Available online at www.sciencedirect.com
ScienceDirect
Mathematics and Computers in Simulation 211 (2023) 241–262
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Original articles

Mathematical modeling of the unemployment problem in a context of financial crisis

Eric Rostand Njike-Tchaptchet, Calvin Tadmon*

Department of Mathematics and Computer Science, Faculty of Science, University of Dschang, P.O Box 67, Cameroon

Received 26 October 2022; received in revised form 6 February 2023; accepted 10 April 2023
Available online 18 April 2023

Abstract

In this paper, we formulate a new system of nonlinear ordinary differential equations to study the unemployment problem in a context of financial crisis. We first prove the existence of a unique positive equilibrium. Then, using an appropriate Lyapunov function under some specified conditions, we prove the global stability of the unique positive equilibrium. We also propose and compare two control strategies with the objective to improve, at the lowest cost, the employment rate. Our results suggest that, in order to reduce the unemployment rate, it is better for a government to assist unemployed people in building their own business which will allow them to further create new vacancies than to assist self-employed individuals to create new vacancies. Numerical simulations are presented to substantiate the theoretical results.

Keywords: Mathematical modeling; Financial crisis; Unemployment; Lyapunov function; Optimal control

1. Introduction

Unemployment is an extremely severe social and economic problem, arising from the difference between demand and supply of the labor market and sometimes emphasized by population's growth. The unemployed population can be defined as the portion of able citizens who desire to work (known as the active population) but, unfortunately, due to insufficient supply, are deprived from working. The increasing tendency of unemployed population in a country significantly affects the other factors, such as the income per person, health costs, quality of healthcare and poverty. This can seriously hamper the growth of the national economy. In fact, unemployed people have low spending potential and pay no income tax to boost the national economy. In an economy, banks act as intermediary channels through which funds are transferred from savers to investors [18]. Therefore, the stability of the financial system is essential to insure new investments that can allow to create more jobs.

In the literature, several authors have proposed mathematical models to analyze the unemployment problem [1–3, 5,14,15,19–23,25,27]. These authors did not account for the impact of financial crisis on the unemployment problem. Using empirical approaches, some authors studied the link between financial crisis and unemployment [4,6,9–11] and established the negative impact of financial crisis on employment. A theoretical model has been proposed in [26] to study the impact of financial crisis on unemployment and economic growth. In the present paper, we propose

* Corresponding author.
E-mail addresses: njikeeric@gmail.com (E.R. Njike-Tchaptchet), tadmoncalvin@gmail.com (C. Tadmon).

https://doi.org/10.1016/j.matcom.2023.04.014

a mathematical model to study the unemployment problem in a context of financial crisis. Our work differs from those in [1–3,5,14,15,19–23,25,27] as we take into account the state health of the financial system. In opposite to the approach proposed in [26], we account for the self-employment.

Our contribution to the literature on unemployment is threefold: (1) formulation and mathematical analysis of an ordinary differential equation (ODE) model describing the behavior of unemployed people in the context of a financial crisis; (2) design and resolution of two optimal control problems that can be useful to public authorities to define and implement policies aiming to reduce the unemployment rate; (3) numerical simulation of theoretical results.

The paper is organized as follows: Section 2 describes the formulation of the mathematical model. Section 3 deals with the mathematical analysis of the model. In Section 4, we formulate and solve two optimal control problems with the objective of reducing, at the lowest cost, the number of unemployed individuals. Section 5 is devoted to numerical simulations. For concluding the work, the main findings are outlined in Section 6.

2. Model formulation

In this Section, we build the mathematical model under appropriate assumptions.

2.1. The model of the unemployment problem

To study the unemployment problem, we consider the local labor market in which unemployed individuals (sellers) and firms or government (buyers) have to search for each other in order to find a match between unemployed's skills and vacancy profile. We assume that the active population is divided in two categories: The unemployed and the employed persons. For a given time $t \geq 0$, we denote by $U(t)$ and $E(t)$ the numbers of unemployed and employed individuals at time t, respectively. We denote by $V(t)$ the number of available vacancies at time t. To built the model, we make the following assumptions:

- Each unemployed individual is qualified and actively in search of a job.
- Each employed individual can fill one and only one job.
- Unemployed individuals and available vacancies are homogeneously distributed.
- In order to be recruited as an employee, each unemployed person has to go through a recruitment process, the purpose of which is to check the suitability of the unemployed person's skills for the position applied for. The matching function at time $t \geq 0$ between the unemployed individuals and vacancies is $U(t)V(t)$.
- There is no emigration of unemployed and employed individuals.
- The rate of change of the number of unemployed persons $U(t)$ depends on the following five factors:
  (i) The number A $(A > 0)$ of new unemployed individuals entering the labor market per unit of time.
  (ii) The number $kU(t)V(t)$ of unemployed individuals recruited per unit of time to fill vacancies. Here k is the rate at which the unemployed individuals become employed.
  (iii) The fraction $\mu U(t)$ of unemployed individuals who leave the labor marked per unit of time due to death. Here, $\mu \in (0, 1)$ is the death rate of the population.
  (iv) The fraction $\beta E(t)$ of employed individuals who lose their jobs as a result of job cuts per unit of time. $\beta \in [0, 1)$ is the likelihood of losing one's job as a result of job cuts.
  (v) The fraction $\lambda U(t)$ of unemployed individuals who set up their own businesses per unit of time. $\lambda \in (0, 1)$ is the rate at which unemployed individuals become self-employed.

  The total change in the number of unemployed individuals per unit of time is summarized as:

$$\dot{U}(t) = A - kU(t)V(t) + \beta E(t) - (\mu + \lambda)U(t). \tag{1}$$

- The rate of change in the number of employed individuals depends on the following five factors:
  (i) The number $kU(t)V(t)$ of unemployed individuals recruited per unit of time to fill vacancies.
  (ii) The fraction $\beta E(t)$ of employed individuals who lose their jobs as a result of job cuts per unit of time.
  (iii) The fraction $\lambda U(t)$ of unemployed individuals who set up their own businesses per unit of time.
  (iv) The fraction $\alpha E(t)$ of employed individuals retired per unit of time. $\alpha \in (0, 1)$ represents the rate of retirement.

(v) The fraction $\mu E(t)$ of employed individuals who leave their jobs per unit of time due to death.

The total change in the number of employed individuals per unit of time is summarized as:

$$\dot{E}(t) = kU(t)V(t) - (\beta + \alpha + \mu)E(t) + \lambda U(t). \tag{2}$$

- The rate of change in the number of vacancies depends on the following six factors:

(i) The number $\sigma U(t)$ of new jobs created by the government and firms per unit of time. Here, $\sigma \in (0, 1)$ represents the rate of creating new vacancies.
(ii) The number $\lambda q U(t)$ of new jobs created per unit of time by self-employed entrepreneurs. Here, q is the average number of new vacancies created by each self-employed individual.
(iii) The number $\alpha E(t)$ of employed individuals retired per unit of time and leaving their positions vacant.
(iv) The number $\mu E(t)$ of dead employed individuals per unit of time and leaving their positions vacant.
(v) The fraction $\delta V(t)$ of vacancies that are canceled for budgetary reasons. Here $\delta \in (0, 1)$ denotes the diminution rate of vacancies due to lack of funds.
(vi) The number $kU(t)V(t)$ of vacancies filled per unit of time after the recruitment process.

The total change in vacancies per unit of time is summarized as:

$$\dot{V}(t) = (\lambda q + \sigma)U(t) + (\alpha + \mu)E(t) - \delta V(t) - kU(t)V(t). \tag{3}$$

- We make the following additional hypotheses on model's parameters

1. $\lambda > \mu$. This assumption is interpreted as follows: as $\frac{1}{\mu}$ represents the average lifetime of an individual in the considered population, we must have $\frac{1}{\lambda} < \frac{1}{\mu}$. In fact, to start a personal business, one has to be alive.
2. $\alpha > \mu$. This assumption is interpreted as follows: as $\frac{1}{\alpha}$ represents the average time spent by an employed individual before retirement, we assume that $\frac{1}{\mu} > \frac{1}{\alpha}$. That is, an employed individual is retired when he is still alive.

Putting together Eqs. (1), (2) and (3), we obtain the following system of ODEs modeling the unemployment problem.

$$\begin{cases} \dot{U}(t) = A - kU(t)V(t) + \beta E(t) - (\mu + \lambda)U(t), \\ \dot{E}(t) = kU(t)V(t) - (\beta + \alpha + \mu)E(t) + \lambda U(t), \\ \dot{V}(t) = (\lambda q + \sigma)U(t) + (\alpha + \mu)E(t) - \delta V(t) - kU(t)V(t) \end{cases} \tag{4}$$

2.2. The impact of financial crisis on unemployment

We assume that the banking system consists of two classes: healthy banks and distressed banks. Healthy banks are those with positive indicators in accordance with the banking authority regulator. Distressed banks are those suffering of losses due to default of a financial consideration in loan's portfolio, losses coming from the insufficiency of liquidity necessary to pursue financial intermediation process, losses due to earlier selling of owned securities and losses coming from bank runs. Stress spreads in the financial system via the interbank network.

As banks are intermediary financial institutions through which funds are channeled from savers to investors [18], we assume that the proportion of the income that is saved is a function of healthy bank's density in the banking system. Therefore, new investments, which are financed by savings, depend on the density of healthy banks. In a situation of general bank distress, what is observed in the real economy is the bank run, that is, clients rush to their respective banks to withdraw their money; by so doing they weaken the power of the banks to play their role of investment driver through the process of lending. In order to study the impact of a financial crisis on unemployment, we denote by $\rho \in [0, 1]$ the given density of distressed banks in the banking system and we make the following hypotheses:

- Public sector and firms can recruit only if there are additional investments. Hence, we postulate that the rate of creating new vacancies σ is a decreasing function of the density of distressed banks. We assume that $\sigma(\rho) = \sigma_0(1 - \rho)^{\epsilon_1}$, where $\sigma_0 \in (0, 1)$, and $\epsilon_1 > 0$ measures the efficiency of intermediation activity in affecting the investments.

- The reduction in investments, due to the financial crisis, leads to job cuts. We assume that β, the likelihood of losing one's job as a result of job cuts, is an increasing function of the density of distressed banks. We assume that $\beta(\rho) = \beta_0(1+\gamma\rho)^{\epsilon_1}$. Here, $\beta_0 \in (0, 1)$ is the likelihood of losing one's job due to job cuts in the absence of financial crisis, and γ is a positive real number.
- We also assume that the average number of new vacancies created by each self-employed individual, q, is a decreasing function of the density of distressed banks. That is, $q(\rho) = q_0(1-\rho)^{\epsilon_1}$, where $q_0 > 0$ is the average number of new vacancies created by each self-employed individual in the absence of a financial crisis.

Taking into account the above hypotheses, we obtain the following system of nonlinear ODEs describing the unemployment problem for a given state of financial health.

$$\begin{cases} \dot{U} = A - kUV + \beta_0(1+\gamma\rho)^{\varepsilon_1}E - (\mu+\lambda)U, \\ \dot{E} = kUV - (\beta_0(1+\gamma\rho)^{\varepsilon_1} + \alpha + \mu)E + \lambda U, \\ \dot{V} = \big(\lambda q_0 + \sigma_0\big)(1-\rho)^{\varepsilon_1}U + (\alpha+\mu)E - \delta V - kUV. \end{cases} \tag{5}$$

3. Mathematical analysis of the model

In this Section, we establish the main properties of system (5).

