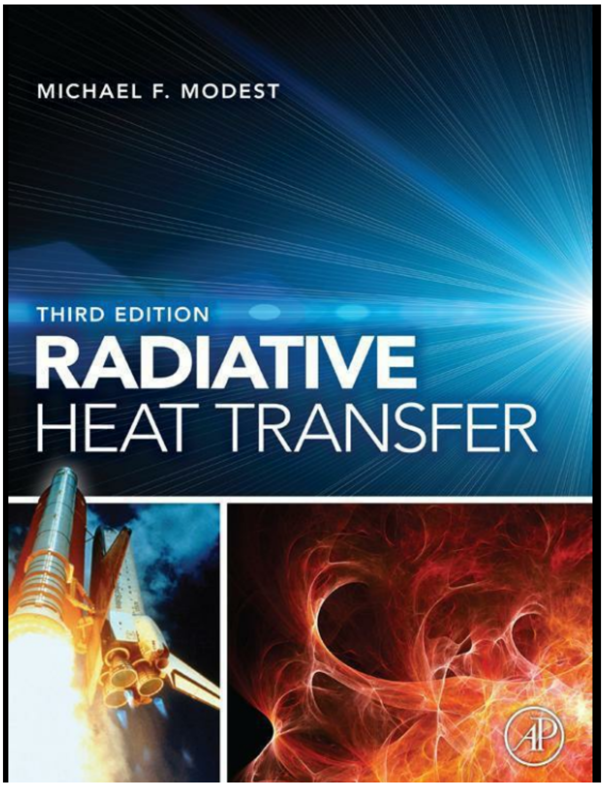
Radiation Heat Transfer (Learning Journal)

Reference:



https://www.sciencedirect.com/book/9780123869449/radiative-heat-transfer

1 – Introduction

The more advanced stuff...

What you should know:

- Basic Conduction: $q^{''} = -k\nabla T$
- Basic Convection: $q^{''} = h\left(T - T_{\infty}\right)$
- Basic Radiation $q^{''} \propto \left(T^4 - T_{\infty}^4\right)$

  Thermal radiation = electromagnetic wave

  Some recap on terms:

  Refractive index $c_n = \frac{c_0}{n}; c_0 = 2.998 * 10^8 \frac{m}{s}$

frequency, ν	(measured in cycles/s = s^{-1} = Hz);
wavelength, λ	(measured in μm = 10^{-6} m or nm = 10^{-9} m);
wavenumber, η	(measured in cm^{-1}); or
angular frequency, ω	(measured in radians/s = s^{-1}).

$$c_0 = \nu\lambda\,(forelectromagneticwaves)$$

$$c_n = \nu\lambda_n$$

$$\epsilon = h\nu$$

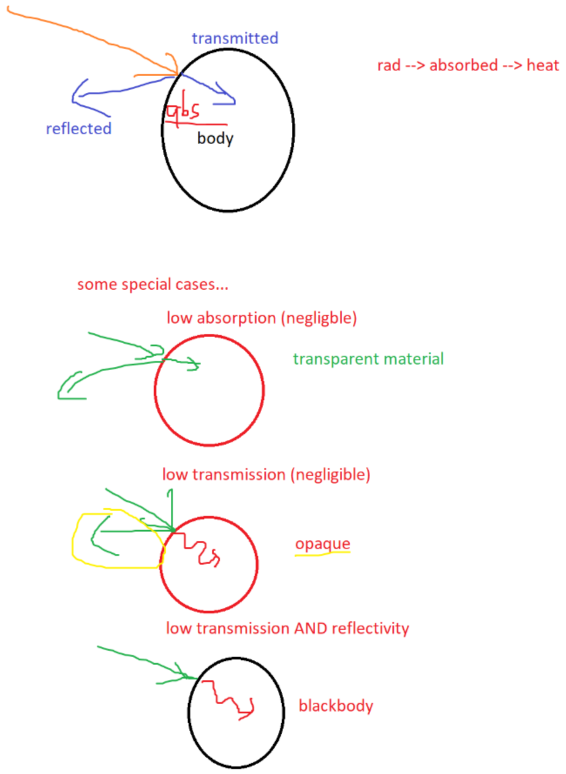
$$\nu = \frac{\omega}{2\pi} = \frac{c}{\lambda} = c\eta$$

$$\eta = \frac{1}{\lambda}\ \ (wavenumber)$$

Wavelength of thermal radiation:

$$10^{-7}\text{m to } 10^2 m$$

What is a blackbody?



We often say, that a blackbody is a good absorber, but absorbers are good emitters...

How do we prove that?

https://en.wikipedia.org/wiki/Kirchhoff′s_law_of_thermal_radiation
For an arbitrary body emitting and absorbing thermal radiation in thermodynamic equilibrium, the emissivity is equal to the absorptivity.

$$\alpha = \varepsilon inthermodynamicequilibrium, onlyradiationheattrf$$

Blackbody Emissions...

What exactly is absorptivity and emissivity?

Before we start, we need to look at emissive power...

spectral emissive power, $E_\nu \equiv$ emitted energy/time/surface area/frequency,
total emissive power, $E \equiv$ emitted energy/time/surface area.

How can we characterize the radiation coming off the body?

$$E_\nu = \frac{energy}{time \cdot Area \cdot frequency}$$

For Total Emissive Power, Integrate across all frequency

$$E = \int_0^\infty E_\nu d\nu$$

Special cases:

- Blackbody
- Monochromatic (one frequency), eg. Laser
- Gray (energy is equally distributed across all frequencies)

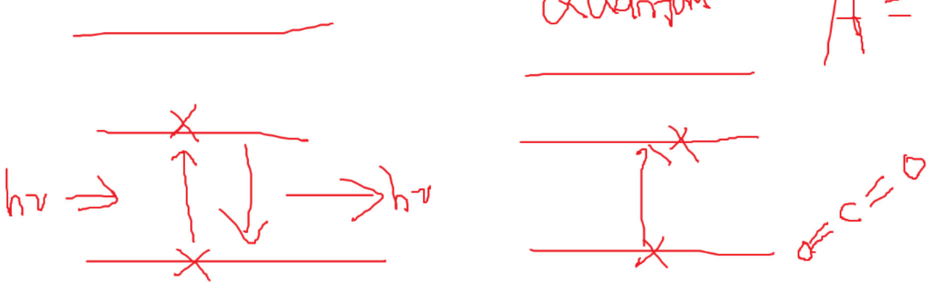
$$E_{b\nu}(T, \nu) = \frac{2\pi h\nu^3 n^2}{c_0^2\left[e^{h\nu/kT} - 1\right]},$$

$$E_{b\nu}\left(T, \nu\right) = \frac{2\pi h\nu^3 n^2}{c_0^2\left[\exp\left(\frac{h\nu}{kT}\right) - 1\right]}$$

Based on quantum physics...

Quantum

$A = \exp\left(-\frac{E}{RT}\right)$



$$E_{b\nu}(T,\nu) = \frac{2\pi h \nu^3 n^2}{c_0^2\left[e^{h\nu/kT}-1\right]},$$

Planck's Law

$$E_{b\nu}\left(T,\nu\right) = \frac{2\pi h\nu^3 n^2}{c_0^2\ \left[\exp\left(\frac{h\nu}{kT}\right)-1\right]}$$

$$E_{b\nu}\left(T,\nu\right) = \frac{2\pi h\nu^3}{c_n^2\ \left[\exp\left(\frac{h\nu}{kT}\right)-1\right]}$$

1) Blackbody emits radiation in all directions equally

Total energy emitted?

$$E_b\left(T\right) = \int_0^\infty \frac{2\pi h\nu^3 n^2}{c_0^2\ \left[\exp\left(\frac{h\nu}{kT}\right)-1\right]} d\nu$$

(1.10)

$$\nu = \frac{c_0}{n\lambda} = \frac{c_0}{n}\eta, \qquad d\nu = -\frac{c_0}{n\lambda^2}\left[1+\frac{\lambda}{n}\frac{dn}{d\lambda}\right]d\lambda = \frac{c_0}{n}\left[1-\frac{\eta}{n}\frac{dn}{d\eta}\right]d\eta,$$

and

$$E_b(T) = \int_0^\infty E_{b\nu}\,d\nu = \int_0^\infty E_{b\lambda}\,d\lambda = \int_0^\infty E_{b\eta}\,d\eta, \qquad \textbf{(1.11)}$$

or

$$E_{b\nu}\,d\nu = -E_{b\lambda}\,d\lambda = E_{b\eta}\,d\eta. \qquad \textbf{(1.12)}$$

$$E_b\left(T\right)=\int_0^\infty E_{b\nu}d\nu=\int_0^\infty \frac{2\pi h\nu^3 n^2}{c_0^2\ \left[\exp\left(\frac{h\nu}{kT}\right)-1\right]}d\nu=\int_0^\infty E_{b\lambda}d\lambda$$

$$E_b\left(T\right)=\int_0^\infty E_{b\nu}d\nu=\int_0^\infty E_{b\lambda}d\lambda$$

$$d\nu=\left(-\frac{c_0}{n\lambda}\right)\left[1+\frac{\lambda}{n}\frac{dn}{d\lambda}\right]d\lambda$$

$$E_b\left(T\right)=\int_\infty^0 \frac{2\pi h\nu^3 n^2}{c_0^2\ \left[\exp\left(\frac{h\nu}{kT}\right)-1\right]}\ \left(-\frac{c_0}{n\lambda}\right)\left[1+\frac{\lambda}{n}\frac{dn}{d\lambda}\right]d\lambda$$

$$E_b\left(T\right)=\int_0^\infty \frac{2\pi h\nu^3 n^2}{c_0^2\ \left[\exp\left(\frac{h\nu}{kT}\right)-1\right]}\ \left(\frac{c_0}{n\lambda}\right)\left[1+\frac{\lambda}{n}\frac{dn}{d\lambda}\right]d\lambda=\int_0^\infty E_{b\lambda}d\lambda$$

$$E_{b\lambda}=\frac{2\pi h\nu^3 n^2}{c_0^2\ \left[\exp\left(\frac{h\nu}{kT}\right)-1\right]}\ \left(\frac{c_0}{n\lambda}\right)\left[1+\frac{\lambda}{n}\frac{dn}{d\lambda}\right]$$

$$\nu=\frac{c_0}{n\lambda}$$

$$E_{b\lambda}=\frac{2\pi h\left(\frac{c_0}{n\lambda}\right)^3 n^2}{c_0^2\ \left[\exp\left(\frac{h\left(\frac{c_0}{n\lambda}\right)}{kT}\right)-1\right]}\ \left(\frac{c_0}{n\lambda}\right)\left[1+\frac{\lambda}{n}\frac{dn}{d\lambda}\right]$$

$$E_b\left(T\right)=\int_0^\infty E_{b\nu}d\nu=\int_0^\infty E_{b\eta}d\eta$$

$$E_b\left(T\right)=\int_0^\infty \frac{2\pi h\nu^3 n^2}{c_0^2\ \left[\exp\left(\frac{h\nu}{kT}\right)-1\right]}d\nu$$

$$d\nu=\frac{c_0}{n}\left[1-\frac{\eta}{n}\left(\frac{dn}{d\eta}\right)\right]d\eta$$

$$E_b\left(T\right)=\int_0^\infty \frac{2\pi h\nu^3 n^2}{c_0^2\ \left[\exp\left(\frac{h\nu}{kT}\right)-1\right]}\frac{c_0}{n}\left[1-\frac{\eta}{n}\left(\frac{dn}{d\eta}\right)\right]d\eta$$

$$\nu=\frac{c_0}{n}\eta$$

$$E_b\left(T\right)=\int_0^\infty \frac{2\pi h\left(\frac{c_0}{n}\eta\right)^3 n^2}{c_0^2\ \left[\exp\left(\frac{h\left(\frac{c_0}{n}\eta\right)}{kT}\right)-1\right]}\frac{c_0}{n}\left[1-\frac{\eta}{n}\left(\frac{dn}{d\eta}\right)\right]d\eta$$

$$E_{b\eta} = \frac{2\pi h\left(\frac{c_0}{n}\eta\right)^3 n^2}{c_0^2\left[\exp\left(\frac{h\left(\frac{c_0}{n}\eta\right)}{kT}\right)-1\right]}\frac{c_0}{n}\left[1-\frac{\eta}{n}\left(\frac{dn}{d\eta}\right)\right]$$

If we have constant refractive index (not always the case, but true for vacuum and gases)

$$E_{b\eta} = \frac{2\pi h\left(\frac{c_0}{n}\eta\right)^3 n^2}{c_0^2\left[\exp\left(\frac{h\left(\frac{c_0}{n}\eta\right)}{kT}\right)-1\right]}\frac{c_0}{n}$$

$$E_{b\lambda} = \frac{2\pi h\left(\frac{c_0}{n\lambda}\right)^3 n^2}{c_0^2\left[\exp\left(\frac{h\left(\frac{c_0}{n\lambda}\right)}{kT}\right)-1\right]}\left(\frac{c_0}{n\lambda}\right)$$

What is the wavelength/frequency/wavenumber where most energy is emitted?

At a fixed temperature T, what is the frequency?

$$E_{b\nu}\left(T,\nu\right) = \frac{2\pi h\nu^3}{c_n\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]}$$

$$\frac{\partial}{\partial\nu}E_{b\nu}\left(T,\nu\right) = 0$$

$$\frac{\partial}{\partial\nu}\frac{2\pi h\nu^3}{c_n\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]} = 0$$

$$\frac{\partial}{\partial\nu}\frac{\nu^3}{c_n\left[\exp\left(\frac{h\nu}{kT}\right)-1\right]} = 0$$

$$\frac{\partial}{\partial\lambda}$$

$$E_{b\lambda}(T,\lambda) = \frac{2\pi hc_0^2}{n^2\lambda^5\left[e^{hc_0/n\lambda kT}-1\right]},$$

$$E_{b\lambda}\left(T,\lambda\right) = \frac{2\pi hc_0^2}{n^2\lambda^5\left[\exp\left(\frac{h\left(\frac{c_0}{n\lambda}\right)}{kT}\right)-1\right]}$$

It is customary to introduce the abbreviations

$$C_1 = 2\pi h c_0^2 = 3.7418 \times 10^{-16}\,\mathrm{W\,m^2},$$
$$C_2 = hc_0/k = 14{,}388\,\mu\mathrm{m\,K} = 1.4388\,\mathrm{cm\,K},$$

so that equation (1.13) may be recast as

$$\frac{E_{b\lambda}}{n^3T^5} = \frac{C_1}{(n\lambda T)^5[e^{C_2/(n\lambda T)} - 1]}, \qquad (n = \text{const}), \qquad (1.15)$$

$$\frac{E_{b\lambda}(T,\lambda)}{n^3T^5} = \frac{2\pi h c_0^2}{T^5 n^5 \lambda^5 \left[\exp\left(\frac{h\left(\frac{c_0}{n\lambda}\right)}{kT}\right) - 1\right]}$$

We do this so we can group the terms

$n\lambda T$ together

$$\frac{E_{b\lambda}(T,\lambda)}{n^3T^5} = \frac{2\pi h c_0^2}{(n\lambda T)^5 \left[\exp\left(\frac{hc_0}{k(n\lambda T)}\right) - 1\right]}$$

$$\frac{\partial}{\partial(n\lambda T)} \frac{E_{b\lambda}(T,\lambda)}{n^3T^5} = \frac{\partial}{\partial(n\lambda T)} \frac{2\pi h c_0^2}{(n\lambda T)^5 \left[\exp\left(\frac{hc_0}{k(n\lambda T)}\right) - 1\right]} = 0$$

$$\frac{\partial}{\partial(n\lambda T)} \frac{2\pi h c_0^2}{(n\lambda T)^5 \left[\exp\left(\frac{hc_0}{k(n\lambda T)}\right) - 1\right]} = 0$$

When we set the derivative to 0, we find the maximum $\frac{E_{b\lambda}(T,\lambda)}{n^3T^5}$, which corresponds to the maximum energy emitted for a particular wavelength.

$$(n\lambda T)_{\max} = C_3 = 2898\,\mu\mathrm{m\,K}.$$

holds constant for all black bodies

https://en.wikipedia.org/wiki/Wien's_displacement_law

So, Planck's law describes **black body emission**, Wien's displacement law describes the wavelength for which maximum energy is emitted in the Planck spectrum.

