Convection BL YouTube

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Part I
Some Links before we start

https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations

Matrices in LaTeX
https://www.overleaf.com/learn/latex/Matrices

Tensors in LaTeX

Navier Stokes Equations
https://www.comsol.com/multiphysics/navier-stokes-equations
https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations

Github
https://github.com/theodoreOnzGit/heatTransferTheory_YouTube

Part II
Hydrodynamics

1 Navier Stokes Equations

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}\ u + \frac{\partial^2}{\partial z^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}\ v + \frac{\partial^2}{\partial z^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - \nu(\frac{\partial^2}{\partial x^2}w + \frac{\partial^2}{\partial y^2}\ w + \frac{\partial^2}{\partial z^2}w) = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + g_z$$

2 Boundary Layer Equations (Laminar)

$$0 = -\frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial y^2}\ u = (-\frac{\partial P}{\partial x})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

2.0.1 Blasius Similarity Solution

$$f' = 0 \ at \ \eta = 0$$

$$f = 0 \ at \ \eta = 0$$

$$f' = 2 \ at \ \eta \to \infty$$

$$f''f + f''' = 0$$

$$u = \frac{u_\infty}{2}f'$$

$$\eta = \frac{1}{2}y\sqrt{\frac{u_\infty}{\nu x}}$$

[not derived previously:]

$$\tau_x = \mu\frac{\partial u}{\partial y} = \rho\nu\frac{u_\infty}{4}\sqrt{\frac{u_\infty}{\nu x}}f''$$

Welty, J., Rorrer, G. L., & Foster, D. G. (2014).
Fundamentals of momentum, heat, and mass transfer. John Wiley & Sons.

2.1 Integral Solution by Theodore Von Karman

Final form of Von Karman equation:

$$\frac{\tau_x}{\rho} = (\frac{\partial u_\infty}{\partial x})\int_0^{\delta_p}(u_\infty - u)\ dy + \frac{\partial}{\partial x}\int_0^{\delta_p}u(u_\infty - u)\ dy$$

3 Solutions to BL

3.1 Von Karman Results (approximate solution)

$$\frac{\delta_p(x)}{x} = \frac{4.64095}{\sqrt{Re_x}}$$

$$C_{fx} \equiv \frac{0.6464}{\sqrt{Re_x}}$$

$$C_{fL} \equiv \frac{1.2928}{\sqrt{Re_L}}$$

3.2 Blasius Results (Exact solution)

$$\frac{\delta_p(x)}{x} = \frac{5}{\sqrt{Re_x}}$$

$$C_{fx} \equiv \frac{0.664}{\sqrt{Re_x}}$$

$$C_{fL} \equiv \frac{1.328}{\sqrt{Re_L}}$$

7.2% off for BL thickness, and 3% off for skin friction coeff. Pretty good!

This shows that Von Karman method is pretty good, if you can't use Blasius results

Welty, J., Rorrer, G. L., & Foster, D. G. (2014).
Fundamentals of momentum, heat, and mass transfer. John Wiley & Sons.

Part III
Energy Equations

$$h \equiv e + PV$$

Enthalpy per unit mass (Specific enthalpy) P is pressure V is specific volume e is specific internal energy

In thermodynamics, U is often used for internal energy, but we avoid using that because it gets confused with u which is x velocity. So we use e instead.

Enthalpy Balance

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = conduction + heat\ generation + dissipation + radiation \tag{1}$$

neglect heat generation and dissipation terms. Also neglect radiation heat transfer

Fourier's law

$$q''_{cond} = -k\nabla T$$

Conduction term:

$$conduction = -\nabla q''_{cond}$$

Heat flux = q''_{cond} ;
Heat flux is heat energy Transferred per unit area

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = -\nabla q''_{cond}$$

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = -\nabla(-k\nabla T)$$

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = \nabla(k\nabla T)$$

heat conduction coeffcient is isotropic (it's a scalar), we also assume k does not change with x,y or z

$$\frac{\partial}{\partial t}h + u\frac{\partial}{\partial x}h + v\frac{\partial}{\partial y}h + w\frac{\partial}{\partial z}h = k\nabla^2 T$$

$$\Delta h = \rho c_p \Delta T$$

$$h - h_{ref} = \rho c_p(T - T_{ref})$$

$$h = \rho c_p(T - T_{ref}) + h_{ref}$$

We note, reference temperature and enthalpy are constant with time, x y z etc.

$$\frac{\partial}{\partial t}\rho c_p T + u\frac{\partial}{\partial x}\rho c_p T + v\frac{\partial}{\partial y}\rho c_p T + w\frac{\partial}{\partial z}\rho c_p T = k\nabla^2 T$$

