Part I
Laplace Transforms

1 definitions

$$\mathcal{L}\left[f(t)\right] = F(s)$$

$$\mathcal{L}\left[f(t)\right] = \int_0^\infty f(t)e^{-st}dt$$

2 derivation

laplace trf of 1

$$\mathcal{L}\left[1\right] = \int_0^\infty e^{-st}dt$$

$$\mathcal{L}\left[1\right] = \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$\mathcal{L}\left[1\right] = 0 - \frac{e^{-s*0}}{-s}$$

$$\mathcal{L}\left[1\right] = \frac{1}{s}$$

laplace trf of exponentials

positive exponentials

$$\mathcal{L}\left[e^{at}\right] = \int_0^\infty e^{at}e^{-st}dt$$

$$\mathcal{L}\left[e^{at}\right] = \int_0^\infty e^{(a-s)t}dt$$

$$\mathcal{L}\left[e^{at}\right] = \left[\frac{e^{(a-s)t}}{(a-s)}\right]_0^\infty$$

$$\mathcal{L}\left[e^{at}\right] = \frac{e^{(a-s)\infty}}{(a-s)} - \frac{e^{(a-s)*0}}{(a-s)}$$

$$\mathcal{L}\left[e^{at}\right] = \frac{e^{(a-s)\infty}}{(a-s)} + \frac{1}{(s-a)}$$

we have this problem:

$$\frac{e^{(a-s)\infty}}{(a-s)}$$

$$a = s \rightarrow \frac{1}{0} = \infty$$

$$a - s > 0 \rightarrow \mathcal{L}\left[e^{at}\right] = \infty$$

$$a - s < 0; s > a; \rightarrow \mathcal{L}\left[e^{at}\right] = \frac{1}{s-a}$$

negative exponentials note that a>0 and is real.

$$\mathcal{L}\left[e^{-at}\right] = \int_0^\infty e^{-at}e^{-st}dt$$

$$\mathcal{L}\left[e^{-at}\right] = \int_0^\infty e^{-(a+s)t}dt$$

$$\mathcal{L}\left[e^{-at}\right] = \left[\frac{e^{-(a+s)t}}{-(a+s)}\right]_0^\infty$$

$$a+s>0$$

$$\mathcal{L}\left[e^{-at}\right] = \frac{e^{-(a+s)\infty}}{-(a+s)} - \frac{e^{-(a+s)*0}}{-(a+s)}$$

$$\mathcal{L}\left[e^{-at}\right] = 0 - \frac{e^{-(a+s)*0}}{-(a+s)}$$

$$\mathcal{L}\left[e^{-at}\right] = \frac{1}{a+s}$$

imaginary exponentials (sines and cosines)

`https://en.wikipedia.org/wiki/Euler%27s_formula`

$$\cos(\omega t) = \frac{\exp(i\omega t) + \exp(-i\omega t)}{2}$$

$$\sin(\omega t) = \frac{\exp(i\omega t) - \exp(-i\omega t)}{2i}$$

$$\mathcal{L}\left[e^{-i\omega t}\right] = \int_0^\infty e^{-i\omega t} e^{-st} dt$$

$$\mathcal{L}\left[e^{-i\omega t}\right] = \int_0^\infty e^{-(i\omega+s)t} dt$$

$$\mathcal{L}\left[e^{-i\omega t}\right] = \left[\frac{e^{-(i\omega+s)t}}{-(i\omega+s)}\right]_0^\infty$$

$$(i\omega + s) > 0$$

$$\mathcal{L}\left[e^{-i\omega t}\right] = \frac{1}{(i\omega + s)}$$

$$\mathcal{L}\left[e^{i\omega t}\right] = \int_0^\infty e^{i\omega t} e^{-st} dt$$

$$\mathcal{L}\left[e^{i\omega t}\right] = \left[\frac{e^{(i\omega-s)t}}{(i\omega - s)}\right]_0^\infty$$

