Fluid Mechanics YouTube

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Part I
Navier Stokes Equations

Compressible N-S equations

$$\frac{\partial}{\partial t}(\rho \vec{u}) + \nabla \bullet (\rho \vec{u} \otimes \vec{u}) = -\nabla p + \mu \nabla^2 \vec{u} + \frac{1}{3}\mu \nabla(\nabla \bullet \vec{u}) + \rho \vec{g}$$

tensor or outer product:

$$\vec{u} \otimes \vec{v} = \vec{u}\vec{v}^T$$

$$\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \otimes \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

$$= \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}$$

Inner product

$$\vec{u} \bullet \vec{v} = (\vec{u}^T \vec{v})^T$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \bullet \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= u_1 v_1 + u_2 v_2 + u_3 v_3$$

Assume incompressible flow:

$$\rho = constant$$

continuity equation

$$\nabla \bullet \vec{u} = 0$$

Incompressible N-S equations

$$\frac{\partial}{\partial t}\vec{u} + (\vec{u} \bullet \nabla)\vec{u} - \nu\nabla^2\vec{u} = -\nabla\frac{P}{\rho_0} + \vec{g}$$

https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations

Matrices in LaTeX
https://www.overleaf.com/learn/latex/Matrices

Tensors in LaTeX

Navier Stokes Equations
https://www.comsol.com/multiphysics/navier-stokes-equations
https://en.wikipedia.org/wiki/Navier%E2%80%93Stokes_equations

Github
https://github.com/theodoreOnzGit/heatTransferTheory_YouTube

First let's deal with:

$$(\vec{u} \bullet \nabla)\vec{u}$$

$$\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \otimes \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \quad (1)$$

$$= \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}$$

Then we do inner product

$$
\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}
$$

$$
\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}
$$

$$
= \begin{pmatrix} u_1\frac{\partial}{\partial x}u_1 + u_2\frac{\partial}{\partial y}u_1 + u_3\frac{\partial}{\partial z}u_1 \\ u_1\frac{\partial}{\partial x}u_2 + u_2\frac{\partial}{\partial y}u_2 + u_3\frac{\partial}{\partial z}u_2 \\ u_1\frac{\partial}{\partial x}u_3 + u_2\frac{\partial}{\partial y}u_3 + u_3\frac{\partial}{\partial z}u_3 \end{pmatrix}
$$

Let's deal with the momentum diffusivity (kinematic viscosity) term:

$$
\nabla^2 = (\nabla \bullet \nabla)
$$

$$
\begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \bullet \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}
$$

$$
\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x}u_1 & \frac{\partial}{\partial x}u_2 & \frac{\partial}{\partial x}u_3 \\ \frac{\partial}{\partial y}u_1 & \frac{\partial}{\partial y}u_2 & \frac{\partial}{\partial y}u_3 \\ \frac{\partial}{\partial z}u_1 & \frac{\partial}{\partial z}u_2 & \frac{\partial}{\partial z}u_3 \end{pmatrix}
$$

$$
= \begin{pmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x}u_1 + \frac{\partial}{\partial y}\frac{\partial}{\partial y}u_1 + \frac{\partial}{\partial z}\frac{\partial}{\partial z}u_1 \\ \frac{\partial}{\partial x}\frac{\partial}{\partial x}u_2 + \frac{\partial}{\partial y}\frac{\partial}{\partial y}u_2 + \frac{\partial}{\partial z}\frac{\partial}{\partial z}u_2 \\ \frac{\partial}{\partial x}\frac{\partial}{\partial x}u_3 + \frac{\partial}{\partial y}\frac{\partial}{\partial y}u_3 + \frac{\partial}{\partial z}\frac{\partial}{\partial z}u_3 \end{pmatrix}
$$

Part II
Boundary Layer Equations

$$
\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u + w\frac{\partial}{\partial z}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}\ u + \frac{\partial^2}{\partial z^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x
$$

$$
\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + w\frac{\partial}{\partial z}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}\ v + \frac{\partial^2}{\partial z^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y
$$

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial x}w + v\frac{\partial}{\partial y}w + w\frac{\partial}{\partial z}w - \nu(\frac{\partial^2}{\partial x^2}w + \frac{\partial^2}{\partial y^2}\ w + \frac{\partial^2}{\partial z^2}w) = -\frac{1}{\rho_0}\frac{\partial P}{\partial z} + g_z$$

Now for 2D what do we do? w=0 everywhere and at all times, $g_z = 0$
We eliminate z terms from the x and y momentum balance

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}\ u + \frac{\partial^2}{\partial z^2}u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}\ v + \frac{\partial^2}{\partial z^2}v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