We assume, volumetric heat capacity does not change with x,y or z and time

$$\rho c_p = volumetric\ heat\ capacity = \frac{kg}{m^3}\frac{J}{kg \bullet k}$$

$$\frac{\partial}{\partial t}\rho c_p T + u\frac{\partial}{\partial x}\rho c_p T + v\frac{\partial}{\partial y}\rho c_p T + w\frac{\partial}{\partial z}\rho c_p T = k\nabla^2 T$$

$$\frac{\partial}{\partial t}T + u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T + w\frac{\partial}{\partial z}T = \frac{k}{\rho c_p}\nabla^2 T$$

$$\frac{\partial}{\partial t}T + u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T + w\frac{\partial}{\partial z}T = \alpha\nabla^2 T$$

$$\alpha = \frac{k}{\rho c_p}$$

$$\frac{\partial}{\partial t}T + u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T + w\frac{\partial}{\partial z}T = \alpha(\frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}\ T + \frac{\partial^2}{\partial z^2}T)$$

Part IV
Thermal Laminar BL Equations

First, we assume, 2D, so z derivatives are zero.

$$\frac{\partial}{\partial t}T + u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T + w\frac{\partial}{\partial z}T = \alpha(\frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}\ T + \frac{\partial^2}{\partial z^2}T)$$

$$\frac{\partial}{\partial t}T + u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}\ T = \alpha(\frac{\partial^2}{\partial x^2}T + \frac{\partial^2}{\partial y^2}\ T)$$

We need to introduce relative temperature

$$T_{rel} = T - T_s$$

$$\frac{\partial}{\partial x}(T - T_s) = \frac{\partial}{\partial x}(T)$$

so long as T_s is constant

$$\frac{\partial}{\partial t}(T - T_s) + u\frac{\partial}{\partial x}(T - T_s) + v\frac{\partial}{\partial y}(T - T_s) = \alpha(\frac{\partial^2}{\partial x^2}(T - T_s) + \frac{\partial^2}{\partial y^2}(T - T_s))$$

Let's see which terms we can neglect

$$\theta = \frac{T - T_s}{T_\infty - T_s}$$

$$u^* = \frac{u}{u_\infty}$$

Let's nondimensionaise temrperature:

$$\frac{\partial}{\partial t}\theta(T_\infty - T_s) + u\frac{\partial}{\partial x}\theta(T_\infty - T_s) + v\frac{\partial}{\partial y}\theta(T_\infty - T_s) = \alpha(\frac{\partial^2}{\partial x^2}\theta(T_\infty - T_s) + \frac{\partial^2}{\partial y^2}\theta(T_\infty - T_s))$$

Divide throughout,

$$\frac{\partial}{\partial t}\theta + u\frac{\partial}{\partial x}\theta + v\frac{\partial}{\partial y}\theta = \alpha(\frac{\partial^2}{\partial x^2}\theta + \frac{\partial^2}{\partial y^2}\theta)$$

$$x^* = \frac{x}{L}$$

$$y^* = \frac{y}{\delta_t}$$

Note: in general, thermal boundary layer is different from momentum boundary layer

$$\frac{\partial}{\partial t}\theta + u\frac{1}{L}\frac{\partial}{\partial x^*}\theta + v\frac{1}{\delta_t}\frac{\partial}{\partial y^*}\theta = \alpha(\frac{1}{L^2}\frac{\partial^2}{\partial(x^*)^2}\theta + \frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta)$$

$$u^* = \frac{u}{u_\infty}$$

$$\frac{\partial}{\partial t}\theta + u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}\theta + v\frac{1}{\delta_t}\frac{\partial}{\partial y^*}\theta = \alpha(\frac{1}{L^2}\frac{\partial^2}{\partial(x^*)^2}\theta + \frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta)$$

We assume no natural convection

$$\frac{\partial}{\partial t}\theta + u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}\theta + v\frac{1}{\delta_t}\frac{\partial}{\partial y^*}\theta = \alpha(\frac{1}{L^2}\frac{\partial^2}{\partial(x^*)^2}\theta + \frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta)$$

If we were to use momentum nondimensionalisation:

$$y^* = \frac{y}{\delta_p}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{u_\infty}{L}\frac{\partial u^*}{\partial x^*} + \frac{v_c}{\delta_p}\frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{u_\infty \delta_p}{v_c L}\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0$$

$$\frac{u_\infty \delta_p}{v_c L} = \mathcal{O}(1)$$

$$\frac{u_\infty \delta_p}{L} = \mathcal{O}(1)v_c$$

we can just set:

$$v_c = \frac{u_\infty \delta_p}{L}\mathcal{O}(1)$$

so that

$$v^* = \frac{v}{v_c} = \mathcal{O}(1)$$

or else

$$v_c = \frac{u_\infty \delta_p}{L}$$

So we substitute this back in:

$$\frac{\partial}{\partial t}\theta + u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}\theta + \frac{u_\infty \delta_p}{L\delta_t}v^*\frac{\partial}{\partial y^*}\theta = \alpha\left(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}\theta + \frac{1}{\delta_t^2}\frac{\partial^2}{\partial (y^*)^2}\theta\right)$$

$$t = \frac{t^*}{t_c}$$

if you talk about hydrodynamic timescales:

$$t_c = \frac{L}{u_\infty}$$

we have to use thermal BL timescales...