Assume

$$i\omega > s$$

$$\mathcal{L}\left[e^{i\omega t}\right] = 0 + \frac{1}{s - i\omega}$$

$$\mathcal{L}\left[e^{i\omega t}\right] = \frac{1}{s - i\omega}$$

now we can do sines and cosines:

$$\cos(\omega t) = \frac{\exp(i\omega t) + \exp(-i\omega t)}{2}$$

$$\mathcal{L}(\cos(\omega t)) = \frac{1}{2}\frac{1}{s - i\omega} + \frac{1}{2}\frac{1}{s + i\omega}$$

$$\mathcal{L}(\cos(\omega t)) = \frac{1}{2}\left[\frac{s + i\omega}{s^2 + \omega^2} + \frac{s - i\omega}{s^2 + \omega^2}\right]$$

$$\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2 + \omega^2}$$

Let's do sines:

$$\sin(\omega t) = \frac{\exp(i\omega t) - \exp(-i\omega t)}{2i}$$

$$\mathcal{L}(\sin(\omega t)) = \frac{1}{2i}\frac{1}{s - i\omega} - \frac{1}{2i}\frac{1}{s + i\omega}$$

$$\mathcal{L}(\sin(\omega t)) = \frac{1}{2i}\left[\frac{s + i\omega}{s^2 + \omega^2} - \frac{s - i\omega}{s^2 + \omega^2}\right]$$

$$\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2 + \omega^2}$$

complex exponentials

$$a < 0; \omega < 0$$

$$\mathcal{L}\left[e^{-(a+i\omega)t}\right] = \int_0^\infty e^{-(a+i\omega)t} e^{-st} dt$$

$$\mathcal{L}\left[e^{-(a+i\omega)t}\right] = \left[\frac{e^{-(a+i\omega+s)t}}{-(a+i\omega+s)}\right]_0^\infty$$

$$a + i\omega + s > 0$$

$$\mathcal{L}\left[e^{-(a+i\omega)t}\right] = \frac{1}{(a+i\omega+s)}$$

$$a < 0; \omega > 0$$

$$\mathcal{L}\left[e^{-(a-i\omega)t}\right] = \int_0^\infty e^{-(a-i\omega)t} e^{-st} dt$$

$$a - i\omega + s > 0$$

$$\mathcal{L}\left[e^{-(a-i\omega)t}\right] = \left[\frac{e^{-(a-i\omega+s)t}}{-(a-i\omega+s)}\right]_0^\infty$$

$$\mathcal{L}\left[e^{-(a-i\omega)t}\right] = \frac{1}{(a-i\omega+s)}$$

So we can apply this to sines and cosines with exponential product.

$$\mathcal{L}\left[e^{-at}sin(\omega t)\right] = \mathcal{L}\left[e^{-at}\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right] = \int_0^\infty e^{-at}e^{-st}\frac{e^{i\omega t} - e^{-i\omega t}}{2i}dt$$

$$\mathcal{L}\left[e^{-at}\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right]$$

$$= \frac{1}{2i}\mathcal{L}(e^{-at+i\omega t}) - \frac{1}{2i}\mathcal{L}(e^{-at-i\omega t})$$

$$= \frac{1}{2i}\frac{1}{(a - i\omega + s)} - \frac{1}{2i}\frac{1}{(a + i\omega + s)}$$

Note:

$$(a - i\omega + s)(a + i\omega + s) = (s^2 + a^2 + \omega^2 + s(a - i\omega) + s(a + i\omega))$$

$$= s^2 + a^2 + \omega^2 + 2as = (s + a)^2 + \omega^2$$

Subs back:

$$= \frac{1}{2i}\frac{a + i\omega + s}{(s + a)^2 + \omega^2} - \frac{1}{2i}\frac{a - i\omega + s}{(s + a)^2 + \omega^2}$$

$$= \frac{1}{2i}\frac{2i\omega}{(s + a)^2 + \omega^2}$$

$$\mathcal{L}\left[e^{-at}sin(\omega t)\right] = \frac{\omega}{(s + a)^2 + \omega^2}$$

Derivatives and integrals

$$\mathcal{L}[\frac{df(t)}{dt}] = \int_0^\infty \frac{df(t)}{dt} \exp(-st)dt$$

Integrate by parts:

$$\mathcal{L}[\frac{df(t)}{dt}] = [f(t)\exp(-st)]_0^\infty - \int_0^\infty f(t)(-s)\exp(-st)dt$$

$$\mathcal{L}[\frac{df(t)}{dt}] = [f(t)\exp(-st)]_0^\infty + s\int_0^\infty f(t)\exp(-st)dt$$

$$\mathcal{L}[\frac{df(t)}{dt}] = [f(t)\exp(-st)]_0^\infty + s\mathcal{L}(f(t))$$

