SLIDES BY

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Tutorial 8: Relations

Q1. Let $A = \{1, 2, \ldots, 10\}$ and $B = \{2, 4, 6, 8, 10, 12, 14\}$.
Define a relation R from A to B by setting

$$x \; R \; y \iff x \text{ is prime and } x \mid y$$

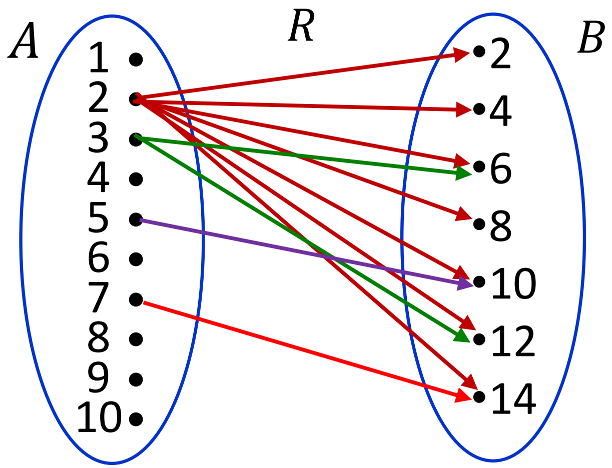
for each $x \in A$ and each $y \in B$. Write down the sets R and R^{-1} in roster notation. Do not use ellipses (. . .) in your answers.

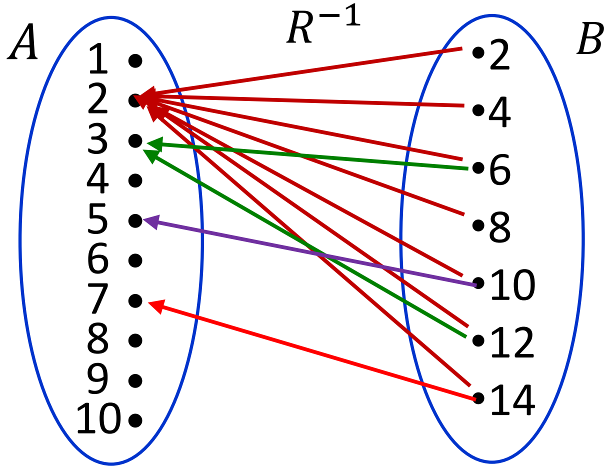
What is R^{-1}?

R^{-1} is the inverse relation of R, i.e.
$R^{-1} = \{(y, x) : (x, y) \in R\}$,
or, $y \; R^{-1} \; x$ iff $x \; R \; y$.

(Note: Unlike functions, every relation has a (unique) inverse.)

(Arrow diagrams shown for illustration purpose only.)





$$R = \{(2,2), (2,4), (2,6), (2,8), (2,10), (2,12), (2,14),(3,6), (3,12), (5,10), (7,14)\}$$

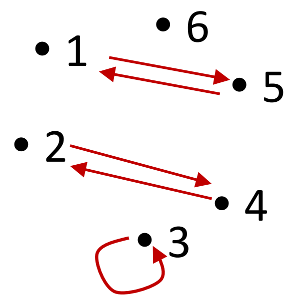
$$R^{-1} = \{(2,2), (4,2), (6,2), (8,2), (10,2), (12,2),(14,2), (6,3), (12,3), (10,5), (14,7)\}$$

Q2. Let R be a relation on set A. Show that R is symmetric $\Leftrightarrow R = R^{-1}$.

(Observation: If R is a symmetric relation on a set A, how would the arrow diagrams for R and R^{-1} look like?)

Eg: Let $A = \{1,2,3,4,5,6\}$ and R be a relation on A s.t. $x\ R\ y \Leftrightarrow x + y = 6$. Show the arrow diagrams for R and R^{-1}.

Tutor: Get students to draw the diagrams themselves, to discover that they are the same.



1. $(\Rightarrow)$
   - 1.1 Suppose R is symmetric.
   - 1.2 $(R \subseteq R^{-1})$
     - 1.2.1 Let $x, y \in A$ such that $(x, y) \in R$.
     - 1.2.2 Then $x\ R\ y$ (by the definition of xRy)
     - 1.2.3 $\therefore y\ R\ x$ (as R is symmetric)
     - 1.2.4 $\therefore x\ R^{-1}\ y$ (by the definition of R^{-1})
     - 1.2.5 $\therefore (x, y) \in R^{-1}$ (by the definition of $xR^{-1}y$)
   - 1.3 $(R \supseteq R^{-1})$
     - 1.3.1 Let $x, y \in A$ such that $(x, y) \in R^{-1}$.
     - 1.3.2 Then $x\ R^{-1}\ y$ (by the definition of $xR^{-1}y$)
     - 1.3.3 $\therefore y\ R\ x$ (by the definition of R^{-1})
     - 1.3.4 $\therefore x\ R\ y$ (as R is symmetric)
     - 1.3.5 $\therefore (x, y) \in R$ (by the definition of xRy)
   - 1.4 $\therefore R = R^{-1}$.

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     - 1.3.4 $\therefore x\ R\ y$ (as R is symmetric)
     - 1.3.5 $\therefore (x, y) \in R$ (by the definition of xRy)
   - 1.4 $\therefore R = R^{-1}$.

2. $(\Leftarrow)$
   - 2.1 Suppose $R = R^{-1}$.
     - 2.1.1 Let $x, y \in A$ such that $x\ R\ y$.
     - 2.1.2 Then $(x, y) \in R$ (by the definition of xRy)
     - 2.1.3 $\therefore (x, y) \in R^{-1}$ (as $R = R^{-1}$)
     - 2.1.4 $\therefore x\ R^{-1}\ y$ (by the definition of $xR^{-1}y$)
     - 2.1.5 $\therefore y\ R\ x$ (by the definition of R^{-1})
   - 2.2 $\therefore R$ is symmetric.

Q3. For each of the following relations on $\mathbb{Q}$, determine if it is (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric, (v) an equivalence relation.

Let A be a set and R a relation on A.

