CS1231S TUTORIAL #8

Cardinality

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Learning objectives of this tutorial

Cardinality

- relationship between bijections and cardinality
- Cantor’s definition of same-cardinality
- the notion of countability
- techniques to show a set is countable
- techniques to show a set is uncountable

Quick summary

Definition (Cantor). A set A is said to have the *same cardinality* as a set B, written as $|A| = |B|$, iff there is a bijection $f: A \to B$.

Definition (Cantor). A set S is *countably infinite* iff $|S| = |\mathbb{Z}^+|$ (or $|\mathbb{N}|$).

Definition (Cantor). A set is *countable* iff it is finite or countably infinite.

Theorem. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

Proposition 9.1. An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \ldots \in B$ in which every element of B appears exactly once.

Lemma 9.2. An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \cdots$ in which every element of B appears.

Theorem 7.4.3. Any subset of any countable set is countable.

Corollary 7.4.4. Any set with an uncountable subset is uncountable.

Proposition 9.3. Every infinite set has a countably infinite subset.

Lemma 9.4. Let A and B be countably infinite sets. Then $A \cup B$ is countable.

Q1. Show that $\mathbb{Z}$ is countable by defining a bijection $g: \mathbb{N} \to \mathbb{Z}$ using a single-line formula.

$$\boldsymbol{g(n) = (-1)^n \left\lfloor \frac{n+1}{2} \right\rfloor}$$

(Other answers possible.)

A function $f: X \to Y$ is **injective** iff $\forall x_1, x_2 \in X\ (f(x_1) = f(x_2) \Rightarrow x_1 = x_2)$.

1. (Injectivity)
   - 1.1. Let $g(a), g(b) \in \mathbb{Z}$ and $g(a) = g(b)$.
   - 1.2. Then $g(a)$ and $g(b)$ must both be non-negative or both negative.
   - 1.3. Case 1: $g(a)$ and $g(b)$ are both non-negative.
     - 1.3.1. Then a and b must be even.
     - 1.3.2. Then $(-1)^a\left\lfloor\frac{a+1}{2}\right\rfloor = (-1)^b\left\lfloor\frac{b+1}{2}\right\rfloor \Rightarrow \left\lfloor\frac{a+1}{2}\right\rfloor = \left\lfloor\frac{b+1}{2}\right\rfloor \Rightarrow \frac{a}{2} = \frac{b}{2} \Rightarrow a = b$.
   - 1.4. Case 2: $g(a)$ and $g(b)$ are both negative.
     - 1.4.1. Then a and b must be odd.
     - 1.4.2. Then $(-1)^a\left\lfloor\frac{a+1}{2}\right\rfloor = (-1)^b\left\lfloor\frac{b+1}{2}\right\rfloor \Rightarrow -\left\lfloor\frac{a+1}{2}\right\rfloor = -\left\lfloor\frac{b+1}{2}\right\rfloor \Rightarrow \frac{a+1}{2} = \frac{b+1}{2} \Rightarrow a = b$.
   - 1.5. In all cases, $a = b$.
   - 1.6. Therefore g is injective.

Q1. Show that $\mathbb{Z}$ is countable by defining a bijection $g: \mathbb{N} \to \mathbb{Z}$ using a single-line formula.

$$\boldsymbol{g(n) = (-1)^n \left\lfloor \frac{n+1}{2} \right\rfloor}$$

(Other answers possible.)

A function $f: X \to Y$ is **surjective** iff $\forall y \in Y\ \exists x \in X\ \big(y = f(x)\big)$.

2. (Surjectivity)
   - 2.1. Let $m \in \mathbb{Z}$. Then m is non-negative or negative.
   - 2.2. Case 1: m is non-negative.
     - 2.2.1. Let $n = 2m$.
     - 2.2.2. Then $n \in \mathbb{N}$ and $g(n) = (-1)^{2m}\left\lfloor \frac{2m+1}{2} \right\rfloor = \frac{2m}{2} = m$.
   - 2.3. Case 2: m is negative.
     - 2.3.1. Let $n = -2m - 1$.
     - 2.3.2. Then $n \in \mathbb{N}$ and $g(n) = (-1)^{-2m-1}\left\lfloor \frac{(-2m-1)+1}{2} \right\rfloor = -\frac{-2m}{2} = m$.
   - 2.4. In all cases, there exists $n \in \mathbb{N}$ such that $g(n) = m$.
   - 2.5. Therefore g is surjective.
3. Therefore g is a bijection from $\mathbb{N}$ to $\mathbb{Z}$.

