Convection BL YouTube

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Part I
Some Links before we start

https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations

Matrices in LaTeX
https://www.overleaf.com/learn/latex/Matrices

Tensors in LaTeX

Navier Stokes Equations
https://www.comsol.com/multiphysics/navier-stokes-equations
https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations

Github
https://github.com/theodoreOnzGit/heatTransferTheory_YouTube

Part II
Hydrodynamics

1 Navier Stokes Equations

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}\ u + \frac{\partial^2}{\partial z^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}\ v + \frac{\partial^2}{\partial z^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - \nu(\frac{\partial^2}{\partial x^2}w + \frac{\partial^2}{\partial y^2}\ w + \frac{\partial^2}{\partial z^2}w) = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + g_z$$

2 Boundary Layer Equations (Laminar)

$$0 = -\frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial y^2}\ u = (-\frac{\partial P}{\partial x})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

2.0.1 Blasius Similarity Solution

$$f' = 0\ at\ \eta = 0$$

$$f = 0\ at\ \eta = 0$$

$$f' = 2\ at\ \eta \to \infty$$

$$f''f + f''' = 0$$

$$u = \frac{u_\infty}{2}f'$$

$$\eta = \frac{1}{2}y\sqrt{\frac{u_\infty}{\nu x}}$$

[not derived previously:]

$$\tau_x = \mu\frac{\partial u}{\partial y} = \rho\nu\frac{u_\infty}{4}\sqrt{\frac{u_\infty}{\nu x}}f''$$

`Welty, J., Rorrer, G. L., & Foster, D. G. (2014).`
`Fundamentals of momentum, heat, and mass transfer. John Wiley & Sons.`

2.1 Integral Solution by Theodore Von Karman

Final form of Von Karman equation:

$$\frac{\tau_x}{\rho} = (\frac{\partial u_\infty}{\partial x})\int_0^{\delta_p}(u_\infty - u)\ dy + \frac{\partial}{\partial x}\int_0^{\delta_p} u(u_\infty - u)\ dy$$

3 Solutions to BL

3.1 Von Karman Results (approximate solution)

$$\frac{\delta_p(x)}{x} = \frac{4.64095}{\sqrt{Re_x}}$$

$$C_{fx} \equiv \frac{0.6464}{\sqrt{Re_x}}$$

$$C_{fL} \equiv \frac{1.2928}{\sqrt{Re_L}}$$

3.2 Blasius Results (Exact solution)

$$\frac{\delta_p(x)}{x} = \frac{5}{\sqrt{Re_x}}$$

$$C_{fx} \equiv \frac{0.664}{\sqrt{Re_x}}$$

$$C_{fL} \equiv \frac{1.328}{\sqrt{Re_L}}$$

7.2% off for BL thickness, and 3% off for skin friction coeff. Pretty good!

This shows that Von Karman method is pretty good, if you can't use Blasius results

`Welty, J., Rorrer, G. L., & Foster, D. G. (2014).`
`Fundamentals of momentum, heat, and mass transfer. John Wiley & Sons.`

Part III
Energy Equations

$$h \equiv e + PV$$

Enthalpy per unit mass (Specific enthalpy) P is pressure V is specific volume e is specific internal energy

In thermodynamics, U is often used for internal energy, but we avoid using that because it gets confused with u which is x velocity. So we use e instead.

Enthalpy Balance

$$\frac{\partial}{\partial t}h+u\frac{\partial}{\partial x}h+v\frac{\partial}{\partial y}h+w\frac{\partial}{\partial z}h = conduction+heat\,generation+dissipation+radiation \tag{1}$$

neglect heat generation and dissipation terms. Also neglect radiation heat transfer

Fourier's law

$$q''_{cond} = -k\nabla T$$

Conduction term:

$$conduction = -\nabla q''_{cond}$$

Heat flux $= q''_{cond}$;

Heat flux is heat energy Transferred per unit area

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = -\nabla q''_{cond}$$

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = -\nabla(-k\nabla T)$$

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = \nabla(k\nabla T)$$

heat conduction coeffcient is isotropic (it's a scalar), we also assume k does not change with x,y or z

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = k\nabla^2 T$$

$$\Delta h = \rho c_p \Delta T$$

$$h - h_{ref} = \rho c_p (T - T_{ref})$$

$$h = \rho c_p (T - T_{ref}) + h_{ref}$$

We note, reference temperature and enthalpy are constant with time, x y z etc.

$$\frac{\partial}{\partial t}\rho c_p T + u\frac{\partial}{\partial x}\rho c_p T + v\frac{\partial}{\partial y}\rho c_p T + w\frac{\partial}{\partial z}\rho c_p T = k\nabla^2 T$$

We assume, volumetric heat capacity does not change with x,y or z and time

$$\rho c_p = volumetric\ heat\ capacity = \frac{kg}{m^3}\frac{J}{kg \bullet k}$$

$$\frac{\partial}{\partial t}\rho c_p T + u\frac{\partial}{\partial x}\rho c_p T + v\frac{\partial}{\partial y}\rho c_p T + w\frac{\partial}{\partial z}\rho c_p T = k\nabla^2 T$$

$$\frac{\partial}{\partial t}T + u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T + w\frac{\partial}{\partial z}T = \frac{k}{\rho c_p}\nabla^2 T$$

$$\frac{\partial}{\partial t}T + u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T + w\frac{\partial}{\partial z}T = \alpha\nabla^2 T$$

$$\alpha = \frac{k}{\rho c_p}$$

$$\frac{\partial}{\partial t}T + u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T + w\frac{\partial}{\partial z}T = \alpha(\frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}\ T + \frac{\partial^2}{\partial z^2}T)$$

Part IV

Thermal Laminar BL Equations

First, we assume, 2D, so z derivatives are zero.

$$\frac{\partial}{\partial t}T + u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T + w\frac{\partial}{\partial z}T = \alpha(\frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}\ T + \frac{\partial^2}{\partial z^2}T)$$

$$\frac{\partial}{\partial t}T + u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T = \alpha(\frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}\ T)$$

We need to introduce relative temperature

$$T_{rel} = T - T_s$$

$$\frac{\partial}{\partial x}(T - T_s) = \frac{\partial}{\partial x}(T)$$

so long as T_s is constant

$$\frac{\partial}{\partial t}(T - T_s) + u\frac{\partial}{\partial x}(T - T_s) + v\frac{\partial}{\partial y}(T - T_s) = \alpha(\frac{\partial^2}{\partial x^2}(T - T_s) + \frac{\partial^2}{\partial y^2}(T - T_s))$$

Let's see which terms we can neglect

$$\theta = \frac{T - T_s}{T_\infty - T_s}$$
$$u^* = \frac{u}{u_\infty}$$

Let's nondimensionaise temrperature:

$$\frac{\partial}{\partial t}\theta(T_\infty - T_s) + u\frac{\partial}{\partial x}\theta(T_\infty - T_s) + v\frac{\partial}{\partial y}\theta(T_\infty - T_s) = \alpha(\frac{\partial^2}{\partial x^2}\theta(T_\infty - T_s) + \frac{\partial^2}{\partial y^2}\theta(T_\infty - T_s))$$

Divide throughout,

$$\frac{\partial}{\partial t}\theta + u\frac{\partial}{\partial x}\theta + v\frac{\partial}{\partial y}\theta = \alpha(\frac{\partial^2}{\partial x^2}\theta + \frac{\partial^2}{\partial y^2}\theta)$$

$$x^* = \frac{x}{L}$$

$$y^* = \frac{y}{\delta_t}$$

Note: in general, thermal boundary layer is different from momentum boundary layer

$$\frac{\partial}{\partial t}\theta + u\frac{1}{L}\frac{\partial}{\partial x^*}\theta + v\frac{1}{\delta_t}\frac{\partial}{\partial y^*}\theta = \alpha(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}\theta + \frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta)$$

$$u^* = \frac{u}{u_\infty}$$

$$\frac{\partial}{\partial t}\theta + u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}\theta + v\frac{1}{\delta_t}\frac{\partial}{\partial y^*}\theta = \alpha(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}\theta + \frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta)$$

We assume no natural convection

$$\frac{\partial}{\partial t}\theta + u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}\theta + v\frac{1}{\delta_t}\frac{\partial}{\partial y^*}\theta = \alpha(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}\theta + \frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta)$$

If we were to use momentum nondimensionalisation:

$$y^* = \frac{y}{\delta_p}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{u_\infty}{L}\frac{\partial u^*}{\partial x^*} + \frac{v_c}{\delta_p}\frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{u_\infty \delta_p}{v_c L}\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{u_\infty \delta_p}{v_c L} = \mathcal{O}(1)$$

$$\frac{u_\infty \delta_p}{L} = \mathcal{O}(1) v_c$$

we can just set:

$$v_c = \frac{u_\infty \delta_p}{L}\mathcal{O}(1)$$

so that

$$v^* = \frac{v}{v_c} = \mathcal{O}(1)$$

or else

$$v_c = \frac{u_\infty \delta_p}{L}$$

So we substitute this back in:

$$\frac{\partial}{\partial t}\theta + u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}\theta + \frac{u_\infty \delta_p}{L\delta_t}v^*\frac{\partial}{\partial y^*}\theta = \alpha(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}\theta + \frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta)$$

$$t = \frac{t^*}{t_c}$$

if you talk about hydrodynamic timescales:

$$t_c = \frac{L}{u_\infty}$$

we have to use thermal BL timescales...