(1) R is reflexive: $\forall x \in A\ (x\ R\ x)$.

(2) R is symmetric: $\forall x, y \in A\ (xRy \Rightarrow yRx)$.

(3) R is transitive: $\forall x, y, z \in A\ (xRy \wedge yRz \Rightarrow xRz)$.

(4) R is antisymmetric: $\forall x, y \in A\ (xRy \wedge yRx \Rightarrow x = y)$.

(5) R is an equivalence relation: R is reflexive, symmetric and transitive.

(a) R is defined by setting $x\ R\ y$ iff $xy \geq 0$ for all $x, y \in \mathbb{Q}$.

✓ Reflexive? $\forall x \in \mathbb{Q}, x \cdot x = x^2 \geq 0$

✓ Symmetric? $\forall x, y \in \mathbb{Q}, xy \geq 0 \ \Rightarrow yx = xy \geq 0$ (by commutativity of $\times$)

✗ Transitive? Counter-example: $1\ R\ 0$ and $0\ R\ (-1)$ but $1 \not R\ (-1)$.

✗ Antisymmetric? Counter-example: $1\ R\ 2$ and $2\ R\ 1$ but $1 \neq 2$.

✗ Equivalence relation? R is not transitive.

Q3. For each of the following relations on $\mathbb{Q}$, determine if it is (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric, (v) an equivalence relation.

Let A be a set and R a relation on A.
(1) R is reflexive: $\forall x \in A\ (x\ R\ x)$.
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(4) R is antisymmetric: $\forall x, y \in A\ (xRy \wedge yRx \Rightarrow x = y)$.
(5) R is an equivalence relation: R is reflexive, symmetric and transitive.

(b) S is defined by setting $x\ S\ y$ iff $xy > 0$ for all $x, y \in \mathbb{Q}$.

Reflexive? ✗ Counter–example: $0 \not S 0$

Symmetric? ✓ $\forall x, y \in \mathbb{Q}, xy > 0 \ \Rightarrow yx = xy > 0$ (by commutativity of $\times$)

Transitive? ✓

1. Let $x, y, z \in \mathbb{Q}$ such that $xy > 0$ and $yz > 0$.
2. Then $xy \cdot yz = xz \cdot y^2 > 0$ (by associativity and commutativity of $\times$)
3. Either $xz > 0 \wedge y^2 > 0$ or $xz < 0 \wedge y^2 < 0$ (by T25, midterm test Q21)
4. But $y^2 \geq 0\ \forall y \in \mathbb{Q}$, so the second case is a contradiction.
5. Therefore, $xz > 0$ (by elimination and specialisation)

Antisymmetric? ✗ Counter-example: $1\ S\ 2$ and $2\ S\ 1$ but $1 \neq 2$.

Equivalence relation? ✗ S is not reflexive.

Q3. For each of the following relations on $\mathbb{Q}$, determine if it is (i) reflexive, (ii) symmetric, (iii) transitive, (iv) antisymmetric, (v) an equivalence relation.

Let A be a set and R a relation on A.
(1) R is reflexive: $\forall x \in A\ (x\ R\ x)$.
(2) R is symmetric: $\forall x, y \in A\ (xRy \Rightarrow yRx)$.
(3) R is transitive: $\forall x, y, z \in A\ (xRy \wedge yRz \Rightarrow xRz)$.
(4) R is antisymmetric: $\forall x, y \in A\ (xRy \wedge yRx \Rightarrow x = y)$.
(5) R is an equivalence relation: R is reflexive, symmetric and transitive.

(c) T is defined by setting
$x\ T\ y$ iff $|x - y| \leq 2$ for all $x, y \in \mathbb{Q}$.

✓ Reflexive? $\forall x \in \mathbb{Q}, |x - x| = 0 \leq 2$.

✓ Symmetric?
1. $\forall x \in \mathbb{Q}, |x| = |-x|$.
2. $\therefore |x - y| \leq 2 \Rightarrow |-(x - y)| = |y - x| \leq 2$.

✗ Transitive? Counter-example: $-2\ T\ 0$ and $0\ T\ 2$ but $-2\ \not{T}\ 2$.

✗ Antisymmetric? Counter-example: $1\ T\ 2$ and $2\ T\ 1$ but $1 \neq 2$.

✗ Equivalence relation? T is not transitive.

Q4. Define a relation R on $\mathbb{Q}$ as follows

$$x \, R \, y \iff x - y \in \mathbb{Z}$$

(a) Show that R is an equivalence relation.

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1. (Reflexivity) Let $x \in \mathbb{Q}$, then $x - x = 0 \in \mathbb{Z}$. So $x \, R \, x$.
2. (Symmetry)
   2.1 Let $x, y \in \mathbb{Q}$ such that $x \, R \, y$.
   2.2 Then $x - y \in \mathbb{Z}$ (by the definition of R)
   2.3 So $y - x = (-1)(x - y) \in \mathbb{Z}$ (closure of integers under $\times$)
   2.4 Hence, $y \, R \, x$ (by the definition of R)
3. (Transitivity)
   3.1 Let $x, y, z \in \mathbb{Q}$ such that $x \, R \, y$ and $y \, R \, z$.
   3.2 Then $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$ (by the definition of R)
   3.3 So $x - z = (x - y) + (y - z) \in \mathbb{Z}$ (by closure of integers under $+$)
   3.4 Hence, $x \, R \, z$ (by the definition of R)
4. Since R is reflexive, symmetric and transitive, it is an equivalence relation.

Q4. Define a relation R on $\mathbb{Q}$ as follows
$$x\,R\,y \iff x - y \in \mathbb{Z}$$

(b) Find an element a in the equivalence class $\left[\frac{37}{7}\right]$ that satisfies $0 \le a < 1$.

Idea: Look for a rational number r such that $\frac{37}{7} - r$ is an integer.

Many possible answers.

