

# Fluid Mechanics YouTube

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February 26, 2021

## Part I

# Navier Stokes Equations

Compressible N-S equations

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \nabla \bullet (\rho \vec{u} \otimes \vec{u}) = -\nabla p + \mu \nabla^2 \vec{u} + \frac{1}{3} \mu \nabla (\nabla \bullet \vec{u}) + \rho \vec{g}$$

tensor or outer product:

$$\vec{u} \otimes \vec{v} = \vec{u} \vec{v}^T$$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \otimes \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

$$= \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}$$

Inner product

$$\vec{u} \bullet \vec{v} = (\vec{u}^T \vec{v})^T$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \bullet \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3$$

Assume incompressible flow:

$$\rho = \text{constant}$$

continuity equation

$$\nabla \bullet \vec{u} = 0$$

Incompressible N-S equations

$$\frac{\partial}{\partial t} \vec{u} + (\vec{u} \bullet \nabla) \vec{u} - \nu \nabla^2 \vec{u} = -\nabla \frac{P}{\rho_0} + \vec{g}$$

[https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes\\_equations](https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations)

Matrices in LaTeX

<https://www.overleaf.com/learn/latex/Matrices>

Tensors in LaTeX

Navier Stokes Equations

<https://www.comsol.com/multiphysics/navier-stokes-equations>

[https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes\\_equations](https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations)

Github

[https://github.com/theodoreOnzGit/heatTransferTheory\\_YouTube](https://github.com/theodoreOnzGit/heatTransferTheory_YouTube)

First let's deal with:

$$(\vec{u} \bullet \nabla) \vec{u}$$

$$\begin{aligned} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \otimes \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} &= \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial}{\partial x} u_1 & \frac{\partial}{\partial x} u_2 & \frac{\partial}{\partial x} u_3 \\ \frac{\partial}{\partial y} u_1 & \frac{\partial}{\partial y} u_2 & \frac{\partial}{\partial y} u_3 \\ \frac{\partial}{\partial z} u_1 & \frac{\partial}{\partial z} u_2 & \frac{\partial}{\partial z} u_3 \end{pmatrix} \end{aligned} \quad (1)$$

Then we do inner product

$$\begin{aligned}
& \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial}{\partial x} u_1 & \frac{\partial}{\partial x} u_2 & \frac{\partial}{\partial x} u_3 \\ \frac{\partial}{\partial y} u_1 & \frac{\partial}{\partial y} u_2 & \frac{\partial}{\partial y} u_3 \\ \frac{\partial}{\partial z} u_1 & \frac{\partial}{\partial z} u_2 & \frac{\partial}{\partial z} u_3 \end{pmatrix} \\
& (u_1 \quad u_2 \quad u_3) \begin{pmatrix} \frac{\partial}{\partial x} u_1 & \frac{\partial}{\partial x} u_2 & \frac{\partial}{\partial x} u_3 \\ \frac{\partial}{\partial y} u_1 & \frac{\partial}{\partial y} u_2 & \frac{\partial}{\partial y} u_3 \\ \frac{\partial}{\partial z} u_1 & \frac{\partial}{\partial z} u_2 & \frac{\partial}{\partial z} u_3 \end{pmatrix} \\
& = \begin{pmatrix} u_1 \frac{\partial}{\partial x} u_1 + u_2 \frac{\partial}{\partial y} u_1 + u_3 \frac{\partial}{\partial z} u_1 \\ u_1 \frac{\partial}{\partial x} u_2 + u_2 \frac{\partial}{\partial y} u_2 + u_3 \frac{\partial}{\partial z} u_2 \\ u_1 \frac{\partial}{\partial x} u_3 + u_2 \frac{\partial}{\partial y} u_3 + u_3 \frac{\partial}{\partial z} u_3 \end{pmatrix}
\end{aligned}$$

Let's deal with the momentum diffusivity (kinematic viscosity) term:

$$\nabla^2 = (\nabla \bullet \nabla)$$

$$\begin{aligned}
& \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial}{\partial x} u_1 & \frac{\partial}{\partial x} u_2 & \frac{\partial}{\partial x} u_3 \\ \frac{\partial}{\partial y} u_1 & \frac{\partial}{\partial y} u_2 & \frac{\partial}{\partial y} u_3 \\ \frac{\partial}{\partial z} u_1 & \frac{\partial}{\partial z} u_2 & \frac{\partial}{\partial z} u_3 \end{pmatrix} \\
& \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} u_1 & \frac{\partial}{\partial x} u_2 & \frac{\partial}{\partial x} u_3 \\ \frac{\partial}{\partial y} u_1 & \frac{\partial}{\partial y} u_2 & \frac{\partial}{\partial y} u_3 \\ \frac{\partial}{\partial z} u_1 & \frac{\partial}{\partial z} u_2 & \frac{\partial}{\partial z} u_3 \end{pmatrix} \\
& = \begin{pmatrix} \frac{\partial}{\partial x} \frac{\partial}{\partial x} u_1 + \frac{\partial}{\partial y} \frac{\partial}{\partial y} u_1 + \frac{\partial}{\partial z} \frac{\partial}{\partial z} u_1 \\ \frac{\partial}{\partial x} \frac{\partial}{\partial x} u_2 + \frac{\partial}{\partial y} \frac{\partial}{\partial y} u_2 + \frac{\partial}{\partial z} \frac{\partial}{\partial z} u_2 \\ \frac{\partial}{\partial x} \frac{\partial}{\partial x} u_3 + \frac{\partial}{\partial y} \frac{\partial}{\partial y} u_3 + \frac{\partial}{\partial z} \frac{\partial}{\partial z} u_3 \end{pmatrix}
\end{aligned}$$

## Part II

# Boundary Layer Equations

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u + w \frac{\partial}{\partial z} u - \nu \left( \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u + \frac{\partial^2}{\partial z^2} u \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t} v + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v + w \frac{\partial}{\partial z} v - \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v + \frac{\partial^2}{\partial z^2} v \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + g_y$$

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - \nu(\frac{\partial^2}{\partial x^2}w + \frac{\partial^2}{\partial y^2}w + \frac{\partial^2}{\partial z^2}w) = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + g_z$$

Now for 2D what do we do?  $w=0$  everywhere and at all times,  $g_z = 0$   
We eliminate  $z$  terms from the  $x$  and  $y$  momentum balance