For simplicity, we may not want to consider transient timescales in flat plate BL analysis

We assume steady state

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$$u^* \frac{u_\infty}{L} \frac{\partial}{\partial x^*} \theta + \frac{u_\infty \delta_p}{L \delta_t} v^* \frac{\partial}{\partial y^*} \theta = \alpha \left(\frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2} \theta + \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

$$u^* \frac{\partial}{\partial x^*} \theta + \frac{\delta_p}{\delta_t} v^* \frac{\partial}{\partial y^*} \theta = \alpha \frac{L}{u_\infty} \left(\frac{1}{L^2} \frac{\partial^2}{\partial (x^*)^2} \theta + \frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

We can make this observation:

$$L^2 >> \delta_t^2$$

$$u^* \frac{\partial}{\partial x^*} \theta + \frac{\delta_p}{\delta_t} v^* \frac{\partial}{\partial y^*} \theta = \alpha \frac{L}{u_\infty} \left(\frac{1}{\delta_t^2} \frac{\partial^2}{\partial (y^*)^2} \theta \right)$$

assume steady state, and performing BL analysis: Also, no dissipation or heat generation

$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} \ T = \alpha \left(\frac{\partial^2}{\partial y^2} \ T \right)$$

Part V
Solution Procedures - Constant Temp Forced Conv BL

$$u \frac{\partial}{\partial x} T + v \frac{\partial}{\partial y} T = \alpha \frac{\partial^2}{\partial y^2} T$$

$$0 = -\frac{\partial P}{\partial y}$$

$$u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u - \nu \frac{\partial^2}{\partial y^2} \ u = \left(-\frac{\partial P}{\partial x} \right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Solution Procedures:

- Similarity Solution (Blasius Style) [GOLD STANDARD]

- Integral Solution (Von Karman Style)

- Computational Fluid Dynamics (MultiPhysics)

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4 back to similarity Solution

In case of no pressure gradient,

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}T = \alpha\frac{\partial^2}{\partial y^2}T$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = \nu\frac{\partial^2}{\partial y^2}u$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

And nondimensionalising

$$f' = 0 \; at \; \eta = 0$$

$$f = 0 \; at \; \eta = 0$$

$$f' = 2 \; at \; \eta \to \infty$$

$$f''f + f''' = 0$$

$$u = \frac{u_\infty}{2}f'$$

$$\eta = \frac{1}{2}y\sqrt{\frac{u_\infty}{\nu x}}$$

[not derived previously:]

$$\tau_x = \mu\frac{\partial u}{\partial y} = \rho\nu\frac{u_\infty}{4}\sqrt{\frac{u_\infty}{\nu x}}f''$$

4.0.1 Reynold's analogy

Hey these equations look so similar

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}T = \alpha\frac{\partial^2}{\partial y^2}T$$

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u = \nu\frac{\partial^2}{\partial y^2}u$$

Let's copy/paste the solution... (under certain constraints)

$$\nu = \alpha$$

$$Pr = \frac{\nu}{\alpha} = 1$$

If you wanna change up the equations:

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}T = \frac{\nu}{Pr}\frac{\partial^2}{\partial y^2}T$$

If Pr=1,

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}T = \nu\frac{\partial^2}{\partial y^2}T$$

Use the relative temperature trick, and change the variable to θ

$$u\frac{\partial}{\partial x}\theta + v\frac{\partial}{\partial y}\theta = \nu\frac{\partial^2}{\partial y^2}\theta$$

If we nondimensionalise the momentum equation

$$u\frac{\partial}{\partial x}u^*u_\infty + v\frac{\partial}{\partial y}u^*u_\infty = \nu\frac{\partial^2}{\partial y^2}u^*u_\infty$$

$$u\frac{\partial}{\partial x}u^* + v\frac{\partial}{\partial y}u^* = \nu\frac{\partial^2}{\partial y^2}u^*$$

We already proved we can reduce the momentum BL equations as follows:

$$f''f + f''' = 0$$

And we note that we can set:

$$\theta = u^*$$

Note that

$$u = \frac{u_\infty}{2}f'$$

$$u^* = \frac{f'}{2}$$

Under Reynold's analogy we can say:

$$\theta = \frac{f'}{2}$$

We can pretty much adapt the Blasius solution to thermal boundary layer... (Reynold's analogy under Pr=1)

We assume that the nondimensional temperature profile and nondimensional BL profile (and BCs) are exactly equal.