1. Note that $\frac{37}{7} = 5 + \frac{2}{7}$.
2. So $\frac{37}{7} - \frac{2}{7} = 5 \in \mathbb{Z}$.
3. Hence $\frac{37}{7}\ R\ \frac{2}{7}$ and therefore $\mathbf{\frac{2}{7}} \in \left[\frac{37}{7}\right]$.

Other answers include: $\frac{9}{7}, \frac{16}{7}, -\frac{5}{7}$, etc., that is, $\frac{37+7k}{7}\ \forall k \in \mathbb{Z}$.

Q4. Define a relation R on $\mathbb{Q}$ as follows

$$x\ R\ y \iff x - y \in \mathbb{Z}$$

(c) Devise a general method to find, for each given equivalence class $[x]$, where $x \in \mathbb{Q}$, an element $a \in [x]$ such that $0 \le a < 1$.

1. Let $x \in \mathbb{Q}$, then $\exists m, n \in \mathbb{Z}, n \neq 0$ such that $x = \frac{m}{n}$.
2. Without loss of generality, assume $n > 0$. (If $n < 0$ we rewrite the fraction $\frac{m}{n}$ by multiplying top and bottom by -1 to make the denominator positive).
3. Let $a = \frac{m \textbf{ mod } n}{n}$.
4. Then $0 \le a < 1$ (since $0 \le m \textbf{ mod } n < n$ by definition of $\textbf{mod}$)
5. Also, since $m = n(m \textbf{ div } n) + (m \textbf{ mod } n)$, we have

$$x - a = \frac{m}{n} - \frac{m \textbf{ mod } n}{n} = \frac{m - m \textbf{ mod } n}{n} = m \textbf{ div } n \in \mathbb{Z}$$

6. Thus $x\ R\ a$ (by definition of R) and so $a \in [x]$ (by definition of equivalence class)

Q5. Let A, B be nonempty sets and $f: A \to B$ be a surjection.
Show that $\mathcal{C}$ is a partition on A where

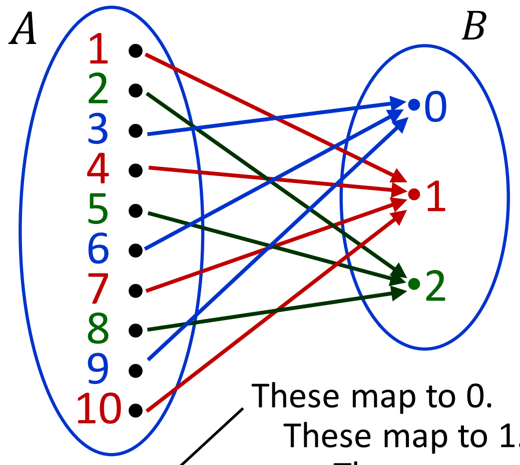
$$\mathcal{C} = \{\{x \in A: f(x) = y\} : y \in B\}$$

Observation:
(1) The components of $\mathcal{C}$ are pairwise disjoint.
(2) Union of all the components of $\mathcal{C}$ is A.

Definition 9.3.1
A partition of a set A is a set $\mathcal{C}$ of nonempty subsets of A s.t.
(≥ 1) $\forall x \in A\ \exists S \in \mathcal{C}\ (x \in S)$; and
(≤ 1) $\forall x \in A\ \forall S, S' \in \mathcal{C}\ (x \in S \wedge x \in S' \Rightarrow S = S')$.
Elements of a partition are called components of the partition.

1. We show that each component of $\mathcal{C}$ is nonempty.
   1.1 Let $S \in \mathcal{C}$.
   1.2 Then $\exists y_0 \in B$ such that $S = \{x \in A: f(x) = y_0\}$ (by the definition of $\mathcal{C}$)
   1.3 Then $\exists x_0 \in A$ such that $f(x_0) = y_0$ (by surjectivity of f)
   1.4 So $x_0 \in S$ and thus S is nonempty.
2. (≥ 1)
   2.1 Let $x_0 \in A$.
   2.2 Define $y_0 = f(x_0)$ and $S = \{x \in A: f(x) = y_0\} \in \mathcal{C}$
   2.3 Then $x_0 \in S$ (as $f(x_0) = y_0$)
3. (≤ 1)
   3.1 Let $x_0 \in A$ and $S, S' \in \mathcal{C}$ such that $x_0 \in S$ and $x_0 \in S'$.
   3.2 Then $\exists y, y' \in B$ such that $S = \{x \in A: f(x) = y\}$ and $S' = \{x \in A: f(x) = y'\}$
   (by the definition of $\mathcal{C}$)
   3.3 Then $f(x_0) = y$ and $f(x_0) = y'$ (as $x_0 \in S$ and $x_0 \in S'$)
   3.4 This implies $y = y'$ (by the functionality of f) and so $S = S'$.
4. From (1), (2) and (3), $\mathcal{C}$ is a partition on A.

Eg:
Let $A = \{1,2,3,4,5,6,7,8,9,10\}, B = \{0,1,2\}$,
and $f: A \to B$ be defined as $f(x) = x \bmod 3$.



These map to 0.
These map to 1.
These map to 2.

Then $\{\{1,4,7,10\}, \{2,5,8\}, \{3,6,9\}\}$ is a partition on A.