There is no spatial variation in u and v w.r.t z We have 2D Navier stokes:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu(\frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}\ u) = -\frac{1}{\rho_0}\frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v - \nu(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}\ v) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

continuity equation

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = 0$$

$$\frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = 0$$

1 nondimensionalisation

Order of magnitude
$$\mathcal{O}$$

Scaling for order magnitude comparison

$$u^*, y^* = \mathcal{O}(1)$$

we define:
$$u^* = \frac{u}{u_\infty}$$

$$u = u_\infty u^*$$

$$x^* = \frac{x}{L}$$

$$y^* = \frac{y}{\delta_p}$$

We scale our continutity equation:

$$\frac{\partial}{\partial x^* L} u^* u_\infty + \frac{\partial}{\partial y^* \delta_p} v = 0$$

$$\frac{\partial}{\partial x^* L} u^* u_\infty + \frac{\partial}{\partial y^* \delta_p} v = 0$$

$$\frac{u_\infty}{L} \frac{\partial}{\partial x^*} u^* + \frac{1}{\delta_p} \frac{\partial}{\partial y^*} v = 0$$

$$\frac{\partial}{\partial x^*} u^* + \frac{L}{\delta_p u_\infty} \frac{\partial}{\partial y^*} v = 0$$

$$v^* = \frac{vL}{\delta_p u_\infty} = \mathcal{O}(1)$$

$$v^* = \frac{v}{\frac{u_\infty \delta_p}{L}}$$

$$\frac{\partial}{\partial x^*} u^* + \frac{\partial}{\partial y^*} v^* = 0$$

Now we move on to the NS equations
so we need to scale time:
x lengthscale = L
x velocityscale = u_∞
timescale = $\frac{L}{u_\infty}$

$$t^* = \frac{t}{\frac{L}{u_\infty}}$$

Let's scale the momentum NS equations

$$\frac{\partial}{\partial t} u + u \frac{\partial}{\partial x} u + v \frac{\partial}{\partial y} u - \nu \left(\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial x} + g_x$$

$$\frac{\partial}{\partial t} v + u \frac{\partial}{\partial x} v + v \frac{\partial}{\partial y} v - \nu \left(\frac{\partial^2}{\partial x^2} v + \frac{\partial^2}{\partial y^2} v \right) = -\frac{1}{\rho_0} \frac{\partial P}{\partial y} + g_y$$

Let's do x momentum equations

$$\frac{\partial}{\partial t}u^*u_\infty+u^*u_\infty\frac{\partial}{\partial x^*L}u^*u_\infty+v\frac{\partial}{\partial y}u^*u_\infty-\nu(\frac{\partial^2}{\partial x^2}u^*u_\infty+\frac{\partial^2}{\partial y^2}u^*u_\infty)=-\frac{1}{\rho_0}\frac{\partial P}{\partial x}+g_x$$

$$\frac{\partial}{\partial t}u^*+u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}u^*+v\frac{1}{\delta_p}\frac{\partial}{\partial y^*}u^*-\nu(\frac{1}{L^2}\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{1}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}u^*)=\frac{1}{u_\infty}(-\frac{1}{L\rho_0}\frac{\partial P}{\partial x^*}+g_x)$$

$$\frac{u_\infty}{L}\frac{\partial}{\partial t^*}u^*+u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}u^*+v\frac{1}{\delta_p}\frac{\partial}{\partial y^*}u^*-\nu(\frac{1}{L^2}\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{1}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}u^*)=\frac{1}{u_\infty}(-\frac{1}{L\rho_0}\frac{\partial P}{\partial x^*}+g_x)$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{\nu}{u_\infty L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}u^*)=\frac{L}{u_\infty^2}(-\frac{1}{L\rho_0}\frac{\partial P}{\partial x^*}+g_x)$$

Reynold's number

$$Re_L=\frac{u_\infty L}{\nu}$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}u^*)=\frac{L}{u_\infty^2}(-\frac{1}{L\rho_0}\frac{\partial P}{\partial x^*}+g_x)$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}u^*)=(-\frac{1}{L\rho_0}\frac{L}{u_\infty^2}\frac{\partial P}{\partial x^*}+\frac{L}{u_\infty^2}g_x)$$

Let's scale gravity

$$g_x^*=\frac{g_x}{|g|}=\cos\theta_x=\mathcal{O}(1)$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}u^*)=(-\frac{1}{\rho_0 u_\infty^2}\frac{\partial P}{\partial x^*}+\frac{L|g|}{u_\infty^2}g_x^*)$$

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}u^*)=(-\frac{1}{\rho_0 u_\infty^2}\frac{\partial P}{\partial x^*}+\frac{1}{Fr^2}g_x^*)$$

$$P^* = \frac{P}{\rho_0 u_\infty^2}$$

After nondimensionalisation, our x momentum equation becomes:

$$\frac{\partial}{\partial t^*}u^* + u^*\frac{\partial}{\partial x^*}u^* + v^*\frac{\partial}{\partial y^*}u^* - \frac{1}{Re_L}\left(\frac{\partial^2}{\partial (x^*)^2}u^* + \frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ u^*\right) = \left(-\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2}g_x^*\right)$$

we dimensionalise y momentum eqns

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v - \nu\left(\frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}\ v\right) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

First the x and time terms:

$$\frac{u_\infty}{L}\frac{\partial}{\partial t^*}v + u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}v + v\frac{\partial}{\partial y}v - \nu\left(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v + \frac{\partial^2}{\partial y^2}\ v\right) = -\frac{1}{\rho_0}\frac{\partial P}{\partial y} + g_y$$

