Fluid Mechanics YouTube

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Part I
Navier Stokes Equations

Compressible N-S equations

$$\frac{\partial}{\partial t}(\rho\vec{u}) + \nabla \bullet (\rho\vec{u} \otimes \vec{u}) = -\nabla p + \mu\nabla^2\vec{u} + \frac{1}{3}\mu\nabla(\nabla \bullet \vec{u}) + \rho\vec{g}$$

tensor or outer product:

$$\vec{u} \otimes \vec{v} = \vec{u}\vec{v}^T$$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \ and \ \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \otimes \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

$$= \begin{pmatrix} u_1v_1 & u_1v_2 & u_1v_3 \\ u_2v_1 & u_2v_2 & u_2v_3 \\ u_3v_1 & u_3v_2 & u_3v_3 \end{pmatrix}$$

Inner product

$$\vec{u} \bullet \vec{v} = (\vec{u}^T\vec{v})^T$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \bullet \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= u_1v_1 + u_2v_2 + u_3v_3$$

Assume incompressible flow:

$$\rho = constant$$

continuity equation

$$\nabla \bullet \vec{u} = 0$$

Incompressible N-S equations

$$\frac{\partial}{\partial t}\vec{u} + (\vec{u} \bullet \nabla)\vec{u} - \nu\nabla^2\vec{u} = -\nabla\frac{P}{\rho_0} + \vec{g}$$

`https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations`

`Matrices in LaTeX`
`https://www.overleaf.com/learn/latex/Matrices`

`Tensors in LaTeX`

`Navier Stokes Equations`
`https://www.comsol.com/multiphysics/navier-stokes-equations`
`https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations`

`Github`
`https://github.com/theodoreOnzGit/heatTransferTheory_YouTube`

First let's deal with:

$$(\vec{u} \bullet \nabla)\vec{u}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \otimes \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \tag{1}$$

$$= \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}$$

Then we do inner product

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}$$

$$\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}$$

$$= \begin{pmatrix} u_1\frac{\partial}{\partial x}u_1 + u_2\frac{\partial}{\partial y}u_1 + u_3\frac{\partial}{\partial z}u_1 \\ u_1\frac{\partial}{\partial x}u_2 + u_2\frac{\partial}{\partial y}u_2 + u_3\frac{\partial}{\partial z}u_2 \\ u_1\frac{\partial}{\partial x}u_3 + u_2\frac{\partial}{\partial y}u_3 + u_3\frac{\partial}{\partial z}u_3 \end{pmatrix}$$

Let's deal with the momentum diffusivity (kinematic viscosity) term:

$$\nabla^2 = (\nabla \bullet \nabla)$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x}u_1 + \frac{\partial}{\partial y}\frac{\partial}{\partial y}u_1 + \frac{\partial}{\partial z}\frac{\partial}{\partial z}u_1 \\ \frac{\partial}{\partial x}\frac{\partial}{\partial x}u_2 + \frac{\partial}{\partial y}\frac{\partial}{\partial y}u_2 + \frac{\partial}{\partial z}\frac{\partial}{\partial z}u_2 \\ \frac{\partial}{\partial x}\frac{\partial}{\partial x}u_3 + \frac{\partial}{\partial y}\frac{\partial}{\partial y}u_3 + \frac{\partial}{\partial z}\frac{\partial}{\partial z}u_3 \end{pmatrix}$$

Part II

Boundary Layer Equations

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}\ u + \frac{\partial^2}{\partial z^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}\ v + \frac{\partial^2}{\partial z^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

$$\frac{\partial}{\partial t}w+u\frac{\partial}{\partial x}w+v\frac{\partial}{\partial y}w+w\frac{\partial}{\partial z}w-\nu(\frac{\partial^2}{\partial x^2}w+\frac{\partial^2}{\partial y^2}\ w+\frac{\partial^2}{\partial z^2}w)=-\frac{1}{\rho_0}\frac{\partial P}{\partial z}+g_z$$