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u + \frac{\partial^2}{\partial z^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v + \frac{\partial^2}{\partial z^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

There is no spatial variation in  $u$  and  $v$  w.r.t  $z$  We have 2D Navier stokes:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

continuity equation

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

## 1 nondimensionalisation

Order of magnitude

$$\mathcal{O}$$

Scaling for order magnitude comparison

$$u^*, y^* = \mathcal{O}(1)$$

we define:

$$u^* = \frac{u}{u_\infty}$$

$$u = u_\infty u^*$$

$$x^* = \frac{x}{L}$$

$$y^* = \frac{y}{\delta_p}$$

We scale our continuity equation:

$$\frac{\partial}{\partial x^* L} u^* u_\infty + \frac{\partial}{\partial y^* \delta_p} v = 0$$

$$\frac{\partial}{\partial x^* L} u^* u_\infty + \frac{\partial}{\partial y^* \delta_p} v = 0$$

$$\frac{u_\infty}{L} \frac{\partial}{\partial x^*} u^* + \frac{1}{\delta_p} \frac{\partial}{\partial y^*} v = 0$$

$$\frac{\partial}{\partial x^*} u^* + \frac{L}{\delta_p u_\infty} \frac{\partial}{\partial y^*} v = 0$$

$$v^* = \frac{v L}{\delta_p u_\infty} = \mathcal{O}(1)$$

$$v^* = \frac{v}{\frac{u_\infty \delta_p}{L}}$$

$$\frac{\partial}{\partial x^*} u^* + \frac{\partial}{\partial y^*} v^* = 0$$

Now we move on to the NS equations

so we need to scale time:

x lengthscale = L

x velocityscale =  $u_\infty$

timescale =  $\frac{L}{u_\infty}$

$$t^* = \frac{t}{\frac{L}{u_\infty}}$$

Let's scale the momentum NS equations

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u - \nu \left( \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t} v + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + g_y$$

Let's do x momentum equations

$$\frac{\partial}{\partial t} u^* u_\infty + u^* u_\infty \frac{\partial}{\partial x^*} L u^* u_\infty + v \frac{\partial}{\partial y} u^* u_\infty - \nu \left( \frac{\partial^2}{\partial x^2} u^* u_\infty + \frac{\partial^2}{\partial y^2} u^* u_\infty \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t} u^* + u^* \frac{u_\infty}{L} \frac{\partial}{\partial x^*} u^* + v \frac{1}{\delta_p} \frac{\partial}{\partial y^*} u^* - \nu \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{1}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* \right) = \frac{1}{u_\infty} \left( -\frac{1}{L \rho_0} \frac{\partial P}{\partial x^*} + g_x \right)$$

$$\frac{u_\infty}{L} \frac{\partial}{\partial t^*} u^* + u^* \frac{u_\infty}{L} \frac{\partial}{\partial x^*} u^* + v \frac{1}{\delta_p} \frac{\partial}{\partial y^*} u^* - \nu \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{1}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* \right) = \frac{1}{u_\infty} \left( -\frac{1}{L \rho_0} \frac{\partial P}{\partial x^*} + g_x \right)$$

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{\nu}{u_\infty L} \left( \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* \right) = \frac{L}{u_\infty^2} \left( -\frac{1}{L \rho_0} \frac{\partial P}{\partial x^*} + g_x \right)$$

Reynold's number

$$Re_L = \frac{u_\infty L}{\nu}$$

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \left( \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* \right) = \frac{L}{u_\infty^2} \left( -\frac{1}{L \rho_0} \frac{\partial P}{\partial x^*} + g_x \right)$$

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \left( \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* \right) = \left( -\frac{1}{L \rho_0} \frac{L}{u_\infty^2} \frac{\partial P}{\partial x^*} + \frac{L}{u_\infty^2} g_x \right)$$

Let's scale gravity

$$g_x^* = \frac{g_x}{|g|} = \cos \theta_x = \mathcal{O}(1)$$

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \left( \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* \right) = \left( -\frac{1}{\rho_0 u_\infty^2} \frac{\partial P}{\partial x^*} + \frac{L|g|}{u_\infty^2} g_x^* \right)$$

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \left( \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* \right) = \left( -\frac{1}{\rho_0 u_\infty^2} \frac{\partial P}{\partial x^*} + \frac{1}{Fr^2} g_x^* \right)$$

$$P^* = \frac{P}{\rho_0 u_\infty^2}$$

After nondimensionalisation, our x momentum equation becomes:

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \left( \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* \right) = \left( -\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2} g_x^* \right)$$

we dimensionalise y momentum eqns

$$\frac{\partial}{\partial t} v + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu \left( \frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + g_y$$

First the x and time terms:

$$\frac{u_\infty}{L} \frac{\partial}{\partial t^*} v + u^* \frac{u_\infty}{L} \frac{\partial}{\partial x^*} v + v \frac{\partial}{\partial y} v - \nu \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2} v + \frac{\partial^2}{\partial y^2} v \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + g_y$$

Second y coordinate terms:

$$\frac{u_\infty}{L} \frac{\partial}{\partial t^*} v + u^* \frac{u_\infty}{L} \frac{\partial}{\partial x^*} v + v \frac{1}{\delta_p} \frac{\partial}{\partial y^*} v - \nu \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2} v + \frac{1}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} v \right) = -\frac{1}{\rho_0} \frac{1}{\delta_p} \frac{\partial P}{\partial y^*} + g_y^* |g|$$

divide by  $\frac{u_\infty^2}{L^2}$

$$\frac{L}{u_\infty} \frac{\partial}{\partial t^*} v + u^* \frac{L}{u_\infty} \frac{\partial}{\partial x^*} v + v \frac{L^2}{u_\infty^2 \delta_p} \frac{\partial}{\partial y^*} v - \nu \frac{L^2}{u_\infty^2} \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2} v + \frac{1}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} v \right) = \frac{L^2}{u_\infty^2} \left( -\frac{1}{\rho_0} \frac{1}{\delta_p} \frac{\partial P}{\partial y^*} + g_y^* |g| \right)$$

divide by  $\delta_p$

$$\frac{L}{u_\infty \delta_p} \frac{\partial}{\partial t^*} v + u^* \frac{L}{u_\infty \delta_p} \frac{\partial}{\partial x^*} v + v \frac{L^2}{u_\infty^2 \delta_p^2} \frac{\partial}{\partial y^*} v - \nu \frac{L^2}{u_\infty^2 \delta_p} \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2} v + \frac{1}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} v \right) = \frac{L^2}{u_\infty^2 \delta_p} \left( -\frac{1}{\rho_0} \frac{1}{\delta_p} \frac{\partial P}{\partial y^*} + g_y^* |g| \right)$$

