CS1231S TUTORIAL #9

Counting 1

Learning objectives of this tutorial

Counting

- Applying multiplication rule, addition rule, difference rule and the inclusion/exclusion rule.
- Applying permutation.
- Learning about circular permutation.
- Solving problems using the Pigeonhole Priciple.

Q1.

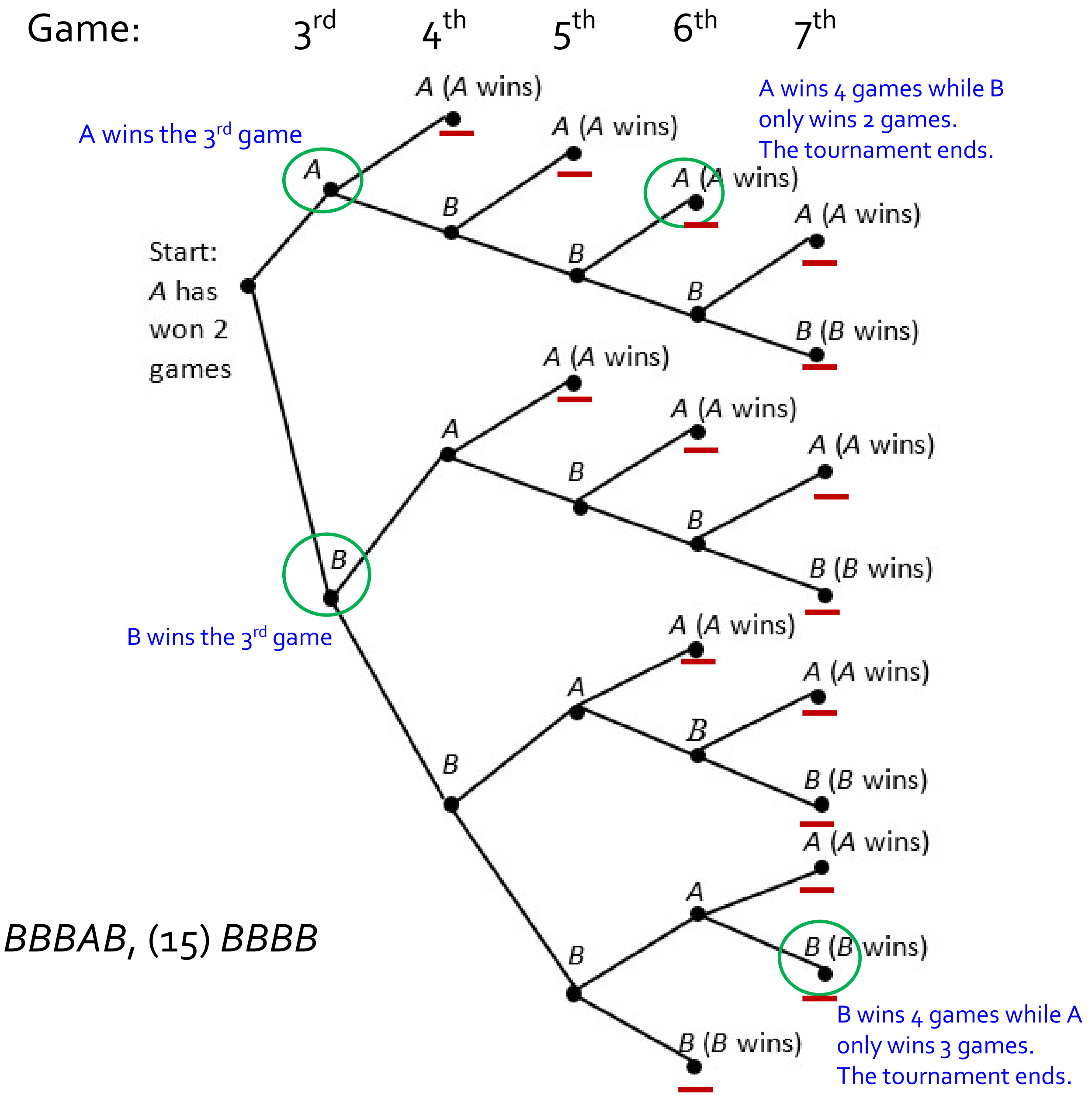
First team to win 4 games wins the tournament. There are two teams *A* and *B*. Team *A* wins the first 2 games. How many ways can the tournament be completed?

We draw the possibility tree for this problem.

At each state (node), we have 2 possible scenarios, A wins the current game or B wins the current game.

There are 15 ways in total.

By listing out:
(1) *AA*, (2) *ABA*, (3) *ABBA*, (4) *ABBBA*, (5) *ABBBB*,
(6) *BAA*, (7) *BABA*, (8) *BABBA*, (9) *BABBB*,
(10) *BBAA*, (11) *BBABA*, (12) *BBABB*, (13) *BBBAA*, (15) *BBBAB*, (15) *BBBB*



Q2. (Past year's exam question.)

Lock with 40 positions. To unlock, rotate clockwise, then counterclockwise, then clockwise. Consecutive numbers not allowed. How many codes are there?

Is the direction (clockwise, counterclockwise) important?

