Fluid Mechanics YouTube

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Part I
Navier Stokes Equations

Compressible N-S equations

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \nabla \bullet (\rho\vec{u} \otimes \vec{u}) = -\nabla p + \mu\nabla^2\vec{u} + \frac{1}{3}\mu\nabla(\nabla \bullet \vec{u}) + \rho\vec{g}$$

tensor or outer product:

$$\vec{u} \otimes \vec{v} = \vec{u}\vec{v}^T$$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \ and \ \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \otimes \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

$$= \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{pmatrix}$$

Inner product

$$\vec{u} \bullet \vec{v} = (\vec{u}^T\vec{v})^T$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \bullet \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= u_1v_1 + u_2v_2 + u_3v_3$$

Assume incompressible flow:

$$\rho = constant$$

continuity equation

$$\nabla \bullet \vec{u} = 0$$

Incompressible N-S equations

$$\frac{\partial}{\partial t}\vec{u} + (\vec{u} \bullet \nabla)\vec{u} - \nu\nabla^2\vec{u} = -\nabla\frac{P}{\rho_0} + \vec{g}$$

`https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations`

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Matrices in LaTeX
https://www.overleaf.com/learn/latex/Matrices

Tensors in LaTeX

Navier Stokes Equations
https://www.comsol.com/multiphysics/navier-stokes-equations
https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations

Github
https://github.com/theodoreOnzGit/heatTransferTheory_YouTube
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First let's deal with:

$$(\vec{u} \bullet \nabla)\vec{u}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \otimes \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}$$

Then we do inner product

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}$$

$$\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}$$

$$= \begin{pmatrix} u_1 \frac{\partial}{\partial x}u_1 + u_2 \frac{\partial}{\partial y}u_1 + u_3 \frac{\partial}{\partial z}u_1 \\ u_1 \frac{\partial}{\partial x}u_2 + u_2 \frac{\partial}{\partial y}u_2 + u_3 \frac{\partial}{\partial z}u_2 \\ u_1 \frac{\partial}{\partial x}u_3 + u_2 \frac{\partial}{\partial y}u_3 + u_3 \frac{\partial}{\partial z}u_3 \end{pmatrix}$$

Let's deal with the momentum diffusivity (kinematic viscosity) term:

$$\nabla^2 = (\nabla \bullet \nabla)$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x}u_1 + \frac{\partial}{\partial y}\frac{\partial}{\partial y}u_1 + \frac{\partial}{\partial z}\frac{\partial}{\partial z}u_1 \\ \frac{\partial}{\partial x}\frac{\partial}{\partial x}u_2 + \frac{\partial}{\partial y}\frac{\partial}{\partial y}u_2 + \frac{\partial}{\partial z}\frac{\partial}{\partial z}u_2 \\ \frac{\partial}{\partial x}\frac{\partial}{\partial x}u_3 + \frac{\partial}{\partial y}\frac{\partial}{\partial y}u_3 + \frac{\partial}{\partial z}\frac{\partial}{\partial z}u_3 \end{pmatrix}$$

Part II

Boundary Layer Equations

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}\ u + \frac{\partial^2}{\partial z^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}\ v + \frac{\partial^2}{\partial z^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - \nu(\frac{\partial^2}{\partial x^2}w + \frac{\partial^2}{\partial y^2}\ w + \frac{\partial^2}{\partial z^2}w) = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + g_z$$

Now for 2D what do we do? w=0 everywhere and at all times, $g_z = 0$
We eliminate z terms from the x and y momentum balance

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}\ u + \frac{\partial^2}{\partial z^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}\ v + \frac{\partial^2}{\partial z^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

There is no spatial variation in u and v w.r.t z We have 2D Navier stokes:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}\ u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}\ v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

continuity equation

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

1 nondimensionalisation

Order of magnitude

$$\mathcal{O}$$

Scaling for order magnitude comparison

$$u^*, y^* = \mathcal{O}(1)$$

we define:

$$u^* = \frac{u}{u_\infty}$$

$$u = u_\infty u^*$$

$$x^* = \frac{x}{L}$$

$$y^* = \frac{y}{\delta_p}$$

We scale our continutity equation:

$$\frac{\partial}{\partial x^* L} u^* u_\infty + \frac{\partial}{\partial y^* \delta_p} v = 0$$

$$\frac{\partial}{\partial x^* L} u^* u_\infty + \frac{\partial}{\partial y^* \delta_p} v = 0$$

$$\frac{u_\infty}{L} \frac{\partial}{\partial x^*} u^* + \frac{1}{\delta_p} \frac{\partial}{\partial y^*} v = 0$$

$$\frac{\partial}{\partial x^*} u^* + \frac{L}{\delta_p u_\infty} \frac{\partial}{\partial y^*} v = 0$$

$$v^* = \frac{vL}{\delta_p u_\infty} = \mathcal{O}(1)$$

$$v^* = \frac{v}{\frac{u_\infty \delta_p}{L}}$$

$$\frac{\partial}{\partial x^*} u^* + \frac{\partial}{\partial y^*} v^* = 0$$

Now we move on to the NS equations
so we need to scale time:
x lengthscale = L
x velocityscale = u_∞
timescale = $\frac{L}{u_\infty}$

$$t^* = \frac{t}{\frac{L}{u_\infty}}$$

Let's scale the momentum NS equations

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u - \nu \left(\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} \ u \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t} v + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu \left(\frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} \ v \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + g_y$$

Let's do x momentum equations

$$\frac{\partial}{\partial t}u^*u_\infty+u^*u_\infty\frac{\partial}{\partial x^*L}u^*u_\infty+v\frac{\partial}{\partial y}u^*u_\infty-\nu(\frac{\partial^2}{\partial x^2}u^*u_\infty+\frac{\partial^2}{\partial y^2}\ u^*u_\infty)=-\frac{1}{\rho_0}\frac{\partial P}{\partial x}+g_x$$

$$\frac{\partial}{\partial t}u^*+u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}u^*+v\frac{1}{\delta_p}\frac{\partial}{\partial y^*}u^*-\nu(\frac{1}{L^2}\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{1}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=\frac{1}{u_\infty}(-\frac{1}{L\rho_0}\frac{\partial P}{\partial x^*}+g_x)$$

$$\frac{u_\infty}{L}\frac{\partial}{\partial t^*}u^*+u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}u^*+v\frac{1}{\delta_p}\frac{\partial}{\partial y^*}u^*-\nu(\frac{1}{L^2}\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{1}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=\frac{1}{u_\infty}(-\frac{1}{L\rho_0}\frac{\partial P}{\partial x^*}+g_x)$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{\nu}{u_\infty L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=\frac{L}{u_\infty^2}(-\frac{1}{L\rho_0}\frac{\partial P}{\partial x^*}+g_x)$$

Reynold's number

$$Re_L=\frac{u_\infty L}{\nu}$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=\frac{L}{u_\infty^2}(-\frac{1}{L\rho_0}\frac{\partial P}{\partial x^*}+g_x)$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=(-\frac{1}{L\rho_0}\frac{L}{u_\infty^2}\frac{\partial P}{\partial x^*}+\frac{L}{u_\infty^2}g_x)$$

Let's scale gravity

$$g_x^*=\frac{g_x}{|g|}=\cos\theta_x=\mathcal{O}(1)$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=(-\frac{1}{\rho_0 u_\infty^2}\frac{\partial P}{\partial x^*}+\frac{L|g|}{u_\infty^2}g_x^*)$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=(-\frac{1}{\rho_0 u_\infty^2}\frac{\partial P}{\partial x^*}+\frac{1}{Fr^2}g_x^*)$$

$$P^* = \frac{P}{\rho_0 u_\infty^2}$$

After nondimensionalisation, our x momentum equation becomes:

$$\frac{\partial}{\partial t^*}u^* + u^*\frac{\partial}{\partial x^*}u^* + v^*\frac{\partial}{\partial y^*}u^* - \frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}u^* + \frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ u^*) = (-\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2}g_x^*)$$

we dimensionalise y momentum eqns

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}\ v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

First the x and time terms:

$$\frac{u_\infty}{L}\frac{\partial}{\partial t^*}v + u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}v + v\frac{\partial}{\partial y}v - \nu(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v + \frac{\partial^2}{\partial y^2}\ v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

Second y coordinate terms:

$$\frac{u_\infty}{L}\frac{\partial}{\partial t^*}v + u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}v + v\frac{1}{\delta_p}\frac{\partial}{\partial y^*}v - \nu(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v + \frac{1}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v) = -\frac{1}{\rho_0}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*} + g_y^*|g|$$

divide by $\frac{u_\infty^2}{L^2}$

$$\frac{L}{u_\infty}\frac{\partial}{\partial t^*}v + u^*\frac{L}{u_\infty}\frac{\partial}{\partial x^*}v + v\frac{L^2}{u_\infty^2\delta_p}\frac{\partial}{\partial y^*}v - \nu\frac{L^2}{u_\infty^2}(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v + \frac{1}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v) = \frac{L^2}{u_\infty^2}(-\frac{1}{\rho_0}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*} + g_y^*|g|)$$

divide by δ_p

$$\frac{L}{u_\infty\delta_p}\frac{\partial}{\partial t^*}v + u^*\frac{L}{u_\infty\delta_p}\frac{\partial}{\partial x^*}v + v\frac{L^2}{u_\infty^2\delta_p^2}\frac{\partial}{\partial y^*}v - \nu\frac{L^2}{u_\infty^2\delta_p}(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v + \frac{1}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v) = \frac{L^2}{u_\infty^2\delta_p}(-\frac{1}{\rho_0}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*} + g_y^*|g|)$$