Combining some terms

$$\frac{\partial}{\partial t^*} v^* + u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* - \nu \frac{L}{u_\infty} \left( \frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{1}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} v^* \right) = \frac{L^2}{u_\infty^2 \delta_p} \left( -\frac{1}{\rho_0} \frac{1}{\delta_p} \frac{\partial P}{\partial y^*} + g_y^* |g| \right)$$

Rearranging

$$\frac{\partial}{\partial t^*} v^* + u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* - \frac{\nu}{u_\infty L} \left( \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} v^* \right) = \left( -\frac{1}{\rho_0} \frac{L^2}{u_\infty^2 \delta_p} \frac{1}{\delta_p} \frac{\partial P}{\partial y^*} + g_y^* |g| \frac{L^2}{u_\infty^2 \delta_p} \right)$$

nondimensionalisng pressure and including the Fr

$$\frac{\partial}{\partial t^*} v^* + u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* - \frac{\nu}{u_\infty L} \left( \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} v^* \right) = \left( -\frac{L^2}{\delta_p^2} \frac{\partial P^*}{\partial y^*} + g_y^* \frac{1}{Fr^2} \frac{L}{\delta_p} \right)$$

Include Re

$$\frac{\partial}{\partial t^*} v^* + u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* - \frac{1}{Re_L} \left( \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} v^* \right) = \left( -\frac{L^2}{\delta_p^2} \frac{\partial P^*}{\partial y^*} + g_y^* \frac{1}{Fr^2} \frac{L}{\delta_p} \right)$$

Review: NS nondimensionalised

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \left( \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* \right) = \left( -\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2} g_x^* \right)$$

$$\frac{\partial}{\partial t^*} v^* + u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* - \frac{1}{Re_L} \left( \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} v^* \right) = \left( -\frac{L^2}{\delta_p^2} \frac{\partial P^*}{\partial y^*} + g_y^* \frac{1}{Fr^2} \frac{L}{\delta_p} \right)$$

$$\frac{\partial}{\partial x^*} u^* + \frac{\partial}{\partial y^*} v^* = 0$$

## 2 How to drop terms?

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \left( \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* \right) = \left( -\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2} g_x^* \right)$$

$$\frac{\partial}{\partial t^*} v^* + u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* - \frac{1}{Re_L} \left( \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} v^* \right) = \left( -\frac{L^2}{\delta_p^2} \frac{\partial P^*}{\partial y^*} + g_y^* \frac{1}{Fr^2} \frac{L}{\delta_p} \right)$$

$$\frac{\partial}{\partial x^*} u^* + \frac{\partial}{\partial y^*} v^* = 0$$

When we want to determine which terms to cancel, we need to know how  $Re_L$  compares with  $\frac{L^2}{\delta_p^2}$

Assumption:

Creeping flow in y direction

$$Re_\delta = \frac{v_c \delta_p}{\nu} = \mathcal{O}(1)$$



How does  $Re_\delta$  compare to  $Re_L$

$$v_c = u_\infty \frac{\delta_p}{L}$$

$$Re_\delta = \frac{u_\infty \frac{\delta_p}{L} \delta_p}{\nu} = \mathcal{O}(1)$$

$$Re_\delta = \frac{u_\infty L}{\nu} \frac{\delta_p^2}{L^2} = \mathcal{O}(1)$$

$$Re_\delta = Re_L \frac{\delta_p^2}{L^2} = \mathcal{O}(1)$$

### 2.0.1 x direction momentum eqn

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \left( \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* \right) = \left( -\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2} g_x^* \right)$$

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{1}{Re_L} \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* = \left( -\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2} g_x^* \right)$$

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{1}{\mathcal{O}(1)} \frac{\partial^2}{\partial (y^*)^2} u^* = \left( -\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2} g_x^* \right)$$

How big is  $Re_L$ ?

$$Re_L = \mathcal{O}\left(\frac{L^2}{\delta_p^2}\right)$$

We assume  $Fr$  is big

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{\mathcal{O}(1)} \frac{\partial^2}{\partial (y^*)^2} u^* = \left( -\frac{\partial P^*}{\partial x^*} \right)$$

### 2.0.2 y direction momentum equation

$$\frac{\partial}{\partial t^*} v^* + u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* - \frac{1}{Re_L} \left( \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} v^* \right) = \left( -\frac{L^2}{\delta_p^2} \frac{\partial P^*}{\partial y^*} + g_y^* \frac{1}{Fr^2} \frac{L}{\delta_p} \right)$$

$$\frac{\partial}{\partial t^*} v^* + u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* - \left( \frac{1}{Re_L} \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{1}{\mathcal{O}(1)} \frac{\partial^2}{\partial (y^*)^2} v^* \right) = \left( -\frac{L^2}{\delta_p^2} \frac{\partial P^*}{\partial y^*} + g_y^* \frac{1}{Fr^2} \frac{L}{\delta_p} \right)$$

$$\left[ \frac{\partial}{\partial t^*} v^* + u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] \frac{1}{Re_L} - \left( \frac{1}{Re_L^2} \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{1}{\mathcal{O}(1) Re_L} \frac{\partial^2}{\partial (y^*)^2} v^* \right) = \frac{1}{Re_L} \left( -\frac{L^2}{\delta_p^2} \frac{\partial P^*}{\partial y^*} + g_y^* \frac{1}{Fr^2} \frac{L}{\delta_p} \right)$$

$$\begin{aligned} & \left[ \frac{\partial}{\partial t^*} v^* + u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] \frac{1}{Re_L} - \left( \frac{1}{Re_L^2} \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{1}{\mathcal{O}(1) Re_L} \frac{\partial^2}{\partial (y^*)^2} v^* \right) \\ &= \left( -\frac{L^2}{\delta_p^2} \frac{1}{Re_L} \frac{\partial P^*}{\partial y^*} + g_y^* \frac{1}{Fr^2} \frac{L}{\delta_p} \frac{1}{Re_L} \right) \end{aligned}$$

Cancelling out...