For simplicity, we may not want to consider transient timescales in flat plate BL analysis

We assume steady state

$$u^* \frac{u_\infty}{L}\frac{\partial}{\partial x^*}\theta + \frac{u_\infty \delta_p}{L\delta_t} v^* \frac{\partial}{\partial y^*}\theta = \alpha(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}\theta + \frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta)$$

$$u^* \frac{\partial}{\partial x^*}\theta + \frac{\delta_p}{\delta_t} v^* \frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}\theta + \frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta)$$

We can make this observation:

$$L^2 >> \delta_t^2$$

$$u^* \frac{\partial}{\partial x^*}\theta + \frac{\delta_p}{\delta_t} v^* \frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta)$$

assume steady state, and performing BL analysis: Also, no dissipation or heat generation

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T = \alpha(\frac{\partial^2}{\partial y^2}\ T)$$

Part V
Solution Procedures - Constant Temp Forced Conv BL

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}T = \alpha\frac{\partial^2}{\partial y^2}T$$

$$0 = -\frac{\partial P}{\partial y}$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial y^2}\ u = (-\frac{\partial P}{\partial x})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Solution Procedures:

- Similarity Solution (Blasius Style) [GOLD STANDARD]
- Integral Solution (Von Karman Style)
- Computational Fluid Dynamics (MultiPhysics)

4 back to similarity Solution

In case of no pressure gradient,

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}T = \alpha\frac{\partial^2}{\partial y^2}T$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = \nu\frac{\partial^2}{\partial y^2}\ u$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

And nondimensionalising

$$f' = 0\ at\ \eta = 0$$

$$f = 0\ at\ \eta = 0$$

$$f' = 2\ at\ \eta \to \infty$$

$$f''f + f''' = 0$$

$$u = \frac{u_\infty}{2}f'$$

$$\eta = \frac{1}{2}y\sqrt{\frac{u_\infty}{\nu x}}$$

[not derived previously:]

$$\tau_x = \mu\frac{\partial u}{\partial y} = \rho\nu\frac{u_\infty}{4}\sqrt{\frac{u_\infty}{\nu x}}f''$$

4.0.1 Reynold's analogy

Hey these equations look so similar

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}T = \alpha\frac{\partial^2}{\partial y^2}T$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = \nu\frac{\partial^2}{\partial y^2}\ u$$

Let's copy/paste the solution... (under certain constraints)

$$\nu = \alpha$$

$$Pr = \frac{\nu}{\alpha} = 1$$

If you wanna change up the equations:

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}T = \frac{\nu}{Pr}\frac{\partial^2}{\partial y^2}T$$

If Pr=1,

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}T = \nu\frac{\partial^2}{\partial y^2}T$$

Use the relative temperature trick, and change the variable to θ

$$u\frac{\partial}{\partial x}\theta + v\frac{\partial}{\partial y}\theta = \nu\frac{\partial^2}{\partial y^2}\theta$$

If we nondimensionalise the momentum equation

$$u\frac{\partial}{\partial x}u^*u_\infty + v\frac{\partial}{\partial y}u^*u_\infty = \nu\frac{\partial^2}{\partial y^2}u^*u_\infty$$

$$u\frac{\partial}{\partial x}u^* + v\frac{\partial}{\partial y}u^* = \nu\frac{\partial^2}{\partial y^2}u^*$$

We already proved we can reduce the momentum BL equations as follows:

$$f''f + f''' = 0$$

And we note that we can set:

$$\theta = u^*$$

Note that

$$u = \frac{u_\infty}{2}f'$$

$$u^* = \frac{f'}{2}$$

Under Reynold's analogy we can say:

$$\theta = \frac{f'}{2}$$

We can pretty much adapt the Blasius solution to thermal boundary layer... (Reynold's analogy under Pr=1)

We assume that the nondimensional temperature profile and nondimensional BL profile (and BCs) are exactly equal.

Based on that, we can get the temperature profile and heat flux

$$\theta = \frac{f'}{2}$$

$$T - T_s = (T_\infty - T_s)\frac{f'}{2}$$

What about heat flux:

$$q'' = -k\frac{\partial T}{\partial y}$$

note:

$$\frac{\partial u}{\partial y} = \frac{u_\infty}{4}\sqrt{\frac{u_\infty}{\nu x}} f''$$

$$\frac{\partial u^* u_\infty}{\partial y} = \frac{u_\infty}{4}\sqrt{\frac{u_\infty}{\nu x}} f''$$

$$\frac{\partial u^*}{\partial y} = \frac{1}{4}\sqrt{\frac{u_\infty}{\nu x}} f''$$

under reynold's analogy

$$u^* = \theta$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{4}\sqrt{\frac{u_\infty}{\nu x}} f''$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{4}\sqrt{\frac{u_\infty}{\nu x}} f''$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{4}\sqrt{\frac{u_\infty x}{\nu x^2}} f''$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{4}\sqrt{Re_x}\frac{1}{x} f''$$

$$\frac{\partial (T - T_s)}{\partial y} = \frac{T_\infty - T_s}{4}\sqrt{Re_x}\frac{1}{x} f''$$

$$\frac{\partial T}{\partial y} = \frac{T_\infty - T_s}{4}\sqrt{Re_x}\frac{1}{x}f''$$

Heat flux:

$$q'' = -k\frac{\partial T}{\partial y}$$

$$q'' = -\frac{k}{x}\frac{T_\infty - T_s}{4}\sqrt{Re_x}f''$$

Heat trf Coefficient and Nu

$$q'' = -h(T_\infty - T_s)$$

Substitute

$$h(T_\infty - T_s) = \frac{k}{x}\frac{T_\infty - T_s}{4}\sqrt{Re_x}f''$$

$$h = \frac{k}{x}\frac{f''}{4}\sqrt{Re_x}$$

Define Nusselt Number:

$$Nu_x \equiv \frac{hx}{k}$$

$$Nu_x = \frac{f''}{4}\sqrt{Re_x}$$

What is the value of f" (look at textbook, or solve it yourself) at y=0

$$f'' = 1.328$$

$$Nu_x = \frac{1.328}{4}\sqrt{Re_x} = 0.332\sqrt{Re_x}$$

5 What if Pr is not 1?

5.1 Similarity Solution by Pohlhausen

`Bejan, A. (2013). Convection heat transfer. John wiley & sons.`
`Welty, J., Rorrer, G. L., & Foster, D. G. (2014).`
`Fundamentals of momentum, heat, and mass transfer. John Wiley & Sons.`

From Blasius Solution
And nondimensionalising

$$f' = 0 \ at \ \eta = 0$$

$$f = 0 \ at \ \eta = 0$$

$$f' = 2 \ at \ \eta \to \infty$$

$$f''f + f''' = 0$$

$$u = \frac{u_\infty}{2} f'$$

Some useful derivatives previously derived (Bejan, 2013,Welty et al., 2014)
From hydrodynamic BL:

$$\frac{\partial}{\partial y^*}\eta = \frac{\partial}{\partial y^*}\frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}} = \frac{\eta}{y^*}$$

$$\frac{\partial}{\partial x^*}\eta = \frac{\partial}{\partial x^*}\frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}$$

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}T = \alpha\frac{\partial^2}{\partial y^2}T$$

We nondimensionalised the energy equation during scaling analysis...

$$u^*\frac{\partial}{\partial x^*}\theta + \frac{\delta_p}{\delta_t}v^*\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta)$$

We just need to convert it to include similarity variables...
We start by making substitutions

$$u^* = \frac{f'}{2}$$

$$v = \frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}(\eta f' - f)$$

$$v_c = \frac{u_\infty \delta_p}{L}$$

$$v^*\frac{u_\infty \delta_p}{L} = -\frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}(\eta f' - f)$$

$$v^* = \frac{1}{2}\frac{L}{\delta_p}\sqrt{\frac{\nu}{xu_\infty}}(\eta f' - f)$$

$$v^* = \frac{1}{2}\frac{L}{\delta_p}Re_x^{-\frac{1}{2}}(\eta f' - f)$$

subs into:

$$u^*\frac{\partial}{\partial x^*}\theta + \frac{\delta_p}{\delta_t}v^*\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta)$$

$$\frac{f'}{2}\frac{\partial}{\partial x^*}\theta + \frac{\delta_p}{\delta_t}\left[-\frac{1}{2}\frac{L}{\delta_p}Re_x^{\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta)$$

$$\frac{f'}{2}\frac{\partial}{\partial x^*}\theta + \frac{L}{\delta_t}\left[-\frac{1}{2}Re_x^{\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta)$$

Now we need to substitute out y^* and x^* for η

$$y^* = \frac{y}{\delta_t}$$

$$x^* = \frac{x}{L}$$

we note x^* is the same definition as the nondimensionalised x^* used in deriving hydrodynamic BL, so we use it again...