3.1. Positivity, boundedness and global existence

The following theorem is about the positivity, the boundedness and the global existence of a unique solution to (5).

Theorem 3.1. *Let U_0, E_0 and V_0 be given initial data for system (5) such that $U_0 > 0$, $E_0 > 0$ and $V_0 > 0$.*

(i) *System (5) admits a unique continuous solution $(U(t), E(t), V(t))$ defined for all $t \in [0, t_f)$, where $t_f \in (0, \infty)$. Furthermore, we have $U(t) > 0$, $E(t) > 0$ and $V(t) > 0$ for all $t \in [0, t_f)$.*

(ii) *The set Ω defined by*

$$\Omega = \left\{ (U, E, V) \in \mathbb{R}^3 \; \frac{A}{\alpha+\mu} \le U, E \le \frac{A}{\mu},\; \frac{A\mu}{\delta\mu + kA} \le V \le \frac{A(\alpha+\mu)\max(\lambda q_0 + \sigma_0, \alpha+\mu)}{\mu\big(\delta(\alpha+\mu) + kA\big)} \right\} \tag{6}$$

is positively invariant for system (5).

(iii) *The solution $(U(t), E(t), V(t))$ of system (5) is defined for all $t \ge 0$.*

Proof.

(i) Consider the function Φ defined from $\mathbb{R}^3$ to $\mathbb{R}^3$ by

$$\Phi(U, E, V) = \begin{pmatrix} A - kUV + \beta_0(1+\gamma\rho)^{\varepsilon_1}E - (\mu+\lambda)U \\ kUV - (\beta_0(1+\gamma\rho)^{\varepsilon_1} + \alpha + \mu)E + \lambda U \\ \big(\lambda q_0 + \sigma_0\big)(1-\rho)^{\varepsilon_1}U + (\alpha+\mu)E - \delta V - kUV \end{pmatrix}.$$

The function Φ is continuously differentiable on $\mathbb{R}^3$ and therefore is locally Lipschitz. Using the Cauchy-Lipschitz Theorem [17], the Cauchy problem for system (5) admits a unique local solution. That is, there exists a positive real number t_f such that $(U(t), E(t), V(t))$ is defined for all $t \in [0, t_f)$.

Let U_0, E_0 and V_0 be given initial data such that $U_0 > 0$, $E_0 > 0$ and $V_0 > 0$.

To prove that $U(t) > 0$ for all $t \in [0, t_f)$, we assume that there exist some $t \in (0, t_f)$ such that $U(t) < 0$. Let $t_1 \in (0, t_f)$ be the first time such that $U(t_1) < 0$. Hence, as U is a continuous function of time, there exists $\bar{t}_1 \in (0, t_1)$ such that

$$U(\bar{t}_1) = 0, \quad E(\bar{t}_1) \ge 0, \quad \text{and } V(\bar{t}_1) \ge 0.$$

Using the first equation of (5), we obtain for $t = \bar{t}_1$

$$\dot{U}(\bar{t}_1) = A + \beta_0(1+\gamma y(\bar{t}_1))^{\varepsilon_1}E(\bar{t}_1) > 0. \tag{7}$$

(7) implies that the function $U : t \mapsto U(t)$ cannot have negative values. The positivity of $E(t)$ and $V(t)$ are proven similarly.

(ii) Let $t \geq 0$.

Adding the right-hand side of the first and second equations of (5) leads to

$$\dot{U}(t) + \dot{E}(t) = A - \mu U(t) - (\alpha + \mu)E(t). \tag{8}$$

Let $M(t) = U(t) + E(t)$ denoting the labor force, we deduce from (8) and the positivity of $U(t)$ and $E(t)$ that

$$A - (\alpha + \mu)M(t) \leq \dot{M}(t) \leq A - \mu M(t). \tag{9}$$

Using the comparison Theorem [16] for the first and second inequalities of (9) leads to

$$\liminf_{t\to\infty} M(t) \geq \frac{A}{\alpha + \mu}, \quad \limsup_{t\to\infty} M(t) \leq \frac{A}{\mu}.$$

That is, $\frac{A}{\alpha+\mu} \leq U(t) \leq \frac{A}{\mu}$ and $\frac{A}{\alpha+\mu} \leq E(t) \leq \frac{A}{\mu}$.

Using the fourth equation of (5) and the previous inequalities, we have

$$A - (\delta + \frac{kA}{\mu})V(t) \leq \dot{V}(t) \leq \frac{A}{\mu}\max(\lambda q_0 + \sigma_0, \alpha + \mu) - (\delta + \frac{kA}{\alpha + \mu})V(t). \tag{10}$$

Using once more the comparison Theorem [16] for the first and second inequalities of (10), we obtain

$$\liminf_{t\to\infty} V(t) \geq \frac{A\mu}{\delta\mu + kA}, \quad \limsup_{t\to\infty} V(t) \leq \frac{A(\alpha + \mu)\max(\lambda q_0 + \sigma_0, \alpha + \mu)}{\mu\big(\delta(\alpha + \mu) + kA\big)}.$$

That is, $\frac{A\mu}{\delta\mu+kA} \leq V(t) \leq \frac{A(\alpha+\mu)\max(\lambda q_0+\sigma_0,\alpha+\mu)}{\mu\left(\delta(\alpha+\mu)+kA\right)}$.

(iii) From (ii) the solution $(U(t), E(t), V(t))$ is contained in a compact subset of $\mathbb{R}^3$ and therefore, is globally defined. □

3.2. Existence and stability of equilibria

In this section, we investigate the existence and stability of stationary solutions to (5). The following result is about the existence of equilibria for system (5).

Theorem 3.2. *System (5) admits a unique equilibrium, $Q_* = (U_*, E_*, V_*)$, in Ω, where*

$$\begin{aligned} E_* &= \frac{A-\mu U_*}{\alpha+\mu}, \\ V_* &= \frac{A+\left((\lambda q_0+\sigma_0)(1-\rho)^{\epsilon_1}-\mu\right)U_*}{\delta+kU_*}, \\ U_* &= \frac{\sqrt{\Delta_1}-\left[\beta_0(1+\gamma\rho)^{\epsilon_1}(\mu\delta-kA)+(\alpha+\mu)\delta(\lambda+\mu)\right]}{2k\left((\alpha+\mu)\left[(\lambda q_0+\sigma_0)(1-\rho)^{\epsilon}+\lambda\right]+\beta_0(1+\gamma\rho)^{\epsilon_1}\mu\right)}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \Delta_1 = &\left[\beta_0(1+\gamma\rho)^{\epsilon_1}(\mu\delta - kA) + (\alpha+\mu)\delta(\lambda+\mu)\right]^2 \\ &+4A\delta\big(\beta_0(1+\gamma\rho)^{\epsilon_1} + \alpha + \mu\big)k\big[(\alpha+\mu)\big((\lambda q_0+\sigma_0)(1-\rho)^{\epsilon_1}+\lambda\big) + \beta_0(1+\gamma\rho)^{\epsilon_1}\mu\big]. \end{aligned}$$

Proof. A point (U_*, E_*, V_*) is an equilibrium of (5) if it is solution to the following system of algebraic equations

$$\begin{cases} A - kUV + \beta_0(1+\gamma\rho)^{\varepsilon_1}E - (\mu+\lambda)U = 0, \\ kUV - (\beta_0(1+\gamma\rho)^{\varepsilon_1} + \alpha + \mu)E + \lambda U = 0, \\ \big(\lambda q_0 + \sigma_0\big)(1-\rho)^{\varepsilon_1}U + (\alpha+\mu)E - \delta V - kUV = 0. \end{cases} \tag{12}$$

By adding the right-hand side of the first and second equations in (12), we obtain

$$A - \mu U - (\alpha + \mu)E = 0.$$

That is,

$$E_* = \frac{A - \mu U_*}{\alpha + \mu}. \tag{13}$$

From the third equation of (12), we get

$$V_* = \frac{A + (\lambda q_0 + \sigma_0 - \mu)(1-\rho)^{\varepsilon_1} U_*}{\delta + k U_*}. \tag{14}$$

Using (13) and (14) in the second equation of (12), we obtain the second order polynomial equation

$$A_1 U_*^2 + A_2 U_* + A_3 = 0, \tag{15}$$

where $A_1 = (\alpha+\mu)k(\lambda(q_0+\sigma_0)(1-\rho)^{\varepsilon_1} + \lambda) + \beta_0(1+\gamma\rho)^{\varepsilon_1}\mu k$, $A_2 = \beta_0(1+\gamma\rho)^{\varepsilon_1}(\mu\delta - kA) + (\alpha+\mu)\delta(\lambda+\mu)$ and $A_3 = -(\beta_0(1+\gamma\rho)^{\varepsilon_1} + \alpha + \mu)A\delta$. Eq. (15) admits a unique economically feasible solution given by

$$U_* = \frac{\sqrt{\Delta_1} - \left[\beta_0(1+\gamma\rho)^{\epsilon_1}(\mu\delta - kA) + (\alpha+\mu)\delta(\lambda+\mu)\right]}{2k\Big((\alpha+\mu)\big[(\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon} + \lambda\big] + \beta_0(1+\gamma\rho)^{\epsilon_1}\mu\Big)},$$

with

$$\begin{aligned}\Delta_1 = &\Big(\beta_0(1+\gamma\rho)^{\epsilon_1}(\mu\delta - kA) + (\alpha+\mu)\delta(\lambda+\mu)\Big)^2 \\ &+4kA\delta\big(\beta_0(1+\gamma\rho)^{\epsilon_1} + \alpha + \mu\big)\Big((\alpha+\mu)\big((\lambda q_0+\sigma_0)(1-\rho)^{\epsilon_1} + \lambda\big) + \beta_0(1+\gamma\rho)^{\epsilon_1}\mu\Big) \quad \square\end{aligned}$$

The equilibrium Q_* can be interpreted as the stationary state of the number of unemployed individuals, the number of employed individuals and the number of vacancies corresponding to the given level of the financial crisis. The following results are about the local stability of the equilibrium point.