Based on that, we can get the temperature profile and heat flux

$$\theta = \frac{f'}{2}$$

$$T - T_s = (T_\infty - T_s)\frac{f'}{2}$$

What about heat flux:

$$q'' = -k\frac{\partial T}{\partial y}$$

note:

$$\frac{\partial u}{\partial y} = \frac{u_\infty}{4}\sqrt{\frac{u_\infty}{\nu x}}f''$$

$$\frac{\partial u^* u_\infty}{\partial y} = \frac{u_\infty}{4}\sqrt{\frac{u_\infty}{\nu x}}f''$$

$$\frac{\partial u^*}{\partial y} = \frac{1}{4}\sqrt{\frac{u_\infty}{\nu x}}f''$$

under reynold's analogy

$$u^* = \theta$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{4}\sqrt{\frac{u_\infty}{\nu x}}f''$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{4}\sqrt{\frac{u_\infty}{\nu x}}f''$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{4}\sqrt{\frac{u_\infty x}{\nu x^2}}f''$$

$$\frac{\partial \theta}{\partial y} = \frac{1}{4}\sqrt{Re_x}\frac{1}{x}f''$$

$$\frac{\partial (T - T_s)}{\partial y} = \frac{T_\infty - T_s}{4}\sqrt{Re_x}\frac{1}{x}f''$$

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$$\frac{\partial T}{\partial y} = \frac{T_\infty - T_s}{4}\sqrt{Re_x}\frac{1}{x}f''$$

Heat flux:

$$q'' = -k\frac{\partial T}{\partial y}$$

$$q'' = -\frac{k}{x}\frac{T_\infty - T_s}{4}\sqrt{Re_x}f''$$

Heat trf Coefficient and Nu

$$q'' = -h(T_\infty - T_s)$$

Substitute

$$h(T_\infty - T_s) = \frac{k}{x}\frac{T_\infty - T_s}{4}\sqrt{Re_x}f''$$

$$h = \frac{k}{x}\frac{f''}{4}\sqrt{Re_x}$$

Define Nusselt Number:

$$Nu_x \equiv \frac{hx}{k}$$

$$Nu_x = \frac{f''}{4}\sqrt{Re_x}$$

What is the value of f" (look at textbook, or solve it yourself)
at y=0

$$f'' = 1.328$$

$$Nu_x = \frac{1.328}{4}\sqrt{Re_x} = 0.332\sqrt{Re_x}$$

5 What if Pr is not 1?

5.1 Similarity Solution by Pohlhausen

Bejan, A. (2013). Convection heat transfer. John wiley & sons.
Welty, J., Rorrer, G. L., & Foster, D. G. (2014).
Fundamentals of momentum, heat, and mass transfer. John Wiley & Sons.

From Blasius Solution
And nondimensionalising

$$f' = 0 \ at \ \eta = 0$$

$$f = 0 \ at \ \eta = 0$$

$$f' = 2 \ at \ \eta \to \infty$$

$$f''f + f''' = 0$$

$$u = \frac{u_\infty}{2} f'$$

Some useful derivatives previously derived (Bejan, 2013,Welty et al., 2014)
From hydrodynamic BL:

$$\frac{\partial}{\partial y^*}\eta = \frac{\partial}{\partial y^*}\frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}} = \frac{\eta}{y^*}$$

$$\frac{\partial}{\partial x^*}\eta = \frac{\partial}{\partial x^*}\frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}$$

$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y}T = \alpha\frac{\partial^2}{\partial y^2}T$$

We nondimensionalised the energy equation during scaling analysis...

$$u^*\frac{\partial}{\partial x^*}\theta + \frac{\delta_p}{\delta_t}v^*\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta)$$

We just need to convert it to include similarity variables...
We start by making substitutions

$$u^* = \frac{f'}{2}$$

$$v = \frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}(\eta f' - f)$$

$$v_c = \frac{u_\infty \delta_p}{L}$$

$$v^*\frac{u_\infty \delta_p}{L} = -\frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}(\eta f' - f)$$

$$v^* = \frac{1}{2}\frac{L}{\delta_p}\sqrt{\frac{\nu}{xu_\infty}}(\eta f' - f)$$

$$v^* = \frac{1}{2}\frac{L}{\delta_p}Re_x^{-\frac{1}{2}}(\eta f' - f)$$

subs into:

$$u^*\frac{\partial}{\partial x^*}\theta + \frac{\delta_p}{\delta_t}v^*\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta)$$

$$\frac{f'}{2}\frac{\partial}{\partial x^*}\theta + \frac{\delta_p}{\delta_t}\left[-\frac{1}{2}\frac{L}{\delta_p}Re_x^{\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta)$$

$$\frac{f'}{2}\frac{\partial}{\partial x^*}\theta + \frac{L}{\delta_t}\left[-\frac{1}{2}Re_x^{\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta)$$

Now we need to substitute out y^* and x^* for η

$$y^* = \frac{y}{\delta_t}$$

$$x^* = \frac{x}{L}$$

we note x^* is the same definition as the nondimensionalised x^* used in deriving hydrodynamic BL, so we use it again...