Solid Angles and Directional Dependence

Direction is $\hat{s}$

The normal area projected in the direction $\hat{s}$ is $dA_j^{''} \rightarrow normalareareceivingtheradiation$

$$dA \rightarrow areaemittingtheradiation$$

$$dA_j^{''} = dA_j \cos\theta_j$$

Solid angle?

$$solidangle = \frac{dA_j^{''}}{r^2} = \frac{dA_j \cos\theta_j}{r^2}$$

$$d\Omega = \frac{dA_j \cos\theta_j}{r^2} = \frac{(rsin\theta d\psi) * (rd\theta)}{r^2} = \sin\theta d\theta d\psi$$

$$I\left(T, \theta, \psi, \nu\right) = I\left(T, \Omega, \nu\right)$$

Just note the difference between spectral intensity and total intensity.

$$E = \int_{d\Omega} d\Omega I\left(r, \overrightarrow{s}\right)$$

$$E_\nu = \int_{d\Omega} d\Omega I\left(r, \overrightarrow{s}, \nu\right)$$

Spectral intensity: per unit **Normal area**

Spectral emissive power: per unit **surface area**

$$E\left(r, \theta, \psi\right) dA = I\left(r, \theta, \psi\right)\ dA_p$$

$$dA_p = dA \cos\theta$$

$$E\left(r, \theta, \psi\right) = I\left(r, \theta, \psi\right) \cos\theta$$

(Lambert's law)

Unit of solid angle: steradians.

$$\Omega_{hemisphere} = \frac{A}{r^2} = \frac{\frac{4\pi r^2}{2}}{r^2} = 2\pi steradians$$

Emissivity and absorptivity

By definition:

Emissivity: energy emitted by a real surface compared to a blackbody

Absorptivity: energy absorbed by a real surface compared to a blackbody

Spectral emissivity

$$\epsilon\left(T,\theta,\psi,\nu\right) = \frac{E_{\nu}\left(T,\theta,\psi,\nu\right)}{E_{\nu b}\left(T,\theta,\psi,\nu\right)} = \frac{I_{\nu}\left(T,\theta,\psi,\nu\right)\cos\theta}{I_{\nu b}\left(T,\theta,\psi,\nu\right)\cos\theta} = \frac{I_{\nu}\left(T,\theta,\psi,\nu\right)}{I_{\nu b}\left(T,\theta,\psi,\nu\right)}$$

Spectral absorptivity

$$\alpha\left(T,\theta,\psi,\nu\right) = \frac{E_{\nu ABS}\left(T,\theta,\psi,\nu\right)}{E_{\nu bABS}\left(T,\theta,\psi,\nu\right)} = \frac{I_{\nu ABS}\left(T,\theta,\psi,\nu\right)\cos\theta}{I_{\nu bABS}\left(T,\theta,\psi,\nu\right)\cos\theta} = \frac{I_{\nu ABS}\left(T,\theta,\psi,\nu\right)}{I_{\nu bABS}\left(T,\theta,\psi,\nu\right)}$$

Radiative Heat Transfer Between Surfaces

Case 1: black surfaces

Case 2: non black surfaces

Case 2.1: grey surfaces

Case 2.2 grey diffuse surfaces

The simplest case

Between two black surfaces…

$$I_{b\lambda} = \frac{2\pi h c_0^2}{n^2\lambda^5\left[\exp\left(\frac{h\left(\frac{c_0}{n\lambda}\right)}{kT}\right) - 1\right]}\frac{1}{\cos\theta}$$

$$E\left(r,\theta,\psi\right) = I\left(r,\theta,\psi\right)\cos\theta$$

In total, since spectrum is not important, we use:

$$E_{b\lambda} = \frac{2\pi h c_0^2}{n^2\lambda^5\left[\exp\left(\frac{h\left(\frac{c_0}{n\lambda}\right)}{kT}\right) - 1\right]}$$

$$I_b = n^2\sigma T^4\,\frac{1}{\cos\theta}\ \text{(power per unit solid angle per unit normal area)}$$

We want to consider the geometric effects on radiation heat transfer...

<u>We need to consider, view factors. next video</u>

Principle: So we need to, in general, integrate the heat transfer between the infinitesimally small areas until we get full heat transfer say between A1 and A2...

1) How do we find heat transfer between dA1 and dA2?

$$I_{b\lambda} = \frac{power}{solidangle\ *\ normal\ area\ *\ wavelength}$$

$$I_b = \int_0^\infty I_{b\lambda} d\lambda = \frac{power}{solid\ angle *\ normal\ area}$$

Heat transfer

$$dq_{12} = I_b * (\text{normal area}) * (solidangle)$$

Normal area of dA2 is $dA_2\cos\theta_2$

Normal area of dA1 is $dA_1\cos\theta_1$

$$dq_{12} = I_b * (dA_1\cos\theta_1) * (solidangle)$$

$$solidangle = \frac{dA_{normal}}{r^2}$$

$$dq_{12} = I_b * (dA_1\cos\theta_1) * \left(\frac{dA_{normal}}{r^2}\right)$$

$$dq_{12} = I_b * (dA_1\cos\theta_1) * \left(\frac{dA_2\cos\theta_2}{r^2}\right)$$

$$q_{12} = \int_{A_1}\int_{A_2} I_b * (\cos\theta_1) * \left(\frac{\cos\theta_2}{r^2}\right) dA_2 dA_1$$

How to find I_b

Total energy output?

$$\frac{power}{area} \ * \ area = n^2 \sigma T^4 \ * \ dA_3 = E_b\left(T\right) \ * \ dA_3$$

Express total energy output in terms of intensity?

$$I_b\left(T\right) = \frac{power}{normalarea \ * \ solidangle}$$

$$power = I_b\left(T\right) \ * \ normalarea \ * \ solidangle$$

$$power = \int_{\Omega} I_b\left(T\right) \ * \ dA_3 \cos\theta_3 \ * \ \frac{dA_4 \cos\theta_4}{r^2}$$

$$d\Omega = \frac{dA_4 \cos\theta_4}{r^2} = \sin\theta_3 \ d\theta_3 \ d\psi_3$$

$$power = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} I_b\left(T\right) * \ dA_3 \cos\theta_3 * \sin\theta_3 \ d\theta_3 \ d\psi_3$$

$$E_b\left(T\right) * \ dA_3 = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} I_b\left(T\right) * \ dA_3 \cos\theta_3 * \sin\theta_3 \ d\theta_3 \ d\psi_3$$

$$E_b\left(T\right) = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} I_b\left(T\right) \cos\theta_3 \sin\theta_3 \ d\theta_3 \ d\psi_3$$

According to Kirchhoff's law, blackbody intensity is direction independent…

$$E_b\left(T\right) = I_b\left(T\right) \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \cos\theta_3 \sin\theta_3 \ d\theta_3 \ d\psi_3$$

$$E_b\left(T\right) = I_b\left(T\right) 2\pi \int_0^{\frac{\pi}{2}} \cos\theta_3 \sin\theta_3 \ d\theta_3$$

$$E_b\left(T\right) = I_b\left(T\right) 2\pi \left(\frac{1}{2}\right)$$

$$E_b\left(T\right) = I_b\left(T\right) \pi$$

$$I_b\left(T\right) = \frac{E_b\left(T\right)}{\pi} = \frac{n^2 \sigma T^4}{\pi}$$

But there could be some confusion, what is this???

$$E\left(r,\theta,\psi\right)=I\left(r,\theta,\psi\right)\cos\theta$$

$$E\left(r,\theta,\psi\right)=\frac{power}{area*\ solidangle}$$

$$E_b\left(T\right)=\frac{power}{area}$$

$$E\left(r,\theta,\psi\right)=\frac{power}{area*\ solidangle}$$

$$I\left(r,\theta,\psi\right)=\frac{power}{normalarea*\ solidangle}$$

$$\frac{power}{\text{solid angle}}=E\left(r,\theta,\psi\right)\ dA_3=I\left(r,\theta,\psi\right)\ dA_3\cos\theta_3$$

$$E\left(r,\theta,\psi\right)=I\left(r,\theta,\psi\right)\ \ \cos\theta_3$$

$$E\left(r,\theta,\psi\right)\rightarrow takesintoaccountsolidangle$$

$E_b\left(T\right)$ doesn't take into account solid angle

Substitute back into our heat transfer expression:

$$q_{12}=\int_{A_1}\int_{A_2} I_b\left(T\right)*\left(\cos\theta_1\right)*\left(\frac{\cos\theta_2}{r^2}\right)dA_2dA_1$$

$$I_b\left(T\right)=\frac{n^2\sigma T^4}{\pi}$$

$$q_{12}=\int_{A_1}\int_{A_2}\frac{n^2\sigma T_1^4}{\pi}*\left(\cos\theta_1\right)*\left(\frac{\cos\theta_2}{r^2}\right)dA_2dA_1$$

$$\left(\frac{\cos\theta_2}{r^2}\right)dA_2=d\Omega=\sin\theta_2\ d\theta_2\ d\psi_2$$

Let's say you wanted to account for the shape factor or view factor of the heat transfer…

We account for the shape using this thing called view factors.

Diffuse view factor, configuration factor, shape factor or angle factor

Diffuse assumption…

Applies to:

- Blackbodies
- Diffuse surfaces
- Gray or otherwise

$$q_{12} = \int_{A_1} \int_{A_2} \frac{n^2 \sigma T_1^4}{\pi} * (\cos\theta_1) * \left(\frac{\cos\theta_2}{r^2}\right) dA_2 dA_1$$

$$\text{view factor}_{1-2} = \frac{q_{12}}{\text{total heat transfer from 1}} = \frac{\int_{A_1} \int_{A_2} \frac{n^2\sigma T_1^4}{\pi} * (\cos\theta_1) * \left(\frac{\cos\theta_2}{r^2}\right) dA_2 dA_1}{E_b(T) * \ A_1}$$

$$\text{view factor}_{1-2} = \frac{q_{12}}{\text{total heat transfer from 1}} = \frac{\int_{A_1} \int_{A_2} \frac{n^2\sigma T_1^4}{\pi} * (\cos\theta_1) * \left(\frac{\cos\theta_2}{r^2}\right) dA_2 dA_1}{n^2 \sigma T_1^4 * \ A_1}$$

If we have constant T1 across the board,

$$\text{view factor}_{1-2} = \frac{q_{12}}{\text{total heat transfer from 1}} = \frac{\int_{A_1} \int_{A_2} \frac{1}{\pi} * (\cos\theta_1) * \left(\frac{\cos\theta_2}{r^2}\right) dA_2 dA_1}{A_1}$$

$$F_{1-2} = \frac{1}{A_1} \int_{A_1} \int_{A_2} (\cos\theta_1) * \left(\frac{\cos\theta_2}{\pi r^2}\right) dA_2 dA_1$$

$$F_{1-2} = \frac{1}{A_1} \int_{A_1} \int_{A_2} (\cos\theta_1) * \left(\frac{\cos\theta_2}{\pi S^2}\right) dA_2 dA_1$$

Let's say we want view factor from dA1 to dA2

$$dq_{12} = I_b * (dA_1 \cos\theta_1) * \left(\frac{dA_2 \cos\theta_2}{S^2}\right)$$

$$I_b = \frac{E_b(T)}{\pi}$$

Total energy leaving dA1 $= E_b(T) \ dA_1$

$$F_{dA_1-dA_2} = \frac{\frac{E_b(T)}{\pi} (dA_1 \cos\theta_1) \left(\frac{dA_2 \cos\theta_2}{S^2}\right)}{E_b(T) \ dA_1}$$

$$F_{dA_1-dA_2} = (\cos\theta_1) \left(\frac{dA_2 \cos\theta_2}{\pi S^2}\right)$$

Just for exercise, dA1 to A2

$$q_{dA_1-dA_2} = \int_{A_2} I_b * (dA_1 \cos\theta_1) * \left(\frac{dA_2 \cos\theta_2}{S^2}\right)$$

$$F_{dA_1-A_2} = \frac{\int_{A_2} I_b * (dA_1 \cos\theta_1) * \left(\frac{dA_2 \cos\theta_2}{S^2}\right)}{E_b\,(T)\;\; dA_1}$$

$$F_{dA_1-A_2} = \int_{A_2} (\cos\theta_1) \left(\frac{dA_2 \cos\theta_2}{\pi S^2}\right)$$

Just for the other exercise A1 to dA2

$$q_{dA_1-dA_2} = \int_{A_1} I_b * (dA_1 \cos\theta_1) * \left(\frac{dA_2 \cos\theta_2}{S^2}\right)$$

$$F_{A_1-dA_2} = \frac{\int_{A_1} \frac{E_b(T)}{\pi} * (dA_1 \cos\theta_1) * \left(\frac{dA_2 \cos\theta_2}{S^2}\right)}{E_b\,(T)\,A_1}$$

$$F_{A_1-dA_2} = \frac{\int_{A_1} (dA_1 \cos\theta_1) \left(\frac{dA_2 \cos\theta_2}{\pi S^2}\right)}{A_1}$$

After view factors...

<u>What then is the application or relevance?</u>

We want to study heat trf between real surfaces...

$$F_{12} = \frac{q_{1-2}}{E_b\,(T)\,A_1}$$

$$q_{1-2} = (E_b\,(T)\,A_1)\;\; F_{12}$$

So what is more representative of a real system?

- Maybe surfaces are gray (but diffuse)
- Maybe there is incoming radiation from outside...
- There is heat radiation from outside, and heat transfer from A2 to A1
- Conduction and convection (later on)
- There can be a participating medium (air or such)

For now we want to, based on heat trf from A1 to A2, A2 to A1 and heat trf from external source, develop an energy balance to determine net heat transfer from A1

$$q\left(r\right)\ dA_1 = E_b\left(T_1, r\right)\ dA_1 - H_0\left(r\right)\ dA_1 - \int_{A_2} E_b\left(T_2, r^{'}\right)\ dF_{dA_2-dA_1}\ dA_2$$

$$q\left(r\right) = E_b\left(T_1, r\right) - H_0\left(r\right) - \int_{A_2} E_b\left(T_2, r^{'}\right)\ dF_{dA_2-dA_1}\frac{dA_2}{dA_1}$$

Reciprocity relation

$$dA_1\ dF_{dA_1-dA_2} = dA_2\ dF_{dA_2-dA_1}$$

$$dF_{dA_1-dA_2} = dF_{dA_2-dA_1}\frac{dA_2}{dA_1}$$

Substituting back, we get,

$$q\left(r\right) = E_b\left(T_1, r\right) - H_0\left(r\right) - \int_{A_2} E_b\left(T_2, r^{'}\right)\ dF_{dA_1-dA_2}$$

The view factor is as such:

$$q\left(r\right) = E_b\left(T_1, r\right) - H_0\left(r\right) - \int_{A_2} E_b\left(T_2, r^{'}\right)\frac{\cos\theta_1\cos\theta_2}{\pi S^2}\ dA_2$$

One simple example of heat trf between black surfaces, before we go onto more complicated heat trf to gray surfaces...

Eg 5.1 in Michael F Modest Textbook.

What is the net heat transfer rate from each surface?

$$q\left(r\right) = E_b\left(T_1, r\right) - H_0\left(r\right) - \int_{A_2} E_b\left(T_2, r^{'}\right)\frac{\cos\theta_1\cos\theta_2}{\pi S^2}\ dA_2$$

$$q_1 = F_{1-2}\left(E_{b1} - E_{b2}\right) + F_{1-3}\left(E_{b1} - E_{b3}\right) + F_{1-4}\left(E_{b1} - E_{b4}\right)$$

$$q_1 = F_{1-2}\left(E_{b1} - E_{b2}\right) + F_{1-4}\left(E_{b1} - E_{b4}\right)$$

$F_{1-2} = F_{1-4}$ symmetry

$$E_{b4} = E_{b2}\left(sametemp\right)$$

$$q_1 = 2F_{1-2}\left(E_{b1} - E_{b2}\right) = q_3$$

Calculation of view factors

$$E_b = n^2 \sigma T^4$$

Derive expressions for heat transfer between gray surfaces (diffuse)

Recap on gray surfaces

Energy balance across dA_1

$$\varepsilon = \frac{\text{heat emitted by surface}}{\text{heat emitted by } black\ surface}$$

For <u>diffuse</u> gray surface (what surfaces are diffuse)

$$q_{net}\ dA_1 = \varepsilon E_b\left(T\right)\ dA_1 + \rho H\left(r\right)\ dA_1 - H\left(r\right)\ dA_1$$

$$\rho H\left(r\right)\ dA_1 - H\left(r\right)\ dA_1 = -\left(1-\rho\right) H\left(r\right)\ dA_1 = -\alpha H\left(r\right)\ dA_1$$

$$\alpha + \rho = 1$$

$$q_{net}\ dA_1 = \varepsilon E_b\left(T\right)\ dA_1 + \rho H\left(r\right)\ dA_1 - H\left(r\right)\ dA_1$$

$$\varepsilon E_b\left(T\right)\ dA_1 + \rho H\left(r\right)\ dA_1 = J\left(r\right)\ dA_1$$

Radiosity $J\left(r\right)$

This is the combined emitted and reflected radiation coming off a surface.

For a black surface, it is a special case of $J\left(r\right) = E_b\left(T\right)$

$$q_{net}\ dA_1 = \varepsilon E_b\left(T\right)\ dA_1 + \rho H\left(r\right)\ dA_1 - H\left(r\right)\ dA_1$$

Replace with radiosity

$$q_{net}\ dA_1 = J\left(r\right)\ dA_1 - H\left(r\right)\ dA_1$$

Next question is what is H(r) for gray surface?

$$H\left(r\right)\ dA_1 = H_0\left(r\right)\ dA_1 + \int_{A_2} J_2\left(r'\right)\ dA_2\ dF_{dA_2-dA_1}$$

$$H\left(r\right)=H_0\left(r\right)+\int_{A_2}J_2\left(r'\right)\ dA_2\frac{dF_{dA_2-dA_1}}{dA_1}$$

$$H\left(r\right)=H_0\left(r\right)+\int_{A_2}J_2\left(r'\right)\ dF_{dA_1-dA_2}$$

$$q_{net}\ dA_1=J\left(r\right)\ dA_1-\left(H_0\left(r\right)+\int_{A_2}J_2\left(r'\right)\ dF_{dA_1-dA_2}\right)\ dA_1$$

$$q_{net}=J_1\left(r\right)-\left(H_0\left(r\right)+\int_{A_2}J_2\left(r'\right)\ dF_{dA_1-dA_2}\right)$$

$$q_{net}=J_1\left(r\right)-H_0\left(r\right)-\int_{A_2}J_2\left(r'\right)\ dF_{dA_1-dA_2}$$

We have a bit of trouble still...