Now for 2D what do we do? w=0 everywhere and at all times, $g_z = 0$
We eliminate z terms from the x and y momentum balance

$$\frac{\partial}{\partial t}u+u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial y}u-\nu(\frac{\partial^2}{\partial x^2}u+\frac{\partial^2}{\partial y^2}\ u+\frac{\partial^2}{\partial z^2}u)=-\frac{1}{\rho_0}\frac{\partial P}{\partial x}+g_x$$

$$\frac{\partial}{\partial t}v+u\frac{\partial}{\partial x}v+v\frac{\partial}{\partial y}v-\nu(\frac{\partial^2}{\partial x^2}v+\frac{\partial^2}{\partial y^2}\ v+\frac{\partial^2}{\partial z^2}v)=-\frac{1}{\rho_0}\frac{\partial P}{\partial y}+g_y$$

There is no spatial variation in u and v w.r.t z We have 2D Navier stokes:

$$\frac{\partial}{\partial t}u+u\frac{\partial}{\partial x}u+v\frac{\partial}{\partial y}u-\nu(\frac{\partial^2}{\partial x^2}u+\frac{\partial^2}{\partial y^2}\ u)=-\frac{1}{\rho_0}\frac{\partial P}{\partial x}+g_x$$

$$\frac{\partial}{\partial t}v+u\frac{\partial}{\partial x}v+v\frac{\partial}{\partial y}v-\nu(\frac{\partial^2}{\partial x^2}v+\frac{\partial^2}{\partial y^2}\ v)=-\frac{1}{\rho_0}\frac{\partial P}{\partial y}+g_y$$

continuity equation

$$\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v+\frac{\partial}{\partial z}w=0$$

$$\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v=0$$

1 nondimensionalisation

Order of magnitude

$$\mathcal{O}$$

Scaling for order magnitude comparison

$$u^*, y^* = \mathcal{O}(1)$$

we define:

$$u^* = \frac{u}{u_\infty}$$

$$u = u_\infty u^*$$

$$x^* = \frac{x}{L}$$

$$y^* = \frac{y}{\delta_p}$$

We scale our continutity equation:

$$\frac{\partial}{\partial x^* L} u^* u_\infty + \frac{\partial}{\partial y^* \delta_p} v = 0$$

$$\frac{\partial}{\partial x^* L} u^* u_\infty + \frac{\partial}{\partial y^* \delta_p} v = 0$$

$$\frac{u_\infty}{L} \frac{\partial}{\partial x^*} u^* + \frac{1}{\delta_p} \frac{\partial}{\partial y^*} v = 0$$

$$\frac{\partial}{\partial x^*} u^* + \frac{L}{\delta_p u_\infty} \frac{\partial}{\partial y^*} v = 0$$

$$v^* = \frac{vL}{\delta_p u_\infty} = \mathcal{O}(1)$$

$$v^* = \frac{v}{\frac{u_\infty \delta_p}{L}}$$

$$\frac{\partial}{\partial x^*} u^* + \frac{\partial}{\partial y^*} v^* = 0$$

Now we move on to the NS equations
so we need to scale time:
x lengthscale = L
x velocityscale = u_∞
timescale = $\frac{L}{u_\infty}$

$$t^* = \frac{t}{\frac{L}{u_\infty}}$$

Let's scale the momentum NS equations

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u - \nu\left(\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2}\ u\right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t} v + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu\left(\frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2}\ v\right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + g_y$$

Let's do x momentum equations

$$\frac{\partial}{\partial t}u^*u_\infty+u^*u_\infty\frac{\partial}{\partial x^*L}u^*u_\infty+v\frac{\partial}{\partial y}u^*u_\infty-\nu(\frac{\partial^2}{\partial x^2}u^*u_\infty+\frac{\partial^2}{\partial y^2}\ u^*u_\infty)=-\frac{1}{\rho_0}\frac{\partial P}{\partial x}+g_x$$

