]\mathbf{x}(n)$$

where

$$d(n) = \begin{cases} +1 & \text{if } \mathbf{x}(n) \text{ belongs to class } \mathscr{C}_1 \\ -1 & \text{if } \mathbf{x}(n) \text{ belongs to class } \mathscr{C}_2 \end{cases}$$

5. *Continuation.* Increment time step n by one and go back to step 2.



Gradient Descent Learning Rates

The Batch Perceptron Algorithm

- The conversion algorithm of the perceptron was presented without reference to a **cost function**.
  - Moreover, the derivation focused on a single-sample correction.
- In the **batch algorithm** we will present two things:
  - Introduce the generalized form of a perceptron cost function
  - Use the cost function to formulate the batch version.

- The cost function is one that permits the application for a **gradient search**. The perceptron cost function is defined as:
  - $J(w) = \sum_{x(n)\in\mathcal{H}}\left(-w^T x(n)d(n)\right)$
  - Where $\mathcal{H}$ is the set of samples x misclassified by a perceptron using w as its weight vector
  - If all the samples are classified correctly, then the set $\mathcal{H}$ is empty and J(w) is zero

The Batch Perceptron Algorithm

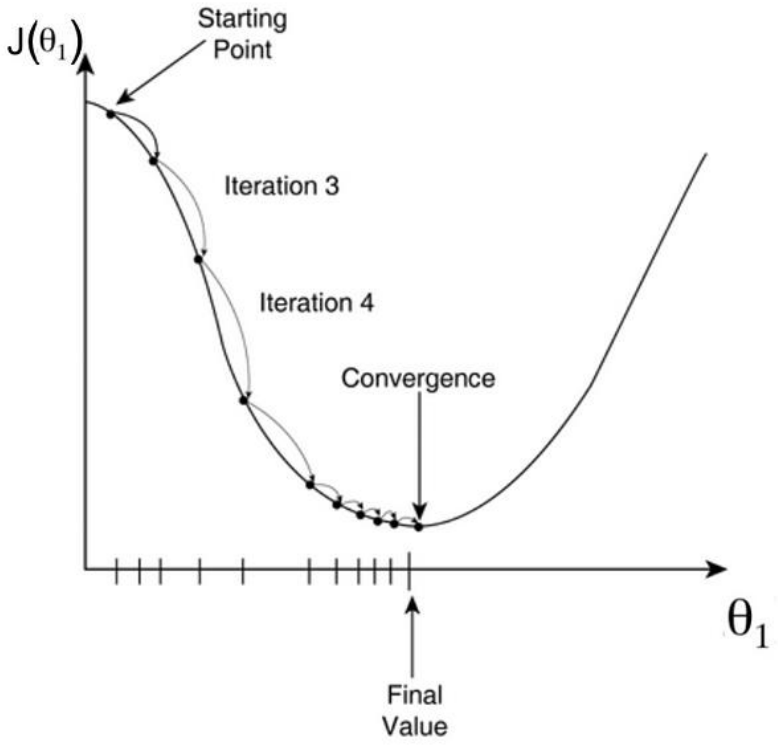
- The cost function $J(w)$ can be differentiated on w which yields the gradient vector:
  - $\nabla J(w) = \sum_{x(n)\in\mathcal{H}}\left(-x(n)d(n)\right)$
  - Where the gradient operator is :
    - $\nabla = \left[\frac{\partial}{\partial w_1}, \frac{\partial}{\partial w_2}, \dots, \frac{\partial}{\partial w_m}\right]^T$
  - In the method of the **steepest descent**, the adjustment of the weight vector w at each timestep of the algorithm is applied in a direction opposite to the gradient vector $\nabla J(w)$ accordingly:
    - $w(n+1) = w(n) - \eta(n)\nabla J(w)$
    - $= w(n) + \eta(n)\sum_{x(n)\in\mathcal{H}}\left(x(n)d(n)\right)$
    - This include the single sample correction version as a special case of the convergence algorithm.
    - The adjustment of the weights a timestep $n+1$ is defined by the sum of all samples misclassified by the weight vector $w(n)$ scaled by the learning rate $\eta(n)$.
    - The algorithm is called batch since at each time step a batch of misclassified samples are used to compute the adjustment.

