

# Convection BL YouTube

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## Part I

### Some Links before we start

[https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes\\_equations](https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations)

Matrices in LaTeX

<https://www.overleaf.com/learn/latex/Matrices>

Tensors in LaTeX

Navier Stokes Equations

<https://www.comsol.com/multiphysics/navier-stokes-equations>

[https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes\\_equations](https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations)

Github

[https://github.com/theodoreOnzGit/heatTransferTheory\\_YouTube](https://github.com/theodoreOnzGit/heatTransferTheory_YouTube)

## Part II

### Hydrodynamics

#### 1 Navier Stokes Equations

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u - \nu\left(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial z^2}u\right) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v - \nu\left(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v + \frac{\partial^2}{\partial z^2}v\right) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - \nu(\frac{\partial^2}{\partial x^2}w + \frac{\partial^2}{\partial y^2}w + \frac{\partial^2}{\partial z^2}w) = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + g_z$$

## 2 Boundary Layer Equations (Laminar)

$$0 = -\frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial y^2}u = (-\frac{\partial P}{\partial x})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

### 2.0.1 Blasius Similarity Solution

$$f' = 0 \text{ at } \eta = 0$$

$$f = 0 \text{ at } \eta = 0$$

$$f' = 2 \text{ at } \eta \rightarrow \infty$$

$$f''f + f''' = 0$$

$$u = \frac{u_\infty}{2}f'$$

$$\eta = \frac{1}{2}y\sqrt{\frac{u_\infty}{\nu x}}$$

[not derived previously:]

$$\tau_x = \mu \frac{\partial u}{\partial y} = \rho \nu \frac{u_\infty}{4} \sqrt{\frac{u_\infty}{\nu x}} f''$$

Welty, J., Rorrer, G. L., & Foster, D. G. (2014).

Fundamentals of momentum, heat, and mass transfer. John Wiley & Sons.

### 2.1 Integral Solution by Theodore Von Karman

Final form of Von Karman equation:

$$\frac{\tau_x}{\rho} = \left(\frac{\partial u_\infty}{\partial x}\right) \int_0^{\delta_p} (u_\infty - u) dy + \frac{\partial}{\partial x} \int_0^{\delta_p} u(u_\infty - u) dy$$

### 3 Solutions to BL

#### 3.1 Von Karman Results (approximate solution)

$$\frac{\delta_p(x)}{x} = \frac{4.64095}{\sqrt{Re_x}}$$

$$C_{fx} \equiv \frac{0.6464}{\sqrt{Re_x}}$$

$$C_{fL} \equiv \frac{1.2928}{\sqrt{Re_L}}$$

#### 3.2 Blasius Results (Exact solution)

$$\frac{\delta_p(x)}{x} = \frac{5}{\sqrt{Re_x}}$$

$$C_{fx} \equiv \frac{0.664}{\sqrt{Re_x}}$$

$$C_{fL} \equiv \frac{1.328}{\sqrt{Re_L}}$$

7.2% off for BL thickness, and 3% off for skin friction coeff. Pretty good!

This shows that Von Karman method is pretty good, if you can't use Blasius results

Welty, J., Rorrer, G. L., & Foster, D. G. (2014).

Fundamentals of momentum, heat, and mass transfer. John Wiley & Sons.

## Part III

# Energy Equations

$$h \equiv e + PV$$

Enthalpy per unit mass (Specific enthalpy) P is pressure V is specific volume e is specific internal energy

In thermodynamics, U is often used for internal energy, but we avoid using that because it gets confused with u which is x velocity. So we use e instead.

Enthalpy Balance

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = \text{conduction} + \text{heat generation} + \text{dissipation} + \text{radiation} \quad (1)$$

neglect heat generation and dissipation terms. Also neglect radiation heat transfer

Fourier's law

$$q''_{cond} = -k\nabla T$$

Conduction term:

$$\text{conduction} = -\nabla q''_{cond}$$

Heat flux =  $q''_{cond}$  ;

Heat flux is heat energy Transferred per unit area

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = -\nabla q''_{cond}$$

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = -\nabla(-k\nabla T)$$

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = \nabla(k\nabla T)$$

heat conduction coefficient is isotropic (it's a scalar), we also assume k does not change with x,y or z

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = k\nabla^2 T$$

$$\Delta h = \rho c_p \Delta T$$

$$h - h_{ref} = \rho c_p (T - T_{ref})$$

$$h = \rho c_p (T - T_{ref}) + h_{ref}$$

We note, reference temperature and enthalpy are constant with time, x y z etc.

$$\frac{\partial}{\partial t}\rho c_p T + u\frac{\partial}{\partial x}\rho c_p T + v\frac{\partial}{\partial y}\rho c_p T + w\frac{\partial}{\partial z}\rho c_p T = k\nabla^2 T$$

We assume, volumetric heat capacity does not change with x,y or z and time

$$\rho c_p = \text{volumetric heat capacity} = \frac{kg}{m^3} \frac{J}{kg \bullet K}$$

$$\frac{\partial}{\partial t} \rho c_p T + u \frac{\partial}{\partial x} \rho c_p T + v \frac{\partial}{\partial y} \rho c_p T + w \frac{\partial}{\partial z} \rho c_p T = k \nabla^2 T$$

$$\frac{\partial}{\partial t} T + u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T + w \frac{\partial}{\partial z} T = \frac{k}{\rho c_p} \nabla^2 T$$

$$\frac{\partial}{\partial t} T + u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T + w \frac{\partial}{\partial z} T = \alpha \nabla^2 T$$

$$\alpha = \frac{k}{\rho c_p}$$

$$\frac{\partial}{\partial t} T + u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T + w \frac{\partial}{\partial z} T = \alpha \left( \frac{\partial^2}{\partial x^2} T + \frac{\partial^2}{\partial y^2} T + \frac{\partial^2}{\partial z^2} T \right)$$

## Part IV

# Thermal Laminar BL Equations

First, we assume, 2D, so z derivatives are zero.

$$\frac{\partial}{\partial t} T + u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T + w \frac{\partial}{\partial z} T = \alpha \left( \frac{\partial^2}{\partial x^2} T + \frac{\partial^2}{\partial y^2} T + \frac{\partial^2}{\partial z^2} T \right)$$

$$\frac{\partial}{\partial t} T + u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \alpha \left( \frac{\partial^2}{\partial x^2} T + \frac{\partial^2}{\partial y^2} T \right)$$

We need to introduce relative temperature

$$T_{rel} = T - T_s$$

$$\frac{\partial}{\partial x} (T - T_s) = \frac{\partial}{\partial x} (T)$$

so long as  $T_s$  is constant

$$\frac{\partial}{\partial t} (T - T_s) + u \frac{\partial}{\partial x} (T - T_s) + v \frac{\partial}{\partial y} (T - T_s) = \alpha \left( \frac{\partial^2}{\partial x^2} (T - T_s) + \frac{\partial^2}{\partial y^2} (T - T_s) \right)$$

Let's see which terms we can neglect

$$\theta = \frac{T - T_s}{T_\infty - T_s}$$

$$u^* = \frac{u}{u_\infty}$$

Let's nondimensionalise temperature:

$$\frac{\partial}{\partial t}\theta(T_\infty - T_s) + u \frac{\partial}{\partial x}\theta(T_\infty - T_s) + v \frac{\partial}{\partial y}\theta(T_\infty - T_s) = \alpha \left( \frac{\partial^2}{\partial x^2}\theta(T_\infty - T_s) + \frac{\partial^2}{\partial y^2}\theta(T_\infty - T_s) \right)$$

Divide throughout,

$$\frac{\partial}{\partial t}\theta + u \frac{\partial}{\partial x}\theta + v \frac{\partial}{\partial y}\theta = \alpha \left( \frac{\partial^2}{\partial x^2}\theta + \frac{\partial^2}{\partial y^2}\theta \right)$$

$$x^* = \frac{x}{L}$$

$$y^* = \frac{y}{\delta_t}$$

Note: in general, thermal boundary layer is different from momentum boundary layer

$$\frac{\partial}{\partial t}\theta + u \frac{1}{L} \frac{\partial}{\partial x^*}\theta + v \frac{1}{\delta_t} \frac{\partial}{\partial y^*}\theta = \alpha \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2}\theta + \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2}\theta \right)$$

$$u^* = \frac{u}{u_\infty}$$

$$\frac{\partial}{\partial t}\theta + u^* \frac{u_\infty}{L} \frac{\partial}{\partial x^*}\theta + v \frac{1}{\delta_t} \frac{\partial}{\partial y^*}\theta = \alpha \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2}\theta + \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2}\theta \right)$$

We assume no natural convection

$$\frac{\partial}{\partial t}\theta + u^* \frac{u_\infty}{L} \frac{\partial}{\partial x^*}\theta + v \frac{1}{\delta_t} \frac{\partial}{\partial y^*}\theta = \alpha \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2}\theta + \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2}\theta \right)$$

If we were to use momentum nondimensionalisation:

$$y^* = \frac{y}{\delta_p}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{u_\infty}{L} \frac{\partial u^*}{\partial x^*} + \frac{v_c}{\delta_p} \frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{u_\infty \delta_p}{v_c L} \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{u_\infty \delta_p}{v_c L} = \mathcal{O}(1)$$

$$\frac{u_\infty \delta_p}{L} = \mathcal{O}(1) v_c$$

we can just set:

$$v_c = \frac{u_\infty \delta_p}{L} \mathcal{O}(1)$$

so that

$$v^* = \frac{v}{v_c} = \mathcal{O}(1)$$

or else

$$v_c = \frac{u_\infty \delta_p}{L}$$

So we substitute this back in:

$$\frac{\partial}{\partial t} \theta + u^* \frac{u_\infty}{L} \frac{\partial}{\partial x^*} \theta + \frac{u_\infty \delta_p}{L \delta_t} v^* \frac{\partial}{\partial y^*} \theta = \alpha \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2} \theta + \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

$$t = \frac{t^*}{t_c}$$

if you talk about hydrodynamic timescales:

$$t_c = \frac{L}{u_\infty}$$

we have to use thermal BL timescales...

For simplicity, we may not want to consider transient timescales in flat plate BL analysis

We assume steady state

$$u^* \frac{u_\infty}{L} \frac{\partial}{\partial x^*} \theta + \frac{u_\infty \delta_p}{L \delta_t} v^* \frac{\partial}{\partial y^*} \theta = \alpha \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2} \theta + \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

$$u^* \frac{\partial}{\partial x^*} \theta + \frac{\delta_p}{\delta_t} v^* \frac{\partial}{\partial y^*} \theta = \alpha \frac{L}{u_\infty} \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2} \theta + \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

We can make this observation:

$$L^2 \gg \delta_t^2$$

$$u^* \frac{\partial}{\partial x^*} \theta + \frac{\delta_p}{\delta_t} v^* \frac{\partial}{\partial y^*} \theta = \alpha \frac{L}{u_\infty} \left( \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

assume steady state, and performing BL analysis: Also, no dissipation or heat generation

$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \alpha \left( \frac{\partial^2}{\partial y^2} T \right)$$

## Part V

# Solution Procedures - Constant Temp Forced Conv BL

$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \alpha \frac{\partial^2}{\partial y^2} T$$

$$0 = - \frac{\partial P}{\partial y}$$

$$u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u - \nu \frac{\partial^2}{\partial y^2} u = \left( - \frac{\partial P}{\partial x} \right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Solution Procedures:

- Similarity Solution (Blasius Style) [GOLD STANDARD]
- Integral Solution (Von Karman Style)
- Computational Fluid Dynamics (MultiPhysics)



## 4 back to similarity Solution

In case of no pressure gradient,

$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \alpha \frac{\partial^2}{\partial y^2} T$$

$$u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u = \nu \frac{\partial^2}{\partial y^2} u$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

And nondimensionalising

$$f' = 0 \text{ at } \eta = 0$$

$$f = 0 \text{ at } \eta = 0$$

$$f' = 2 \text{ at } \eta \rightarrow \infty$$

$$f'' f + f''' = 0$$

$$u = \frac{u_\infty}{2} f'$$

$$\eta = \frac{1}{2} y \sqrt{\frac{u_\infty}{\nu x}}$$

[not derived previously:]

$$\tau_x = \mu \frac{\partial u}{\partial y} = \rho \nu \frac{u_\infty}{4} \sqrt{\frac{u_\infty}{\nu x}} f''$$

### 4.0.1 Reynold's analogy

Hey these equations look so similar

$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \alpha \frac{\partial^2}{\partial y^2} T$$

$$u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u = \nu \frac{\partial^2}{\partial y^2} u$$

Let's copy/paste the solution... (under certain constraints)

$$\nu = \alpha$$

$$Pr = \frac{\nu}{\alpha} = 1$$

If you wanna change up the equations:

$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \frac{\nu}{Pr} \frac{\partial^2}{\partial y^2} T$$

If  $Pr=1$ ,

$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \nu \frac{\partial^2}{\partial y^2} T$$

Use the relative temperature trick, and change the variable to  $\theta$

$$u \frac{\partial}{\partial x} \theta + v \frac{\partial}{\partial y} \theta = \nu \frac{\partial^2}{\partial y^2} \theta$$

If we nondimensionalise the momentum equation

$$u \frac{\partial}{\partial x} u^* u_\infty + v \frac{\partial}{\partial y} u^* u_\infty = \nu \frac{\partial^2}{\partial y^2} u^* u_\infty$$

$$u \frac{\partial}{\partial x} u^* + v \frac{\partial}{\partial y} u^* = \nu \frac{\partial^2}{\partial y^2} u^*$$

We already proved we can reduce the momentum BL equations as follows:

$$f'' f + f''' = 0$$

And we note that we can set:

$$\theta = u^*$$

Note that

$$u = \frac{u_\infty}{2} f'$$

$$u^* = \frac{f'}{2}$$

Under Reynold's analogy we can say:

$$\theta = \frac{f'}{2}$$

We can pretty much adapt the Blasius solution to thermal boundary layer... (Reynold's analogy under  $Pr=1$ )

We assume that the nondimensional temperature profile and nondimensional BL profile (and BCs) are exactly equal.