$$\frac{\partial}{\partial t}u^*+u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}u^*+v\frac{1}{\delta_p}\frac{\partial}{\partial y^*}u^*-\nu(\frac{1}{L^2}\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{1}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=\frac{1}{u_\infty}(-\frac{1}{L\rho_0}\frac{\partial P}{\partial x^*}+g_x)$$

$$\frac{u_\infty}{L}\frac{\partial}{\partial t^*}u^*+u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}u^*+v\frac{1}{\delta_p}\frac{\partial}{\partial y^*}u^*-\nu(\frac{1}{L^2}\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{1}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=\frac{1}{u_\infty}(-\frac{1}{L\rho_0}\frac{\partial P}{\partial x^*}+g_x)$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{\nu}{u_\infty L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=\frac{L}{u_\infty^2}(-\frac{1}{L\rho_0}\frac{\partial P}{\partial x^*}+g_x)$$

Reynold's number

$$Re_L=\frac{u_\infty L}{\nu}$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=\frac{L}{u_\infty^2}(-\frac{1}{L\rho_0}\frac{\partial P}{\partial x^*}+g_x)$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=(-\frac{1}{L\rho_0}\frac{L}{u_\infty^2}\frac{\partial P}{\partial x^*}+\frac{L}{u_\infty^2}g_x)$$

Let's scale gravity

$$g_x^*=\frac{g_x}{|g|}=\cos\theta_x=\mathcal{O}(1)$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=(-\frac{1}{\rho_0u_\infty^2}\frac{\partial P}{\partial x^*}+\frac{L|g|}{u_\infty^2}g_x^*)$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ u^*)=(-\frac{1}{\rho_0u_\infty^2}\frac{\partial P}{\partial x^*}+\frac{1}{Fr^2}g_x^*)$$

$$P^* = \frac{P}{\rho_0 u_\infty^2}$$

After nondimensionalisation, our x momentum equation becomes:

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ u^*) = (-\frac{\partial P^*}{\partial x^*}+\frac{1}{Fr^2}g_x^*)$$

we dimensionalise y momentum eqns

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}\ v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

First the x and time terms:

$$\frac{u_\infty}{L}\frac{\partial}{\partial t^*}v + u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}v + v\frac{\partial}{\partial y}v - \nu(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v + \frac{\partial^2}{\partial y^2}\ v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

Second y coordinate terms:

$$\frac{u_\infty}{L}\frac{\partial}{\partial t^*}v+u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}v+v\frac{1}{\delta_p}\frac{\partial}{\partial y^*}v-\nu(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v+\frac{1}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v) = -\frac{1}{\rho_0}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*}+g_y^*|g|$$

divide by $\frac{u_\infty^2}{L^2}$

$$\frac{L}{u_\infty}\frac{\partial}{\partial t^*}v+u^*\frac{L}{u_\infty}\frac{\partial}{\partial x^*}v+v\frac{L^2}{u_\infty^2\delta_p}\frac{\partial}{\partial y^*}v-\nu\frac{L^2}{u_\infty^2}(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v+\frac{1}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v) = \frac{L^2}{u_\infty^2}(-\frac{1}{\rho_0}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*}+g_y^*|g|)$$

divide by δ_p

$$\frac{L}{u_\infty\delta_p}\frac{\partial}{\partial t^*}v+u^*\frac{L}{u_\infty\delta_p}\frac{\partial}{\partial x^*}v+v\frac{L^2}{u_\infty^2\delta_p^2}\frac{\partial}{\partial y^*}v-\nu\frac{L^2}{u_\infty^2\delta_p}(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v+\frac{1}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v) = \frac{L^2}{u_\infty^2\delta_p}(-\frac{1}{\rho_0}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*}+g_y^*|g|)$$

Combining some terms

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\nu\frac{L}{u_\infty}(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{1}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*) = \frac{L^2}{u_\infty^2\delta_p}(-\frac{1}{\rho_0}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*}+g_y^*|g|)$$

Rearranging

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{\nu}{u_\infty L}(\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*) = (-\frac{1}{\rho_0}\frac{L^2}{u_\infty^2\delta_p}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*}+g_y^*|g|\frac{L^2}{u_\infty^2\delta_p})$$

nondimensionalisng pressure and including the Fr

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{\nu}{u_\infty L}(\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*)=(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