We might know $E_b\left(T\right)$ and that will give us, the intensity... but we don't know the radiosity necessarily.

$$q\left(r\right)=\varepsilon\left(r\right)\ E_b\left(r\right)-\alpha\left(r\right)\left(H_0\left(r\right)+\int_{A_2}J_2\left(r'\right)\ dF_{dA_1-dA_2}\right)$$

$$q_{net}\ dA_1=\varepsilon E_b\left(T\right)\ dA_1+\rho H\left(r\right)\ dA_1-H\left(r\right)\ dA_1$$

$$q_{net}=\varepsilon E_b\left(T\right)-\alpha H\left(r\right)=J\left(r\right)-H\left(r\right)$$

$$q_{net}=\varepsilon E_b\left(T\right)-\alpha H\left(r\right)$$

$$q_{net}=J\left(r\right)-H\left(r\right)$$

Make $H\left(r\right)$ the subject

$$H\left(r\right)=J\left(r\right)-q_{net}$$

$$q_{net}=\varepsilon E_b\left(T\right)-\alpha\left(J\left(r\right)-q_{net}\right)$$

$$q_{net}=\varepsilon E_b\left(T\right)-\alpha J\left(r\right)+\alpha q_{net}$$

$$q_{net}\left(1-\alpha\right)=\varepsilon E_b\left(T\right)-\alpha J\left(r\right)$$

Note $\alpha=\varepsilon$ (assumption of gray diffuse surface)

$$q_{net}\left(1-\varepsilon\right)=\varepsilon E_b\left(T\right)-\varepsilon J\left(r\right)$$

$$q_{net}=\frac{\varepsilon}{\left(1-\varepsilon\right)}\left(E_b\left(T\right)-J\left(r\right)\right)$$

We want to eliminate $J\left(r\right)$ from:

$$q\left(r\right)=\varepsilon\left(r\right)\ E_b\left(r\right)-\alpha\left(r\right)\left(H_0\left(r\right)+\int_{A_2}J_2\left(r'\right)\ dF_{dA_1-dA_2}\right)$$

$$q_{net}=\frac{\varepsilon}{\left(1-\varepsilon\right)}E_b\left(T\right)-\frac{\varepsilon}{\left(1-\varepsilon\right)}J\left(r\right)$$

$$\frac{\varepsilon}{\left(1-\varepsilon\right)}J\left(r\right)=\frac{\varepsilon}{\left(1-\varepsilon\right)}E_b\left(T\right)-q_{net}$$

$$J\left(r\right)=E_b\left(T\right)-\frac{1-\varepsilon}{\varepsilon}q_{net}$$

Now we are ready to substitute

$$J_2\left(r'\right)=E_{b2}\left(r'\right)-\frac{1-\varepsilon_2}{\varepsilon_2}q_{2net}\left(r^{'}\right)$$

$$q\left(r\right)=\varepsilon\left(r\right)\ E_b\left(r\right)-\alpha\left(r\right)\left(H_0\left(r\right)+\int_{A_2}\left\{E_{b2}\left(r'\right)-\frac{1-\varepsilon_2}{\varepsilon_2}q_{2net}\left(r^{'}\right)\right\}\ dF_{dA_1-dA_2}\right)$$

Okay so $\varepsilon=\alpha$

$$\frac{q\left(r\right)}{\varepsilon\left(r\right)}=E_b\left(r\right)-H_0\left(r\right)-\left(\int_{A_2}\left\{E_{b2}\left(r'\right)-\frac{1-\varepsilon_2}{\varepsilon_2}q_{2net}\left(r^{'}\right)\right\}\ dF_{dA_1-dA_2}\right)$$

$$-\int_{A_2}\left\{E_{b2}\left(r^{'}\right)-\frac{1-\varepsilon_2}{\varepsilon_2}q_{2net}\left(r^{'}\right)\right\}\ dF_{dA_1-dA_2}=\int_{A_2}\left\{\frac{1-\varepsilon_2}{\varepsilon_2}q_{2net}\left(r^{'}\right)\right\}\ dF_{dA_1-dA_2}-\int_{A_2}E_{b2}\left(r^{'}\right)\ dF_{d}$$

$$\frac{q\left(r\right)}{\varepsilon\left(r\right)}=E_b\left(r\right)-H_0\left(r\right)+\int_{A_2}\left\{\frac{1-\varepsilon_2}{\varepsilon_2}q_{2net}\left(r^{'}\right)\right\}\ dF_{dA_1-dA_2}-\int_{A_2}E_{b2}\left(r^{'}\right)\ dF_{dA_1-dA_2}$$

$$\frac{q\left(r\right)}{\varepsilon\left(r\right)}-\int_{A_2}\left\{\frac{1-\varepsilon_2}{\varepsilon_2}q_{2net}\left(r^{'}\right)\right\}\ dF_{dA_1-dA_2}+H_0\left(r\right)=E_b\left(r\right)-\int_{A_2}E_{b2}\left(r^{'}\right)\ dF_{dA_1-dA_2}$$

Electrical Network Analogy (5.4 of Michael F Modest Textbook)

$$q_{net}=\frac{\varepsilon}{\left(1-\varepsilon\right)}\left(E_b\left(T\right)-J\left(r\right)\right)$$

$$q_{net}A=\frac{\varepsilon A}{\left(1-\varepsilon\right)}\left(E_b\left(T\right)-J\left(r\right)\right)$$

$$q_{net}A=\frac{\left(E_b\left(T\right)-J\left(r\right)\right)}{\frac{\left(1-\varepsilon\right)}{\varepsilon A}}$$

"current" $q_{net}A$

Potential difference $= E_b\left(T\right)-J\left(r\right)$

Resistance $= \frac{(1-\varepsilon)}{\varepsilon A}$

$$F_{i-j}=F_{j-i}\frac{A_j}{A_i}$$

$$q_i=J_i-H_{i0}-\sum_{j=1}^{N}J_jF_{i-j}$$

$$q_1A=A\left(J_1-0-J_2\left(F_{1-2}\right)\right)$$

$$F_{1-2}=1$$

$$q_1A=\frac{J_1-J_2}{\frac{1}{A}}$$

$$J_iA_i=\sum_{j=1}^{N}F_{i-j}J_iA_i$$

$$J_i=\sum_{j=1}^{N}F_{i-j}J_i$$

$$q_i = \sum_{j=1}^{N} F_{i-j} J_i - H_{i0} - \sum_{j=1}^{N} J_j F_{i-j}$$

$$q_i = \sum_{j=1}^{N} F_{i-j} \left(J_i - J_j\right) - H_{i0}$$

$$q_1 A = \left(J_1 - J_2\right)\left(AF_{1-2}\right)$$

$$q_1 A = \frac{J_1 - J_2}{\frac{1}{(AF_{1-2})}} = \frac{J_1 - J_2}{\frac{1}{(A)}}$$

$$q_{net} A = \frac{\left(E_b\left(T\right) - J\left(r\right)\right)}{\frac{(1-\varepsilon)}{\varepsilon A}}$$

$$q_1 A_1 = \frac{E_{b1}\left(T\right) - E_{b2}\left(T\right)}{\frac{(1-\varepsilon_1)}{\varepsilon_1 A_1} + \frac{(1-\varepsilon_2)}{\varepsilon_2 A_2} + \frac{1}{(A_1 F_{1-2})}}$$

$$q_1 = \frac{E_{b1}\left(T\right) - E_{b2}\left(T\right)}{\frac{(1-\varepsilon_1)}{\varepsilon_1} + \frac{(1-\varepsilon_2)}{\varepsilon_2} + \frac{1}{(F_{1-2})}}$$

$$q_1 = \frac{E_{b1}\left(T\right) - E_{b2}\left(T\right)}{\frac{(1-\varepsilon_1)}{\varepsilon_1} + \frac{(1-\varepsilon_2)}{\varepsilon_2} + \frac{1}{(1)}}$$

https://www.khanacademy.org/science/electrical-engineering/ee-circuit-analysis-topic/ee-resistor-circuits/a/ee-delta-wye-resistor-networks

$$R_1 = \frac{\frac{1}{A_2 F_{2-1}} * \frac{1}{A_1 F_{1-3}}}{\frac{1}{A_1 F_{1-3}} + \frac{1}{A_2 F_{2-1}} + \frac{1}{A_3 F_{3-2}}}$$

$$R_2 = \frac{\frac{1}{A_2 F_{2-1}} * \frac{1}{A_3 F_{3-2}}}{\frac{1}{A_1 F_{1-3}} + \frac{1}{A_2 F_{2-1}} + \frac{1}{A_3 F_{3-2}}}$$

$$R_3 = \frac{\frac{1}{A_1 F_{1-3}} * \frac{1}{A_3 F_{3-2}}}{\frac{1}{A_1 F_{1-3}} + \frac{1}{A_2 F_{2-1}} + \frac{1}{A_3 F_{3-2}}}$$

$$q_3 = \frac{J_c - E_{b3}}{R_3 + \frac{1-\varepsilon_3}{A_3 \varepsilon_3}}$$

$$q_2 = \frac{J_c - E_{b2}}{R_2 + \frac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

$$q_1 = \frac{J_c - E_{b1}}{R_1 + \frac{1-\varepsilon_1}{A_1\varepsilon_1}}$$

$$q_2 = \frac{J_c - J_2}{R_2}$$

$$q_2 = \frac{J_2 - E_{b2}}{\frac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

Unknowns: J2, Jc, q1,q2,q3

Assume all resistances are known

5 eqns, 5 unknowns

$$q_2 = \frac{J_c - J_2}{R_2}$$

$$q_2 = \frac{J_2 - E_{b2}}{\frac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

$$q_2 = \frac{J_c - E_{b2}}{R_2 + \frac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

$$\frac{J_c - E_{b2}}{R_2 + \frac{1-\varepsilon_2}{A_2\varepsilon_2}} = \frac{J_2 - E_{b2}}{\frac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

$$\frac{J_2 - E_{b2}}{\frac{1-\varepsilon_2}{A_2\varepsilon_2}} = \frac{J_c - J_2}{R_2}$$

Make J2 the subject so as to eliminate it

$$J_2 - E_{b2} = \frac{1-\varepsilon_2}{R_2 A_2 \varepsilon_2}\left(J_c - J_2\right)$$

$$J_2 - E_{b2} = \left(J_c \frac{1-\varepsilon_2}{R_2 A_2 \varepsilon_2} - J_2 \frac{1-\varepsilon_2}{R_2 A_2 \varepsilon_2}\right)$$

$$J_2 \left(1 + \frac{1-\varepsilon_2}{R_2 A_2 \varepsilon_2}\right) = E_{b2} + J_c \frac{1-\varepsilon_2}{R_2 A_2 \varepsilon_2}$$

$$J_2 = \frac{E_{b2} + J_c \frac{1-\varepsilon_2}{R_2 A_2 \varepsilon_2}}{\left(1 + \frac{1-\varepsilon_2}{R_2 A_2 \varepsilon_2}\right)}$$

Substitute back to find Jc

$$\frac{J_c - E_{b2}}{R_2 + \frac{1-\varepsilon_2}{A_2\varepsilon_2}} = \frac{J_2 - E_{b2}}{\frac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

$$\frac{J_c - E_{b2}}{R_2 + \frac{1-\varepsilon_2}{A_2\varepsilon_2}} = \frac{E_{b2} + J_c\frac{1-\varepsilon_2}{R_2A_2\varepsilon_2} - E_{b2}\left(1 + \frac{1-\varepsilon_2}{R_2A_2\varepsilon_2}\right)}{\frac{1-\varepsilon_2}{A_2\varepsilon_2}\left(1 + \frac{1-\varepsilon_2}{R_2A_2\varepsilon_2}\right)}$$

Let's do this

$$K_1 = R_2 + \frac{1-\varepsilon_2}{A_2\varepsilon_2}$$

$$K_2 = \frac{1-\varepsilon_2}{R_2A_2\varepsilon_2}$$

$$K_3 = \left(1 + \frac{1-\varepsilon_2}{R_2A_2\varepsilon_2}\right)$$

$$K_4 = \frac{1-\varepsilon_2}{A_2\varepsilon_2}\left(1 + \frac{1-\varepsilon_2}{R_2A_2\varepsilon_2}\right)$$

And we look at a neater equation:

$$\frac{J_c - E_{b2}}{K_1} = \frac{E_{b2} + J_cK_2 - E_{b2}K_3}{K_4}$$

$$J_c - E_{b2} = \frac{E_{b2} + J_cK_2 - E_{b2}K_3}{\frac{K_4}{K_1}}$$

$$J_c - E_{b2} = \frac{K_1}{K_4}\left(E_{b2} + J_cK_2 - E_{b2}K_3\right)$$

$$J_c - E_{b2} = \left(\frac{K_1}{K_4}E_{b2} + J_c\frac{K_1}{K_4}K_2 - E_{b2}\frac{K_1}{K_4}K_3\right)$$

$$J_c - J_c\frac{K_1}{K_4}K_2 = \left(\frac{K_1}{K_4}E_{b2} + E_{b2} - E_{b2}\frac{K_1}{K_4}K_3\right)$$

$$J_c\left(1 - \frac{K_1}{K_4}K_2\right) = E_{b2}\left(\frac{K_1}{K_4} + 1 - \frac{K_1}{K_4}K_3\right)$$

$$J_c = \frac{E_{b2}\left(\frac{K_1}{K_4} + 1 - \frac{K_1}{K_4}K_3\right)}{\left(1 - \frac{K_1}{K_4}K_2\right)}$$

Where

$$K_1 = R_2 + \frac{1-\varepsilon_2}{A_2\varepsilon_2}$$

$$K_2 = \frac{1-\varepsilon_2}{R_2 A_2 \varepsilon_2}$$

$$K_3 = \left(1 + \frac{1-\varepsilon_2}{R_2 A_2 \varepsilon_2}\right)$$

$$K_4 = \frac{1-\varepsilon_2}{A_2\varepsilon_2}\left(1 + \frac{1-\varepsilon_2}{R_2 A_2 \varepsilon_2}\right)$$

With Jc found, we can substitute back into q1 q2 and q3

$$q_3 = \frac{J_c - E_{b3}}{R_3 + \frac{1-\varepsilon_3}{A_3\varepsilon_3}}$$

$$q_2 = \frac{J_c - E_{b2}}{R_2 + \frac{1-\varepsilon_2}{A_2\varepsilon_2}}$$

$$q_1 = \frac{J_c - E_{b1}}{R_1 + \frac{1-\varepsilon_1}{A_1\varepsilon_1}}$$

Radiation Transfer Equation (RTE)

How do we deal with participating mediums?

So what happens when radiation goes through a participating medium?

Attenuation (weakening effect)

- Outscattering
- Absorption

Augmentation (strengthening effect)

- Inscattering
- Emission

What quantity is best used to describe the radiation?

Direction, solid angle and frequency are important to radiation...

$$I_\lambda$$

We use intensity in general to describe radiation transport. All the direction (we take energy per unit **normal** area), solid angle and frequency etc etc is taken into account when we talk about intensity.

We can do a quick energy balance (absorption)

$$energy\ accumulated = (I|_s - I|_{s+ds})\ \ dA$$

$$ds * dA * \frac{\text{rate of energy absorbed}}{\text{volume}} = (I|_s - I|_{s+ds})dA$$

$$\frac{\text{rate of energy absorbed}}{\text{volume}} = \frac{(I|_s - I|_{s+ds})}{ds}$$

$$\frac{(\text{rate of energy absorbed})}{volume} = -\frac{dI}{ds}$$

$$\frac{dI}{ds} = -\frac{(\text{rate of energy absorbed})}{volume}$$

We can assume the rate of energy absorbed is proportional to $I|_s$

$$\frac{dI_\lambda}{ds} = -I_\lambda|_s * \kappa_\lambda$$

$$\frac{dI_\lambda}{ds}|_{absorption} = -I_\lambda \kappa_\lambda$$

(this is attenuation due to absorption)

We can think of outscattering in a similar

$$\frac{dI_\lambda}{ds}|_{outscattering} = -I_\lambda \sigma_\lambda$$

It is convenient to combine both...

$$\frac{dI_\lambda}{ds}|_{extinction} = \frac{dI_\lambda}{ds}|_{absorption} + \frac{dI_\lambda}{ds}|_{outscattering} = -I_\lambda \sigma_\lambda + \kappa_\lambda I_\lambda$$

$$\sigma_\lambda + \kappa_\lambda = \beta_\lambda \ \text{ (extinction coefficient)}$$

$$\frac{dI_\lambda}{ds}|_{attenuation} \ = -\beta_\lambda I_\lambda$$

$$\frac{dI_\lambda}{I_\lambda} = -\beta_\lambda \; ds$$

$$\ln \frac{I_\lambda \left(s^* = s\right)}{I_\lambda \left(s^* = 0\right)} = -\int_0^s \beta_\lambda \; ds^*$$

$$\frac{I_\lambda \left(s^* = s\right)}{I_\lambda \left(s^* = 0\right)} = \exp\left(-\int_0^s \beta_\lambda \; ds^*\right)$$

$$I_\lambda \left(s^* = s\right) = I_\lambda \left(s^* = 0\right) \; exp\left(-\int_0^s \beta_\lambda \; ds^*\right)$$

If β_λ was constant

$$I_\lambda \left(s^* = s\right) = I_\lambda \left(s^* = 0\right) \; exp\left(-\beta_\lambda s\right)$$

This quantity is going to pop up very often...

$$\tau_\lambda = \int_0^s \beta_\lambda \; ds^*$$

This is known as optical thickness τ_λ

So optical thickness is what really matters when it comes to attenuation of radiation, not the actual length.

$$I_\lambda \left(s^* = s\right) = I_\lambda \left(s^* = 0\right) \; exp\left(-\tau_\lambda\right)$$

We don't quite like to use ds as much

$$\frac{dI_\lambda}{ds}|_{attenuation} \; = -\beta_\lambda I_\lambda$$

We prefer to use $d\tau_\lambda$

We want to next describe the augmentation bit...

Emission bit...

And then the inscattering bit

We need to tackle the emission bit first...

$$\frac{dI_\lambda}{ds}|_{emission} = j_\lambda$$

$$j_\lambda = \kappa_\lambda I_{b\lambda}$$

$$\frac{dI_\lambda}{ds}|_{emission} = \kappa_\lambda I_{b\lambda}$$

$$I_{b\lambda} = \frac{E_{b\lambda}}{\pi} = \frac{n^2\sigma T^4}{\pi}$$

$$\kappa_\lambda \rightarrow absorptioncoefficient$$

Let's consider inscattering

For inscattering we can start by doing an energy balance

$$Intensity = \frac{power}{normal\ area *\ solid\ angle *\ wavelength}$$

Power contributed to by $I_\lambda|_s$

$$I_\lambda|_s \text{ dA d}\Omega d\lambda$$

Power contributed to by $I_\lambda|_{s+ds}$

$$I_\lambda|_{s+ds} \text{ dA d}\Omega d\lambda$$

Power contributed to the system by the third ray?

$$I_\lambda\left(j\right)$$

The energy coming in is:

$$I_\lambda\left(j\right) d\lambda d\Omega_j\ dA\cos\theta$$

How much energy is scattered?