Combining some terms

$$\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^* - \nu\frac{L}{u_\infty}(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v^* + \frac{1}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*) = \frac{L^2}{u_\infty^2\delta_p}(-\frac{1}{\rho_0}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*} + g_y^*|g|)$$

Rearranging

$$\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^* - \frac{\nu}{u_\infty L}(\frac{\partial^2}{\partial (x^*)^2}v^* + \frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*) = (-\frac{1}{\rho_0}\frac{L^2}{u_\infty^2\delta_p}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*} + g_y^*|g|\frac{L^2}{u_\infty^2\delta_p})$$

nondimensionalisng pressure and including the Fr

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{\nu}{u_\infty L}(\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*)=(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

Include Re

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*)=(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

Review: NS nondimensionalised

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ u^*)=(-\frac{\partial P^*}{\partial x^*}+\frac{1}{Fr^2}g_x^*)$$

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*)=(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

$$\frac{\partial}{\partial x^*}u^*+\frac{\partial}{\partial y^*}v^*=0$$

2 How to drop terms?

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ u^*)=(-\frac{\partial P^*}{\partial x^*}+\frac{1}{Fr^2}g_x^*)$$

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*)=(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

$$\frac{\partial}{\partial x^*}u^*+\frac{\partial}{\partial y^*}v^*=0$$

When we want to determine which terms to cancel, we need to know how Re_L compares with $\frac{L^2}{\delta_p^2}$

Assumption:

Creeping flow in y direction

$$Re_\delta=\frac{v_c\delta_p}{\nu}=\mathcal{O}(1)$$

How does Re_δ compare to Re_L

$$v_c = u_\infty \frac{\delta_p}{L}$$

$$Re_\delta = \frac{u_\infty \frac{\delta_p}{L} \delta_p}{\nu} = \mathcal{O}(1)$$

$$Re_\delta = \frac{u_\infty L}{\nu} \frac{\delta_p^2}{L^2} = \mathcal{O}(1)$$

$$Re_\delta = Re_L \frac{\delta_p^2}{L^2} = \mathcal{O}(1)$$

2.0.1 x direction momentum eqn

$$\frac{\partial}{\partial t^*}u^* + u^* \frac{\partial}{\partial x^*}u^* + v^* \frac{\partial}{\partial y^*}u^* - \frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}u^* + \frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ u^*) = (-\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2}g_x^*)$$

$$\frac{\partial}{\partial t^*}u^* + u^* \frac{\partial}{\partial x^*}u^* + v^* \frac{\partial}{\partial y^*}u^* - \frac{1}{Re_L}\frac{\partial^2}{\partial (x^*)^2}u^* + \frac{1}{Re_L}\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ u^* = (-\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2}g_x^*)$$

$$\frac{\partial}{\partial t^*}u^* + u^* \frac{\partial}{\partial x^*}u^* + v^* \frac{\partial}{\partial y^*}u^* - \frac{1}{Re_L}\frac{\partial^2}{\partial (x^*)^2}u^* + \frac{1}{\mathcal{O}(1)}\frac{\partial^2}{\partial (y^*)^2}\ u^* = (-\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2}g_x^*)$$

How big is Re_L?

$$Re_L = \mathcal{O}(\frac{L^2}{\delta_p^2})$$

We assume Fr is big

$$\frac{\partial}{\partial t^*}u^* + u^* \frac{\partial}{\partial x^*}u^* + v^* \frac{\partial}{\partial y^*}u^* - + \frac{1}{\mathcal{O}(1)}\frac{\partial^2}{\partial (y^*)^2}\ u^* = (-\frac{\partial P^*}{\partial x^*})$$

2.0.2 y direction momentum equation

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*)=(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-(\frac{1}{Re_L}\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{1}{\mathcal{O}(1)}\frac{\partial^2}{\partial (y^*)^2}\ v^*)=(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

$$[\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*]\frac{1}{Re_L}-(\frac{1}{Re_L^2}\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{1}{\mathcal{O}(1)Re_L}\frac{\partial^2}{\partial (y^*)^2}\ v^*)=\frac{1}{Re_L}(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

$$[\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^*]\frac{1}{Re_L} - (\frac{1}{Re_L^2}\frac{\partial^2}{\partial (x^*)^2}v^* + \frac{1}{\mathcal{O}(1)Re_L}\frac{\partial^2}{\partial (y^*)^2}\ v^*)$$

$$= (-\frac{L^2}{\delta_p^2}\frac{1}{Re_L}\frac{\partial P^*}{\partial y^*} + g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p}\frac{1}{Re_L})$$

Cancelling out...

$$[\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^*]\frac{1}{Re_L} - (\frac{1}{Re_L^2}\frac{\partial^2}{\partial (x^*)^2}v^* + \frac{1}{\mathcal{O}(1)Re_L}\frac{\partial^2}{\partial (y^*)^2}\ v^*)$$

$$= (-\frac{1}{\mathcal{O}(1)}\frac{\partial P^*}{\partial y^*} + g_y^*\frac{1}{Fr^2}\frac{L^2}{\delta_p^2}\frac{1}{Re_L}\frac{\delta_p}{L})$$

Simplifying

$$[\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^*]\frac{1}{Re_L} - (\frac{1}{Re_L^2}\frac{\partial^2}{\partial (x^*)^2}v^* + \frac{1}{\mathcal{O}(1)Re_L}\frac{\partial^2}{\partial (y^*)^2}\ v^*)$$

$$= \frac{1}{\mathcal{O}(1)}(-\frac{\partial P^*}{\partial y^*} + g_y^*\frac{1}{Fr^2}\frac{\delta_p}{L})$$

For large Re_L

$$0 = (-\frac{\partial P^*}{\partial y^*} + g_y^*\frac{1}{Fr^2}\frac{\delta_p}{L})$$

$$g_y^*\frac{1}{Fr^2}\frac{\delta_p}{L} = \frac{\partial P^*}{\partial y^*}$$

Only if g=0,

$$0 = -\frac{\partial P^*}{\partial y^*}$$

Now we have our BL equations:

$$0 = -\frac{\partial P^*}{\partial y^*}$$

$$\frac{\partial}{\partial t^*}u^* + u^*\frac{\partial}{\partial x^*}u^* + v^*\frac{\partial}{\partial y^*}u^* - \frac{1}{\mathcal{O}(1)}\frac{\partial^2}{\partial (y^*)^2}\ u^* = (-\frac{\partial P^*}{\partial x^*})$$

redimensionalise to obtain the laminar BL equations:

$$0 = -\frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial (y)^2}\ u = (-\frac{\partial P}{\partial x})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

3 Solutions to the BL equations laminar

How to solve?

- 1st Similarity solution
- 2nd Von Karman Solution (Integral solution - approximate)
- 3rd numerical (CFD)

3.1 similarity solution (Blasius solution)

$$0 = -\frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial (y)^2}\ u = (-\frac{\partial P}{\partial x})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Similarity solution $\rightarrow$ combine variables to convert PDE to ODE
Making 2 assumptions before we continue:
1) steady state 2) no pressure gradient

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial y^2}\ u = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

How to combine variables to make life easier for us to solve?
introduce the streamfunction (ψ):

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

Note: streamfunction only works for 2D fluid flow
Substitute into 2D continuity equation,

$$\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x} = 0$$

Substitute into the 2D x momentum equation

$$(\frac{\partial \psi}{\partial y})\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial y}) + (-\frac{\partial \psi}{\partial x})\frac{\partial}{\partial y}(\frac{\partial \psi}{\partial y}) - \nu\frac{\partial^2}{\partial y^2}(\frac{\partial \psi}{\partial y}) = 0$$

$$(\frac{\partial \psi}{\partial y})\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial y}) - (\frac{\partial \psi}{\partial x})(\frac{\partial^2 \psi}{\partial y^2}) - \nu\frac{\partial^3}{\partial y^3}\psi = 0$$

We need to compress the number of variables further to 1 indep variable (η) and 1 dependent variable (f(η))

$$\eta = \eta(x, y)$$

$$f = f(\eta)$$

Before we continue, BCs first!
1 BC in x dir for u

$$x = 0, y \neq 0, u = u_\infty$$

2 BCs in y dir for u

no slip

$$u = 0 \ at \ y = 0$$

$$y \to \infty; \ u \to u_\infty$$

1 BC in y direction for v

no slip

$$v = 0 \ at \ y = 0$$

3.1.1 similarity transform

How do we start to get these "combo parameters" aka similarity variables?