$$\begin{aligned} & \left[ \frac{\partial}{\partial t^*} v^* + u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] \frac{1}{Re_L} - \left( \frac{1}{Re_L^2} \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{1}{\mathcal{O}(1) Re_L} \frac{\partial^2}{\partial (y^*)^2} v^* \right) \\ &= \left( -\frac{1}{\mathcal{O}(1)} \frac{\partial P^*}{\partial y^*} + g_y^* \frac{1}{Fr^2} \frac{L^2}{\delta_p^2} \frac{1}{Re_L} \frac{\delta_p}{L} \right) \end{aligned}$$

Simplifying

$$\begin{aligned} & \left[ \frac{\partial}{\partial t^*} v^* + u^* \frac{\partial}{\partial x^*} v^* + v^* \frac{\partial}{\partial y^*} v^* \right] \frac{1}{Re_L} - \left( \frac{1}{Re_L^2} \frac{\partial^2}{\partial (x^*)^2} v^* + \frac{1}{\mathcal{O}(1) Re_L} \frac{\partial^2}{\partial (y^*)^2} v^* \right) \\ &= \frac{1}{\mathcal{O}(1)} \left( -\frac{\partial P^*}{\partial y^*} + g_y^* \frac{1}{Fr^2} \frac{\delta_p}{L} \right) \end{aligned}$$

For large  $Re_L$

$$\begin{aligned} 0 &= \left( -\frac{\partial P^*}{\partial y^*} + g_y^* \frac{1}{Fr^2} \frac{\delta_p}{L} \right) \\ g_y^* \frac{1}{Fr^2} \frac{\delta_p}{L} &= \frac{\partial P^*}{\partial y^*} \end{aligned}$$

Only if  $g=0$ ,

$$0 = -\frac{\partial P^*}{\partial y^*}$$

Now we have our BL equations:

$$0 = -\frac{\partial P^*}{\partial y^*}$$

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{\mathcal{O}(1)} \frac{\partial^2}{\partial (y^*)^2} u^* = \left(-\frac{\partial P^*}{\partial x^*}\right)$$

redimensionalise to obtain the laminar BL equations:

$$0 = -\frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u - \nu \frac{\partial^2}{\partial (y)^2} u = \left(-\frac{\partial P}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

### 3 Solutions to the BL equations laminar

How to solve?

- 1st Similarity solution
- 2nd Von Karman Solution (Integral solution - approximate)
- 3rd numerical (CFD)

#### 3.1 similarity solution (Blasius solution)

$$0 = -\frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u - \nu \frac{\partial^2}{\partial (y)^2} u = \left(-\frac{\partial P}{\partial x}\right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Similarity solution  $\rightarrow$  combine variables to convert PDE to ODE

Making 2 assumptions before we continue:

1) steady state 2) no pressure gradient

$$u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u - \nu \frac{\partial^2}{\partial y^2} u = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

How to combine variables to make life easier for us to solve?

introduce the streamfunction ( $\psi$ ):

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

Note: streamfunction only works for 2D fluid flow

Substitute into 2D continuity equation,

$$\frac{\partial}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y} \frac{\partial \psi}{\partial x} = 0$$

Substitute into the 2D x momentum equation

$$\left(\frac{\partial \psi}{\partial y}\right) \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y}\right) + \left(-\frac{\partial \psi}{\partial x}\right) \frac{\partial}{\partial y} \left(\frac{\partial \psi}{\partial y}\right) - \nu \frac{\partial^2}{\partial y^2} \left(\frac{\partial \psi}{\partial y}\right) = 0$$

$$\left(\frac{\partial \psi}{\partial y}\right) \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y}\right) - \left(\frac{\partial \psi}{\partial x}\right) \left(\frac{\partial^2 \psi}{\partial y^2}\right) - \nu \frac{\partial^3}{\partial y^3} \psi = 0$$

We need to compress the number of variables further to 1 indep variable ( $\eta$ ) and 1 dependent variable ( $f(\eta)$ )

$$\eta = \eta(x, y)$$

$$f = f(\eta)$$

Before we continue, BCs first!

1 BC in x dir for u

$$x = 0, y \neq 0, u = u_\infty$$

2 BCs in y dir for u

no slip

$$u = 0 \text{ at } y = 0$$

$$y \rightarrow \infty; u \rightarrow u_\infty$$

1 BC in y direction for v

no slip

$$v = 0 \text{ at } y = 0$$

### 3.1.1 similarity transform

How do we start to get these "combo parameters" aka similarity variables?

<https://ntrs.nasa.gov/citations/20050028493>

Based on Blasius's paper (translated by NACA) it's good to nondimensionalise to find these similarity variables

$$\psi^* = \frac{\psi}{\psi_0}$$

Let's nondimensionalise the momentum equations:

$$\left(\frac{\partial \psi}{\partial y}\right) \frac{\partial}{\partial x} \left(\frac{\partial \psi}{\partial y}\right) - \left(\frac{\partial \psi}{\partial x}\right) \left(\frac{\partial^2 \psi}{\partial y^2}\right) - \nu \frac{\partial^3 \psi}{\partial y^3} = 0$$

$$\frac{\psi_0^2}{\delta_p^2 L} \left(\frac{\partial \psi^*}{\partial y^*}\right) \frac{\partial}{\partial x^*} \left(\frac{\partial \psi^*}{\partial y^*}\right) - \frac{\psi_0^2}{L \delta_p^2} \left(\frac{\partial \psi^*}{\partial x^*}\right) \left(\frac{\partial^2 \psi^*}{\partial (y^*)^2}\right) - \frac{\nu \psi_0}{\delta_p^3} \frac{\partial^3 \psi^*}{\partial (y^*)^3} = 0$$

$$\left(\frac{\partial \psi^*}{\partial y^*}\right) \frac{\partial}{\partial x^*} \left(\frac{\partial \psi^*}{\partial y^*}\right) - \left(\frac{\partial \psi^*}{\partial x^*}\right) \left(\frac{\partial^2 \psi^*}{\partial (y^*)^2}\right) - \frac{\nu \psi_0}{\delta_p^3} \frac{\delta_p^2 L}{\psi_0^2} \frac{\partial^3 \psi^*}{\partial (y^*)^3} = 0$$

$$\left(\frac{\partial \psi^*}{\partial y^*}\right) \frac{\partial}{\partial x^*} \left(\frac{\partial \psi^*}{\partial y^*}\right) - \left(\frac{\partial \psi^*}{\partial x^*}\right) \left(\frac{\partial^2 \psi^*}{\partial (y^*)^2}\right) - \frac{\nu}{\delta_p} \frac{L}{\psi_0} \frac{\partial^3 \psi^*}{\partial (y^*)^3} = 0$$

dimensionless group:

$$\frac{\nu L}{\delta_p \psi_0} = \mathcal{O}(1)$$

We want the equations to be nondimensionalised exactly.