We apply:

$$\frac{dI_\lambda}{ds}|_{outscattering} = -I_\lambda\sigma_\lambda$$

$$\frac{dI_\lambda}{ds}|_{outscattering} *\ d\lambda d\Omega_j\ dA\cos\theta = -I_\lambda\sigma_\lambda *\ d\lambda d\Omega_j\ dA\cos\theta$$

$$\frac{dI_\lambda}{ds}|_{outscattering} *\ d\lambda d\Omega_j\ dA\cos\theta = -I_\lambda\left(j\right)\sigma_\lambda *\ d\lambda d\Omega_j\ dA\cos\theta$$

$$dI_\lambda|_{outscattering} * \ d\lambda d\Omega_j \ dA\cos\theta = -I_\lambda\left(j\right)\sigma_\lambda * \ d\lambda d\Omega_j \ dA\cos\theta \ ds$$

$$dI_\lambda|_{outscattering} * \ d\lambda d\Omega_j \ dA\cos\theta = -I_\lambda\left(j\right)\sigma_\lambda * \ d\lambda d\Omega_j \ dA\cos\theta\frac{ds}{\cos\theta}$$

$$dI_\lambda|_{outscattering} * \ d\lambda d\Omega_j \ dA\cos\theta = -I_\lambda\left(j\right)\sigma_\lambda d\lambda d\Omega_j \ \text{dA ds}$$

How much energy is scattered into direction i?

$$\left(I_\lambda\hat{s}_j\right)\sigma_\lambda d\lambda d\Omega_j \ \text{dA ds} * \Phi\left(\hat{s}_j,\hat{s}_i\right) * \frac{d\Omega_i}{4\pi}$$

Our ultimate reason for going thru all that calculation: find an expression for the inscattering term

To find the total energy being inscattered, we need to integrate across all $d\Omega_j$

$$\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)\sigma_\lambda d\lambda d\Omega_j dAds * \Phi\left(\hat{s}_j,\hat{s}_i\right) * \frac{d\Omega_i}{4\pi}$$

$$d\lambda dAds\frac{d\Omega_i}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)\sigma_\lambda \ \Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_j$$

We can assume scattering coefficient is constant with respect to direction

$$d\lambda dAdsd\Omega_i\frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right) \ \Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_j$$

If inscattering were considered,

$$I_\lambda|_{s+ds} \ \text{dA d}\Omega_i d\lambda - I_\lambda|_s \ \text{dA d}\Omega_i d\lambda = d\lambda dAdsd\Omega_i\frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right) \ \Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_j$$

$$I_\lambda|_{s+ds} - I_\lambda|_s = \ ds \ \frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right) \ \Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_j$$

$$\frac{I_\lambda|_{s+ds} - I_\lambda|_s}{ds} = \ \frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right) \ \Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_j$$

$$\frac{dI}{ds}|_{inscattering} = \frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right) \ \Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_j$$

A first form of our RTE

$$\frac{dI}{ds} = emission + inscatteringterm - outscatteringterm - absorptionterm$$

$$\frac{dI}{ds} = \kappa_\lambda I_{b\lambda} + \frac{\sigma_\lambda}{4\pi} \int_{\Omega_j = 4\pi} I_\lambda \left(\hat{s}_j\right) \ \Phi \left(\hat{s}_j, \hat{s}_i\right) d\Omega_j - \left(\sigma_\lambda + \kappa_\lambda\right) I_\lambda$$

$$\frac{dI}{ds} = \kappa_\lambda I_{b\lambda} + \frac{\sigma_\lambda}{4\pi} \int_{\Omega_j = 4\pi} I_\lambda \left(\hat{s}_j\right) \ \Phi \left(\hat{s}_j, \hat{s}_i\right) d\Omega_j - \beta_\lambda I_\lambda$$

Solve Problem with the RTE

To solve analytically, we need to make MANY simplifications, but we want to solve it, at least computationally.

Numerical methods spherical harmonics, or discrete ordinate methods, or even simplified spherical harmonics

We want to start with analytical solutions for extremely simple cases.

We start with:

$$\frac{dI}{ds} = \kappa_\lambda I_{b\lambda} + \frac{\sigma_\lambda}{4\pi} \int_{\Omega_j = 4\pi} I_\lambda \left(\hat{s}_j\right) \ \Phi \left(\hat{s}_j, \hat{s}_i\right) d\Omega_j - \beta_\lambda I_\lambda$$

We want to find:

heat transfer rate,

- Heat flux
- Divergence of heat flux

  We want to simplify the RTE
- Get rid of time dependence
- Include optical thickness τ_λ

  Let's find time dependence

$$\frac{dI}{ds}$$

$$dI = \frac{\partial I}{\partial t} dt + \frac{\partial I}{\partial s} \ ds$$

Divide throughout by ds

$$\frac{dI}{ds} = \frac{\partial I}{\partial t}\frac{dt}{ds} + \frac{\partial I}{\partial s}$$

What is $\frac{dt}{ds}$?

$$\frac{dI}{ds} = \frac{\partial I}{\partial t}\frac{dt}{ds} + \frac{\partial I}{\partial s}$$

$\frac{ds}{dt}$ is the speed of the radiation

$$\frac{ds}{dt} = c\,(\text{in that medium}) = \frac{c_0}{n}$$

$$\frac{dI}{ds} = \frac{\partial I}{\partial t}\frac{n}{c_0} + \frac{\partial I}{\partial s}$$

$$\frac{dI}{ds} = \frac{\partial I}{\partial t}\frac{1}{c} + \frac{\partial I}{\partial s}$$

$$c_0 = 2.998 * 10^8 \frac{m}{s}$$

$$\frac{dI}{ds} = \frac{\partial I}{\partial t}\frac{1}{c} + \frac{\partial I}{\partial s}$$

So usually, $\frac{\partial I}{\partial t}$ « c, except for a few special cases, eg. short pulsed lasers with ps or fs transience

So for the the most part, unless we are dealing with extremely fast laser pulses or other special cases,

We can ignore $\frac{\partial I}{\partial t}\frac{1}{c}$

$$\frac{dI}{ds} = \frac{\partial I}{\partial s}$$

Without time dependence:

$$\frac{\partial I}{\partial s} = \kappa_\lambda I_{b\lambda} + \frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)\ \Phi\left(\hat{s}_j, \hat{s}_i\right) d\Omega_j - \beta_\lambda I_\lambda$$

For optical thickness:

$$\tau_\lambda = \int_0^s \beta_\lambda\ ds^*$$

$$\frac{\partial I}{\partial s} = \frac{\partial I}{\partial \tau_\lambda} * \frac{\partial \tau_\lambda}{\partial s}$$

$$\frac{\partial \tau_\lambda}{\partial s} = \beta_\lambda$$

$$\frac{\partial I\left(\tau_\lambda\right)}{\partial \tau_\lambda}\beta_\lambda = \kappa_\lambda I_{b\lambda} + \frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\tau_\lambda\right)\ \Phi\left(\hat{s}_j,\hat{s}_i\right) d\Omega_j - \beta_\lambda I_\lambda$$

$$\frac{\partial I\left(\tau_\lambda\right)}{\partial \tau_\lambda} = \frac{\kappa_\lambda}{\beta_\lambda}\ I_{b\lambda} + \frac{\frac{\sigma_\lambda}{\beta_\lambda}}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\tau_\lambda\right)\ \Phi\left(\hat{s}_j,\hat{s}_i\right) d\Omega_j - I_\lambda$$

$$\frac{\sigma_\lambda}{\beta_\lambda} = \frac{\sigma_\lambda}{\sigma_\lambda+\kappa_\lambda} = \omega_\lambda$$

albedo (how much of the beam attenuation is due to scattering as compared to absorption)

$$\frac{\partial I\left(\tau_\lambda\right)}{\partial \tau_\lambda} = \frac{\kappa_\lambda}{\beta_\lambda}\ I_{b\lambda} + \frac{\omega_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\tau_\lambda\right)\ \Phi\left(\hat{s}_j,\hat{s}_i\right) d\Omega_j - I_\lambda$$

$$\frac{\kappa_\lambda}{\beta_\lambda} = \frac{\kappa_\lambda}{\sigma_\lambda+\kappa_\lambda} = \frac{\kappa_\lambda+\sigma_\lambda-\sigma_\lambda}{\sigma_\lambda+\kappa_\lambda} = 1-\omega_\lambda$$

$$\frac{\partial I\left(\tau_\lambda\right)}{\partial \tau_\lambda} = \left(1-\omega_\lambda\right) I_{b\lambda} + \frac{\omega_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\tau_\lambda\right)\ \Phi\left(\hat{s}_j,\hat{s}_i\right) d\Omega_j - I_\lambda$$

$$S\left(\tau_\lambda,\hat{s}\right) = \left(1-\omega_\lambda\right) I_{b\lambda} + \frac{\omega_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\tau_\lambda\right)\ \Phi\left(\hat{s}_j,\hat{s}_i\right) d\Omega_j$$

$$\frac{\partial I\left(\tau_\lambda\right)}{\partial \tau_\lambda} = S\left(\tau_\lambda,\hat{s}\right) - I_\lambda$$

To solve this:

$$\frac{dI_\lambda}{ds} = \hat{s}\cdot\nabla I_\lambda$$

$$\begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial}{\partial x} I_\lambda \\ \frac{\partial}{\partial y} I_\lambda \\ \frac{\partial}{\partial z} I_\lambda \end{pmatrix}$$

$$s_x = \frac{\partial x}{\partial s}$$

$$\frac{dI_\lambda}{ds} = \frac{1}{ds}\left(\frac{\partial}{\partial x} I_\lambda dx + \frac{\partial}{\partial y}\ I_\lambda dy + \frac{\partial}{\partial z} I_\lambda\ dz\right)$$

$$\frac{dI_\lambda}{ds} = \left(\frac{\partial}{\partial x} I_\lambda \frac{dx}{ds} + \frac{\partial}{\partial y}\ I_\lambda \frac{dy}{ds} + \frac{\partial}{\partial z} I_\lambda \frac{dz}{ds}\right)$$

$$\frac{dI_\lambda}{ds} = \left(s_x \frac{\partial}{\partial x} I_\lambda + s_y \frac{\partial}{\partial y}\ I_\lambda + s_z \frac{\partial}{\partial z} I_\lambda\right)$$

Solving the RTE

$$\frac{\partial I\left(\tau_\lambda\right)}{\partial \tau_\lambda} = S\left(\tau_\lambda, \hat{s}\right) - I_\lambda$$

$$\frac{d}{d\tau_\lambda} I_\lambda\left(\tau_\lambda\right) + I_\lambda\left(\tau_\lambda\right) = S\left(\tau_\lambda,\ \hat{s}\right)$$

$$\frac{d}{d\tau_\lambda}\left(I_\lambda e^{\tau_\lambda}\right) = S\left(\tau_\lambda,\ \hat{s}\right) e^{\tau_\lambda}$$

$$\int_{I_\lambda e^{\tau_\lambda}|_{\tau_\lambda=0}}^{I_\lambda e^{\tau_\lambda}|_{\tau_\lambda=\tau_\lambda}} \left(dI_\lambda e^{\tau_\lambda}\right) = \int_{\tau_\lambda=0}^{\tau_\lambda=\tau_\lambda} S(\tau_\lambda, \hat{s}) e^{\tau_\lambda} d\tau_\lambda$$

$$\left[I_\lambda e^{\tau_\lambda}\right]|_{\tau_\lambda=\tau_\lambda} - \left[I_\lambda e^{\tau_\lambda}\right]|_{\tau_\lambda=0} = \int_{\tau_\lambda=0}^{\tau_\lambda=\tau_\lambda} S\left(\tau_\lambda,\ \hat{s}\right) e^{\tau_\lambda} d\tau_\lambda$$

$$I_\lambda\left(\tau_\lambda\right) e^{\tau_\lambda} - I_\lambda\left(\tau_\lambda = 0\right) = \int_{\tau_\lambda=0}^{\tau_\lambda=\tau_\lambda} S\left(\tau_\lambda,\ \hat{s}\right) e^{\tau_\lambda} d\tau_\lambda$$

$$I_\lambda\left(\tau_\lambda\right) e^{\tau_\lambda} = I_\lambda\left(\tau_\lambda = 0\right) + \int_{\tau_\lambda=0}^{\tau_\lambda=\tau_\lambda} S\left(\tau_\lambda,\ \hat{s}\right) e^{\tau_\lambda} d\tau_\lambda$$

$$I_\lambda\left(\tau_\lambda\right) = I_\lambda\left(\tau_\lambda = 0\right) e^{-\tau_\lambda} + e^{-\tau_\lambda} \int_{\tau_\lambda=0}^{\tau_\lambda=\tau_\lambda} S\left(\tau_\lambda,\ \hat{s}\right) e^{\tau_\lambda} d\tau_\lambda$$

$$I_\lambda\left(\tau_\lambda\right) = I_\lambda\left(\tau_\lambda = 0\right) e^{-\tau_\lambda} + e^{-\tau_\lambda} \int_{\tau_\lambda^{'}=0}^{\tau_\lambda^{'}=\tau_\lambda} S\left(\tau_\lambda^{'},\ \hat{s}\right) e^{\tau_\lambda^{'}} d\tau_\lambda^{'}$$

$$I_\lambda\left(\tau_\lambda^{'} = \tau_\lambda\right) = I_\lambda\left(\tau_\lambda^{'} = 0\right) e^{-\tau_\lambda} + \int_{\tau_\lambda^{'}=0}^{\tau_\lambda^{'}=\tau_\lambda} S\left(\tau_\lambda^{'},\ \hat{s}\right) e^{-\left(\tau_\lambda - \tau_\lambda^{'}\right)} d\tau_\lambda^{'}$$

We want to solve for important quantities:

What are we interested in when we solve the RTE?

- Heat flux (wall)
- Divergence of heat flux (within the fluid)

How to calc heat flux (in principle)

Soln to RTE:

$$I_\lambda\left(\tau_\lambda^{'}=\tau_\lambda\right)=I_\lambda\left(\tau_\lambda^{'}=0\right)e^{-\tau_\lambda}+\int_{\tau_\lambda^{'}=0}^{\tau_\lambda^{'}=\tau_\lambda}S\left(\tau_\lambda^{'},\ \hat{s}\right)e^{-\left(\tau_\lambda-\tau_\lambda^{'}\right)}d\tau_\lambda^{'}$$

How do we relate intensity to heat flux??

$$q^{''}=\frac{power}{area}$$

$$I=\frac{power}{solidangle\cdot normalarea}$$

If area $= dA$

Normal area$=\ dA\cos\theta$

$$power=q^{''}*\ dA=I*\ solid\ angle*\ normal\ area$$

$$power=q^{''}*\ dA=I*\ solid\ angle*\ normal\ area$$

$$I*\ solidangle*\ normalarea=\int_{hemisphere}\mathrm{I}\ \mathrm{d}\Omega\ dA\cos\theta$$

$$I*\ solid\ angle*\ normal\ area=\int_{hemisphere}I\ \ \cos\theta d\Omega\ dA$$

$$q^{''}=\int_{hemisphere}I\ \ \cos\theta d\Omega$$

$$q^{''}=\int_{2\pi}I\left(\theta,\phi\right)\cos\theta d\Omega$$

$$d\Omega=\sin\theta^{'}d\theta^{\prime}d\phi$$

Question is $\theta^{'}=\theta$?

$$\theta=polarangle$$

If $\theta^{'}=polarangle$

$$\theta=polarangle\left(cftochapter1.5\right)$$

$$q^{''} = \int_{2\pi} I\left(\theta,\phi\right)\cos\theta d\Omega \ = \int_0^{2\pi}\int_0^{\frac{\pi}{2}} I \ \left(\theta,\phi\right)\cos\theta\sin\theta d\theta d\phi$$

Key points of comparison

1) $\hat{n}\cdot\hat{s} = \cos\theta$
2) They integrate over 4pi rather than 2pi
3) They included spectral dependence

$$q_\lambda^{''} = \int_{4\pi} I_\lambda\left(\theta,\phi\right)\cos\theta d\Omega \ = \int_0^{2\pi}\int_0^{\pi} I_\lambda\left(\theta,\phi\right)\cos\theta\sin\theta d\theta d\phi$$

$$I_\lambda\left(\tau_\lambda^{'} = \tau_\lambda\right) = I_\lambda\left(\tau_\lambda^{'} = 0\right)e^{-\tau_\lambda} + \int_{\tau_\lambda^{'}=0}^{\tau_\lambda^{'}=\tau_\lambda} S\left(\tau_\lambda^{'},\ \hat{s}\right)e^{-\left(\tau_\lambda-\tau_\lambda^{'}\right)}d\tau_\lambda^{'}$$

$$q_\lambda^{''}\left(scalar\right) = \int_{4\pi} I_\lambda\left(\theta,\phi\right)\cos\theta d\Omega \ = \int_0^{2\pi}\int_0^{\pi}\left(I_\lambda\left(\tau_\lambda^{'} = 0\right)e^{-\tau_\lambda} + \int_{\tau_\lambda^{'}=0}^{\tau_\lambda^{'}=\tau_\lambda} S\left(\tau_\lambda^{'},\ \hat{s}\right)e^{-\left(\tau_\lambda-\tau_\lambda^{'}\right)}d\tau_\lambda^{'}\right)\cos\theta$$

Now to calc divergence of heat flux

This is volumetric heat output per unit volume

$$\nabla\cdot q_\lambda^{''}\left(vector\right) = \nabla\cdot\int_{4\pi} I_\lambda\left(\theta,\phi\right)\cos\theta d\Omega \ = \nabla\cdot\int_0^{2\pi}\int_0^{\pi} I_\lambda\left(\theta,\phi\right)\cos\theta\sin\theta d\theta d\phi$$

$$\nabla\cdot\overrightarrow{q_\lambda^{''}} = \nabla\cdot\int_{4\pi} I_\lambda\left(\hat{s}\right)\ \hat{s}\ d\Omega$$

$$\int_0^{2\pi}\int_0^{\pi} I_\lambda\left(\theta,\phi\right)\hat{n}\cdot\hat{s}\sin\theta d\theta d\phi = \hat{n}\cdot\int_0^{2\pi}\int_0^{\pi} I_\lambda\left(\theta,\phi\right)\hat{s}\sin\theta d\theta d\phi$$

So we find that

$$\overrightarrow{q_\lambda^{''}} = \int_0^{2\pi}\int_0^{\pi} I_\lambda\left(\theta,\phi\right)\hat{s}\sin\theta d\theta d\phi$$

substitute

$$\left(I_\lambda\left(\tau_\lambda^{'}=0\right)e^{-\tau_\lambda}+\int_{\tau_\lambda^{'}=0}^{\tau_\lambda^{'}=\tau_\lambda}S\left(\tau_\lambda^{'},\ \hat{s}\right)e^{-\left(\tau_\lambda-\tau_\lambda^{'}\right)}d\tau_\lambda^{'}\right)$$

$$\overrightarrow{q_\lambda^{''}}=\int_0^{2\pi}\int_0^{\pi}\left(I_\lambda\left(\tau_\lambda^{'}=0\right)e^{-\tau_\lambda}+\int_{\tau_\lambda^{'}=0}^{\tau_\lambda^{'}=\tau_\lambda}S\left(\tau_\lambda^{'},\ \hat{s}\right)e^{-\left(\tau_\lambda-\tau_\lambda^{'}\right)}d\tau_\lambda^{'}\right)\hat{s}\sin\theta d\theta d\phi$$

But if we want to solve analytically, we may want to find simpler expressions for $\nabla\cdot q_\lambda^{''}$

We start by doing this

To get the divergence expression:

Integrate of 4pi

$$\int_{4\pi}\hat{s}\cdot\nabla I_\lambda\ d\Omega=\nabla\cdot\int_{4\pi}I_\lambda\hat{s}\ d\Omega$$

so this will be divergence of heat flux if:

$$q^{''}\left(vector\right)=\int_{4\pi}I_\lambda\hat{s}\ d\Omega$$

But why is the heat flux:

$$q^{''}\left(scalar\right)=\int_{2\pi}I\left(\theta,\phi\right)\cos\theta d\Omega\ =\int_{2\pi}I\ \ \left(\theta,\phi\right)\hat{n}\cdot\hat{s}d\Omega$$

So which is correct?