Q2.

Let B be a countably infinite set and C be a finite set.
Show that $B \cup C$ is countable
(a) by using the sequence argument.

1. Apply Lemma 9.2 to obtain a sequence $b_0, b_1, b_2, \ldots$ in which every element of B appears.
2. Suppose $|C| = n \in \mathbb{N}$. We may write $C = \{c_0, c_1, \ldots, c_{n-1}\}$.
3. Then $c_0, c_1, \ldots, c_{n-1}, b_0, b_1, b_2, \ldots$ is a sequence in which every element of $B \cup C$ appears.
4. So $B \cup C$ is countable by Lemma 9.2.

Lemma 9.2. An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \ldots$ in which every element of B appears.

Q2.

Let B be a countably infinite set and C be a finite set.
Show that $B \cup C$ is countable
(b) by defining a bijection $g: \mathbb{N} \to B \cup C$.

1. As B is a countably infinite set, we have a bijection $f: \mathbb{N} \to B$.
2. Remove all elements in C that are in B. After removal, $C = \{c_0, c_1, \ldots, c_{k-1}\}$.
3. Define a function $g: \mathbb{N} \to B \cup C$ such that

$$g(i) = \begin{cases} c_i & \text{if } i < k; \\ f(i-k) & \text{otherwise.} \end{cases}$$

4. As the c_i's are distinct, $g(i) = g(j) \Rightarrow c_i = c_j \Rightarrow i = j$. Hence g is injective from $\{0,1,\cdots,k-1\}$ to C.
5. For every c_i, there exists an i such that $g(i) = c_i$. Hence g is surjective from $\{0,1,\cdots,k-1\}$ to C.
6. Therefore g is a bijection between $\{0,1,\cdots,k-1\}$ and C.
7. g is a bijection between $\{k, k+1, \cdots\}$ and B as f is bijective between $\{0,1,2,\cdots\}$ and B.
8. Therefore g is a bijection between $\mathbb{N}$ and $B \cup C$.

Q3. (a) Suppose $A_1, A_2, \cdots$ are finite sets. Then $\bigcup_{i=1}^{n} A_i$ is finite for any $n \geq 2$.

Proof:

"We will prove by induction on n. Since A_1 and A_2 are finite, then $A_1 \cup A_2$ is finite, so the claim is true for $n = 2$. Now suppose the claim is true for $n = k$, so $\bigcup_{i=1}^{k} A_i$ is finite. Let $A_{k+1} = \emptyset$.

Then $\bigcup_{i=1}^{k+1} A_i = \left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1} = \bigcup_{i=1}^{k} A_i$ which is finite by the induction hypothesis, so the claim is true for $n = k + 1$. Therefore, the claim is true for all $n \geq 2$."

What is wrong with this "proof"?

There is an implicit universal quantification on $A_1, A_2, \cdots$, i.e. we have to prove the claim is true for all possible $A_1, A_2, \cdots$, so we cannot just consider the special case $A_{k+1} = \emptyset$.

Q3. (b) Prove the following is false.

"Suppose $A_1, A_2, \cdots$ are finite sets. Then $\bigcup_{i=1}^{\infty} A_i$ is finite."

Let $A_i = \{i\}$ for all $i \geq 1$.

Then $\bigcup_{i=1}^{\infty} A_i = \mathbb{Z}^+$ which is infinite.

Q4. Suppose $A_1, A_2, A_3, \cdots$ are countable sets.

(a) Prove, by induction, that $\bigcup_{i=1}^{n} A_i$ is countable for any $n \in \mathbb{Z}^+$.