Theorem 3.3. *The equilibrium Q_* is locally asymptotically stable.*

Proof. The Jacobian matrix $J(Q_*)$ of system (5) evaluated at the point Q_* is defined by

$$J(Q_*) = \begin{pmatrix} -kV_* - (\mu+\lambda) & -\beta_0(1+\gamma\rho)^{\epsilon_1} & -kU_* \\ kV_* + \lambda & -\big(\beta_0(1+\gamma\rho)^{\epsilon_1} + \alpha + \mu\big) & kU_* \\ (\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1} - kV_* & \alpha + \mu & -\delta - kU_* \end{pmatrix}. \tag{16}$$

The characteristic equation, with unknown w, associated to (16) is given by

$$w^3 + a_2 w^2 + a_1 w + a_0 = 0, \tag{17}$$

where, $a_0 = k(\alpha+\mu)(\lambda q_0+\sigma_0)(1-\rho)^{\epsilon_1}U_* + (\delta + kU_*)\big((\alpha+\mu)\lambda + \beta_0(1+\gamma\rho)^{\epsilon_1}\mu\big) + \delta(\alpha+\mu)\mu + \delta(\alpha+\mu)kV_*$, $a_1 = (\lambda q_0+\sigma_0)(1-\rho)^{\epsilon_1}kU_* + (\alpha+2\mu+\lambda)(\delta+kU_*) + k(\delta+\alpha+\mu)V_* + (\alpha+\mu)(\mu+\lambda) + \beta_0(1+\gamma\rho)^{\epsilon_1}\mu$, $a_2 = \delta + \alpha + 2\mu + \lambda + k(U_* + V_*)$. The coefficients a_0, a_1 and a_2 are positive and we have

$$\begin{aligned} a_2a_1 - a_0 = &\left[(kU_*+\delta)(\alpha+2\mu+\lambda) + \delta kV_*\right] + kU_*\big(\lambda q_0 + \sigma_0\big)(1-\rho)^{\epsilon_1}\big(\delta + \mu + \lambda + kU_* + kV_*\big) \\ &+ \big(\alpha + 2\mu + \lambda + kV_*\big)\big[(\alpha+\mu)(kV_* + \mu + \lambda) + \beta_0(1+\gamma\rho)^{\epsilon_1}\mu\big] + k^2(\alpha+\mu)U_*V_* \\ &> 0.\end{aligned}$$

Thus, from the Routh–Hurwitz criterion, we deduce that all the solutions of Eq. (17) have negative real parts. Therefore, the equilibrium Q_* is locally asymptotically stable. $\square$

The following theorem is about the global stability of the equilibrium Q_*.

Theorem 3.4. *If the following conditions*

$$\begin{aligned} &\beta_0(1+\gamma\rho)^{\epsilon_1}E_* \geq \tfrac{(\lambda q_0+\sigma_0)(1-\rho)^{\epsilon_1}kU_*^2}{\delta} \\ &\tfrac{\delta\mu}{k} \geq \big(\lambda + (\lambda q_0+\sigma_0)(1-\rho)^{\epsilon_1} + kV_*\big)U_* \end{aligned} \tag{18}$$

are satisfied, then the equilibrium Q_* is globally asymptotically stable.

Proof. Consider the following positive definite function corresponding to system (5), for the equilibrium Q_*:

$$L = \big(U - U_* - U_* \ln \frac{U}{U_*}\big) + m_1\big(E - E_* - E_* \ln \frac{E}{E_*}\big) + m_2\big(V - V_* - V_* \ln \frac{V}{V_*}\big), \tag{19}$$

where m_1 and m_2 are arbitrary constants to be chosen suitably later. Differentiating (19) with respect to time, along the solutions of system (5), we get

$$\begin{aligned}\frac{dL}{dt} &= \big(1 - \frac{U_*}{U}\big)\dot U + m_1\big(1 - \frac{E_*}{E}\big)\dot E + m_2\big(1 - \frac{V_*}{V}\big)\dot V \\ &= \big(1 - \frac{U_*}{U}\big)\Big[A\big(1 - \frac{U}{U_*}\big) - kU_*V_*\big(\frac{UV}{U_*V_*} - \frac{U}{U_*}\big) + \beta_0(1+\gamma\rho)^{\varepsilon_1}\big(\frac{E}{E_*} - \frac{U}{U_*}\big)\Big] \\ &+ m_1\big(1 - \frac{E_*}{E}\big)\Big[kU_*V_*\big(\frac{UV}{U_*V_*} - \frac{E}{E_*}\big) + \lambda U_*\big(\frac{U}{U_*} - \frac{E}{E_*}\big)\Big] \\ &+ m_2\big(1 - \frac{V_*}{V}\big)\Big[(\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1}U_*\big(\frac{U}{U_*} - \frac{V}{V_*}\big) + (\alpha+\mu)E_*\big(\frac{E}{E_*} - \frac{V}{V_*}\big) - kU_*V_*\big(\frac{UV}{U_*V_*} - \frac{V}{V_*}\big)\Big].\end{aligned} \tag{20}$$

Expanding (20) and choosing $m_1 = 1 + \frac{kU_*}{\delta}$ and $m_2 = \frac{kU_*}{\delta}$, we obtain the following relation:

$$\begin{aligned}\frac{dL}{dt} &= \Big(\frac{\delta\mu}{k} - \big(\lambda + (\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1} + kV_*\big)U_*\Big)\Big(2 - \frac{U}{U_*} - \frac{U_*}{U}\Big) \\ &+ \Big(\beta_0(1+\gamma\rho)^{\epsilon_1}E_* - \frac{(\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1}U_*^2}{\delta}\Big)\Big(2 - \frac{EU_*}{E_*U} - \frac{UE_*}{U_*E}\Big) \\ &+ \Big(\alpha + \mu + \frac{\beta_0(1+\gamma\rho)^{\epsilon_1}kU_*}{\delta}\Big)\Big(3 - \frac{E}{E_*} - \frac{U_*}{U} - \frac{UE_*}{U_*E}\Big) \\ &+ (\alpha+\mu)\frac{kU_*E_*}{\delta}\Big(3 - \frac{U_*}{U} - \frac{UVE_*}{U_*V_*E} - \frac{EV_*}{E_*V}\Big) \\ &+ \frac{(\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1}kU_*^2}{\delta}\Big(4 - \frac{U_*}{U} - \frac{UVE_*}{U_*V_*E} - \frac{EU_*}{E_*U} - \frac{UV_*}{U_*V}\Big).\end{aligned}$$

Thus, by using the arithmetic-geometric mean inequality and conditions (18), it follows that $\frac{dL}{dt} \le 0$. Moreover, $\frac{dL}{dt} = 0$ holds if and only if $U = U_*$, $E = E_*$, $V = V_*$. Consequently, L is a Lyapunov function and Q_* is globally asymptotically stable. □

The global asymptotic stability of the equilibrium means that whatever is the initial state of employment, at long-term, the number of unemployed will converge to a stationary state.

Let us give an economic interpretation of the inequalities in (18).

- Condition $\beta_0(1+\gamma\rho)^{\epsilon_1}E_* \ge \frac{(\lambda q_0+\sigma_0)(1-\rho)^{\epsilon_1}kU_*^2}{\delta}$ can be rewritten as

$$\Big(\frac{E_*}{U_*}\Big)\Big(\frac{1}{(\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1}}\Big) \ge \Big(\frac{kU_*}{\delta}\Big)\Big(\frac{1}{\beta_0(1+\gamma\rho)^{\epsilon_1}}\Big). \tag{21}$$

  Inequality (21) means that, at equilibrium, the weighted average time for creating a new vacancy is greater than or equal to the weighted average time to lose a job due to job cuts. The coefficients $\frac{E_*}{U_*}$ and $\frac{kU_*}{\delta}$ are the employability rate and the rate of filling vacancies, respectively.
- Condition $\frac{\delta\mu}{k} \ge \big(\lambda + (\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1} + kV_*\big)U_*$ can be rewritten as

$$\frac{1}{\lambda + (\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1} + kV_*} \ge \Big(\frac{1}{\mu}\Big)\Big(\frac{kU_*}{\delta}\Big). \tag{22}$$

  Inequality (22) means that, at equilibrium, the average cumulated time to get a job either from self-employed or recruited workers is greater than or equal to the weighted average life expectancy.

Remark 3.1. The positivity of the equilibrium Q_* means that the number of unemployed persons cannot be reduced to zero. Therefore, there will always exist unemployed individuals in a given economy.

3.3. Impact of some parameters on the unemployment rate

Due to the global stability of equilibrium of model (5), it can be interesting to study the impact of some parameters on the unemployment rate at equilibrium. In fact, reducing the equilibrium value of the unemployment rate will entail the reduction of the unemployment level at long-term. For a given time $t \geq 0$, the unemployment rate $\tau(t)$ is defined as

$$\tau(t) = \frac{U(t)}{E(t) + U(t)},$$

where $U(t)$ and $E(t)$ satisfy system (5). Given the density $\rho \in [0, 1]$ of the distressed banks in the banking system, denote by $\tau^* = \frac{U^*}{U^*+E^*}$ the equilibrium value of the unemployment rate.

Proposition 3.1.

$$\frac{\partial \tau^*}{\partial \rho} > 0, \tag{23}$$

$$\frac{\partial \tau^*}{\partial \lambda} < 0, \tag{24}$$

$$\frac{\partial \tau^*}{\partial k} < 0. \tag{25}$$

Proof. Let us establish the inequality (23).

We have

$$\frac{\partial \tau^*(\rho)}{\partial \rho} = \frac{\frac{\partial U^*}{\partial \rho} E^* - \frac{\partial E^*}{\partial \rho} U^*}{(U^* + E^*)^2},$$

where $\frac{\partial U^*(\rho)}{\partial \rho}$ and $\frac{\partial E^*(\rho)}{\partial \rho}$ are obtained by solving the following system of algebraic equations

$$\begin{cases} -(kV^* + \mu + \lambda)\frac{\partial U^*}{\partial \rho} + \beta_0(1+\gamma\rho)^{\epsilon_1}\frac{\partial E^*}{\partial \rho} - kU^*\frac{\partial V^*}{\partial \rho} = -\beta_0\epsilon_1\gamma(1+\gamma\rho)^{\epsilon_1-1}E^*, \\ (kV^* + \lambda)\frac{\partial U^*}{\partial \rho} - \big(\beta_0(1+\gamma\rho)^{\epsilon_1} + \alpha + \mu\big)\frac{\partial E^*}{\partial \rho} + kU^*\frac{\partial V^*}{\partial \rho} = \beta_0\epsilon_1\gamma(1+\gamma\rho)^{\epsilon_1-1}E^*, \\ \big((\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1} - kV^*\big)\frac{\partial U^*}{\partial \rho} + (\alpha+\mu)\frac{\partial E^*}{\partial \rho} - (\delta + kU^*)\frac{\partial V^*}{\partial \rho} = \epsilon_1(\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1-1}U^*. \end{cases} \tag{26}$$

Adding the left-hand side of the first two equations of (26), we get

$$\frac{\partial U^*}{\partial \rho} = -\Big(\frac{\alpha+\mu}{\mu}\Big)\frac{\partial E^*}{\partial \rho}. \tag{27}$$

Eq. (27) means that $\frac{\partial U^*}{\partial \rho}$ and $\frac{\partial E^*}{\partial \rho}$ vary in opposite directions. Using (27) in the second and third equations of (26), we obtain

$$\frac{\partial U^*}{\partial \rho} = \frac{(\alpha+\mu)\big[(\delta + kU^*)(\beta_0\epsilon_1\gamma(1+\gamma\rho)^{\epsilon_1-1}E^*) + kU^*\epsilon_1(\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1-1}U^*\big]}{(\alpha+\mu)\big[\delta kV^* + \delta\lambda + \lambda kU^* + \mu\lambda + (\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1}kU^*\big] + \beta_0\mu(1+\gamma\rho)^{\epsilon_1}(\delta + kV^*)} > 0. \tag{28}$$

As $\frac{\partial U^*}{\partial \rho} > 0$, using (27) once more, we deduce that $\frac{\partial \tau^*(\rho)}{\partial \rho} > 0$. To obtain the inequalities (24) and (25), we solve the following systems of algebraic equations

$$\begin{cases} -(kV^* + \mu + \lambda)\frac{\partial U^*}{\partial \lambda} + \beta_0(1+\gamma\rho)^{\epsilon_1}\frac{\partial E^*}{\partial \lambda} - kU^*\frac{\partial V^*}{\partial \lambda} = U^*, \\ (kV^* + \lambda)\frac{\partial U^*}{\partial \lambda} - \big(\beta_0(1+\gamma\rho)^{\epsilon_1} + \alpha + \mu\big)\frac{\partial E^*}{\partial \lambda} + kU^*\frac{\partial V^*}{\partial \lambda} = -U^*, \\ \big((\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1} - kV^*\big)\frac{\partial U^*}{\partial \lambda} + (\alpha+\mu)\frac{\partial E^*}{\partial \lambda} - (\delta + kU^*)\frac{\partial V^*}{\partial \lambda} = -q_0(1-\rho)^{\epsilon_1}U^* \end{cases} \tag{29}$$

and

$$\begin{cases} -(kV^* + \mu + \lambda)\frac{\partial U^*}{\partial k} + \beta_0(1+\gamma\rho)^{\epsilon_1}\frac{\partial E^*}{\partial k} - kU^*\frac{\partial V^*}{\partial k} = U^*V^*, \\ (kV^* + \lambda)\frac{\partial U^*}{\partial k} - \big(\beta_0(1+\gamma\rho)^{\epsilon_1} + \alpha + \mu\big)\frac{\partial E^*}{\partial k} + kU^*\frac{\partial V^*}{\partial k} = -U^*V^*, \\ \big((\lambda q_0 + \sigma_0)(1-\rho)^{\epsilon_1} - kV^*\big)\frac{\partial U^*}{\partial k} + (\alpha+\mu)\frac{\partial E^*}{\partial k} - (\delta + kU^*)\frac{\partial V^*}{\partial k} = U^*V^*, \end{cases} \tag{30}$$

respectively.