Gradient Descent

- Fundamental algorithm in machine learning and optimization.
- Used to train various models.
- How it works:
  - Imagine you're lost in a mountainous landscape and trying to find the valley (minimum point).
  - Gradient descent would be like taking small steps downhill, always following the steepest slope you can feel.
- Here's the gist:
  - **Imagine a function as a landscape**: The function's output (like error) varies depending on its inputs. High points represent areas with high error (think of them as hills).
  - **Start at a random point**: This is your initial guess for the model's parameters (weights).
  - **Calculate the slope (gradient)**: This tells you how much and in which direction the error changes when you move a little bit.
  - **Take a small step downhill**: Adjust the parameters slightly in the direction of the steepest descent.
  - **Repeat**: Keep calculating the slope and taking small steps downhill, until you reach a point where the slope is almost zero (a minimum).

Gradient Descent – A pre-look

- Different types of gradient descent:
  - **Batch gradient descent**: Considers the entire dataset for each step, can be slow for large datasets.
  - **Stochastic gradient descent**: Considers only one data point at a time, faster but can be noisy.
  - **Mini-batch gradient descent**: Considers a small batch of data points at a time, balances speed and stability.
- Gradient descent is powerful, but it's not perfect:
  - Can get stuck in local minima: Imagine being stuck in a small valley instead of the main one. You'd need a good starting point or momentum to escape.
  - Requires differentiable **functions**: Not all functions are smooth enough for gradient descent to work.
  - Can be sensitive to learning rate: A too-high rate might overshoot the minimum, too-low might get stuck.



Cost Function – "One Half Mean Squared Error":

$$J(\theta_0, \theta_1) = \frac{1}{2m}\sum_{i=1}^{m}\left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right)^2$$

Objective:

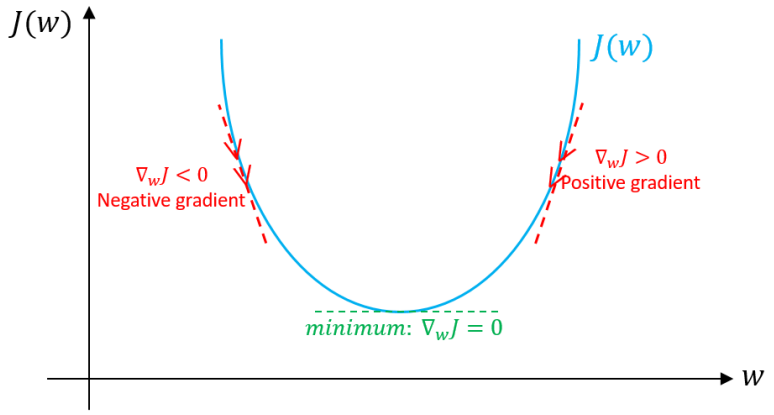
$$\min_{\theta_0, \theta_1} J(\theta_0, \theta_1)$$

Derivatives:

$$\frac{\partial}{\partial\theta_0}J(\theta_0, \theta_1) = \frac{1}{m}\sum_{i=1}^{m}\left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right)$$

$$\frac{\partial}{\partial\theta_1}J(\theta_0, \theta_1) = \frac{1}{m}\sum_{i=1}^{m}\left(h_\theta\left(x^{(i)}\right) - y^{(i)}\right)\cdot x^{(i)}$$

Gradient Descent



Similarity:

- Both are iterative optimization algorithms used to minimize an error function.
- Both update the parameters (weights) of a model based on the gradient of the error function.
- Both work well for linear problems or can be used with non-linear problems if combined with other techniques like feature engineering or neural networks.