Based on that, we can get the temperature profile and heat flux

$$\theta = \frac{f'}{2}$$

$$T - T_s = (T_\infty - T_s) \frac{f'}{2}$$

What about heat flux:

$$q'' = -k \frac{\partial T}{\partial y}$$

note:

$$\frac{\partial u}{\partial y} = \frac{u_\infty}{4} \sqrt{\frac{u_\infty}{\nu x}} f''$$

$$\frac{\partial u^* u_\infty}{\partial y} = \frac{u_\infty}{4} \sqrt{\frac{u_\infty}{\nu x}} f''$$

$$\frac{\partial u^*}{\partial y} = \frac{1}{4} \sqrt{\frac{u_\infty}{\nu x}} f''$$

under reynold's analogy

$$u^* = \theta$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{4} \sqrt{\frac{u_\infty}{\nu x}} f''$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{4} \sqrt{\frac{u_\infty}{\nu x}} f''$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{4} \sqrt{\frac{u_\infty x}{\nu x^2}} f''$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{4} \sqrt{Re_x} \frac{1}{x} f''$$

$$\frac{\partial(T - T_s)}{\partial y} = \frac{T_\infty - T_s}{4} \sqrt{Re_x} \frac{1}{x} f''$$

$$\frac{\partial T}{\partial y} = \frac{T_{\infty} - T_s}{4} \sqrt{Re_x} \frac{1}{x} f''$$

Heat flux:

$$q'' = -k \frac{\partial T}{\partial y}$$

$$q'' = -\frac{k}{x} \frac{T_{\infty} - T_s}{4} \sqrt{Re_x} f''$$

Heat trf Coefficient and Nu

$$q'' = -h(T_{\infty} - T_s)$$

Substitute

$$h(T_{\infty} - T_s) = \frac{k}{x} \frac{T_{\infty} - T_s}{4} \sqrt{Re_x} f''$$

$$h = \frac{k}{x} \frac{f''}{4} \sqrt{Re_x}$$

Define Nusselt Number:

$$Nu_x \equiv \frac{hx}{k}$$

$$Nu_x = \frac{f''}{4} \sqrt{Re_x}$$

What is the value of  $f''$  (look at textbook, or solve it yourself)  
at  $y=0$

$$f'' = 1.328$$

$$Nu_x = \frac{1.328}{4} \sqrt{Re_x} = 0.332 \sqrt{Re_x}$$

## 5 What if Pr is not 1?

### 5.1 Similarity Solution by Pohlhausen

Bejan, A. (2013). Convection heat transfer. John Wiley & sons.

Welty, J., Rorrer, G. L., & Foster, D. G. (2014).

Fundamentals of momentum, heat, and mass transfer. John Wiley & Sons.

From Blasius Solution  
And nondimensionalising

$$f' = 0 \text{ at } \eta = 0$$

$$f = 0 \text{ at } \eta = 0$$

$$f' = 2 \text{ at } \eta \rightarrow \infty$$

$$f''f + f''' = 0$$

$$u = \frac{u_\infty}{2} f'$$

Some useful derivatives previously derived (Bejan, 2013, Welty et al., 2014)

From hydrodynamic BL:

$$\frac{\partial}{\partial y^*} \eta = \frac{\partial}{\partial y^*} \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} = \frac{\eta}{y^*}$$

$$\frac{\partial}{\partial x^*} \eta = \frac{\partial}{\partial x^*} \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}}$$

$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \alpha \frac{\partial^2}{\partial y^2} T$$

We nondimensionalised the energy equation during scaling analysis...

$$u^* \frac{\partial}{\partial x^*} \theta + \frac{\delta_p}{\delta_t} v^* \frac{\partial}{\partial y^*} \theta = \alpha \frac{L}{u_\infty} \left( \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

We just need to convert it to include similarity variables...

We start by making substitutions

$$u^* = \frac{f'}{2}$$

$$v = \frac{1}{2} \sqrt{\frac{\nu u_\infty}{x}} (\eta f' - f)$$

$$v_c = \frac{u_\infty \delta_p}{L}$$

$$v^* \frac{u_\infty \delta_p}{L} = -\frac{1}{2} \sqrt{\frac{\nu u_\infty}{x}} (\eta f' - f)$$

$$v^* = \frac{1}{2} \frac{L}{\delta_p} \sqrt{\frac{\nu}{xu_\infty}} (\eta f' - f)$$

$$v^* = \frac{1}{2} \frac{L}{\delta_p} Re_x^{-\frac{1}{2}} (\eta f' - f)$$

subs into:

$$u^* \frac{\partial}{\partial x^*} \theta + \frac{\delta_p}{\delta_t} v^* \frac{\partial}{\partial y^*} \theta = \alpha \frac{L}{u_\infty} \left( \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

$$\frac{f'}{2} \frac{\partial}{\partial x^*} \theta + \frac{\delta_p}{\delta_t} \left[ -\frac{1}{2} \frac{L}{\delta_p} Re_x^{\frac{1}{2}} (\eta f' - f) \right] \frac{\partial}{\partial y^*} \theta = \alpha \frac{L}{u_\infty} \left( \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

$$\frac{f'}{2} \frac{\partial}{\partial x^*} \theta + \frac{L}{\delta_t} \left[ -\frac{1}{2} Re_x^{\frac{1}{2}} (\eta f' - f) \right] \frac{\partial}{\partial y^*} \theta = \alpha \frac{L}{u_\infty} \left( \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

Now we need to substitute out  $y^*$  and  $x^*$  for  $\eta$

$$y^* = \frac{y}{\delta_t}$$

$$x^* = \frac{x}{L}$$

we note  $x^*$  is the same definition as the nondimensionalised  $x^*$  used in deriving hydrodynamic BL, so we use it again...

So we can use:

$$\frac{\partial}{\partial x^*} \eta = \frac{\partial}{\partial x^*} \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}}$$

$$= \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} \frac{-1}{2x^*} = \frac{\eta}{-2x^*}$$

Replace left most term...

$$\frac{\partial}{\partial x^*} = \frac{\partial \eta}{\partial x^*} \frac{\partial}{\partial \eta} = -\frac{\eta}{2x^*} \frac{\partial}{\partial \eta}$$

$$\frac{f'}{2} \frac{\partial}{\partial x^*} \theta + \frac{L}{\delta_t} \left[ \frac{1}{2} Re_x^{-\frac{1}{2}} (\eta f' - f) \right] \frac{\partial}{\partial y^*} \theta = \alpha \frac{L}{u_\infty} \left( \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

$$\frac{f'}{2}(-\frac{\eta}{2x^*})\frac{\partial}{\partial\eta}\theta + \frac{L}{\delta_t}\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}\left(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta\right)$$

$$\frac{f'}{2}(-\frac{\eta}{2x^*})\theta' + \frac{L}{\delta_t}\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}\left(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta\right)$$

$$\frac{f'}{2}(-\frac{\eta}{2x^*})\theta' + L\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y}\theta = \alpha\frac{L}{u_\infty}\left(\frac{\partial^2}{\partial y^2}\theta\right)$$

What about the y terms?

$$\eta = \frac{1}{2}\frac{y}{x}(Re_x)^{\frac{1}{2}} = \frac{1}{2}y\sqrt{\frac{u_\infty}{\nu x}}$$

$$\frac{\partial}{\partial y} = \frac{\partial\eta}{\partial y}\frac{\partial}{\partial\eta}$$

$$\frac{\partial\eta}{\partial y} = \frac{1}{2}\sqrt{\frac{u_\infty}{\nu x}} = \frac{\eta}{y}$$

For RHS...

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\eta}{\partial y}\frac{\partial\theta}{\partial\eta}\right) = \frac{\partial}{\partial y}\left(\frac{\partial\eta}{\partial y}\theta'\right)$$

$$= \frac{\partial\theta'}{\partial y}\frac{\partial\eta}{\partial y} + \frac{\partial^2\eta}{\partial y^2}\theta'$$

$$\frac{\partial^2\eta}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\eta}{y}\right) = 0$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial\theta'}{\partial y}\frac{\partial\eta}{\partial y} = \frac{\partial\eta}{\partial y}\frac{\partial\theta'}{\partial\eta}\frac{\partial\eta}{\partial y} = \theta''\left(\frac{\eta^2}{y^2}\right)$$

Substitution back in:

$$\frac{f'}{2}(-\frac{\eta}{2x^*})\theta' + L\left[-\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\eta}{y}\theta' = \alpha\frac{L}{u_\infty}\left(\frac{\eta^2}{y^2}\theta''\right)$$

tidying up:

$$\left(-\frac{\eta f'}{4x^*}\right)\theta' + L\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\eta}{y}\theta' = \alpha\frac{L}{u_\infty}\left(\frac{\eta^2}{y^2}\theta''\right)$$

$$\begin{aligned}
(-\frac{\eta f'}{4x^*})\theta' + \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}(\eta f' - f)\theta' &= \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'') \\
(-\frac{\eta f'}{4x^*})\theta' + \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}(\eta f'\theta' - f\theta') &= \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'') \\
(-\frac{\eta f'}{4x^*})\theta' + \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}\eta f'\theta' - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' &= \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'') \\
\theta'\eta f'(-\frac{1}{4x^*} + \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' &= \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')
\end{aligned}$$

simplifying...

$$\begin{aligned}
\theta'\eta f'(-\frac{1}{4x^*} + \frac{1}{4}\sqrt{\frac{u_\infty}{\nu x}}LRe_x^{-\frac{1}{2}}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' &= \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'') \\
\theta'\eta f'(-\frac{1}{4x^*} + \frac{1}{4x}\sqrt{Re_x}LRe_x^{-\frac{1}{2}}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' &= \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'') \\
\theta'\eta f'(-\frac{1}{4x^*} + \frac{L}{4x}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' &= \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'') \\
\theta'\eta f'(-\frac{1}{4x^*} + \frac{1}{4x^*}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' &= \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'') \\
-\frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' &= \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'') \\
-\frac{1}{2}\frac{1}{2}\frac{1}{x}Re_x^{\frac{1}{2}}LRe_x^{-\frac{1}{2}}f\theta' &= \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'') \\
-\frac{1}{4}\frac{1}{x^*}f\theta' &= \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'') \\
-\frac{1}{4}\frac{1}{x^*}f\theta' &= \alpha\frac{L}{u_\infty}(\frac{1}{4}\frac{Re_x}{x^2}\theta'') \\
-\frac{1}{4}\frac{1}{x^*}f\theta' &= \alpha\frac{1}{u_\infty}(\frac{1}{4}\frac{Re_x}{xx^*}\theta'')
\end{aligned}$$



$$-\frac{1}{4}f\theta' = \alpha \frac{1}{u_\infty} \left( \frac{1}{4} \frac{Re_x}{x} \theta'' \right)$$

$$-f\theta' = \alpha \frac{1}{u_\infty} \left( \frac{Re_x}{x} \theta'' \right)$$

$$-f\theta' = \alpha \frac{1}{u_\infty} \left( \frac{u_\infty x}{x\nu} \theta'' \right)$$

$$-f\theta' = \frac{\alpha}{\nu} \theta''$$

$$-f\theta' = \frac{1}{Pr} \theta''$$

$$\theta'' + Pr f\theta' = 0$$

## 5.2 Similarity Solution BCs

2 Bcs:

$$y \rightarrow \infty ; T \rightarrow T_\infty$$

$$\eta \rightarrow \infty ; \theta \rightarrow 1$$

Reminder

$$\theta = \frac{T - T_s}{T_\infty - T_s}$$

$$y = 0; T = T_s$$

$$\eta = 0 ; \theta = 0$$

## 5.3 solving the equation

$$\theta'' + Pr f\theta' = 0$$

$$\frac{\partial \theta'}{\partial \eta} + Pr f\theta' = 0$$

$$\frac{\partial \theta'}{\partial \eta} = -Pr f\theta'$$

$$\frac{\partial \theta'}{\partial \eta} = -Pr f \theta'$$

$$\frac{1}{\theta'} \frac{\partial \theta'}{\partial \eta} = -Pr f$$

$$\int_0^\infty \frac{1}{\theta'} \frac{\partial \theta'}{\partial \eta} d\eta = - \int_0^\infty Pr f d\eta$$

$$\int_{\theta'|_{\eta=0}}^{\theta'|_{\eta=\infty}} \frac{1}{\theta'} d\theta' = - \int_0^\infty Pr f(\eta) d\eta$$

$$\log_e \left( \frac{\theta'|_{\eta=\infty}}{\theta'|_{\eta=0}} \right) = - \int_0^\infty Pr f(\eta) d\eta$$

$$\frac{\theta'|_{\eta=\infty}}{\theta'|_{\eta=0}} = \exp \left( - \int_0^\infty Pr f(\eta) d\eta \right)$$

What if  $\eta$  is not infinity,

$$\int_{\theta'|_{\eta'=0}}^{\theta'|_{\eta'=\eta}} \frac{1}{\theta'} d\theta' = - \int_0^\eta Pr f(\eta') d\eta'$$

$$\log_e \left( \frac{\theta'|_{\eta'=\infty}}{\theta'|_{\eta=0}} \right) = - \int_0^\infty Pr f(\eta) d\eta$$

$$\frac{\theta'|_{\eta'=\eta}}{\theta'|_{\eta=0}} = \exp \left( - \int_0^\eta Pr f(\eta') d\eta' \right)$$

$$\theta'|_{\eta} = \theta'|_{\eta=0} \exp \left( - \int_0^\eta Pr f(\eta') d\eta' \right)$$

Integrate again,

$$\int_0^\eta \theta'(\eta') d\eta' = \int_0^\eta \theta'|_{\eta'=0} \exp \left( - \int_0^{\eta'} Pr f(\eta') d\eta' \right) d\eta'$$

$$\theta(\eta) = \int_0^\eta \theta'|_{\eta=0} \exp \left( - \int_0^{\eta'} Pr f(\eta') d\eta' \right) d\eta'$$

use other conventional dummy variables to avoid confusion

$$\theta(\eta) = \int_{\gamma=0}^{\gamma=\eta} \theta'|_{\eta'=0} \exp \left( - \int_{\beta=0}^{\beta=\eta} Pr f(\beta) d\beta \right) \gamma$$