https://ntrs.nasa.gov/citations/20050028493

Based on Blasius's paper (translated by NACA) it's good to nondimnesionalise to find these similarity variables

$$\psi^* = \frac{\psi}{\psi_0}$$

Let's nondimensionalise the momentum equations:

$$(\frac{\partial \psi}{\partial y})\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial y}) - (\frac{\partial \psi}{\partial x})(\frac{\partial^2 \psi}{\partial y^2}) - \nu\frac{\partial^3}{\partial y^3}\psi = 0$$

$$\frac{\psi_0^2}{\delta_p^2 L}(\frac{\partial \psi^*}{\partial y^*})\frac{\partial}{\partial x^*}(\frac{\partial \psi^*}{\partial y^*}) - \frac{\psi_0^2}{L\delta_p^2}(\frac{\partial \psi^*}{\partial x^*})(\frac{\partial^2 \psi^*}{\partial (y^*)^2}) - \frac{\nu\psi_0}{\delta_p^3}\frac{\partial^3}{\partial (y^*)^3}\psi^* = 0$$

$$(\frac{\partial \psi^*}{\partial y^*})\frac{\partial}{\partial x^*}(\frac{\partial \psi^*}{\partial y^*}) - (\frac{\partial \psi^*}{\partial x^*})(\frac{\partial^2 \psi^*}{\partial (y^*)^2}) - \frac{\nu\psi_0}{\delta_p^3}\frac{\delta_p^2 L}{\psi_0^2}\frac{\partial^3}{\partial (y^*)^3}\psi^* = 0$$

$$(\frac{\partial \psi^*}{\partial y^*})\frac{\partial}{\partial x^*}(\frac{\partial \psi^*}{\partial y^*}) - (\frac{\partial \psi^*}{\partial x^*})(\frac{\partial^2 \psi^*}{\partial (y^*)^2}) - \frac{\nu}{\delta_p}\frac{L}{\psi_0}\frac{\partial^3}{\partial (y^*)^3}\psi^* = 0$$

dimensionless group:

$$\frac{\nu L}{\delta_p \psi_0} = \mathcal{O}(1)$$

We want the equations to be nondimensionalised exactly.

If we want the equations looks like:

$$(\frac{\partial \psi^*}{\partial y^*})\frac{\partial}{\partial x^*}(\frac{\partial \psi^*}{\partial y^*}) - (\frac{\partial \psi^*}{\partial x^*})(\frac{\partial^2 \psi^*}{\partial (y^*)^2}) - \frac{\partial^3}{\partial (y^*)^3}\psi^* = 0$$

$$\frac{\nu L}{\delta_p \psi_0} = 1$$

$$\psi_0 = \frac{\nu L}{\delta_p}$$

Otherwise:

$$\psi_0 = \mathcal{O}(1)\frac{\nu L}{\delta_p}$$

How do we get rid of dependent variables δ_p?

$$Re_\delta = \frac{v_c \delta_p}{\nu} = \mathcal{O}(1)$$

$$Re_\delta = \frac{u_\infty L}{\nu}\frac{\delta_p^2}{L^2} = \mathcal{O}(1)$$

From continututy equation

$$v_c = \frac{u_\infty \delta_p}{L}$$

(not too helpful)
What's helpful is to use the physics of the BL ie creeping flow in BL
The other assumption:

$$Re_\delta = 1$$

$$Re_\delta = \frac{u_\infty}{\nu L}\delta_p^2$$

$$\delta_p = \sqrt{Re_\delta}\sqrt{\frac{\nu L}{u_\infty}}$$

Substitute back:

$$\psi_0 = \frac{\nu L}{\delta_p} = \frac{\nu L}{\sqrt{Re_\delta}\sqrt{\frac{\nu L}{u_\infty}}}$$

$$\psi_0 = \frac{\nu L}{\sqrt{\frac{\nu L}{u_\infty}}}\frac{1}{\sqrt{Re_\delta}}$$

$$\psi_0 = \sqrt{u_\infty L\nu}\frac{1}{\sqrt{Re_\delta}}$$

Let's see our nondimensionalised streamfunction:

$$\psi^* = \frac{\psi}{\psi_0} = \sqrt{Re_\delta}\frac{\psi}{\sqrt{u_\infty L\nu}}$$

If $Re_\delta = 1$

$$\psi^* = \frac{\psi}{\psi_0} = \frac{\psi}{\sqrt{u_\infty L\nu}}$$

What about our independent variable? We need to combine the x and y coordinate variables

we'll use

$$Re_\delta = \frac{u_\infty}{\nu L}\delta_p^2$$

$$\delta_p = \sqrt{Re_\delta}\sqrt{\frac{\nu L}{u_\infty}}$$

If we want x and y explicitly,

$$x = x^* L$$

$$y = y^* \delta_p$$

$$\frac{y}{y^*} = \sqrt{Re_\delta}\sqrt{\frac{\nu\frac{x}{x^*}}{u_\infty}}$$

$$y = \sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}\sqrt{\frac{\nu x}{u_\infty}}$$

$$y = \sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}\sqrt{\frac{\nu x}{u_\infty}}$$

$$y\sqrt{\frac{u_\infty}{\nu x}} = \sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}} = \mathcal{O}(1)$$

And we have dimensionless stream function

$$\psi^* = \frac{\psi}{\psi_0} = \frac{\psi}{\sqrt{u_\infty L \nu}}$$

$$\psi^* = \frac{\psi}{\sqrt{u_\infty \frac{x}{x^*} \nu}}$$

$$\psi^* = \sqrt{x^*}\frac{\psi}{\sqrt{u_\infty x \nu}}$$

If $Re_\delta \neq 1$

$$\psi^* = \frac{\psi}{\psi_0} = \sqrt{Re_\delta}\frac{\psi}{\sqrt{u_\infty L \nu}}$$

$$\psi^* = \frac{\psi}{\psi_0} = \sqrt{Re_\delta x^*}\frac{\psi}{\sqrt{u_\infty x \nu}}$$

from Blasius's paper:

$$\eta = \frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}} = \frac{1}{2}y\sqrt{\frac{u_\infty}{\nu x}}$$

$$f(\eta) = \frac{\psi^*}{\sqrt{x^* Re_\delta}} = \frac{\psi}{\sqrt{u_\infty x \nu}}$$

We start transforming variables

$$(\frac{\partial \psi^*}{\partial y^*})\frac{\partial}{\partial x^*}(\frac{\partial \psi^*}{\partial y^*}) - (\frac{\partial \psi^*}{\partial x^*})(\frac{\partial^2 \psi^*}{\partial (y^*)^2}) - \frac{\partial^3}{\partial (y^*)^3}\psi^* = 0$$

Use chain rule

$$\frac{\partial \psi^*}{\partial y^*} = \frac{\partial f}{\partial \eta}\frac{\partial \psi^*}{\partial f}\frac{\partial \eta}{\partial y^*}$$

$$\frac{\partial f(\eta)}{\partial \psi^*} = \frac{1}{\sqrt{x^* Re_\delta}}\frac{\partial}{\partial \psi^*}\psi^* = \frac{1}{\sqrt{x^* Re_\delta}}$$

[CORRECTION:]

$$\psi^* = f(\eta)\sqrt{x^*}\sqrt{Re_\delta}$$

So that:

$$\frac{\partial \psi^*}{\partial y^*} = \frac{\partial}{\partial y^*}f(\eta)\sqrt{x^*}\sqrt{Re_\delta}$$

Assume Re_δ is constant with respect to both x and y,

$$\frac{\partial\psi^*}{\partial y^*} = \sqrt{x^*}\sqrt{Re_\delta}\frac{\partial}{\partial y^*}f(\eta)$$

$$\frac{\partial\psi^*}{\partial y^*} = \sqrt{x^*}\sqrt{Re_\delta}\frac{\partial\eta}{\partial y^*}\frac{\partial}{\partial\eta}f(\eta) = \sqrt{x^*}\sqrt{Re_\delta}\frac{\partial\eta}{\partial y^*}f'$$

[END OF CORRECTION]

$$\frac{\partial}{\partial y^*}\eta = \frac{\partial}{\partial y^*}\frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}} = \frac{\eta}{y^*}$$

$$\frac{\partial\psi^*}{\partial y^*} = \frac{\partial f}{\partial\eta}\sqrt{x^*Re_\delta}\frac{\eta}{y^*} = f'\sqrt{x^*Re_\delta}\frac{\eta}{y^*}$$

substitute:

$$\eta = \frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}$$

$$\frac{\partial\psi^*}{\partial y^*} = \frac{\partial f}{\partial\eta}\sqrt{x^*Re_\delta}\frac{\eta}{y^*} = f'\sqrt{x^*Re_\delta}\frac{1}{y^*}\frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}} = f'\frac{Re_\delta}{2}$$