If we want the equations looks like:

$$\left(\frac{\partial \psi^*}{\partial y^*}\right) \frac{\partial}{\partial x^*} \left(\frac{\partial \psi^*}{\partial y^*}\right) - \left(\frac{\partial \psi^*}{\partial x^*}\right) \left(\frac{\partial^2 \psi^*}{\partial (y^*)^2}\right) - \frac{\partial^3 \psi^*}{\partial (y^*)^3} = 0$$

$$\frac{\nu L}{\delta_p \psi_0} = 1$$

$$\psi_0 = \frac{\nu L}{\delta_p}$$

Otherwise:

$$\psi_0 = \mathcal{O}(1) \frac{\nu L}{\delta_p}$$

How do we get rid of dependent variables  $\delta_p$ ?

$$Re_\delta = \frac{v_c \delta_p}{\nu} = \mathcal{O}(1)$$

$$Re_\delta = \frac{u_\infty L}{\nu} \frac{\delta_p^2}{L^2} = \mathcal{O}(1)$$

From continuity equation

$$v_c = \frac{u_\infty \delta_p}{L}$$

(not too helpful)

What's helpful is to use the physics of the BL ie creeping flow in BL

The other assumption:

$$Re_\delta = 1$$

$$Re_\delta = \frac{u_\infty}{\nu L} \delta_p^2$$

$$\delta_p = \sqrt{Re_\delta} \sqrt{\frac{\nu L}{u_\infty}}$$

Substitute back:

$$\psi_0 = \frac{\nu L}{\delta_p} = \frac{\nu L}{\sqrt{Re_\delta} \sqrt{\frac{\nu L}{u_\infty}}}$$

$$\psi_0 = \frac{\nu L}{\sqrt{\frac{\nu L}{u_\infty}}} \frac{1}{\sqrt{Re_\delta}}$$

$$\psi_0 = \sqrt{u_\infty L \nu} \frac{1}{\sqrt{Re_\delta}}$$

Let's see our nondimensionalised streamfunction:

$$\psi^* = \frac{\psi}{\psi_0} = \sqrt{Re_\delta} \frac{\psi}{\sqrt{u_\infty L \nu}}$$

If  $Re_\delta = 1$

$$\psi^* = \frac{\psi}{\psi_0} = \frac{\psi}{\sqrt{u_\infty L \nu}}$$

What about our independent variable? We need to combine the x and y coordinate variables  
we'll use

$$Re_\delta = \frac{u_\infty}{\nu L} \delta_p^2$$

$$\delta_p = \sqrt{Re_\delta} \sqrt{\frac{\nu L}{u_\infty}}$$

If we want x and y explicitly,

$$x = x^* L$$

$$y = y^* \delta_p$$

$$\frac{y}{y^*} = \sqrt{Re_\delta} \sqrt{\frac{\nu \frac{x}{x^*}}{u_\infty}}$$

$$y = \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} \sqrt{\frac{\nu x}{u_\infty}}$$

$$y = \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} \sqrt{\frac{\nu x}{u_\infty}}$$

$$y \sqrt{\frac{u_\infty}{\nu x}} = \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} = \mathcal{O}(1)$$

And we have dimensionless stream function

$$\psi^* = \frac{\psi}{\psi_0} = \frac{\psi}{\sqrt{u_\infty L \nu}}$$

$$\psi^* = \frac{\psi}{\sqrt{u_\infty \frac{x}{x^*} \nu}}$$

$$\psi^* = \sqrt{x^*} \frac{\psi}{\sqrt{u_\infty x \nu}}$$

If  $Re_\delta \neq 1$

$$\psi^* = \frac{\psi}{\psi_0} = \sqrt{Re_\delta} \frac{\psi}{\sqrt{u_\infty L \nu}}$$

$$\psi^* = \frac{\psi}{\psi_0} = \sqrt{Re_\delta x^*} \frac{\psi}{\sqrt{u_\infty x \nu}}$$

from Blasius's paper:

$$\eta = \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} = \frac{1}{2} y \sqrt{\frac{u_\infty}{\nu x}}$$

$$f(\eta) = \frac{\psi^*}{\sqrt{x^* Re_\delta}} = \frac{\psi}{\sqrt{u_\infty x \nu}}$$

We start transforming variables

$$\left(\frac{\partial \psi^*}{\partial y^*}\right) \frac{\partial}{\partial x^*} \left(\frac{\partial \psi^*}{\partial y^*}\right) - \left(\frac{\partial \psi^*}{\partial x^*}\right) \left(\frac{\partial^2 \psi^*}{\partial (y^*)^2}\right) - \frac{\partial^3}{\partial (y^*)^3} \psi^* = 0$$

Use chain rule

$$\begin{aligned} \frac{\partial \psi^*}{\partial y^*} &= \frac{\partial f}{\partial \eta} \frac{\partial \psi^*}{\partial f} \frac{\partial \eta}{\partial y^*} \\ \frac{\partial f(\eta)}{\partial \psi^*} &= \frac{1}{\sqrt{x^* Re_\delta}} \frac{\partial}{\partial \psi^*} \psi^* = \frac{1}{\sqrt{x^* Re_\delta}} \end{aligned}$$

[CORRECTION:]

$$\psi^* = f(\eta) \sqrt{x^*} \sqrt{Re_\delta}$$

So that:

$$\frac{\partial \psi^*}{\partial y^*} = \frac{\partial}{\partial y^*} f(\eta) \sqrt{x^*} \sqrt{Re_\delta}$$



Assume  $Re_\delta$  is constant with respect to both x and y,

$$\frac{\partial \psi^*}{\partial y^*} = \sqrt{x^*} \sqrt{Re_\delta} \frac{\partial}{\partial y^*} f(\eta)$$

$$\frac{\partial \psi^*}{\partial y^*} = \sqrt{x^*} \sqrt{Re_\delta} \frac{\partial \eta}{\partial y^*} \frac{\partial}{\partial \eta} f(\eta) = \sqrt{x^*} \sqrt{Re_\delta} \frac{\partial \eta}{\partial y^*} f'$$