(both area correct if you define q” correctly)

$$q^{''}=\overrightarrow{q_\lambda^{''}}\cdot\hat{n}=\int_{4\pi}I_\lambda\left(\hat{s}\right)\ \hat{n}\cdot\hat{s}\ d\Omega$$

$$\overrightarrow{q_\lambda^{''}}\cdot\hat{n}=\int_{4\pi}I_\lambda\left(\hat{s}\right)\ \hat{n}\cdot\hat{s}\ d\Omega$$

$$\overrightarrow{q_\lambda^{''}}\cdot\hat{n}=\hat{n}\cdot\int_{4\pi}I_\lambda\left(\hat{s}\right)\ \hat{s}\ d\Omega$$

Commutative law works here!

$$\hat{n}\cdot\overrightarrow{q_\lambda^{''}}=\hat{n}\cdot\int_{4\pi}I_\lambda\left(\hat{s}\right)\ \hat{s}\ d\Omega$$

Comparing, we find that the heat flux vector is:

$$\overrightarrow{q''_\lambda} = \int_{4\pi} I_\lambda(\hat{s}) \;\hat{s}\; d\Omega$$

We need to differentiate heat flux from heat flux vector

So we have:

$$\hat{n} \cdot \overrightarrow{q''_\lambda} = \hat{n} \cdot \int_{4\pi} I_\lambda(\hat{s}) \;\hat{s}\; d\Omega$$

And comparing the vectors, we find that

$$\overrightarrow{q''_\lambda} = \int_{4\pi} I_\lambda(\hat{s}) \;\hat{s}\; d\Omega$$

<u>So back to our original problem:</u>

We wanted to find divergence of heat flux

$$\nabla \cdot \overrightarrow{q''_\lambda} = \nabla \cdot \int_{4\pi} I_\lambda \hat{s}\; d\Omega = \int_{4\pi} \hat{s} \cdot \nabla I_\lambda \; d\Omega$$

To get the divergence expression:

Integrate of 4pi

$$\nabla \cdot \overrightarrow{q''_\lambda} = \int_{4\pi} \hat{s} \cdot \nabla I_\lambda \; d\Omega = \nabla \cdot \int_{4\pi} I_\lambda \hat{s}\; d\Omega$$

So we need to integrate the RTE expression on the right hand side...

$$\frac{dI(\hat{s}_i)}{ds} = \hat{s} \cdot \nabla I_\lambda = \kappa_\lambda I_{b\lambda} + \frac{\sigma_\lambda}{4\pi} \int_{\Omega_j = 4\pi} I_\lambda(\hat{s}_j) \;\; \Phi(\hat{s}_j, \hat{s}_i)\, d\Omega_j - \beta_\lambda I_\lambda$$

So integrate RHS with respect to solid angle

$$\int_{4\pi} \left(\kappa_\lambda I_{b\lambda} + \frac{\sigma_\lambda}{4\pi} \int_{\Omega_j = 4\pi} I_\lambda(\hat{s}_j) \;\; \Phi(\hat{s}_j, \hat{s}_i)\, d\Omega_j - \beta_\lambda I_\lambda \right) d\Omega_i$$

Stay tuned for part ii

Term 1:

$$\int_{4\pi} (\kappa_\lambda I_{b\lambda})\, d\Omega_i = 4\pi \kappa_\lambda I_{b\lambda}$$

Term 2:

$$\int_{4\pi}\left(\frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)\ \Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_j\right)d\Omega_i=\frac{\sigma_\lambda}{4\pi}\int_{4\pi}\left(\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)\ \Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_j\right)d\Omega_i$$

$$\frac{\sigma_\lambda}{4\pi}\int_{4\pi}\left(\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)\ \Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_j\right)d\Omega_i=\frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)\int_{4\pi}\Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_i\ \ \mathrm{d}\Omega_j$$

Term 3:

$$-\int_{4\pi}\left(\beta_\lambda I_\lambda\left(\hat{s}_i\right)\right)d\Omega_i$$

$$4\pi\kappa_\lambda I_{b\lambda}+\frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)\int_{4\pi}\Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_i\ \ \mathrm{d}\Omega_j-\int_{4\pi}\left(\beta_\lambda I_\lambda\left(\hat{s}_i\right)\right)d\Omega_i$$

Consider:

$$\int_{4\pi}\Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_i\equiv 4\pi$$

$$\frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)4\pi\ d\Omega_j=\sigma_\lambda\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)\ d\Omega_j$$

$$I_\lambda=\frac{power}{normalarea\cdot solidangle\cdot wavelength}$$

$$G_\lambda=\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)\ d\Omega_j=\frac{power}{normalarea\cdot wavelength}$$

(incident radiation function)

$$\frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)4\pi\ d\Omega_j=\sigma_\lambda\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)\ d\Omega_j=\sigma_\lambda\ G_\lambda$$

$$\sigma_\lambda=\sigma_{s\lambda}$$

Let's take a look back at our RHS

$$4\pi\kappa_\lambda I_{b\lambda} + \frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\hat{s}_j\right)\int_{4\pi}\Phi\left(\hat{s}_j,\hat{s}_i\right)d\Omega_i \ \ \mathrm{d}\Omega_j - \int_{4\pi}\left(\beta_\lambda I_\lambda\left(\hat{s}_i\right)\right)d\Omega_i$$

$$4\pi\kappa_\lambda I_{b\lambda} + \sigma_\lambda\ G_\lambda - \int_{4\pi}\left(\beta_\lambda I_\lambda\left(\hat{s}_i\right)\right)d\Omega_i$$

$$4\pi\kappa_\lambda I_{b\lambda} + \sigma_\lambda\ G_\lambda - \beta_\lambda\int_{4\pi}\left(I_\lambda\left(\hat{s}_i\right)\right)d\Omega_i$$

$$\int_{4\pi}\left(I_\lambda\left(\hat{s}_i\right)\right)d\Omega_i = G_\lambda$$

$$4\pi\kappa_\lambda I_{b\lambda} + \sigma_\lambda\ G_\lambda - \beta_\lambda G_\lambda$$

$$4\pi\kappa_\lambda I_{b\lambda} + \left(\sigma_\lambda - \beta_\lambda\right)\ G_\lambda$$

$$\beta_\lambda = \sigma_\lambda + \kappa_\lambda$$

$$4\pi\kappa_\lambda I_{b\lambda} - \kappa_\lambda\ G_\lambda = \kappa_\lambda\left(4\pi I_{b\lambda} - G_\lambda\right)$$

$$\nabla\cdot q_r^{''} = \kappa_\lambda\left(4\pi I_{b\lambda} - G_\lambda\right)$$

Special case: radiative equilibrium.

Divergence of heat flux:

$$\rho c_p\frac{\partial T}{\partial t} = -\nabla\cdot q_{cond} - \nabla\cdot q_{rad} - \nabla\cdot q_{conv} + \dot{q}_{gen}$$

$$q_{cond} = -k\nabla T$$

Moving on to solve RTE for 1D cases – analytically

Most simple of analytical solutions.

Simplifications of solving RTE

- 1D (plane, cylindrical or spherical)
- Gray (no spectral dependence)

So here is our RTE

$$\frac{\partial I(\tau_\lambda)}{\partial \tau_\lambda} = (1-\omega_\lambda) I_{b\lambda} + \frac{\omega_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda(\tau_\lambda)\ \Phi(\hat{s}_j,\hat{s}_i)\, d\Omega_j - I_\lambda$$

$$S(\tau_\lambda,\hat{s}) = (1-\omega_\lambda) I_{b\lambda} + \frac{\omega_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda(\tau_\lambda)\ \Phi(\hat{s}_j,\hat{s}_i)\, d\Omega_j$$

We start with:

$$\hat{s}\cdot\nabla I_\lambda = \frac{dI_\lambda}{ds} = \kappa_\lambda I_{b\lambda} + \frac{\sigma_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda(\hat{s}_j)\ \Phi_\lambda(\hat{s}_j,\hat{s}_i)\, d\Omega_j - \beta_\lambda I_\lambda$$

Integrate across spectrum ie we do $\int_0^\infty \ldots d\lambda$

$$\hat{s}\cdot\nabla I = \frac{dI}{ds} = \kappa I_b + \frac{\sigma}{4\pi}\int_{\Omega_j=4\pi} I(\hat{s}_j)\ \Phi(\hat{s}_j,\hat{s}_i)\, d\Omega_j - \beta I$$

If we were to solve...

$$\frac{dI(\tau)}{d\tau} = (1-\omega) I_b + \frac{\omega}{4\pi}\int_{\Omega_j=4\pi} I(\tau)\ \Phi(\hat{s}_j,\hat{s}_i)\, d\Omega_j - I$$

If we have defined

$$S(\tau) = (1-\omega) I_b + \frac{\omega}{4\pi}\int_{\Omega_j=4\pi} I(\tau)\ \Phi(\hat{s}_j,\hat{s}_i)\, d\Omega_j$$

And then we integrate:

$$I(r,\hat{s}) = I_\omega(\hat{s})\, e^{-\tau_s} + \int_0^{\tau_s} S\left(\tau_s^{'},\hat{s}\right) e^{-\left(\tau_s-\tau_s^{'}\right)} d\tau_s^{'}$$

Important things: how do we get heat flux?

$$q^{''} = \int_{4\pi} I(\theta,\phi)\cos\theta d\Omega$$

To calc heat flux, we need an expression for I in terms of theta and phi

$$q^{''} = \int_0^{2\pi}\int_0^{\pi} I(\theta,\phi)\cos\theta\sin\theta d\theta d\phi$$

So what is $I(\theta,\phi)$?

What is our shape we are dealing with?

0.0.0.1 1-D Plane

In 1D plane geometry, we see optical thickness τ vary according to the direction $\hat{s}$

And of course, the direction $\hat{s}$ can be described in terms of angle θ and ϕ

So how do we get τ in terms of θ and ϕ or $\hat{s}$

Consider this diagram:

τ_s depends on the polar angle θ

What about ϕ (azimuthal angle)

$$\tau_s = \frac{\tau}{\cos\theta}$$

τ_s only depends on polar angle and not the azimuthal angle

$$I\left(r,\hat{s}\right) = I_\omega\left(\hat{s}\right)e^{-\tau_s} + \int_0^{\tau_s} S\left(\tau_s^{'},\hat{s}\right)e^{-\left(\tau_s-\tau_s^{'}\right)}d\tau_s^{'}$$

We can state the coordinates r, $\hat{s}$ using τ and θ,ϕ

R position vector. Since this is 1D problem, only one coordinate matters...

R can be represented by τ

What about $\hat{s}$?

$\hat{s}$ is determined by θ (polar angle) and ϕ (azimuthal angle)

$$I\left(\tau,\theta,\phi\right) = I_\omega\left(\theta,\phi\right)e^{-\tau_s(\tau,\theta)} + \int_0^{\tau_s(\tau,\theta)} S\left(\tau_s^{'}\left(\tau^{'},\theta\right),\theta,\phi\right)e^{-\left(\tau_s\left(\tau^{'},\theta\right)-\tau_s^{'}\left(\tau^{'},\theta\right)\right)}d\tau_s^{'}\left(\left(\tau^{'},\theta\right)\right)$$

So three coordinates are important...

To simplify further, we assume:

Azimuthal symmetry: ie no dependence on ϕ (azimuthal angle)

$$I\left(\tau,\theta\right) = I_\omega\left(\theta\right)e^{-\tau_s(\tau,\theta)} + \int_0^{\tau_s(\tau,\theta)} S\left(\tau_s^{'}\left(\tau^{'},\theta\right),\theta\right)e^{-\left(\tau_s(\tau,\theta)-\tau_s^{'}\left(\tau^{'},\theta\right)\right)}d\tau_s^{'}\left(\left(\tau^{'},\theta\right)\right)$$

So only two coordinates are important here...

If the plates are emit isotropically...

$$I_\omega = constant\left(notdependenton\theta\right)$$

$$I(\tau,\theta) = I_\omega e^{-\tau_s(\tau,\theta)} + \int_0^{\tau_s(\tau,\theta)} S\left(\tau_s^{'}\left(\tau^{'},\theta\right),\theta\right) e^{-\left(\tau_s(\tau,\theta)-\tau_s^{'}\left(\tau^{'},\theta\right)\right)} d\tau_s^{'}\left(\left(\tau^{'},\theta\right)\right)$$

I_ω *is* I_b if blackbody surface, J if gray surface

So what is the source term?

Normally without azimuthal symmetry

$$S(\tau,\theta,\phi) = (1-\omega)\, I_b(\tau,\theta,\phi) + \frac{\omega}{4\pi}\int_{\Omega_j=4\pi} I\left(\tau_{s_j}\left(\tau^{'},\ \theta_j,\phi_j\right)\right)\ \Phi\left(\hat{s}_j(\theta_j,\phi_j),\hat{s}_i(\theta,\phi)\right) d\Omega_j$$

With azimuthal symmetry we take out ϕ dependence of all things including the phase function Φ

With uniform medium temp:

$$S(\tau,\theta,\phi) = (1-\omega)\, I_b + \frac{\omega}{4\pi}\int_{\Omega_j=4\pi} I\left(\tau_{s_j}\left(\tau^{'},\ \theta_j,\phi_j\right)\right)\ \Phi\left(\hat{s}_j(\theta_j,\phi_j),\hat{s}_i(\theta,\phi)\right) d\Omega_j$$

Otherwise we can apply the azimuthal symmetry assumption, assumes the medium is scattering in a very specific sort of way.

$$S(\tau,\theta) = (1-\omega)\, I_b(\tau,\theta) + \frac{\omega}{4\pi}\int_{\Omega_j=4\pi} I\left(\tau_{s_j}\left(\tau^{'},\ \theta_j\right)\right)\ \Phi\left(\hat{s}_j(\theta_j),\hat{s}_i(\theta)\right) d\Omega_j$$

If we have isotropic scattering, then, it looks as if:

$$\Phi\left(\hat{s}_j(\theta_j),\hat{s}_i(\theta)\right) = 1$$

$$S(\tau,\theta) = (1-\omega)\, I_b(\tau,\theta) + \frac{\omega}{4\pi}\int_{\Omega_j=4\pi} I\left(\tau_{s_j}\left(\tau^{'},\ \theta_j\right)\right)\ d\Omega_j$$

$$S(\tau,\theta) = (1-\omega)\, I_b(\tau,\theta) + \frac{\omega}{4\pi} G(\tau)$$

This is where we wanna start for our analytical solution.

So again, what are we interested in finding?