Second y coordinate terms:

$$\frac{u_\infty}{L}\frac{\partial}{\partial t^*}v + u^*\frac{u_\infty}{L}\frac{\partial}{\partial x^*}v + v\frac{1}{\delta_p}\frac{\partial}{\partial y^*}v - \nu\left(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v + \frac{1}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v\right) = -\frac{1}{\rho_0}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*} + g_y^*|g|$$

divide by $\frac{u_\infty^2}{L^2}$

$$\frac{L}{u_\infty}\frac{\partial}{\partial t^*}v + u^*\frac{L}{u_\infty}\frac{\partial}{\partial x^*}v + v\frac{L^2}{u_\infty^2\delta_p}\frac{\partial}{\partial y^*}v - \nu\frac{L^2}{u_\infty^2}\left(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v + \frac{1}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v\right) = \frac{L^2}{u_\infty^2}\left(-\frac{1}{\rho_0}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*} + g_y^*|g|\right)$$

divide by δ_p

$$\frac{L}{u_\infty\delta_p}\frac{\partial}{\partial t^*}v + u^*\frac{L}{u_\infty\delta_p}\frac{\partial}{\partial x^*}v + v\frac{L^2}{u_\infty^2\delta_p^2}\frac{\partial}{\partial y^*}v - \nu\frac{L^2}{u_\infty^2\delta_p}\left(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v + \frac{1}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v\right) = \frac{L^2}{u_\infty^2\delta_p}\left(-\frac{1}{\rho_0}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*} + g_y^*|g|\right)$$

Combining some terms

$$\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^* - \nu\frac{L}{u_\infty}\left(\frac{1}{L^2}\frac{\partial^2}{\partial (x^*)^2}v^* + \frac{1}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*\right) = \frac{L^2}{u_\infty^2\delta_p}\left(-\frac{1}{\rho_0}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*} + g_y^*|g|\right)$$

Rearranging

$$\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^* - \frac{\nu}{u_\infty L}\left(\frac{\partial^2}{\partial (x^*)^2}v^* + \frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial (y^*)^2}\ v^*\right) = \left(-\frac{1}{\rho_0}\frac{L^2}{u_\infty^2\delta_p}\frac{1}{\delta_p}\frac{\partial P}{\partial y^*} + g_y^*|g|\frac{L^2}{u_\infty^2\delta_p}\right)$$

nondimensionalisng pressure and including the Fr

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{\nu}{u_\infty L}\left(\frac{\partial^2}{\partial(x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\,v^*\right)=\left(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p}\right)$$

Include Re

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{1}{Re_L}\left(\frac{\partial^2}{\partial(x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\,v^*\right)=\left(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p}\right)$$

Review: NS nondimensionalised

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}\left(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\,u^*\right)=\left(-\frac{\partial P^*}{\partial x^*}+\frac{1}{Fr^2}g_x^*\right)$$

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{1}{Re_L}\left(\frac{\partial^2}{\partial(x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\,v^*\right)=\left(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p}\right)$$

$$\frac{\partial}{\partial x^*}u^*+\frac{\partial}{\partial y^*}v^*=0$$

2 How to drop terms?

$$\frac{\partial}{\partial t^*}u^*+u^*\frac{\partial}{\partial x^*}u^*+v^*\frac{\partial}{\partial y^*}u^*-\frac{1}{Re_L}\left(\frac{\partial^2}{\partial(x^*)^2}u^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\,u^*\right)=\left(-\frac{\partial P^*}{\partial x^*}+\frac{1}{Fr^2}g_x^*\right)$$

$$\frac{\partial}{\partial t^*}v^*+u^*\frac{\partial}{\partial x^*}v^*+v^*\frac{\partial}{\partial y^*}v^*-\frac{1}{Re_L}\left(\frac{\partial^2}{\partial(x^*)^2}v^*+\frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\,v^*\right)=\left(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*}+g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p}\right)$$

$$\frac{\partial}{\partial x^*}u^*+\frac{\partial}{\partial y^*}v^*=0$$

When we want to determine which terms to cancel, we need to know how Re_L compares with $\frac{L^2}{\delta_p^2}$

Assumption:

Creeping flow in y direction

$$Re_\delta=\frac{v_c\delta_p}{\nu}=\mathcal{O}(1)$$

How does Re_δ compare to Re_L

$$v_c = u_\infty \frac{\delta_p}{L}$$

$$Re_\delta = \frac{u_\infty \frac{\delta_p}{L} \delta_p}{\nu} = \mathcal{O}(1)$$

$$Re_\delta = \frac{u_\infty L}{\nu} \frac{\delta_p^2}{L^2} = \mathcal{O}(1)$$

$$Re_\delta = Re_L \frac{\delta_p^2}{L^2} = \mathcal{O}(1)$$

2.0.1   x direction momentum eqn

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \left(\frac{\partial^2}{\partial (x^*)^2} u^* + \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* \right) = \left(-\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2} g_x^* \right)$$

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{1}{Re_L} \frac{L^2}{\delta_p^2} \frac{\partial^2}{\partial (y^*)^2} u^* = \left(-\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2} g_x^* \right)$$

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - \frac{1}{Re_L} \frac{\partial^2}{\partial (x^*)^2} u^* + \frac{1}{\mathcal{O}(1)} \frac{\partial^2}{\partial (y^*)^2} u^* = \left(-\frac{\partial P^*}{\partial x^*} + \frac{1}{Fr^2} g_x^* \right)$$