$$\mathcal{L}[\frac{df(t)}{dt}] = [f(t \to \infty)\exp(-s\infty) - f(t=0) * \exp(-s * 0)] + s\mathcal{L}(f(t))$$

$$f(t) \neq \exp(st)$$

and $f(t)$ doesn't go to infinity faster than $\exp(-st)$, s>0

$$\mathcal{L}[\frac{df(t)}{dt}] = [0 - f(t=0)] + s\mathcal{L}(f(t))$$

$$\mathcal{L}[\frac{df(t)}{dt}] = s\mathcal{L}(f(t)) - f(t=0)$$

We can use this transform for integrals as long as u have the BC

$$\mathcal{L}[f(t)] = s\mathcal{L}(\int f(t)dt) - \int f(t)dt|_{t=0})$$

$$\mathcal{L}(\int f(t)dt) = \frac{1}{s}\left[[\mathcal{L}[f(t)] + \int f(t)dt|_{t=0}]\right]$$

We can see in laplace domain, integrating means multiply by $\frac{1}{s}$ and differentiating is multiplying by s. Of course we have BCs.

Let's do some examples,

$$\mathcal{L}[t]$$

Note:

$$\frac{dt}{dt} = 1$$

We can use this:

$$\mathcal{L}(\int f(t)dt) = \frac{1}{s}\left[[\mathcal{L}[f(t)] + \int f(t)dt|_{t=0}]\right]$$

in this case $f(t) = 1$

$$\mathcal{L}(t) = \frac{1}{s}\left[[\mathcal{L}[1] + \int 1dt|_{t=0}]\right]$$

$$\mathcal{L}(t) = \frac{1}{s}\left[[\mathcal{L}[1] + t|_{t=0}]\right]$$

$$\mathcal{L}(t) = \frac{1}{s}\left[[\mathcal{L}[1] + 0]\right]$$

$$\mathcal{L}(t) = \frac{1}{s}\left[[\frac{1}{s}]\right] = \frac{1}{s^2}$$

Let's do

$$\mathcal{L}[t^2]$$

Let's use the integration formula, say f(t) = $2t$

$$\mathcal{L}(\int f(t)dt) = \frac{1}{s}\left[[\mathcal{L}[f(t)] + \int f(t)dt|_{t=0}]\right]$$

$$\mathcal{L}(t^2) = \frac{1}{s}\left[[2\mathcal{L}[t] + t^2_{t=0}]\right]$$

$$\mathcal{L}(t^2) = \frac{1}{s}[2\frac{1}{s^2}] = \frac{2}{s^3}$$

Part II
applications examples

3 ODEs of time varying systems

linear time invariant (LTI) systems

$$y'' + 2y' + y = \sin(\omega t)$$

$$y' + 3y = \exp(-2t) + 4$$

nonlinear:

$$(y'')^2 + 2y' + y = \sin(\omega t)$$

2nd order ODE:

$$\mathcal{L}(y'' + 2y' + y) = \mathcal{L}(\sin(\omega t)$$

$$\mathcal{L}(y'' + 2y' + y) = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}(y'') + y(s) + 2[sy(s) - y(t=0)] = \frac{\omega}{s^2 + \omega^2}$$

$$\mathcal{L}(y'') = s\mathcal{L}(y'(t)) - y'(t=0)$$

$$\mathcal{L}(y'') = s(sy(s) - y(t=0)) - y'(t=0)$$
$$\mathcal{L}(y'') = s^2y(s) - sy(t=0) - y'(t=0)$$

subs back:

$$s^2y(s) - sy(t=0) - y'(t=0) + y(s) + 2[sy(s) - y(t=0)] = \frac{\omega}{s^2+\omega^2}$$

$$s^2y(s) + y(s) + [2sy(s) - 2y(t=0)] = \frac{\omega}{s^2+\omega^2} + sy(t=0) + y'(t=0)$$

$$s^2y(s) + y(s) + 2sy(s) = \frac{\omega}{s^2+\omega^2} + sy(t=0) + y'(t=0) + 2y(t=0)$$

$$s^2y(s) + y(s) + 2sy(s) = \frac{\omega}{s^2+\omega^2} + (2+s)y(t=0) + y'(t=0)$$