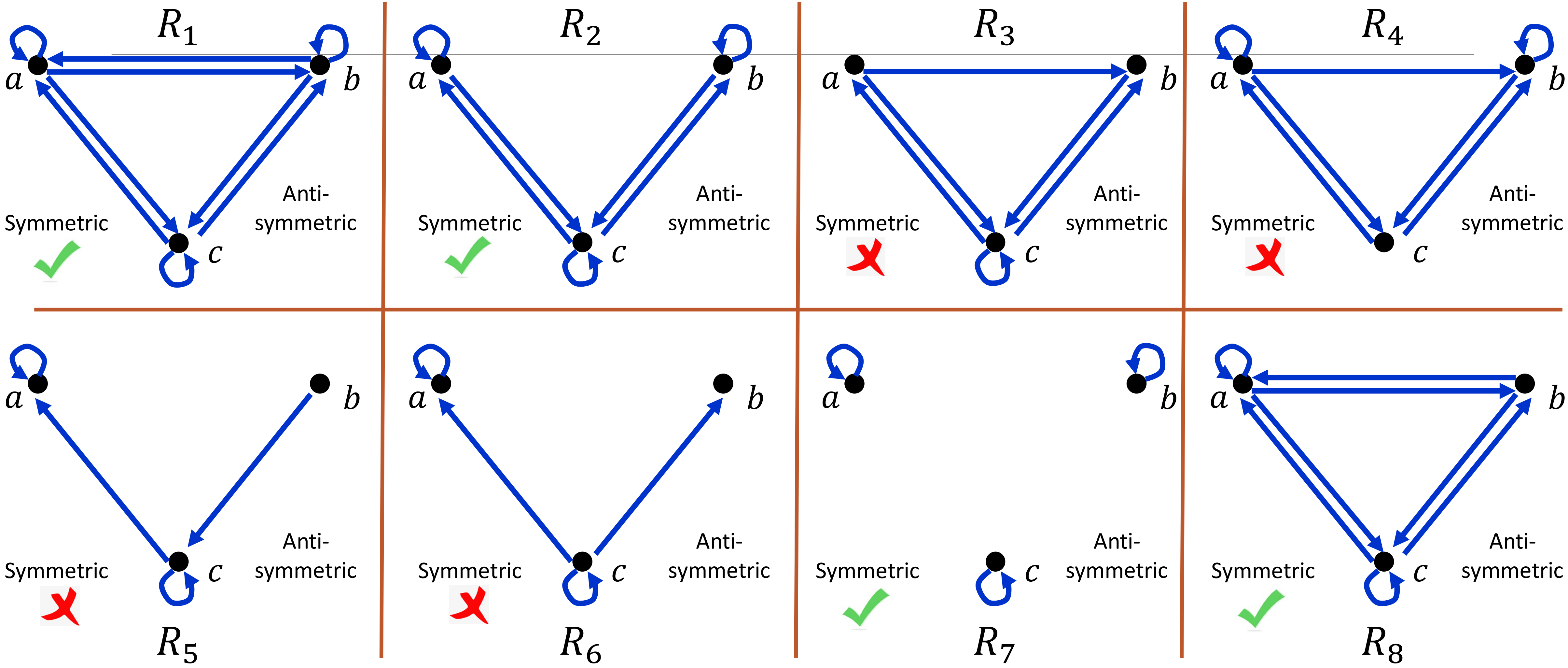
That is, the preimages of 0,1,2 form the respective components of the partition.

Partial orders

R is symmetric: $\forall x, y \in A\ (xRy \Rightarrow yRx)$.

R is antisymmetric: $\forall x, y \in A\ (xRy \wedge yRx \Rightarrow x = y)$.

Definition 9.4.1
Let A be a set and R be a relation on A.
R is a partial order if R is reflexive, antisymmetric and transitive.



Partial orders

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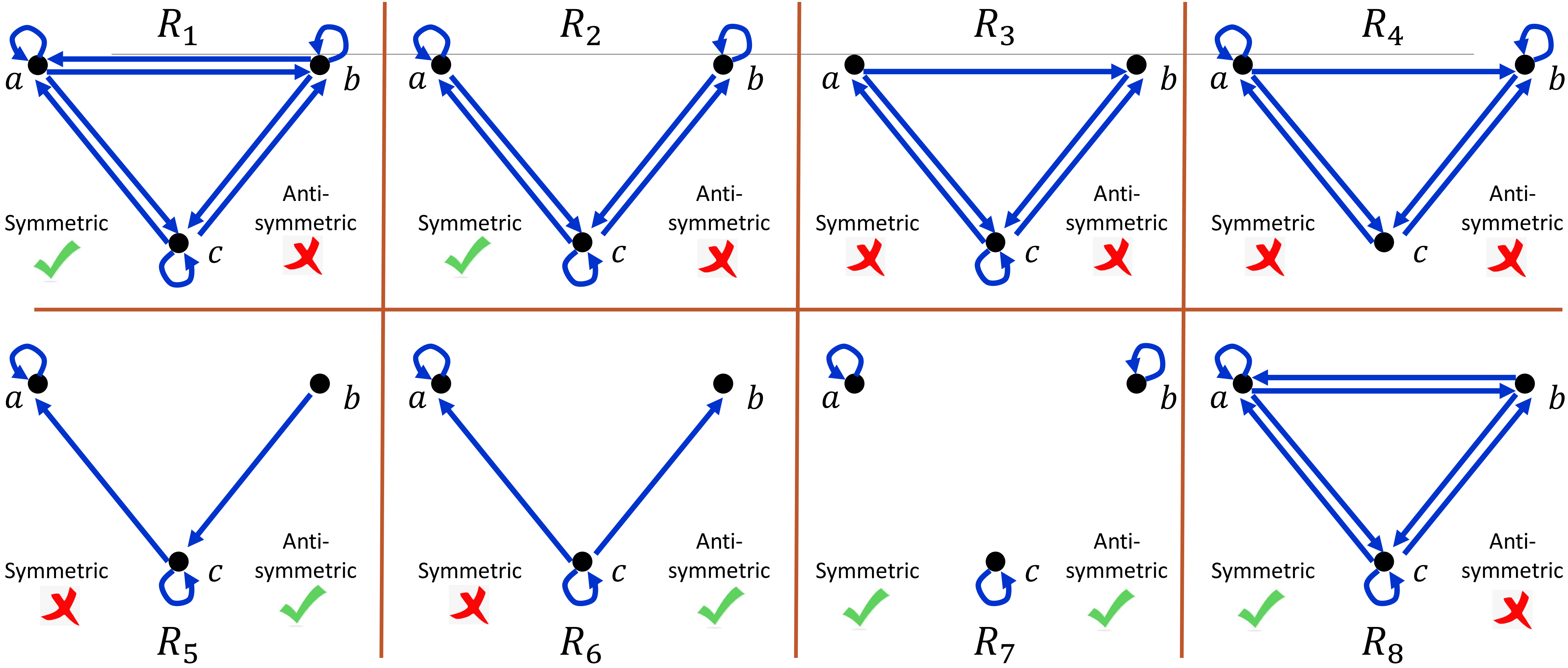


Q6. Considering the "divides" relation of each of the following sets of integers, draw a Hasse diagram and find all largest, smallest, maximal and minimal elements, and a linearization.