Solving system (29), we obtain

$$\frac{\partial U^*}{\partial \lambda} = -\Big(\frac{\alpha+\mu}{\mu}\Big)\frac{\partial E^*}{\partial \lambda}$$
$$= \frac{-(\alpha+\mu)\big(\delta + k(1+q_0(1-\rho)^{\epsilon_1})U^*\big)U^*}{(\alpha+\mu)\big[(kV^*+\lambda)\delta + \lambda k U^* + \delta\mu + (\lambda q_0 + \sigma_0)kU^*(1-\rho)^{\epsilon_1}\big] + \beta_0(1+\gamma\rho)^{\epsilon_1}\mu(\delta + kU^*)} < 0.$$

Inequality (24) follows.

From system (30), we obtain

$$\frac{\partial U^*}{\partial k} = -\Big(\frac{\alpha+\mu}{\mu}\Big)\frac{\partial E^*}{\partial \lambda}$$
$$= \frac{-(\alpha+\mu)\delta U^* V^*}{(\alpha+\mu)\big[(kV^*+\lambda)\delta + \lambda k U^* + \delta\mu + (\lambda q_0 + \sigma_0)kU^*(1-\rho)^{\epsilon_1}\big] + \beta_0(1+\gamma\rho)^{\epsilon_1}\mu(\delta + kU^*)} < 0.$$

Therefore, we deduce inequality (25).

We obtain the inequalities (23), (24) and (25) by doing the sensitivity analysis of the equilibrium $\tau^*(\rho)$ with respect to ρ, λ and k, respectively. □

Let us give some interpretations of Proposition 3.1.

Inequality (23), means that the unemployment rate is an increasing function of ρ. This underlines the negative impact of financial crisis on employment. Therefore, to improve the employment rate, the government must reduce the density of stressed banks in the banking system.

Inequality (24) means that the unemployment rate is a decreasing function of λ. That is, self-employment has a negative impact on unemployment or a positive impact on employment. Thus, to improve the employment rate, the government can encourage self-employment.

Inequality (25) means that the unemployment rate is a decreasing function of k. This means that to increase the employment rate, the government can improve the success rate in recruitment process, for example by creating job placement agencies with the aim to identify job offers and, for a given offer, to direct the jobseekers whose profiles best correspond to the offer.

Fig. 1 gives, for a fixed value of $A = 3000$ and $\epsilon_1 = 1$, the influence of parameters variations on the behavior of unemployed individuals. This Figure highlights the importance of the financial health of the banking system. The existing vacancies have a significant impact on the behavior of the system. This is illustrated by (i) the rate at which new vacancies are created by government and private sectors (σ_0); (ii) the diminution rate of vacancies (δ) and (iii) the rate at which employed individuals retire (α). The parameters k and λ reflecting the rate at which unemployed individuals become employed is also an important factor in addressing the unemployment problem.

Table 1 gives the range of values used to study the influence of parameters on the unemployment problem. The ranges of parameters k, β_0, μ, α, σ and δ have been chosen in such a way that they are compatible with values used in [5]. For the parameter q_0 representing the average number of new vacancies created by a self-employed individual, we assume that due to limited financial capacities, each self-employed individual creates between one and two vacancies after two years (24 months) of operation. To choose the range of the parameter λ, which represents the average time taken by an unemployed individual to launch his own activities, we assume that the average time required to put in place well operating lucrative activities is between one and 36 months. For the parameter ρ, representing the density of distressed banks in the banking system, we assume that due to the existence of regulators, its lowest value is greater than one per cent.

Fig. 2 illustrates the joint dependence of the equilibrium unemployment rate on the parameters ρ and λ. This Figure can be interpreted as follows:

(i) The negative impact of financial crisis on employment is exacerbated if there is no self-employment.

(ii) High self-employment rate has positive impact on employment even in a financial crisis context.

(iii) Low level of self-employment is unfavorable for employment even in the absence of financial crisis.

Fig. 3 illustrates the joint dependence of the equilibrium unemployment rate on the parameters ρ and k. This Figure can be interpreted as follows:

Table 1
Range of parameters used to study the influence of parameters on the unemployment problem.

Parameter	Range	Source
k	$(10^{-5}, 10^{-2})$	[5]
β_0	$(10^{-4}, 10^{-1})$	[5]
λ	$(1/36, 1)$	Assumed
μ	$(10^{-3}, 10^{-1})$	[5]
α	(0.0027,0.005)	[5]
q_0	$(1/24, 1/12)$	Assumed
σ	$(10^{-3}, 2\times10^{-1})$	[5]
δ	$(10^{-3}, 10^{-1})$	[5]
ρ	$(10^{-2}, 1)$	Assumed

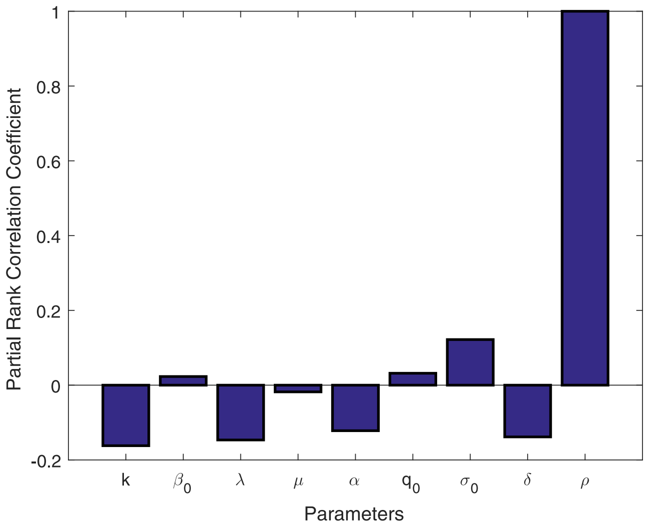

Fig. 1. Influence of parameters variation on the unemployment problem.

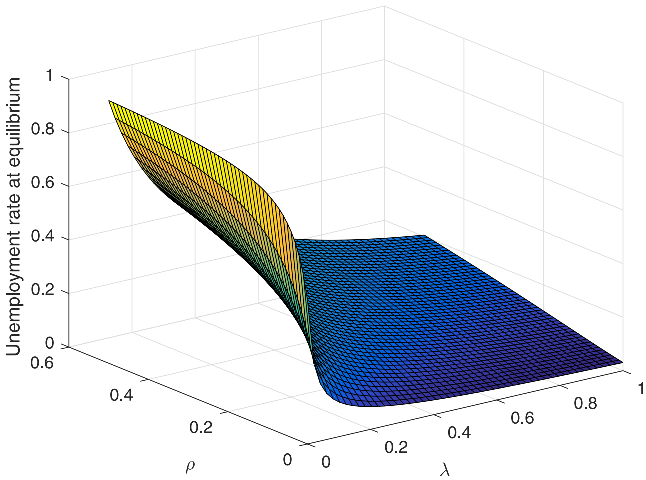

Fig. 2. Evolution of the equilibrium unemployment rate as a function of λ and ρ for the following values of parameters: $k = 1.08\times10^{-5}$, $\beta_0 = 0.01$, $\mu = 0.032$, $\alpha = 0.035$, $q_0 = 1/24$, $\delta = 0.075$, $\sigma = 0.01$.

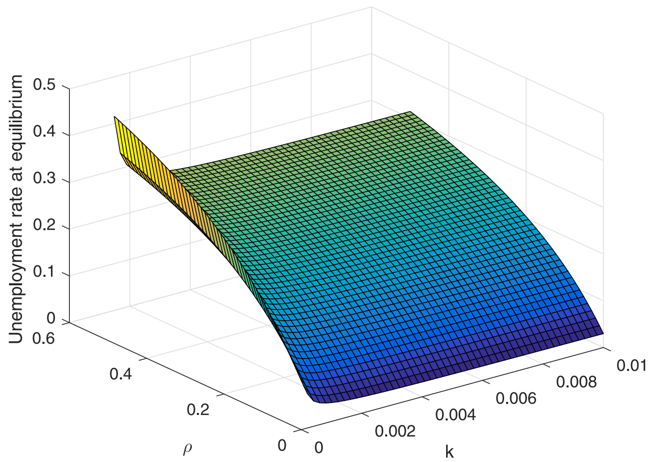

Fig. 3. Evolution of the equilibrium unemployment rate as a function of k and ρ, for the following values of parameters: $\lambda = 0.15$, $\beta_0 = 0.01$, $\mu = 0.032$, $\alpha = 0.035$, $q_0 = 1/24$, $\delta = 0.075$, $\sigma = 0.01$.

(i) The negative impact of a financial crisis on employment is exacerbated if there is no recruitment in public and private sectors. This is due to job cuts as a consequence of the financial crisis.

(ii) High values of success rate to recruitment process have low impact on unemployment rate in the context of a financial crisis. In fact, recruitments are balanced with job cuts due to a financial crisis.

Using the above analysis on the influence of some parameters on the unemployment rate, we can infer that, to solve the unemployment problem, apart from insuring the financial stability, the government can focus on the assistance of self-employed people in order to allow them to create more vacancies. It can also improve the process of matching vacancies with jobseekers qualifications.

In the next section, based on the above study, we define some optimal control problems to study the impact that government policies can have on the reduction of the unemployment rate.

4. Optimal control problems

In this section, we propose and compare two strategies that reduce the number of unemployed persons based on optimal control problems. The first strategy is as follows:

(S_{11}) Promote self-employment by accompanying the unemployed individuals in the process of creating businesses.

(S_{12}) Improve the success rate in the search for employment.

The second strategy is as follows:

(S_{21}) Support existing self-employees to create more vacancies.

(S_{22}) Improve the success rate in the search for employment.