So we can use:

$$\frac{\partial}{\partial x^*}\eta = \frac{\partial}{\partial x^*}\frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}$$

$$= \frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}\frac{-1}{2x^*} = \frac{\eta}{-2x^*}$$

Replace left most term...

$$\frac{\partial}{\partial x^*} = \frac{\partial \eta}{\partial x^*}\frac{\partial}{\partial \eta} = -\frac{\eta}{2x^*}\frac{\partial}{\partial \eta}$$

$$\frac{f'}{2}\frac{\partial}{\partial x^*}\theta + \frac{L}{\delta_t}\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta)$$

$$\frac{f'}{2}(-\frac{\eta}{2x^*})\frac{\partial}{\partial\eta}\theta + \frac{L}{\delta_t}\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta)$$

$$\frac{f'}{2}(-\frac{\eta}{2x^*})\theta' + \frac{L}{\delta_t}\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta)$$

$$\frac{f'}{2}(-\frac{\eta}{2x^*})\theta' + L\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y}\theta = \alpha\frac{L}{u_\infty}(\frac{\partial^2}{\partial y^2}\theta)$$

What about the y terms?

$$\eta = \frac{1}{2}\frac{y}{x}(Re_x)^{\frac{1}{2}} = \frac{1}{2}y\sqrt{\frac{u_\infty}{\nu x}}$$

$$\frac{\partial}{\partial y} = \frac{\partial\eta}{\partial y}\frac{\partial}{\partial\eta}$$

$$\frac{\partial\eta}{\partial y} = \frac{1}{2}\sqrt{\frac{u_\infty}{\nu x}} = \frac{\eta}{y}$$

For RHS...

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial\eta}{\partial y}\frac{\partial\theta}{\partial\eta}) = \frac{\partial}{\partial y}(\frac{\partial\eta}{\partial y}\theta')$$

$$= \frac{\partial\theta'}{\partial y}\frac{\partial\eta}{\partial y} + \frac{\partial^2\eta}{\partial y^2}\theta'$$

$$\frac{\partial^2\eta}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\eta}{y}) = 0$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial\theta'}{\partial y}\frac{\partial\eta}{\partial y} = \frac{\partial\eta}{\partial y}\frac{\partial\theta'}{\partial\eta}\frac{\partial\eta}{\partial y} = \theta''(\frac{\eta^2}{y^2})$$

Substitution back in:

$$\frac{f'}{2}(-\frac{\eta}{2x^*})\theta' + L\left[-\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\eta}{y}\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

tidying up:

$$(-\frac{\eta f'}{4x^*})\theta' + L\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\eta}{y}\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$(-\frac{\eta f'}{4x^*})\theta' + \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}(\eta f' - f)\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$(-\frac{\eta f'}{4x^*})\theta' + \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}(\eta f'\theta' - f\theta') = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$(-\frac{\eta f'}{4x^*})\theta' + \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}\eta f'\theta' - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$\theta'\eta f'(-\frac{1}{4x^*} + \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

simplifying...

$$\theta'\eta f'(-\frac{1}{4x^*} + \frac{1}{4}\sqrt{\frac{u_\infty}{\nu x}}LRe_x^{-\frac{1}{2}}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$\theta'\eta f'(-\frac{1}{4x^*} + \frac{1}{4x}\sqrt{Re_x}LRe_x^{-\frac{1}{2}}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$\theta'\eta f'(-\frac{1}{4x^*} + \frac{L}{4x}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$\theta'\eta f'(-\frac{1}{4x^*} + \frac{1}{4x^*}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$-\frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$-\frac{1}{2}\frac{1}{2}\frac{1}{x}Re_x^{\frac{1}{2}}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$-\frac{1}{4}\frac{1}{x^*}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$-\frac{1}{4}\frac{1}{x^*}f\theta' = \alpha\frac{L}{u_\infty}(\frac{1}{4}\frac{Re_x}{x^2}\theta'')$$

$$-\frac{1}{4}\frac{1}{x^*}f\theta' = \alpha\frac{1}{u_\infty}(\frac{1}{4}\frac{Re_x}{xx^*}\theta'')$$

$$-\frac{1}{4}f\theta' = \alpha\frac{1}{u_\infty}(\frac{1}{4}\frac{Re_x}{x}\theta'')$$

$$-f\theta' = \alpha\frac{1}{u_\infty}(\frac{Re_x}{x}\theta'')$$

$$-f\theta' = \alpha\frac{1}{u_\infty}(\frac{u_\infty x}{x\nu}\theta'')$$

$$-f\theta' = \frac{\alpha}{\nu}\theta''$$

$$-f\theta' = \frac{1}{Pr}\theta''$$

$$\theta'' + Pr\ f\theta' = 0$$

5.2 Similarity Solution BCs

2 Bcs:

$$y \to \infty\ ;\ T \to T_\infty$$

$$\eta \to \infty\ ;\ \theta \to 1$$

Reminder

$$\theta = \frac{T - T_s}{T_\infty - T_s}$$

$$y = 0; T = T_s$$

$$\eta = 0\ ; \theta = 0$$

5.3 solving the equation

$$\theta'' + Pr\ f\theta' = 0$$

$$\frac{\partial\theta'}{\partial\eta} + Pr\ f\theta' = 0$$

$$\frac{\partial\theta'}{\partial\eta} = -Pr\ f\theta'$$

$$\frac{\partial\theta'}{\partial\eta} = -Pr\ f\theta'$$

$$\frac{1}{\theta'}\frac{\partial\theta'}{\partial\eta} = -Pr\ f$$

$$\int_0^\infty \frac{1}{\theta'}\frac{\partial\theta'}{\partial\eta}d\eta = -\int_0^\infty Pr\ f d\eta$$

$$\int_{\theta'|_{\eta=0}}^{\theta'|_{\eta=\infty}} \frac{1}{\theta'}d\theta' = -\int_0^\infty Pr\ f(\eta)d\eta$$

$$\log_e(\frac{\theta'|_{\eta=\infty}}{\theta'|_{\eta=0}}) = -\int_0^\infty Pr\ f(\eta)d\eta$$

$$\frac{\theta'|_{\eta=\infty}}{\theta'|_{\eta=0}} = \exp(-\int_0^\infty Pr\ f(\eta)d\eta)$$

What if η is not infinity,

$$\int_{\theta'|_{\eta'=0}}^{\theta'|_{\eta'=\eta}} \frac{1}{\theta'}d\theta' = -\int_0^\eta Pr\ f(\eta')d\eta'$$

$$\log_e(\frac{\theta'|_{\eta'=\infty}}{\theta'|_{\eta=0}}) = -\int_0^\infty Pr\ f(\eta)d\eta$$

$$\frac{\theta'|_{\eta'=\eta}}{\theta'|_{\eta=0}} = \exp(-\int_0^\eta Pr\ f(\eta')d\eta')$$

$$\theta'|_\eta = \theta'|_{\eta=0}\exp(-\int_0^\eta Pr\ f(\eta')d\eta')$$

Integrate again,

$$\int_0^\eta \theta'(\eta')d\eta' = \int_0^\eta \theta'|_{\eta'=0}\exp(-\int_0^\eta Pr\ f(\eta')d\eta')d\eta'$$

$$\theta(\eta) = \int_0^\eta \theta'|_{\eta=0}\exp(-\int_0^\eta Pr\ f(\eta')d\eta')d\eta'$$

use other conventional dummy variables to avoid confusion

$$\theta(\eta) = \int_{\gamma=0}^{\gamma=\eta} \theta'|_{\eta'=0}\exp(-\int_{\beta=0}^{\beta=\eta} Pr\ f(\beta)d\beta)\gamma$$

One problem: we don't know what $\theta'|_{\eta'=0}$ is
we use the boundary condition:

$$\theta = 1 \ at \ \eta = infinity$$

the dummy variable representing η is γ

$$\theta(\eta=\infty) = \int_{\gamma=0}^{\gamma=\infty} \theta'|_{\eta=0} \exp(-\int_{\beta=0}^{\beta=\eta} Pr \ f(\beta) d\beta)\gamma$$

$$1 = \int_{\gamma=0}^{\gamma=\infty} \theta'|_{\eta=0} \exp(-\int_{\beta=0}^{\beta=\eta} Pr \ f(\beta) d\beta)\gamma$$

$$\theta'(0)^{-1} = \int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\eta} Pr \ f(\beta) d\beta)\gamma$$

$$\theta'(0) = \left[\int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\eta} Pr \ f(\beta) d\beta)\gamma\right]^{-1}$$

using correct mathematical symbols:

$$\theta'(0) = \left[\int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\gamma} Pr \ f(\beta) d\beta)\gamma\right]^{-1}$$

$$\theta(\eta) = \theta'(0) \int_{\gamma=0}^{\gamma=\eta} \exp(-Pr \int_{\beta=0}^{\beta=\gamma} \ f(\beta) d\beta)\gamma$$

We want to find out heat flux...

$$q'' = -k \frac{\partial T}{\partial y}|_{y=0}$$

$$q'' = -k \frac{\partial (T - T_s)}{\partial y}|_{y=0}$$

$$q'' = -k(T_\infty - T_s) \frac{\partial \theta}{\partial y}|_{y=0}$$

$$h = \frac{q''}{-(T_\infty - T_s)} = k \frac{\partial \theta}{\partial y}|_{y=0}$$

$$\frac{h}{k} = \frac{\partial \theta}{\partial y}|_{y=0}$$

$$\frac{h}{k} = \frac{\partial\eta}{\partial y}\frac{\partial\theta}{\partial\eta}|_{y=0}$$

$$\frac{h}{k} = \frac{\eta}{y}\frac{\partial\theta}{\partial\eta}|_{y=0}$$

$$\frac{h}{k} = \frac{1}{2}\sqrt{\frac{u_\infty}{\nu x}}\frac{\partial\theta}{\partial\eta}|_{y=0}$$

$$\frac{h}{k} = \frac{1}{2}\sqrt{\frac{u_\infty}{\nu x}}\theta'(0)$$

$$\frac{h}{k} = \frac{1}{2}\sqrt{\frac{u_\infty x}{\nu x^2}}\theta'(0)$$

$$\frac{hx}{k} = \frac{1}{2}\sqrt{Re_x}\theta'(0)$$

$$Nu_x = \frac{1}{2}\sqrt{Re_x}\theta'(0)$$

$$Nu_x = \frac{1}{2}\sqrt{Re_x}\left[\int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\gamma} Pr\ f(\beta)d\beta)\gamma\right]^{-1}$$