Feature	Gradient Descent	Perceptron
Objective function	Any differentiable function	Misclassification error (for binary classification)
Learning rule	Adjusts weights proportionally to the gradient	Updates weights only if prediction is wrong
Convergence guarantee	No guaranteed convergence (can get stuck in local minima)	Guaranteed convergence for linearly separable problems
Number of updates per iteration	Updates all weights at once	Updates only weights of misclassified inputs
Applications	Wider range of tasks, including regression, multi-class classification, etc.	Simple binary classification problems

Perceptron Vs Gradient Descent

Problem Solving

- Given the below dataset

Training Example	x_1	x_2	Class
a.	0	1	-1
b.	2	0	-1
c.	1	1	+1

1. Construct by hand a Perceptron which correctly classifies the above data. Use your knowledge of plane geometry to choose appropriate values for the weights w_0, w_1 and w_2.
2. Demonstrate the Perceptron Learning Algorithm on the above data, using a learning rate of 1.0 and initial weight values of

$$w_0 = -1.5,\ w_1 = 0,\ w_2 = 2$$

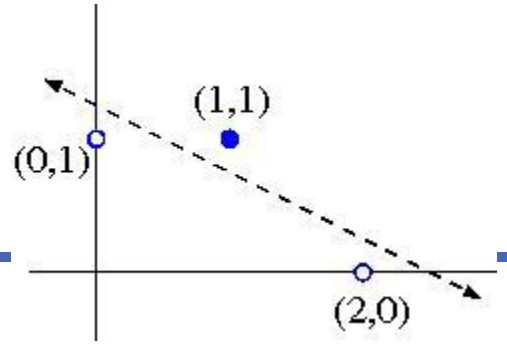
In your answer, you should clearly indicate the new weight values at the end of each training step.

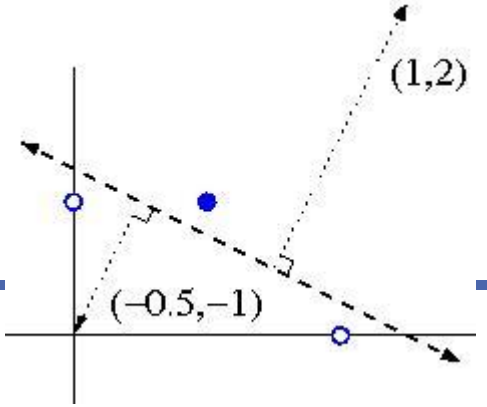
Data Analysis

Training Example	x_1	x_2	Class
a.	0	1	-1
b.	2	0	-1
c.	1	1	+1

- Solution (part a)
- First → plot the data in the 2D space (figure bottom left)
- This line has slope -1/2 and x_2-intersect 5/4, so its equation is:
- $x_2 = 5/4 - x_1/2$, i.e. $2x_1 + 4x_2 - 5 = 0$.
- Taking account of which side is positive, this corresponds to these weights:
- $w_0 = -5$, $w_1 = 2$, $w_2 = 4$
- Alternatively, we can derive weights $w_1=1$ and $w_2=2$ by drawing a vector normal to the separating line, in the direction pointing towards the positive data points:
- The bias weight w_0 can then be found by computing the dot product of the normal vector with a perpendicular vector from the separating line to the origin. In this case $w_0 = 1(-0.5) + 2(-1) = -2.5$
- (Note: these weights differ from the previous ones by a normalizing constant, which is fine for a Perceptron)

Data Analysis





- Solution (part b)

Training Example	x_1	x_2	Class
a.	0	1	-1
b.	2	0	-1
c.	1	1	+1

Iteration	w_0	w_1	w_2	Training Example	x_1	x_2	Class	$s=w_0+w_1x_1+w_2x_2$	Action
1	-1.5	0	2	a.	0	1	-	+0.5	Subtract
2	-2.5	0	1	b.	2	0	-	-2.5	None
3	-2.5	0	1	c.	1	1	+	-1.5	Add
4	-1.5	1	2	a.	0	1	-	+0.5	Subtract
5	-2.5	1	1	b.	2	0	-	-0.5	None
6	-2.5	1	1	c.	1	1	+	-0.5	Add
7	-1.5	2	2	a.	0	1	-	+0.5	Subtract
8	-2.5	2	1	b.	2	0	-	+1.5	Subtract
9	-3.5	0	1	c.	1	1	+	-2.5	Add
10	-2.5	1	2	a.	0	1	-	-0.5	None
11	-2.5	1	2	b.	2	0	-	-0.5	None
12	-2.5	1	2	c.	1	1	+	+0.5	None