One problem: we don't know what  $\theta'|_{\eta'=0}$  is  
we use the boundary condition:

$$\theta = 1 \text{ at } \eta = \text{infinity}$$

the dummy variable representing  $\eta$  is  $\gamma$

$$\theta(\eta = \infty) = \int_{\gamma=0}^{\gamma=\infty} \theta'|_{\eta=0} \exp(-\int_{\beta=0}^{\beta=\eta} Pr f(\beta) d\beta) \gamma$$

$$1 = \int_{\gamma=0}^{\gamma=\infty} \theta'|_{\eta=0} \exp(-\int_{\beta=0}^{\beta=\eta} Pr f(\beta) d\beta) \gamma$$

$$\theta'(0)^{-1} = \int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\eta} Pr f(\beta) d\beta) \gamma$$

$$\theta'(0) = \left[ \int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\eta} Pr f(\beta) d\beta) \gamma \right]^{-1}$$

using correct mathematical symbols:

$$\theta'(0) = \left[ \int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\gamma} Pr f(\beta) d\beta) \gamma \right]^{-1}$$

$$\theta(\eta) = \theta'(0) \int_{\gamma=0}^{\gamma=\eta} \exp(-Pr \int_{\beta=0}^{\beta=\gamma} f(\beta) d\beta) \gamma$$

We want to find out heat flux...

$$q'' = -k \frac{\partial T}{\partial y} \Big|_{y=0}$$

$$q'' = -k \frac{\partial (T - T_s)}{\partial y} \Big|_{y=0}$$

$$q'' = -k(T_\infty - T_s) \frac{\partial \theta}{\partial y} \Big|_{y=0}$$

$$h = \frac{q''}{-(T_\infty - T_s)} = k \frac{\partial \theta}{\partial y} \Big|_{y=0}$$

$$\frac{h}{k} = \frac{\partial \theta}{\partial y} \Big|_{y=0}$$

$$\frac{h}{k} = \frac{\partial \eta}{\partial y} \frac{\partial \theta}{\partial \eta} \Big|_{y=0}$$

$$\frac{h}{k} = \frac{\eta}{y} \frac{\partial \theta}{\partial \eta} \Big|_{y=0}$$

$$\frac{h}{k} = \frac{1}{2} \sqrt{\frac{u_\infty}{\nu x}} \frac{\partial \theta}{\partial \eta} \Big|_{y=0}$$

$$\frac{h}{k} = \frac{1}{2} \sqrt{\frac{u_\infty}{\nu x}} \theta'(0)$$

$$\frac{h}{k} = \frac{1}{2} \sqrt{\frac{u_\infty x}{\nu x^2}} \theta'(0)$$

$$\frac{hx}{k} = \frac{1}{2} \sqrt{Re_x} \theta'(0)$$

$$Nu_x = \frac{1}{2} \sqrt{Re_x} \theta'(0)$$

$$Nu_x = \frac{1}{2} \sqrt{Re_x} \left[ \int_{\gamma=0}^{\gamma=\infty} \exp\left(- \int_{\beta=0}^{\beta=\gamma} Pr f(\beta) d\beta\right) \gamma \right]^{-1}$$

In textbook expression:

$$Nu_x = \sqrt{Re_x} \left[ \int_{\gamma=0}^{\gamma=\infty} \exp\left(- \int_{\beta=0}^{\beta=\gamma} \frac{1}{2} Pr f(\beta) d\beta\right) \gamma \right]^{-1}$$

What is the value of:

$$\left[ \int_{\gamma=0}^{\gamma=\infty} \exp\left(- \int_{\beta=0}^{\beta=\gamma} \frac{1}{2} Pr f(\beta) d\beta\right) \gamma \right]^{-1}$$

Pohlhausen worked it out for  $Pr > 0.5$ :

$$\left[ \int_{\gamma=0}^{\gamma=\infty} \exp\left(- \int_{\beta=0}^{\beta=\gamma} \frac{1}{2} Pr f(\beta) d\beta\right) \gamma \right]^{-1} \approx 0.332 Pr^{\frac{1}{3}}$$

Liquid metals  $Pr < 0.1$

Air  $Pr = 0.72$

Water  $Pr = 7$

Pr number says that for liquids of same viscosity, a higher prandtl number means lower thermal diffusivity

Thermal diffusivity intuition, high thermal conductivity means high thermal diffusivity, all else constant.

$$Nu_x = 0.332 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

$$Nu_L = \frac{1}{L} \int_0^L Nu_x dx$$

$$Nu_L = \frac{1}{L} \int_0^L 0.332 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}} dx$$

$$Nu_L = Pr^{\frac{1}{3}} \frac{1}{L} \int_0^L 0.332 Re_x^{\frac{1}{2}} dx$$

$$Nu_L = 0.664 Re_L^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

Note: For the above solutions, use film temperature to evaluate thermal properties!

$$T_{film} = \frac{T_s + T_{\infty}}{2}$$

eg.  $c_p, \mu$  etc..

## 6 Integral Solutions to Thermal BL Constant Temp Flat Plate

In the momentum BL, we considered a control volume in the BL to get:

$$\frac{\tau_x}{\rho} = \left( \frac{\partial u_{\infty}}{\partial x} \right) \int_0^{\delta_p} (u_{\infty} - u) dy + \frac{\partial}{\partial x} \int_0^{\delta_p} u(u_{\infty} - u) dy$$

Now for thermal BL, we can also consider control volume too (Welty et al., 2014).

We can consider again four sides of the control volume and perform an enthalpy balance, neglecting fluid kinetic energy:

$$\dot{Q} - \dot{W} = \text{net enthalpy flows out} + \Delta \text{ enthalpy w.r.t time}$$

W is Work done by system, if Q+W then W is work done on system

$$\dot{Q} = q'' dA$$

$$q''_{wall} = -k \frac{\partial T}{\partial y} \big|_{y=0}$$

$$dA = l_z \Delta x$$

For conduction, ignore conduction within fluid. (important assumption)

2nd assumption, no work done  $W=0$ ,

Let's consider the enthalpy flows and assume steady state..

Consider net enthalpy flows in:

neglecting kinetic energy

inflow of enthalpy through left

$$= \int_0^{\delta_t} \rho u h l_z dy|_x$$

$$= \int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_x$$

outflow of enthalpy through the right

$$= \int_0^{\delta_t} \rho u h l_z dy|_{x+\Delta x}$$

$$= \int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_{x+\Delta x}$$

inflow of enthalpy through the top

Specific enthalpy:

$$h_\infty = c_p (T_\infty - T_{ref})$$

mass flowrate into system:

$$\dot{m} = \int_0^{\delta_t} \rho u l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u l_z dy|_x$$

put this together to find enthalpy inflow through top:

$$\dot{m} h_\infty = \left( \int_0^{\delta_t} \rho u l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u l_z dy|_x \right) c_p (T_\infty - T_{ref})$$

Let's subs everything back:

$\dot{Q} - \dot{W} = \text{net enthalpy flows out} + \Delta \text{ enthalpy w.r.t time}$

$$-k \frac{\partial T}{\partial y}|_{y=0} l_z \Delta x - 0 = -\left(\int_0^{\delta_t} \rho u l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u l_z dy|_x\right) c_p (T_\infty - T_{ref})$$

$$+ \int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_x + 0$$

divide throughout by  $\Delta x$  take limit  $\Delta x \rightarrow 0$

$$-k \frac{\partial T}{\partial y}|_{y=0} l_z = -\frac{1}{\Delta x} \left(\int_0^{\delta_t} \rho u l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u l_z dy|_x\right) c_p (T_\infty - T_{ref})$$

$$+ \frac{1}{\Delta x} \left[\int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_x\right]$$

Assume  $T_\infty$  constant with x

$$-k \frac{\partial T}{\partial y}|_{y=0} l_z = -\frac{\partial}{\partial x} \left(\int_0^{\delta_t} \rho u l_z dy\right) c_p (T_\infty - T_{ref})$$

$$+ \frac{\partial}{\partial x} \left[\int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy\right]$$

Drop out  $T_{ref}$  after bring  $c_p T_\infty$  into integral:

$$-k \frac{\partial T}{\partial y}|_{y=0} l_z = -\frac{\partial}{\partial x} \left(\int_0^{\delta_t} \rho u l_z dy c_p T_\infty\right)$$

$$+ \frac{\partial}{\partial x} \left[\int_0^{\delta_t} \rho u c_p T l_z dy\right]$$

Finally get rid of  $l_z$

$$-k \frac{\partial T}{\partial y}|_{y=0} = -\frac{\partial}{\partial x} \left(\int_0^{\delta_t} \rho u dy c_p T_\infty\right) + \frac{\partial}{\partial x} \left[\int_0^{\delta_t} \rho u c_p T dy\right]$$

$$-k \frac{\partial T}{\partial y}|_{y=0} = -\frac{\partial}{\partial x} \left(\int_0^{\delta_t} \rho u dy c_p T_\infty\right) + \frac{\partial}{\partial x} \left[\int_0^{\delta_t} \rho u c_p T dy\right]$$

Tidy up equation:

$$-k \frac{\partial T}{\partial y} \Big|_{y=0} = \frac{\partial}{\partial x} \left[ \int_0^{\delta_t} \rho u c_p (T - T_\infty) dy \right]$$

y is integrated out so the integral on the RHS does not depend on y

$$-k \frac{\partial T}{\partial y} \Big|_{y=0} = \frac{d}{dx} \left[ \int_0^{\delta_t} \rho u c_p (T - T_\infty) dy \right]$$

You can even bring out  $\rho c_p$ :

$$-k \frac{\partial T}{\partial y} \Big|_{y=0} = \rho c_p \frac{d}{dx} \left[ \int_0^{\delta_t} u (T - T_\infty) dy \right]$$

## 6.1 The Hardcore Integral Solution

But in other textbooks, von karman's integral approach literally integrates the BL equations over y (Bejan, 2013).

we start at:

$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \alpha \left( \frac{\partial^2}{\partial y^2} T \right)$$

and for we put back u and v into the differential using product rule.

$$\frac{\partial}{\partial x} (uT) = u \frac{\partial}{\partial x} T + T \frac{\partial}{\partial x} u$$

$$u \frac{\partial}{\partial x} T = \frac{\partial}{\partial x} (uT) - T \frac{\partial}{\partial x} u$$

$$\frac{\partial}{\partial y} (vT) = v \frac{\partial}{\partial y} T + T \frac{\partial}{\partial y} v$$

$$v \frac{\partial}{\partial y} T = \frac{\partial}{\partial y} (vT) - T \frac{\partial}{\partial y} v$$

subs back in:

$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \alpha \left( \frac{\partial^2}{\partial y^2} T \right)$$

$$\frac{\partial}{\partial x} (uT) - T \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} (vT) - T \frac{\partial}{\partial y} v = \alpha \left( \frac{\partial^2}{\partial y^2} T \right)$$



$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) - T(\frac{\partial}{\partial y}v + \frac{\partial}{\partial x}u) = \alpha(\frac{\partial^2}{\partial y^2} T)$$

Now we consider continuity equation

$$\frac{\partial}{\partial y}v + \frac{\partial}{\partial x}u = 0$$

And finally get...

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \alpha(\frac{\partial^2}{\partial y^2} T)$$

Now that we have these equations we integrate across y in the BL.

$$\begin{aligned} \int_0^Y \frac{\partial}{\partial x}(uT)dy + \int_0^Y \frac{\partial}{\partial y}(vT)dy &= \int_0^Y \alpha(\frac{\partial^2}{\partial y^2} T)dy \\ \int_0^Y \frac{\partial}{\partial x}(uT)dy + (vT)|_Y - (vT)|_0 &= \alpha \int_0^Y (\frac{\partial^2}{\partial y^2} T)dy \\ \int_0^Y \frac{\partial}{\partial x}(uT)dy + (vT)|_Y - (vT)|_0 &= \alpha[\frac{\partial T}{\partial y}|_Y - \frac{\partial T}{\partial y}|_0] \end{aligned}$$

Notice there is a partial derivative with x, we will need to use Leibniz's Rule

[https://en.wikipedia.org/wiki/Leibniz\\_integral\\_rule](https://en.wikipedia.org/wiki/Leibniz_integral_rule)

Consider the integral

$$\int_{y1=a(x)}^{y2=b(x)} f(x, y)dy$$

$$\frac{d}{dx} \left( \int_{y1=a(x)}^{y2=b(x)} f(x, y)dy \right) = f(x, y = b(x)) \frac{d}{dx} b(x) - f(x, y = a(x)) \frac{d}{dx} a(x) + \int_{y1=a(x)}^{y2=b(x)} \frac{\partial}{\partial x} f(x, y)dy$$

We can apply it, we let  $f(x, y) = uT$

$$\frac{d}{dx} \left( \int_{y1=a(x)}^{y2=b(x)} uT dy \right) = uT(x, y = b(x)) \frac{d}{dx} b(x) - uT(x, y = a(x)) \frac{d}{dx} a(x) + \int_{y1=a(x)}^{y2=b(x)} \frac{\partial}{\partial x} uT dy$$

substitute in  $y_1=0, y_2=Y$

$$\frac{d}{dx} \left( \int_0^{y_2=Y} uT dy \right) = uT(x, y=Y) \frac{d}{dx} Y(x) - uT(x, y=0) \frac{d}{dx} 0 + \int_0^{y_2=Y} \frac{\partial}{\partial x} uT dy$$

$$\frac{d}{dx} \left( \int_0^Y uT dy \right) = uT(x, y=Y) \frac{d}{dx} Y(x) + \int_0^Y \frac{\partial}{\partial x} uT dy$$

$$\int_0^Y \frac{\partial}{\partial x} uT dy = \frac{d}{dx} \left( \int_0^Y uT dy \right) - uT(x, y=Y) \frac{d}{dx} Y(x)$$

substitute back into main equation:

$$\int_0^Y \frac{\partial}{\partial x} (uT) dy + (vT)|_Y - (vT)|_0 = \alpha \left[ \frac{\partial T}{\partial y} |_Y - \frac{\partial T}{\partial y} |_0 \right]$$

$$\frac{d}{dx} \left( \int_0^Y uT dy \right) - uT(x, y=Y) \frac{d}{dx} Y(x) + (vT)|_Y - (vT)|_0 = \alpha \left[ \frac{\partial T}{\partial y} |_Y - \frac{\partial T}{\partial y} |_0 \right]$$

Then consider getting  $v$  ( $y$  velocity) by integrating continuity equation (Bejan, 2013).