In textbook expression:

$$Nu_x = \sqrt{Re_x}\left[\int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\gamma} \frac{1}{2}Pr\ f(\beta)d\beta)\gamma\right]^{-1}$$

What is the value of:

$$\left[\int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\gamma} \frac{1}{2}Pr\ f(\beta)d\beta)\gamma\right]^{-1}$$

Pohlhausen worked it out for Pr>0.5:

$$\left[\int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\gamma} \frac{1}{2}Pr\ f(\beta)d\beta)\gamma\right]^{-1} \approx 0.332Pr^{\frac{1}{3}}$$

Liquid metals $Pr < 0.1$
Air $Pr = 0.72$
Water $Pr = 7$

Pr number says that for liquids of same viscosity, a higher prandtl number means lower thermal diffusivity

Thermal diffusivity intuition, high thermal conductivity means high thermal diffusivity, all else constant.

$$Nu_x = 0.332 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

$$Nu_L = \frac{1}{L}\int_0^L Nu_x dx$$

$$Nu_L = \frac{1}{L}\int_0^L 0.332 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}} dx$$

$$Nu_L = Pr^{\frac{1}{3}} \frac{1}{L}\int_0^L 0.332 Re_x^{\frac{1}{2}} dx$$

$$Nu_L = 0.664 Re_L^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

Note: For the above solutions, use film temperature to evaluate thermal properties!

$$T_{film} = \frac{T_s + T_\infty}{2}$$

eg. c_p, μ etc..

6 Integral Solutions to Thermal BL Constant Temp Flat Plate

In the momentum BL, we considered a control volume in the BL to get:

$$\frac{\tau_x}{\rho} = (\frac{\partial u_\infty}{\partial x})\int_0^{\delta_p} (u_\infty - u)\ dy + \frac{\partial}{\partial x}\int_0^{\delta_p} u(u_\infty - u)\ dy$$

Now for thermal BL, we can also consider control volume too (Welty et al., 2014).

We can consider again four sides of the control volume and perform an enthalpy balance, neglecting fluid kinetic energy:

$$\dot{Q} - \dot{W} = net\ enthalpy\ flows\ out + \Delta\ enthalpy\ w.r.t\ time$$

W is Work done by system, if Q+W then W is work done on system

$$\dot{Q} = q'' dA$$

$$q''_{wall} = -k\frac{\partial T}{\partial y}|_{y=0}$$

$$dA = l_z \Delta x$$

For conduction, ignore conduction within fluid. (important assumption) 2nd assumption, no work done W=0,

Let's consider the enthalpy flows and assume steady state..

Consider net enthalpy flows in:

neglecting kinetic energy

inflow of enthalpy through left

$$= \int_0^{\delta_t} \rho u h l_z dy|_x$$

$$= \int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_x$$

outflow of enthalpy through the right

$$= \int_0^{\delta_t} \rho u h l_z dy|_{x+\Delta x}$$

$$= \int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_{x+\Delta x}$$

inflow of enthalpy through the top

Specific enthalpy:

$$h_\infty = c_p (T_\infty - T_{ref})$$

mass flowrate into system:

$$\dot{m} = \int_0^{\delta_t} \rho u l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u l_z dy|_x$$

put this toegther to find enthalpy inflow through top:

$$\dot{m} h_\infty = (\int_0^{\delta_t} \rho u l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u l_z dy|_x) c_p (T_\infty - T_{ref})$$

Let's subs everything back:

$$\dot{Q} - \dot{W} = net\ enthalpy\ flows\ out + \Delta\ enthalpy\ w.r.t\ time$$

$$-k\frac{\partial T}{\partial y}|_{y=0} l_z \Delta x - 0 = -(\int_0^{\delta_t} \rho u l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u l_z dy|_x) c_p (T_\infty - T_{ref})$$

$$+\int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_x + 0$$

divide throughout by Δx take limit $\Delta x \to 0$

$$-k\frac{\partial T}{\partial y}|_{y=0} l_z = -\frac{1}{\Delta x}(\int_0^{\delta_t} \rho u l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u l_z dy|_x) c_p (T_\infty - T_{ref})$$

$$+\frac{1}{\Delta x}[\int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_x]$$

Assume T_∞ constant with x

$$-k\frac{\partial T}{\partial y}|_{y=0} l_z = -\frac{\partial}{\partial x}(\int_0^{\delta_t} \rho u l_z dy|) c_p (T_\infty - T_{ref})$$

$$+\frac{\partial}{\partial x}[\int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy]$$

Drop out T_{ref} after bring $c_p T_\infty$ into integral:

$$-k\frac{\partial T}{\partial y}|_{y=0} l_z = -\frac{\partial}{\partial x}(\int_0^{\delta_t} \rho u l_z dy c_p T_\infty)$$

$$+\frac{\partial}{\partial x}[\int_0^{\delta_t} \rho u c_p T l_z dy]$$

Finally get rid of l_z

$$-k\frac{\partial T}{\partial y}|_{y=0} = -\frac{\partial}{\partial x}(\int_0^{\delta_t} \rho u dy c_p T_\infty) + \frac{\partial}{\partial x}[\int_0^{\delta_t} \rho u c_p T dy]$$

$$-k\frac{\partial T}{\partial y}|_{y=0} = -\frac{\partial}{\partial x}(\int_0^{\delta_t} \rho u dy c_p T_\infty) + \frac{\partial}{\partial x}[\int_0^{\delta_t} \rho u c_p T dy]$$

Tidy up equation:

$$-k\frac{\partial T}{\partial y}|_{y=0} = \frac{\partial}{\partial x}[\int_0^{\delta_t} \rho u c_p (T - T_\infty) dy]$$

y is integrated out so the integral on the RHS does not depend on y

$$-k\frac{\partial T}{\partial y}|_{y=0} = \frac{d}{dx}[\int_0^{\delta_t} \rho u c_p (T - T_\infty) dy]$$

You can even bring out ρc_p:

$$-k\frac{\partial T}{\partial y}|_{y=0} = \rho c_p \frac{d}{dx}[\int_0^{\delta_t} u (T - T_\infty) dy]$$

6.1 The Hardcore Integral Solution

But in other textbooks, von karman's integral approach literally integrates the BL equations over y (Bejan, 2013).

we start at:

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T = \alpha(\frac{\partial^2}{\partial y^2}\ T)$$

and for we put back u and v into the differential using product rule.

$$\frac{\partial}{\partial x}(uT) = u\frac{\partial}{\partial x}T + T\frac{\partial}{\partial x}u$$

$$u\frac{\partial}{\partial x}T = \frac{\partial}{\partial x}(uT) - T\frac{\partial}{\partial x}u$$

$$\frac{\partial}{\partial y}(vT) = v\frac{\partial}{\partial y}T + T\frac{\partial}{\partial y}v$$

$$v\frac{\partial}{\partial y}T = \frac{\partial}{\partial y}(vT) - T\frac{\partial}{\partial y}v$$

subs back in:

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T = \alpha(\frac{\partial^2}{\partial y^2}\ T)$$

$$\frac{\partial}{\partial x}(uT) - T\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}(vT) - T\frac{\partial}{\partial y}v = \alpha(\frac{\partial^2}{\partial y^2}\ T)$$

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) - T(\frac{\partial}{\partial y}v + \frac{\partial}{\partial x}u) = \alpha(\frac{\partial^2}{\partial y^2}\ T)$$

Now we consider continuity equation

$$\frac{\partial}{\partial y}v + \frac{\partial}{\partial x}u = 0$$

And finally get...

$$\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT) = \alpha(\frac{\partial^2}{\partial y^2}\ T)$$

Now that we have these equations we integrate across y in the BL.

$$\int_0^Y \frac{\partial}{\partial x}(uT)dy + \int_0^Y \frac{\partial}{\partial y}(vT)dy = \int_0^Y \alpha(\frac{\partial^2}{\partial y^2}\ T)dy$$

$$\int_0^Y \frac{\partial}{\partial x}(uT)dy + (vT)|_Y - (vT)|_0 = \alpha \int_0^Y (\frac{\partial^2}{\partial y^2}\ T)dy$$

$$\int_0^Y \frac{\partial}{\partial x}(uT)dy + (vT)|_Y - (vT)|_0 = \alpha[\frac{\partial T}{\partial y}|_Y - \frac{\partial T}{\partial y}|_0]$$

Notice there is a partial derivative with x, we will need to use Leibniz's Rule

`https://en.wikipedia.org/wiki/Leibniz_integral_rule`

Consider the integral

$$\int_{y1=a(x)}^{y2=b(x)} f(x,y)dy$$

$$\frac{d}{dx}\left(\int_{y1=a(x)}^{y2=b(x)} f(x,y)dy\right) = f(x, y=b(x))\frac{d}{dx}b(x) - f(x, y=a(x))\frac{d}{dx}a(x) + \int_{y1=a(x)}^{y2=b(x)} \frac{\partial}{\partial x}f(x,y)dy$$

We can apply it, we let f(x,y) $= uT$

$$\frac{d}{dx}\left(\int_{y1=a(x)}^{y2=b(x)} uTdy\right) = uT(x, y=b(x))\frac{d}{dx}b(x) - uT(x, y=a(x))\frac{d}{dx}a(x) + \int_{y1=a(x)}^{y2=b(x)} \frac{\partial}{\partial x}uTdy$$

substitute in y1=0, y2=Y

$$\frac{d}{dx}\left(\int_0^{y2=Y} uTdy\right) = uT(x,y=Y)\frac{d}{dx}Y(x) - uT(x,y=0)\frac{d}{dx}0 + \int_0^{y2=Y} \frac{\partial}{\partial x}uTdy$$

$$\frac{d}{dx}\left(\int_0^{Y} uTdy\right) = uT(x,y=Y)\frac{d}{dx}Y(x) + \int_0^{Y} \frac{\partial}{\partial x}uTdy$$

$$\int_0^{Y} \frac{\partial}{\partial x}uTdy = \frac{d}{dx}\left(\int_0^{Y} uTdy\right) - uT(x,y=Y)\frac{d}{dx}Y(x)$$

substitute back into main equation:

$$\int_0^{Y} \frac{\partial}{\partial x}(uT)dy + (vT)|_Y - (vT)|_0 = \alpha[\frac{\partial T}{\partial y}|_Y - \frac{\partial T}{\partial y}|_0]$$

$$\frac{d}{dx}\left(\int_0^{Y} uTdy\right) - uT(x,y=Y)\frac{d}{dx}Y(x) + (vT)|_Y - (vT)|_0 = \alpha[\frac{\partial T}{\partial y}|_Y - \frac{\partial T}{\partial y}|_0]$$

Then consider getting v (y velocity) by integrating continuity equation (Bejan, 2013).