[END OF CORRECTION]

$$\frac{\partial}{\partial y^*} \eta = \frac{\partial}{\partial y^*} \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} = \frac{\eta}{y^*}$$

$$\frac{\partial \psi^*}{\partial y^*} = \frac{\partial f}{\partial \eta} \sqrt{x^* Re_\delta} \frac{\eta}{y^*} = f' \sqrt{x^* Re_\delta} \frac{\eta}{y^*}$$

substitute:

$$\eta = \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}}$$

$$\frac{\partial \psi^*}{\partial y^*} = \frac{\partial f}{\partial \eta} \sqrt{x^* Re_\delta} \frac{\eta}{y^*} = f' \sqrt{x^* Re_\delta} \frac{1}{y^*} \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} = f' \frac{Re_\delta}{2}$$

So this becomes:

$$(f' \frac{Re_\delta}{2}) \frac{\partial}{\partial x^*} (f' \frac{Re_\delta}{2}) - (\frac{\partial \psi^*}{\partial x^*}) (\frac{\partial}{\partial (y^*)} f' \frac{Re_\delta}{2}) - \frac{\partial^2}{\partial (y^*)^2} f' \frac{Re_\delta}{2} = 0$$

$$(f' \frac{Re_\delta}{2}) \frac{\partial}{\partial x^*} (f') - (\frac{\partial \psi^*}{\partial x^*}) (\frac{\partial}{\partial (y^*)} f') - \frac{\partial^2}{\partial (y^*)^2} f' = 0$$

Now for higher order derivatives, note that:

$$\frac{\partial}{\partial y^*} (\frac{\eta}{y^*}) = \frac{y^* \frac{\partial \eta}{\partial y^*} - \eta \frac{\partial y^*}{\partial y^*}}{(y^*)^2}$$

note that

$$\frac{\partial}{\partial y^*} \eta = \frac{\eta}{y^*}$$

$$\frac{\partial}{\partial y^*} (\frac{\eta}{y^*}) = \frac{y^* \frac{\eta}{y^*} - \eta}{(y^*)^2} = 0$$

What does this tell us?

$$\frac{\eta}{y^*}$$

is constant with respect to  $y^*$

$$\frac{\partial}{\partial y^*} f' = \frac{\partial \eta}{\partial y^*} f'' = \frac{\eta}{y^*} f''$$

Then we have

$$\begin{aligned} \frac{\partial^2}{\partial (y^*)^2} f' &= \frac{\partial}{\partial (y^*)} \left( \frac{\partial \eta}{\partial y^*} \frac{\partial}{\partial \eta} f' \right) = \frac{\partial}{\partial (y^*)} \left( \frac{\partial \eta}{\partial y^*} f'' \right) \\ &= \frac{\eta^2}{(y^*)^2} \frac{\partial}{\partial \eta} f'' = \frac{\eta^2}{(y^*)^2} f''' \end{aligned}$$

We can substitute our expressions back:

$$\begin{aligned} (f' \frac{Re_\delta}{2}) \frac{\partial}{\partial x^*} (f') - (\frac{\partial \psi^*}{\partial x^*}) (\frac{\partial}{\partial (y^*)} f') - \frac{\partial^2}{\partial (y^*)^2} f' &= 0 \\ (f' \frac{Re_\delta}{2}) \frac{\partial}{\partial x^*} (f') - (\frac{\partial \psi^*}{\partial x^*}) \frac{\eta}{y^*} f'' - \frac{\eta^2}{(y^*)^2} f''' &= 0 \end{aligned}$$

Now let's deal with the  $x^*$  terms

$$\begin{aligned} \frac{\partial}{\partial x^*} &= \frac{\partial \eta}{\partial x^*} \frac{\partial}{\partial \eta} \\ \eta &= \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} \\ \frac{\partial}{\partial x^*} \eta &= \frac{\partial}{\partial x^*} \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} \\ &= \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} \frac{-1}{2x^*} = \frac{\eta}{-2x^*} \\ (f' \frac{Re_\delta}{2}) \frac{\eta}{-2x^*} (f'') - (\frac{\partial \psi^*}{\partial x^*}) \frac{\eta}{y^*} f'' - \frac{\eta^2}{(y^*)^2} f''' &= 0 \end{aligned}$$

Now for the derivative:

$$\psi^* = f(\eta) \sqrt{x^*} \sqrt{Re_\delta}$$

$$\frac{\partial \psi^*}{\partial x^*} = \frac{\partial}{\partial x^*} f(\eta) \sqrt{x^*} \sqrt{Re_\delta}$$

$$\frac{\partial \psi^*}{\partial x^*} = \sqrt{Re_\delta} [\sqrt{x^*} \frac{\partial}{\partial x^*} f(\eta) + f(\eta) \frac{\partial}{\partial x^*} \sqrt{x^*}]$$

$$\frac{\partial \psi^*}{\partial x^*} = \sqrt{Re_\delta} [\sqrt{x^*} \frac{-\eta}{2x^*} f' + f(\eta) \frac{1}{2\sqrt{x^*}}]$$

$$(f' \frac{Re_\delta}{2}) \frac{\eta}{-2x^*} (f'') - (\sqrt{Re_\delta} [\sqrt{x^*} \frac{-\eta}{2x^*} f' + f(\eta) \frac{1}{2\sqrt{x^*}}]) \frac{\eta}{y^*} f'' - \frac{\eta^2}{(y^*)^2} f''' = 0$$