1. Let $P(n)$ means "$\bigcup_{i=1}^{n} A_i$ is countable ".
2. Basis step: $\bigcup_{i=1}^{1} A_i = A_1$ is countable, so $P(1)$ is true.
3. Induction step: Suppose $P(k)$ is true.
   - 3.1. $\bigcup_{i=1}^{k+1} A_i = \left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}$.
   - 3.2. Since $\bigcup_{i=1}^{k} A_i$ is countable (by induction hypothesis) and A_{k+1} is countable, so $\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}$ is countable (by Lemma 9.4).
   - 3.3. Hence $P(k+1)$ is true.
4. Therefore $\bigcup_{i=1}^{n} A_i$ is countable for any $n \in \mathbb{Z}^+$ by MI.

Lemma 9.4. Let A and B be countably infinite sets. Then $A \cup B$ is countable.

Q4. Suppose $A_1, A_2, A_3, \cdots$ are countable sets.

(b) Does (a) prove that $\bigcup_{i=1}^{\infty} A_i$ is countable?

No.

Question 3(b) shows that a proof that $\bigcup_{i=1}^{n} A_i$ is finite for every $n \geq 2$ does not imply $\bigcup_{i=1}^{\infty} A_i$ is finite. Similarly here, a proof that $\bigcup_{i=1}^{n} A_i$ is countable for every $n \geq 1$ does not imply that $\bigcup_{i=1}^{\infty} A_i$ is countable

Note that $\bigcup_{i=1}^{\infty} A_i$ is indeed countable, just that it cannot be proved using the approach in part (a). We will prove it in the next question.

Q5. Let S_i be a countably infinite set for each $i \in \mathbb{Z}^+$. Prove that $\bigcup_{i\in\mathbb{Z}^+} S_i$ is countable.

$$\begin{array}{lccccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
S_4 & b_{4,1}, & b_{4,2}, & b_{4,3}, & b_{4,4}, & b_{4,5}, & b_{4,6}, & \ldots \\
S_3 & b_{3,1}, & b_{3,2}, & b_{3,3}, & b_{3,4}, & b_{3,5}, & b_{3,6}, & \ldots \\
S_2 & b_{2,1}, & b_{2,2}, & b_{2,3}, & b_{2,4}, & b_{2,5}, & b_{2,6}, & \ldots \\
S_1 & b_{1,1}, & b_{1,2}, & b_{1,3}, & b_{1,4}, & b_{1,5}, & b_{1,6}, & \ldots
\end{array}$$

Lemma 9.2. An infinite set B is countable iff there is a sequence in which every element of B appears.

Q5.

Let S_i be a countably infinite set for each $i \in \mathbb{Z}^+$. Prove that $\bigcup_{i\in\mathbb{Z}^+} S_i$ is countable.

Lemma 9.2. An infinite set B is countable iff there is a sequence $b_0, b_1, b_2, \ldots$ in which every element of B appears.

1. $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable (lecture 9 slide 29).
2. Hence there is a bijection $f\colon \mathbb{Z}^+ \to \mathbb{Z}^+ \times \mathbb{Z}^+$.
3. For each $i \in \mathbb{Z}^+$, since S_i is countable, apply Lemma 9.2 to find a sequence $b_{i,1}, b_{i,2}, b_{i,3}, \ldots$ in which every element of S_i appears.
4. Define a sequence $c_1, c_2, c_3, \ldots$ by setting each $c_k = b_{i,j}$, where $(i,j) = f(k)$.
5. In view of Lemma 9.2, it suffices to show that every element of $\bigcup_{i\in\mathbb{Z}^+} S_i$ appears in the sequence $c_1, c_2, c_3, \ldots$.
   - 5.1. Take $x \in \bigcup_{i\in\mathbb{Z}^+} S_i$.
   - 5.2. The definition of $\bigcup_{i\in\mathbb{Z}^+} S_i$ gives $i \in \mathbb{Z}^+$ such that $x \in S_i$.
   - 5.3. So line 3 tells us x appears in the sequence $b_{i,1}, b_{i,2}, b_{i,3}, \ldots$.
   - 5.4. Let $j \in \mathbb{Z}^+$ such that $x = b_{i,j}$.
   - 5.5. From the surjectivity of f, we obtain $k \in \mathbb{Z}^+$ such that $f(k) = (i,j)$.
   - 5.6. Then $x = b_{i,j} = c_k$ by the definition of c_k.