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\int_0^{Y} \frac{\partial u}{\partial x}dy + \int_0^{Y} \frac{\partial v}{\partial y}dy = 0$$

$$\int_0^{Y} \frac{\partial u}{\partial x}dy + v|_Y - v|_0 = 0$$

$$v|_Y = v|_0 - \int_0^{Y} \frac{\partial u}{\partial x}dy$$

We considered Leibniz's rule before using f=uT,

$$\int_0^{Y} \frac{\partial}{\partial x}uTdy = \frac{d}{dx}\left(\int_0^{Y} uTdy\right) - uT(x,y=Y)\frac{d}{dx}Y(x)$$

now consider f(x,y)=u(x,y)

$$\int_0^{Y} \frac{\partial}{\partial x}udy = \frac{d}{dx}\left(\int_0^{Y} udy\right) - u(x,y=Y)\frac{d}{dx}Y(x)$$

$$v|_Y = v|_0 - (\frac{d}{dx}\left(\int_0^Y udy\right) - u(x, y = Y)\frac{d}{dx}Y(x))$$

$$v|_Y = v|_0 - \frac{d}{dx}\left(\int_0^Y udy\right) + u(x, y = Y)\frac{d}{dx}Y(x)$$

Let's substitute this back into our energy equation:

$$\frac{d}{dx}\left(\int_0^Y uTdy\right) - uT(x, y = Y)\frac{d}{dx}Y(x) + (vT)|_Y - (vT)|_0 = \alpha[\frac{\partial T}{\partial y}|_Y - \frac{\partial T}{\partial y}|_0]$$

$$\frac{d}{dx}\left(\int_0^Y uTdy\right) - uT(x, y = Y)\frac{d}{dx}Y(x) + v|_Y T|_Y - (vT)|_0 = \alpha[\frac{\partial T}{\partial y}|_Y - \frac{\partial T}{\partial y}|_0]$$

$$\frac{d}{dx}\left(\int_0^Y uTdy\right) - uT(x, y = Y)\frac{d}{dx}Y(x) + (v|_0 - \frac{d}{dx}\left(\int_0^Y udy\right) + u(x, y = Y)\frac{d}{dx}Y(x))T|_Y - (vT)|_0$$

$$= \alpha[\frac{\partial T}{\partial y}|_Y - \frac{\partial T}{\partial y}|_0]$$

$$\frac{d}{dx}\left(\int_0^Y uTdy\right) - uT(x, y = Y)\frac{d}{dx}Y(x) + v|_0 T|_Y - \frac{d}{dx}\left(\int_0^Y udy\right)T|_Y + u(x, y = Y)T|_Y\frac{d}{dx}Y(x) - (vT)|_0$$

$$= \alpha[\frac{\partial T}{\partial y}|_Y - \frac{\partial T}{\partial y}|_0]$$

$$\frac{d}{dx}\left(\int_0^Y uTdy\right) + v|_0 T|_Y - \frac{d}{dx}\left(\int_0^Y udy\right)T|_Y - (vT)|_0 = \alpha[\frac{\partial T}{\partial y}|_Y - \frac{\partial T}{\partial y}|_0]$$

Now we need to introduce some BCs to tidy things up:

- no slip
- $Y = \delta_t$
- conduction at surface >> conduction at BL

Firstly no slip v=0 at y=0:

$$\frac{d}{dx}\left(\int_0^Y uTdy\right) - \frac{d}{dx}\left(\int_0^Y udy\right)T|_Y = \alpha[\frac{\partial T}{\partial y}|_Y - \frac{\partial T}{\partial y}|_0]$$

now, Y = $\delta_t(x)$

$$\frac{d}{dx}\left(\int_0^{\delta_t} uTdy\right) - \frac{d}{dx}\left(\int_0^{\delta_t} udy\right)T|_{\delta_t} = \alpha[\frac{\partial T}{\partial y}|_{\delta_t} - \frac{\partial T}{\partial y}|_0]$$

note: $T_\infty = T|_{\delta_t}$

$$\frac{d}{dx}\left(\int_0^{\delta_t} uTdy\right) - \frac{d}{dx}\left(\int_0^{\delta_t} udy\right)T_\infty = \alpha[\frac{\partial T}{\partial y}|_{\delta_t} - \frac{\partial T}{\partial y}|_0]$$

in general, $T_\infty = T_\infty(x)$

$$\frac{d}{dx}\left(\int_0^{\delta_t} uTdy\right) - T_\infty(x)\frac{d}{dx}\left(\int_0^{\delta_t} udy\right) = \alpha[\frac{\partial T}{\partial y}|_{\delta_t} - \frac{\partial T}{\partial y}|_0]$$

Apply product rule:

$$\frac{d}{dx}\left(T_\infty(x)\int_0^{\delta_t} udy\right) = T_\infty(x)\frac{d}{dx}\left(\int_0^{\delta_t} udy\right) + \left(\frac{dT_\infty(x)}{dx}\right)\int_0^{\delta_t} udy$$

multiply all by -1, im going to just use T_∞ instead of $T_\infty(x)$

$$\frac{d}{dx}\left(T_\infty\int_0^{\delta_t} udy\right) = T_\infty\frac{d}{dx}\left(\int_0^{\delta_t} udy\right) + \left(\frac{dT_\infty}{dx}\right)\int_0^{\delta_t} udy$$

$$-\frac{d}{dx}\left(T_\infty\int_0^{\delta_t} udy\right) = -T_\infty\frac{d}{dx}\left(\int_0^{\delta_t} udy\right) - \left(\frac{dT_\infty}{dx}\right)\int_0^{\delta_t} udy$$

$$-\frac{d}{dx}\left(T_\infty\int_0^{\delta_t} udy\right) + \left(\frac{dT_\infty}{dx}\right)\int_0^{\delta_t} udy = -T_\infty\frac{d}{dx}\left(\int_0^{\delta_t} udy\right)$$

$$-T_\infty\frac{d}{dx}\left(\int_0^{\delta_t} udy\right) = -\frac{d}{dx}\left(T_\infty\int_0^{\delta_t} udy\right) + \left(\frac{dT_\infty}{dx}\right)\int_0^{\delta_t} udy$$

Substitute back:

$$\frac{d}{dx}\left(\int_0^{\delta_t} uT dy\right) - T_\infty(x)\frac{d}{dx}\left(\int_0^{\delta_t} udy\right) = \alpha[\frac{\partial T}{\partial y}|_{\delta_t} - \frac{\partial T}{\partial y}|_0]$$

$$\frac{d}{dx}\left(\int_0^{\delta_t} uT dy\right) - \frac{d}{dx}\left(T_\infty \int_0^{\delta_t} udy\right) + \left(\frac{dT_\infty}{dx}\right)\int_0^{\delta_t} udy = \alpha[\frac{\partial T}{\partial y}|_{\delta_t} - \frac{\partial T}{\partial y}|_0]$$

bring T_∞ into integral

$$\frac{d}{dx}\left(\int_0^{\delta_t} uT dy\right) - \frac{d}{dx}\left(\int_0^{\delta_t} uT_\infty dy\right) + \left(\frac{dT_\infty}{dx}\right)\int_0^{\delta_t} udy = \alpha[\frac{\partial T}{\partial y}|_{\delta_t} - \frac{\partial T}{\partial y}|_0]$$

$$\frac{d}{dx}\left(\int_0^{\delta_t} u(T - T_\infty) dy\right) + \left(\frac{dT_\infty}{dx}\right)\int_0^{\delta_t} udy = \alpha[\frac{\partial T}{\partial y}|_{\delta_t} - \frac{\partial T}{\partial y}|_0]$$

on RHS we assume heat conduction at BL negligible compared to heat conduction at surface

$$[\frac{\partial T}{\partial y}|_{\delta_t} - \frac{\partial T}{\partial y}|_0] \approx -\frac{\partial T}{\partial y}|_0$$

So we are left with our integral BL equation:

$$\frac{d}{dx}\left(\int_0^{\delta_t} u(T - T_\infty) dy\right) + \left(\frac{dT_\infty}{dx}\right)\int_0^{\delta_t} udy = -\alpha\frac{\partial T}{\partial y}|_0$$

note: $\alpha = \frac{k}{\rho c_p}$

Now assume a power series (cubic) expression for temperature and velocity...