(a) $A = \{1, 2, 4, 5, 10, 15, 20\}$

Note that the "divides" relation on integers is a partial order.

Definition 9.5.1

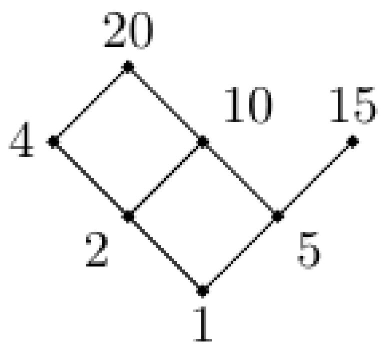
Let $\preccurlyeq$ be a partial order on a set A, and $c \in A$.

(1) c is a minimal element if $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.

(2) c is a maximal element if $\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x)$.

(3) c is the smallest (minimum) element if $\forall x \in A\ (c \preccurlyeq x)$.

(4) c is the largest (maximum) element if $\forall x \in A\ (x \preccurlyeq c)$.



Largest element	No largest
Smallest element	1
Maximal elements	20 and 15
Minimal elements	1
Linearization	$\leq$ on A (many other answers)

Q6. Considering the "divides" relation of each of the following sets of integers, draw a Hasse diagram and find all largest, smallest, maximal and minimal elements, and a linearization.

(b) $B = \{2, 3, 4, 6, 8, 9, 12, 18\}$

Definition 9.5.1

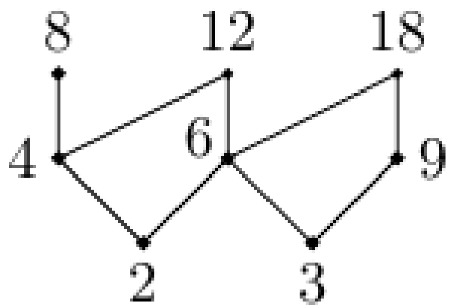
Let $\preccurlyeq$ be a partial order on a set A, and $c \in A$.

(1) c is a minimal element if $\forall x \in A\ (x \preccurlyeq c \Rightarrow c = x)$.

(2) c is a maximal element if $\forall x \in A\ (c \preccurlyeq x \Rightarrow c = x)$.

(3) c is the smallest (minimum) element if $\forall x \in A\ (c \preccurlyeq x)$.

(4) c is the largest (maximum) element if $\forall x \in A\ (x \preccurlyeq c)$.



Largest element	No largest
Smallest element	No smallest
Maximal elements	8, 12 and 18
Minimal elements	2 and 3
Linearization	$\leq$ on B (many other answers)

Q7. Let $\preccurlyeq$ be a partial order on a set P, and $a, b \in P$

- a, b are **comparable** if $a \preccurlyeq b$ or $b \preccurlyeq a$.
- a, b are **compatible** if $\exists c \in P$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

(a) Is it true that in all partially ordered sets, any two comparable elements are compatible?

YES

1. Let $a, b \in P$ be comparable.
2. Then either $a \preccurlyeq b$ or $b \preccurlyeq a$ (by definition of comparability)
3. Case 1: $a \preccurlyeq b$
   - 3.1 Let $c = b$.
   - 3.2 $\exists c \in P$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$ (since $b \preccurlyeq b$ by reflexivity).
   - 3.3 Therefore a and b are compatible (by the definition of compatibility)
4. Case 2: $b \preccurlyeq a$
   - 4.1 Let c = a
   - 4.2 $\exists c \in P$ such that $b \preccurlyeq c$ and $a \preccurlyeq c$ (since $a \preccurlyeq a$ by reflexivity).
   - 4.3 Therefore a and b are compatible (by the definition of compatibility)
5. In all cases a and b are compatible.

Q7. Let $\preccurlyeq$ be a partial order on a set P, and $a, b \in P$

- a, b are **comparable** if $a \preccurlyeq b$ or $b \preccurlyeq a$.
- a, b are **compatible** if $\exists c \in P$ such that $a \preccurlyeq c$ and $b \preccurlyeq c$.

(b) Is it true that in all partially ordered sets, any two compatible elements are comparable?