Data Analysis

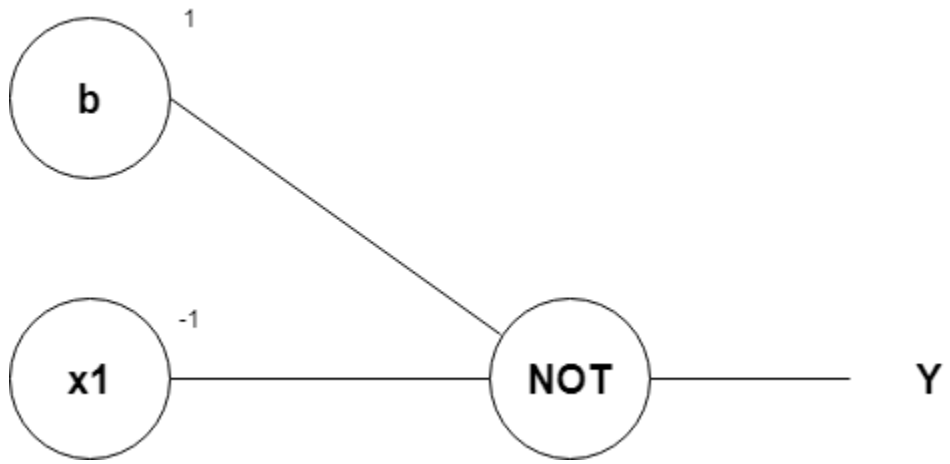
- Consider a McCulloch-Pitts Neural Network. Design one that implements a 1 input NOT Gate

Input 1	Output
0	1
1	0

Problem solving session

- Consider a McCulloch-Pitts Neural Network. Design one that implements a 1 input NOT Gate

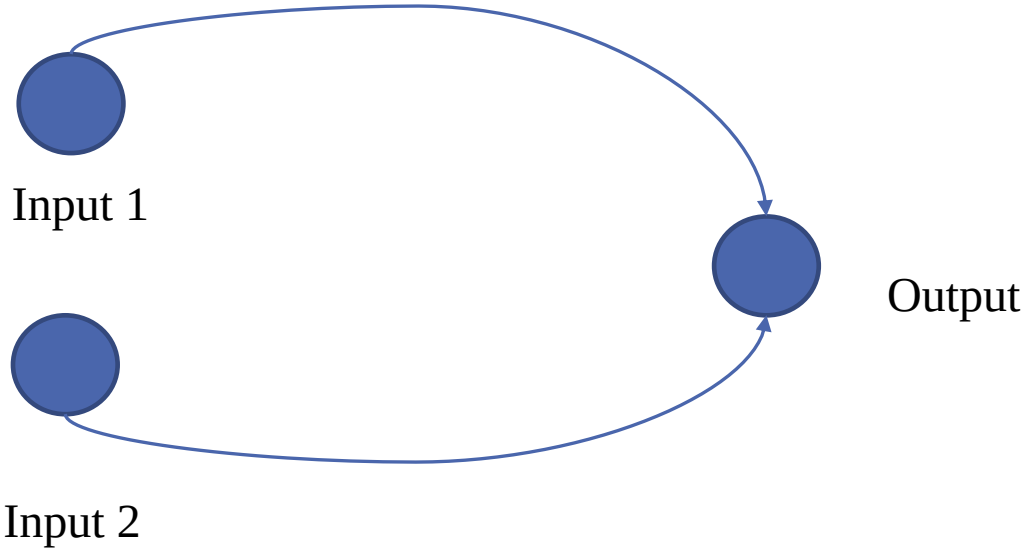
Input 1	Output
0	1
1	0



Problem solving session

- Consider a McCulloch-Pitts Neural Network. Design one that implements a 2 input AND Gate