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\int_0^Y \frac{\partial u}{\partial x} dy + \int_0^Y \frac{\partial v}{\partial y} dy = 0$$

$$\int_0^Y \frac{\partial u}{\partial x} dy + v|_Y - v|_0 = 0$$

$$v|_Y = v|_0 - \int_0^Y \frac{\partial u}{\partial x} dy$$

We considered Leibniz's rule before using  $f=uT$ ,

$$\int_0^Y \frac{\partial}{\partial x} uT dy = \frac{d}{dx} \left( \int_0^Y uT dy \right) - uT(x, y=Y) \frac{d}{dx} Y(x)$$

now consider  $f(x,y)=u(x,y)$

$$\int_0^Y \frac{\partial}{\partial x} u dy = \frac{d}{dx} \left( \int_0^Y u dy \right) - u(x, y=Y) \frac{d}{dx} Y(x)$$

$$v|_Y = v|_0 - \left(\frac{d}{dx} \left( \int_0^Y u dy \right) - u(x, y = Y) \frac{d}{dx} Y(x)\right)$$

$$v|_Y = v|_0 - \frac{d}{dx} \left( \int_0^Y u dy \right) + u(x, y = Y) \frac{d}{dx} Y(x)$$

Let's substitute this back into our energy equation:

$$\frac{d}{dx} \left( \int_0^Y u T dy \right) - u T(x, y = Y) \frac{d}{dx} Y(x) + (v T)|_Y - (v T)|_0 = \alpha \left[ \frac{\partial T}{\partial y} \Big|_Y - \frac{\partial T}{\partial y} \Big|_0 \right]$$

$$\frac{d}{dx} \left( \int_0^Y u T dy \right) - u T(x, y = Y) \frac{d}{dx} Y(x) + v|_Y T|_Y - (v T)|_0 = \alpha \left[ \frac{\partial T}{\partial y} \Big|_Y - \frac{\partial T}{\partial y} \Big|_0 \right]$$

$$\begin{aligned} \frac{d}{dx} \left( \int_0^Y u T dy \right) - u T(x, y = Y) \frac{d}{dx} Y(x) + \left( v|_0 - \frac{d}{dx} \left( \int_0^Y u dy \right) + u(x, y = Y) \frac{d}{dx} Y(x) \right) T|_Y - (v T)|_0 \\ = \alpha \left[ \frac{\partial T}{\partial y} \Big|_Y - \frac{\partial T}{\partial y} \Big|_0 \right] \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} \left( \int_0^Y u T dy \right) - u T(x, y = Y) \frac{d}{dx} Y(x) + v|_0 T|_Y - \frac{d}{dx} \left( \int_0^Y u dy \right) T|_Y + u(x, y = Y) T|_Y \frac{d}{dx} Y(x) - (v T)|_0 \\ = \alpha \left[ \frac{\partial T}{\partial y} \Big|_Y - \frac{\partial T}{\partial y} \Big|_0 \right] \end{aligned}$$

$$\frac{d}{dx} \left( \int_0^Y u T dy \right) + v|_0 T|_Y - \frac{d}{dx} \left( \int_0^Y u dy \right) T|_Y - (v T)|_0 = \alpha \left[ \frac{\partial T}{\partial y} \Big|_Y - \frac{\partial T}{\partial y} \Big|_0 \right]$$

Now we need to introduce some BCs to tidy things up:

- no slip
- $Y = \delta_t$
- conduction at surface  $\gg$  conduction at BL

Firstly no slip  $v=0$  at  $y=0$ :

$$\frac{d}{dx} \left( \int_0^Y u T dy \right) - \frac{d}{dx} \left( \int_0^Y u dy \right) T|_Y = \alpha \left[ \frac{\partial T}{\partial y} \Big|_Y - \frac{\partial T}{\partial y} \Big|_0 \right]$$

now,  $Y = \delta_t(x)$

$$\frac{d}{dx} \left( \int_0^{\delta_t} u T dy \right) - \frac{d}{dx} \left( \int_0^{\delta_t} u dy \right) T|_{\delta_t} = \alpha \left[ \frac{\partial T}{\partial y} \Big|_{\delta_t} - \frac{\partial T}{\partial y} \Big|_0 \right]$$

note:  $T_\infty = T|_{\delta_t}$

$$\frac{d}{dx} \left( \int_0^{\delta_t} u T dy \right) - \frac{d}{dx} \left( \int_0^{\delta_t} u dy \right) T_\infty = \alpha \left[ \frac{\partial T}{\partial y} \Big|_{\delta_t} - \frac{\partial T}{\partial y} \Big|_0 \right]$$

in general,  $T_\infty = T_\infty(x)$

$$\frac{d}{dx} \left( \int_0^{\delta_t} u T dy \right) - T_\infty(x) \frac{d}{dx} \left( \int_0^{\delta_t} u dy \right) = \alpha \left[ \frac{\partial T}{\partial y} \Big|_{\delta_t} - \frac{\partial T}{\partial y} \Big|_0 \right]$$

Apply product rule:

$$\frac{d}{dx} \left( T_\infty(x) \int_0^{\delta_t} u dy \right) = T_\infty(x) \frac{d}{dx} \left( \int_0^{\delta_t} u dy \right) + \left( \frac{dT_\infty(x)}{dx} \right) \int_0^{\delta_t} u dy$$

multiply all by -1, im going to just use  $T_\infty$  instead of  $T_\infty(x)$

$$\frac{d}{dx} \left( T_\infty \int_0^{\delta_t} u dy \right) = T_\infty \frac{d}{dx} \left( \int_0^{\delta_t} u dy \right) + \left( \frac{dT_\infty}{dx} \right) \int_0^{\delta_t} u dy$$

$$-\frac{d}{dx} \left( T_\infty \int_0^{\delta_t} u dy \right) = -T_\infty \frac{d}{dx} \left( \int_0^{\delta_t} u dy \right) - \left( \frac{dT_\infty}{dx} \right) \int_0^{\delta_t} u dy$$

$$-\frac{d}{dx} \left( T_\infty \int_0^{\delta_t} u dy \right) + \left( \frac{dT_\infty}{dx} \right) \int_0^{\delta_t} u dy = -T_\infty \frac{d}{dx} \left( \int_0^{\delta_t} u dy \right)$$

$$-T_\infty \frac{d}{dx} \left( \int_0^{\delta_t} u dy \right) = -\frac{d}{dx} \left( T_\infty \int_0^{\delta_t} u dy \right) + \left( \frac{dT_\infty}{dx} \right) \int_0^{\delta_t} u dy$$

Substitute back:

$$\frac{d}{dx} \left( \int_0^{\delta_t} uT dy \right) - T_\infty(x) \frac{d}{dx} \left( \int_0^{\delta_t} u dy \right) = \alpha \left[ \frac{\partial T}{\partial y} \Big|_{\delta_t} - \frac{\partial T}{\partial y} \Big|_0 \right]$$

$$\frac{d}{dx} \left( \int_0^{\delta_t} uT dy \right) - \frac{d}{dx} \left( T_\infty \int_0^{\delta_t} u dy \right) + \left( \frac{dT_\infty}{dx} \right) \int_0^{\delta_t} u dy = \alpha \left[ \frac{\partial T}{\partial y} \Big|_{\delta_t} - \frac{\partial T}{\partial y} \Big|_0 \right]$$

bring  $T_\infty$  into integral

$$\frac{d}{dx} \left( \int_0^{\delta_t} uT dy \right) - \frac{d}{dx} \left( \int_0^{\delta_t} uT_\infty dy \right) + \left( \frac{dT_\infty}{dx} \right) \int_0^{\delta_t} u dy = \alpha \left[ \frac{\partial T}{\partial y} \Big|_{\delta_t} - \frac{\partial T}{\partial y} \Big|_0 \right]$$

$$\frac{d}{dx} \left( \int_0^{\delta_t} u(T - T_\infty) dy \right) + \left( \frac{dT_\infty}{dx} \right) \int_0^{\delta_t} u dy = \alpha \left[ \frac{\partial T}{\partial y} \Big|_{\delta_t} - \frac{\partial T}{\partial y} \Big|_0 \right]$$

on RHS we assume heat conduction at BL negligible compared to heat conduction at surface

$$\left[ \frac{\partial T}{\partial y} \Big|_{\delta_t} - \frac{\partial T}{\partial y} \Big|_0 \right] \approx - \frac{\partial T}{\partial y} \Big|_0$$

So we are left with our integral BL equation:

$$\frac{d}{dx} \left( \int_0^{\delta_t} u(T - T_\infty) dy \right) + \left( \frac{dT_\infty}{dx} \right) \int_0^{\delta_t} u dy = -\alpha \frac{\partial T}{\partial y} \Big|_0$$

note:  $\alpha = \frac{k}{\rho c_p}$

Now assume a power series (cubic) expression for temperature and velocity...

Recall for velocity:

$$\frac{u}{u_\infty} = \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \quad (2)$$

with the BCs:

$$u = 0 \text{ at } y = 0$$

$$u = u_\infty \text{ at } y = \delta_p$$

$$\frac{\partial u}{\partial y} = 0 \text{ at } y = \delta_p$$

$$\frac{\partial^2 u}{\partial y^2} = 0 \text{ at } y = 0$$

and assuming

$$u = a + by + cy^2 + dy^3$$

Near the wall, wall shear stress is constant.

Now we apply the same thing for temperature:

$$T - T_s = a + by + cy^2 + dy^3$$

with the BCs:

$$T - T_s = 0 \text{ at } y = 0$$

$$T - T_s = T_\infty - T_s \text{ at } y = \delta_t$$

$$\frac{\partial(T - T_s)}{\partial y} = 0 \text{ at } y = \delta_t$$

$$\frac{\partial^2(T - T_s)}{\partial y^2} = 0 \text{ at } y = 0$$

Last BC is saying, near the wall, heat flux is constant..

Now if we do the same thing for temperature profile, you will get:

$$\frac{T - T_s}{T_\infty - T_s} = \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3$$

$$T = T_s + (T_\infty - T_s) \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right)$$

we will then need to consider ratio of thermal BL and momentum BL.

Let's try substituting these back into our integral BL equation.

So in case we need: We already know that from hydrodynamic BL integral and similarity solution

$$\frac{\delta_p}{x} = \frac{4.64}{\sqrt{Re_x}}$$

$$\frac{\delta_p}{x} = \frac{5}{\sqrt{Re_x}}$$

Let's substitute back.

$$\frac{d}{dx} \left( \int_0^{\delta_t} u(T - T_\infty) dy \right) + \left( \frac{dT_\infty}{dx} \right) \int_0^{\delta_t} u dy = -\alpha \frac{\partial T}{\partial y} \Big|_0$$

Assume  $T_\infty$  is constant w.r.t x

$$\frac{d}{dx} \left( \int_0^{\delta_t} u(T - T_\infty) dy \right) = -\alpha \frac{\partial T}{\partial y} \Big|_0$$

Subs the Temp and Velocity profiles:

$$\frac{d}{dx} \left( \int_0^{\delta_t} u_\infty \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) (T - T_\infty) dy \right) = -\alpha \frac{\partial T}{\partial y} \Big|_0$$

$$\frac{d}{dx} \left( \int_0^{\delta_t} u_\infty \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) (T - T_\infty) dy \right) = -\alpha \frac{\partial T}{\partial y} \Big|_0$$

subs:

$$T = T_s + (T_\infty - T_s) \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right)$$

$$\frac{d}{dx} \left( \int_0^{\delta_t} u_\infty \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( (T_s + (T_\infty - T_s) \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right)) - T_\infty \right) dy \right) = -\alpha \frac{\partial T}{\partial y} \Big|_0$$

$$\frac{d}{dx} \left( \int_0^{\delta_t} u_\infty \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( (T_s - T_\infty + (T_\infty - T_s) \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right)) \right) dy \right) = -\alpha \frac{\partial T}{\partial y} \Big|_0$$

$$\frac{d}{dx} \left( \int_0^{\delta_t} u_\infty \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) (T_s - T_\infty - (T_s - T_\infty) \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right)) dy \right) = -\alpha \frac{\partial T}{\partial y} \Big|_0$$

$$\frac{d}{dx} \left( \int_0^{\delta_t} u_\infty (T_s - T_\infty) \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( 1 - \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right) \right) dy \right) = -\alpha \frac{\partial T}{\partial y} \Big|_0$$

$$u_\infty (T_s - T_\infty) \frac{d}{dx} \left( \int_0^{\delta_t} \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( 1 - \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right) \right) dy \right) = -\alpha \frac{\partial T}{\partial y} \Big|_0$$

divide both sides by  $(T_s - T_\infty)$ ,

$$u_\infty \frac{d}{dx} \left( \int_0^{\delta_t} \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( 1 - \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right) \right) dy \right) = -\alpha \left( \frac{1}{(T_s - T_\infty)} \right) \frac{\partial T - T_s}{\partial y} \Big|_0$$

$$u_\infty \frac{d}{dx} \left( \int_0^{\delta_t} \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( 1 - \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right) \right) dy \right) = \alpha \left( \frac{1}{(T_\infty - T_s)} \right) \frac{\partial T - T_s}{\partial y} \Big|_0$$

$$u_\infty \frac{d}{dx} \left( \int_0^{\delta_t} \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( 1 - \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right) \right) dy \right) = \alpha \frac{\partial \theta}{\partial y} \Big|_0$$