How big is Re_L?

$$Re_L = \mathcal{O}\left(\frac{L^2}{\delta_p^2} \right)$$

We assume Fr is big

$$\frac{\partial}{\partial t^*} u^* + u^* \frac{\partial}{\partial x^*} u^* + v^* \frac{\partial}{\partial y^*} u^* - + \frac{1}{\mathcal{O}(1)} \frac{\partial^2}{\partial (y^*)^2} u^* = \left(-\frac{\partial P^*}{\partial x^*} \right)$$

2.0.2 y direction momentum equation

$$\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^* - \frac{1}{Re_L}\left(\frac{\partial^2}{\partial(x^*)^2}v^* + \frac{L^2}{\delta_p^2}\frac{\partial^2}{\partial(y^*)^2}\ v^*\right) = \left(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*} + g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p}\right)$$

$$\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^* - \left(\frac{1}{Re_L}\frac{\partial^2}{\partial(x^*)^2}v^* + \frac{1}{\mathcal{O}(1)}\frac{\partial^2}{\partial(y^*)^2}\ v^*\right) = \left(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*} + g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p}\right)$$

$$\left[\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^*\right]\frac{1}{Re_L} - \left(\frac{1}{Re_L^2}\frac{\partial^2}{\partial(x^*)^2}v^* + \frac{1}{\mathcal{O}(1)Re_L}\frac{\partial^2}{\partial(y^*)^2}\ v^*\right) = \frac{1}{Re_L}\left(-\frac{L^2}{\delta_p^2}\frac{\partial P^*}{\partial y^*} + g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p}\right)$$

$$\left[\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^*\right]\frac{1}{Re_L} - \left(\frac{1}{Re_L^2}\frac{\partial^2}{\partial(x^*)^2}v^* + \frac{1}{\mathcal{O}(1)Re_L}\frac{\partial^2}{\partial(y^*)^2}\ v^*\right)$$

$$= \left(-\frac{L^2}{\delta_p^2}\frac{1}{Re_L}\frac{\partial P^*}{\partial y^*} + g_y^*\frac{1}{Fr^2}\frac{L}{\delta_p}\frac{1}{Re_L}\right)$$

Cancelling out...

$$\left[\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^*\right]\frac{1}{Re_L} - \left(\frac{1}{Re_L^2}\frac{\partial^2}{\partial(x^*)^2}v^* + \frac{1}{\mathcal{O}(1)Re_L}\frac{\partial^2}{\partial(y^*)^2}\ v^*\right)$$

$$= \left(-\frac{1}{\mathcal{O}(1)}\frac{\partial P^*}{\partial y^*} + g_y^*\frac{1}{Fr^2}\frac{L^2}{\delta_p^2}\frac{1}{Re_L}\frac{\delta_p}{L}\right)$$

Simplifying

$$\left[\frac{\partial}{\partial t^*}v^* + u^*\frac{\partial}{\partial x^*}v^* + v^*\frac{\partial}{\partial y^*}v^*\right]\frac{1}{Re_L} - \left(\frac{1}{Re_L^2}\frac{\partial^2}{\partial(x^*)^2}v^* + \frac{1}{\mathcal{O}(1)Re_L}\frac{\partial^2}{\partial(y^*)^2}\ v^*\right)$$

$$= \frac{1}{\mathcal{O}(1)}\left(-\frac{\partial P^*}{\partial y^*} + g_y^*\frac{1}{Fr^2}\frac{\delta_p}{L}\right)$$

For large Re_L

$$0 = \left(-\frac{\partial P^*}{\partial y^*} + g_y^*\frac{1}{Fr^2}\frac{\delta_p}{L}\right)$$

$$g_y^*\frac{1}{Fr^2}\frac{\delta_p}{L} = \frac{\partial P^*}{\partial y^*}$$

Only if g=0,

$$0 = -\frac{\partial P^*}{\partial y^*}$$

Now we have our BL equations:

$$0 = -\frac{\partial P^*}{\partial y^*}$$

$$\frac{\partial}{\partial t^*}u^* + u^*\frac{\partial}{\partial x^*}u^* + v^*\frac{\partial}{\partial y^*}u^* - \frac{1}{\mathcal{O}(1)}\frac{\partial^2}{\partial(y^*)^2}\ u^* = (-\frac{\partial P^*}{\partial x^*})$$

redimensionalise to obtain the laminar BL equations:

$$0 = -\frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial(y)^2}\ u = (-\frac{\partial P}{\partial x})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

3   Solutions to the BL equations laminar

How to solve?