Input 1	Input 2	Output
0	0	0
0	1	0
1	0	0
1	1	1

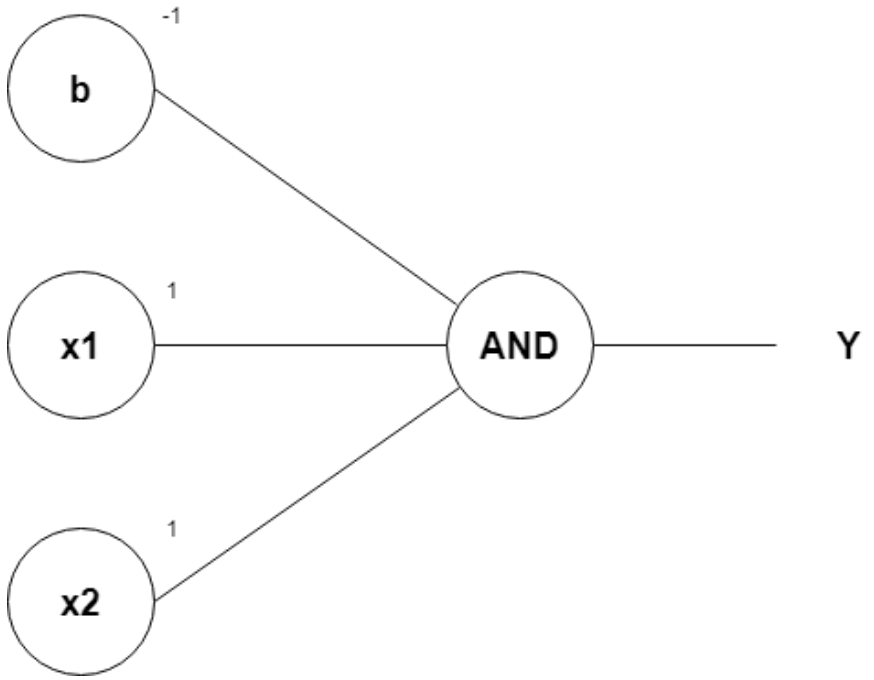


- We can deduce weights based on inference:
  - The only place the neuron fires is when the inputs are (1, 1)
  - Given the formula is
  - $v_k = \sum_{j=1}^{m} w_{kj}\, x_j$ and $y(k) = \begin{cases} 1, & v_k \geq 0 \\ 0, & v_k < 0 \end{cases}$

Problem solving session

- Consider a McCulloch-Pitts Neural Network. Design one that implements a 2 input AND Gate