So we can use:

$$\frac{\partial}{\partial x^*}\eta = \frac{\partial}{\partial x^*}\frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}$$

$$= \frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}\frac{-1}{2x^*} = \frac{\eta}{-2x^*}$$

Replace left most term...

$$\frac{\partial}{\partial x^*} = \frac{\partial\eta}{\partial x^*}\frac{\partial}{\partial\eta} = -\frac{\eta}{2x^*}\frac{\partial}{\partial\eta}$$

$$\frac{f'}{2}\frac{\partial}{\partial x^*}\theta + \frac{L}{\delta_t}\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta)$$

$$\frac{f'}{2}(-\frac{\eta}{2x^*})\frac{\partial}{\partial\eta}\theta + \frac{L}{\delta_t}\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta)$$

$$\frac{f'}{2}(-\frac{\eta}{2x^*})\theta' + \frac{L}{\delta_t}\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y^*}\theta = \alpha\frac{L}{u_\infty}(\frac{1}{\delta_t^2}\frac{\partial^2}{\partial(y^*)^2}\theta)$$

$$\frac{f'}{2}(-\frac{\eta}{2x^*})\theta' + L\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\partial}{\partial y}\theta = \alpha\frac{L}{u_\infty}(\frac{\partial^2}{\partial y^2}\theta)$$

What about the y terms?

$$\eta = \frac{1}{2}\frac{y}{x}(Re_x)^{\frac{1}{2}} = \frac{1}{2}y\sqrt{\frac{u_\infty}{\nu x}}$$

$$\frac{\partial}{\partial y} = \frac{\partial\eta}{\partial y}\frac{\partial}{\partial\eta}$$

$$\frac{\partial\eta}{\partial y} = \frac{1}{2}\sqrt{\frac{u_\infty}{\nu x}} = \frac{\eta}{y}$$

For RHS...

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\partial\eta}{\partial y}\frac{\partial\theta}{\partial\eta}) = \frac{\partial}{\partial y}(\frac{\partial\eta}{\partial y}\theta')$$

$$= \frac{\partial\theta'}{\partial y}\frac{\partial\eta}{\partial y} + \frac{\partial^2\eta}{\partial y^2}\theta'$$

$$\frac{\partial^2\eta}{\partial y^2} = \frac{\partial}{\partial y}(\frac{\eta}{y}) = 0$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial\theta'}{\partial y}\frac{\partial\eta}{\partial y} = \frac{\partial\eta}{\partial y}\frac{\partial\theta'}{\partial\eta}\frac{\partial\eta}{\partial y} = \theta''(\frac{\eta^2}{y^2})$$

Substitution back in:

$$\frac{f'}{2}(-\frac{\eta}{2x^*})\theta' + L\left[-\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\eta}{y}\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

tidying up:

$$(-\frac{\eta f'}{4x^*})\theta' + L\left[\frac{1}{2}Re_x^{-\frac{1}{2}}(\eta f' - f)\right]\frac{\eta}{y}\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

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$$(-\frac{\eta f'}{4x^*})\theta' + \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}(\eta f' - f)\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$(-\frac{\eta f'}{4x^*})\theta' + \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}(\eta f'\theta' - f\theta') = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$(-\frac{\eta f'}{4x^*})\theta' + \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}\eta f'\theta' - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$\theta'\eta f'(-\frac{1}{4x^*} + \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

simplifying...

$$\theta'\eta f'(-\frac{1}{4x^*} + \frac{1}{4}\sqrt{\frac{u_\infty}{\nu x}}LRe_x^{-\frac{1}{2}}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$\theta'\eta f'(-\frac{1}{4x^*} + \frac{1}{4x}\sqrt{Re_x}LRe_x^{-\frac{1}{2}}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$\theta'\eta f'(-\frac{1}{4x^*} + \frac{L}{4x}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$\theta'\eta f'(-\frac{1}{4x^*} + \frac{1}{4x^*}) - \frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$-\frac{1}{2}\frac{\eta}{y}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$-\frac{1}{2}\frac{1}{2}\frac{1}{x}Re_x^{\frac{1}{2}}LRe_x^{-\frac{1}{2}}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$-\frac{1}{4}\frac{1}{x^*}f\theta' = \alpha\frac{L}{u_\infty}(\frac{\eta^2}{y^2}\theta'')$$

$$-\frac{1}{4}\frac{1}{x^*}f\theta' = \alpha\frac{L}{u_\infty}(\frac{1}{4}\frac{Re_x}{x^2}\theta'')$$

$$-\frac{1}{4}\frac{1}{x^*}f\theta' = \alpha\frac{1}{u_\infty}(\frac{1}{4}\frac{Re_x}{xx^*}\theta'')$$

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$$-\frac{1}{4}f\theta' = \alpha\frac{1}{u_\infty}(\frac{1}{4}\frac{Re_x}{x}\theta'')$$

$$-f\theta' = \alpha\frac{1}{u_\infty}(\frac{Re_x}{x}\theta'')$$

$$-f\theta' = \alpha\frac{1}{u_\infty}(\frac{u_\infty x}{x\nu}\theta'')$$

$$-f\theta' = \frac{\alpha}{\nu}\theta''$$

$$-f\theta' = \frac{1}{Pr}\theta''$$

$$\theta'' + Pr\ f\theta' = 0$$

5.2 Similarity Solution BCs

2 Bcs:

$$y \to \infty\ ;\ T \to T_\infty$$

$$\eta \to \infty\ ;\ \theta \to 1$$

Reminder

$$\theta = \frac{T - T_s}{T_\infty - T_s}$$

$$y = 0; T = T_s$$

$$\eta = 0\ ; \theta = 0$$

5.3 solving the equation

$$\theta'' + Pr\ f\theta' = 0$$

$$\frac{\partial\theta'}{\partial\eta} + Pr\ f\theta' = 0$$

$$\frac{\partial\theta'}{\partial\eta} = -Pr\ f\theta'$$

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$$\frac{\partial \theta'}{\partial \eta} = -Pr \ f\theta'$$