Include Re

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*)=(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

Review: NS nondimensionalised

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ u^*)=(-\frac{\partial P^*}{\partial x^*}+\frac{1}{Fr^2}g_x^*)$$

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*)=(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

$$\frac{\partial}{\partial x^*}u^*+\frac{\partial}{\partial y^*}v^*=0$$

2 How to drop terms?

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ u^*)=(-\frac{\partial P^*}{\partial x^*}+\frac{1}{Fr^2}g_x^*)$$

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*)=(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

$$\frac{\partial}{\partial x^*}u^*+\frac{\partial}{\partial y^*}v^*=0$$

When we want to determine which terms to cancel, we need to know how Re_L compares with $\frac{L^2}{\delta_p^2}$

Assumption:

Creeping flow in y direction

$$Re_\delta=\frac{v_c\delta_p}{\nu}=\mathcal{O}(1)$$

How does Re_δ compare to Re_L

$$v_c = u_\infty \frac{\delta_p}{L}$$

$$Re_\delta = \frac{u_\infty \frac{\delta_p}{L} \delta_p}{\nu} = \mathcal{O}(1)$$

$$Re_\delta = \frac{u_\infty L}{\nu} \frac{\delta_p^2}{L^2} = \mathcal{O}(1)$$

$$Re_\delta = Re_L \frac{\delta_p^2}{L^2} = \mathcal{O}(1)$$

2.0.1 x direction momentum eqn

$$\frac{\partial}{\partial t^*}u^* + u^* \frac{\partial}{\partial x^*}u^* + v^* \frac{\partial}{\partial y^*}u^* - \frac{1}{Re_L}(\frac{\partial^2}{\partial (x^*)^2}u^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2}\ u^*) = (-\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2} g_x^*)$$

$$\frac{\partial}{\partial t^*}u^* + u^* \frac{\partial}{\partial x^*}u^* + v^* \frac{\partial}{\partial y^*}u^* - \frac{1}{Re_L} \frac{\partial^2}{\partial (x^*)^2}u^* + \frac{1}{Re_L} \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2}\ u^* = (-\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2} g_x^*)$$

$$\frac{\partial}{\partial t^*}u^* + u^* \frac{\partial}{\partial x^*}u^* + v^* \frac{\partial}{\partial y^*}u^* - \frac{1}{Re_L} \frac{\partial^2}{\partial (x^*)^2}u^* + \frac{1}{\mathcal{O}(1)} \frac{\partial^2}{\partial (y^*)^2}\ u^* = (-\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2} g_x^*)$$

How big is Re_L?

$$Re_L = \mathcal{O}(\frac{L^2}{\delta_p^2})$$

We assume Fr is big

$$\frac{\partial}{\partial t^*}u^* + u^* \frac{\partial}{\partial x^*}u^* + v^* \frac{\partial}{\partial y^*}u^* - + \frac{1}{\mathcal{O}(1)} \frac{\partial^2}{\partial (y^*)^2}\ u^* = (-\frac{\partial P^*}{\partial x^*})$$

2.0.2 y direction momentum equation

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ v^*)=(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-(\frac{1}{Re_L}\frac{\partial^2}{\partial(x^*)^2}v^*+\frac{1}{\mathcal{O}(1)}\frac{\partial^2}{\partial(y^*)^2}\ v^*)=(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

$$[\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*]\frac{1}{Re_L}-(\frac{1}{Re_L^2}\frac{\partial^2}{\partial(x^*)^2}v^*+\frac{1}{\mathcal{O}(1)Re_L}\frac{\partial^2}{\partial(y^*)^2}\ v^*)=\frac{1}{Re_L}(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p})$$

$$[\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*]\frac{1}{Re_L}-(\frac{1}{Re_L^2}\frac{\partial^2}{\partial(x^*)^2}v^*+\frac{1}{\mathcal{O}(1)Re_L}\frac{\partial^2}{\partial(y^*)^2}\ v^*)$$

$$=(-\frac{L^2}{\delta_p^2}\frac{1}{Re_L}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p}\frac{1}{Re_L})$$