So this becomes:

$$(f'\frac{Re_\delta}{2})\frac{\partial}{\partial x^*}(f'\frac{Re_\delta}{2}) - (\frac{\partial\psi^*}{\partial x^*})(\frac{\partial}{\partial(y^*)}f'\frac{Re_\delta}{2}) - \frac{\partial^2}{\partial(y^*)^2}f'\frac{Re_\delta}{2} = 0$$

$$(f'\frac{Re_\delta}{2})\frac{\partial}{\partial x^*}(f') - (\frac{\partial\psi^*}{\partial x^*})(\frac{\partial}{\partial(y^*)}f') - \frac{\partial^2}{\partial(y^*)^2}f' = 0$$

Now for higher order derivatives, note that:

$$\frac{\partial}{\partial y^*}(\frac{\eta}{y^*}) = \frac{y^*\frac{\partial\eta}{\partial y^*} - \eta\frac{\partial y^*}{\partial y^*}}{(y^*)^2}$$

note that

$$\frac{\partial}{\partial y^*}\eta = \frac{\eta}{y^*}$$

$$\frac{\partial}{\partial y^*}(\frac{\eta}{y^*}) = \frac{y^*\frac{\eta}{y^*} - \eta}{(y^*)^2} = 0$$

What does this tell us?

$$\frac{\eta}{y^*}$$

is constant with respect to y^*

$$\frac{\partial}{\partial y^*}f' = \frac{\partial \eta}{\partial y^*}f'' = \frac{\eta}{y^*}f''$$

Then we have

$$\frac{\partial^2}{\partial (y^*)^2}f' = \frac{\partial}{\partial (y^*)}(\frac{\partial \eta}{\partial y^*}\frac{\partial}{\partial \eta}f') = \frac{\partial}{\partial (y^*)}(\frac{\partial \eta}{\partial y^*}f'')$$

$$= \frac{\eta^2}{(y^*)^2}\frac{\partial}{\partial \eta}f'' = \frac{\eta^2}{(y^*)^2}f'''$$

We can substitute our expressions back:

$$(f'\frac{Re_\delta}{2})\frac{\partial}{\partial x^*}(f') - (\frac{\partial \psi^*}{\partial x^*})(\frac{\partial}{\partial (y^*)}f') - \frac{\partial^2}{\partial (y^*)^2}f' = 0$$

$$(f'\frac{Re_\delta}{2})\frac{\partial}{\partial x^*}(f') - (\frac{\partial \psi^*}{\partial x^*})\frac{\eta}{y^*}f'' - \frac{\eta^2}{(y^*)^2}f''' = 0$$

Now let's deal with the x^* terms

$$\frac{\partial}{\partial x^*} = \frac{\partial \eta}{\partial x^*}\frac{\partial}{\partial \eta}$$

$$\eta = \frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}$$

$$\frac{\partial}{\partial x^*}\eta = \frac{\partial}{\partial x^*}\frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}$$

$$= \frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}\frac{-1}{2x^*} = \frac{\eta}{-2x^*}$$

$$(f'\frac{Re_\delta}{2})\frac{\eta}{-2x^*}(f'') - (\frac{\partial \psi^*}{\partial x^*})\frac{\eta}{y^*}f'' - \frac{\eta^2}{(y^*)^2}f''' = 0$$

Now for the derivative:

$$\psi^* = f(\eta)\sqrt{x^*}\sqrt{Re_\delta}$$

$$\frac{\partial \psi^*}{\partial x^*} = \frac{\partial}{\partial x^*} f(\eta)\sqrt{x^*}\sqrt{Re_\delta}$$

$$\frac{\partial \psi^*}{\partial x^*} = \sqrt{Re_\delta}[\sqrt{x^*}\frac{\partial}{\partial x^*} f(\eta) + f(\eta)\frac{\partial}{\partial x^*}\sqrt{x^*}]$$

$$\frac{\partial \psi^*}{\partial x^*} = \sqrt{Re_\delta}[\sqrt{x^*}\frac{-\eta}{2x^*} f' + f(\eta)\frac{1}{2\sqrt{x^*}}]$$

$$(f'\frac{Re_\delta}{2})\frac{\eta}{-2x^*}(f'') - (\sqrt{Re_\delta}[\sqrt{x^*}\frac{-\eta}{2x^*} f' + f(\eta)\frac{1}{2\sqrt{x^*}}])\frac{\eta}{y^*} f'' - \frac{\eta^2}{(y^*)^2} f''' = 0$$

Let's substitute

$$\frac{\eta}{y^*} = \frac{1}{2}\frac{\sqrt{Re_\delta}}{\sqrt{x^*}}$$

$$(f'\frac{Re_\delta}{2})\frac{\eta}{-2x^*}(f'') - (\sqrt{Re_\delta}[\sqrt{x^*}\frac{-\eta}{2x^*} f' + f(\eta)\frac{1}{2\sqrt{x^*}}])\frac{1}{2}\frac{\sqrt{Re_\delta}}{\sqrt{x^*}} f'' - (\frac{1}{2}\frac{\sqrt{Re_\delta}}{\sqrt{x^*}})^2 f''' = 0$$

$$(f'\frac{Re_\delta}{2})\frac{\eta}{-2x^*}(f'') - (\sqrt{Re_\delta}[\sqrt{x^*}\frac{-\eta}{2x^*} f' + f(\eta)\frac{1}{2\sqrt{x^*}}])\frac{1}{2}\frac{\sqrt{Re_\delta}}{\sqrt{x^*}} f'' - (\frac{1}{4}\frac{Re_\delta}{x^*}) f''' = 0$$

$$(f'\frac{Re_\delta}{2})\frac{\eta}{-2x^*}(f'') - (Re_\delta[\frac{-\eta}{2x^*} f' + f(\eta)\frac{1}{2x^*}])\frac{1}{2} f'' - (\frac{1}{4}\frac{Re_\delta}{x^*}) f''' = 0$$

$$-(f'\frac{Re_\delta}{2})\frac{\eta}{2x^*}(f'') - [\frac{Re_\delta}{2} f''\frac{-\eta}{2x^*} f' + Re_\delta\frac{1}{2} f'' f(\eta)\frac{1}{2x^*}] - (\frac{1}{4}\frac{Re_\delta}{x^*}) f''' = 0$$

$$-[Re_\delta\frac{1}{2} f'' f(\eta)\frac{1}{2x^*}] - (\frac{1}{4}\frac{Re_\delta}{x^*}) f''' = 0$$

$$f'' f + f''' = 0$$

And we're done!
Now to transform the BCs:

$$u = 0 \ at \ y = 0$$

$$v = 0 \ at \ y = 0$$

$$u \to u_\infty \ at \ y \to \infty$$

$$\eta = \frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}} = \frac{1}{2}y\sqrt{\frac{u_\infty}{\nu x}}$$

$$f(\eta) = \frac{\psi^*}{\sqrt{x^* Re_\delta}} = \frac{\psi}{\sqrt{u_\infty x \nu}}$$

$$u = 0 \ at \ \eta = 0$$

$$v = 0 \ at \ \eta = 0$$

$$u \to u_\infty \ at \ \eta \to \infty$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y} f(\eta)\sqrt{u_\infty x \nu}$$

$$u = \sqrt{u_\infty x \nu}\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta} f(\eta) = \sqrt{u_\infty x \nu}\frac{1}{2}\sqrt{\frac{u_\infty}{\nu x}} f'$$

$$u = \frac{u_\infty}{2} f'$$

$$u = u_\infty \to f' = 2$$

$$v = -\frac{\partial \psi}{\partial x}$$

$$v = -\frac{\partial}{\partial x} f(\eta)\sqrt{u_\infty x \nu}$$

$$v = -[f(\eta)\frac{\partial}{\partial x}\sqrt{u_\infty x \nu} + \sqrt{u_\infty x \nu}\frac{\partial}{\partial x} f(\eta)]$$

$$v = -[f(\eta)\frac{\partial}{\partial x}\sqrt{u_\infty x \nu} + \sqrt{u_\infty x \nu}\frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} f(\eta)]$$

some steps later...

$$v = -\frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}(\eta f' - f)$$

$$f' = 0 \ at \ \eta = 0$$

$$f = 0 \; at \; \eta = 0$$

$$f' = 2 \; at \; \eta \to \infty$$

$$f''f + f''' = 0$$

3.1.2 how to solve Blasius's equation

$$f' = 0 \; at \; \eta = 0$$

$$f = 0 \; at \; \eta = 0$$

$$f' = 2 \; at \; \eta \to \infty$$

$$f''f + f''' = 0$$

Series solution (aka Frobenius method)

`https://mathworld.wolfram.com/FrobeniusMethod.html`
`http://naca.central.cranfield.ac.uk/reports/1950/naca-tm-1256.pdf`

Assumes:

$$f = \sum_{n=0}^{\infty} a_n \eta^n$$

$$f' = \sum_{n=0}^{\infty} a_{n+1}(n+1)\eta^n$$

$$f'' = \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)\eta^n$$

$$f''' = \sum_{n=0}^{\infty} a_{n+3}(n+1)(n+2)(n+3)\eta^n$$

Runge Kutta Methods numerical methods...