$$
\begin{array}{llllll}
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
S_3\colon & b_{3,1}, & b_{3,2}, & b_{3,3}, & b_{3,4}, & \ldots \\
S_2\colon & b_{2,1}, & b_{2,2}, & b_{2,3}, & b_{2,4}, & \ldots \\
S_1\colon & b_{1,1}, & b_{1,2}, & b_{1,3}, & b_{1,4}, & \ldots
\end{array}
$$

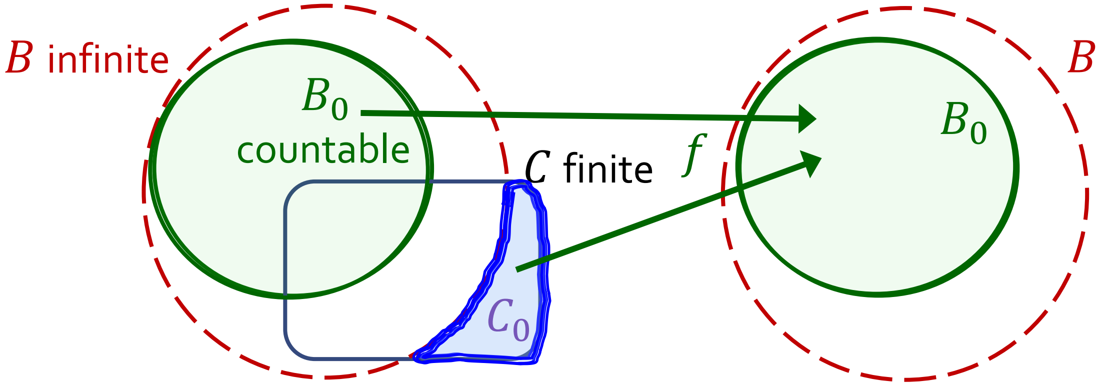
Definition. A function $f\colon A \to B$ is *surjective* iff $\forall y \in B\ \exists x \in A\ (y = f(x))$.

Q6. Let B be an infinite set and C be a finite set. Define a bijection $B \cup C \to B$.

Q2. Let B be a countably infinite set and C a finite set. Then $B \cup C$ is countable.

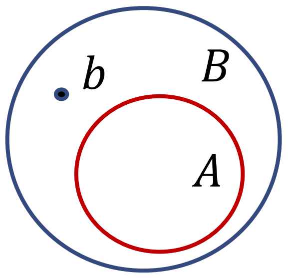
1. Use Proposition 9.3 to find a countably infinite subset $B_0 \subseteq B$.
2. Let $C_0 = C \setminus B$, so that C_0 is finite.
3. Then $B_0 \cup C_0$ is countable by Q2.
4. Hence $|B_0 \cup C_0| = |\mathbb{Z}^+| = |B_0|$ by the definition of countably infinite set.
5. Hence there is a bijection $f : B_0 \cup C_0 \to B_0$.
6. Define $g : B \cup C \to B$ as follows: for each $x \in B \cup C$,
$$g(x) = \begin{cases} f(x), & \text{if } x \in B_0 \cup C_0; \\ x, & \text{otherwise,} \end{cases}$$
7. g is the required bijection.

Proposition 9.3. Every infinite set has a countably infinite subset.



Q7. Prove that a set B is infinite if and only if there is $A \subsetneq B$ such that $|A| = |B|$.

1. ("Only if")
   1.1. Suppose B is infinite.
   1.2. Take any $b \in B$.
   1.3. Define $A = B \setminus \{b\}$.
   1.4. Then $A \subsetneq B$.
   1.5. As B is infinite, we know A is infinite too.
   1.6. So $|B| = |A \cup \{b\}| = |A|$ by question 6.



Q6. Let B be an infinite set and C a finite set. Then there is a bijection $B \cup C \to B$. So $|B \cup C| = |B|$.