We are interested in finding $q_r^{''}$ (heat flux scalar) and $\nabla\cdot\overrightarrow{q_r^{''}}$

First we want to find our heat flux scalar

$$q_r^{''} = \int_0^{2\pi}\int_0^{\pi} I(r,\theta,\phi)\cos\theta\sin\theta d\theta d\phi$$

We first want to find expression of I For 1D case, azimuthal symmetry,

$$I(\tau,\theta) = I_\omega e^{-\tau_s(\tau,\theta)} + \int_0^{\tau_s(\tau,\theta)} S\left(\tau_s^{'}\left(\tau^{'},\theta\right),\theta\right) e^{-\left(\tau_s(\tau,\theta)-\tau_s^{'}\left(\tau^{'},\theta\right)\right)} d\tau_s^{'}\left(\left(\tau^{'},\theta\right)\right)$$

So we need to substitute:

$$\tau_s = \frac{\tau}{\cos\theta}$$

$$I(\tau,\theta) = I_\omega e^{-\frac{\tau}{\cos\theta}} + \int_0^{\frac{\tau}{\cos\theta}} S\left(\tau_s^{'}\left(\tau^{'},\theta\right),\theta\right) e^{-\left(\frac{\tau}{\cos\theta}-\tau_s^{'}\left(\tau^{'},\theta\right)\right)} d\tau_s^{'}\left(\left(\tau^{'},\theta\right)\right)$$

What about $\tau_s^{'}$? And $d\tau_s^{'}$

$$\tau_s^{'} = \frac{\tau'}{\cos\theta}$$

$$d\tau_s^{'} = \frac{d\tau^{'}}{\cos\theta}$$

$$I(\tau,\theta) = I_\omega e^{-\frac{\tau}{\cos\theta}} + \int_0^{\tau} S\left(\tau_s^{'}\left(\tau^{'},\theta\right),\theta\right) e^{-\left(\frac{\tau}{\cos\theta}-\frac{\tau'}{\cos\theta}\right)} \frac{d\tau^{'}}{\cos\theta}$$

We see $\cos\theta$ pop up a lot for 1D parallel plane case...

$$\mu = \cos\theta$$

The only way θ matters is via $\cos\theta$ or $d\cos\theta = -\sin\theta d\theta$

So usually, we can just substitute all θ relations with μ

$$I(\tau,\mu) = I_\omega e^{-\frac{\tau}{\mu}} + \int_0^{\tau} S\left(\tau_s^{'}\left(\tau^{'},\mu\right),\mu\right) e^{-\left(\frac{\tau}{\mu}-\frac{\tau'}{\mu}\right)} \frac{d\tau^{'}}{\mu}$$

We need to substitute into our $q_r^{''}$ relation...

$$q_r^{''} = \int_0^{2\pi}\int_0^{\pi} I(r,\theta,\phi)\cos\theta\sin\theta d\theta d\phi$$

We can start by simplifying... (use azimuthal symmetry assumption, and apply 1D case)

$$q_r^{''} = \int_0^{2\pi}\int_0^{\pi} I(\tau,\theta)\cos\theta\sin\theta d\theta d\phi$$

$$q_r^{''} = 2\pi \int_0^{\pi} I(\tau, \theta) \cos\theta \sin\theta d\theta$$

Next thing, we want to get things in terms of μ

$$q_r^{''} = 2\pi \int_0^{\pi} I(\tau, \theta) \mu \sin\theta d\theta$$

What is $\sin\theta d\theta$

$$\frac{d}{d\theta} \cos\theta = -\sin\theta$$

$$\frac{d}{d\theta}\mu = -\sin\theta$$

$$d\mu = -\sin\theta d\theta$$

$$-d\mu = \sin\theta d\theta$$

$$q_r^{''} = 2\pi \int_0^{\pi} I(\tau, \theta) \mu (-d\mu)$$

Then we need to change the limits of integration...

What is $\cos\theta$ when $\theta = 0$

Cos 0 is 1

Cos π is -1

$$q_r^{''} = 2\pi \int_1^{-1} I(\tau, \theta) \mu (-d\mu)$$

$$q_r^{''} = 2\pi \int_{\mu=-1}^{\mu=1} I(\tau, \mu) \mu d\mu$$

So we can substitute:

$$I(\tau, \mu) = I_\omega e^{-\frac{\tau}{\mu}} + \int_0^{\tau} S\left(\tau_s^{'}\left(\tau^{'}, \mu\right), \mu\right) e^{-\left(\frac{\tau}{\mu} - \frac{\tau^{'}}{\mu}\right)} \frac{d\tau^{'}}{\mu}$$

$$q_r^{''} = 2\pi \int_{\mu=-1}^{\mu=1} \left(I_\omega e^{-\frac{\tau}{\mu}} + \int_0^{\tau} S\left(\tau_s^{'}\left(\tau^{'}, \mu\right), \mu\right) e^{-\left(\frac{\tau}{\mu} - \frac{\tau^{'}}{\mu}\right)} \frac{d\tau^{'}}{\mu} \right) \mu d\mu$$

Next step: we do integration with μ

We have 2 cases to consider...

One is $0 < \theta < \frac{\pi}{2}$

$$I^{+}(\tau,\mu) = I_{\omega}e^{-\frac{\tau}{\mu}} + \int_{0}^{\tau} S\left(\tau_s^{'}\left(\tau^{'},\mu\right),\mu\right)e^{-\left(\frac{\tau}{\mu}-\frac{\tau'}{\mu}\right)}\frac{d\tau^{'}}{\mu}$$

And second is $\frac{\pi}{2} < \theta < \pi$

$$I^{-}(\tau,\mu)$$

The shortest distance from the top plane to wherever the point in question is...

This distance is: $\tau_L - \tau$

And then

$$\tau_s = \frac{\tau_L - \tau}{\cos A}$$

What is the relation between A and theta?

$$A = \pi - \theta$$

$$\cos A = \cos(\pi - \theta)$$

What is the relationship between $\cos\theta$ and $\cos(\pi - \theta)$

$$\cos A = \cos(\pi - \theta) = -\cos\theta$$

$$\tau_s = \frac{\tau_L - \tau}{\cos A} = \frac{\tau_L - \tau}{-\cos\theta} = \frac{\tau - \tau_L}{\cos\theta}$$

We can take a look at I^{+}

$$I^{+}(\tau,\mu) = I_{\omega}e^{-\frac{\tau}{\mu}} + \int_{0}^{\tau} S\left(\tau_s^{'}\left(\tau^{'},\mu\right),\mu\right)e^{-\left(\frac{\tau}{\mu}-\frac{\tau'}{\mu}\right)}\frac{d\tau^{'}}{\mu}$$

We can form an equivalent for I^{-}

$$I^{-}(\tau,\mu) = I_{\omega}e^{-\frac{\tau-\tau_L}{\mu}} + \int_{0}^{\tau} S\left(\tau_s^{'}\left(\tau^{'},\mu\right),\mu\right)e^{-\left(\frac{\tau-\tau_L}{\mu}-\frac{\tau^{'}-\tau_L}{\mu}\right)}\frac{d\tau^{'}}{\mu}$$

$$I^{-}(\tau,\mu)=I_{\omega}e^{-\frac{\tau-\tau_L}{\mu}}+\int_0^{\tau}S\left(\tau_s^{'}\left(\tau^{'},\mu\right),\mu\right)e^{-\left(\frac{\tau}{\mu}-\frac{\tau^{'}}{\mu}\right)}\frac{d\tau^{'}}{\mu}$$

$$I^{-}(\tau,\mu)=I_{\omega}e^{-\frac{\tau-\tau_L}{\mu}}+\int_0^{\tau}S\left(\tau_s^{'}\left(\tau^{'},\mu\right),\mu\right)e^{-\left(\frac{\tau}{\mu}-\frac{\tau^{'}}{\mu}\right)}\frac{d\tau^{'}}{\mu}$$

$$\tau^{'}\to\tau_L-\tau^{'}$$

~~We need to take note to do these in terms of μ~~

$$\frac{\tau^{'}}{\mu}\to\frac{\tau^{'}-\tau_L}{\mu}$$

So what about $d\tau^{'}$?

$\frac{d\tau^{'}}{\mu}$ was originally to describe $d\tau_s^{'}$

$$I^{-}(\tau,\mu)=I_{\omega}e^{-\frac{\tau-\tau_L}{\mu}}+\int_{\tau_L}^{\tau}S\left(\tau_s^{'}\left(\tau^{'},\mu\right),\mu\right)e^{-\left(\frac{\tau}{\mu}-\frac{\tau^{'}}{\mu}\right)}\frac{d\tau^{'}}{\mu}$$

So back to our scalar heat flux:

$$q_r^{''}=2\pi\int_{\mu=-1}^{\mu=1}\left(I_{\omega}e^{-\frac{\tau}{\mu}}+\int_0^{\tau}S\left(\tau_s^{'}\left(\tau^{'},\mu\right),\mu\right)e^{-\left(\frac{\tau}{\mu}-\frac{\tau^{'}}{\mu}\right)}\frac{d\tau^{'}}{\mu}\right)\mu d\mu$$

$$I^{+}(\tau,\mu)=I_{bottom}e^{-\frac{\tau}{\mu}}+\int_0^{\tau}S\left(\tau_s^{'}\left(\tau^{'},\mu\right),\mu\right)e^{-\left(\frac{\tau}{\mu}-\frac{\tau^{'}}{\mu}\right)}\frac{d\tau^{'}}{\mu}\ ;0<\mu<1$$

$$I^{-}(\tau,\mu)=I_{top}e^{-\frac{\tau-\tau_L}{\mu}}+\int_{\tau_L}^{\tau}S\left(\tau_s^{'}\left(\tau^{'},\mu\right),\mu\right)e^{-\left(\frac{\tau}{\mu}-\frac{\tau^{'}}{\mu}\right)}\frac{d\tau^{'}}{\mu};-1<\mu<0$$

$$q_r^{''}=2\pi\int_{\mu=-1}^{\mu=1}\left(I_{\omega}e^{-\frac{\tau}{\mu}}+\int_0^{\tau}S\left(\tau_s^{'}\left(\tau^{'},\mu\right),\mu\right)e^{-\left(\frac{\tau}{\mu}-\frac{\tau^{'}}{\mu}\right)}\frac{d\tau^{'}}{\mu}\right)\mu d\mu$$

$$q_r^{''}=2\pi\left[\int_{\mu=0}^{\mu=1}\left(I^{+}\right)\mu d\mu+\int_{\mu=-1}^{\mu=0}\left(I^{-}\right)\mu d\mu\right]$$

$$q_r^{''}=2\pi\left[\int_{\mu=0}^{\mu=1}\left(I_{bottom}e^{-\frac{\tau}{\mu}}+\int_0^{\tau}S\left(\tau_s^{'}\left(\tau^{'},\mu\right),\mu\right)e^{-\left(\frac{\tau}{\mu}-\frac{\tau^{'}}{\mu}\right)}\frac{d\tau^{'}}{\mu}\right)\mu d\mu+\int_{\mu=-1}^{\mu=0}\left(I_{top}e^{-\frac{\tau-\tau_L}{\mu}}+\int_{\tau_L}^{\tau}S\left(\tau_s^{'}\right.\right.\right.$$

$$q_r^{''} = 2\pi \left[\int_{\mu=0}^{\mu=1} \left(I_{bottom} e^{-\frac{\tau}{\mu}} \mu + \int_0^{\tau} S\left(\tau_s^{'}\left(\tau^{'}, \mu\right), \mu\right) e^{-\left(\frac{\tau}{\mu} - \frac{\tau^{'}}{\mu}\right)} d\tau^{'} \right) \, d\mu + \int_{\mu=-1}^{\mu=0} \left(I_{top} e^{-\frac{\tau - \tau_L}{\mu}} \mu + \int_{\tau_L}^{\tau} S\left(\tau \right.\right.\right.$$

Without further simplification, even the analytical solution is left in **implicit** form.

Now we want to find divergence of heat flux

$$\nabla \cdot \overrightarrow{q_r^{''}}$$

$$\nabla \cdot q_r^{''} = \kappa_\lambda \left(4\pi I_{b\lambda} - G_\lambda\right)$$

The only thing we need to consider is G_λ or G

$$G \equiv \int_{4\pi} I_{received} \, d\Omega$$

$$G \equiv \int_0^{2\pi} \int_0^{\pi} I_{received} \sin\theta d\theta d\phi$$

Under azimuthal symmetry (independent of phi) and 1D case

$$G = 2\pi \int_{-1}^{1} I_{received} d\mu$$

$$G = 2\pi \left[\int_{\mu=0}^{\mu=1} \left(I_{bottom} e^{-\frac{\tau}{\mu}} + \int_0^{\tau} S\left(\tau_s^{'}\left(\tau^{'}, \mu\right), \mu\right) e^{-\left(\frac{\tau}{\mu} - \frac{\tau^{'}}{\mu}\right)} \frac{d\tau^{'}}{\mu} \right) d\mu + \int_{\mu=-1}^{\mu=0} \left(I_{top} e^{-\frac{\tau - \tau_L}{\mu}} + \int_{\tau_L}^{\tau} S\left(\tau_s^{'}\left(\right.\right.\right.\right.$$

Even for G, the analytical solution is left in implicit form...

And divergence is expressed as:

$$\nabla \cdot q_r^{''} = \kappa_\lambda \left(4\pi I_{b\lambda} - G_\lambda\right)$$

$$\nabla \cdot q_r^{''} = \kappa_\lambda \left(4\pi I_{b\lambda} - 2\pi \left[\int_{\mu=0}^{\mu=1} \left(I_{bottom} e^{-\frac{\tau}{\mu}} + \int_0^{\tau} S\left(\tau_s^{'}\left(\tau^{'}, \mu\right), \mu\right) e^{-\left(\frac{\tau}{\mu} - \frac{\tau^{'}}{\mu}\right)} \frac{d\tau^{'}}{\mu} \right) d\mu + \int_{\mu=-1}^{\mu=0} \left(I_{top} e^{-\frac{\tau-\tau}{\mu}} \right.\right.\right.$$

Meaning u can't solve this, except for making more simplifying assumptions

Numerical Methods

So how does one solve the RTE?

$$\frac{\partial I\left(\tau_\lambda\right)}{\partial \tau_\lambda} = \left(1-\omega_\lambda\right) I_{b\lambda} + \frac{\omega_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda\left(\tau_\lambda\right)\ \Phi\left(\hat{s}_j, \hat{s}_i\right) d\Omega_j - I_\lambda$$

Where we have our Albedo,

$$\frac{\sigma_\lambda}{\beta_\lambda} = \frac{\sigma_\lambda}{\sigma_\lambda + \kappa_\lambda} = \omega_\lambda$$

albedo (how much of the beam attenuation is due to scattering as compared to absorption)

And our optical thickness

$$\tau_\lambda = \int_0^s \beta_\lambda\ ds^*$$

Already the RTE is hard to solve even for 1D cases. How can we generalise it for more complex geometry?

Why is it hard? Because of the scattering term and direction dependence.

To deal with this term, we need some approximation methods.

In summary we have two classes of methods commonly used in numerical solvers:

1) Discrete ordinates (S_N type discretisation)
2) Spherical harmonics (P_N type discretisation)

Now what does this discretisation do?

- convert complicated integro-differential equations into multiple PDEs

Nevertheless, even these methods are computationally intensive, so we can simplify some of the radiation cases where the media is optically thin or thick (small or big τ)

Without further adieu let's begin...

Simplifying Angular Dependnece: The Moment Method

Modest, M. F. (2013). *Radiative heat transfer.* Academic press.

http://www.jicfus.jp/jp/wp-content/uploads/2011/12/hanawa-111205.pdf

This is also known as the Milne Eddington approximation.

Now why are we learning this?

- The moment method is important as a baseline to understand spherical harmonics

  So what's this about?

  Let's go back to our RTE

$$\frac{\partial I_\lambda(\tau_\lambda)}{\partial \tau_\lambda} = (1-\omega_\lambda) I_{b\lambda} + \frac{\omega_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda(\tau_\lambda)\ \Phi(\hat{s}_j, \hat{s}_i)\, d\Omega_j - I_\lambda(\tau_\lambda)$$

  And for simplicity's sake we take a gray medium approximation,

  Or equivalently, we evaluate on a spectral basis (ie, we don't consider multiple transport equations showing multiple parts of the spectrum, but only one set corresponding to one wavelength),

  Since I evaluate one part of the spectrum, I leave the notation unchanged

$$\frac{\partial I_\lambda(\tau_\lambda)}{\partial \tau_\lambda} = (1-\omega_\lambda) I_{b\lambda} + \frac{\omega_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda(\tau_\lambda)\ \Phi(\hat{s}_j, \hat{s}_i)\, d\Omega_j - I_\lambda(\tau_\lambda)$$

  Now what can we do about this? The angular dependence is rather annoying?

  Here's an idea, let's integrate the intesity over all angles and try to get a transport equation out of that...

$$newquantity \rightarrow \int_{-1}^{1} I(\tau_\lambda)\, d\mu$$

  Where $\mu = \cos\theta$

  So integrating over all directions kind of gets rid of directional dependence...

  Doesn't really make physical sense to the equation, but it is a good <u>mathematical tool</u> to get rid of directional dependence.

  So let's integrate with $d\mu$ through the entire equation..

$$\frac{\partial I_\lambda(\tau_{s\lambda})}{\partial \tau_{s\lambda}} = (1-\omega_\lambda) I_{b\lambda} + \frac{\omega_\lambda}{4\pi}\int_{\Omega_j=4\pi} I_\lambda(\tau_{s\lambda})\ \Phi(\hat{s}_j, \hat{s}_i)\, d\Omega_j - I_\lambda(\tau_{s\lambda})$$

  To simplify things, we take a 1D plane parallel case, the above equation simplifies to:

$$\mu\frac{\partial I_\lambda(\tau_\lambda)}{\partial \tau_\lambda} = (1-\omega_\lambda) I_{b\lambda} + \frac{\omega_\lambda}{2}\int_{-1}^{1} I_\lambda(\tau_\lambda, \mu)\ \Phi(\mu, \mu_i)\, d\mu_i - I_\lambda(\tau_\lambda)$$

  What happened here? We integrated out the azimuthal part of the solid angle (ϕ) because we assumed azimuthal symmetry.