Where  $\theta = \frac{T - T_s}{T_\infty - T_s}$

Let's evaluate the integral:

$$\int_0^{\delta_t} \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( 1 - \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right) \right) dy$$

let's substitute the integral for dy with  $d \frac{y}{\delta_t}$

$$dy = \delta_t d \frac{y}{\delta_t}$$

note:  $\delta_t$  and  $\delta_p$  are constant w.r.t y

subs the change of variable:

$$\begin{aligned} & \int_0^{\delta_t} \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( 1 - \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right) \right) dy \\ &= \int_0^{\delta_t} \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( 1 - \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right) \right) d \frac{y}{\delta_t} \delta_t \\ &= \delta_t \int_0^1 \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( 1 - \left( \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3 \right) \right) d \frac{y}{\delta_t} \end{aligned}$$

let's replace  $\beta = \frac{y}{\delta_t}$ ,  $y = \beta \delta_t$

$$\begin{aligned} &= \delta_t \int_0^1 \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( 1 - \left( \frac{3}{2} \beta - \frac{1}{2} \beta^3 \right) \right) d\beta \\ &= \delta_t \int_0^1 \left( \frac{3}{2} \frac{y}{\delta_p} - \frac{1}{2} \left( \frac{y}{\delta_p} \right)^3 \right) \left( 1 - \frac{3}{2} \beta + \frac{1}{2} \beta^3 \right) d\beta \end{aligned}$$



$$\begin{aligned}
&= \delta_t \int_0^1 \left( \frac{3}{2} \frac{\beta \delta_t}{\delta_p} - \frac{1}{2} \left( \frac{\beta \delta_t}{\delta_p} \right)^3 \right) \left( 1 - \frac{3}{2} \beta + \frac{1}{2} \beta^3 \right) d\beta \\
&= \delta_t \int_0^1 \frac{3}{2} \frac{\beta \delta_t}{\delta_p} \left( 1 - \frac{3}{2} \beta + \frac{1}{2} \beta^3 \right) - \frac{1}{2} \left( \frac{\beta \delta_t}{\delta_p} \right)^3 \left( 1 - \frac{3}{2} \beta + \frac{1}{2} \beta^3 \right) d\beta \\
&= \delta_t \int_0^1 \frac{3}{2} \frac{\beta \delta_t}{\delta_p} \left( 1 - \frac{3}{2} \beta + \frac{1}{2} \beta^3 \right) d\beta - \delta_t \int_0^1 \frac{1}{2} \left( \frac{\beta \delta_t}{\delta_p} \right)^3 \left( 1 - \frac{3}{2} \beta + \frac{1}{2} \beta^3 \right) d\beta
\end{aligned}$$

[careless mistake! multiplied in  $\beta$  not  $\beta^3$ ]

$$\begin{aligned}
&= \delta_t \frac{\delta_t}{\delta_p} \int_0^1 \frac{3}{2} \beta \left( 1 - \frac{3}{2} \beta + \frac{1}{2} \beta^3 \right) d\beta - \delta_t \left( \frac{\delta_t}{\delta_p} \right)^3 \int_0^1 \frac{1}{2} \beta^3 \left( 1 - \frac{3}{2} \beta + \frac{1}{2} \beta^3 \right) d\beta \\
&= \delta_t \frac{\delta_t}{\delta_p} \int_0^1 \frac{3}{2} \left( \beta - \frac{3}{2} \beta^2 + \frac{1}{2} \beta^4 \right) d\beta - \delta_t \left( \frac{\delta_t}{\delta_p} \right)^3 \int_0^1 \frac{1}{2} \left( \beta^3 - \frac{3}{2} \beta^4 + \frac{1}{2} \beta^6 \right) d\beta \\
&= \frac{3}{2} \delta_t \frac{\delta_t}{\delta_p} \int_0^1 \left( \beta - \frac{3}{2} \beta^2 + \frac{1}{2} \beta^4 \right) d\beta - \frac{1}{2} \delta_t \left( \frac{\delta_t}{\delta_p} \right)^3 \int_0^1 \left( \beta^3 - \frac{3}{2} \beta^4 + \frac{1}{2} \beta^6 \right) d\beta \\
&= \frac{3}{2} \delta_t \frac{\delta_t}{\delta_p} \int_0^1 \left( \beta - \frac{3}{2} \beta^2 + \frac{1}{2} \beta^4 \right) d\beta - \frac{1}{2} \delta_t \left( \frac{\delta_t}{\delta_p} \right)^3 \int_0^1 \left( \beta^3 - \frac{3}{2} \beta^4 + \frac{1}{2} \beta^6 \right) d\beta \\
&= \frac{3}{2} \delta_t \frac{\delta_t}{\delta_p} \frac{1}{10} - \frac{1}{2} \delta_t \left( \frac{\delta_t}{\delta_p} \right)^3 \frac{3}{140} \\
&= \frac{3}{20} \delta_t \frac{\delta_t}{\delta_p} - \frac{3}{280} \delta_t \left( \frac{\delta_t}{\delta_p} \right)^3
\end{aligned}$$

Substituting the integral back:

$$\begin{aligned}
u_\infty \frac{d}{dx} \left( \frac{3}{20} \delta_t \frac{\delta_t}{\delta_p} - \frac{3}{280} \delta_t \left( \frac{\delta_t}{\delta_p} \right)^3 \right) &= \alpha \frac{\partial \theta}{\partial y} \Big|_0 \\
u_\infty \frac{d}{dx} \left( \frac{3}{20} \frac{\delta_t^2}{\delta_p} - \frac{3}{280} \left( \frac{\delta_t^4}{\delta_p^3} \right) \right) &= \alpha \frac{\partial \theta}{\partial y} \Big|_0
\end{aligned}$$

in approximate method, we consider the ratio  $\frac{\delta_t}{\delta_p}$

if  $\frac{\delta_t}{\delta_p} \gg 1$  we can ignore first term, otherwise, if  $\frac{\delta_t}{\delta_p} \ll 1$  we can ignore the second term

$$\frac{\delta_t}{\delta_p} \gg 1$$

$$u_\infty \frac{d}{dx} \left( -\frac{3}{280} \left( \frac{\delta_t^4}{\delta_p^3} \right) \right) = \alpha \frac{\partial \theta}{\partial y} \Big|_0$$

$$\frac{\delta_t}{\delta_p} \ll 1$$

$$u_\infty \frac{d}{dx} \left( \frac{3}{20} \frac{\delta_t^2}{\delta_p} \right) = \alpha \frac{\partial \theta}{\partial y} \Big|_0$$

Now we can substitute  $\theta$

$$\theta = \frac{T - T_s}{T_\infty - T_s} = \frac{3}{2} \frac{y}{\delta_t} - \frac{1}{2} \left( \frac{y}{\delta_t} \right)^3$$

$$\frac{\partial}{\partial y} \theta = \frac{3}{2} \frac{1}{\delta_t} - \frac{1}{2} \left( \frac{3y^2}{\delta_t^3} \right)$$

now set  $y=0$

$$\frac{\partial}{\partial y} \theta \Big|_{y=0} = \frac{3}{2\delta_t}$$

subs back in:

$$\frac{\delta_t}{\delta_p} \gg 1$$

$$u_\infty \frac{d}{dx} \left( -\frac{3}{280} \left( \frac{\delta_t^4}{\delta_p^3} \right) \right) = \alpha \frac{3}{2\delta_t}$$

$$\frac{\delta_t}{\delta_p} \ll 1$$

$$u_\infty \frac{d}{dx} \left( \frac{3}{20} \frac{\delta_t^2}{\delta_p} \right) = \alpha \frac{3}{2\delta_t}$$

note: from momentum integral equation

$$\delta_p = x \frac{4.64}{\sqrt{Re_x}} = 4.64 \sqrt{\frac{\nu x}{u_\infty}}$$

Try substituting in latter case

$$\frac{\delta_t}{\delta_p} \ll 1$$

$$u_\infty \frac{d}{dx} \left( \frac{3}{20} \frac{\delta_t^2}{\delta_p} \right) = \alpha \frac{3}{2\delta_t}$$

$$u_\infty \frac{d}{dx} \left( \frac{3}{20} \frac{\delta_t^2}{4.64 \sqrt{\frac{\nu x}{u_\infty}}} \right) = \alpha \frac{3}{2\delta_t}$$

$$\frac{u_\infty^{1.5}}{\sqrt{\nu}} \frac{3}{20 * 4.64} \frac{d}{dx} \left( \frac{\delta_t^2}{\sqrt{x}} \right) = \alpha \frac{3}{2\delta_t}$$

using quotient rule [careless mistake here, forgot to multiply by 2]:

$$\frac{d}{dx} \left( \frac{\delta_t^2}{\sqrt{x}} \right) = \frac{2\sqrt{x}\delta_t \frac{d\delta_t}{dx} - \delta_t^2(0.5) \frac{1}{\sqrt{x}}}{x}$$

substitute back:

$$\frac{u_\infty^{1.5}}{\sqrt{\nu}} \frac{3}{20 * 4.64} \frac{2\sqrt{x}\delta_t \frac{d\delta_t}{dx} - \delta_t^2(0.5) \frac{1}{\sqrt{x}}}{x} = \alpha \frac{3}{2\delta_t}$$

$$\frac{u_\infty^{1.5}}{\sqrt{\nu}} \frac{3}{20 * 4.64} \left( \frac{2}{\sqrt{x}} \delta_t \frac{d\delta_t}{dx} - \delta_t^2(0.5) \frac{1}{x\sqrt{x}} \right) = \alpha \frac{3}{2\delta_t}$$

$$\frac{u_\infty^{1.5}}{\sqrt{\nu}} \frac{3}{20 * 4.64} \left( \frac{2}{\sqrt{x}} \delta_t^2 \frac{d\delta_t}{dx} - \delta_t^3(0.5) \frac{1}{x\sqrt{x}} \right) = \alpha \frac{3}{2}$$

$$\frac{d\delta_t^3}{dx} = 3\delta_t^2 \frac{d\delta_t}{dx}$$

$$\frac{1}{3} \frac{d\delta_t^3}{dx} = \delta_t^2 \frac{d\delta_t}{dx}$$

substitute back:

$$\frac{u_\infty^{1.5}}{\sqrt{\nu}} \frac{3}{20 * 4.64} \left( \frac{1}{\sqrt{x}} \frac{2}{3} \frac{d\delta_t^3}{dx} - \delta_t^3(0.5) \frac{1}{x\sqrt{x}} \right) = \alpha \frac{3}{2}$$

1st order linear ODE in  $\delta_t^3$

$$\frac{u_\infty^{1.5}}{\sqrt{\nu}} \left( \frac{1}{\sqrt{x}} \frac{2}{3} \frac{d\delta_t^3}{dx} - \delta_t^3(0.5) \frac{1}{x\sqrt{x}} \right) = \alpha \frac{20 * 4.64}{3} \frac{3}{2}$$

$$\frac{1}{\sqrt{x}} \frac{2}{3} \frac{d\delta_t^3}{dx} - \delta_t^3(0.5) \frac{1}{x\sqrt{x}} = \alpha \frac{10 * 4.64}{1} \frac{\sqrt{\nu}}{u_\infty^{1.5}}$$

$$\frac{d\delta_t^3}{dx} - \delta_t^3(0.75) \frac{1}{x} = \alpha \frac{15 * 4.64}{1} \frac{\sqrt{\nu}}{u_\infty^{1.5}} \sqrt{x}$$

Let

$$A = \alpha \frac{15 * 4.64}{1} \frac{\sqrt{\nu}}{u_\infty^{1.5}}$$

$$\frac{d\delta_t^3}{dx} - \delta_t^3 \frac{0.75}{x} = A\sqrt{x}$$

[Alternate solution: substitute later, may get better solution (less careless mistake)]

$$u_\infty \frac{d}{dx} \left( \frac{3}{20} \frac{\delta_t^2}{\delta_p} \right) = \alpha \frac{3}{2\delta_t}$$

$$u_\infty \frac{d}{dx} \left( \frac{3}{20} \delta_p \frac{\delta_t^2}{\delta_p^2} \right) = \alpha \frac{3}{2\delta_t}$$

$$u_\infty \frac{d}{dx} \left( \frac{3}{20} \delta_p \frac{\delta_t^2}{\delta_p^2} \right) = \alpha \frac{3}{2\delta_t}$$

let  $\phi = \frac{\delta_t}{\delta_p}$

$$u_\infty \frac{d}{dx} \left( \frac{3}{20} \delta_p \phi^2 \right) = \alpha \frac{3}{2\delta_t}$$

$$u_\infty \frac{d}{dx} (\delta_p \phi^2) = \alpha \frac{10}{\delta_t}$$

Product rule:

$$\phi^2 \frac{d}{dx} \delta_p + \delta_p \frac{d}{dx} \phi^2 = \alpha \frac{10}{u_\infty \delta_t}$$

$$\delta_t \phi^2 \frac{d}{dx} \delta_p + \delta_t \delta_p \frac{d}{dx} \phi^2 = \alpha \frac{10}{u_\infty}$$

sub  $\delta_t = \phi \delta_p$

$$\delta_p \phi^3 \frac{d}{dx} \delta_p + \phi \delta_p^2 \frac{d}{dx} \phi^2 = \alpha \frac{10}{u_\infty}$$

$$\delta_p \phi^3 \frac{d}{dx} \delta_p + 2\phi^2 \delta_p^2 \frac{d}{dx} \phi = \alpha \frac{10}{u_\infty}$$

$$\phi^3 \delta_p \frac{d}{dx} \delta_p + 2\delta_p^2 \phi^2 \frac{d}{dx} \phi^2 = \alpha \frac{10}{u_\infty}$$

$$\phi^3 \delta_p \frac{d}{dx} \delta_p + \delta_p^2 \frac{2}{3} \frac{d}{dx} \phi^3 = \alpha \frac{10}{u_\infty}$$

$$\delta_p = x \frac{4.64}{\sqrt{Re_x}} = \frac{4.64\sqrt{\nu x}}{\sqrt{u_\infty}}$$

$$\delta_p = x \frac{4.64}{\sqrt{Re_x}} = \frac{4.64\sqrt{\nu x}}{\sqrt{u_\infty}}$$

$$\delta_p^2 = x \frac{4.64^2}{\sqrt{Re_x}} = \frac{4.64^2 \nu x}{u_\infty}$$

$$\frac{d}{dx} \delta_p = \frac{1}{2} \frac{4.64\sqrt{\nu}}{\sqrt{u_\infty x}}$$

$$\delta_p \frac{d}{dx} \delta_p = \frac{1}{2} \frac{4.64\sqrt{\nu x}}{\sqrt{u_\infty}} \frac{4.64\sqrt{\nu}}{\sqrt{u_\infty x}}$$

$$\delta_p \frac{d}{dx} \delta_p = \frac{1}{2} \frac{4.64^2 \nu}{u_\infty}$$

subs back:

$$\phi^3 \delta_p \frac{d}{dx} \delta_p + \delta_p^2 \frac{2}{3} \frac{d}{dx} \phi^3 = \alpha \frac{10}{u_\infty}$$

$$\phi^3 \frac{1}{2} \frac{4.64^2 \nu}{u_\infty} + \frac{4.64^2 \nu x}{u_\infty} \frac{2}{3} \frac{d}{dx} \phi^3 = \alpha \frac{10}{u_\infty}$$

$$\phi^3 \frac{1}{2} \frac{4.64^2 \nu}{1} + \frac{4.64^2 \nu x}{1} \frac{2}{3} \frac{d}{dx} \phi^3 = \alpha \frac{10}{1}$$

$$\phi^3 \frac{1}{2} + x \frac{2}{3} \frac{d}{dx} \phi^3 = \frac{10}{4.64^2} \frac{\alpha}{\nu}$$

$$\phi^3 \frac{3}{2} + 2x \frac{d}{dx} \phi^3 = \frac{30}{4.64^2} \frac{1}{Pr}$$

$$\phi^3 + \frac{4}{3} x \frac{d}{dx} \phi^3 = \frac{20}{4.64^2} \frac{1}{Pr}$$

First order ODE:

<https://mathworld.wolfram.com/First-OrderOrdinaryDifferentialEquation.html>

$$y' + p(x)y = q(x)$$

$$y = \frac{\int e^{\int^x p(x')dx'} q(x) dx + c}{e^{\int^x p(x')dx'}}$$

Replace x by dummy variables x'

$$\frac{d\delta_t^3}{dx'} - \delta_t^3 \frac{0.75}{x'} = A\sqrt{x'}$$

$$p(x') = -\frac{0.75}{x'}$$

$$q(x') = A\sqrt{x'}$$

[second careless mistake: wrong choice of integrating factor]

$$\int^x p(x')dx' = \int^x \frac{-0.75}{x'} dx' = -0.75 \ln(x)$$

$$\exp(0.75 \ln(x)) = -\exp(0.75)x$$

$$\int e^{\int^x p(x')dx'} q(x) dx = - \int \exp(0.75)x A\sqrt{x} dx$$

$$= -A \exp(0.75) \int x^{1.5} dx$$

$$= -A \exp(0.75) \frac{x^{2.5}}{2.5}$$

$$\delta_t^3 = \frac{-A \exp(0.75) \frac{x^{2.5}}{2.5} + c}{-\exp(0.75)x}$$

$$\delta_t^3 = \frac{-A \exp(0.75) \frac{x^{2.5}}{2.5}}{-\exp(0.75)x} + \frac{c}{\exp(0.75)x}$$

$$\delta_t^3 = A \frac{x^{1.5}}{2.5} + \frac{c}{-\exp(0.75)x}$$

[wrong... ODE solving gone wrong...]

<https://www.wolframalpha.com/input/?i=solve+y%27+--+0.75+y%2Ft+%3DA+sqrt%28t%29>

According to wolfram:

$$y = \frac{4}{3}Ax^{1.5} + Cx^{0.75}$$

apply BC, x=0  $\delta_t = 0$ , c=0

$$\delta_t^3 = \frac{4}{3}Ax^{1.5}$$

$$\delta_t = \mathcal{O}(\sqrt{x})$$

$$\delta_t^3 = \alpha \frac{15 * 4.64}{1} \frac{\sqrt{\nu}}{u_\infty^{1.5}} x^{1.5} \frac{4}{3}$$

$$\frac{\delta_p}{x} = \frac{4.64}{\sqrt{Re_x}}$$

$$\delta_t^3 = \frac{\alpha}{\nu} \frac{15 * 4.64}{1} \frac{\nu^{1.5}}{u_\infty^{1.5}} \frac{4}{3} x^{1.5}$$

$$\frac{x}{\sqrt{Re_x}} = \frac{\sqrt{\nu x}}{\sqrt{u_\infty}}$$

$$\delta_t^3 = \frac{\alpha}{\nu} \frac{15 * 4.64}{1} \frac{x^3}{Re_x^{1.5}} \frac{4}{3}$$

$$\frac{\delta_p}{4.64} = \frac{x}{\sqrt{Re_x}}$$

$$\delta_t^3 = \frac{1}{Pr} \frac{15 * 4.64}{1} \left( \frac{\delta_p}{4.64} \right)^3 \frac{4}{3}$$

$$\frac{\delta_t^3}{\delta_p^3} = \frac{1}{Pr} \frac{15 * 4.64}{4.64^3} \frac{4}{3}$$

$$\frac{\delta_t^3}{\delta_p^3} = \frac{1}{Pr} 0.92895$$

$$\frac{\delta_t}{\delta_p} = 0.97573 \frac{1}{Pr^{1/3}}$$

[redundant to use similarity soln and wrong number] If you use the similarity solution, you will get this:

$$\frac{\delta_p}{x} = \frac{5}{\sqrt{Re_x}}$$

$$\delta_t^3 = \frac{1}{Pr} \frac{30 * 5}{1} \left(\frac{\delta_p}{5}\right)^3 \frac{1}{2.5}$$

$$\frac{\delta_t}{\delta_p} = 0.78297 \frac{1}{Pr^{1/3}}$$

[redundant to use similarity soln and wrong number]

What's our heat flux?

$$\frac{\partial}{\partial y} \theta|_{y=0} = \frac{3}{2\delta_t}$$

$$\frac{\partial}{\partial y} \theta|_{y=0} = \frac{3}{2\delta_p * 0.97573 \frac{1}{Pr^{1/3}}}$$

$$\frac{\partial}{\partial y} \theta|_{y=0} = \frac{3}{2\delta_p * 0.97573} * Pr^{1/3}$$

$$\delta_p = \frac{4.64x}{\sqrt{Re_x}}$$

$$\frac{\partial}{\partial y} \theta|_{y=0} = \frac{3}{2 \frac{4.64x}{\sqrt{Re_x}} * 0.97573} * Pr^{1/3}$$

$$\frac{\partial}{\partial y} \theta|_{y=0} = \frac{3}{2 * 4.64x * 0.97573} * Pr^{1/3} Re^{1/2}$$

$$Nu_x = \frac{hx}{k}$$

$$q''_{wall \rightarrow fluid} = -k \frac{\partial T}{\partial y} = -h(T_\infty - T_s)$$

$$-k \frac{\partial T}{\partial y} = -h(T_\infty - T_s)$$

$$\frac{\partial T}{\partial y} = \frac{h}{k}(T_\infty - T_s)$$



$$\theta = \frac{T - T_s}{T_\infty - T_s}$$

$$\frac{\partial \theta}{\partial y} = \frac{h}{k}$$

$$\frac{h}{k}|_{y=0} = \frac{3}{2 * 4.64x * 0.97573} * Pr^{1/3} Re^{1/2}$$

$$\frac{hx}{k}|_{y=0} = \frac{3}{2 * 4.64 * 0.97573} * Pr^{1/3} Re^{1/2}$$

$$Nu_x|_{y=0} = \frac{3}{2 * 4.64 * 0.97573} * Pr^{1/3} Re^{1/2}$$

$$Nu_x|_{y=0} = 0.3313 Pr^{1/3} Re_x^{1/2}$$

$$Nu_L|_{y=0} = 0.663 Pr^{1/3} Re_L^{1/2}$$

## Part VI

# Natural Convection

## 7 Simplifying N-S equations and Energy Equations

Mass, momentum and energy balances:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial z^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v + \frac{\partial^2}{\partial z^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - \nu(\frac{\partial^2}{\partial x^2}w + \frac{\partial^2}{\partial y^2}w + \frac{\partial^2}{\partial z^2}w) = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + g_z$$

$$\frac{\partial}{\partial t}T + u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}T + w\frac{\partial}{\partial z}T = \alpha(\frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}T + \frac{\partial^2}{\partial z^2}T)$$

Note: compressibility applies here! How can we deal with this?

$$\frac{\partial}{\partial x}\rho u + \frac{\partial}{\partial y}\rho v + \frac{\partial}{\partial z}\rho w = 0$$

(note: steady state only)

$$u\frac{\partial}{\partial x}\rho + v\frac{\partial}{\partial y}\rho + w\frac{\partial}{\partial z}\rho + \rho\left[\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w\right] = 0$$

$$u\frac{\partial}{\partial x}\rho + v\frac{\partial}{\partial y}\rho + w\frac{\partial}{\partial z}\rho + \rho\left[\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w\right] = 0$$

We do term by term analysis:

$$\begin{aligned} & u\frac{\partial}{\partial x}\rho + \rho\frac{\partial u}{\partial x} \\ &= \mathcal{O}(u_{max}\frac{0.02\rho}{\delta_t}) + \mathcal{O}(\rho\frac{u_{max}}{\delta_t}) \\ & u\frac{\partial}{\partial x}\rho + \rho\frac{\partial u}{\partial x} \approx \rho\frac{\partial u}{\partial x} \end{aligned}$$

The change in density is considered small in comparison to original fluid density... (Boussinesq approximation)

Therefore,

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

With this equation, we can still use essentially the same form of the (mostly) incompressible momentum equations:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial z^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v + \frac{\partial^2}{\partial z^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - \nu(\frac{\partial^2}{\partial x^2}w + \frac{\partial^2}{\partial y^2}w + \frac{\partial^2}{\partial z^2}w) = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + g_z$$

For simplicity, we assume a 2D type flow, where convection goes upwards in the positive y axis. This is to say,  $w=0$  everywhere,  $g_z = 0$

$$0 = -\frac{1}{\rho_0}\frac{\partial P}{\partial z}$$

we can also apply the same for the x and y momentum eqns

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial z^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v + \frac{\partial^2}{\partial z^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

No variation of u and v in z direction Thus resulting in:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

With gravity being only in y direction,

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

With steady state assumption:

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x}$$

$$u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

Thus we have our full set of hydrodynamic equations to begin:

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

$$u \frac{\partial}{\partial x}u + v \frac{\partial}{\partial y}u - \nu \left( \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial x}$$

$$u \frac{\partial}{\partial x}v + v \frac{\partial}{\partial y}v - \nu \left( \frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + g_y$$

$$\frac{\partial}{\partial t}T + u \frac{\partial}{\partial x}T + v \frac{\partial}{\partial y}T + w \frac{\partial}{\partial z}T = \alpha \left( \frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}T + \frac{\partial^2}{\partial z^2}T \right)$$

We make the assumption that the change in density is so small it does not significantly impact thermal diffusivity. Again if we expand things out via product rule, the terms with change in density are assumed small compared to other terms.

$$u \frac{\partial}{\partial x}T + v \frac{\partial}{\partial y}T = \alpha \left( \frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}T \right)$$

## 8 Boussinesq approximation and dimensionless analysis

Now where does the buoyancy force come from?

It comes from surrounding fluid of heavier density "pushing" the lighter fluid upwards.

We expect buoyancy force to come from the pressure term. How do we then put this into our equations?

Consider this dimensionless analysis that as long as we are within the boundary layer, (Bejan, 2013):

$$\mathcal{O}(x) = \delta_T$$

$$\mathcal{O}(y) = H \text{ (wall height)}$$

$$H \gg \delta_T$$

(why thermal boundary layer and not momentum boundary layer? because the underlying driving force is temperature, that's the more important BL in natural convection)

Also static pressure should change only with y, not x.

What's the expression then?

$$P_\infty(y) = (L - y)\rho_\infty(-g_y)$$

L is the length from the "surface" so to speak.

$$\frac{dP_\infty}{dy} = -\rho_\infty(-g_y) = -\rho_\infty g$$

substitute this back and we get:

$$u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + g_y$$

$$u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = -\frac{1}{\rho_0} (-\rho_\infty(-g_y)) + g_y$$

of course  $g_y$  is negative, we can simply state

$$g_y = -g$$

$$u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = -\frac{1}{\rho_0} (-\rho_\infty g) - g$$

Note that the  $\rho_0$  here refers to fluid density at the BL, not in the freestream.

The change in  $\rho_0$  was small compared to other terms. But in this case, on the RHS, the only driving force for the fluid is this density change, so it cannot be neglected as before.

$$\rho u \frac{\partial}{\partial x} v + \rho v \frac{\partial}{\partial y} v - \rho \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = (\rho_\infty g) - \rho g$$

$$\rho u \frac{\partial}{\partial x} v + \rho v \frac{\partial}{\partial y} v - \rho \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = (\rho_\infty - \rho)g$$

Here's where the boussinesq approximation comes in, only applies for small density changes due to temperature, not pressure. Constant static pressure is assumed (Bejan, 2013)

$$\rho = \rho_\infty(1 - \beta(T - T_\infty))$$

This is a linear relationship for a constant  $\beta$  value (thermal expansion coefficient).

$$\beta = -\frac{1}{\rho} \left( \frac{\partial \rho}{\partial T} \right)_P$$

When we substitute all this in, we are ready to start deriving the equations.

$$\rho u \frac{\partial}{\partial x} v + \rho v \frac{\partial}{\partial y} v - \rho \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = \rho_\infty \beta (T - T_\infty) g$$

Once again, for the sake of simplification, density changes are not important except that the buoyancy forces are accounted for, therefore,

$$\rho \approx \rho_\infty$$

$$u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = \beta (T - T_\infty) g$$

This substitution is known as the boussinesq approximation.

First we start with continuity

$$\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0$$

$$x = x^* \delta_T$$

$$y = y^* H$$

What's the characteristic velocity scale? We don't have a  $u_\infty$  to really look to here...

Okay, so maybe, it's easier to start with energy equations because this is our driving force! In forced convection, the driving force is sort of implicit in the momentum and mass conservation equations (freestream velocity is driving force).

$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \alpha \left( \frac{\partial^2}{\partial x^2} T + \frac{\partial^2}{\partial y^2} T \right)$$

So now, we can nondimensionalise T, x and y. u and v will remain.