Let's substitute

$$\frac{\eta}{y^*} = \frac{1}{2} \frac{\sqrt{Re_\delta}}{\sqrt{x^*}}$$

$$(f' \frac{Re_\delta}{2}) \frac{\eta}{-2x^*} (f'') - (\sqrt{Re_\delta} [\sqrt{x^*} \frac{-\eta}{2x^*} f' + f(\eta) \frac{1}{2\sqrt{x^*}}]) \frac{1}{2} \frac{\sqrt{Re_\delta}}{\sqrt{x^*}} f'' - (\frac{1}{2} \frac{\sqrt{Re_\delta}}{\sqrt{x^*}})^2 f''' = 0$$

$$(f' \frac{Re_\delta}{2}) \frac{\eta}{-2x^*} (f'') - (\sqrt{Re_\delta} [\sqrt{x^*} \frac{-\eta}{2x^*} f' + f(\eta) \frac{1}{2\sqrt{x^*}}]) \frac{1}{2} \frac{\sqrt{Re_\delta}}{\sqrt{x^*}} f'' - (\frac{1}{4} \frac{Re_\delta}{x^*}) f''' = 0$$

$$(f' \frac{Re_\delta}{2}) \frac{\eta}{-2x^*} (f'') - (Re_\delta [\frac{-\eta}{2x^*} f' + f(\eta) \frac{1}{2x^*}]) \frac{1}{2} f'' - (\frac{1}{4} \frac{Re_\delta}{x^*}) f''' = 0$$

$$-(f' \frac{Re_\delta}{2}) \frac{\eta}{2x^*} (f'') - [\frac{Re_\delta}{2} f'' \frac{-\eta}{2x^*} f' + Re_\delta \frac{1}{2} f'' f(\eta) \frac{1}{2x^*}] - (\frac{1}{4} \frac{Re_\delta}{x^*}) f''' = 0$$

$$-[Re_\delta \frac{1}{2} f'' f(\eta) \frac{1}{2x^*}] - (\frac{1}{4} \frac{Re_\delta}{x^*}) f''' = 0$$

$$f'' f + f''' = 0$$

And we're done!

Now to transform the BCs:

$$u = 0 \text{ at } y = 0$$

$$v = 0 \text{ at } y = 0$$

$$u \rightarrow u_\infty \text{ at } y \rightarrow \infty$$

$$\eta = \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} = \frac{1}{2} y \sqrt{\frac{u_\infty}{\nu x}}$$

$$f(\eta) = \frac{\psi^*}{\sqrt{x^* Re_\delta}} = \frac{\psi}{\sqrt{u_\infty x \nu}}$$

$$u = 0 \text{ at } \eta = 0$$

$$v = 0 \text{ at } \eta = 0$$

$$u \rightarrow u_\infty \text{ at } \eta \rightarrow \infty$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y} f(\eta) \sqrt{u_\infty x \nu}$$

$$u = \sqrt{u_\infty x \nu} \frac{\partial \eta}{\partial y} \frac{\partial}{\partial \eta} f(\eta) = \sqrt{u_\infty x \nu} \frac{1}{2} \sqrt{\frac{u_\infty}{\nu x}} f'$$

$$u = \frac{u_\infty}{2} f'$$

$$u = u_\infty \rightarrow f' = 2$$

$$v = -\frac{\partial \psi}{\partial x}$$

$$v = -\frac{\partial}{\partial x} f(\eta) \sqrt{u_\infty x \nu}$$

$$v = -[f(\eta) \frac{\partial}{\partial x} \sqrt{u_\infty x \nu} + \sqrt{u_\infty x \nu} \frac{\partial}{\partial x} f(\eta)]$$

$$v = -[f(\eta) \frac{\partial}{\partial x} \sqrt{u_\infty x \nu} + \sqrt{u_\infty x \nu} \frac{\partial \eta}{\partial x} \frac{\partial}{\partial \eta} f(\eta)]$$

some steps later...

$$v = -\frac{1}{2} \sqrt{\frac{\nu u_\infty}{x}} (\eta f' - f)$$

$$f' = 0 \text{ at } \eta = 0$$

$$f = 0 \text{ at } \eta = 0$$

$$f' = 2 \text{ at } \eta \rightarrow \infty$$

$$f''f + f''' = 0$$

### 3.1.2 how to solve Blasius's equation

$$f' = 0 \text{ at } \eta = 0$$

$$f = 0 \text{ at } \eta = 0$$

$$f' = 2 \text{ at } \eta \rightarrow \infty$$

$$f''f + f''' = 0$$

### Series solution (aka Frobenius method)

<https://mathworld.wolfram.com/FrobeniusMethod.html>

<http://naca.central.cranfield.ac.uk/reports/1950/naca-tm-1256.pdf>

Assumes:

$$f = \sum_{n=0}^{\infty} a_n \eta^n$$

$$f' = \sum_{n=0}^{\infty} a_{n+1} (n+1) \eta^n$$

$$f'' = \sum_{n=0}^{\infty} a_{n+2} (n+1)(n+2) \eta^n$$

$$f''' = \sum_{n=0}^{\infty} a_{n+3} (n+1)(n+2)(n+3) \eta^n$$

### Runge Kutta Methods numerical methods...

[https://en.wikipedia.org/wiki/Runge%E2%80%93Kutta\\_methods](https://en.wikipedia.org/wiki/Runge%E2%80%93Kutta_methods)

<https://projects.exeter.ac.uk/fluidflow/Courses/FluidDynamics3211-2/BoundaryLayers/rk>

[https://www.researchgate.net/publication/259772650\\_Numerical\\_Approximations\\_of\\_Blasius](https://www.researchgate.net/publication/259772650_Numerical_Approximations_of_Blasius)

### 3.2 Integral Solution by Theodore Von Karman

We are interested in  $\tau_x$  which is wall shear stress for x direction

we perform force balance on the BL in x dir. We look at the diagram and see the trapezium