- 1st Similarity solution

- 2nd Von Karman Solution (Integral solution - approximate)

- 3rd numerical (CFD)

3.1   similarity solution (Blasius solution)

$$0 = -\frac{\partial P}{\partial y}$$

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial(y)^2}\ u = (-\frac{\partial P}{\partial x})$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Similarity solution $\rightarrow$ combine variables to convert PDE to ODE
Making 2 assumptions before we continue:
1) steady state 2) no pressure gradient

$$u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - \nu\frac{\partial^2}{\partial y^2}\,u = 0$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

How to combine variables to make life easier for us to solve?
introduce the streamfunction (ψ):

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

Note: streamfunction only works for 2D fluid flow
Substitute into 2D continuity equation,

$$\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x} = 0$$

Substitute into the 2D x momentum equation

$$\left(\frac{\partial \psi}{\partial y}\right)\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \left(-\frac{\partial \psi}{\partial x}\right)\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) - \nu\frac{\partial^2}{\partial y^2}\left(\frac{\partial \psi}{\partial y}\right) = 0$$

$$\left(\frac{\partial \psi}{\partial y}\right)\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) - \left(\frac{\partial \psi}{\partial x}\right)\left(\frac{\partial^2 \psi}{\partial y^2}\right) - \nu\frac{\partial^3}{\partial y^3}\psi = 0$$

We need to compress the number of variables further to 1 indep variable
(η) and 1 dependent variable $(f(\eta))$

$$\eta = \eta(x, y)$$

$$f = f(\eta)$$

Before we continue, BCs first!
1 BC in x dir for u

$$x = 0, y \neq 0, u = u_\infty$$

2 BCs in y dir for u

no slip

$$u = 0 \ at \ y = 0$$

$$y \to \infty; \ u \to u_\infty$$

1 BC in y direction for v

   no slip

$$v = 0 \ at \ y = 0$$

3.1.1 similarity transform

How do we start to get these "combo parameters" aka similarity variables?

`https://ntrs.nasa.gov/citations/20050028493`

Based on Blasius's paper (translated by NACA) it's good to nondimnesionalise to find these similarity variables

$$\psi^* = \frac{\psi}{\psi_0}$$

Let's nondimensionalise the momentum equations:

$$(\frac{\partial \psi}{\partial y})\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial y}) - (\frac{\partial \psi}{\partial x})(\frac{\partial^2 \psi}{\partial y^2}) - \nu \frac{\partial^3}{\partial y^3}\psi = 0$$

$$\frac{\psi_0^2}{\delta_p^2 L}(\frac{\partial \psi^*}{\partial y^*})\frac{\partial}{\partial x^*}(\frac{\partial \psi^*}{\partial y^*}) - \frac{\psi_0^2}{L\delta_p^2}(\frac{\partial \psi^*}{\partial x^*})(\frac{\partial^2 \psi^*}{\partial (y^*)^2}) - \frac{\nu\psi_0}{\delta_p^3}\frac{\partial^3}{\partial (y^*)^3}\psi^* = 0$$

$$(\frac{\partial \psi^*}{\partial y^*})\frac{\partial}{\partial x^*}(\frac{\partial \psi^*}{\partial y^*}) - (\frac{\partial \psi^*}{\partial x^*})(\frac{\partial^2 \psi^*}{\partial (y^*)^2}) - \frac{\nu\psi_0}{\delta_p^3}\frac{\delta_p^2 L}{\psi_0^2}\frac{\partial^3}{\partial (y^*)^3}\psi^* = 0$$

$$(\frac{\partial \psi^*}{\partial y^*})\frac{\partial}{\partial x^*}(\frac{\partial \psi^*}{\partial y^*}) - (\frac{\partial \psi^*}{\partial x^*})(\frac{\partial^2 \psi^*}{\partial (y^*)^2}) - \frac{\nu}{\delta_p}\frac{L}{\psi_0}\frac{\partial^3}{\partial (y^*)^3}\psi^* = 0$$

dimensionless group:

$$\frac{\nu L}{\delta_p \psi_0} = \mathcal{O}(1)$$

We want the equations to be nondimensionalised exactly.
If we want the equations looks like:

$$(\frac{\partial \psi^*}{\partial y^*})\frac{\partial}{\partial x^*}(\frac{\partial \psi^*}{\partial y^*}) - (\frac{\partial \psi^*}{\partial x^*})(\frac{\partial^2 \psi^*}{\partial (y^*)^2}) - \frac{\partial^3}{\partial (y^*)^3}\psi^* = 0$$

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$$\frac{\nu L}{\delta_p \psi_0} = 1$$

$$\psi_0 = \frac{\nu L}{\delta_p}$$

Otherwise:

$$\psi_0 = \mathcal{O}(1)\frac{\nu L}{\delta_p}$$

How do we get rid of dependent variables δ_p?

$$Re_\delta = \frac{v_c \delta_p}{\nu} = \mathcal{O}(1)$$

$$Re_\delta = \frac{u_\infty L}{\nu}\frac{\delta_p^2}{L^2} = \mathcal{O}(1)$$

From continutity equation

$$v_c = \frac{u_\infty \delta_p}{L}$$

(not too helpful)

What's helpful is to use the physics of the BL ie creeping flow in BL

The other assumption:

$$Re_\delta = 1$$

$$Re_\delta = \frac{u_\infty}{\nu L}\delta_p^2$$

$$\delta_p = \sqrt{Re_\delta}\sqrt{\frac{\nu L}{u_\infty}}$$

Substitute back:

$$\psi_0 = \frac{\nu L}{\delta_p} = \frac{\nu L}{\sqrt{Re_\delta}\sqrt{\frac{\nu L}{u_\infty}}}$$

$$\psi_0 = \frac{\nu L}{\sqrt{\frac{\nu L}{u_\infty}}}\frac{1}{\sqrt{Re_\delta}}$$