$$\frac{1}{\theta'}\frac{\partial \theta'}{\partial \eta} = -Pr \ f$$

$$\int_0^\infty \frac{1}{\theta'}\frac{\partial \theta'}{\partial \eta}d\eta = -\int_0^\infty Pr \ fd\eta$$

$$\int_{\theta'|_{\eta=0}}^{\theta'|_{\eta=\infty}} \frac{1}{\theta'}d\theta' = -\int_0^\infty Pr \ f(\eta)d\eta$$

$$\log_e(\frac{\theta'|_{\eta=\infty}}{\theta'|_{\eta=0}}) = -\int_0^\infty Pr \ f(\eta)d\eta$$

$$\frac{\theta'|_{\eta=\infty}}{\theta'|_{\eta=0}} = \exp(-\int_0^\infty Pr \ f(\eta)d\eta)$$

What if η is not infinity,

$$\int_{\theta'|_{\eta'=0}}^{\theta'|_{\eta'=\eta}} \frac{1}{\theta'}d\theta' = -\int_0^\eta Pr \ f(\eta')d\eta'$$

$$\log_e(\frac{\theta'|_{\eta'=\infty}}{\theta'|_{\eta=0}}) = -\int_0^\infty Pr \ f(\eta)d\eta$$

$$\frac{\theta'|_{\eta'=\eta}}{\theta'|_{\eta=0}} = \exp(-\int_0^\eta Pr \ f(\eta')d\eta')$$

$$\theta'|_\eta = \theta'|_{\eta=0} \exp(-\int_0^\eta Pr \ f(\eta')d\eta')$$

Integrate again,

$$\int_0^\eta \theta'(\eta')d\eta' = \int_0^\eta \theta'|_{\eta'=0} \exp(-\int_0^\eta Pr \ f(\eta')d\eta')d\eta'$$

$$\theta(\eta) = \int_0^\eta \theta'|_{\eta=0} \exp(-\int_0^\eta Pr \ f(\eta')d\eta')d\eta'$$

use other conventional dummy variables to avoid confusion

$$\theta(\eta) = \int_{\gamma=0}^{\gamma=\eta} \theta'|_{\eta'=0} \exp(-\int_{\beta=0}^{\beta=\eta} Pr \ f(\beta)d\beta)\gamma$$

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One problem: we don't know what $\theta'|_{\eta'=0}$ is
we use the boundary condition:

$$\theta = 1 \ at \ \eta = infinity$$

the dummy variable representing η is γ

$$\theta(\eta = \infty) = \int_{\gamma=0}^{\gamma=\infty} \theta'|_{\eta=0} \exp(-\int_{\beta=0}^{\beta=\eta} Pr \ f(\beta)d\beta)\gamma$$

$$1 = \int_{\gamma=0}^{\gamma=\infty} \theta'|_{\eta=0} \exp(-\int_{\beta=0}^{\beta=\eta} Pr \ f(\beta)d\beta)\gamma$$

$$\theta'(0)^{-1} = \int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\eta} Pr \ f(\beta)d\beta)\gamma$$

$$\theta'(0) = \left[\int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\eta} Pr \ f(\beta)d\beta)\gamma\right]^{-1}$$

using correct mathematical symbols:

$$\theta'(0) = \left[\int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\gamma} Pr \ f(\beta)d\beta)\gamma\right]^{-1}$$

$$\theta(\eta) = \theta'(0) \int_{\gamma=0}^{\gamma=\eta} \exp(-Pr \int_{\beta=0}^{\beta=\gamma} f(\beta)d\beta)\gamma$$

We want to find out heat flux...

$$q'' = -k\frac{\partial T}{\partial y}|_{y=0}$$

$$q'' = -k\frac{\partial(T - T_s)}{\partial y}|_{y=0}$$

$$q'' = -k(T_\infty - T_s)\frac{\partial\theta}{\partial y}|_{y=0}$$

$$h = \frac{q''}{-(T_\infty - T_s)} = k\frac{\partial\theta}{\partial y}|_{y=0}$$

$$\frac{h}{k} = \frac{\partial\theta}{\partial y}|_{y=0}$$

$$\frac{h}{k} = \frac{\partial \eta}{\partial y} \frac{\partial \theta}{\partial \eta}\Big|_{y=0}$$

$$\frac{h}{k} = \frac{\eta}{y} \frac{\partial \theta}{\partial \eta}\Big|_{y=0}$$

$$\frac{h}{k} = \frac{1}{2}\sqrt{\frac{u_\infty}{\nu x}} \frac{\partial \theta}{\partial \eta}\Big|_{y=0}$$

$$\frac{h}{k} = \frac{1}{2}\sqrt{\frac{u_\infty}{\nu x}} \theta'(0)$$

$$\frac{h}{k} = \frac{1}{2}\sqrt{\frac{u_\infty x}{\nu x^2}} \theta'(0)$$

$$\frac{hx}{k} = \frac{1}{2}\sqrt{Re_x} \theta'(0)$$

$$Nu_x = \frac{1}{2}\sqrt{Re_x} \theta'(0)$$

$$Nu_x = \frac{1}{2}\sqrt{Re_x}\left[\int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\gamma} Pr\ f(\beta)d\beta)\gamma\right]^{-1}$$