Secondly we changed the coordinates, so instead of using the optical thickness which the beam actually travels through (τ_s)), we use τ which is the optical thickness between the point in question and the surface.

$$\tau = \tau_s \cos\theta = \mu\tau_s$$

$$\mu = \cos\theta$$

Using this substitution, the term of the left hand side changes to $\mu\frac{\partial I_\lambda(\tau_\lambda)}{\partial\tau_\lambda}$ becomes

$$\int_{-1}^{1} \mu\frac{\partial I_\lambda(\tau_\lambda)}{\partial\tau_\lambda}d\mu = \int_{-1}^{1}(1-\omega_\lambda)\, I_{b\lambda}d\mu + \int_{-1}^{1}\frac{\omega_\lambda}{2}\int_{-1}^{1} I_\lambda(\tau_\lambda,\mu)\ \Phi(\mu,\mu_i)\, d\mu_i d\mu - \int_{-1}^{1} I_\lambda(\tau_\lambda)\, d\mu$$

Now to make things nicer for us, let's multiply by 2π since π will appear a lot...

$$2\pi\int_{-1}^{1} \mu\frac{\partial I_\lambda(\tau_\lambda)}{\partial\tau_\lambda}d\mu = 2\pi\int_{-1}^{1}(1-\omega_\lambda)\, I_{b\lambda}d\mu + 2\pi\int_{-1}^{1}\frac{\omega_\lambda}{2}\int_{-1}^{1} I_\lambda(\tau_\lambda,\mu)\ \Phi(\mu,\mu_i)\, d\mu_i d\mu - 2\pi\int_{-1}^{1} I_\lambda(\tau_\lambda$$

Now that we've done this, how do we simplify things?

Well, we define a new quantity that doesn't depend on angle since we've integrate the angle θ in terms of $\mu = \cos\theta$ out,

Let's call:

$$I_0 = 2\pi\int_{-1}^{1} I_\lambda(\tau_\lambda)\, d\mu$$

And bringing the integral in,

$$2\pi\int_{-1}^{1} \mu\frac{\partial I_\lambda(\tau_\lambda)}{\partial\tau_\lambda}d\mu = 2\pi\int_{-1}^{1}(1-\omega_\lambda)\, I_{b\lambda}d\mu + 2\pi\int_{-1}^{1}\frac{\omega_\lambda}{2}\int_{-1}^{1} I_\lambda(\tau_\lambda,\mu)\ \Phi(\mu,\mu_i)\, d\mu_i d\mu - I_0$$

However we note that there is another annoying term:

$$2\pi\int_{-1}^{1} \mu\frac{\partial I_\lambda(\tau_\lambda)}{\partial\tau_\lambda}d\mu$$

We shall call this

$$I_1 = 2\pi\int_{-1}^{1} \mu I_\lambda(\tau_\lambda)\, d\mu$$

$$\frac{\partial I_1}{\partial\tau_\lambda} = 2\pi(\int_{-1}^{1} \mu\frac{\partial}{\partial\tau_\lambda} I_\lambda(\tau_\lambda) d\mu)$$

I_0 and I_1 are called the zeroth and first moments respectively. This is why the method is known as the "moment method"

Then we integrate $\int_{-1}^{1}(1-\omega_\lambda) I_{b\lambda} d\mu = 2I_{b\lambda}(1-\omega_\lambda)$

$$\frac{\partial I_1}{\partial \tau_\lambda} = 2\pi\left(2I_{b\lambda}(1-\omega_\lambda)\right) + 2\pi \int_{-1}^{1} \frac{\omega_\lambda}{2} \int_{-1}^{1} I_\lambda(\tau_\lambda, \mu)\ \Phi(\mu, \mu_i)\, d\mu_i d\mu - I_0$$

$$\frac{\partial I_1}{\partial \tau_\lambda} = 2\pi\left(2I_{b\lambda}(1-\omega_\lambda)\right) + I_0 \frac{\omega_\lambda}{2} \int_{-1}^{1} \Phi(\mu, \mu_i)\, d\mu_i - I_0$$

Now what can we do about the phase function?

Remember how we define phase function...

$$\int_{4\pi} \Phi(\hat{s}_j, \hat{s}_i)\, d\Omega_i \equiv 4\pi$$

Equivalently, after applying azimuthal symmetry

$$\frac{1}{2} \int_{-1}^{1} \Phi(\mu, \mu_i)\, d\mu_i = 1$$

$$\frac{\partial I_1}{\partial \tau_\lambda} = 2\pi\left(2I_{b\lambda}(1-\omega_\lambda)\right) + I_0\ \omega_\lambda - I_0$$

$$\frac{\partial I_1}{\partial \tau_\lambda} = 2\pi\left(2I_{b\lambda}(1-\omega_\lambda)\right) + I_0\ \omega_\lambda - I_0$$

$$\frac{\partial I_1}{\partial \tau_\lambda} = \left(4\pi I_{b\lambda}(1-\omega_\lambda)\right) + I_0\ \omega_\lambda - I_0$$

$$\frac{\partial I_1}{\partial \tau_\lambda} = \left(4\pi I_{b\lambda}(1-\omega_\lambda)\right) - I_0(1-\omega_\lambda)$$

$$\frac{\partial I_1}{\partial \tau_\lambda} = \left(4\pi I_{b\lambda} - I_0\right)(1-\omega_\lambda)$$

Now what use is this?

If we integrate out the angles, the idea is that we can perform the derivative of:

$$I_0 = 2\pi \int_{-1}^{1} I_\lambda(\tau_\lambda)\, d\mu$$

$$\frac{\partial}{\partial \mu} I_0 = 2\pi I_\lambda(\tau_\lambda)$$

Provided of course you know how I_0 changes with μ , you could in theory solve for $I_\lambda(\tau_\lambda)$

I suppose that's why you would call it the differential method as well.

Ok great...

How then do we find $I_0(\mu)$? We need another equation to connect I_1

This equation is not helpful to help us find angular dependence:

$$\frac{\partial I_1}{\partial \tau_\lambda} = (4\pi I_{b\lambda} - I_0)(1-\omega_\lambda)$$

Only perhaps the Total incoming intensity of sorts.

You'd probably need <u>more</u> equations to help us find that.

So consider repeating the above steps only starting integration after multiplying μ first.

So we repeat the integration process:

$$\int_{-1}^{1} \mu \frac{\partial I_\lambda(\tau_\lambda)}{\partial \tau_\lambda} d\mu = \int_{-1}^{1} (1-\omega_\lambda) I_{b\lambda} d\mu + \int_{-1}^{1} \frac{\omega_\lambda}{2} \int_{-1}^{1} I_\lambda(\tau_\lambda, \mu)\ \Phi(\mu, \mu_i)\, d\mu_i d\mu - \int_{-1}^{1} I_\lambda(\tau_\lambda)\, d\mu$$

Except that we multiply μ in first

$$\int_{-1}^{1} \mu^2 \frac{\partial I_\lambda(\tau_\lambda)}{\partial \tau_\lambda} d\mu = \int_{-1}^{1} (1-\omega_\lambda) I_{b\lambda} \mu d\mu + \int_{-1}^{1} \frac{\omega_\lambda}{2} \int_{-1}^{1} I_\lambda(\tau_\lambda, \mu)\ \Phi(\mu, \mu_i)\, d\mu_i \mu d\mu - \int_{-1}^{1} I_\lambda(\tau_\lambda)\, \mu d\mu$$

Now multiply with 2π and integrate again.

$$2\pi \int_{-1}^{1} \mu^2 \frac{\partial I_\lambda(\tau_\lambda)}{\partial \tau_\lambda} d\mu = 2\pi \int_{-1}^{1} (1-\omega_\lambda) I_{b\lambda} \mu d\mu + 2\pi \int_{-1}^{1} \frac{\omega_\lambda}{2} \int_{-1}^{1} I_\lambda(\tau_\lambda, \mu)\ \Phi(\mu, \mu_i)\, d\mu_i \mu d\mu - 2\pi \int_{-1}^{1} I_\lambda$$

And like before we define:

$$I_2 = 2\pi \int_{-1}^{1} I_\lambda(\tau_\lambda)\, \mu^2 d\mu$$

This is called the second intensity moment, we had the first and zeroth just now...

Now we can just define intensity moments in general:

$$I_k = 2\pi \int_{-1}^{1} I_\lambda(\tau_\lambda)\, \mu^k d\mu$$

So hopefully this shows you what moments are about.

Now let's integrate this:

$$2\pi \int_{-1}^{1} \frac{\partial I_\lambda(\tau_\lambda)}{\partial \tau_\lambda} \mu^2 d\mu = 2\pi \int_{-1}^{1} (1-\omega_\lambda) I_{b\lambda} \mu d\mu + 2\pi \int_{-1}^{1} \frac{\omega_\lambda}{2} \int_{-1}^{1} I_\lambda(\tau_\lambda, \mu)\ \Phi(\mu, \mu_i)\, d\mu_i \mu d\mu - 2\pi \int_{-1}^{1} I_\lambda$$

Let's get the easy bits out of the way:

$$\frac{\partial I_2}{\partial \tau_\lambda} = 2\pi \int_{-1}^{1} (1 - \omega_\lambda) I_{b\lambda} \mu d\mu + 2\pi \int_{-1}^{1} \frac{\omega_\lambda}{2} \int_{-1}^{1} I_\lambda (\tau_\lambda, \mu) \ \Phi (\mu, \mu_i) d\mu_i \mu d\mu - I_1$$

All right now, we try to integrate w.r.t μ

$$\frac{\partial I_2}{\partial \tau_\lambda} = 2\pi \int_{-1}^{1} (1 - \omega_\lambda) I_{b\lambda} \mu d\mu + 2\pi \int_{-1}^{1} \frac{\omega_\lambda}{2} \int_{-1}^{1} I_\lambda (\tau_\lambda, \mu) \ \Phi (\mu, \mu_i) d\mu_i \mu d\mu - I_1$$

$$\int_{-1}^{1} \mu d\mu = \left[\frac{\mu^2}{2}\right]_{-1}^{1} = 0$$

That makes things quite easy

We very quickly see that this equation gets rid of essentially 2 terms

$$\frac{\partial I_2}{\partial \tau_\lambda} = -I_1$$

So we have these two equations:

$$\frac{\partial I_1}{\partial \tau_\lambda} = (4\pi I_{b\lambda} - I_0)(1 - \omega_\lambda)$$

$$\frac{\partial I_2}{\partial \tau_\lambda} = -I_1$$

That's 3 equations with two unknowns.

Where is out last equation?

Well, we can assume radiation intensity to be isotropic (two flux)

Meaning

$$I = I^+ for 0 \leq \mu \leq 1$$

$$I = I^- for - 1 \leq \mu \leq 0$$

This is where our simplification comes in: **assuming a two flux model**

How does this generate our 3$^{\text{rd}}$ equation?

We can substitute this in here:

$$I_k = 2\pi \int_{-1}^{1} I_\lambda (\tau_\lambda) \mu^k d\mu$$

$$I_k = 2\pi \left[\int_{-1}^{0} I_\lambda\left(\tau_\lambda\right)\mu^k d\mu + \int_{0}^{1} I_\lambda\left(\tau_\lambda\right)\mu^k d\mu \right]$$

Now substitute the above relations,

$$I_k = 2\pi \left[I^- \int_{-1}^{0} \mu^k d\mu + I^+ \int_{0}^{1} \mu^k d\mu \right]$$

Integration yields,

$$I_k = 2\pi \left[I^- \ \left[\frac{\mu^{k+1}}{k+1} \right]_{\mu=-1}^{0} + I^+ \left[\frac{\mu^{k+1}}{k+1} \right]_{\mu=0}^{1} \right]$$

We can factorise out $k+1$ and the inside terms reduces to:

$$I_k = \frac{2\pi}{k+1} \ \left[I^- \ (-1)^k + I^+ \right]$$

For I_2 specifically,

$$I_2 = \frac{2\pi}{2+1} \ \left[I^- \ (-1)^2 + I^+ \right]$$

$$I_2 = \frac{2\pi}{3} \ \left[I^- + I^+ \right]$$

Okay now what? Let's see how to do I^- , I^+

$$I_0 = 2\pi \int_{-1}^{1} I_\lambda\left(\tau_\lambda\right) d\mu$$

$$I_0 = 2\pi \int_{-1}^{0} I_\lambda\left(\tau_\lambda\right) d\mu + 2\pi \int_{0}^{1} I_\lambda\left(\tau_\lambda\right) d\mu$$

$$I_0 = 2\pi \left[I^+ + I^- \right]$$

Thus our 3$^{\text{rd}}$ equation becomes:

$$I_2 = \frac{I_0}{3}$$

We also note some interesting properties of I_0 and I_1

$$I_0 = 2\pi \int_{-1}^{1} I_\lambda\left(\tau_\lambda\right) d\mu = G$$

$$I_1 = 2\pi \int_{-1}^{1} I_\lambda\left(\tau_\lambda\right) \mu d\mu = q$$

This relates to physical quantities we really care about...

Thus we can transform:

$$\frac{\partial I_1}{\partial \tau_\lambda} = (4\pi I_{b\lambda} - I_0)(1 - \omega_\lambda)$$

$$\frac{\partial I_2}{\partial \tau_\lambda} = -I_1$$

Using

$$I_2 = \frac{I_0}{3}$$

into

$$\frac{\partial I_1}{\partial \tau_\lambda} = (4\pi I_{b\lambda} - I_0)(1 - \omega_\lambda)$$

$$\frac{\partial I_0}{\partial \tau_\lambda} = -3I_1$$

These become coupled ODEs cos only τ_λ is the variable of interest

$$\frac{dI_1}{d\tau_\lambda} = (4\pi I_{b\lambda} - I_0)(1 - \omega_\lambda)$$

$$\frac{dI_0}{d\tau_\lambda} = -3I_1$$

Now substituting the above relations:

$$\frac{dq}{d\tau_\lambda} = (4\pi I_{b\lambda} - G)(1 - \omega_\lambda)$$

$$\frac{dG}{d\tau_\lambda} = -3q$$

What BCs shall we use?

We use radiosities so we can account for gray boundaries

At $\tau = 0$ (bottom plate)

$$I^+ = \frac{J_1}{\pi}$$

At $\tau = \tau_L$ (top plate)

$$I^- = \frac{J_2}{\pi}$$

Now we see:

$$G = 2\pi \int_{-1}^{1} I_\lambda \left(\tau_\lambda\right) d\mu = 2\pi \left[I^+ + I^-\right]$$

$$q = 2\pi \int_{-1}^{1} I_\lambda \left(\tau_\lambda\right) \mu d\mu = 2\pi \left[I^+ \left[\frac{\mu^2}{2}\right]_0^1 - I^- \left[\frac{\mu^2}{2}\right]_{-1}^0\right] = \pi \left[I^+ - I^-\right]$$

We can solve for I^+ and I^- in terms of G and q, we assume J_1 and J_2 are known
At $\tau = 0$ (bottom plate)

$$I^+ = \frac{J_1}{\pi}$$

$$G + 2q = 4J_1$$

At $\tau = \tau_L$ (top plate)

$$I^- = \frac{J_2}{\pi}$$

$$G - 2q = 4J_2$$

With these two BCs and systems of coupled ODEs, one can consider the above system solvable, easier compared to integrating things out completely.

With these, you can essentially solve for the intensity field

$$G = I_0 = 2\pi \int_{-1}^{1} I_\lambda \left(\tau_\lambda\right) d\mu$$

$$I_\lambda \left(\tau_\lambda\right) = \frac{1}{2\pi} \frac{\partial}{\partial \mu} G$$

Though this isn't completely necessary...
It's better that since we know:

$$I = I^+ for 0 \leq \mu \leq 1$$

$$I = I^- for - 1 \leq \mu \leq 0$$

If we solve for G and q in terms of τ

$$G\left(\tau\right) = 2\pi \left[I^+ \left(\tau\right) + I^- \left(\tau\right)\right]$$

$$q(\tau) = \pi \left[I^{+}(\tau) - I^{-}(\tau)\right]$$

We will then be able to get $I^{+}(\tau)$ and $I^{-}(\tau)$ everywhere

The intensity field $I(\tau)$ can then be constructed

$$I(\tau, \mu) = I^{+}(\tau)\, for 0 \leq \mu \leq 1$$

$$I(\tau, \mu) = I^{-}(\tau)\, for - 1 \leq \mu \leq 0$$

How can we construct this out into some mathematical function?

First let's get our moment out:

$$G + 2q = 2\pi \left[2I^{+}\right]$$

$$G + 2q = 4\pi I^{+}$$

Likewise

$$G - 2q = 4\pi I^{-}$$

$$I^{+} = \frac{1}{4\pi}\left[G + 2q\right]$$

$$I^{-} = \frac{1}{4\pi}\left[G - 2q\right]$$

See a pattern? G is constant, but the second term changes with direction

So we have a direction independent term and a direction dependent term

Our radiation field can be represented in something like:

$$I(\tau, \mu) = \frac{1}{4\pi}\left[G + directiondependentterm\right]$$

The direction dependent term takes a value 2q when $0 \leq \mu \leq 1$

And a value -2q when $-1 \leq \mu \leq 0$

What can be this term?

Well we have 2q being our magnitude term...

And for the other term, suppose we have some vector...

Usually we can do our dot product...

Let s be our vector for the ray direction.

And n be our vector going from the bottom plate to top plate (for 1D, the length coordinate can be x)

$$\overrightarrow{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$dotprocuct = \overrightarrow{n} \cdot \overrightarrow{s} = |\overrightarrow{n}||\overrightarrow{s}| \cos\theta = |\overrightarrow{n}||\overrightarrow{s}|\mu$$

Unfortunately, the magnitude is adjusted by μ

We can simply adjust that using:

$$|\overrightarrow{n} \cdot \overrightarrow{s}| = |\overrightarrow{n}||\overrightarrow{s}||\mu|$$

Now if we want the function to be as what we were talking about, we can use:

$$\frac{\overrightarrow{n} \cdot \overrightarrow{s}}{|\overrightarrow{n} \cdot \overrightarrow{s}|} = \frac{\mu}{|\mu|}$$

Except for $\mu = 0$, this is pretty good. And most of the time we're not interested in $\mu = 0$ for plane parallel case

Hence we can use a radiation field like so:

$$I(\tau, \mu) = \frac{1}{4\pi} \left[G(\tau) + 2q(\tau) \frac{\mu}{|\mu|} \right]$$

Or

$$I(\tau, \mu) = \frac{1}{4\pi} \left[G(\tau) + 2q(\tau) \frac{\overrightarrow{n} \cdot \overrightarrow{s}}{|\overrightarrow{n} \cdot \overrightarrow{s}|} \right]$$

In terms of moments:

$$I(\tau, \mu) = \frac{1}{4\pi} \left[I_0 + 2I_1 \frac{\overrightarrow{n} \cdot \overrightarrow{s}}{|\overrightarrow{n} \cdot \overrightarrow{s}|} \right]$$

Now this is relatively simple for 1D case. How about 3D?