Input 1	Input 2	Output
0	0	0
0	1	0
1	0	0
1	1	1

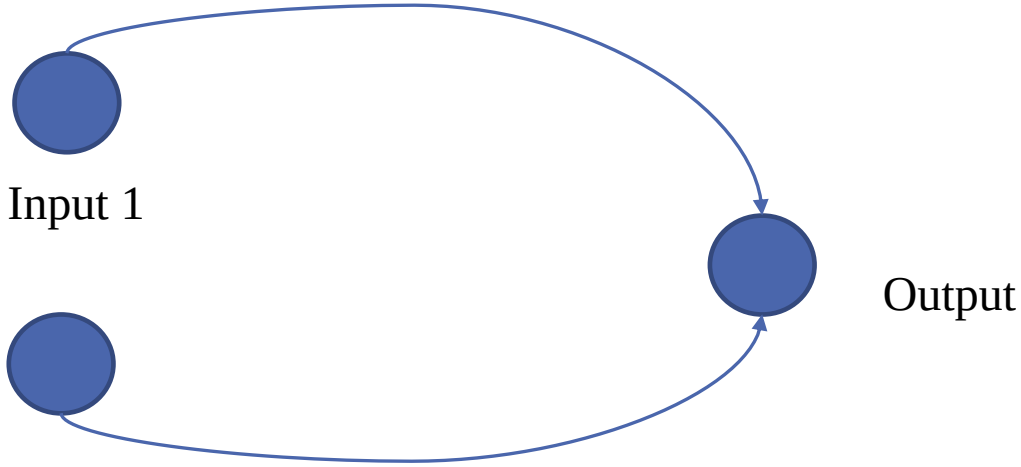


- We can deduce weights based on inference:
  - The only place the neuron fires is when the inputs are (1, 1)
  - Given the formula is
  - $v_k = \sum_{j=1}^{m} w_{kj}\, x_j$ and $y(k) = \begin{cases} 1, & v_k \geq 0 \\ 0, & v_k < 0 \end{cases}$
  - We can solve the equations by placing the values of x1 and x2 and the output to get 4 equations with 3 unknowns. We end up with w1=1 and w2=1 and Threshold=2

Problem solving session

- Consider a McCulloh Pitts Neural Network. Design one that implements a 2 input AND Not Gate

Input 1	Input 2	Output
0	0	0
0	1	0
1	0	1
1	1	0



- We can deduce weights based on inference:
  - The only place the neuron fires is when the inputs are (1, 0)
  - Given the formula is
  - $v_k = \sum_{j=1}^{m} w_{kj}\, x_j$ and $y(k) = \begin{cases} 1, & v_k \geq 0 \\ 0, & v_k < 0 \end{cases}$

Problem solving session

- Consider a McCulloch-Pitts Neural Network. Design one that implements a 2 input AND Not Gate

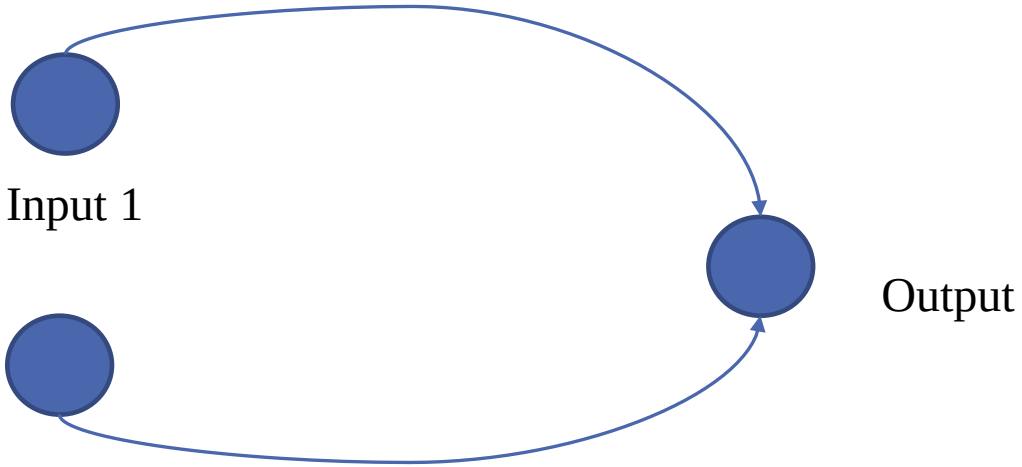
Input 1	Input 2	Output
0	0	0
0	1	0
1	0	1
1	1	0

- We can deduce weights based on inference:
  - The only place the neuron fires is when the inputs are (1, 1)
  - Given the formula is
  - $v_k = \sum_{j=1}^{m} w_{kj}\, x_j$ and $y(k) = \begin{cases} 1, & v_k \geq 0 \\ 0, & v_k < 0 \end{cases}$
  - We can solve the equations by placing the values of x1 and x2 and the output to get 4 equations with 3 unknowns. We end up with w1=2 and w2=-1 and Threshold=2

Problem solving session

- Consider a McCulloch-Pitts Neural Network. Design one that implements a 2 input OR