In textbook expression:

$$Nu_x = \sqrt{Re_x}\left[\int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\gamma} \frac{1}{2}Pr\ f(\beta)d\beta)\gamma\right]^{-1}$$

What is the value of:

$$\left[\int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\gamma} \frac{1}{2}Pr\ f(\beta)d\beta)\gamma\right]^{-1}$$

Pohlhausen worked it out for Pr>0.5:

$$\left[\int_{\gamma=0}^{\gamma=\infty} \exp(-\int_{\beta=0}^{\beta=\gamma} \frac{1}{2}Pr\ f(\beta)d\beta)\gamma\right]^{-1} \approx 0.332 Pr^{\frac{1}{3}}$$

Liquid metals $Pr < 0.1$
Air $Pr = 0.72$
Water $Pr = 7$

Pr number says that for liquids of same viscosity, a higher prandtl number means lower thermal diffusivity

Thermal diffusivity intuition, high thermal conductivity means high thermal diffusivity, all else constant.

$$Nu_x = 0.332 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

$$Nu_L = \frac{1}{L} \int_0^L Nu_x dx$$

$$Nu_L = \frac{1}{L} \int_0^L 0.332 Re_x^{\frac{1}{2}} Pr^{\frac{1}{3}} dx$$

$$Nu_L = Pr^{\frac{1}{3}} \frac{1}{L} \int_0^L 0.332 Re_x^{\frac{1}{2}} dx$$

$$Nu_L = 0.664 Re_L^{\frac{1}{2}} Pr^{\frac{1}{3}}$$

Note: For the above solutions, use film temperature to evaluate thermal properties!

$$T_{film} = \frac{T_s + T_\infty}{2}$$

eg. c_p, μ etc..

6 Integral Solutions to Thermal BL Constant Temp Flat Plate

In the momentum BL, we considered a control volume in the BL to get:

$$\frac{\tau_x}{\rho} = (\frac{\partial u_\infty}{\partial x}) \int_0^{\delta_p} (u_\infty - u) \, dy + \frac{\partial}{\partial x} \int_0^{\delta_p} u(u_\infty - u) \, dy$$

Now for thermal BL, we can also consider control volume too (Welty et al., 2014).

We can consider again four sides of the control volume and perform an enthalpy balance, neglecting fluid kinetic energy:

$$\dot{Q} - \dot{W} = net \; enthalpy \; flows \; out + \Delta \; enthalpy \; w.r.t \; time$$

W is Work done by system, if Q+W then W is work done on system

$$\dot{Q} = q''dA$$

$$q''_{wall} = -k\frac{\partial T}{\partial y}|_{y=0}$$

$$dA = l_z \Delta x$$

For conduction, ignore conduction within fluid. (important assumption)
2nd assumption, no work done W=0,
Let's consider the enthalpy flows and assume steady state..
Consider net enthalpy flows in:
neglecting kinetic energy
inflow of enthalpy through left

$$= \int_0^{\delta_t} \rho u h l_z dy|_x$$

$$= \int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_x$$

outflow of enthalpy through the right

$$= \int_0^{\delta_t} \rho u h l_z dy|_{x+\Delta x}$$

$$= \int_0^{\delta_t} \rho u c_p (T - T_{ref}) l_z dy|_{x+\Delta x}$$

inflow of enthalpy through the top
Specific enthalpy:

$$h_\infty = c_p(T_\infty - T_{ref})$$

mass flowrate into system:

$$\dot{m} = \int_0^{\delta_t} \rho u l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u l_z dy|_x$$

put this toegther to find enthalpy inflow through top:

$$\dot{m}h_\infty = (\int_0^{\delta_t} \rho u l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u l_z dy|_x)c_p(T_\infty - T_{ref})$$

Let's subs everything back:

$$\dot{Q} - \dot{W} = net\ enthalpy\ flows\ out + \Delta\ enthalpy\ w.r.t\ time$$

$$-k\frac{\partial T}{\partial y}|_{y=0}l_z\Delta x - 0 = -(\int_0^{\delta_t} \rho u l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u l_z dy|_x)c_p(T_\infty - T_{ref})$$

$$+ \int_0^{\delta_t} \rho u c_p(T - T_{ref})l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u c_p(T - T_{ref})l_z dy|_x + 0$$

divide throughout by Δx take limit $\Delta x \to 0$

$$-k\frac{\partial T}{\partial y}|_{y=0}l_z = -\frac{1}{\Delta x}(\int_0^{\delta_t} \rho u l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u l_z dy|_x)c_p(T_\infty - T_{ref})$$

$$+\frac{1}{\Delta x}[\int_0^{\delta_t} \rho u c_p(T - T_{ref})l_z dy|_{x+\Delta x} - \int_0^{\delta_t} \rho u c_p(T - T_{ref})l_z dy|_x]$$