4.1. The first strategy of control

The problem is posed as follows.

$$\begin{cases} \min_{\{(u,v)\in L^0[0,t_f],[0,1]^2\}} \int_0^{t_f} \eta_1 U(t) + \eta_2 u^2(t) + \eta_3 v^2(t) dt \\ \text{subject to} \\ \dot{U} = A - k(1+u)UV + \beta_0(1+\gamma\rho)^{\varepsilon_1} E - (\mu+\lambda)U - vU, \\ \dot{E} = k(1+u)UV - (\beta_0(1+\gamma\rho)^{\varepsilon_1} + \alpha + \mu)E + \lambda U + vU, \\ \dot{V} = \big((\lambda+v)q_0 + \sigma_0\big)(1-\rho)^{\varepsilon_1} U + (\alpha+\mu)E - \delta V - k(1+u)UV, \\ \big(U(0), E(0), V(0)\big) = \big(U_0, E_0, V_0\big), \end{cases} \tag{31}$$

where $\eta_i,\ i = 1,\ldots,3$, U_0, E_0, V_0 are positive real numbers. The integral represents the general cost of implementing government policies to reduce the unemployment rate in the period $[0, t_f]$. η_2 is the amount of money invested to implement the government policies intended to improve the rate of successful matchings between jobseekers and vacancies. η_3 is the average amount of money spent per unemployed person in order to support the installation as self-employed individual. η_1 represents the relative importance in the cost of the control of the number of unemployed individuals. The state variables $\big(U(t), E(t), V(t)\big)$ belong to $AC\big([0, t_f]; \mathbb{R}^3\big)$, the set of absolutely continuous functions from $[0, t_f]$ to $\mathbb{R}^3$. The control $\big(u(t), v(t)\big)$ belongs to the set of $L^1\big([0, t_f], [0, 1]^2\big)$ of Lebesgue integrable functions from $[0, t_f]$ to $[0, 1]^2$. For a given $t \geq 0$, $u(t)$ is the efficiency rate of the government policies intended to improve the rate of successful matchings between jobseekers and vacancies. $v(t)$ is the fraction of unemployed individuals having benefited from the government initiatives intended to encourage self-employment.

4.1.1. The existence and uniqueness of optimal solution

Problem (31) can be rewritten as follows:

$$\begin{cases} \min_{\{(u,v)\in L^1[0,t_f],[0,1]^2\}} \int_0^{t_f} \mathcal{L}\big(X(t), \mathcal{U}(t)\big)dt \\ \text{subject to} \\ \dot{X}(t) = f\big(X(t), \mathcal{U}(t)\big),\ \ a.e\ t \in [0, t_f], \\ X_0 = (U_0, E_0, V_0), \end{cases} \tag{32}$$

where

$X(t) = \big(U(t), E(t), V(t)\big) \in\ AC\big([0, t_f]; \mathbb{R}^3\big)$,
$\mathcal{U}(t) = \big(u(t), v(t)\big) \in L^1\big([0, t_f], [0, 1]^2\big)$,
$\mathcal{L}\big(X, \mathcal{U}\big) = \eta_1 U + \eta_2 u^2 + \eta_3 v^2$,

and

$$f(X, \mathcal{U}) = \begin{pmatrix} A - k(1+u)UV + \beta_0(1+\gamma\rho)^{\varepsilon_1}E - (\mu+\lambda)U - vU \\ k(1+u)UV - (\beta_0(1+\gamma\rho)^{\varepsilon_1} + \alpha + \mu)E + \lambda U + vU \\ \big((\lambda+v)q_0 + \sigma_0\big)(1-\rho)^{\varepsilon_1}U + (\alpha+\mu)E - \delta V - k(1+u)UV \end{pmatrix}.$$

It is obvious that $\mathcal{L}$ and f are continuous functions of X and $\mathcal{U}$. We say that a pair $\big(X(t), \mathcal{U}(t)\big)$ in $AC\big([0, t_f]; \mathbb{R}^3\big) \times L^1\big(, [0, 1]^2\big)$ is a feasible pair if it satisfies

$$\begin{cases} \dot{X}(t) = f\big(X(t), \mathcal{U}(t)\big),\ \ a.e\ t \in [0, t_f], \\ X_0 = (U_0, E_0, V_0). \end{cases} \tag{33}$$

We denote by $\mathcal{F}$ the set of all feasible pairs.

We begin by examining sufficient conditions for the existence of a solution to the optimal control problem. We refer to the conditions in Theorem III.4.1 and its corresponding Corollary in [12]. The nontrivial requirements from Fleming and Rishel's theorem are

(H_1) The set of all solutions to the following system

$$\begin{cases} \dot{U} = A - k(1+u)UV + \beta_0(1+\gamma\rho)^{\varepsilon_1}E - (\mu+\lambda)U - vU, \\ \dot{E} = k(1+u)UV - (\beta_0(1+\gamma\rho)^{\varepsilon_1} + \alpha + \mu)E + \lambda U + vU, \\ \dot{V} = \big((\lambda+v)q_0 + \sigma_0\big)(1-\rho)^{\varepsilon_1}U + (\alpha+\mu)E - \delta V - k(1+u)UV, \\ \big(U(0), E(0), V(0)\big) = \big(U_0, E_0, V_0\big), \end{cases} \tag{34}$$

with control (u, v) in $L^1\big(, [0, 1]^2\big)$ is nonempty.

(H_2) The state system can be written as a linear function of the control variables with coefficients depending on time and the state variables.

(H_3) The function $\mathcal{L}$ is convex in $L^1\big(, [0, 1]^2\big)$, and additionally satisfies

$$\mathcal{L}\big(X, \mathcal{U}\big) \geq c_0\|\mathcal{U}\|^{c_1} - c_2,$$

where $c_0, c_1 > 0$ and $\|\cdot\|$ is a norm in $\mathbb{R}^2$.

The following theorem is about the existence of optimal solution of problem (31).

Theorem 4.1. *There exists an optimal control pair (u^*, v^*) and corresponding solution $\big(U^*, E^*, V^*\big)$ to the state initial value problem (34) that minimize*

$$\int_0^{t_f} \big(\eta_1 U(t) + \eta_2 u^2(t) + \eta_3 v^2(t)\big)dt.$$

Proof. By proceeding as in the proof of Theorem 3.1, (H_1) is obtained. Condition (H_2) is verified by using the linear dependence of the state equations on controls u and v. To verify condition (H_3), note that $\mathcal{L}$ is a quadratic function of the controls, $\mathcal{L}$ is clearly convex as function of the controls. To prove the bound on the $\mathcal{L}$ we note that by the definition of $L^1\big(, [0, 1]^2\big)$ we have

$$\mathcal{L}\big(X, \mathcal{U}\big) = \eta_1 U + \eta_2 u^2 + \eta_3 v^2 \geq \frac{1}{2}\min(\eta_2, \eta_3)\|(u, v)\|^2,$$

where $\|(u, v)\| = \sqrt{u^2 + v^2}$ is the Euclidian norm in $\mathbb{R}^2$. Therefore, we obtain the existence of (u^*, v^*) and the corresponding optimal trajectory. For a sufficiently small value of t_f, we obtain the uniqueness of the optimal solution [13]. □

4.1.2. The characterization of the optimal controls

We apply Pontryagin's Maximum Principle [24] and convert the optimization problem (31) to the problem of finding the point-wise minimum relative to u and v of the Hamiltonian. All control minimizers in problem (31) are in [0, 1] and therefore are essentially bounded. By the Pontryagin Maximum Principle [24], we address the question of how to identify the solutions predicted by Theorem 4.1. Moreover, our optimal control problem (31) has only fixed initial conditions, with the state variables being free at the final time, that is, $U(t_f)$, $E(t_f)$ and $V(t_f)$ are free. This implies that abnormal minimizers [7] are not possible in our context and we can fix the cost multiplier associated with the Lagrangian to be equal to one. The Hamiltonian of problem (31) is given by

$$\begin{aligned}\mathcal{H}_1\big(X, \mathcal{U}, P\big) = {} & \eta_1 U + \eta_2 u^2 + \eta_3 v^2 + p_1\big[A - k(1+u)UV + \beta_0(1+\gamma\rho)^{\varepsilon_1}E - (\mu+\lambda)U - vU\big] \\ & + p_2\big[k(1+u)UV - (\beta_0(1+\gamma\rho)^{\varepsilon_1} + \alpha + \mu)E + \lambda U + vU\big] \\ & + p_3\big[\big((\lambda+v)q_0 + \sigma_0\big)(1-\rho)^{\varepsilon_1}U + (\alpha+\mu)E - \delta V - k(1+u)UV\big],\end{aligned}$$

with $P = (p_1, p_2, p_3)$.

The following theorem gives necessary optimality conditions for the optimal control problem.

Theorem 4.2. *If $\big(U^*, E^*, V^*, u^*, v^*\big)$ is a minimizer of problem (31), then there exist multipliers $p_1(\cdot)$, $p_2(\cdot)$, $p_3(\cdot)$ in $AC\big([0, t_f]; \mathbb{R}^3\big)$ such that*

$$\left\{\begin{array}{l} \dot p_1(t) = p_1(t)\big(k(1+u(t))V(t) + \mu + \lambda + v(t)\big) - p_2(t)\big(k(1+u(t))V + \lambda + v(t)\big) \\ -p_3(t)\big[\big((\lambda+v(t))q_0 + \sigma_0\big)\big(1-y\big)^{\epsilon_1} - k(1+v(t))V\big] - \eta_1, \\ \dot p_2(t) = -p_1(t)\beta_0(1+\gamma\rho)^{\epsilon_1} + p_2(t)\big(\beta_0(1+\gamma\rho)^{\epsilon_1} + \alpha + \mu\big) - p_3(t)(\alpha+\mu), \\ \dot p_3(t) = p_1(t)k(1+u(t))U(t) - p_2(t)k(1+u(t))U(t) + p_3(t)\big(\delta + k(1+u(t))U\big), \end{array}\right. \tag{35}$$

for almost all $t \in [0, t_f]$, with transversality conditions

$$p_1(t_f) = p_2(t_f) = p_3(t_f) = 0. \tag{36}$$

The optimal control pair for all $t \in [0, t_f]$ is given by

$$u^*(t) = \min\left\{\max\left\{0, \frac{\big(p_1(t) - p_2(t) + p_3(t)\big)kU^*(t)V^*(t)}{2\eta_2}\right\}, 1\right\}, \tag{37}$$

and

$$v^*(t) = \min\left\{\max\left\{0, \frac{\big(p_1(t) - p_2(t) - q_0 p_3(t)(1-\rho)^{\epsilon_1}\big)U^*(t)}{2\eta_3}\right\}, 1\right\}. \tag{38}$$

Proof. The result follows from a direct application of a version of Pontryagin's Maximum Principle for bounded controls [24].