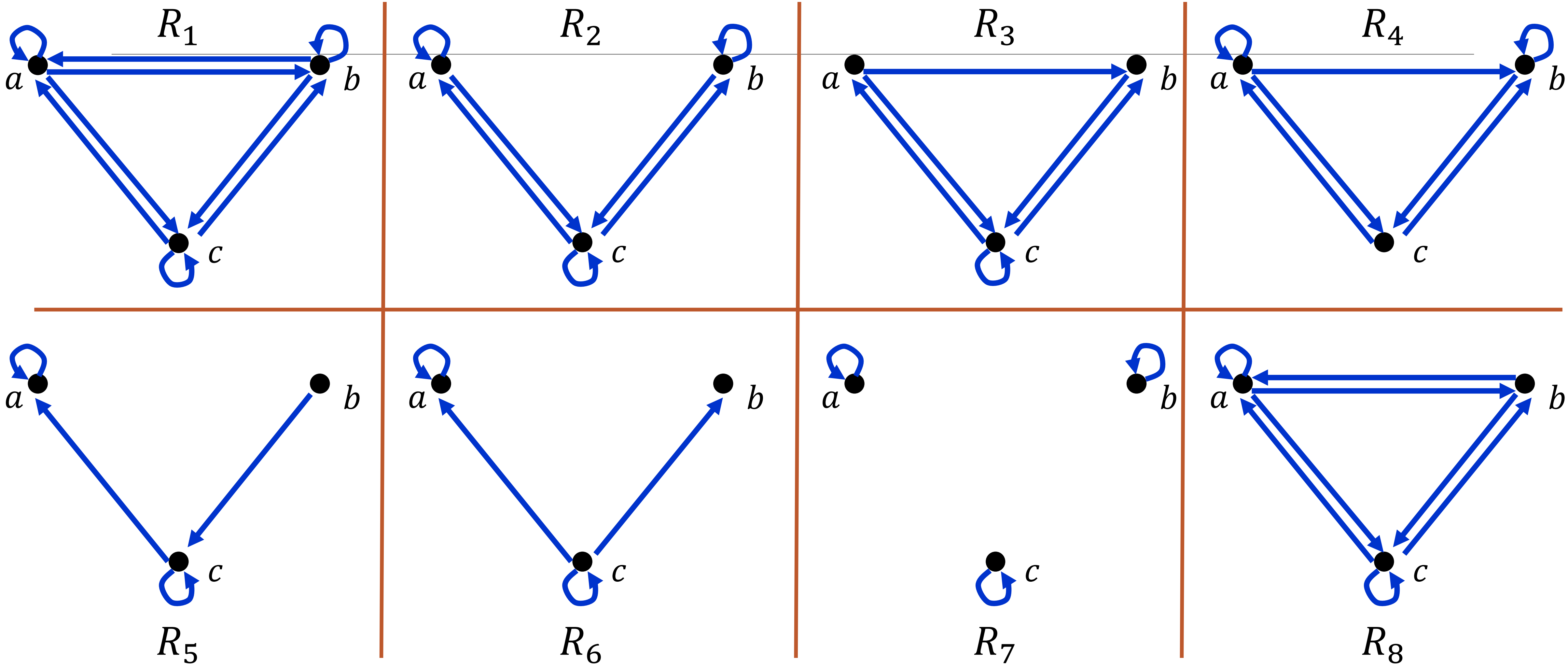
NO

1. Consider the "divides" relation $|$ on $\mathbb{Z}^+$ which is a partial order on $\mathbb{Z}^+$.
2. For $2, 3 \in \mathbb{Z}^+$, $2 \mid 6$ and $3 \mid 6$, so 2 and 3 are compatible.
3. However, $2 \nmid 3$ and $3 \nmid 2$, so 2 and 3 are not comparable.

Given a set $A = \{a, b, c\}$ and a relation defined on A as follows.

Which of the relations below are REFLEXIVE?

Let A be a set and R a relation on A.
R is reflexive: $\forall x \in A\ (x\ R\ x)$.



QUIZ TIME!

Given a set $A = \{a, b, c\}$ and a relation defined on A as follows.

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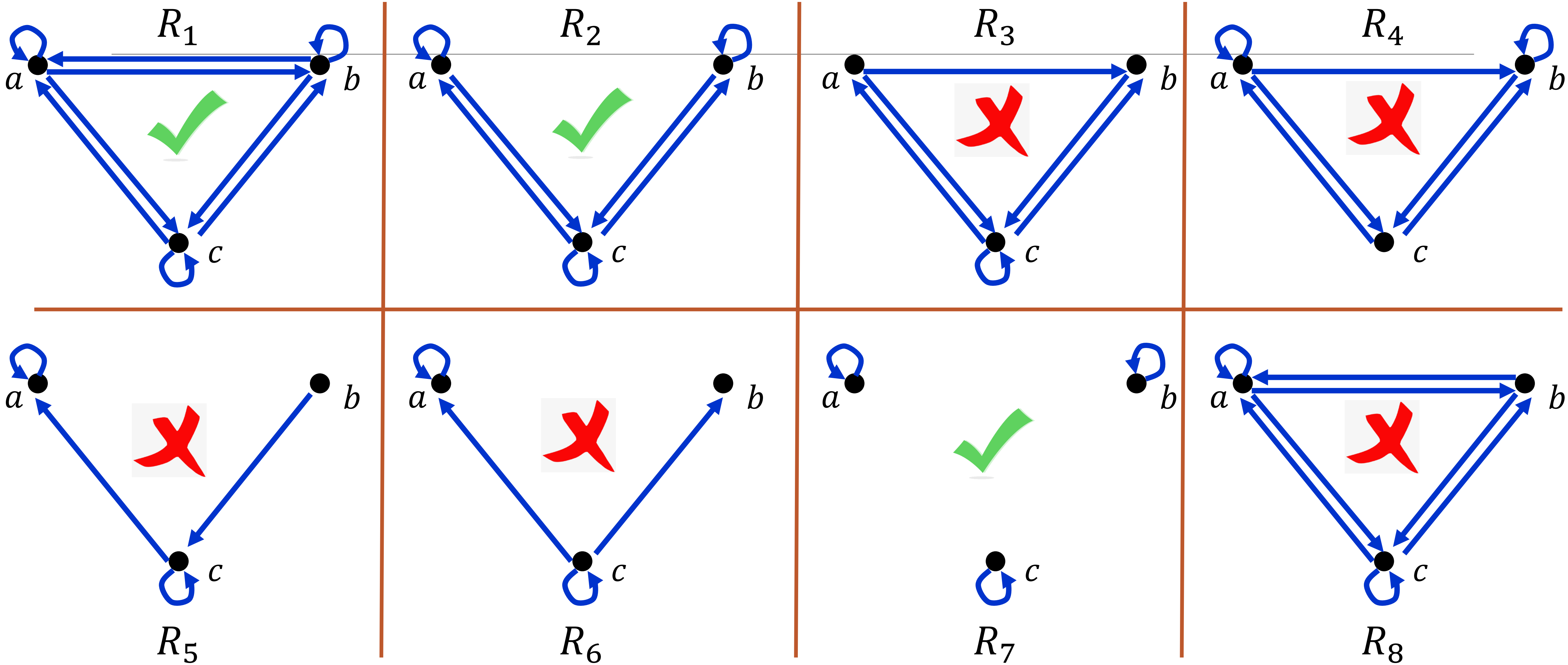
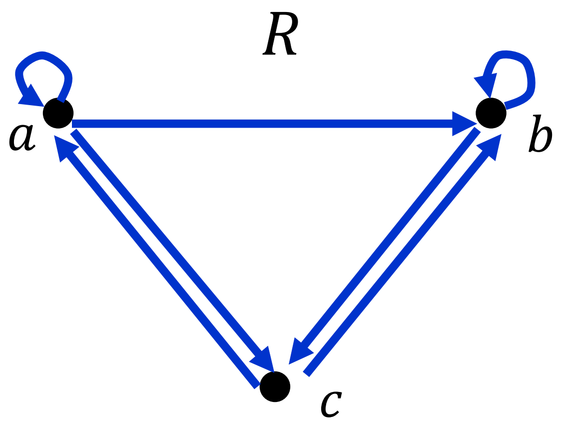


Given a set $A = \{a, b, c\}$ and a relation defined on A as follows.