`https://en.wikipedia.org/wiki/Runge%E2%80%93Kutta_methods`
`https://projects.exeter.ac.uk/fluidflow/Courses/FluidDynamics3211-2/BoundaryLayers/rk`

`https://www.researchgate.net/publication/259772650_Numerical_Approximations_of_Blasiu`

3.2 Integral Solution by Theodore Von Karman

We are interested in τ_x which is wall shear stress for x direction

we perform force balance on the BL in x dir. We look at the diagram and see the trapezium

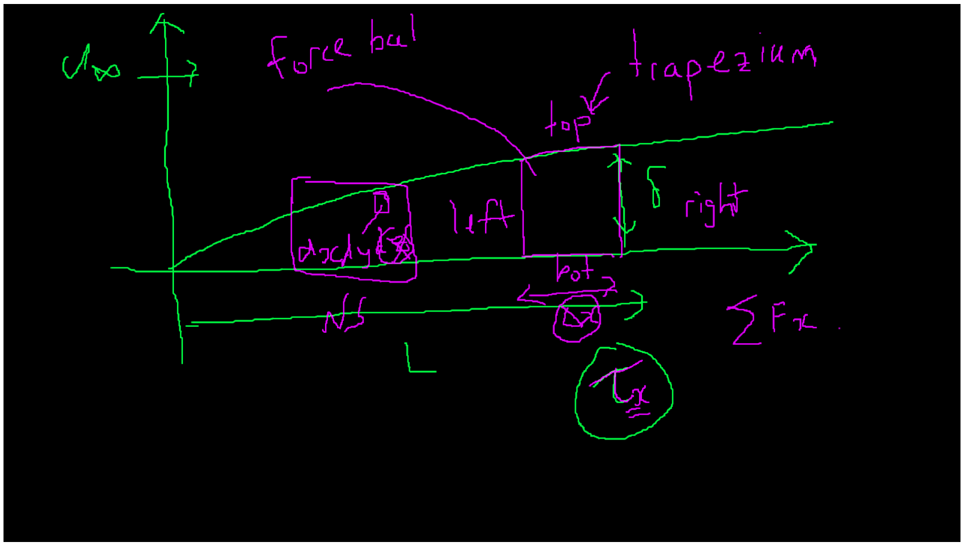

Figure 1: vonkarman BL

$$\sum F_x = top + bottom + left + right$$

In equilibrium

$$\sum F_x = net\ outflow\ of\ momentum\ in\ CV + accumulation\ term$$

$$accumulation\ term = 0$$

$$bottom = -\tau_x l_z \Delta x$$

$$left = (P\delta_p l_z)|_x$$

$$right = -(P\delta_p l_z)|_{x+\Delta x}$$

$$top = \frac{P|_x + P|_{x+\Delta x}}{2}(\delta_{x+\Delta x} - \delta_x)l_z$$

let's take a look at net outflow of momentum:
left side outflows:

$$-\int \rho u^2 dA|_x = -l_z \int_0^{\delta_p} \rho u^2 dy|_x$$

right side outflow:

$$\int \rho u^2 dA|_{x+\Delta x} = l_z \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x}$$

Top side outflow:

$$-\dot{m}_{top} u_\infty$$

Now we need an expression for $\dot{m}_{top}$

$$\dot{m}_{top} = \dot{m}_{right} - \dot{m}_{left}$$

$$\dot{m}_{top} = l_z \int_0^{\delta_p|_{x+\Delta x}} \rho u|_{x+\Delta x} dy - l_z \int_0^{\delta_p|_x} \rho u|_x dy$$

total momentum outlflow

$$net\, outflow\, of\, momentum = -l_z \int_0^{\delta_p} \rho u^2 dy|_x + l_z \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x} - \dot{m}_{top} u_\infty$$

$$net\ outflow\ of\ momentum = -l_z \int_0^{\delta_p} \rho u^2 dy|_x + l_z \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x}$$

$$-(l_z \int_0^{\delta_p|_{x+\Delta x}} \rho u|_{x+\Delta x} dy - l_z \int_0^{\delta_p|_x} \rho u|_x dy) u_\infty$$

Sum of forces:

$$\sum F_x = -\tau_x l_z \Delta x + (P\delta_p l_z)|_x - (P\delta_p l_z)|_{x+\Delta x} + \frac{P|_x + P|_{x+\Delta x}}{2}(\delta_{x+\Delta x} - \delta_x)$$

Equating both:

$$-\tau_x l_z \Delta x + (P\delta_p l_z)|_x - (P\delta_p l_z)|_{x+\Delta x} + \frac{P|_x + P|_{x+\Delta x}}{2}(\delta_{x+\Delta x} - \delta_x) l_z =$$

$$= -l_z \int_0^{\delta_p} \rho u^2 dy|_x + l_z \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x}$$

$$-(l_z \int_0^{\delta_p|_{x+\Delta x}} \rho u|_{x+\Delta x} dy - l_z \int_0^{\delta_p|_x} \rho u|_x dy) u_\infty$$

Cancel out l_z

$$-\tau_x \Delta x + (P\delta_p)|_x - (P\delta_p)|_{x+\Delta x} + \frac{P|_x + P|_{x+\Delta x}}{2}(\delta_{x+\Delta x} - \delta_x) =$$

$$= -\int_0^{\delta_p} \rho u^2 dy|_x + \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x}$$

$$-(\int_0^{\delta_p|_{x+\Delta x}} \rho u|_{x+\Delta x} dy - \int_0^{\delta_p|_x} \rho u|_x dy) u_\infty$$

Divide by Δx limit $\Delta x \to 0$

$$-\tau_x + \frac{(P\delta_p)|_x - (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2}\frac{P|_x + P|_{x+\Delta x}}{\Delta x}(\delta_p|_{x+\Delta x} - \delta_p|_x) =$$

$$= \frac{-\int_0^{\delta_p} \rho u^2 dy|_x + \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x}}{\Delta x}$$

$$-\frac{1}{\Delta x}(\int_0^{\delta_p|_{x+\Delta x}} \rho u|_{x+\Delta x} dy - \int_0^{\delta_p|_x} \rho u|_x dy) u_\infty$$

Take limits first:

$$\lim_{\Delta x \to 0}[-\tau_x + \frac{(P\delta_p)|_x - (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2}\frac{P|_x + P|_{x+\Delta x}}{\Delta x}(\delta_p|_{x+\Delta x} - \delta_p|_x)] =$$

$$= \lim_{x \to 0}[\frac{-\int_0^{\delta_p} \rho u^2 dy|_x + \int_0^{\delta_p} \rho u^2 dy|_{x+\Delta x}}{\Delta x}$$

$$-\frac{1}{\Delta x}(\int_0^{\delta_p|_{x+\Delta x}} \rho u|_{x+\Delta x} dy - \int_0^{\delta_p|_x} \rho u|_x dy)u_\infty]$$

$$\lim_{\Delta x \to 0}[-\tau_x + \frac{(P\delta_p)|_x - (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2}\frac{P|_x + P|_{x+\Delta x}}{\Delta x}(\delta_p|_{x+\Delta x} - \delta_p|_x)] =$$

$$= \frac{\partial}{\partial x}\int_0^{\delta_p} \rho u^2 dy - \frac{\partial}{\partial x}(u_\infty \int_0^{\delta_p} \rho u dy)$$

[careless mistake here! last term on the right hand side (RHS)]

$$\lim_{\Delta x \to 0}[-\tau_x + \frac{(P\delta_p)|_x - (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2}\frac{P|_x + P|_{x+\Delta x}}{\Delta x}(\delta_p|_{x+\Delta x} - \delta_p|_x)] =$$

$$= \frac{\partial}{\partial x}\int_0^{\delta_p} \rho u^2 dy - u_\infty \frac{\partial}{\partial x}(\int_0^{\delta_p} \rho u dy)$$

Left hand side...