Now combine this with the continuity equation we have two equations with two unknowns (order of magnitude so to speak).

$$u \frac{1}{\delta_T} \frac{\partial}{\partial x^*} T + v \frac{1}{H} \frac{\partial}{\partial y^*} T = \alpha \left( \frac{1}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} T + \frac{1}{H^2} \frac{\partial^2}{\partial (y^*)^2} T \right)$$

Now we can also nondimensionalise temperature.

$$\theta = \frac{T - T_\infty}{T_s - T_\infty}$$

$$(T_s - T_\infty)d\theta = d(T - T_\infty) = dT$$

$$u \frac{1}{\delta_T} \frac{\partial}{\partial x^*} T + v \frac{1}{H} \frac{\partial}{\partial y^*} T = \alpha \left( \frac{1}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} T + \frac{1}{H^2} \frac{\partial^2}{\partial (y^*)^2} T \right)$$

$$u \frac{1}{\delta_T} \frac{\partial}{\partial x^*} (T - T_\infty) + v \frac{1}{H} \frac{\partial}{\partial y^*} (T - T_\infty) = \alpha \left( \frac{1}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} (T - T_\infty) + \frac{1}{H^2} \frac{\partial^2}{\partial (y^*)^2} (T - T_\infty) \right)$$

$$u \frac{(T_s - T_\infty)}{\delta_T} \frac{\partial}{\partial x^*} \theta + v \frac{(T_s - T_\infty)}{H} \frac{\partial}{\partial y^*} \theta = \alpha \left( \frac{(T_s - T_\infty)}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} \theta + \frac{(T_s - T_\infty)}{H^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

Now since  $H \gg \delta_T$ , we can assume that

$$\left( \frac{(T_s - T_\infty)}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} \theta + \frac{(T_s - T_\infty)}{H^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right) \approx \left( \frac{(T_s - T_\infty)}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} \theta \right)$$

$$u \frac{(T_s - T_\infty)}{\delta_T} \frac{\partial}{\partial x^*} \theta + v \frac{(T_s - T_\infty)}{H} \frac{\partial}{\partial y^*} \theta = \alpha \left( \frac{(T_s - T_\infty)}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} \theta \right)$$

And since the nondimensionalised terms are all  $\mathcal{O}(1)$

$$\mathcal{O}\left(u \frac{(T_s - T_\infty)}{\delta_T}\right) + \mathcal{O}\left(v \frac{(T_s - T_\infty)}{H}\right) = \mathcal{O}\left(\alpha \frac{(T_s - T_\infty)}{\delta_T^2}\right)$$

So now we can work with the mass balance equations:

$$\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = 0$$

$$x = x^* \delta_T$$

$$y = y^* H$$

$$\frac{1}{\delta_T} \frac{\partial}{\partial x^*} u + \frac{1}{H} \frac{\partial}{\partial y^*} v = 0$$

$$\frac{\mathcal{O}(u)}{\delta_T} \frac{\partial}{\partial x^*} u^* + \frac{\mathcal{O}(v)}{H} \frac{\partial}{\partial y^*} v^* = 0$$

$$u = \mathcal{O}(u) u^*$$

$$\mathcal{O}\left(\frac{u}{\delta_T}\right) = \mathcal{O}\left(\frac{v}{H}\right)$$

Where u and v are typical velocity values in the BL. This is thermal BL  
btw

From this, we find that we cannot just throw out any of the terms here...  
However, we do know that the x and y velocity terms are on the same order  
of magnitude. We can then perform elimination to find:

$$\mathcal{O}\left(\frac{u}{\delta_T}\right) + \mathcal{O}\left(\frac{v}{H}\right) = \mathcal{O}\left(\frac{\alpha}{\delta_T^2}\right)$$

$$2\mathcal{O}\left(\frac{u}{\delta_T}\right) = \mathcal{O}\left(\frac{\alpha}{\delta_T^2}\right)$$

$$2\mathcal{O}(u) = \mathcal{O}\left(\frac{\alpha}{\delta_T}\right)$$

if i eliminate u,

$$2\mathcal{O}(v) = \mathcal{O}\left(\frac{\alpha H}{\delta_T^2}\right)$$

Of course we can easily get rid of the factor of 2, since it is still in the  
same order of magnitude

$$2 = \mathcal{O}(1)$$

$$\mathcal{O}(u) = \mathcal{O}\left(\frac{\alpha}{\delta_T}\right)$$

$$\mathcal{O}(v) = \mathcal{O}\left(\frac{\alpha H}{\delta_T^2}\right)$$

Now let's move on to the momentum equations



$$u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = \beta(T - T_\infty)g$$

Now here, we only nondimensionalise the position and velocity coordinates, for density and viscosity, while they are changing withing the fluid, we choose not to nondimensionalise as the changes are not as significant as the terms contributing to inertial forces, frictional forces and buoyancy forces.

The easiest is to start with the temperature.

$$u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = \beta(T_s - T_\infty)g\theta$$

Then the advection terms,

$$\mathcal{O}(u)\mathcal{O}(v) \frac{1}{\delta_T} u^* \frac{\partial}{\partial x^*} v^* + \mathcal{O}(v)\mathcal{O}(v) \frac{1}{H} v^* \frac{\partial}{\partial y^*} v^* - \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = \beta(T_s - T_\infty)g\theta$$

Then the frictional (viscous) forces

$$\mathcal{O}(u)\mathcal{O}(v) \frac{1}{\delta_T} u^* \frac{\partial}{\partial x^*} v^* + \mathcal{O}(v)\mathcal{O}(v) \frac{1}{H} v^* \frac{\partial}{\partial y^*} v^* - \nu \left( \frac{\mathcal{O}(v)}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{\mathcal{O}(v)}{H^2} \frac{\partial^2}{\partial (y^*)^2} v^* \right) = \beta(T_s - T_\infty)g\theta$$

We make an approximation:

$$\left( \frac{\mathcal{O}(v)}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{\mathcal{O}(v)}{H^2} \frac{\partial^2}{\partial (y^*)^2} v^* \right) \approx \left( \frac{\mathcal{O}(v)}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} v^* \right)$$

So we are left with:

$$\mathcal{O}(u)\mathcal{O}(v) \frac{1}{\delta_T} u^* \frac{\partial}{\partial x^*} v^* + \mathcal{O}(v)\mathcal{O}(v) \frac{1}{H} v^* \frac{\partial}{\partial y^*} v^* - \nu \left( \frac{\mathcal{O}(v)}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} v^* \right) = \beta(T_s - T_\infty)g\theta$$

$$\frac{\alpha}{\delta_T} \mathcal{O}(v) \frac{1}{\delta_T} u^* \frac{\partial}{\partial x^*} v^* + \mathcal{O}(v)\mathcal{O}(v) \frac{1}{H} v^* \frac{\partial}{\partial y^*} v^* - \nu \left( \frac{\mathcal{O}(v)}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} v^* \right) = \beta(T_s - T_\infty)g\theta$$

$$\frac{\alpha}{\delta_T} \frac{\alpha H}{\delta_T^2} \frac{1}{\delta_T} u^* \frac{\partial}{\partial x^*} v^* + \frac{\alpha H}{\delta_T^2} \frac{\alpha H}{\delta_T^2} \frac{1}{H} v^* \frac{\partial}{\partial y^*} v^* - \nu \left( \frac{\frac{\alpha H}{\delta_T^2}}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} v^* \right) = \beta(T_s - T_\infty)g\theta$$

$$\frac{\alpha^2 H}{\delta_T^4} u^* \frac{\partial}{\partial x^*} v^* + \frac{\alpha^2 H}{\delta_T^4} v^* \frac{\partial}{\partial y^*} v^* - \nu \left( \frac{\alpha H}{\delta_T^4} \frac{\partial^2}{\partial (x^*)^2} v^* \right) = \beta (T_s - T_\infty) g \theta$$

Viscous forces are always significant in laminar BL flow.  
Or in its dimensional form,

$$u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu \left( \frac{\partial^2}{\partial x^2} v \right) = \beta (T - T_\infty) g$$

Now if we want to nondimensionalise everything, we found the characteristic velocity scaling to be:

$$\mathcal{O}\left(\frac{u}{\delta_T}\right) = \mathcal{O}\left(\frac{v}{H}\right)$$

$$\mathcal{O}(u) = \mathcal{O}\left(\frac{\alpha}{\delta_T}\right)$$

$$\mathcal{O}(v) = \mathcal{O}\left(\frac{\alpha H}{\delta_T^2}\right)$$

Hence, we can use these to nondimensionalise the velocity scales

$$\mathcal{O}(u) \mathcal{O}(v) \frac{1}{\delta_T} u^* \frac{\partial}{\partial x^*} v^* + \mathcal{O}(v) \mathcal{O}(v) \frac{1}{H} v^* \frac{\partial}{\partial y^*} v^* - \nu \left( \frac{\mathcal{O}(v)}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} v^* \right) = \beta (T_s - T_\infty) g \theta$$

$$\frac{\alpha}{\delta_T} \frac{\alpha H}{\delta_T^2} \frac{1}{\delta_T} u^* \frac{\partial}{\partial x^*} v^* + \frac{\alpha H}{\delta_T^2} \frac{\alpha H}{\delta_T^2} \frac{1}{H} v^* \frac{\partial}{\partial y^*} v^* - \nu \left( \frac{\alpha H}{\delta_T^2} \frac{1}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} v^* \right) = \beta (T_s - T_\infty) g \theta$$

Now collecting terms together,

$$\frac{\alpha^2 H}{\delta_T^4} u^* \frac{\partial}{\partial x^*} v^* + \frac{\alpha^2 H}{\delta_T^4} v^* \frac{\partial}{\partial y^*} v^* - \nu \left( \frac{\alpha H}{\delta_T^4} \frac{\partial^2}{\partial (x^*)^2} v^* \right) = \beta (T_s - T_\infty) g \theta$$

we divide both sides by,  $\frac{\alpha^2 H}{\delta_T^4}$

$$u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* - \left( \frac{\nu}{\alpha} \frac{\partial^2}{\partial (x^*)^2} v^* \right) = \frac{\delta_T^4}{\alpha^2 H} \beta (T_s - T_\infty) g \theta$$

We introduce the prandtl number,

$$u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* - (Pr \frac{\partial^2}{\partial (x^*)^2} v^*) = \frac{\delta_T^4}{\alpha^2 H} \beta(T_s - T_\infty) g \theta$$

Also we notice the ratio  $\frac{\delta_T}{H}$  should be important, it describes the relationship between thermal BL thickness and wall height.

$$u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* - (Pr \frac{\partial^2}{\partial (x^*)^2} v^*) = \frac{\delta_T^4}{\alpha^2 H^4} \beta(T_s - T_\infty) H^3 g \theta$$

$$u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* - (Pr \frac{\partial^2}{\partial (x^*)^2} v^*) = \frac{\delta_T^4}{H^4} \frac{\beta(T_s - T_\infty) H^3 g}{\alpha^2} \theta$$

Divide throughout by Pr,

$$\frac{1}{Pr} \left[ u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] - \frac{\partial^2}{\partial (x^*)^2} v^* = \frac{\delta_T^4}{H^4} \frac{\beta(T_s - T_\infty) H^3 g}{\alpha^2} \frac{1}{Pr} \theta$$

$$\frac{1}{Pr} \left[ u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] - \frac{\partial^2}{\partial (x^*)^2} v^* = \frac{\delta_T^4}{H^4} \frac{\beta(T_s - T_\infty) H^3 g}{\alpha^2} \frac{\alpha}{\nu} \theta$$

$$\frac{1}{Pr} \left[ u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] - \frac{\partial^2}{\partial (x^*)^2} v^* = \frac{\delta_T^4}{H^4} \frac{\beta(T_s - T_\infty) H^3 g}{\alpha \nu} \theta$$

Here we see the occurrence of an important number:

$$Ra_H = \frac{\beta(T_s - T_\infty) H^3 g}{\alpha \nu}$$

This is called the Rayleigh number. Which is THE important term for natural convection, it is dependent on both thermal diffusivity and momentum diffusivity.

$$\frac{1}{Pr} \left[ u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] - \frac{\partial^2}{\partial (x^*)^2} v^* = \frac{\delta_T^4}{H^4} Ra_H \theta$$

It mismashes all the effects in one term. Both thermal and momentum diffusivity, so it's not quite like the Reynold's number.

We can see that Pr is important in whether we consider advection of momentum an important effect. In fact  $\frac{1}{Pr}$  is important in this equation.

We can remove these terms, but what are they small in comparison to? We need another term with  $\frac{1}{Pr}$  coefficient to compare it to.

Normally if Prandtl number is really big, and the Rayleigh number is big, this comparison is not needed. But if the prandtl number is small, we got a problem. So we'll need to do comparisons another way.

$$\frac{1}{Pr} \left[ u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] - \frac{\partial^2}{\partial (x^*)^2} v^* = \frac{\delta_T^4}{H^4} \frac{\beta(T_s - T_\infty) H^3 g}{\alpha \nu} \theta$$

note: Pr is very important, It describes the relationship between momentum BL thickness and thermal BL thickness

$$\frac{\delta_p}{\delta_t} = f(Pr)$$

If Pr low, (liquid metals) let's try this trick

$$\frac{1}{Pr} \left[ u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] - \frac{\partial^2}{\partial (x^*)^2} v^* = \frac{\delta_T^4}{H^4} \frac{\beta(T_s - T_\infty) H^3 g}{\alpha \nu} \frac{Pr}{Pr} \theta$$

$$\frac{1}{Pr} \left[ u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] - \frac{\partial^2}{\partial (x^*)^2} v^* = \frac{\delta_T^4}{H^4} \frac{\beta(T_s - T_\infty) H^3 g}{\alpha^2} \frac{1}{Pr} \theta$$

Only within thermal BL, advection for liquid metals or low Pr fluids, we can neglect momentum transport away from thermal BL through viscous action

$$\begin{aligned} \frac{1}{Pr} \left[ u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] &= \frac{\delta_T^4}{H^4} \frac{\beta(T_s - T_\infty) H^3 g}{\alpha^2} \frac{1}{Pr} \theta \\ \left[ u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] &= \frac{\delta_T^4}{H^4} \frac{\beta(T_s - T_\infty) H^3 g}{\alpha^2} \theta \end{aligned}$$

Now we find this new dimensionless number(Bejan, 2013):

$$Bo = \frac{\beta(T_s - T_\infty) H^3 g}{\alpha^2} = Ra_H Pr$$

This is called the boussinesq number. Important for fluids with low Pr, so that the viscous term can then be neglected.

$$u \frac{(T_s - T_\infty)}{\delta_T} \frac{\partial}{\partial x^*} \theta + v \frac{(T_s - T_\infty)}{H} \frac{\partial}{\partial y^*} \theta = \alpha \left( \frac{(T_s - T_\infty)}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} \theta \right)$$

## 9 solution procedure - integral

Now to solve the equation. First we always need to know what we are solving for, otherwise we will go in circles.