Input 1	Input 2	Output
0	0	0
0	1	1
1	0	1
1	1	1

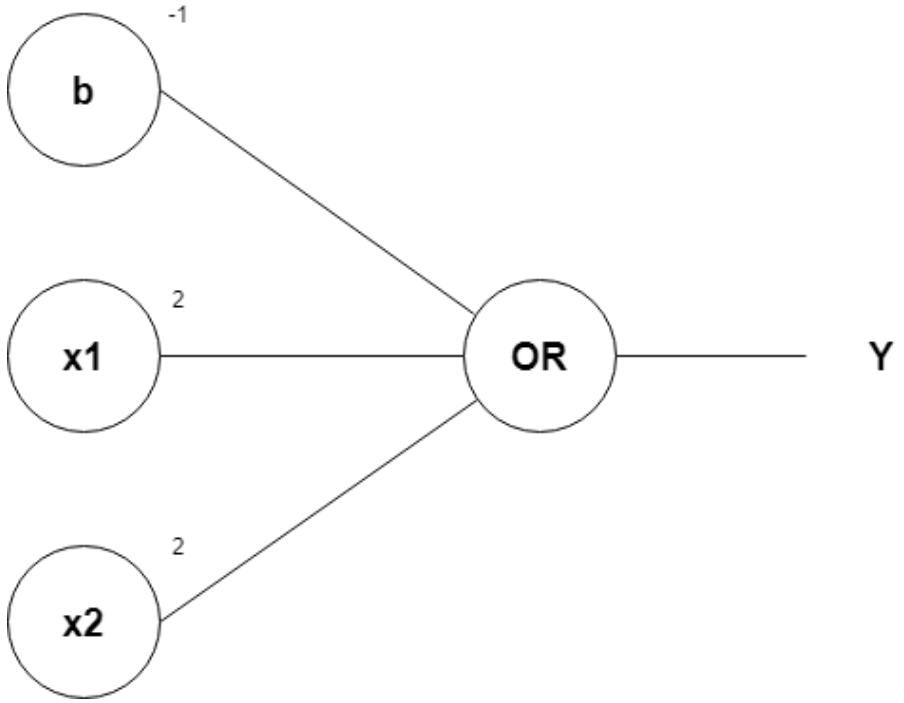


- We can deduce weights based on inference:
  - The only place the neuron does not fire is when the inputs are (0, 0)
  - Given the formula is
  - $v_k = \sum_{j=1}^{m} w_{kj}\, x_j$ and $y(k) = \begin{cases} 1, & v_k \geq 0 \\ 0, & v_k < 0 \end{cases}$

Problem solving session

- Consider a McCulloch-Pitts Neural Network. Design one that implements a 2 input OR

Input 1	Input 2	Output
0	0	0
0	1	1
1	0	1
1	1	1



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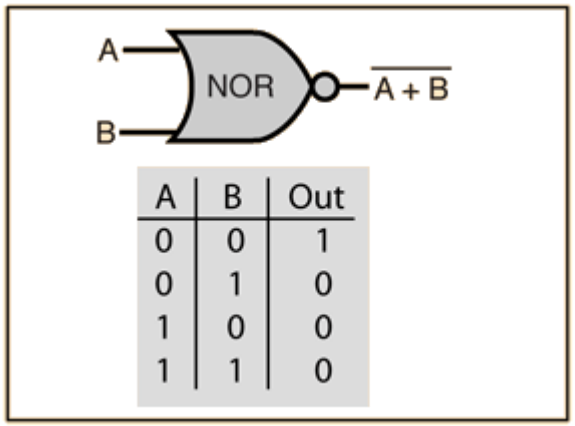
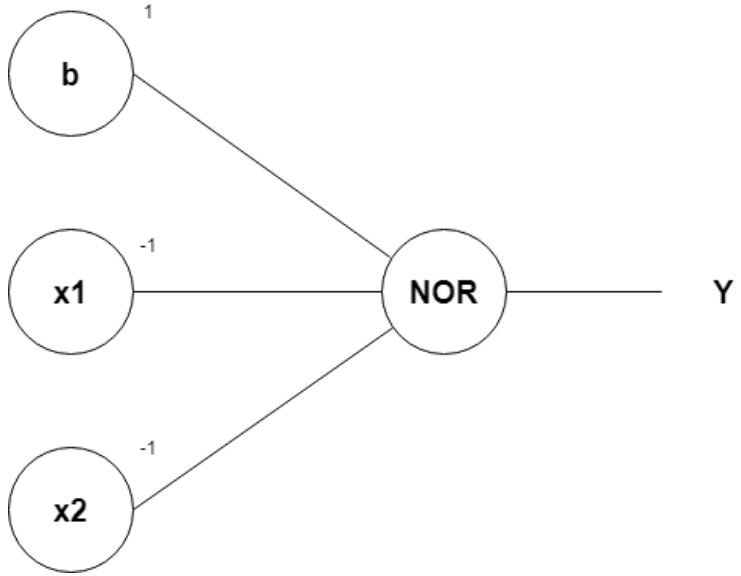
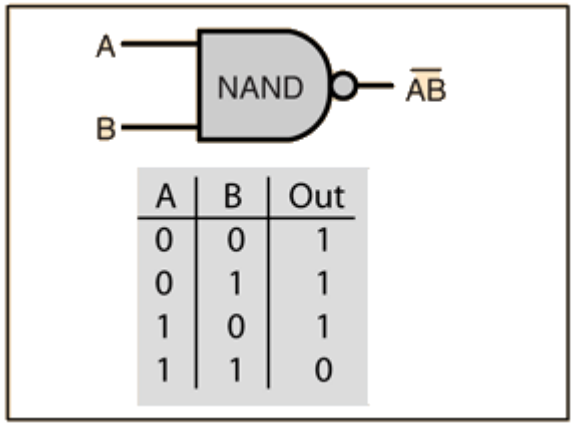
Problem solving session