Assume T_∞ constant with x

$$-k\frac{\partial T}{\partial y}|_{y=0}l_z = -\frac{\partial}{\partial x}(\int_0^{\delta_t} \rho u l_z dy|)c_p(T_\infty - T_{ref})$$

$$+\frac{\partial}{\partial x}[\int_0^{\delta_t} \rho u c_p(T - T_{ref})l_z dy]$$

Drop out T_{ref} after bring $c_p T_\infty$ into integral:

$$-k\frac{\partial T}{\partial y}|_{y=0}l_z = -\frac{\partial}{\partial x}(\int_0^{\delta_t} \rho u l_z dy c_p T_\infty)$$

$$+\frac{\partial}{\partial x}[\int_0^{\delta_t} \rho u c_p T l_z dy]$$

Finally get rid of l_z

$$-k\frac{\partial T}{\partial y}|_{y=0} = -\frac{\partial}{\partial x}(\int_0^{\delta_t} \rho u dy c_p T_\infty) + \frac{\partial}{\partial x}[\int_0^{\delta_t} \rho u c_p T dy]$$

$$-k\frac{\partial T}{\partial y}|_{y=0} = -\frac{\partial}{\partial x}(\int_0^{\delta_t} \rho u dy c_p T_\infty) + \frac{\partial}{\partial x}[\int_0^{\delta_t} \rho u c_p T dy]$$

Tidy up equation:

$$-k\frac{\partial T}{\partial y}|_{y=0} = \frac{\partial}{\partial x}[\int_0^{\delta_t} \rho u c_p (T - T_\infty) dy]$$

y is integrated out so the integral on the RHS does not depend on y

$$-k\frac{\partial T}{\partial y}|_{y=0} = \frac{d}{dx}[\int_0^{\delta_t} \rho u c_p (T - T_\infty) dy]$$

You can even bring out ρc_p:

$$-k\frac{\partial T}{\partial y}|_{y=0} = \rho c_p \frac{d}{dx}[\int_0^{\delta_t} u (T - T_\infty) dy]$$

But in other textbooks, von karman's integral approach literally integrates the BL equations over y (Bejan, 2013).

we start at:
$$u\frac{\partial}{\partial x}T + v\frac{\partial}{\partial y} T = \alpha(\frac{\partial^2}{\partial y^2} T)$$

and for we put back u and v into the differential using product rule.

Now we consider continuity equation

And finally get...

Now that we have these equations we integrate across y in the BL.

Notice there is a partial derivative with x, we will need to use Leibniz's Rule

https://en.wikipedia.org/wiki/Leibniz_integral_rule

Consider the integral

$$\int_{y1=a(x)}^{y2=b(x)} f(x,y) dy$$

$$\frac{d}{dx}\left(\int_{y1=a(x)}^{y2=b(x)} f(x,y) dy\right) = f(x, y=b(x))\frac{d}{dx}b(x) - f(x, y=a(x))\frac{d}{dx}a(x) + \int_{y1=a(x)}^{y2=b(x)} \frac{\partial}{\partial x} f(x,y) dy$$

We can apply it

Then consider getting w (y velocity) by integrating continuity equation (Bejan, 2013).

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Now assume a power series (cubic) expression for temperature and velocity...

Recall for velocity:

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta_p} - \frac{1}{2}(\frac{y}{\delta_p})^3 \qquad (2)$$

we will then need to consider ratio of thermal BL and momentum BL.

We already know that from hydrodynamic BL integral and similarity solution

$$\frac{\delta_p}{x} = \frac{4.64}{\sqrt{Re_x}}$$

$$\frac{\delta_p}{x} = \frac{5}{\sqrt{Re_x}}$$

7   Resources Online

For Momentum BLs:

```
http://web.mit.edu/fluids-modules/www/highspeed_flows/ver2/bl_Chap2.pdf
https://community.dur.ac.uk/suzanne.fielding/teaching/BLT/sec3.pdf
```

```
for Von Karman
https://nptel.ac.in/content/storage2/courses/112104118/lecture-29/29-3_momentum.htm
```

For Thermal BLs:

```
http://mech.sut.ac.ir/People/Courses/18/Chapter3-%20Part2.pdf
http://raops.org.in/epapers/june15_9.pdf
```

Part VI
Github Repo

```
https://github.com/theodoreOnzGit/heatTransferTheory_YouTube
```

Look under convection heat transfer...

Part VII
Bibliography

References

Bejan, A. (2013). *Convection heat transfer*. John wiley & sons.

Welty, J., Rorrer, G. L., & Foster, D. G. (2014). *Fundamentals of momentum, heat, and mass transfer*. John Wiley & Sons.