$$-\tau_x + \lim_{\Delta x \to 0}[\frac{(P\delta_p)|_x - (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2}\frac{P|_x + P|_{x+\Delta x}}{\Delta x}(\delta_p|_{x+\Delta x} - \delta_p|_x)] =$$

$$= \frac{\partial}{\partial x}\int_0^{\delta_p} \rho u^2 dy - u_\infty \frac{\partial}{\partial x}(\int_0^{\delta_p} \rho u dy)$$

look in the terms inside the limit:

$$\lim_{\Delta x \to 0}[\frac{(P\delta_p)|_x - (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2}\frac{P|_x + P|_{x+\Delta x}}{\Delta x}(\delta_p|_{x+\Delta x} - \delta_p|_x)]$$

$$= \lim_{\Delta x \to 0}[\frac{(P\delta_p)|_x - (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2}\frac{2P|_x + P|_{x+\Delta x} - P|_x}{\Delta x}(\delta_p|_{x+\Delta x} - \delta_p|_x)]$$

$$= \lim_{\Delta x \to 0}[\frac{(P\delta_p)|_x - (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2}\frac{P|_{x+\Delta x} - P|_x}{\Delta x}(\delta_p|_{x+\Delta x} - \delta_p|_x) + P|_x \frac{1}{\Delta x}(\delta_p|_{x+\Delta x} - \delta_p|_x)]$$

$$= P\frac{\partial \delta_p}{\partial x} + \lim_{\Delta x \to 0}[\frac{(P\delta_p)|_x - (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2}\frac{P|_{x+\Delta x} - P|_x}{\Delta x}(\delta_p|_{x+\Delta x} - \delta_p|_x)]$$

$$= P\frac{\partial \delta_p}{\partial x} + \lim_{\Delta x \to 0}[\frac{(P\delta_p)|_x - (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2}\frac{P|_{x+\Delta x} - P|_x}{\Delta x}(\delta_p|_{x+\Delta x}) - \frac{1}{2}\delta_p|_x\frac{P|_{x+\Delta x} - P|_x}{\Delta x}]$$

$$= P\frac{\partial \delta_p}{\partial x} + \lim_{\Delta x \to 0}[\frac{(P\delta_p)|_x - (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2}\frac{\delta_p|_{x+\Delta x}P|_{x+\Delta x} - \delta_p|_{x+\Delta x}P|_x}{\Delta x} - \frac{1}{2}\frac{\delta_p|_xP|_{x+\Delta x} - \delta_p|_xP|_x}{\Delta x}]$$

$$= P\frac{\partial \delta_p}{\partial x} + \lim_{\Delta x \to 0}[\frac{(P\delta_p)|_x - (P\delta_p)|_{x+\Delta x}}{\Delta x} + \frac{1}{2}\frac{(\delta_pP)_{x+\Delta x} - \delta_p|_{x+\Delta x}P|_x}{\Delta x} - \frac{1}{2}\frac{\delta_p|_xP|_{x+\Delta x} - (\delta_pP)|_x}{\Delta x}]$$

$$= P\frac{\partial \delta_p}{\partial x} + \lim_{\Delta x \to 0}[\frac{(1.5P\delta_p)|_x - 0.5(P\delta_p)|_{x+\Delta x}}{\Delta x} - \frac{1}{2}\frac{\delta_p|_{x+\Delta x}P|_x}{\Delta x} - \frac{1}{2}\frac{\delta_p|_xP|_{x+\Delta x}}{\Delta x}]$$

$$= P\frac{\partial \delta_p}{\partial x} + \lim_{\Delta x \to 0}[\frac{(P\delta_p)|_x + (0.5P\delta_p)|_x - 0.5(P\delta_p)|_{x+\Delta x}}{\Delta x} - \frac{1}{2}\frac{\delta_p|_{x+\Delta x}P|_x}{\Delta x} - \frac{1}{2}\frac{\delta_p|_xP|_{x+\Delta x}}{\Delta x}]$$

$$= P\frac{\partial \delta_p}{\partial x} - 0.5\frac{\partial P\delta_p}{\partial x} + \lim_{\Delta x \to 0}[\frac{(P\delta_p)|_x}{\Delta x} - \frac{1}{2}\frac{\delta_p|_{x+\Delta x}P|_x}{\Delta x} - \frac{1}{2}\frac{\delta_p|_xP|_{x+\Delta x}}{\Delta x}]$$

$$= P\frac{\partial \delta_p}{\partial x} - 0.5\frac{\partial P\delta_p}{\partial x} + \lim_{\Delta x \to 0}[\frac{(P\delta_p)|_x}{\Delta x} - \frac{1}{2}\frac{\delta_p|_{x+\Delta x}P|_x - \delta_p|_xP|_x + \delta_p|_xP_x}{\Delta x} - \frac{1}{2}\frac{\delta_p|_xP|_{x+\Delta x}}{\Delta x}]$$

$$= P\frac{\partial \delta_p}{\partial x} - 0.5\frac{\partial P\delta_p}{\partial x} + \lim_{\Delta x \to 0}[\frac{0.5(P\delta_p)|_x}{\Delta x} - \frac{1}{2}\frac{\delta_p|_{x+\Delta x}P|_x - \delta_p|_xP|_x}{\Delta x} - \frac{1}{2}\frac{\delta_p|_xP|_{x+\Delta x}}{\Delta x}]$$

$$= P\frac{\partial \delta_p}{\partial x} - 0.5\frac{\partial P\delta_p}{\partial x} - 0.5P\frac{\partial \delta_p}{\partial x} + \lim_{\Delta x \to 0}[-\frac{1}{2}\frac{\delta_p|_xP|_{x+\Delta x} - \delta_p|_xP|_x}{\Delta x}]$$

$$= P\frac{\partial \delta_p}{\partial x} - 0.5\frac{\partial P\delta_p}{\partial x} - 0.5P\frac{\partial \delta_p}{\partial x} - 0.5\delta_p\frac{\partial P}{\partial x}$$

$$= P\frac{\partial \delta_p}{\partial x} - 0.5P\frac{\partial \delta_p}{\partial x} - 0.5\delta_p\frac{\partial P}{\partial x} - 0.5P\frac{\partial \delta_p}{\partial x} - 0.5\delta_p\frac{\partial P}{\partial x}$$

$$= -0.5\delta_p \frac{\partial P}{\partial x} - 0.5\delta_p \frac{\partial P}{\partial x}$$

$$= -\delta_p \frac{\partial P}{\partial x}$$

Substitute back in...

$$-\tau_x - \delta_p \frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\int_0^{\delta_p} \rho u^2 dy - u_\infty \frac{\partial}{\partial x}(\int_0^{\delta_p} \rho u dy)$$

$$-\delta_p \frac{\partial P}{\partial x} = \tau_x + \frac{\partial}{\partial x}\int_0^{\delta_p} \rho u^2 dy - u_\infty \frac{\partial}{\partial x}(\int_0^{\delta_p} \rho u dy)$$

Only assumption is that

$$u_\infty = constant\ w.r.t\ y$$

Bernoulli's equation
[careless mistake here...]
I wrote

$$\frac{\partial P}{\partial x} = \rho u_\infty \frac{\partial u_\infty}{\partial x}$$

When it's actually

$$-\frac{\partial P}{\partial x} = \rho u_\infty \frac{\partial u_\infty}{\partial x}$$

we can assume this holds with or without flat plate...
Substitute back in:

$$\delta_p \rho u_\infty \frac{\partial u_\infty}{\partial x} = \tau_x + \frac{\partial}{\partial x}\int_0^{\delta_p} \rho u^2 dy - u_\infty \frac{\partial}{\partial x}(\int_0^{\delta_p} \rho u dy)$$

constant density fluid:

$$\delta_p u_\infty \frac{\partial u_\infty}{\partial x} = \frac{\tau_x}{\rho} + \frac{\partial}{\partial x}\int_0^{\delta_p} u^2 dy - u_\infty \frac{\partial}{\partial x}(\int_0^{\delta_p} u dy)$$

We'll need to tidy all this up:
2 tricks to use to tidy up equation

$$\frac{\partial}{\partial x} u_\infty^2 \delta_p = \delta_p u_\infty \frac{\partial u_\infty}{\partial x} + u_\infty \frac{\partial \delta_p u_\infty}{\partial x}$$

$$\delta_p = \int_0^{\delta_p} 1 dy = \int_0^{\delta_p} dy$$

$$\frac{\partial}{\partial x}u_\infty^2\int_0^{\delta_p}dy = \int_0^{\delta_p}dyu_\infty\frac{\partial u_\infty}{\partial x} + u_\infty\frac{\partial\int_0^{\delta_p}dyu_\infty}{\partial x}$$

$$\delta_p u_\infty\frac{\partial u_\infty}{\partial x} = \frac{\partial}{\partial x}u_\infty^2\delta_p - u_\infty\frac{\partial\delta_p u_\infty}{\partial x}$$

substitute back in:

$$\frac{\partial}{\partial x}u_\infty^2\delta_p - u_\infty\frac{\partial\delta_p u_\infty}{\partial x} = \frac{\tau_x}{\rho} + \frac{\partial}{\partial x}\int_0^{\delta_p}u^2dy - u_\infty\frac{\partial}{\partial x}(\int_0^{\delta_p}udy)$$

[continue from here... note careless mistake]

1. Partial derivative wrong for mass balance 2. I think i said you can swap derivatives and integrals freely, do not do that. Leibniz's rule applied rather carelessly (don't carelessly swap integrals, leave it as δ_p first) 3. Bernoulli's equation sign is wrong

$$\frac{\partial}{\partial x}u_\infty^2\delta_p - u_\infty\frac{\partial\delta_p u_\infty}{\partial x} = \frac{\tau_x}{\rho} + \frac{\partial}{\partial x}\int_0^{\delta_p}u^2dy - u_\infty\frac{\partial}{\partial x}(\int_0^{\delta_p}udy)$$

Combine terms:

$$\frac{\partial}{\partial x}[u_\infty^2\delta_p - \int_0^{\delta_p}u^2dy] + u_\infty\frac{\partial}{\partial x}[\int_0^{\delta_p}udy - \delta_p u_\infty] = \frac{\tau_x}{\rho}$$