Recall for velocity:

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3 \tag{2}$$

with the BCs:

$$u = 0 \ at \ y = 0$$

$$u = u_\infty \ at \ y = \delta_p$$

$$\frac{\partial u}{\partial y} = 0 \ at \ y = \delta_p$$

$$\frac{\partial^2 u}{\partial y^2} = 0 \ at \ y = 0$$

and assuming

$$u = a + by + cy^2 + dy^3$$

Near the wall, wall shear stress is constant.
Now we apply the same thing for temperature:

$$T - T_s = a + by + cy^2 + dy^3$$

with the BCs:

$$T - T_s = 0 \ at \ y = 0$$

$$T - T_s = T_\infty - T_s \ at \ y = \delta_t$$

$$\frac{\partial (T - T_s)}{\partial y} = 0 \ at \ y = \delta_t$$

$$\frac{\partial^2 (T - T_s)}{\partial y^2} = 0 \ at \ y = 0$$

Last BC is saying, near the wall, heat flux is constant..
Now if we do the same thing for temperature profile, you will get:

$$\frac{T - T_s}{T_\infty - T_s} = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}(\frac{y}{\delta_t})^3$$

$$T = T_s + (T_\infty - T_s)(\frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}(\frac{y}{\delta_t})^3)$$

we will then need to consider ratio of thermal BL and momentum BL.
Let's try substituting these back into our integral BL equation.
So in case we need: We already know that from hydrodynamic BL integral and similarity solution

$$\frac{\delta_p}{x} = \frac{4.64}{\sqrt{Re_x}}$$

$$\frac{\delta_p}{x} = \frac{5}{\sqrt{Re_x}}$$

Let's substitute back.

$$\frac{d}{dx}\left(\int_0^{\delta_t} u(T-T_\infty)dy\right)+\left(\frac{dT_\infty}{dx}\right)\int_0^{\delta_t} udy=-\alpha\frac{\partial T}{\partial y}|_0$$

Assume T_∞ is constant w.r.t x

$$\frac{d}{dx}\left(\int_0^{\delta_t} u(T-T_\infty)dy\right)=-\alpha\frac{\partial T}{\partial y}|_0$$

Subs the Temp and Velocity profiles:

$$\frac{d}{dx}\left(\int_0^{\delta_t} u_\infty(\frac{3}{2}\frac{y}{\delta_p}-\frac{1}{2}(\frac{y}{\delta_p})^3)(T-T_\infty)dy\right)=-\alpha\frac{\partial T}{\partial y}|_0$$

$$\frac{d}{dx}\left(\int_0^{\delta_t} u_\infty(\frac{3}{2}\frac{y}{\delta_p}-\frac{1}{2}(\frac{y}{\delta_p})^3)(T-T_\infty)dy\right)=-\alpha\frac{\partial T}{\partial y}|_0$$

subs:

$$T=T_s+(T_\infty-T_s)(\frac{3}{2}\frac{y}{\delta_t}-\frac{1}{2}(\frac{y}{\delta_t})^3)$$

$$\frac{d}{dx}\left(\int_0^{\delta_t} u_\infty(\frac{3}{2}\frac{y}{\delta_p}-\frac{1}{2}(\frac{y}{\delta_p})^3)((T_s+(T_\infty-T_s)(\frac{3}{2}\frac{y}{\delta_t}-\frac{1}{2}(\frac{y}{\delta_t})^3))-T_\infty)dy\right)=-\alpha\frac{\partial T}{\partial y}|_0$$

$$\frac{d}{dx}\left(\int_0^{\delta_t} u_\infty(\frac{3}{2}\frac{y}{\delta_p}-\frac{1}{2}(\frac{y}{\delta_p})^3)((T_s-T_\infty+(T_\infty-T_s)(\frac{3}{2}\frac{y}{\delta_t}-\frac{1}{2}(\frac{y}{\delta_t})^3)))dy\right)=-\alpha\frac{\partial T}{\partial y}|_0$$

$$\frac{d}{dx}\left(\int_0^{\delta_t} u_\infty(\frac{3}{2}\frac{y}{\delta_p}-\frac{1}{2}(\frac{y}{\delta_p})^3)(T_s-T_\infty-(T_s-T_\infty)(\frac{3}{2}\frac{y}{\delta_t}-\frac{1}{2}(\frac{y}{\delta_t})^3))dy\right)=-\alpha\frac{\partial T}{\partial y}|_0$$

$$\frac{d}{dx}\left(\int_0^{\delta_t} u_\infty(T_s-T_\infty)(\frac{3}{2}\frac{y}{\delta_p}-\frac{1}{2}(\frac{y}{\delta_p})^3)(1-(\frac{3}{2}\frac{y}{\delta_t}-\frac{1}{2}(\frac{y}{\delta_t})^3))dy\right)=-\alpha\frac{\partial T}{\partial y}|_0$$

$$u_\infty(T_s-T_\infty)\frac{d}{dx}\left(\int_0^{\delta_t}(\frac{3}{2}\frac{y}{\delta_p}-\frac{1}{2}(\frac{y}{\delta_p})^3)(1-(\frac{3}{2}\frac{y}{\delta_t}-\frac{1}{2}(\frac{y}{\delta_t})^3))dy\right)=-\alpha\frac{\partial T}{\partial y}|_0$$

divide both sides by (T_s-T_∞),

$$u_\infty \frac{d}{dx}\left(\int_0^{\delta_t} (\frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3)(1-(\frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}(\frac{y}{\delta_t})^3))dy\right) = -\alpha(\frac{1}{(T_s - T_\infty)})\frac{\partial T - T_s}{\partial y}|_0$$

$$u_\infty \frac{d}{dx}\left(\int_0^{\delta_t} (\frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3)(1-(\frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}(\frac{y}{\delta_t})^3))dy\right) = \alpha(\frac{1}{(T_\infty - T_s)})\frac{\partial T - T_s}{\partial y}|_0$$

$$u_\infty \frac{d}{dx}\left(\int_0^{\delta_t} (\frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3)(1-(\frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}(\frac{y}{\delta_t})^3))dy\right) = \alpha\frac{\partial \theta}{\partial y}|_0$$

Where $\theta = \frac{T-T_s}{T_\infty - T_s}$
Let's evaluate the integral:

$$\int_0^{\delta_t} (\frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3)(1-(\frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}(\frac{y}{\delta_t})^3))dy$$

let's substitute the integral for dy with $d\frac{y}{\delta_t}$

$$dy = \delta_t d\frac{y}{\delta_t}$$

note: δ_t and δ_p are constant w.r.t y
subs the change of variable:

$$\int_0^{\delta_t} (\frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3)(1-(\frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}(\frac{y}{\delta_t})^3))dy$$

$$= \int_0^{\delta_t} (\frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3)(1-(\frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}(\frac{y}{\delta_t})^3))d\frac{y}{\delta_t}\delta_t$$

$$= \delta_t \int_0^{1} (\frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3)(1-(\frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}(\frac{y}{\delta_t})^3))d\frac{y}{\delta_t}$$

let's replace $\beta = \frac{y}{\delta_t}$, $y = \beta\delta_t$

$$= \delta_t \int_0^{1} (\frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3)(1-(\frac{3}{2}\beta - \frac{1}{2}\beta)^3)d\beta$$

$$= \delta_t \int_0^{1} (\frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3)(1-\frac{3}{2}\beta + \frac{1}{2}\beta^3)d\beta$$

$$= \delta_t \int_0^1 (\frac{3}{2}\frac{\beta\delta_t}{\delta_p} - \frac{1}{2}(\frac{\beta\delta_t}{\delta_p})^3)(1 - \frac{3}{2}\beta + \frac{1}{2}\beta^3)d\beta$$

$$= \delta_t \int_0^1 \frac{3}{2}\frac{\beta\delta_t}{\delta_p}(1 - \frac{3}{2}\beta + \frac{1}{2}\beta^3) - \frac{1}{2}(\frac{\beta\delta_t}{\delta_p})^3(1 - \frac{3}{2}\beta + \frac{1}{2}\beta^3)d\beta$$

$$= \delta_t \int_0^1 \frac{3}{2}\frac{\beta\delta_t}{\delta_p}(1 - \frac{3}{2}\beta + \frac{1}{2}\beta^3)d\beta - \delta_t \int_0^1 \frac{1}{2}(\frac{\beta\delta_t}{\delta_p})^3(1 - \frac{3}{2}\beta + \frac{1}{2}\beta^3)d\beta$$