2. ("If") (We prove the contrapositive: If B is finite then for all $A \subsetneq B, |A| \neq |B|$.)
   2.1. Suppose B is finite.
   2.2. Take any$A \subsetneq B$.
   2.3. As B is finite, we know A is also finite.
   2.4. There are strictly fewer elements in A than in B.
   2.5. Hence there is no bijection $A \to B$.
   2.6. This means $|A| \neq |B|$ by Equality of Cardinality of Finite Sets Theorem.

Equality of Cardinality of Finite Sets.
Let A and B be any finite sets.
$|A| = |B|$ iff there is a bijection $f: A \to B$.

Q8. Prove that $\mathbb{C}$ is uncountable.

1. $\mathbb{R} \subseteq \mathbb{C}$.
2. We know $\mathbb{R}$ is uncountable (lecture #9 example #5).
3. Therefore $\mathbb{C}$ is uncountable by Corollary 7.4.4.

Corollary 7.4.4. Any set with an uncountable subset is uncountable.

Q9. Let A be a countably infinite set. Prove that $\mathcal{P}(A)$ is uncountable.

1. Suppose not, that is, $\mathcal{P}(A)$ is countable.
2. $\mathcal{P}(A)$ is infinite as A is infinite and $\{a\} \in \mathcal{P}(A)$ for every $a \in A$.
3. By Proposition 9.1, there is a sequence $a_0, a_1, a_2, \cdots \in A$ in which every element of A appears exactly once.
4. By Proposition 9.1, there is a sequence $B_0, B_1, B_2, \cdots \in \mathcal{P}(A)$ in which every element of$\mathcal{P}(A)$ appears exactly once.
5. Now, define $B = \{a_i \colon a_i \notin B_i\}$.

Proposition 9.1. An infinite set B is countable if and only if there is a sequence $b_0, b_1, b_2, \ldots \in B$ in which every element of B appears exactly once.

Q9. Let A be a countably infinite set. Prove that $\wp(A)$ is uncountable.

1. Suppose not, that is, $\mathcal{P}(A)$ is countable.
2. $\mathcal{P}(A)$ is infinite as A is infinite and $\{a\} \in \mathcal{P}(A)$ for every $a \in A$.
3. By Proposition 9.1, there is a sequence $a_0, a_1, a_2, \cdots \in A$ in which every element of A appears exactly once.
4. By Proposition 9.1, there is a sequence $B_0, B_1, B_2, \cdots \in \mathcal{P}(A)$ in which every element of $\mathcal{P}(A)$ appears exactly once.
5. Now, define $B = \{a_i : a_i \notin B_i\}$.
6. Note that $B \in \mathcal{P}(A)$ since $a_0, a_1, a_2, \cdots \in A$.
7. To show $B \neq B_i$ for all $i \in \mathbb{N}$.
   - 7.1. Let $i \in \mathbb{N}$.
   - 7.2. Case 1: If $a_i \notin B_i$, then $a_i \in B$ by the definition of B.
   - 7.3. Case 2: If $a_i \in B_i$, then $a_i \notin B$ by the definition of B.
   - 7.4. In all cases, $B \neq B_i$.
8. Since B is not in the sequence $B_0, B_1, B_2, \cdots \in \mathcal{P}(A)$, this contradicts that $\mathcal{P}(A)$ is countable.
9. Therefore, $\mathcal{P}(A)$ is uncountable.

	a_0	a_1	a_2	a_3	a_4	$\cdots$
B_0	$\notin$	$\in$	$\notin$	$\notin$	$\notin$	$\cdots$
B_1	$\in$	$\notin$	$\in$	$\notin$	$\in$	$\cdots$
B_2	$\notin$	$\in$	$\in$	$\notin$	$\in$	$\cdots$
B_3	$\notin$	$\notin$	$\in$	$\notin$	$\notin$	$\cdots$
B_4	$\in$	$\notin$	$\in$	$\in$	$\notin$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
B	$\in$	$\in$	$\notin$	$\in$	$\in$	$\cdots$

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