Common mistake



It is wrong to say that "a is reflexive", "b is reflexive", "c is not reflexive". We say $a\ R\ a$ (a is related to itself), $c \not R\ c$ (c is not related to itself).

We either say the relation R is reflexive or not reflexive. We don't say an element of A is reflexive or not reflexive.

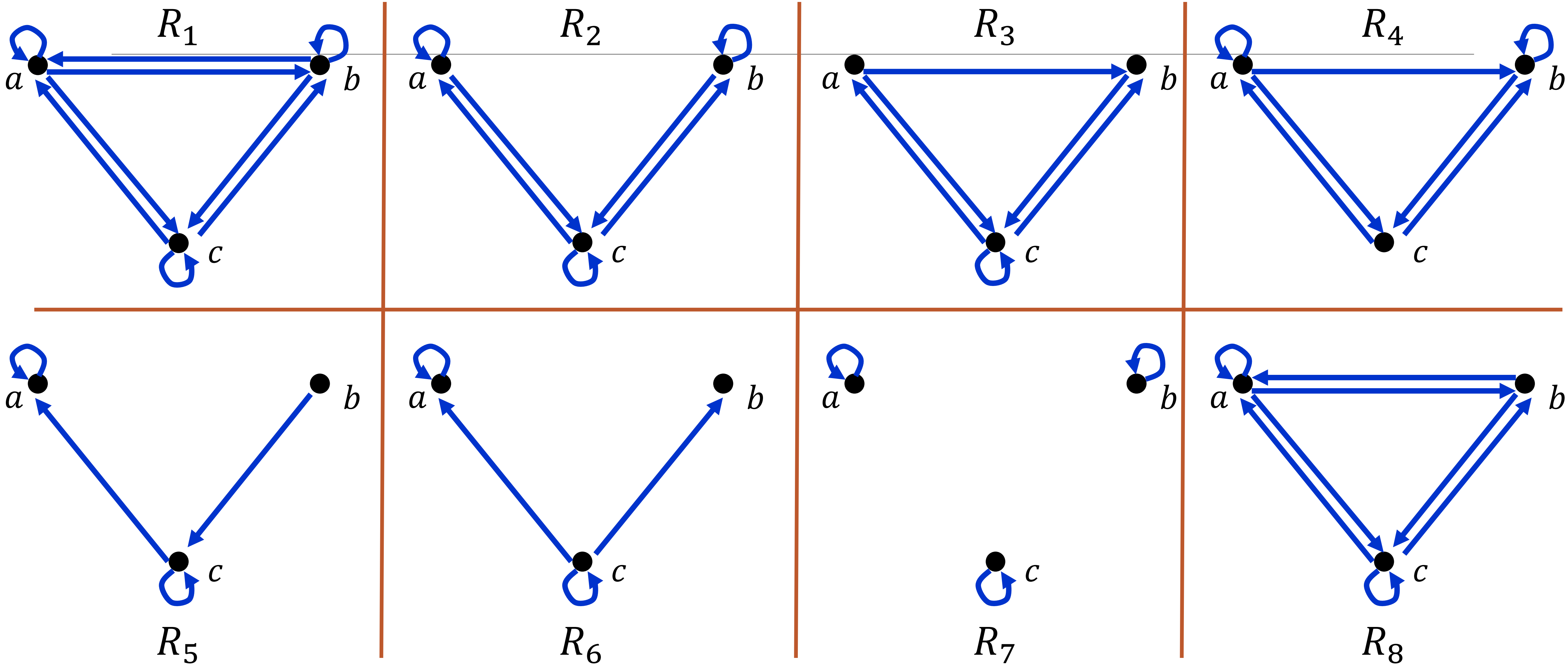
Reflexivity, symmetry and transitivity are properties of relations, not individual elements of A.

QUIZ TIME!

Given a set $A = \{a, b, c\}$ and a relation defined on A as follows.

Which of the relations below are SYMMETRIC?

Let A be a set and R a relation on A.
R is symmetric: $\forall x, y \in A\ (xRy \Rightarrow yRx)$.