[careless mistake! multiplied in β not β^3]

$$= \delta_t \frac{\delta_t}{\delta_p} \int_0^1 \frac{3}{2}\beta(1 - \frac{3}{2}\beta + \frac{1}{2}\beta^3)d\beta - \delta_t(\frac{\delta_t}{\delta_p})^3 \int_0^1 \frac{1}{2}\beta^3(1 - \frac{3}{2}\beta + \frac{1}{2}\beta^3)d\beta$$

$$= \delta_t \frac{\delta_t}{\delta_p} \int_0^1 \frac{3}{2}(\beta - \frac{3}{2}\beta^2 + \frac{1}{2}\beta^4)d\beta - \delta_t(\frac{\delta_t}{\delta_p})^3 \int_0^1 \frac{1}{2}(\beta^3 - \frac{3}{2}\beta^4 + \frac{1}{2}\beta^6)d\beta$$

$$= \frac{3}{2}\delta_t \frac{\delta_t}{\delta_p} \int_0^1 (\beta - \frac{3}{2}\beta^2 + \frac{1}{2}\beta^4)d\beta - \frac{1}{2}\delta_t(\frac{\delta_t}{\delta_p})^3 \int_0^1 (\beta^3 - \frac{3}{2}\beta^4 + \frac{1}{2}\beta^6)d\beta$$

$$= \frac{3}{2}\delta_t \frac{\delta_t}{\delta_p} \int_0^1 (\beta - \frac{3}{2}\beta^2 + \frac{1}{2}\beta^4)d\beta - \frac{1}{2}\delta_t(\frac{\delta_t}{\delta_p})^3 \int_0^1 (\beta^3 - \frac{3}{2}\beta^4 + \frac{1}{2}\beta^6)d\beta$$

$$= \frac{3}{2}\delta_t \frac{\delta_t}{\delta_p}\frac{1}{10} - \frac{1}{2}\delta_t(\frac{\delta_t}{\delta_p})^3 \frac{3}{140}$$

$$= \frac{3}{20}\delta_t \frac{\delta_t}{\delta_p} - \frac{3}{280}\delta_t(\frac{\delta_t}{\delta_p})^3$$

Substituting the integral back:

$$u_\infty \frac{d}{dx}\left(\frac{3}{20}\delta_t \frac{\delta_t}{\delta_p} - \frac{3}{280}\delta_t(\frac{\delta_t}{\delta_p})^3\right) = \alpha \frac{\partial\theta}{\partial y}|_0$$

$$u_\infty \frac{d}{dx}\left(\frac{3}{20}\frac{\delta_t^2}{\delta_p} - \frac{3}{280}(\frac{\delta_t^4}{\delta_p^3})\right) = \alpha \frac{\partial\theta}{\partial y}|_0$$

in approximate method, we consider the ratio $\frac{\delta_t}{\delta_p}$

if $\frac{\delta_t}{\delta_p} >> 1$ we can ignore first term, otherwise, if $\frac{\delta_t}{\delta_p} << 1$ we can ignore the second term

$$\frac{\delta_t}{\delta_p} >> 1$$

$$u_\infty \frac{d}{dx}\left(-\frac{3}{280}(\frac{\delta_t^4}{\delta_p^3})\right) = \alpha\frac{\partial\theta}{\partial y}|_0$$

$$\frac{\delta_t}{\delta_p} << 1$$

$$u_\infty \frac{d}{dx}\left(\frac{3}{20}\frac{\delta_t^2}{\delta_p}\right) = \alpha\frac{\partial\theta}{\partial y}|_0$$

Now we can substitute θ

$$\theta = \frac{T - T_s}{T_\infty - T_s} = \frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}(\frac{y}{\delta_t})^3$$

$$\frac{\partial}{\partial y}\theta = \frac{3}{2}\frac{1}{\delta_t} - \frac{1}{2}(\frac{3y^2}{\delta_t^3})$$

now set y=0

$$\frac{\partial}{\partial y}\theta|_{y=0} = \frac{3}{2\delta_t}$$

subs back in:

$$\frac{\delta_t}{\delta_p} >> 1$$

$$u_\infty \frac{d}{dx}\left(-\frac{3}{280}(\frac{\delta_t^4}{\delta_p^3})\right) = \alpha\frac{3}{2\delta_t}$$

$$\frac{\delta_t}{\delta_p} << 1$$

$$u_\infty \frac{d}{dx}\left(\frac{3}{20}\frac{\delta_t^2}{\delta_p}\right) = \alpha\frac{3}{2\delta_t}$$

note: from momentum integral equation

$$\delta_p = x\frac{4.64}{\sqrt{Re_x}} = 4.64\sqrt{\frac{\nu x}{u_\infty}}$$

Try substituting in latter case

$$\frac{\delta_t}{\delta_p} << 1$$

$$u_\infty \frac{d}{dx}\left(\frac{3}{20}\frac{\delta_t^2}{\delta_p}\right) = \alpha\frac{3}{2\delta_t}$$

$$u_\infty \frac{d}{dx}\left(\frac{3}{20}\frac{\delta_t^2}{4.64\sqrt{\frac{\nu x}{u_\infty}}}\right) = \alpha\frac{3}{2\delta_t}$$

$$\frac{u_\infty^{1.5}}{\sqrt{\nu}}\frac{3}{20 * 4.64}\frac{d}{dx}\left(\frac{\delta_t^2}{\sqrt{x}}\right) = \alpha\frac{3}{2\delta_t}$$

using quotient rule [careless mistake here, forgot to multiply by 2]:

$$\frac{d}{dx}\left(\frac{\delta_t^2}{\sqrt{x}}\right) = \frac{2\sqrt{x}\delta_t\frac{d\delta_t}{dx} - \delta_t^2(0.5)\frac{1}{\sqrt{x}}}{x}$$

substitute back:

$$\frac{u_\infty^{1.5}}{\sqrt{\nu}}\frac{3}{20 * 4.64}\frac{2\sqrt{x}\delta_t\frac{d\delta_t}{dx} - \delta_t^2(0.5)\frac{1}{\sqrt{x}}}{x} = \alpha\frac{3}{2\delta_t}$$

$$\frac{u_\infty^{1.5}}{\sqrt{\nu}}\frac{3}{20 * 4.64}\left(\frac{2}{\sqrt{x}}\delta_t\frac{d\delta_t}{dx} - \delta_t^2(0.5)\frac{1}{x\sqrt{x}}\right) = \alpha\frac{3}{2\delta_t}$$

$$\frac{u_\infty^{1.5}}{\sqrt{\nu}}\frac{3}{20 * 4.64}\left(\frac{2}{\sqrt{x}}\delta_t^2\frac{d\delta_t}{dx} - \delta_t^3(0.5)\frac{1}{x\sqrt{x}}\right) = \alpha\frac{3}{2}$$

$$\frac{d\delta_t^3}{dx} = 3\delta_t^2\frac{d\delta_t}{dx}$$

$$\frac{1}{3}\frac{d\delta_t^3}{dx} = \delta_t^2\frac{d\delta_t}{dx}$$

substitute back:

$$\frac{u_\infty^{1.5}}{\sqrt{\nu}}\frac{3}{20 * 4.64}\left(\frac{1}{\sqrt{x}}\frac{2}{3}\frac{d\delta_t^3}{dx} - \delta_t^3(0.5)\frac{1}{x\sqrt{x}}\right) = \alpha\frac{3}{2}$$

1st order linear ODE in δ_t^3

$$\frac{u_\infty^{1.5}}{\sqrt{\nu}}\left(\frac{1}{\sqrt{x}}\frac{2}{3}\frac{d\delta_t^3}{dx} - \delta_t^3(0.5)\frac{1}{x\sqrt{x}}\right) = \alpha\frac{20 * 4.64}{3}\frac{3}{2}$$

$$\frac{1}{\sqrt{x}}\frac{2}{3}\frac{d\delta_t^3}{dx} - \delta_t^3(0.5)\frac{1}{x\sqrt{x}} = \alpha\frac{10*4.64}{1}\frac{\sqrt{\nu}}{u_\infty^{1.5}}$$

$$\frac{d\delta_t^3}{dx} - \delta_t^3(0.75)\frac{1}{x} = \alpha\frac{15*4.64}{1}\frac{\sqrt{\nu}}{u_\infty^{1.5}}\sqrt{x}$$

Let

$$A = \alpha\frac{15*4.64}{1}\frac{\sqrt{\nu}}{u_\infty^{1.5}}$$

$$\frac{d\delta_t^3}{dx} - \delta_t^3\frac{0.75}{x} = A\sqrt{x}$$

[Alternate solution: substitute later, may get better solution (less careless mistake)]

$$u_\infty\frac{d}{dx}\left(\frac{3}{20}\frac{\delta_t^2}{\delta_p}\right) = \alpha\frac{3}{2\delta_t}$$

$$u_\infty\frac{d}{dx}\left(\frac{3}{20}\delta_p\frac{\delta_t^2}{\delta_p^2}\right) = \alpha\frac{3}{2\delta_t}$$

$$u_\infty\frac{d}{dx}\left(\frac{3}{20}\delta_p\frac{\delta_t^2}{\delta_p^2}\right) = \alpha\frac{3}{2\delta_t}$$

let $\phi = \frac{\delta_t}{\delta_p}$

$$u_\infty\frac{d}{dx}\left(\frac{3}{20}\delta_p\phi^2\right) = \alpha\frac{3}{2\delta_t}$$

$$u_\infty\frac{d}{dx}\left(\delta_p\phi^2\right) = \alpha\frac{10}{\delta_t}$$

Product rule:

$$\phi^2\frac{d}{dx}\delta_p + \delta_p\frac{d}{dx}\phi^2 = \alpha\frac{10}{u_\infty\delta_t}$$

$$\delta_t\phi^2\frac{d}{dx}\delta_p + \delta_t\delta_p\frac{d}{dx}\phi^2 = \alpha\frac{10}{u_\infty}$$

sub $\delta_t = \phi\delta_p$

$$\delta_p\phi^3\frac{d}{dx}\delta_p + \phi\delta_p^2\frac{d}{dx}\phi^2 = \alpha\frac{10}{u_\infty}$$

$$\delta_p\phi^3\frac{d}{dx}\delta_p + 2\phi^2\delta_p^2\frac{d}{dx}\phi = \alpha\frac{10}{u_\infty}$$

$$\phi^3\delta_p\frac{d}{dx}\delta_p + 2\delta_p^2\phi^2\frac{d}{dx}\phi^2 = \alpha\frac{10}{u_\infty}$$