Cancelling out...

$$[\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*]\frac{1}{Re_L}-(\frac{1}{Re_L^2}\frac{\partial^2}{\partial(x^*)^2}v^*+\frac{1}{\mathcal{O}(1)Re_L}\frac{\partial^2}{\partial(y^*)^2}\ v^*)$$

$$=(-\frac{1}{\mathcal{O}(1)}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L^2}{\delta_p^2}\frac{1}{Re_L}\frac{\delta_p}{L})$$

Simplifying

$$[\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*]\frac{1}{Re_L}-(\frac{1}{Re_L^2}\frac{\partial^2}{\partial(x^*)^2}v^*+\frac{1}{\mathcal{O}(1)Re_L}\frac{\partial^2}{\partial(y^*)^2}\ v^*)$$

$$=\frac{1}{\mathcal{O}(1)}(-\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{\delta_p}{L})$$

For large Re_L

$$0=(-\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{\delta_p}{L})$$

$$g_y^*\frac{1}{Fr^2}\frac{\delta_p}{L}=\frac{\partial P^*}{\partial y^*}$$

Only if g=0,

$$0 = -\frac{\partial P^*}{\partial y^*}$$

Now we have our BL equations:

$$0 = -\frac{\partial P^*}{\partial y^*}$$

$$\frac{\partial}{\partial t^*}u^* + u^*\frac{\partial}{\partial x^*}u^* + v^*\frac{\partial}{\partial y^*}u^* - \frac{1}{\mathcal{O}(1)}\frac{\partial^2}{\partial (y^*)^2}\ u^* = (-\frac{\partial P^*}{\partial x^*})$$

redimensionalise to obtain the laminar BL equations:

$$0 = -\frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial (y)^2}\ u = (-\frac{\partial P}{\partial x})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

3 Solutions to the BL equations laminar

How to solve?

- 1st Similarity solution
- 2nd Von Karman Solution (Integral solution - approximate)
- 3rd numerical (CFD)

3.1 similarity solution (Blasius solution)

$$0 = -\frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial (y)^2}\ u = (-\frac{\partial P}{\partial x})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Similarity solution → combine variables to convert PDE to ODE
Making 2 assumptions before we continue:
1) steady state 2) no pressure gradient

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial y^2}\ u = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

How to combine variables to make life easier for us to solve?
introduce the streamfunction (ψ):

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

Note: streamfunction only works for 2D fluid flow
Substitute into 2D continuity equation,

$$\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x} = 0$$

Substitute into the 2D x momentum equation

$$(\frac{\partial \psi}{\partial y})\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial y}) + (-\frac{\partial \psi}{\partial x})\frac{\partial}{\partial y}(\frac{\partial \psi}{\partial y}) - \nu\frac{\partial^2}{\partial y^2}(\frac{\partial \psi}{\partial y}) = 0$$

$$(\frac{\partial \psi}{\partial y})\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial y}) - (\frac{\partial \psi}{\partial x})(\frac{\partial^2 \psi}{\partial y^2}) - \nu\frac{\partial^3}{\partial y^3}\psi = 0$$

We need to compress the number of variables further to 1 indep variable (η) and 1 dependent variable (f(η))

$$\eta = \eta(x, y)$$

$$f = f(\eta)$$

Before we continue, BCs first!
1 BC in x dir for u
2 BCs in y dir for u
no slip

$$u = 0\ at\ y = 0$$

1 BC in y direction for v
no slip

$$v = 0 \; at \; y = 0$$

4 Resources Online

http://web.mit.edu/fluids-modules/www/highspeed_flows/ver2/bl_Chap2.pdf
https://community.dur.ac.uk/suzanne.fielding/teaching/BLT/sec3.pdf

Part III
Github Repo

https://github.com/theodoreOnzGit/heatTransferTheory_YouTube

Look under convection heat transfer...