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$$\psi_0 = \sqrt{u_\infty L \nu} \frac{1}{\sqrt{Re_\delta}}$$

Let's see our nondimensionalised streamfunction:

$$\psi^* = \frac{\psi}{\psi_0} = \sqrt{Re_\delta} \frac{\psi}{\sqrt{u_\infty L \nu}}$$

If $Re_\delta = 1$

$$\psi^* = \frac{\psi}{\psi_0} = \frac{\psi}{\sqrt{u_\infty L \nu}}$$

What about our independent variable? We need to combine the x and y coordinate variables

we'll use

$$Re_\delta = \frac{u_\infty}{\nu L} \delta_p^2$$

$$\delta_p = \sqrt{Re_\delta} \sqrt{\frac{\nu L}{u_\infty}}$$

If we want x and y explicitly,

$$x = x^* L$$

$$y = y^* \delta_p$$

$$\frac{y}{y^*} = \sqrt{Re_\delta} \sqrt{\frac{\nu \frac{x}{x^*}}{u_\infty}}$$

$$y = \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} \sqrt{\frac{\nu x}{u_\infty}}$$

$$y = \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} \sqrt{\frac{\nu x}{u_\infty}}$$

$$y \sqrt{\frac{u_\infty}{\nu x}} = \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} = \mathcal{O}(1)$$

And we have dimensionless stream function

$$\psi^* = \frac{\psi}{\psi_0} = \frac{\psi}{\sqrt{u_\infty L \nu}}$$

$$\psi^* = \frac{\psi}{\sqrt{u_\infty \frac{x}{x^*} \nu}}$$

$$\psi^* = \sqrt{x^*} \frac{\psi}{\sqrt{u_\infty x \nu}}$$

If $Re_\delta \neq 1$

$$\psi^* = \frac{\psi}{\psi_0} = \sqrt{Re_\delta} \frac{\psi}{\sqrt{u_\infty L \nu}}$$

$$\psi^* = \frac{\psi}{\psi_0} = \sqrt{Re_\delta x^*} \frac{\psi}{\sqrt{u_\infty x \nu}}$$

from Blasius's paper:

$$\eta = \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} = \frac{1}{2} y \sqrt{\frac{u_\infty}{\nu x}}$$

$$f(\eta) = \frac{\psi^*}{\sqrt{x^* Re_\delta}} = \frac{\psi}{\sqrt{u_\infty x \nu}}$$

We start transforming variables

$$\left(\frac{\partial \psi^*}{\partial y^*}\right) \frac{\partial}{\partial x^*} \left(\frac{\partial \psi^*}{\partial y^*}\right) - \left(\frac{\partial \psi^*}{\partial x^*}\right) \left(\frac{\partial^2 \psi^*}{\partial (y^*)^2}\right) - \frac{\partial^3}{\partial (y^*)^3} \psi^* = 0$$

Use chain rule

$$\frac{\partial \psi^*}{\partial y^*} = \frac{\partial f}{\partial \eta} \frac{\partial \psi^*}{\partial f} \frac{\partial \eta}{\partial y^*}$$

$$\frac{\partial f(\eta)}{\partial \psi^*} = \frac{1}{\sqrt{x^* Re_\delta}} \frac{\partial}{\partial \psi^*} \psi^* = \frac{1}{\sqrt{x^* Re_\delta}}$$

[CORRECTION:]

$$\psi^* = f(\eta) \sqrt{x^*} \sqrt{Re_\delta}$$

So that:

$$\frac{\partial \psi^*}{\partial y^*} = \frac{\partial}{\partial y^*} f(\eta) \sqrt{x^*} \sqrt{Re_\delta}$$

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Assume Re_δ is constant with respect to both x and y,

$$\frac{\partial \psi^*}{\partial y^*} = \sqrt{x^*}\sqrt{Re_\delta}\frac{\partial}{\partial y^*}f(\eta)$$

$$\frac{\partial \psi^*}{\partial y^*} = \sqrt{x^*}\sqrt{Re_\delta}\frac{\partial \eta}{\partial y^*}\frac{\partial}{\partial \eta}f(\eta) = \sqrt{x^*}\sqrt{Re_\delta}\frac{\partial \eta}{\partial y^*}f'$$

[END OF CORRECTION]

$$\frac{\partial}{\partial y^*}\eta = \frac{\partial}{\partial y^*}\frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}} = \frac{\eta}{y^*}$$

$$\frac{\partial \psi^*}{\partial y^*} = \frac{\partial f}{\partial \eta}\sqrt{x^* Re_\delta}\frac{\eta}{y^*} = f'\sqrt{x^* Re_\delta}\frac{\eta}{y^*}$$

substitute:

$$\eta = \frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}}$$

$$\frac{\partial \psi^*}{\partial y^*} = \frac{\partial f}{\partial \eta}\sqrt{x^* Re_\delta}\frac{\eta}{y^*} = f'\sqrt{x^* Re_\delta}\frac{1}{y^*}\frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}} = f'\frac{Re_\delta}{2}$$