We have some questions of interest.

- What is the Nu (heat transfer coeff)?
- Velocity profile
- BL thickness of thermal and momentum BL

Remember, we may not want to solve the equation in totality, but rather go for special cases, eg. high or low Pr to make life easier for ourselves.

For example, in high Pr fluids, we want to just neglect the momentum advection terms, making life easier for ourselves.

### 9.1 High Pr fluids

In this case, very simply, we eliminate the momentum advection terms.

$$\begin{aligned} \frac{1}{Pr} \left[ u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] - \frac{\partial^2}{\partial (x^*)^2} v^* &= \frac{\delta_T^4}{H^4} Ra_H \theta \\ - \frac{\partial^2}{\partial (x^*)^2} v^* &= \frac{\delta_T^4}{H^4} Ra_H \theta \end{aligned}$$

From this we can see that:

$$\frac{\delta_T^4}{H^4} Ra_H = \mathcal{O}(1)$$

(remember all dimensionless terms are of the order of magnitude of 1)

$$\delta_T = \mathcal{O}(H Ra_H^{-\frac{1}{4}})$$

Another way to write this is:

$$\begin{aligned} \delta_T &= \mathcal{O}(1) H Ra_H^{-\frac{1}{4}} \\ \delta_T &= \mathcal{O}(1) H \left( \frac{\beta(T_s - T_\infty) H^3 g}{\alpha \nu} \right)^{-\frac{1}{4}} \\ \delta_T &= \mathcal{O}(1) H^{1/4} \left( \frac{\beta(T_s - T_\infty) g}{\alpha \nu} \right)^{-\frac{1}{4}} \end{aligned}$$

Based on this, we can guess the velocity profile. Given that

$$v = \mathcal{O}\left(\frac{aH}{\delta_T^2}\right)$$

$$v = \mathcal{O}(1) \frac{aH}{\delta_T^2}$$

...

$$v \sim H^{0.5}$$

What about the Nusselt number or heat transfer coefficient?

We need to then look at our energy equations:

$$u \frac{(T_s - T_\infty)}{\delta_T} \frac{\partial}{\partial x^*} \theta + v \frac{(T_s - T_\infty)}{H} \frac{\partial}{\partial y^*} \theta = \alpha \left( \frac{(T_s - T_\infty)}{\delta_T^2} \frac{\partial^2}{\partial (x^*)^2} \theta \right)$$

The heat transfer in the x direction is governed by,

$$\frac{\partial T}{\partial x} = \mathcal{O}\left(u \frac{(T_s - T_\infty)}{\delta_T}\right)$$

Recall:

$$u = \mathcal{O}\left(\frac{\alpha}{\delta_T}\right)$$

$$\frac{\partial T}{\partial x} = \mathcal{O}\left(\frac{\alpha}{\delta_T} \frac{(T_s - T_\infty)}{\delta_T}\right)$$

$$\frac{\partial T}{\partial x} = \mathcal{O}\left(\alpha \frac{(T_s - T_\infty)}{\delta_T^2}\right)$$

$$v = \mathcal{O}\left(\frac{aH}{\delta_T^2}\right)$$

$$\frac{\partial T}{\partial y} = \mathcal{O}(1) v \frac{(T_s - T_\infty)}{H}$$

$$\frac{\partial T}{\partial y} = \mathcal{O}(1) \frac{aH}{\delta_T^2} \frac{(T_s - T_\infty)}{H}$$

$$\frac{\partial T}{\partial y} = \mathcal{O}(1) \frac{a(T_s - T_\infty)}{\delta_T^2}$$

$$u^* \frac{\partial}{\partial x^*} \theta + v^* \frac{\partial}{\partial y^*} \theta = \frac{\partial^2}{\partial (x^*)^2} \theta$$

With this we can see how the temperature gradient scales as we go up the wall.

Why is all this important? Because we assume a velocity or temperature profile shape when we do integral solution analysis.

So how can we get a temperature profile across x? It's more or less qualitative. Just like for the original von karman solution, a polynomial solution was assumed.

We can try looking at the energy equations and come up with some qualitative shapes:

Consider the energy equation, hold y constant.

$$u^* \frac{\partial}{\partial x^*} \theta + v^* \frac{\partial}{\partial y^*} \theta = \frac{\partial^2}{\partial (x^*)^2} \theta$$

As we go further away from the wall, intuitively u and v increase. Or more correctly advection increases. Up to some certain maximum point, and then falls off.

For temperature, there is conduction effect but with some advection which effectively acts like a heat sink.

In Bejan's textbook, the recommended temp profile is(Bejan, 2013):

$$T - T_\infty = (T_s - T_\infty) \exp\left(-\frac{x}{\delta_T}\right)$$

$$\theta = \exp\left(-\frac{x}{\delta_T}\right)$$

If we were to substitute this back,

$$\frac{1}{Pr} \left[ u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] - \frac{\partial^2}{\partial (x^*)^2} v^* = \frac{\delta_T^4}{H^4} Ra_H \theta$$

$$-\frac{\partial^2}{\partial (x^*)^2} v^* = \frac{\delta_T^4}{H^4} Ra_H \theta$$

$$-\frac{\partial^2}{\partial (x^*)^2} v^* = \frac{\delta_T^4}{H^4} Ra_H \exp\left(-\frac{x}{\delta_T}\right)$$

$$-\frac{\partial^2}{\partial(x^*)^2}v^* = \mathcal{O}(1)\exp(-\frac{x}{\delta_T})$$

$$v^* = f(\exp(-\frac{x^*}{\delta_T}))$$

In Bejan's textbook, the recommended velocity profile is (Bejan, 2013):

$$v(x, y) = v_Y(y) \exp(-\frac{x}{\delta_p})(1 - \exp(-\frac{x}{\delta_T}))$$

Now this is just an estimate and assumption, based on the rough shapes of the BL, there are no exact reasoning why there must be exponential decay shapes.

BUT.... one can think of it this way, consider the control volume in the BL,

For exponential equations, we will have the form

$$\frac{dT}{dx} = -kT$$

There's conduction in and out (right term), and advection in and out (left terms)

along x axis, velocity is likely to go in negative x direction. Fluid flows up, has to come in from somewhere. Also will draw in cold fluid and release hot fluid at temperature T.

Along y axis, velocity goes in positive y direction, again, draw in cold fluid, release hot fluid at temperature T.

Basically, we do have a heat loss term in the form kT.

Note that there is a heat flux term

Now for integral solutions, we need to integrate momentum and energy equations across BL in x direction:

$$u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu (\frac{\partial^2}{\partial x^2} v) = \beta(T_s - T_\infty)g\theta$$

$$\int_0^{x_\infty} dx u \frac{\partial}{\partial x} v + \int_0^{x_\infty} dx v \frac{\partial}{\partial y} v - \int_0^{x_\infty} dx \nu (\frac{\partial^2}{\partial x^2} v) = \int_0^{x_\infty} dx \beta(T_s - T_\infty)g\theta$$

The choice of  $x_\infty$  is such that the conditions are close enough to freestream. Ie  $u = u_\infty = 0$ ,  $T = T_\infty$ .

We can do the same for energy equation



$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \alpha \left( \frac{\partial^2}{\partial x^2} T \right)$$

$$\int_0^{x_\infty} dx \, u \frac{\partial}{\partial x} T + \int_0^{x_\infty} dx \, v \frac{\partial}{\partial y} T = \int_0^{x_\infty} dx \, \alpha \left( \frac{\partial^2}{\partial x^2} T \right)$$

We can start by sorting out the non advection terms first, these are easier. We assume the thermal diffusivity, expansion and momentum diffusivity take on average values at film temperature  $T_f = \frac{T_s + T_\infty}{2}$

For the energy equation, we can combine back the terms:

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) - T \left( \frac{\partial}{\partial y} v + \frac{\partial}{\partial x} u \right) = \alpha \left( \frac{\partial^2}{\partial x^2} T \right)$$

Fortunately,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \alpha \left( \frac{\partial^2}{\partial x^2} T \right)$$

$$\int_0^{x_\infty} dx \, \frac{\partial}{\partial x}(uT) + \int_0^{x_\infty} dx \, \frac{\partial}{\partial y}(vT) = \int_0^{x_\infty} dx \, \alpha \left( \frac{\partial^2}{\partial x^2} T \right)$$

$$(uT)|_\infty - (uT)|_0 + \int_0^{x_\infty} dx \, \frac{\partial}{\partial y}(vT) = \int_0^{x_\infty} dx \, \alpha \left( \frac{\partial^2}{\partial x^2} T \right)$$

$$\int_0^{x_\infty} dx \, \frac{\partial}{\partial y}(vT) = \int_0^{x_\infty} dx \, \alpha \left( \frac{\partial^2}{\partial x^2} T \right)$$

$$\int_0^{x_\infty} dx \, \frac{\partial}{\partial y}(vT) = \alpha \left( \frac{\partial T}{\partial x} \Big|_\infty - \frac{\partial T}{\partial x} \Big|_0 \right)$$

$$\int_0^{x_\infty} dx \, \frac{\partial}{\partial y}(vT) = \alpha \left( - \frac{\partial T}{\partial x} \Big|_0 \right)$$

And we'll have to use Leibniz's rule again...

$$\frac{d}{dx} \left( \int_{y1=a(x)}^{y2=b(x)} f(x, y) dy \right) = f(x, y=b(x)) \frac{d}{dx} b(x) - f(x, y=a(x)) \frac{d}{dx} a(x) + \int_{y1=a(x)}^{y2=b(x)} \frac{\partial}{\partial x} f(x, y) dy$$

Note that our x and y is swapped around

$$\frac{d}{dy} \left( \int_{x1=a(y)}^{x2=b(y)} f(y, x) dx \right) = f(y, x = b(y)) \frac{d}{dy} b(y) - f(y, x = a(y)) \frac{d}{dy} a(y) + \int_{x1=a(y)}^{x2=b(y)} \frac{\partial}{\partial y} f(y, x) dx$$

$$x1 = 0 ; a(y) = 0$$

$$x2 = \delta_p(y) ; b(y) = \delta_p$$

$$f(y, x) = vT$$

Substitute into leibniz's rule:

$$\frac{d}{dy} \left( \int_{x1=0}^{x2=\delta_p(y)} vT dx \right) = vT(y, x = \delta_p(y)) \frac{d}{dy} \delta_p(y) - vT(y, x = 0) \frac{d}{dy} 0 + \int_{x1=0}^{x2=\delta_p(y)} \frac{\partial}{\partial y} vT dx$$

$$\frac{d}{dy} \left( \int_{x1=0}^{x2=\delta_p(y)} vT dx \right) = vT(y, x = \delta_p(y)) \frac{d}{dy} \delta_p(y) + \int_{x1=0}^{x2=\delta_p(y)} \frac{\partial}{\partial y} vT dx$$

$$v = 0 \text{ at } \delta_p$$

$$\frac{d}{dy} \left( \int_{x1=0}^{x2=\delta_p(y)} vT dx \right) = \int_{x1=0}^{x2=\delta_p(y)} \frac{\partial}{\partial y} vT dx$$

Change some variables to make it easier to nondimensionalise

$$\frac{d}{dy} \left( \int_{x1=0}^{x2=\delta_p(y)} v(T - T_\infty) dx \right) = \int_{x1=0}^{x2=\delta_p(y)} \frac{\partial}{\partial y} v(T - T_\infty) dx = \int_{x1=0}^{x2=\delta_p(y)} \frac{\partial}{\partial y} vT dx$$

subs back in energy equation:

$$\int_0^{x_\infty} dx \frac{\partial}{\partial y} (vT) = \alpha \left( -\frac{\partial T}{\partial x} \Big|_0 \right)$$

$$\frac{d}{dy} \left( \int_{x1=0}^{x2=\delta_p(y)} v(T - T_\infty) dx \right) = \alpha \left( -\frac{\partial T}{\partial x} \Big|_0 \right)$$

$$\frac{d}{dy} \left( \int_0^{\delta_p(y)} v(T_\infty - T) dx \right) = \alpha \frac{\partial T}{\partial x} \Big|_{x=0}$$

## Part VII

# Resources Online

For Momentum BLs:

[http://web.mit.edu/fluids-modules/www/highspeed\\_flows/ver2/bl\\_Chap2.pdf](http://web.mit.edu/fluids-modules/www/highspeed_flows/ver2/bl_Chap2.pdf)

<https://community.dur.ac.uk/suzanne.fielding/teaching/BLT/sec3.pdf>

for Von Karman

[https://nptel.ac.in/content/storage2/courses/112104118/lecture-29/29-3\\_momentum.htm](https://nptel.ac.in/content/storage2/courses/112104118/lecture-29/29-3_momentum.htm)

For Thermal BLs:

<http://mech.sut.ac.ir/People/Courses/18/Chapter3-%20Part2.pdf>

[http://raops.org.in/epapers/june15\\_9.pdf](http://raops.org.in/epapers/june15_9.pdf)

<https://nptel.ac.in/content/storage2/courses/112101001/downloads/lec25.pdf>

## Part VIII

# Github Repo

[https://github.com/theodoreOnzGit/heatTransferTheory\\_YouTube](https://github.com/theodoreOnzGit/heatTransferTheory_YouTube)

Look under convection heat transfer...

## Part IX

# Bibliography

## References

Bejan, A. (2013). *Convection heat transfer*. John Wiley & sons.

Welty, J., Rorrer, G. L., & Foster, D. G. (2014). *Fundamentals of momentum, heat, and mass transfer*. John Wiley & Sons.