The optimality conditions dictate that

$$\frac{\partial \mathcal{H}_1}{\partial u} = \frac{\partial \mathcal{H}_1}{\partial v} = 0$$

for the optimal pair (u^*, v^*) in the interior of the interval $[0, 1]$, and this condition is simplified in Eqs. (37) and (38) considering the bounds 0 and 1. □

4.2. The second strategy of control

The problem is posed as follows:

$$\begin{cases} \min_{\{(u,v)\in L^1[0,t_f],[0,1]\}} \int_0^{t_f} \big(\eta_1 U(t) + \eta_2 u^2(t) + \eta_4 v^2(t)\big)dt \\ \text{subject to} \\ \dot{U} = A - k(1+u)UV + \beta_0(1+\gamma\rho)^{\varepsilon_1} E - (\mu+\lambda)U, \\ \dot{E} = k(1+u)UV - (\beta_0(1+\gamma\rho)^{\varepsilon_1} + \alpha + \mu)E + \lambda U, \\ \dot{V} = \big(\lambda(q_0+v)+\sigma_0\big)(1-\rho)^{\varepsilon_1} U + (\alpha+\mu)E - \delta V - k(1+u)UV, \\ \big(U(0), E(0), V(0)\big) = \big(U_0, E_0, V_0\big), \end{cases} \tag{39}$$

where η_i, $i = 1, \ldots, 3$, U_0, E_0, V_0 are positive real numbers. The integral represents the general cost of implementation of government policies to reduce the unemployment rate in the period $[0, t_f]$. The constants η_1 and η_2 are defined as in problem (31). η_4 is the average amount of money spent as the assistance to a self-employed individual for the creation of one vacancy. The state variables $\big(U(t), E(t), V(t)\big)$ belong to $AC\big([0, t_f]; \mathbb{R}^3\big)$, the set of absolutely continuous functions from $[0, t_f]$ to $\mathbb{R}^3$. The control $\big(u(t), v(t)\big)$ belongs to the set of $L^1\big([0, t_f], [0, 1]^2\big)$ of Lebesgue integrable functions from $[0, t_f]$ to $[0, 1]^2$. For a given $t \geq 0$, $u(t)$ is defined as for problem (31), and $v(t)$ is the average number of additional vacancies created by each self-employed person after the assistance of the government.

4.2.1. Existence and uniqueness of the optimal solution

Let us consider the following initial state value problem associated to (39):

$$\begin{cases} \dot{U} = A - k(1+u)UV + \beta_0(1+\gamma\rho)^{\varepsilon_1} E - (\mu+\lambda)U, \\ \dot{E} = k(1+u)UV - (\beta_0(1+\gamma\rho)^{\varepsilon_1} + \alpha + \mu)E + \lambda U, \\ \dot{V} = \big(\lambda(q_0+v)+\sigma_0\big)(1-\rho)^{\varepsilon_1} U + (\alpha+\mu)E - \delta V - k(1+u)UV, \\ \big(U(0), E(0), V(0)\big) = \big(U_0, E_0, V_0\big). \end{cases} \tag{40}$$

The following theorem is about the existence of an optimal control pair for problem (39).

Theorem 4.3. *There exists an optimal control pair (u^*, v^*) for problem (39), and the corresponding solution $\big(U^*, E^*, V^*\big)$ to the state initial value problem (34) that minimizes*

$$\int_0^{t_f} \big(\eta_1 U(t) + \eta_2 u^2(t) + \eta_4 v^2(t)\big)dt.$$

Proof. Using the same development as in the proof of Theorem 4.1, one can verify that the hypotheses (H_1), (H_2) and (H_3) are satisfied. Therefore, we obtain the existence of (u^*, v^*) and the corresponding optimal trajectory. For a sufficiently small value of t_f, we obtain the uniqueness of the optimal solution [13]. □

4.2.2. The characterization of the optimal controls

The Hamiltonian of problem (39) is given by

$$\begin{aligned} \mathcal{H}_2\big(X, \mathcal{U}, P\big) = {} & \eta_1 U + \eta_2 u^2 + \eta_4 v^2 + p_1\big[A - k(1+u)UV + \beta_0(1+\gamma\rho)^{\varepsilon_1} E - (\mu+\lambda)U\big] \\ & + p_2\big[k(1+u)UV - (\beta_0(1+\gamma\rho)^{\varepsilon_1} + \alpha + \mu)E + \lambda U\big] \\ & + p_3\big[\big(\lambda(q_0+v)+\sigma_0\big)(1-\rho)^{\varepsilon_1} U + (\alpha+\mu)E - \delta V - k(1+u)UV\big], \end{aligned}$$

with $P = (p_1, p_2, p_3)$.

The following theorem gives necessary optimality conditions for the optimal control problem.

Table 2
Values of the parameters used to perform simulations of the unemployment problem.

Parameter	Value	Source
A	3000	[5]
k	1.08×10^{-5}	[5]
β_0	0.01	[5]
λ	0.15	Assumed
μ	0.032	[5]
α	0.035	[5]
q_0	1/24	Assumed
σ	0.01	Assumed
ϵ_1	1	Assumed
δ	0.075	[5]
η_1	1	Assumed
η_2	10^{-2}	Assumed
η_3	2×10^{-5}	Assumed
η_4	2×10^{-5}	Assumed
Times unit	1 month	Assumed

Theorem 4.4. *If $\left(U^*, E^*, V^*, u^*, v^*\right)$ is a minimizer of problem (39), then there exist multipliers $p_1(\cdot)$, $p_2(\cdot)$ and $p_3(\cdot)$ in $AC\left([0, t_f]; \mathbb{R}^3\right)$ such that*

$$\begin{cases} \dot{p}_1(t) = p_1(t)\big(k(1+u(t))V(t) + \mu + \lambda\big) - p_2(t)\big(k(1+u(t))V + \lambda\big) \\ -p_3(t)\big[\big(\lambda(q_0 + v(t)) + \sigma_0\big)\big(1-\rho\big)^{\epsilon_1} - k(1+u(t))V\big] - \eta_1, \\ \dot{p}_2(t) = -p_1(t)\beta_0(1+\gamma\rho)^{\epsilon_1} + p_2(t)\big(\beta_0(1+\gamma\rho)^{\epsilon_1} + \alpha + \mu\big) - p_3(t)(\alpha + \mu), \\ \dot{p}_3(t) = p_1(t)k(1+u(t))U(t) - p_2(t)k(1+u(t))U(t) + p_3(t)\big(\delta + k(1+u(t))U\big), \end{cases} \tag{41}$$

for almost all $t \in [0, t_f]$, with transversality conditions

$$p_1(t_f) = p_2(t_f) = p_3(t_f) = 0. \tag{42}$$

The optimal control pair for all $t \in [0, t_f]$ is given by

$$u^*(t) = \min\left\{\max\left\{0, \frac{\big(p_2(t) - p_3(t) + p_4(t)\big)kU^*(t)V^*(t)}{2\eta_2}\right\}, 1\right\}, \tag{43}$$

and

$$v^*(t) = \min\left\{\max\left\{0, \frac{-\lambda(1-\rho)U^*(t)p_4(t)}{2\eta_4}\right\}, 1\right\}. \tag{44}$$

Proof. The proof is similar to that of Theorem 4.1. □

5. Numerical simulations

In this section, using ODE 45 routine and the shooting method implemented on MATLAB software, we now carry out numerical simulations to illustrate the theoretical results obtained. Table 2 gives the values of the parameters used to perform simulations of the unemployment problem.

5.1. The impact of financial crisis on employment

Figs. 4 and 5 illustrate the impact of the health of the financial system on employment. These pictures confirm the stylized fact underlining the negative impact of a financial crisis on employment as underlined in [4,8–11,26].

Fig. 6 shows the long-term behavior of system (5). It is an illustration of the global asymptotic stability of the equilibrium.

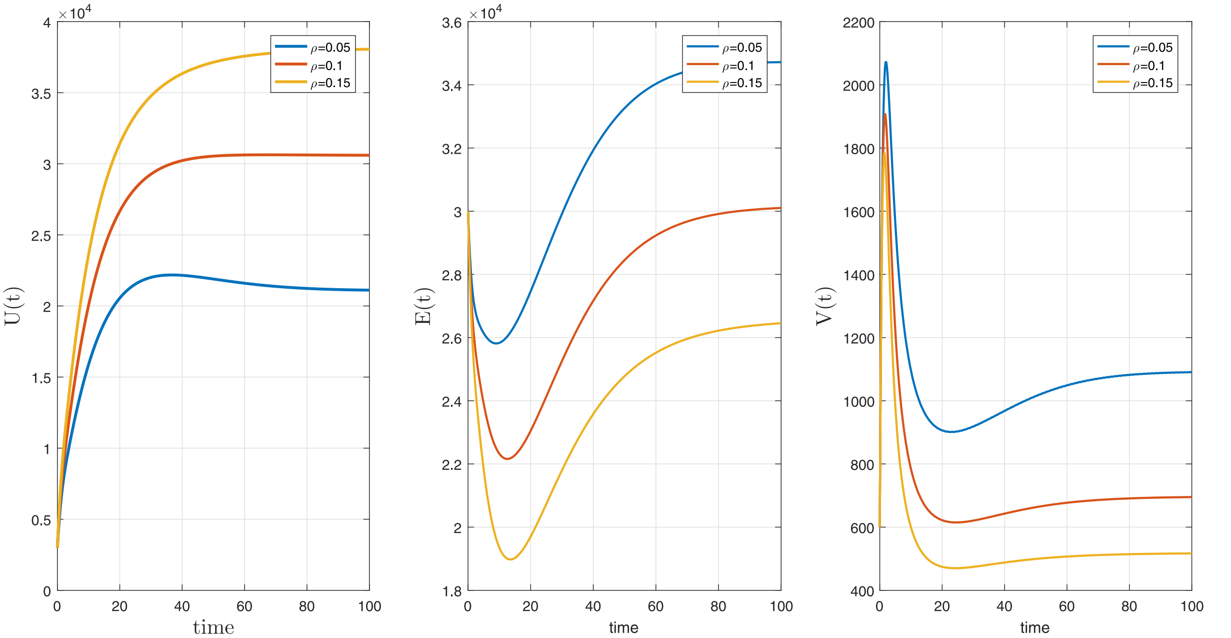

Fig. 4. The impact of a financial crisis on unemployment. Low level of a financial crisis is positive for employment and vacancies.

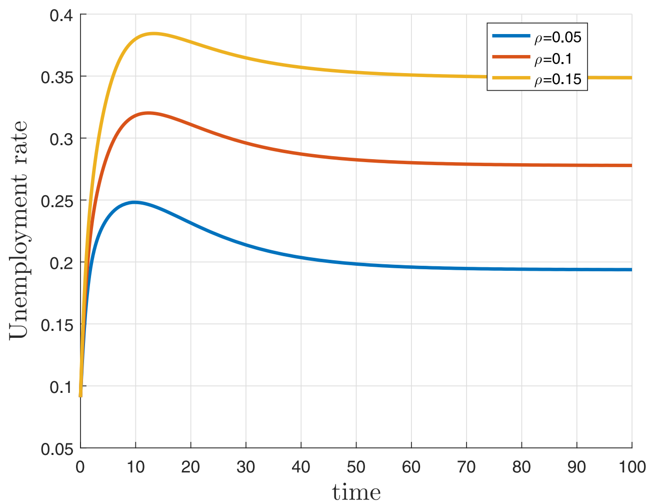

Fig. 5. The impact of financial crisis on the unemployment rate. Low level of financial crisis leads to low level of the unemployment rate.

5.2. The optimal control problems

For a sufficiently small time, we proved in Section 4, Theorem 4.1 and Theorem 4.3, the existence and uniqueness of optimal solutions for problems (31) and (39), respectively. We also gave the characterizations of optimal controls in Theorem 4.2 and Theorem 4.4. In what follows, we use the shooting method implemented on MATLAB software to illustrate these results.