So this becomes:

$$(f'\frac{Re_\delta}{2})\frac{\partial}{\partial x^*}(f'\frac{Re_\delta}{2}) - (\frac{\partial \psi^*}{\partial x^*})(\frac{\partial}{\partial (y^*)}f'\frac{Re_\delta}{2}) - \frac{\partial^2}{\partial (y^*)^2}f'\frac{Re_\delta}{2} = 0$$

$$(f'\frac{Re_\delta}{2})\frac{\partial}{\partial x^*}(f') - (\frac{\partial \psi^*}{\partial x^*})(\frac{\partial}{\partial (y^*)}f') - \frac{\partial^2}{\partial (y^*)^2}f' = 0$$

Now for higher order derivatives, note that:

$$\frac{\partial}{\partial y^*}\left(\frac{\eta}{y^*}\right) = \frac{y^*\frac{\partial \eta}{\partial y^*} - \eta\frac{\partial y^*}{\partial y^*}}{(y^*)^2}$$

note that

$$\frac{\partial}{\partial y^*}\eta = \frac{\eta}{y^*}$$

$$\frac{\partial}{\partial y^*}\left(\frac{\eta}{y^*}\right) = \frac{y^*\frac{\eta}{y^*} - \eta}{(y^*)^2} = 0$$

What does this tell us?

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$$\frac{\eta}{y^*}$$

is constant with respect to y^*

$$\frac{\partial}{\partial y^*} f' = \frac{\partial \eta}{\partial y^*} f'' = \frac{\eta}{y^*} f''$$

Then we have

$$\frac{\partial^2}{\partial (y^*)^2} f' = \frac{\partial}{\partial (y^*)} \left(\frac{\partial \eta}{\partial y^*} \frac{\partial}{\partial \eta} f' \right) = \frac{\partial}{\partial (y^*)} \left(\frac{\partial \eta}{\partial y^*} f'' \right)$$

$$= \frac{\eta^2}{(y^*)^2} \frac{\partial}{\partial \eta} f'' = \frac{\eta^2}{(y^*)^2} f'''$$

We can substitute our expressions back:

$$\left(f' \frac{Re_\delta}{2} \right) \frac{\partial}{\partial x^*} (f') - \left(\frac{\partial \psi^*}{\partial x^*} \right) \left(\frac{\partial}{\partial (y^*)} f' \right) - \frac{\partial^2}{\partial (y^*)^2} f' = 0$$

$$\left(f' \frac{Re_\delta}{2} \right) \frac{\partial}{\partial x^*} (f') - \left(\frac{\partial \psi^*}{\partial x^*} \right) \frac{\eta}{y^*} f'' - \frac{\eta^2}{(y^*)^2} f''' = 0$$

Now let's deal with the x^* terms

$$\frac{\partial}{\partial x^*} = \frac{\partial \eta}{\partial x^*} \frac{\partial}{\partial \eta}$$

$$\eta = \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}}$$

$$\frac{\partial}{\partial x^*} \eta = \frac{\partial}{\partial x^*} \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}}$$

$$= \frac{1}{2} \sqrt{Re_\delta} \frac{y^*}{\sqrt{x^*}} \frac{-1}{2x^*} = \frac{\eta}{-2x^*}$$

$$\left(f' \frac{Re_\delta}{2} \right) \frac{\eta}{-2x^*} (f'') - \left(\frac{\partial \psi^*}{\partial x^*} \right) \frac{\eta}{y^*} f'' - \frac{\eta^2}{(y^*)^2} f''' = 0$$

Now for the derivative:

$$\psi^* = f(\eta) \sqrt{x^*} \sqrt{Re_\delta}$$

$$\frac{\partial \psi^*}{\partial x^*} = \frac{\partial}{\partial x^*} f(\eta) \sqrt{x^*} \sqrt{Re_\delta}$$

$$\frac{\partial \psi^*}{\partial x^*} = \sqrt{Re_\delta} [\sqrt{x^*} \frac{\partial}{\partial x^*} f(\eta) + f(\eta) \frac{\partial}{\partial x^*} \sqrt{x^*}]$$

$$\frac{\partial \psi^*}{\partial x^*} = \sqrt{Re_\delta} [\sqrt{x^*} \frac{-\eta}{2x^*} f' + f(\eta) \frac{1}{2\sqrt{x^*}}]$$

$$(f' \frac{Re_\delta}{2}) \frac{\eta}{-2x^*} (f'') - (\sqrt{Re_\delta} [\sqrt{x^*} \frac{-\eta}{2x^*} f' + f(\eta) \frac{1}{2\sqrt{x^*}}]) \frac{\eta}{y^*} f'' - \frac{\eta^2}{(y^*)^2} f''' = 0$$

