Part I
Laplace Transforms

1 definitions

$$\mathcal{L}\left[f(t)\right] = F(s)$$
$$\mathcal{L}\left[f(t)\right] = \int_0^\infty f(t)e^{-st}dt$$

2 derivation

laplace trf of 1

$$\mathcal{L}\left[1\right] = \int_0^\infty e^{-st}dt$$

$$\mathcal{L}\left[1\right] = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$\mathcal{L}\left[1\right] = 0 - \frac{e^{-s*0}}{-s}$$

$$\mathcal{L}\left[1\right] = \frac{1}{s}$$

laplace trf of exponentials

positive exponentials

$$\mathcal{L}\left[e^{at}\right] = \int_0^\infty e^{at} e^{-st} dt$$

$$\mathcal{L}\left[e^{at}\right] = \int_0^\infty e^{(a-s)t} dt$$

$$\mathcal{L}\left[e^{at}\right] = \left[\frac{e^{(a-s)t}}{(a-s)}\right]_0^\infty$$

$$\mathcal{L}\left[e^{at}\right] = \frac{e^{(a-s)\infty}}{(a-s)} - \frac{e^{(a-s)*0}}{(a-s)}$$

$$\mathcal{L}\left[e^{at}\right] = \frac{e^{(a-s)\infty}}{(a-s)} + \frac{1}{(s-a)}$$

we have this problem:

$$\frac{e^{(a-s)\infty}}{(a-s)}$$

$$a = s \to \frac{1}{0} = \infty$$

$$a - s > 0 \to \mathcal{L}\left[e^{at}\right] = \infty$$

$$a - s < 0; s > a; \to \mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

negative exponentials note that a>0 and is real.

$$\mathcal{L}\left[e^{-at}\right] = \int_0^\infty e^{-at}e^{-st}dt$$

$$\mathcal{L}\left[e^{-at}\right] = \int_0^\infty e^{-(a+s)t}dt$$

$$\mathcal{L}\left[e^{-at}\right] = \left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_0^\infty$$

$$a + s > 0$$

$$\mathcal{L}\left[e^{-at}\right] = \frac{e^{-(a+s)\infty}}{-(a+s)} - \frac{e^{-(a+s)*0}}{-(a+s)}$$

$$\mathcal{L}\left[e^{-at}\right] = 0 - \frac{e^{-(a+s)*0}}{-(a+s)}$$

$$\mathcal{L}\left[e^{-at}\right] = \frac{1}{a+s}$$

imaginary exponentials (sines and cosines)

`https://en.wikipedia.org/wiki/Euler%27s_formula`

$$\cos(\omega t) = \frac{\exp(i\omega t) + \exp(-i\omega t)}{2}$$

$$\sin(\omega t) = \frac{\exp(i\omega t) - \exp(-i\omega t)}{2i}$$

$$\mathcal{L}\left[e^{-i\omega t}\right] = \int_0^\infty e^{-i\omega t} e^{-st} dt$$

$$\mathcal{L}\left[e^{-i\omega t}\right] = \int_0^\infty e^{-(i\omega+s)t} dt$$

$$\mathcal{L}\left[e^{-i\omega t}\right] = \left[\frac{e^{-(i\omega+s)t}}{-(i\omega+s)}\right]_0^\infty$$

$$(i\omega + s) > 0$$

$$\mathcal{L}\left[e^{-i\omega t}\right] = \frac{1}{(i\omega + s)}$$

$$\mathcal{L}\left[e^{i\omega t}\right] = \int_0^\infty e^{i\omega t} e^{-st} dt$$

$$\mathcal{L}\left[e^{i\omega t}\right] = \left[\frac{e^{(i\omega-s)t}}{(i\omega - s)}\right]_0^\infty$$