Fig. 7 gives the profiles of optimal controls for the first strategy. On Fig. 8, we illustrate the impact of the first strategy of control on the system. The controls reduce significantly the number of unemployed individuals and increase the number of employed individuals. The controls also have impact on the number of vacancies. The controlled path of the number of vacancies firstly decreases and further increases above the uncontrolled paths. This

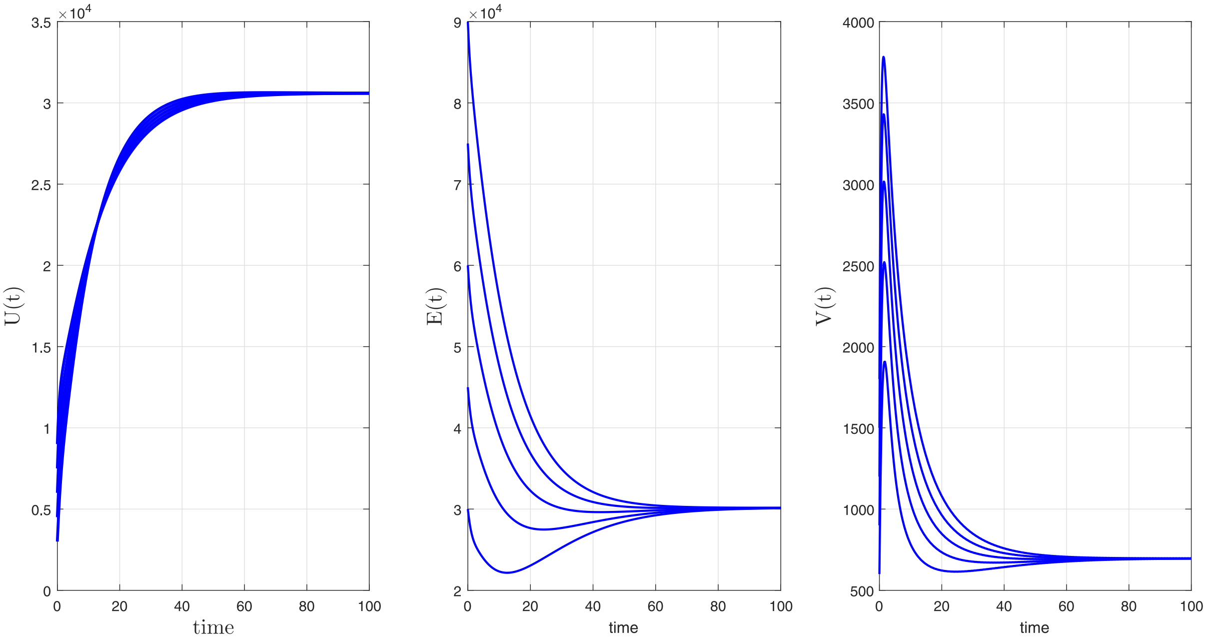

Fig. 6. The long term behavior of the system: global asymptotic stability of the equilibrium.

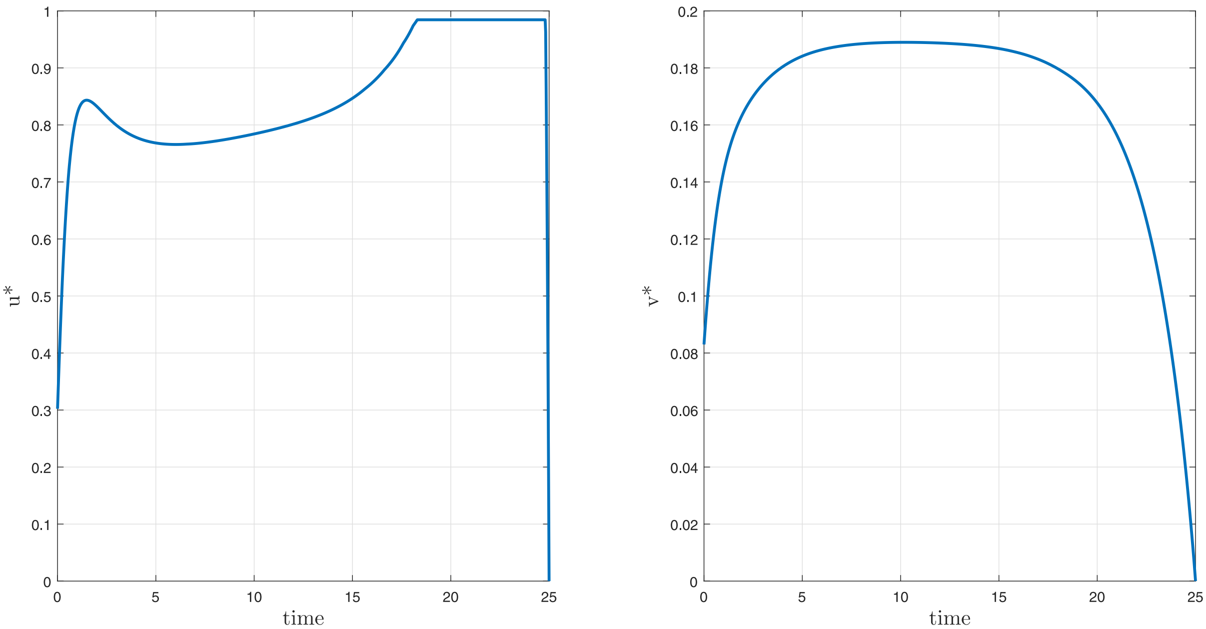

Fig. 7. Optimal control profiles for the first strategy with $\rho = 0.1$ and $t_f = 25$ months.

can be interpreted as follows: (i) the reduction of the number of vacancies is due to the impact of the implementation of government strategy to enhance the matching rate between jobseekers and vacancies; (ii) the increase of the controlled number of vacancies above the uncontrolled one is the result of the additional jobs created by self-employed persons who benefited from the governmental policies. Fig. 9 illustrates the impact of the first strategy of control on the unemployment rate. For our set of parameters, one can note the difference of about 10% between the controlled and uncontrolled unemployment rates.

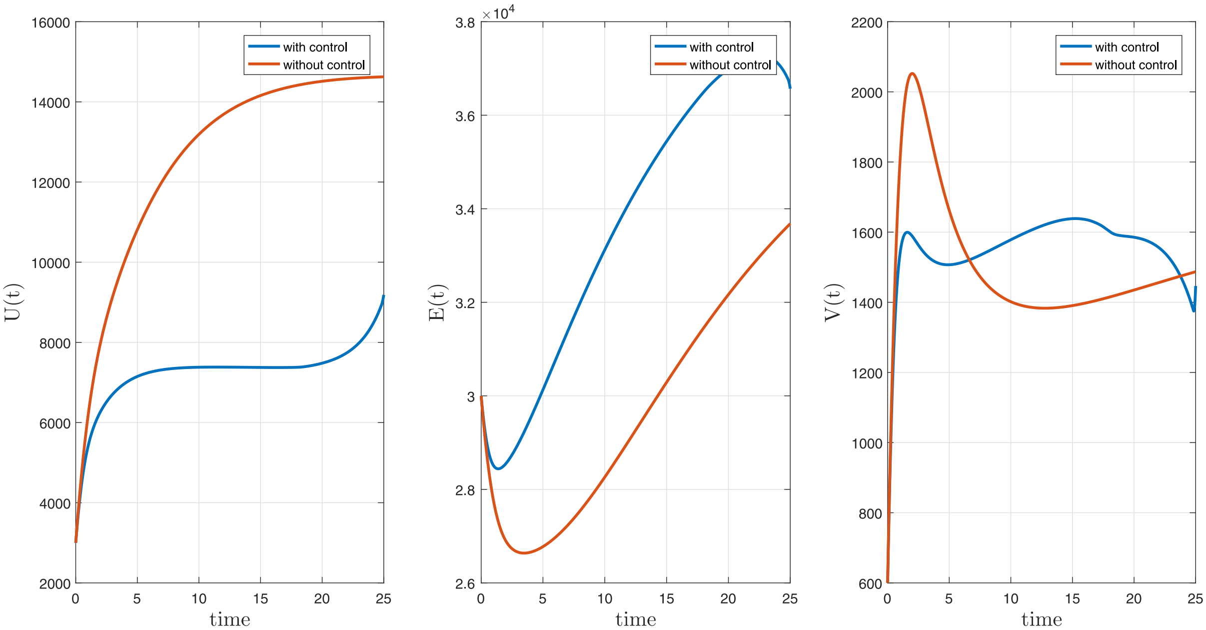

Fig. 8. Optimal behavior of the unemployment problem for the first strategy of control with $\rho = 0.1$ and $t_f = 25$ months.

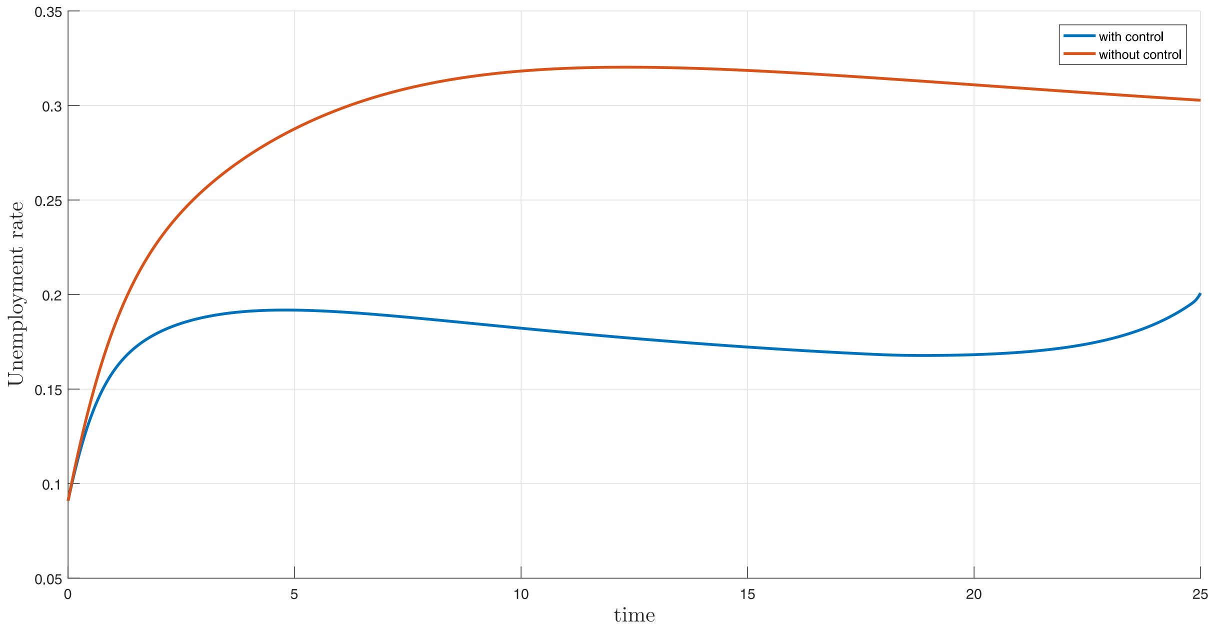

Fig. 9. Impact of the first strategy of control on the unemployment rate. $\rho = 0.1$ and $t_f = 25$ months.