$$y(s)(s^2+2s+1) = \frac{\omega}{s^2+\omega^2} + (2+s)y(t=0) + y'(t=0)$$

$$y(s) = \frac{\omega}{s^2+\omega^2}\frac{1}{(s^2+2s+1)} + \frac{(2+s)y(t=0) + y'(t=0)}{s^2+2s+1}$$

$$y(s) = \frac{\omega}{s^2+\omega^2}\frac{1}{(s^2+2s+1)} + \frac{(2+s)y(t=0)}{s^2+2s+1} + \frac{y'(t=0)}{s^2+2s+1}$$

$$y(t) = \mathcal{L}^{-1}(\frac{\omega}{s^2+\omega^2}\frac{1}{(s^2+2s+1)} + \frac{(2+s)y(t=0)}{s^2+2s+1} + \frac{y'(t=0)}{s^2+2s+1})$$

$$y(t) = \mathcal{L}^{-1}(\frac{\omega}{s^2+\omega^2}\frac{1}{(s+1)^2} + \frac{(2+s)y(t=0)}{(s+1)^2} + \frac{y'(t=0)}{(s+1)^2})$$

Note:

$$\mathcal{L}[e^{-t}] = \frac{1}{s+1}$$

Partial fraction:

$$\frac{\omega}{s^2+\omega^2}\frac{1}{(s+1)^2} = \frac{Bs+C}{s^2+\omega^2} + \frac{D}{s+1} + \frac{E}{(s+1)^2}$$

B, C, D and E are constants

$$\frac{\omega}{s^2+\omega^2}\frac{1}{(s+1)^2} = \frac{Bs+C}{s^2+\omega^2} + \frac{D}{s+1} + \frac{E}{(s+1)^2}$$

`https://www.wolframalpha.com/input/?i=partial+fraction+1%2F`

From wolfram:

$$\frac{1}{s^2+\omega^2}\frac{1}{(s+1)^2} = \frac{-\omega^2-2s+1}{(\omega^2+1)^2}\frac{1}{\omega^2+s^2}+\frac{2}{(\omega^2+1)^2}\frac{1}{s+1}+\frac{1}{\omega^2+1}\frac{1}{(s+1)^2}$$

multiply by ω

$$\frac{\omega}{s^2+\omega^2}\frac{1}{(s+1)^2} = \frac{-\omega^2-2s+1}{(\omega^2+1)^2}\frac{\omega}{\omega^2+s^2}+\frac{2\omega}{(\omega^2+1)^2}\frac{1}{s+1}+\frac{\omega}{\omega^2+1}\frac{1}{(s+1)^2}$$

$$E = \frac{\omega}{\omega^2+1}$$

$$D = \frac{2\omega}{(\omega^2+1)^2}$$

$$B = \frac{-2\omega}{(\omega^2+1)^2}$$

$$C = \frac{\omega(1-\omega^2)}{(\omega^2+1)^2}$$

So we can reduce the first term to this:

$$\frac{\omega}{s^2+\omega^2}\frac{1}{(s+1)^2} = \frac{Bs+C}{s^2+\omega^2}+\frac{D}{s+1}+\frac{E}{(s+1)^2}$$

How do we inverse laplace $\frac{1}{(s+1)^2}$?

Frequency Shift

$$\mathcal{L}[e^{-at}f(t)] = \int_0^\infty e^{-at}f(t)e^{-st}dt$$

$$\mathcal{L}[e^{-at}f(t)] = \int_0^\infty f(t)e^{-(s+a)t}dt$$

Compare this with:

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt = F(s)$$

So we can say:

$$\mathcal{L}[e^{-at}f(t)] = \int_0^\infty f(t)e^{-(s+a)t}dt = F(s+a)$$

let's say

$$\mathcal{L}(t^2) = \frac{2}{s^3}$$

$$\mathcal{L}(t^2\exp(-at)) = \frac{2}{(s+a)^3}$$

back to the qn Now back to: $\frac{1}{(s+1)^2}$

$$\mathcal{L}(t) = \frac{1}{s^2}$$

$$\mathcal{L}(te^{-at}) = \frac{1}{(s+1)^2}$$
$$te^{-at} = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^2}\right]$$

back to main qn:

$$y(t) = \mathcal{L}^{-1}(\frac{\omega}{s^2+\omega^2}\frac{1}{(s+1)^2} + \frac{(2+s)y(t=0)}{(s+1)^2} + \frac{y'(t=0)}{(s+1)^2})$$