Replace δ_p with:

$$\delta_p = \int_0^{\delta_p}1dy = \int_0^{\delta_p}dy$$

Assume

$$\frac{\partial}{\partial y}U_\infty = 0$$

$$\frac{\partial}{\partial x}[\int_0^{\delta_p}u_\infty^2dy - \int_0^{\delta_p}u^2dy] + u_\infty\frac{\partial}{\partial x}[-\int_0^{\delta_p}u_\infty dy + \int_0^{\delta_p}udy] = \frac{\tau_x}{\rho}$$

$$\frac{\partial}{\partial x}\int_0^{\delta_p}(u_\infty^2 - u^2)dy + u_\infty\frac{\partial}{\partial x}\int_0^{\delta_p}(u - u_\infty)dy = \frac{\tau_x}{\rho}$$

get a $a^2 - b^2 = (a+b)(a-b)$ expansion

$$\frac{\partial}{\partial x}\int_0^{\delta_p}(u_\infty - u)(u+u_\infty)dy + u_\infty\frac{\partial}{\partial x}\int_0^{\delta_p}(u-u_\infty)dy = \frac{\tau_x}{\rho}$$

separate out first integral

$$\frac{\partial}{\partial x}\int_0^{\delta_p}(u_\infty - u)udy + \frac{\partial}{\partial x}\int_0^{\delta_p}(u_\infty - u)u_\infty dy + u_\infty\frac{\partial}{\partial x}\int_0^{\delta_p}(u-u_\infty)dy = \frac{\tau_x}{\rho}$$

Notice the last integral, it is the odd one out in terms of derivatives, let's change it using product rule

$$u_\infty\frac{\partial}{\partial x}\int_0^{\delta_p}(u-u_\infty)dy$$

$$= \frac{\partial}{\partial x}[u_\infty\int_0^{\delta_p}(u-u_\infty)dy] - (\frac{\partial u_\infty}{\partial x})\int_0^{\delta_p}(u-u_\infty)dy$$

We can substitute this back in:

$$\frac{\partial}{\partial x}\int_0^{\delta_p}(u_\infty - u)udy + \frac{\partial}{\partial x}\int_0^{\delta_p}(u_\infty - u)u_\infty dy + \frac{\partial}{\partial x}[u_\infty\int_0^{\delta_p}(u-u_\infty)dy]]$$

$$-(\frac{\partial u_\infty}{\partial x})\int_0^{\delta_p}(u-u_\infty)dy = \frac{\tau_x}{\rho}$$

Now we can bring the u_∞ into the dy integral

$$\frac{\partial}{\partial x}\int_0^{\delta_p}(u_\infty - u)udy + \frac{\partial}{\partial x}\int_0^{\delta_p}(u_\infty - u)u_\infty dy + \frac{\partial}{\partial x}[\int_0^{\delta_p}u_\infty(u-u_\infty)dy]]$$

$$-(\frac{\partial u_\infty}{\partial x})\int_0^{\delta_p}(u-u_\infty)dy = \frac{\tau_x}{\rho}$$

Terms inside cancel each other out...

$$\frac{\partial}{\partial x}\int_0^{\delta_p}(u_\infty - u)udy + \frac{\partial}{\partial x}\int_0^{\delta_p}(u_\infty - u)u_\infty dy - \frac{\partial}{\partial x}[\int_0^{\delta_p}u_\infty(u_\infty-u)dy]]$$

$$-(\frac{\partial u_\infty}{\partial x})\int_0^{\delta_p}(u-u_\infty)dy = \frac{\tau_x}{\rho}$$

bring minus sign in,

$$\frac{\partial}{\partial x}\int_0^{\delta_p}(u_\infty - u)udy + (\frac{\partial u_\infty}{\partial x})\int_0^{\delta_p}(u_\infty - u)dy = \frac{\tau_x}{\rho}$$

Final form of Von Karman equation:

$$\frac{\tau_x}{\rho} = (\frac{\partial u_\infty}{\partial x})\int_0^{\delta_p}(u_\infty - u)\ dy + \frac{\partial}{\partial x}\int_0^{\delta_p} u(u_\infty - u)\ dy$$

4 Solutions to BL for Von Karman

Pohlhausen solution:

$$u = a + by + cy^2 + dy^3$$

Consider boundary conditions
no slip:

$$u = 0\ at\ y = 0$$

This implies a=0

$$u = by + cy^2 + dy^3$$

$$y \to \infty\ u = u_\infty$$

$$y \to \delta_p\ u \to u_\infty$$

approximation:

$$y = \delta_p\ u = u_\infty$$

$$u_\infty = b\delta_p + c\delta_p^2 + d\delta_p^3 \tag{2}$$

exact:

$$y = \delta_p\ u = 0.99u_\infty$$

inviscid flow near δ_p and shear stress is 0 there

$$\frac{\partial u}{\partial y} = 0\ at\ y = \delta_p$$

$$\frac{\partial u}{\partial y} = b + 2cy + 3dy^2$$

$$0 = b + 2c\delta_p + 3d\delta_p^2 \tag{3}$$

Last BC:

$$\frac{\partial^2 u}{\partial y^2} = 0 \ at \ y = 0$$

in other words

$$\frac{\partial \tau_x}{\partial y} = 0$$

$$\frac{\partial^2 u}{\partial y^2} = 2c + 6dy$$

$$0 = 2c$$

$$c = 0$$

2 equations left:

$$0 = b + 3d\delta_p^2$$

$$u_\infty = b\delta_p + d\delta_p^3$$

$$b = -3d\delta_p^2$$

$$u_\infty = -3d\delta_p^3 + d\delta_p^3$$

$$u_\infty = -2d\delta_p^3$$

$$d = -\frac{u_\infty}{2\delta_p^3} \tag{4}$$

$$b = -3\delta_p^2(-\frac{u_\infty}{2\delta_p^3})$$

$$b = \frac{3}{2\delta_p} u_\infty \tag{5}$$

Pohlhausen velocity profile

$$u = \frac{3}{2\delta_p} u_\infty y - \frac{u_\infty}{2\delta_p^3} y^3$$

if we wanted to include the 0.99 u_∞

$$u = 0.99(\frac{3}{2\delta_p} u_\infty y - \frac{u_\infty}{2\delta_p^3} y^3)$$

substitute back in von karman equation

$$\frac{\tau_x}{\rho} = (\frac{\partial u_\infty}{\partial x}) \int_0^{\delta_p} (u_\infty - u)\ dy + \frac{\partial}{\partial x} \int_0^{\delta_p} u(u_\infty - u)\ dy$$

under constant freestream velocity:

$$\frac{\tau_x}{\rho} = \frac{\partial}{\partial x} \int_0^{\delta_p} u(u_\infty - u)\ dy$$

$$\frac{\tau_x}{\rho} = \frac{\partial}{\partial x} \int_0^{\delta_p} (\frac{3}{2\delta_p} u_\infty y - \frac{u_\infty}{2\delta_p^3} y^3)(u_\infty - (\frac{3}{2\delta_p} u_\infty y - \frac{u_\infty}{2\delta_p^3} y^3))\ dy$$

note that δ_p only changes with x, not y

$$\frac{\tau_x}{\rho} = \frac{\partial}{\partial x}(\frac{3}{2\delta_p} u_\infty \frac{\delta_p^2}{2} - \frac{u_\infty}{2\delta_p^3} \frac{\delta_p^4}{4})(u_\infty \delta_p - (\frac{3}{2\delta_p} u_\infty \frac{\delta_p^2}{2} - \frac{u_\infty}{2\delta_p^3} \frac{\delta_p^4}{4}))$$

$$\frac{\tau_x}{\rho} = \frac{\partial}{\partial x}(\frac{3}{2\delta_p} u_\infty \frac{\delta_p^2}{2} - \frac{u_\infty}{2\delta_p^3} \frac{\delta_p^4}{4})(u_\infty \delta_p - \frac{3}{2\delta_p} u_\infty \frac{\delta_p^2}{2} + \frac{u_\infty}{2\delta_p^3} \frac{\delta_p^4}{4})$$

$$\frac{\tau_x}{\rho} = \frac{\partial}{\partial x}(u_\infty \delta_p(\frac{3}{2\delta_p} u_\infty \frac{\delta_p^2}{2} - \frac{u_\infty}{2\delta_p^3} \frac{\delta_p^4}{4}) - \frac{3}{2\delta_p} u_\infty \frac{\delta_p^2}{2}(\frac{3}{2\delta_p} u_\infty \frac{\delta_p^2}{2} - \frac{u_\infty}{2\delta_p^3} \frac{\delta_p^4}{4}) + \frac{u_\infty}{2\delta_p^3} \frac{\delta_p^4}{4}(\frac{3}{2\delta_p} u_\infty \frac{\delta_p^2}{2} - \frac{u_\infty}{2\delta_p^3} \frac{\delta_p^4}{4}))$$

$$\frac{\tau_x}{\rho} = \frac{\partial}{\partial x}((\frac{3}{2\delta_p} u_\infty^2 \frac{\delta_p^3}{2} - \frac{u_\infty^2}{2\delta_p^3} \frac{\delta_p^5}{4})$$

$$-\frac{3}{2\delta_p} u_\infty \frac{\delta_p^2}{2}(\frac{3}{2\delta_p} u_\infty \frac{\delta_p^2}{2} - \frac{u_\infty}{2\delta_p^3} \frac{\delta_p^4}{4}) + \frac{u_\infty}{2\delta_p^3} \frac{\delta_p^4}{4}(\frac{3}{2\delta_p} u_\infty \frac{\delta_p^2}{2} - \frac{u_\infty}{2\delta_p^3} \frac{\delta_p^4}{4}))$$

(not very efficient...)