We note that for I_1 , we integrated across $-1 \leq \mu \leq 1$, and $\mu = \cos\theta$ and θ is the angle between ray direction and the x direction. So this is only concerning x direction.

With y and z direction, we need to consider the angles between the y and z direction as well.

And depending with respect to which angle we integrate, we will get different I_1

So we need to specify a vector that can capture this information.

Consider

$$I(\tau,\mu) = \frac{1}{4\pi}\left[I_0(\tau) + 2I_1(\tau)\frac{\overrightarrow{n}\cdot\overrightarrow{s}}{|\overrightarrow{n}\cdot\overrightarrow{s}|}\right]$$

And replace $I_1\overrightarrow{n}$ with $\overrightarrow{I_1}$

$$\overrightarrow{I_1} = \begin{pmatrix} I_{1x} \\ I_{1y} \\ I_{1z} \end{pmatrix}$$

Not only that, we need to replace τ and μ with more general coordinates, ie. r and $\hat{s}$

$$I(r,\hat{s}) = \frac{1}{4\pi}\left[I_0(r) + 2\overrightarrow{I_1}(r)\cdot\hat{s}\right]$$

So in 3D, this is the first moment approximation of the moment method. And this is how our radiation field is reconstructed. (or at least conceptually)

So in general we can construct radiation fields as such:

$$I(r,\hat{s}) = \left[I_0 + \overrightarrow{I_1}(r)\cdot\hat{s}\right]$$

I_0 here is NOT the first moment, but is a constant containing the first moment. Same for I_1

It gets rapidly more complex as you go for higher order moments, we get tensors and such

$$I(r,\hat{s}) = \left[I_0 + \overrightarrow{I_1}(r)\cdot\hat{s} + \overrightarrow{I_2}(r):\hat{s}\hat{s} + \ldots\right]$$

$\overrightarrow{I_2}(r)$ contains 2^{nd} moment terms and is a tensor.

If we expand this out

$$\overrightarrow{I_2}(r):\hat{s}\hat{s} = I_{2xx}\left(\hat{s}\cdot\hat{i}\right)\left(\hat{s}\cdot\hat{i}\right)+I_{2xy}\left(\hat{s}\cdot\hat{i}\right)\left(\hat{s}\cdot\hat{j}\right)+I_{2xz}\left(\hat{s}\cdot\hat{i}\right)\left(\hat{s}\cdot\hat{k}\right)+\ldots+I_{2zz}\left(\hat{s}\cdot\hat{k}\right)\left(\hat{s}\cdot\hat{k}\right)$$

So essentially this is what the moment method is about.

The general idea of such methods is that we can express a complicated direction dependent flux as a sum of smaller, simpler vectors to calculate,

The magnitude of the constants I_{2xx} , I_{1x} etc, are dependent on position, and the $\hat{s}$ data captures the direction.

And we can calculate the position dependent coefficients with use of more coupled equations.

Spherical Harmonics

$$I\left(r,\hat{s}\right)=\left[I_0\left(r\right)+\overrightarrow{I_1}\left(r\right)\cdot\hat{s}+\overrightarrow{I_2}\left(r\right):\hat{s}\hat{s}+\ldots\right]$$

Now the method of moments is generic, but the idea is that you can reconstruct a radiation field solution using a series of vectors.

Likewise you can construct

$$I\left(r,\hat{s}\right)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}I_l^m\left(r\right)Y_l^m\left(\hat{s}\right)$$

What is this?

So our moments or coefficients are those captured in those $I_l^m\left(r\right)$, these give the magnitude of each of the spherical harmonics functions.

What are spherical harmonics?

- Remember the 1D simple harmonic oscillation equation?
  - $\frac{d^2x}{dt^2}=-\omega^2x$
  - The general solution is in the form

$$x=\sum_{n=1}^{\infty}A_n\sin nx+\sum_{m=1}^{\infty}B_m\cos mx$$

Now the equivalent of this equation in 3D spherical coordinates is the spherical harmonics:

https://mathworld.wolfram.com/SphericalHarmonic.html

in 3D you can write

$$\nabla^2x+\omega^2x=0$$

This is known as Helmholtz differential equation.

Now in spherical coordinates:

$$\nabla^2x=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial x}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial x}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2x}{\partial\phi^2}$$

Note that in this notation, θ is the polar angle and ϕ is the azimuthal angle. Sometimes the notation may flip, but it's important to know which is polar and which is azimuthal.

A common way to solve for this 3D Laplacian type equation is to use separation of variables

This means

$$x = \Theta\left(\theta\right)\Phi\left(\phi\right)R\left(r\right)$$

Where the function is a product of a polar angle function $\Theta\left(\theta\right)$, azimuthal angle function $\Phi\left(\phi\right)$ and a radius based function $R\left(r\right)$

https://mathworld.wolfram.com/LaplacesEquationSphericalCoordinates.html

Substituting this into our harmonics equation results in:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial x}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial x}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 x}{\partial\phi^2}=-\omega^2 x$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Theta\left(\theta\right)\Phi\left(\phi\right)R\left(r\right)}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta\left(\theta\right)\Phi\left(\phi\right)R\left(r\right)}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Theta\left(\theta\right)\Phi\left(\phi\right)R\left(r\right)}{\partial\phi^2}=-\omega$$

Now we can factorise out...

$$\Theta\left(\theta\right)\Phi\left(\phi\right)\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R\left(r\right)}{\partial r}\right)+\frac{\Phi\left(\phi\right)R\left(r\right)}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta\left(\theta\right)}{\partial\theta}\right)+\frac{\Theta\left(\theta\right)R\left(r\right)}{r^2\sin^2\theta}\frac{\partial^2\Phi\left(\phi\right)}{\partial\phi^2}=-\omega^2\Theta\left(\theta\right)\Phi\left(\phi\right)R\left(\right.$$

Now divide throughout by $\Theta\left(\theta\right)\Phi\left(\phi\right)R\left(r\right)$

$$\frac{1}{R\left(r\right)}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R\left(r\right)}{\partial r}\right)+\frac{1}{\Theta\left(\theta\right)r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta\left(\theta\right)}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta\Phi\left(\phi\right)}\frac{\partial^2\Phi\left(\phi\right)}{\partial\phi^2}=-\omega^2$$

Now let's multiply throughout by r^2

$$\frac{1}{R\left(r\right)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R\left(r\right)}{\partial r}\right)+\frac{1}{\Theta\left(\theta\right)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta\left(\theta\right)}{\partial\theta}\right)+\frac{1}{\sin^2\theta\Phi\left(\phi\right)}\frac{\partial^2\Phi\left(\phi\right)}{\partial\phi^2}=-r^2\omega^2$$

You will notice that you can separate the terms into angle dependent and R dependent terms

$$\frac{1}{\Theta\left(\theta\right)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta\left(\theta\right)}{\partial\theta}\right)+\frac{1}{\sin^2\theta\Phi\left(\phi\right)}\frac{\partial^2\Phi\left(\phi\right)}{\partial\phi^2}=-r^2\omega^2-\frac{1}{R\left(r\right)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R\left(r\right)}{\partial r}\right)$$

How do we proceed? The only way for this above equation to be solved

Equate RHS and LHS to a constant

$$\frac{1}{\Theta(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)+\frac{1}{\sin^2\theta\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=-r^2\omega^2-\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)=constant$$

Let's call this constant K_1

$$\frac{1}{\Theta(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)+\frac{1}{\sin^2\theta\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=K_1$$

$$-r^2\omega^2-\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)=K_1$$

You then get an ODE for $R(r)$ and can solve for that. But that's not the interesting bit

The interesting bit is the one with angular dependence, which can further be separated into two more equations:

$$\frac{1}{\Theta(\theta)\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)+\frac{1}{\sin^2\theta\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=K_1$$

Multiply by $\sin^2\theta$

$$\frac{\sin\theta}{\Theta(\theta)}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)+\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=K_1\sin^2\theta$$

Now we can group some terms together...

$$\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=K_1\sin^2\theta-\frac{\sin\theta}{\Theta(\theta)}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta(\theta)}{\partial\theta}\right)$$

Once more, we can get the left and right to equal some constant.

We can call this constant $-m^2$

Why? It makes the azimuthal angle equation relatively easy to solve

$$\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}=-m^2$$

$$\Phi(\phi)=\sum_{m=-\infty}^{\infty}A_m\cos m\phi+B_m\sin m\phi$$

This can also be expressed in terms of exponentials

$$\Phi(\phi) = \sum_{m=-\infty}^{\infty} C_m e^{im\theta}$$

The polar angle equation is more interesting still to solve

$$K_1 \sin^2\theta - \frac{\sin\theta}{\Theta(\theta)} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Theta(\theta)}{\partial\theta}\right) = -m^2$$

$$K_1 \sin^2\theta + m^2 = \frac{\sin\theta}{\Theta(\theta)} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Theta(\theta)}{\partial\theta}\right)$$

How then to solve this?

This actually can be converted into a form of Legendre's ODE:

https://mathworld.wolfram.com/AssociatedLegendreDifferentialEquation.html

This is Legendre's ODE:

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right] + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0$$

Or

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left[l(l+1) - \frac{m^2}{1-x^2}\right] y = 0$$

Which has well known solutions known as Legendre Polynomials. More on that later.

Let's convert:

$$K_1 + m^2 = \frac{\sin\theta}{\Theta(\theta)} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Theta(\theta)}{\partial\theta}\right)$$

Into a Legendre type equation. We can expand out the derivative first

$$\frac{\sin\theta}{\Theta(\theta)} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\Theta(\theta)}{\partial\theta}\right) = \frac{\sin\theta}{\Theta(\theta)} \sin\theta \frac{\partial^2\Theta(\theta)}{\partial\theta^2} + \frac{\sin\theta}{\Theta(\theta)} \cos\theta \frac{\partial\Theta(\theta)}{\partial\theta}$$

$$= \frac{\sin^2\theta}{\Theta(\theta)} \frac{\partial^2\Theta(\theta)}{\partial\theta^2} + \frac{\sin\theta\cos\theta}{\Theta(\theta)} \frac{\partial\Theta(\theta)}{\partial\theta}$$

Let's substitute back

$$K_1 \sin^2\theta + m^2 = \frac{\sin^2\theta}{\Theta(\theta)} \frac{\partial^2\Theta(\theta)}{\partial\theta^2} + \frac{\sin\theta\cos\theta}{\Theta(\theta)} \frac{\partial\Theta(\theta)}{\partial\theta}$$

Divide throughout by $\sin^2\theta$

$$K_1 + \frac{m^2}{\sin^2\theta} = \frac{1}{\Theta(\theta)}\frac{\partial^2\Theta(\theta)}{\partial\theta^2} + \frac{\cos\theta}{\Theta(\theta)\sin\theta}\frac{\partial\Theta(\theta)}{\partial\theta}$$

All right, we don't quite have a legendre looking equation here, but let's make the substitution

$$x = \cos\theta$$

$$\frac{dx}{d\theta} = -\sin\theta$$

$$d\theta = -\frac{dx}{\sin\theta}$$

Now let's start replacing term by term starting with the $\sin^2\theta = 1 - \cos^2\theta = 1 - x^2$

$$K_1 + \frac{m^2}{1-x^2} = \frac{1}{\Theta(\theta)}\frac{\partial^2\Theta(\theta)}{\partial\theta^2} + \frac{\cos\theta}{\Theta(\theta)\sin\theta}\frac{\partial\Theta(\theta)}{\partial\theta}$$

$$\frac{\partial^2\Theta(\theta)}{\partial\theta^2} = \frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}\Theta(\theta)\right) = -\sin\theta\frac{\partial}{\partial x}\left(-\sin\theta\frac{\partial}{\partial x}\Theta(x)\right)$$

$$= \sin\theta\left[\sin\theta\frac{\partial}{\partial x}\frac{\partial}{\partial x}\Theta(x) + \frac{\partial}{\partial x}\Theta(x)\frac{\partial}{\partial x}\sin\theta\right]$$

$$= \sin^2\theta\frac{\partial^2}{\partial x^2}\Theta(x) + \sin\theta\frac{\partial\Theta}{\partial x}\frac{\partial\sin\theta}{\partial x}$$

What then is this? how can we differentiate: $\frac{\partial\sin\theta}{\partial x}$

Since $x = \cos\theta$

$$\frac{dx}{d\theta} = -\sin\theta$$

$$\frac{\partial\sin\theta}{\partial x} = \frac{d\sin\theta}{d\theta}\frac{d\theta}{dx} = \frac{d\sin\theta}{d\theta}\left(-\frac{1}{\sin\theta}\right) = -\frac{\cos\theta}{\sin\theta}$$

Now things can cancel out really nicely once we substitute back in:

$$\frac{1}{\Theta(\theta)}\frac{\partial^2\Theta(\theta)}{\partial\theta^2} = \sin^2\theta\frac{\partial^2}{\partial x^2}\Theta(x) + \sin\theta\frac{\partial\Theta(x)}{\partial x}\frac{\partial\sin\theta}{\partial x}$$

$$\frac{1}{\Theta\left(\theta\right)}\frac{\partial^2\Theta\left(\theta\right)}{\partial\theta^2}=\sin^2\theta\frac{\partial^2}{\partial x^2}\Theta\left(x\right)+\sin\theta\frac{\partial\Theta\left(x\right)}{\partial x}\left(-\frac{\cos\theta}{\sin\theta}\right)$$

$$\frac{1}{\Theta\left(\theta\right)}\frac{\partial^2\Theta\left(\theta\right)}{\partial\theta^2}=\sin^2\theta\frac{\partial^2}{\partial x^2}\Theta\left(x\right)-\cos\theta\frac{\partial\Theta\left(x\right)}{\partial x}$$

Converting in terms of x,

$$\frac{1}{\Theta\left(\theta\right)}\frac{\partial^2\Theta\left(\theta\right)}{\partial\theta^2}=\left(1-x^2\right)\frac{\partial^2}{\partial x^2}\Theta\left(x\right)-x\frac{\partial\Theta\left(x\right)}{\partial x}$$

For the final term, we know:

$$-\sin\theta\frac{\partial}{\partial x}\Theta\left(x\right)=\frac{\partial\Theta\left(\theta\right)}{\partial\theta}$$

So,

$$\frac{\cos\theta}{\Theta\left(\theta\right)\sin\theta}\frac{\partial\Theta\left(\theta\right)}{\partial\theta}=-\sin\theta\frac{\cos\theta}{\Theta\left(\theta\right)\sin\theta}\frac{\partial}{\partial x}\Theta\left(x\right)=-\frac{\cos\theta}{\Theta\left(\theta\right)}\frac{\partial\Theta\left(x\right)}{\partial x}=-\frac{x}{\Theta\left(\theta\right)}\frac{\partial\Theta\left(x\right)}{\partial x}$$

Substituting into:

$$K_1+\frac{m^2}{1-x^2}=\frac{1}{\Theta\left(\theta\right)}\frac{\partial^2\Theta\left(\theta\right)}{\partial\theta^2}+\frac{\cos\theta}{\Theta\left(\theta\right)\sin\theta}\frac{\partial\Theta\left(\theta\right)}{\partial\theta}$$

$$K_1+\frac{m^2}{1-x^2}=\frac{1}{\Theta\left(\theta\right)}\frac{\partial^2\Theta\left(\theta\right)}{\partial\theta^2}-\frac{x}{\Theta\left(\theta\right)}\frac{\partial\Theta\left(x\right)}{\partial x}$$

And the other term:

$$\frac{1}{\Theta\left(\theta\right)}\frac{\partial^2\Theta\left(\theta\right)}{\partial\theta^2}=\left(1-x^2\right)\frac{\partial^2}{\partial x^2}\Theta\left(x\right)-x\frac{\partial\Theta\left(x\right)}{\partial x}$$

$$K_1+\frac{m^2}{1-x^2}=\left(1-x^2\right)\frac{\partial^2}{\partial x^2}\Theta\left(x\right)-x\frac{\partial\Theta\left(x\right)}{\partial x}-\frac{x}{\Theta\left(\theta\right)}\frac{\partial\Theta\left(x\right)}{\partial x}$$

$$K_1+\frac{m^2}{1-x^2}=\left(1-x^2\right)\frac{\partial^2}{\partial x^2}\Theta\left(x\right)-2x\frac{\partial\Theta\left(x\right)}{\partial x}$$

Now we've completely gotten rid of θ and this almost looks like the legendre polynomial

$$\left(1-x^2\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\left[l\left(l+1\right)-\frac{m^2}{1-x^2}\right]y=0$$

To use legendre polynomials to solve here, we need to assume:

$$K_1 = -l\,(l+1)$$

Where l is an integer

$$-l\,(l+1) + \frac{m^2}{1-x^2} = \left(1-x^2\right)\frac{\partial^2\Theta\,(x)}{\partial x^2} - 2x\frac{\partial\Theta\,(x)}{\partial x}$$

Thus with $x = \cos\theta$ we have converted the polar angle equation into a Legendre ODE, for which the solution is a legendre polynomial $P_l^m\,(x)$

The general solution is:

$$\Theta\,(x) = C_1\ P_l^m\,(x) + C_2\ Q_l^m\,(x)$$

$Q_l^m\,(x)$ is a legendre function for non integer values of l . We can generally ignore those because they reach undefined values at $x = \pm 1$. Remember, $x = \cos\theta$ so we only have x going from -1 to 1.

Thus, for us, l can only take integer values, so we use legendre polynomials

$$\Theta\,(x) = C_1\ P_l^m\,(x)$$

Why is it only integer?