Let's substitute

$$\frac{\eta}{y^*} = \frac{1}{2} \frac{\sqrt{Re_\delta}}{\sqrt{x^*}}$$

$$(f' \frac{Re_\delta}{2}) \frac{\eta}{-2x^*} (f'') - (\sqrt{Re_\delta} [\sqrt{x^*} \frac{-\eta}{2x^*} f' + f(\eta) \frac{1}{2\sqrt{x^*}}]) \frac{1}{2} \frac{\sqrt{Re_\delta}}{\sqrt{x^*}} f'' - (\frac{1}{2} \frac{\sqrt{Re_\delta}}{\sqrt{x^*}})^2 f''' = 0$$

$$(f' \frac{Re_\delta}{2}) \frac{\eta}{-2x^*} (f'') - (\sqrt{Re_\delta} [\sqrt{x^*} \frac{-\eta}{2x^*} f' + f(\eta) \frac{1}{2\sqrt{x^*}}]) \frac{1}{2} \frac{\sqrt{Re_\delta}}{\sqrt{x^*}} f'' - (\frac{1}{4} \frac{Re_\delta}{x^*}) f''' = 0$$

$$(f' \frac{Re_\delta}{2}) \frac{\eta}{-2x^*} (f'') - (Re_\delta [\frac{-\eta}{2x^*} f' + f(\eta) \frac{1}{2x^*}]) \frac{1}{2} f'' - (\frac{1}{4} \frac{Re_\delta}{x^*}) f''' = 0$$

$$-(f' \frac{Re_\delta}{2}) \frac{\eta}{2x^*} (f'') - [\frac{Re_\delta}{2} f'' \frac{-\eta}{2x^*} f' + Re_\delta \frac{1}{2} f'' f(\eta) \frac{1}{2x^*}] - (\frac{1}{4} \frac{Re_\delta}{x^*}) f''' = 0$$

$$-[Re_\delta \frac{1}{2} f'' f(\eta) \frac{1}{2x^*}] - (\frac{1}{4} \frac{Re_\delta}{x^*}) f''' = 0$$

$$f'' f + f''' = 0$$

And we're done!
Now to transform the BCs:

$$u = 0 \ at \ y = 0$$

$$v = 0 \; at \; y = 0$$

$$u \to u_\infty \; at \; y \to \infty$$

$$\eta = \frac{1}{2}\sqrt{Re_\delta}\frac{y^*}{\sqrt{x^*}} = \frac{1}{2}y\sqrt{\frac{u_\infty}{\nu x}}$$

$$f(\eta) = \frac{\psi^*}{\sqrt{x^* Re_\delta}} = \frac{\psi}{\sqrt{u_\infty x \nu}}$$

$$u = 0 \; at \; \eta = 0$$

$$v = 0 \; at \; \eta = 0$$

$$u \to u_\infty \; at \; \eta \to \infty$$

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}f(\eta)\sqrt{u_\infty x \nu}$$

$$u = \sqrt{u_\infty x \nu}\frac{\partial \eta}{\partial y}\frac{\partial}{\partial \eta}f(\eta) = \sqrt{u_\infty x \nu}\frac{1}{2}\sqrt{\frac{u_\infty}{\nu x}}f'$$

$$u = \frac{u_\infty}{2}f'$$

$$u = u_\infty \to f' = 2$$

$$v = -\frac{\partial \psi}{\partial x}$$

$$v = -\frac{\partial}{\partial x}f(\eta)\sqrt{u_\infty x \nu}$$

$$v = -[f(\eta)\frac{\partial}{\partial x}\sqrt{u_\infty x \nu} + \sqrt{u_\infty x \nu}\frac{\partial}{\partial x}f(\eta)]$$

$$v = -[f(\eta)\frac{\partial}{\partial x}\sqrt{u_\infty x \nu} + \sqrt{u_\infty x \nu}\frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta}f(\eta)]$$

some steps later...

$$v = -\frac{1}{2}\sqrt{\frac{\nu u_\infty}{x}}(\eta f' - f)$$

$$f' = 0 \; at \; \eta = 0$$

$$f = 0 \ at \ \eta = 0$$

$$f' = 2 \ at \ \eta \to \infty$$

$$f''f + f''' = 0$$

3.1.2 how to solve Blasius's equation

$$f' = 0 \ at \ \eta = 0$$

$$f = 0 \ at \ \eta = 0$$

$$f' = 2 \ at \ \eta \to \infty$$

$$f''f + f''' = 0$$

Series solution (aka Frobenius method)

https://mathworld.wolfram.com/FrobeniusMethod.html
http://naca.central.cranfield.ac.uk/reports/1950/naca-tm-1256.pdf

Assumes:

$$f = \sum_{n=0}^{\infty} a_n \eta^n$$

$$f' = \sum_{n=0}^{\infty} a_{n+1}(n+1)\eta^n$$

$$f'' = \sum_{n=0}^{\infty} a_{n+2}(n+1)(n+2)\eta^n$$

$$f''' = \sum_{n=0}^{\infty} a_{n+3}(n+1)(n+2)(n+3)\eta^n$$

Runge Kutta Methods   numerical methods...

https://en.wikipedia.org/wiki/Runge%E2%80%93Kutta_methods
https://projects.exeter.ac.uk/fluidflow/Courses/FluidDynamics3211-2/BoundaryLayers/rk

https://www.researchgate.net/publication/259772650_Numerical_Approximations_of_Blasiu

4   Resources Online

http://web.mit.edu/fluids-modules/www/highspeed_flows/ver2/bl_Chap2.pdf
https://community.dur.ac.uk/suzanne.fielding/teaching/BLT/sec3.pdf

Part III
Github Repo

https://github.com/theodoreOnzGit/heatTransferTheory_YouTube

Look under convection heat transfer...