The more efficient method:

$$\frac{\tau_x}{\rho} = \frac{\partial}{\partial x}\int_0^{\delta_p} (\frac{3}{2}u_\infty\frac{y}{\delta_p} - \frac{u_\infty}{2}(\frac{y}{\delta_p})^3)(u_\infty - (\frac{3}{2}u_\infty(\frac{y}{\delta_p}) - \frac{u_\infty}{2}(\frac{y}{\delta_p})^3))\ dy$$

$$\frac{\tau_x}{\rho} = u_\infty^2\frac{\partial}{\partial x}\int_0^{\delta_p} (\frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3)(1 - (\frac{3}{2}(\frac{y}{\delta_p}) - \frac{1}{2}(\frac{y}{\delta_p})^3))\ dy$$

Use chain rule:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$dy = \frac{dy}{du}du$$

$$u = \frac{y}{\delta_p}$$

$$du = \frac{1}{\delta_p}dy$$

$$\delta_p du = dy$$

Change integration variable:

$$u = \frac{y}{\delta_p}$$

$$\frac{\tau_x}{\rho} = u_\infty^2\frac{\partial}{\partial x}\delta_p\int_0^{1} (\frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3)(1 - (\frac{3}{2}(\frac{y}{\delta_p}) - \frac{1}{2}(\frac{y}{\delta_p})^3))\ d\frac{y}{\delta_p}$$

$$\frac{\tau_x}{\rho} = u_\infty^2\frac{\partial}{\partial x}\delta_p\int_0^{1} (\frac{3}{2}u - \frac{1}{2}(u)^3)(1 - (\frac{3}{2}u) + \frac{1}{2}(u)^3)\ du$$

$$\frac{\tau_x}{\rho}|_{wall} = 0.139286u_\infty^2\frac{\partial}{\partial x}\delta_p$$

From textbook:

$$\frac{\tau_x}{\rho}|_{wall} = \frac{39}{280}u_\infty^2\frac{\partial}{\partial x}\delta_p$$

Substitute in shear stress

$$\tau_x|_{wall} = \rho\nu\frac{\partial u}{\partial y}|_{y=0}$$

shear stress

$$\frac{\partial u}{\partial y} = \frac{3}{2\delta_p}u_\infty + 3(-\frac{u_\infty}{2\delta_p^3})y^2$$

at y=0 we find $\tau_x|_{y=0}$

$$\frac{\partial u}{\partial y} = \frac{3}{2\delta_p}u_\infty$$

$$\frac{\tau_x|_{wall}}{\rho} = \nu\frac{\partial u}{\partial y}|_{y=0} = \nu\frac{3}{2\delta_p}u_\infty$$

Substitute back:

$$\nu\frac{3}{2\delta_p}u_\infty = \frac{39}{280}u_\infty^2\frac{\partial}{\partial x}\delta_p$$

$$\nu\frac{1}{\delta_p} = \frac{13}{140}u_\infty\frac{\partial}{\partial x}\delta_p$$

$$\nu\frac{1}{\delta_p} = \frac{13}{140}u_\infty\frac{d}{dx}\delta_p$$

$$\nu\frac{140}{13}dx = u_\infty\delta_p d\delta_p$$

$$\nu\frac{140}{13}\int_0^x dx = u_\infty\int_0^{\delta_p(x)}\delta_p d\delta_p$$

In proper math notation, we have to use dummy variables

$$\nu\frac{140}{13}\int_0^x dx' = u_\infty\int_0^{\delta_p(x)}\delta_p' d\delta_p'$$

Integrating:

$$\nu\frac{140}{13}x = u_\infty\frac{\delta_p(x)^2}{2}$$

$$\nu\frac{280}{13}x = u_\infty\delta_p(x)^2$$

$$\delta_p(x) = \sqrt{\frac{280}{13}\frac{\nu x}{u_\infty}}$$

$$\delta_p(x) = \sqrt{\frac{280}{13}\frac{\nu}{u_\infty x}x^2}$$

$$\delta_p(x)\frac{1}{x} = \sqrt{\frac{280}{13}\frac{\nu}{u_\infty x}}$$

$$\frac{\delta_p(x)}{x} = \sqrt{\frac{280}{13}\frac{1}{Re_x}}$$

$$\frac{\delta_p(x)}{x} = 4.64095\sqrt{\frac{1}{Re_x}}$$

Where $Re_x = \frac{u_\infty x}{\nu}$
To get shear stress

$$\frac{\tau_x|_{wall}}{\rho} = \nu\frac{3}{2\delta_p}u_\infty$$

$$\frac{\tau_x|_{wall}}{\rho} = \nu\frac{3}{2 * 4.64095\sqrt{\frac{1}{Re_x}}x}u_\infty$$

Skin coefficient of friction (local)

$$C_{fx} \equiv \frac{\tau_x|_{wall}}{\frac{1}{2}\rho u_\infty^2}$$

$$C_{fx} \equiv \frac{2}{u_\infty^2}\nu\frac{3}{2 * 4.64095\sqrt{\frac{1}{Re_x}}x}u_\infty$$

$$C_{fx} \equiv \frac{0.6464}{u_\infty}\nu\frac{1}{\sqrt{\frac{1}{Re_x}}x}$$

$$C_{fx} \equiv \frac{0.6464}{1}\frac{1}{\sqrt{\frac{1}{Re_x}}Re_x}$$

$$C_{fx} \equiv \frac{0.6464}{\sqrt{Re_x}}$$

Average skin friction coefficient:

$$C_{fL} \equiv \frac{1}{L}\int_0^L C_{fx}dx$$

$$C_{fL} \equiv \frac{1}{L}\int_0^L \frac{0.6464}{\sqrt{Re_x}}dx$$

$$C_{fL} \equiv \frac{1}{L}\int_0^L \frac{0.6464\sqrt{\nu}}{\sqrt{u_\infty x}}dx$$

$$C_{fL} \equiv \frac{0.6464\sqrt{\nu}}{\sqrt{u_\infty}L}\int_0^L \frac{1}{\sqrt{x}}dx$$

$$C_{fL} \equiv \frac{0.6464\sqrt{\nu}}{\sqrt{u_\infty}L}\int_0^L \frac{1}{\sqrt{x}}dx$$

$$C_{fL} \equiv \frac{0.6464\sqrt{\nu}}{\sqrt{u_\infty}L}2\sqrt{L}$$

$$C_{fL} \equiv \frac{0.6464\sqrt{\nu}}{\sqrt{u_\infty}L}2\sqrt{L}$$

$$C_{fL} \equiv \frac{1.2928\sqrt{\nu}}{\sqrt{u_\infty L}}$$

$$C_{fL} \equiv \frac{1.2928}{\sqrt{Re_L}}$$

4.0.1 Solution Comparison to Similarity Solution

Von Karman Results (approximate solution)

$$\frac{\delta_p(x)}{x} = \frac{4.64095}{\sqrt{Re_x}}$$

$$C_{fx} \equiv \frac{0.6464}{\sqrt{Re_x}}$$

$$C_{fL} \equiv \frac{1.2928}{\sqrt{Re_L}}$$

Blasius Results (Exact solution)

$$\frac{\delta_p(x)}{x} = \frac{5}{\sqrt{Re_x}}$$

$$C_{fx} \equiv \frac{0.664}{\sqrt{Re_x}}$$

$$C_{fL} \equiv \frac{1.328}{\sqrt{Re_L}}$$

7.2% off for BL thickness, and 3% off for skin friction coeff. Pretty good!

This shows that Von Karman method is pretty good, if you can't use Blasius results

`Welty, J., Rorrer, G. L., & Foster, D. G. (2014).`
`Fundamentals of momentum, heat, and mass transfer. John Wiley & Sons.`

5 Resources Online

`http://web.mit.edu/fluids-modules/www/highspeed_flows/ver2/bl_Chap2.pdf`
`https://community.dur.ac.uk/suzanne.fielding/teaching/BLT/sec3.pdf`

`for Von Karman`
`https://nptel.ac.in/content/storage2/courses/112104118/lecture-29/29-3_momentum.htm`

Part III
Github Repo

`https://github.com/theodoreOnzGit/heatTransferTheory_YouTube`

Look under convection heat transfer...