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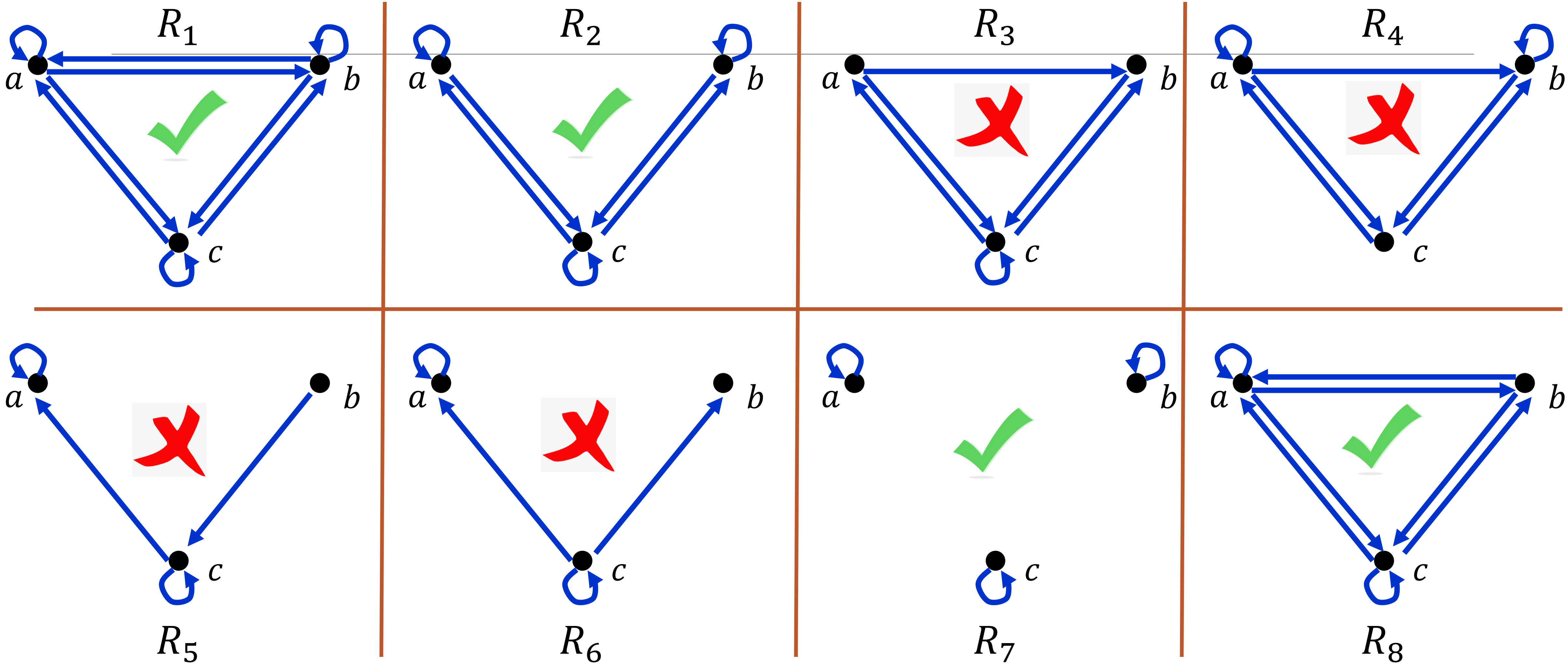
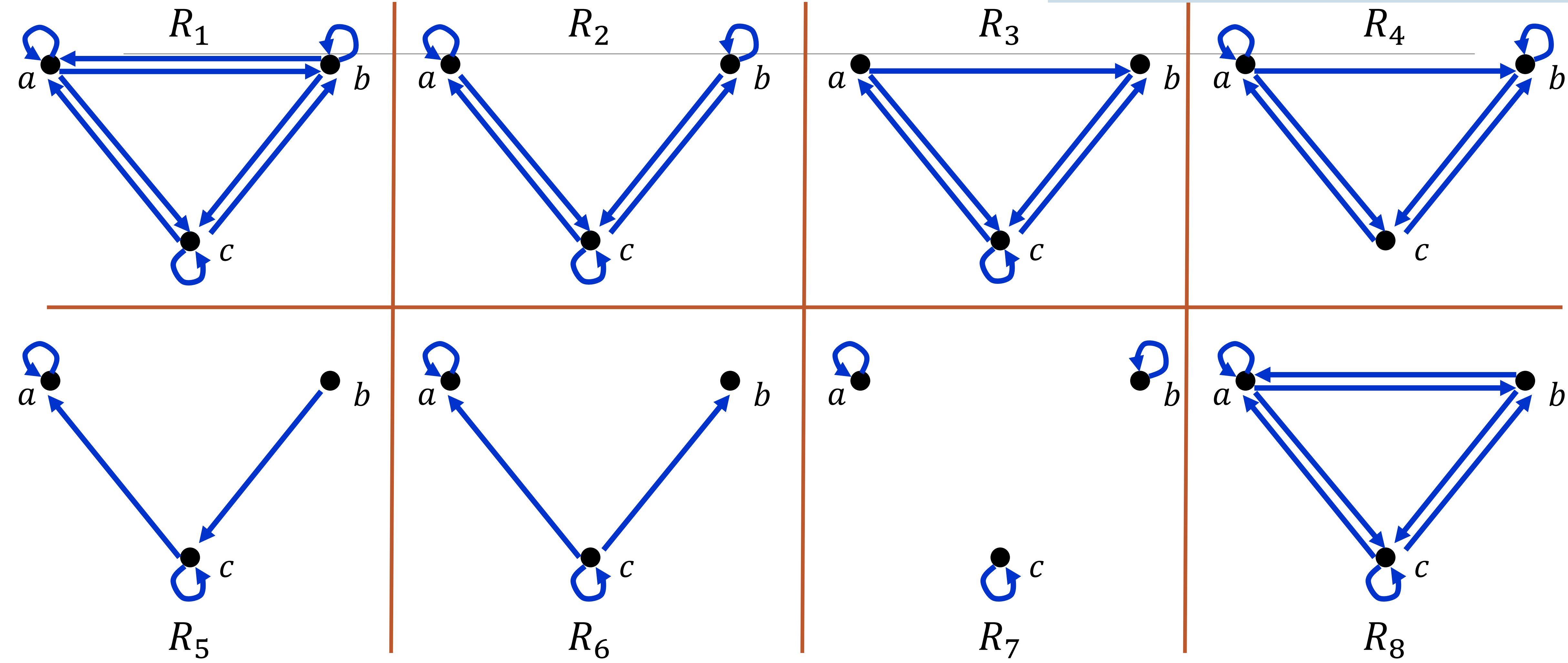


Given a set $A = \{a, b, c\}$ and a relation defined on A as follows.

Which of the relations below are TRANSITIVE?

Let A be a set and R a relation on A.
R is transitive:

$$\forall x, y, z \in A \ (xRy \land yRz \Rightarrow xRz).$$



Given a set $A = \{a, b, c\}$ and a relation defined on A as follows.

Which of the relations below are TRANSITIVE?

Let A be a set and R a relation on A.
R is transitive:

$$\forall x, y, z \in A \ (xRy \wedge yRz \Rightarrow xRz).$$