Assume

$$i\omega > s$$

$$\mathcal{L}\left[e^{i\omega t}\right] = 0 + \frac{1}{s - i\omega}$$

$$\mathcal{L}\left[e^{i\omega t}\right] = \frac{1}{s - i\omega}$$

now we can do sines and cosines:

$$\cos(\omega t) = \frac{\exp(i\omega t) + \exp(-i\omega t)}{2}$$

$$\mathcal{L}(\cos(\omega t)) = \frac{1}{2}\frac{1}{s - i\omega} + \frac{1}{2}\frac{1}{s + i\omega}$$

$$\mathcal{L}(\cos(\omega t)) = \frac{1}{2}\left[\frac{s + i\omega}{s^2 + \omega^2} + \frac{s - i\omega}{s^2 + \omega^2}\right]$$

$$\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2 + \omega^2}$$

Let's do sines:

$$\sin(\omega t) = \frac{\exp(i\omega t) - \exp(-i\omega t)}{2i}$$

$$\mathcal{L}(\sin(\omega t)) = \frac{1}{2i}\frac{1}{s - i\omega} - \frac{1}{2i}\frac{1}{s + i\omega}$$

$$\mathcal{L}(\sin(\omega t)) = \frac{1}{2i}\left[\frac{s + i\omega}{s^2 + \omega^2} - \frac{s - i\omega}{s^2 + \omega^2}\right]$$

$$\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2 + \omega^2}$$

complex exponentials

$$a < 0; \omega < 0$$

$$\mathcal{L}\left[e^{-(a+i\omega)t}\right] = \int_0^\infty e^{-(a+i\omega)t}e^{-st}dt$$

$$\mathcal{L}\left[e^{-(a+i\omega)t}\right] = \left[\frac{e^{-(a+i\omega+s)t}}{-(a+i\omega+s)}\right]_0^\infty$$

$$a + i\omega + s > 0$$

$$\mathcal{L}\left[e^{-(a+i\omega)t}\right] = \frac{1}{(a+i\omega+s)}$$

$$a < 0; \omega > 0$$

$$\mathcal{L}\left[e^{-(a-i\omega)t}\right] = \int_0^\infty e^{-(a-i\omega)t}e^{-st}dt$$

$$a - i\omega + s > 0$$

$$\mathcal{L}\left[e^{-(a-i\omega)t}\right] = \left[\frac{e^{-(a-i\omega+s)t}}{-(a-i\omega+s)}\right]_0^\infty$$

$$\mathcal{L}\left[e^{-(a-i\omega)t}\right] = \frac{1}{(a-i\omega+s)}$$

So we can apply this to sines and cosines with exponential product.

$$\mathcal{L}\left[e^{-at}sin(\omega t)\right] = \mathcal{L}\left[e^{-at}\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right] = \int_0^\infty e^{-at}e^{-st}\frac{e^{i\omega t} - e^{-i\omega t}}{2i}dt$$

$$\mathcal{L}\left[e^{-at}\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right]$$

$$= \frac{1}{2i}\mathcal{L}(e^{-at+i\omega t}) - \frac{1}{2i}\mathcal{L}(e^{-at-i\omega t})$$

$$= \frac{1}{2i}\frac{1}{(a - i\omega + s)} - \frac{1}{2i}\frac{1}{(a + i\omega + s)}$$

Note:

$$(a - i\omega + s)(a + i\omega + s) = (s^2 + a^2 + \omega^2 + s(a - i\omega) + s(a + i\omega))$$

$$= s^2 + a^2 + \omega^2 + 2as = (s + a)^2 + \omega^2$$

Subs back:

$$= \frac{1}{2i}\frac{a + i\omega + s}{(s + a)^2 + \omega^2} - \frac{1}{2i}\frac{a - i\omega + s}{(s + a)^2 + \omega^2}$$

$$= \frac{1}{2i}\frac{2i\omega}{(s + a)^2 + \omega^2}$$

$$\mathcal{L}\left[e^{-at} sin(\omega t)\right] = \frac{\omega}{(s+a)^2 + \omega^2}$$

Part II
Appendix

3 Font Sizes

https://www.overleaf.com/learn/latex/Questions/How_do_I_adj