- Think to other gates like

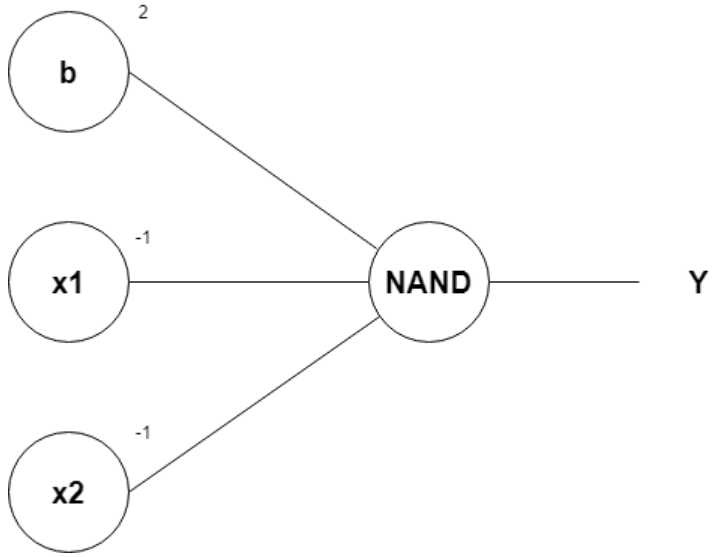


A	B	Out
0	0	1
0	1	0
1	0	0
1	1	0





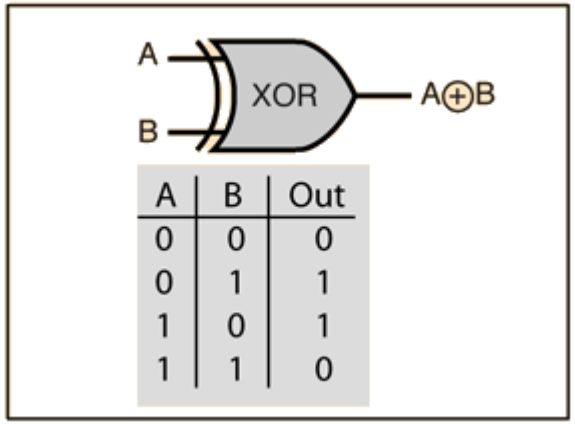
A	B	Out
0	0	1
0	1	1
1	0	1
1	1	0



Problem solving session

Problem solving session

- What about the following



A	B	Out
0	0	0
0	1	1
1	0	1
1	1	0