$$\phi^3\delta_p\frac{d}{dx}\delta_p + \delta_p^2\frac{2}{3}\frac{d}{dx}\phi^3 = \alpha\frac{10}{u_\infty}$$

$$\delta_p = x\frac{4.64}{\sqrt{Re_x}} = \frac{4.64\sqrt{\nu x}}{\sqrt{u_\infty}}$$

$$\delta_p = x\frac{4.64}{\sqrt{Re_x}} = \frac{4.64\sqrt{\nu x}}{\sqrt{u_\infty}}$$

$$\delta_p^2 = x\frac{4.64^2}{\sqrt{Re_x}} = \frac{4.64^2\nu x}{u_\infty}$$

$$\frac{d}{dx}\delta_p = \frac{1}{2}\frac{4.64\sqrt{\nu}}{\sqrt{u_\infty x}}$$

$$\delta_p\frac{d}{dx}\delta_p = \frac{1}{2}\frac{4.64\sqrt{\nu x}}{\sqrt{u_\infty}}\frac{4.64\sqrt{\nu}}{\sqrt{u_\infty x}}$$

$$\delta_p\frac{d}{dx}\delta_p = \frac{1}{2}\frac{4.64^2\nu}{u_\infty}$$

subs back:

$$\phi^3\delta_p\frac{d}{dx}\delta_p + \delta_p^2\frac{2}{3}\frac{d}{dx}\phi^3 = \alpha\frac{10}{u_\infty}$$

$$\phi^3\frac{1}{2}\frac{4.64^2\nu}{u_\infty} + \frac{4.64^2\nu x}{u_\infty}\frac{2}{3}\frac{d}{dx}\phi^3 = \alpha\frac{10}{u_\infty}$$

$$\phi^3\frac{1}{2}\frac{4.64^2\nu}{1} + \frac{4.64^2\nu x}{1}\frac{2}{3}\frac{d}{dx}\phi^3 = \alpha\frac{10}{1}$$

$$\phi^3\frac{1}{2} + x\frac{2}{3}\frac{d}{dx}\phi^3 = \frac{10}{4.64^2}\frac{\alpha}{\nu}$$

$$\phi^3\frac{3}{2} + 2x\frac{d}{dx}\phi^3 = \frac{30}{4.64^2}\frac{1}{Pr}$$

$$\phi^3 + \frac{4}{3}x\frac{d}{dx}\phi^3 = \frac{20}{4.64^2}\frac{1}{Pr}$$

First order ODE:

https://mathworld.wolfram.com/First-OrderOrdinaryDifferentialEquation.html

$$y' + p(x)y = q(x)$$

$$y = \frac{\int e^{\int^x p(x')dx'} q(x)dx + c}{e^{\int^x p(x')dx'}}$$

Replace x by dummy variables x'

$$\frac{d\delta_t^3}{dx'} - \delta_t^3 \frac{0.75}{x'} = A\sqrt{x'}$$

$$p(x') = -\frac{0.75}{x'}$$

$$q(x') = A\sqrt{x'}$$

[second careless mistake: wrong choice of integrating factor]

$$\int^x p(x')dx' = \int^x \frac{-0.75}{x'} dx' = -0.75\ln(x)$$

$$\exp(0.75\ln(x)) = -\exp(0.75)x$$

$$\int e^{\int^x p(x')dx'} q(x)dx = -\int \exp(0.75)xA\sqrt{x}dx$$

$$= -A\exp(0.75)\int x^{1.5}dx$$

$$= -A\exp(0.75)\frac{x^{2.5}}{2.5}$$

$$\delta_t^3 = \frac{-A\exp(0.75)\frac{x^{2.5}}{2.5} + c}{-\exp(0.75)x}$$

$$\delta_t^3 = \frac{-A\exp(0.75)\frac{x^{2.5}}{2.5}}{-\exp(0.75)x} + \frac{c}{\exp(0.75)x}$$

$$\delta_t^3 = A\frac{x^{1.5}}{2.5} + \frac{c}{-\exp(0.75)x}$$

[wrong... ODE solving gone wrong...]

https://www.wolframalpha.com/input/?i=solve+y%27+-+0.75+y%2Ft+%3DA+sqrt%28t%29

According to wolfram:

$$y = \frac{4}{3}Ax^{1.5} + Cx^{0.75}$$

apply BC, x=0 $\delta_t = 0$, c=0

$$\delta_t^3 = \frac{4}{3}Ax^{1.5}$$

$$\delta_t = \mathcal{O}(\sqrt{x})$$

$$\delta_t^3 = \alpha\frac{15 * 4.64}{1}\frac{\sqrt{\nu}}{u_\infty^{1.5}}x^{1.5}\frac{4}{3}$$

$$\frac{\delta_p}{x} = \frac{4.64}{\sqrt{Re_x}}$$

$$\delta_t^3 = \frac{\alpha}{\nu}\frac{15 * 4.64}{1}\frac{\nu^{1.5}}{u_\infty^{1.5}}\frac{4}{3}x^{1.5}$$

$$\frac{x}{\sqrt{Re_x}} = \frac{\sqrt{\nu x}}{\sqrt{u_\infty}}$$

$$\delta_t^3 = \frac{\alpha}{\nu}\frac{15 * 4.64}{1}\frac{x^3}{Re_x^{1.5}}\frac{4}{3}$$

$$\frac{\delta_p}{4.64} = \frac{x}{\sqrt{Re_x}}$$

$$\delta_t^3 = \frac{1}{Pr}\frac{15 * 4.64}{1}(\frac{\delta_p}{4.64})^3\frac{4}{3}$$

$$\frac{\delta_t^3}{\delta_p^3} = \frac{1}{Pr}\frac{15 * 4.64}{4.64^3}\frac{4}{3}$$

$$\frac{\delta_t^3}{\delta_p^3} = \frac{1}{Pr}0.92895$$

$$\frac{\delta_t}{\delta_p} = 0.97573\frac{1}{Pr^{1/3}}$$

[redundant to use similarity soln and wrong number] If you use the similarity solution, you will get this:

$$\frac{\delta_p}{x} = \frac{5}{\sqrt{Re_x}}$$

$$\delta_t^3 = \frac{1}{Pr}\frac{30*5}{1}(\frac{\delta_p}{5})^3\frac{1}{2.5}$$

$$\frac{\delta_t}{\delta_p} = 0.78297\frac{1}{Pr^{1/3}}$$

[redundant to use similarity soln and wrong number]

What's our heat flux?

$$\frac{\partial}{\partial y}\theta|_{y=0} = \frac{3}{2\delta_t}$$

$$\frac{\partial}{\partial y}\theta|_{y=0} = \frac{3}{2\delta_p * 0.97573\frac{1}{Pr^{1/3}}}$$

$$\frac{\partial}{\partial y}\theta|_{y=0} = \frac{3}{2\delta_p * 0.97573} * Pr^{1/3}$$

$$\delta_p = \frac{4.64x}{\sqrt{Re_x}}$$

$$\frac{\partial}{\partial y}\theta|_{y=0} = \frac{3}{2\frac{4.64x}{\sqrt{Re_x}} * 0.97573} * Pr^{1/3}$$

$$\frac{\partial}{\partial y}\theta|_{y=0} = \frac{3}{2 * 4.64x * 0.97573} * Pr^{1/3}Re^{1/2}$$

$$Nu_x = \frac{hx}{k}$$

$$q''_{wall\rightarrow fluid} = -k\frac{\partial T}{\partial y} = -h(T_\infty - T_s)$$

$$-k\frac{\partial T}{\partial y} = -h(T_\infty - T_s)$$

$$\frac{\partial T}{\partial y} = \frac{h}{k}(T_\infty - T_s)$$

$$\theta = \frac{T - T_s}{T_\infty - T_s}$$

$$\frac{\partial \theta}{\partial y} = \frac{h}{k}$$

$$\frac{h}{k}|_{y=0} = \frac{3}{2 * 4.64x * 0.97573} * Pr^{1/3} Re^{1/2}$$

$$\frac{hx}{k}|_{y=0} = \frac{3}{2 * 4.64 * 0.97573} * Pr^{1/3} Re^{1/2}$$

$$Nu_x|_{y=0} = \frac{3}{2 * 4.64 * 0.97573} * Pr^{1/3} Re^{1/2}$$

$$Nu_x|_{y=0} = 0.3313 Pr^{1/3} Re_x^{1/2}$$

$$Nu_L|_{y=0} = 0.663 Pr^{1/3} Re_L^{1/2}$$

7 Resources Online

For Momentum BLs:

```
http://web.mit.edu/fluids-modules/www/highspeed_flows/ver2/bl_Chap2.pdf
https://community.dur.ac.uk/suzanne.fielding/teaching/BLT/sec3.pdf

for Von Karman
https://nptel.ac.in/content/storage2/courses/112104118/lecture-29/29-3_momentum.htm
```

For Thermal BLs:

```
http://mech.sut.ac.ir/People/Courses/18/Chapter3-%20Part2.pdf
http://raops.org.in/epapers/june15_9.pdf

https://nptel.ac.in/content/storage2/courses/112101001/downloads/lec25.pdf
```

Part VI
Github Repo

```
https://github.com/theodoreOnzGit/heatTransferTheory_YouTube
```

Look under convection heat transfer...

Part VII
Bibliography

References

Bejan, A. (2013). *Convection heat transfer*. John wiley & sons.

Welty, J., Rorrer, G. L., & Foster, D. G. (2014). *Fundamentals of momentum, heat, and mass transfer*. John Wiley & Sons.