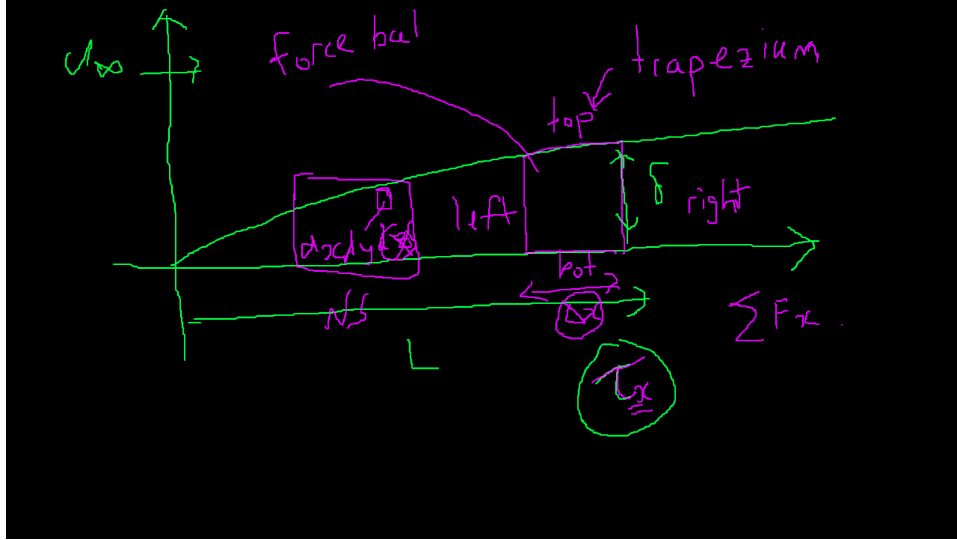


Figure 1: vonkarman BL

$$\sum F_x = top + bottom + left + right$$

In equilibrium

$$\sum F_x = net\ outflow\ of\ momentum\ in\ CV + accumulation\ term$$

$$accumulation\ term = 0$$

$$bottom = \tau_x l_z \Delta x$$

$$left = (P \delta_p l_z)|_x$$

$$right = (P \delta_p l_z)|_{x+\Delta x}$$

$$top = \frac{P|_x + P|_{x+\Delta x}}{2}(\delta_{x+\Delta x} - \delta_x)l_z$$

let's take a look at net outflow of momentum:

left side outflows:

$$-\int \rho u^2 dA|_x = -l_z \int_0^{\delta_p} \rho u^2 dy|_x$$

right side outflow:

$$\int \rho u^2 dA|_{x+\Delta x} = l_z \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x}$$

Top side outflow:

$$-\dot{m}_{top}u_\infty$$

Now we need an expression for  $\dot{m}_{top}$

$$\dot{m}_{top} = \dot{m}_{right} - \dot{m}_{left}$$

$$\dot{m}_{top} = l_z \int_0^{\delta_p|_{x+\Delta x}} \rho u|_{x+\Delta x} dy - l_z \int_0^{\delta_p|_x} \rho u|_x dy$$

total momentum outflow

$$net\ outflow\ of\ momentum = -l_z \int_0^{\delta_p} \rho u^2 dy|_x + l_z \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x} - \dot{m}_{top}u_\infty$$

$$net\ outflow\ of\ momentum = -l_z \int_0^{\delta_p} \rho u^2 dy|_x + l_z \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x}$$

$$-(l_z \int_0^{\delta_p|_{x+\Delta x}} \rho u|_{x+\Delta x} dy - l_z \int_0^{\delta_p|_x} \rho u|_x dy)u_\infty$$

Sum of forces:

$$\sum F_x = \tau_x l_z \Delta x + (P\delta_p l_z)|_x + (P\delta_p l_z)|_{x+\Delta x} + \frac{P|_x + P|_{x+\Delta x}}{2}(\delta_{x+\Delta x} - \delta_x)$$

Equating both:

$$\begin{aligned}
& \tau_x l_z \Delta x + (P \delta_p l_z)|_x + (P \delta_p l_z)|_{x+\Delta x} + \frac{P|_x + P|_{x+\Delta x}}{2} (\delta_{x+\Delta x} - \delta_x) l_z = \\
& = -l_z \int_0^{\delta_p} \rho u^2 dy|_x + l_z \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x} \\
& - (l_z \int_0^{\delta_p|_{x+\Delta x}} \rho u|_{x+\Delta x} dy - l_z \int_0^{\delta_p|x} \rho u|_x dy) u_\infty
\end{aligned}$$

Cancel out  $l_z$

$$\begin{aligned}
& \tau_x \Delta x + (P \delta_p)|_x + (P \delta_p)|_{x+\Delta x} + \frac{P|_x + P|_{x+\Delta x}}{2} (\delta_{x+\Delta x} - \delta_x) = \\
& = - \int_0^{\delta_p} \rho u^2 dy|_x + \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x} \\
& - \left( \int_0^{\delta_p|_{x+\Delta x}} \rho u|_{x+\Delta x} dy - \int_0^{\delta_p|x} \rho u|_x dy \right) u_\infty
\end{aligned}$$

Divide by  $\Delta x$  limit  $\Delta x \rightarrow 0$

$$\begin{aligned}
& \tau_x + \frac{(P \delta_p)|_x + (P \delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2} \frac{P|_x + P|_{x+\Delta x}}{\Delta x} (\delta_{p|x+\Delta x} - \delta_{p|x}) = \\
& = \frac{- \int_0^{\delta_p} \rho u^2 dy|_x + \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x}}{\Delta x} \\
& - \frac{1}{\Delta x} \left( \int_0^{\delta_p|_{x+\Delta x}} \rho u|_{x+\Delta x} dy - \int_0^{\delta_p|x} \rho u|_x dy \right) u_\infty
\end{aligned}$$

Take limits first:

$$\begin{aligned}
& \lim_{x \rightarrow 0} \left[ \tau_x + \frac{(P \delta_p)|_x + (P \delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2} \frac{P|_x + P|_{x+\Delta x}}{\Delta x} (\delta_{p|x+\Delta x} - \delta_{p|x}) \right] = \\
& = \lim_{x \rightarrow 0} \left[ \frac{- \int_0^{\delta_p} \rho u^2 dy|_x + \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x}}{\Delta x} \right]
\end{aligned}$$



$$\begin{aligned}
& -\frac{1}{\Delta x} \left( \int_0^{\delta_p|_{x+\Delta x}} \rho u|_{x+\Delta x} dy - \int_0^{\delta_p|x} \rho u|_x dy \right) u_\infty] \\
\lim_{x \rightarrow 0} [\tau_x + \frac{(P\delta_p)|_x + (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2} \frac{P|_x + P|_{x+\Delta x}}{\Delta x} (\delta_p|_{x+\Delta x} - \delta_p|x)] = \\
& = \frac{\partial}{\partial x} \int_0^{\delta_p} \rho u^2 dy - \frac{\partial}{\partial x} (u_\infty \int_0^{\delta_p} \rho u dy)
\end{aligned}$$

## 4 Resources Online

[http://web.mit.edu/fluids-modules/www/highspeed\\_flows/ver2/bl\\_Chap2.pdf](http://web.mit.edu/fluids-modules/www/highspeed_flows/ver2/bl_Chap2.pdf)  
<https://community.dur.ac.uk/suzanne.fielding/teaching/BLT/sec3.pdf>

## Part III

# Github Repo

[https://github.com/theodore0nzGit/heatTransferTheory\\_YouTube](https://github.com/theodore0nzGit/heatTransferTheory_YouTube)

Look under convection heat transfer...