Fig. 10 gives the optimal control profiles for the second strategy. Fig. 11 illustrates the impact of the second strategy of control on the system. We note the limited impact on the number of unemployed individuals, the number of employed individuals and the number of vacancies. The observations of Fig. 11 are confirmed On Fig. 12 with the small difference between the controlled and uncontrolled unemployment rates. Fig. 13 gives a comparison between

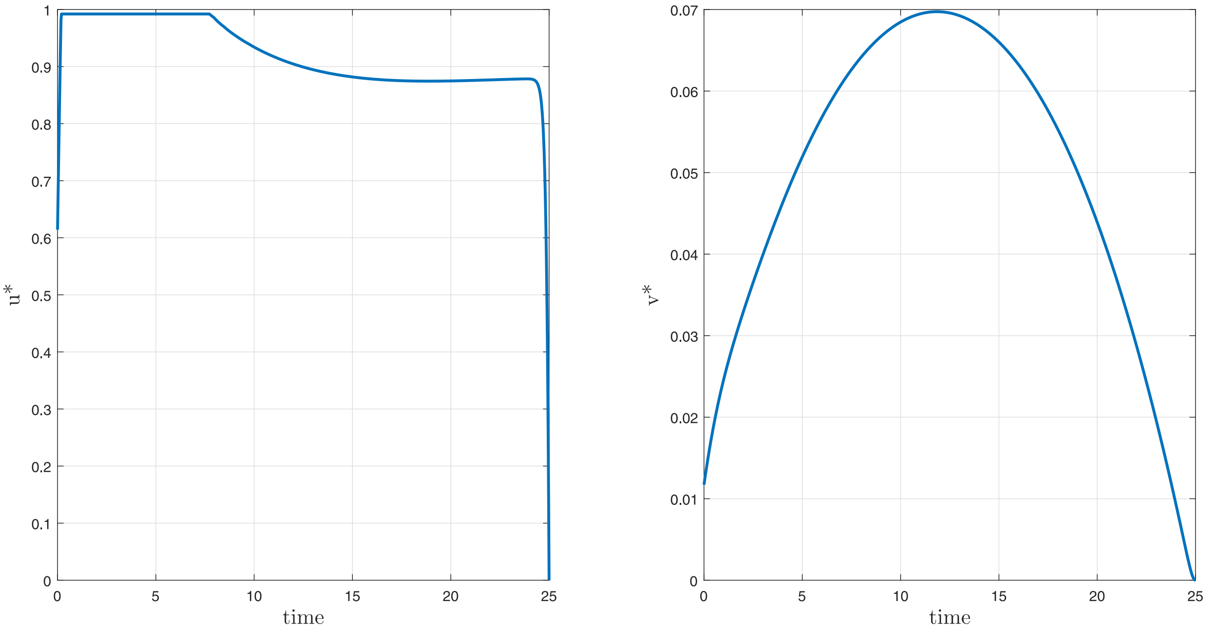

Fig. 10. Optimal control profiles for the second strategy with $\rho = 0.1$ and $t_f = 25$ months.

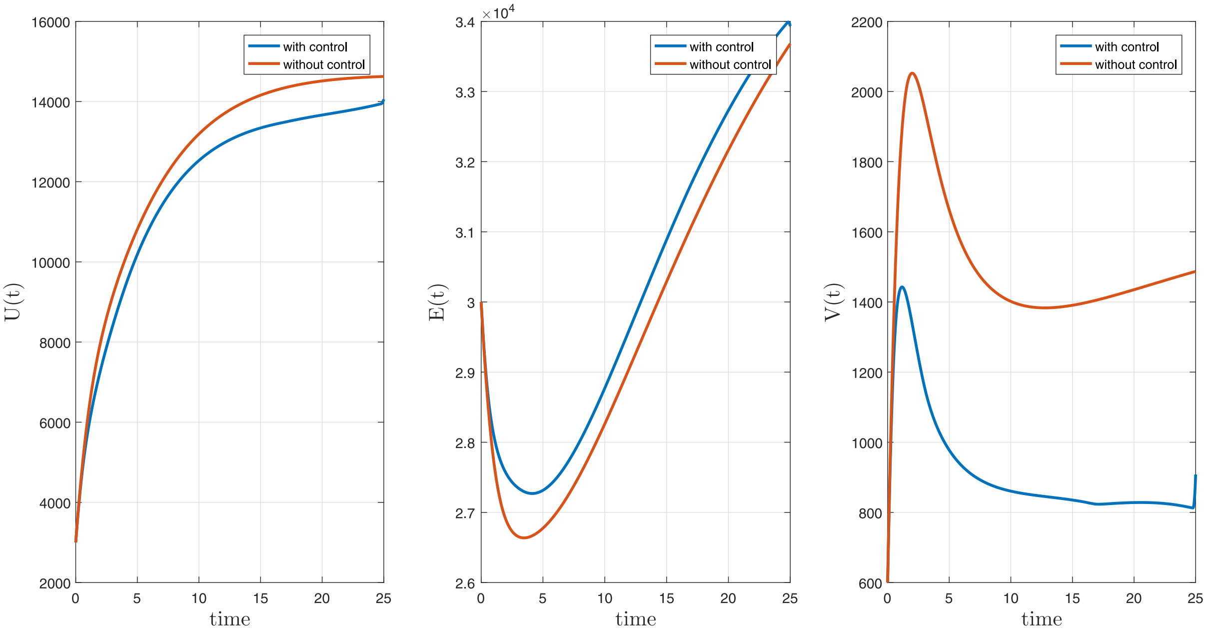

Fig. 11. Optimal behavior of the unemployment problem for the second strategy of control with $\rho = 0.1$ and $t_f = 25$ months.

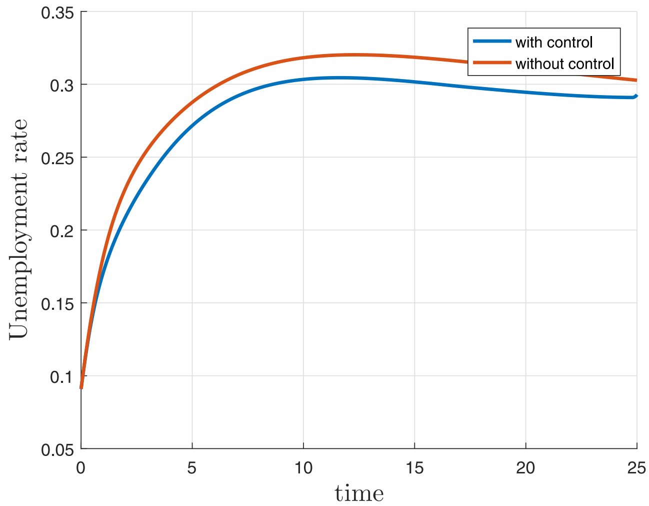

Fig. 12. Impact of the second strategy of control on the unemployment rate. $\rho = 0.1$ and $t_f = 25$ months.

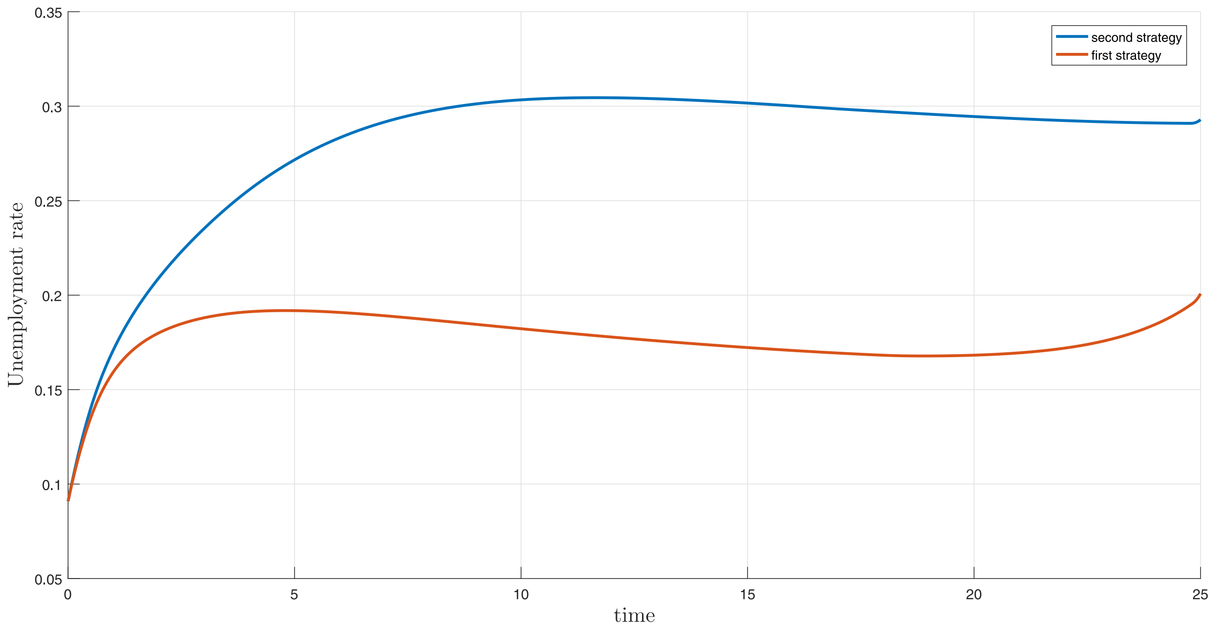

Fig. 13. Comparison of the impact of the first and second strategies of control on the unemployment rate. $\rho = 0.1$ and $t_f = 25$ months.

the first and second strategies of control. It is immediate that, for our set of parameters, the first strategy of control leads to best results than the second control strategy. This suggests that in order to reduce unemployment, it is more efficient for a government to assist unemployed individuals in building their own business which will allow them to further create new vacancies than to assist self-employed individuals to create new vacancies.

6. Conclusion

In this paper, we built a nonlinear system of ordinary differential equations to study the unemployment problem in a context of financial crisis. After the mathematical analysis of the model, our main findings can be summarized as follows:

- We obtained, under some conditions, the global asymptotic stability of the stationary solution to the system of ODEs. This result gives the long term behavior of the unemployment problem.
- We studied and illustrated the join dependence of the long-term unemployment rate on the financial health of the banking system and the matchings rate between the jobseekers and vacancies. This allowed us to obtain the following results:
  - **(i)** The negative impact of financial crisis on employment is exacerbated if there is no recruitment in public and private sectors. This is due to job cuts as a consequence of financial crisis.
  - **(ii)** High values of success rate to recruitment process have low impact on unemployment rate in the context of financial crisis. In fact, recruitment are balanced with job cuts due to a financial crisis.
- We studied and illustrated the joint dependence of the long term unemployment rate on the state of financial health of the banking system and the rate at which unemployed people create their own business. This allowed us to have the following results:
  - **(i)** The negative impact of the financial crisis on employment is exacerbated if there is no self-employment.
  - **(ii)** High self-employment rate has positive impact on employment even in the financial crisis context.
  - **(iii)** Low level of self-employment is unfavorable to employment even in the absence of a financial crisis.
- In order to reduce the number of unemployed individuals, we defined two strategies of control. The comparison of these strategies suggests that, in order to reduce unemployment, it is more efficient for a government to assist unemployed persons in building their own business which will allow them to further create new vacancies than to assist self-employed individuals to create new vacancies.

In our approach, we did not take into account the specific skills of unemployed individuals; this constitutes our next challenge.

CRediT authorship contribution statement

Eric Rostand Njike-Tchaptchet: Writing – original draft, Literature review, Simulations. **Calvin Tadmon:** Conceptualization, Supervision, Perusal of the whole work.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Our manuscript has no associated data.

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