$$\frac{2+s}{(s+1)^2} = \frac{s+1+1}{(s+1)^2} = \frac{1}{s+1} + \frac{1}{(s+1)^2}$$

$$y(t) = \mathcal{L}^{-1}(\frac{\omega}{s^2+\omega^2}\frac{1}{(s+1)^2} + \frac{(2+s)y(t=0)}{(s+1)^2} + \frac{y'(t=0)}{(s+1)^2})$$

$$= \mathcal{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\frac{1}{(s+1)^2} + y(t=0)\left[\frac{1}{(s+1)^2} + \frac{1}{s+1}\right] + \frac{y'(t=0)}{(s+1)^2}\right)$$

$$= \mathcal{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\frac{1}{(s+1)^2}\right) + y(t=0)\left[te^{-t} + e^{-t}\right] + y'(t=0)te^{-t}$$

$$= \mathcal{L}^{-1}(\frac{Bs}{(s^2+\omega^2)} + \frac{C}{(s^2+\omega^2)} + \frac{D}{s+1} + \frac{E}{(s+1)^2}) + y(t=0)\left[te^{-t} + e^{-t}\right]$$

$$+y'(t=0)te^{-t}$$

$$= De^{-t} + Ete^{-t} + y(t=0)\left[te^{-t} + e^{-t}\right]$$
$$+y'(t=0)te^{-t} + B\cos(\omega t) + \frac{C}{\omega}\sin(\omega t)$$

$$E = \frac{\omega}{\omega^2+1}$$
$$D = \frac{2\omega}{(\omega^2+1)^2}$$
$$B = \frac{-2\omega}{(\omega^2+1)^2}$$
$$C = \frac{\omega(1-\omega^2)}{(\omega^2+1)^2}$$

Leaving it as it is...
Example 2: first order ODE

$$y' + 3y = \exp(-2t) + 4$$

$$sY(s) - y(t=0) + 3Y(s) = \frac{1}{s+2} + \frac{4}{s}$$

$$sY(s) + 3Y(s) = \frac{1}{s+2} + \frac{4}{s} + y(t=0)$$

$$Y(s)(s+3) = \frac{1}{s+2} + \frac{4}{s} + y(t=0)$$

$$Y(s) = \frac{1}{s+2}\frac{1}{(s+3)} + \frac{4}{s}\frac{1}{(s+3)} + y(t=0)\frac{1}{s+3}$$

$$Y(s) = \frac{1}{s+2}\frac{1}{(s+3)} + \frac{4}{s}\frac{1}{(s+3)} + y(t=0)\frac{1}{s+3}$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\frac{1}{(s+3)} + \frac{4}{s}\frac{1}{(s+3)} + y(t=0)\frac{1}{s+3}\right)$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+2}\frac{1}{(s+3)} + \frac{4}{s}\frac{1}{(s+3)}\right) + y(t=0)\exp(-3t)$$

wolfram partial fraction

$$\frac{1}{s+2}\frac{1}{(s+3)} = \frac{1}{s+2} - \frac{1}{s+3}$$

note: careless mistake in video, i forgot to times 4

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s+2} - \frac{1}{s+3} + \frac{4}{s}\frac{1}{(s+3)}\right) + y(t=0)\exp(-3t)$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{4}{s}\frac{1}{(s+3)}\right) + y(t=0)\exp(-3t) + \exp(-2t) - \exp(-3t)$$

$$y(t) = \mathcal{L}^{-1}\left(\frac{4/3}{s} - \frac{1/3}{s+3}\right) + y(t=0)\exp(-3t) + \exp(-2t) - \exp(-3t)$$

$$y(t) = \frac{4}{3} - \frac{1}{3}e^{-3t} + y(t=0)\exp(-3t) + \exp(-2t) - \exp(-3t)$$

$$y(t) = \frac{4}{3} - \frac{1}{3}e^{-3t} + [y(t=0) - 1]\exp(-3t) + \exp(-2t)$$

$$y(t) = \frac{4}{3} + \left[y(t=0) - \frac{4}{3}\right]\exp(-3t) + \exp(-2t)$$

Part III
Appendix

4 Font Sizes

`https://www.overleaf.com/learn/latex/Questions/How_do_I_adj`