No, we only care about what is the position we need to arrive at

There are 40 possible positions that the first number can be.

Since the second number must be different than the first one, there are 39 (40 - 1) possible positions that the second number can be.

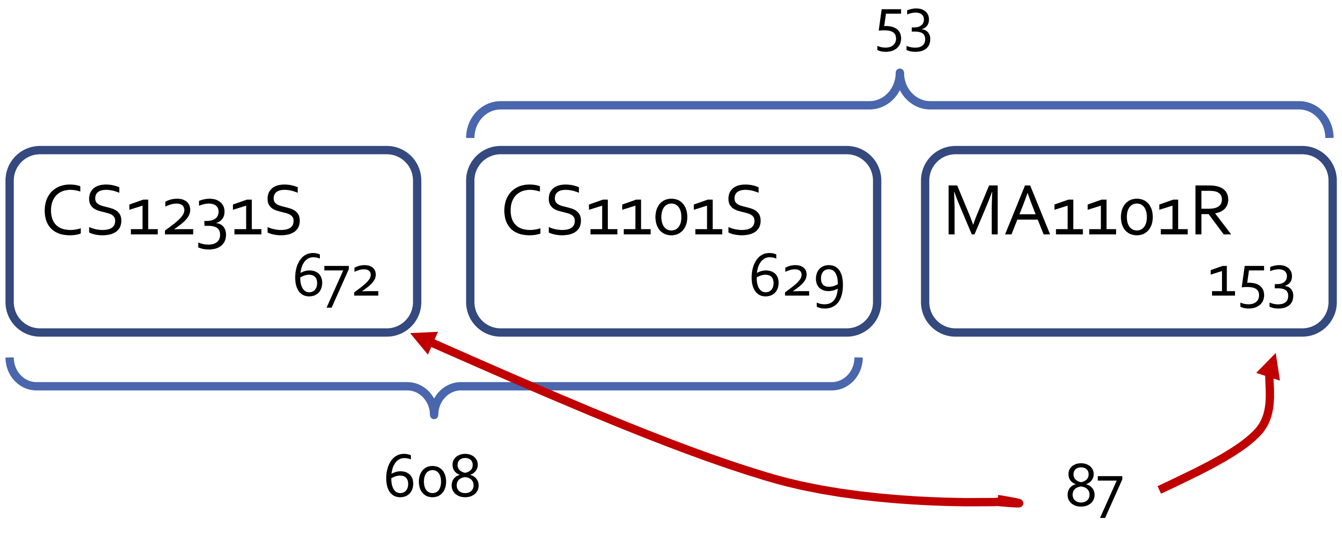
Since the third number must be different than the second one, there are 39 (40 - 1) possible positions that the third number can be.

By Theorem 9.2.1 The multiplication rule,
there are 40 x 39 x 39 = 60840 combinations.

Q3. CS students: 789

All 3 modules: 46

How many CS students are not taking any of these 3 modules?



What theorem should you use?

Theorem 9.3.3 Inclusion/Exclusion Rule

1. Let A, B and C be the sets of CS students taking CS1231S, CS1101S and MA1101R respectively.
2. $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
   $= 672 + 629 + 153 - 608 - 87 - 53 + 46 = 752$
3. $|\bar{A} \cap \bar{B} \cap \bar{C}| = \left|\overline{(A \cup B \cup C)}\right| = |U| - |A \cup B \cup C| = 789 - 752 = \mathbf{37}$.
4. There are **37** CS students who are not taking any of these three modules.

Q4. Among all permutations of n positive integers from 1 through n, where $n \geq 3$, how many of them have integers 1, 2 or 3 in the correct position?

Theorem 9.3.3 Inclusion/Exclusion Rule again!

Let P_k be the permutations with integer k in the correct position.

What is $|P_1|$? $|P_2|$? $|P_3|$? $|P_1| = |P_2| = |P_3| = (n-1)!$

What is $|P_1 \cap P_2|$? $|P_2 \cap P_3|$? $|P_1 \cap P_3|$?

$$|P_1 \cap P_2| = |P_2 \cap P_3| = |P_1 \cap P_3| = (n-2)!$$

What is $|P_1 \cap P_2 \cap P_3|$? $|P_1 \cap P_2 \cap P_3| = (n-3)!$

By the inclusion/exclusion rule (theorem 9.3.3),

$$\begin{aligned}|P_1 \cup P_2 \cup P_3| &= 3(n-1)! - 3(n-2)! + (n-3)! \\ &= (3n^2 - 12n + 13)(n-3)!\end{aligned}$$

Q5.

Given n boxes numbered 1 to n, each box to be filled with a white ball or a blue ball. At least one box contains a white ball and boxes containing white balls must be consecutively numbers.

What is the total number of ways this can be done?

Case 1: A white ball in a box. n

Case 2: White balls in 2 consecutive boxes. $n-1$

Case 3: White balls in 3 consecutive boxes. $n-2$

⋮

Case n: White balls in n consecutive boxes. 1

What rule to apply? Addition rule.

1. For k $(1 \le k \le n)$ consecutively numbered boxes that contain white balls, there are $n-k+1$ ways.

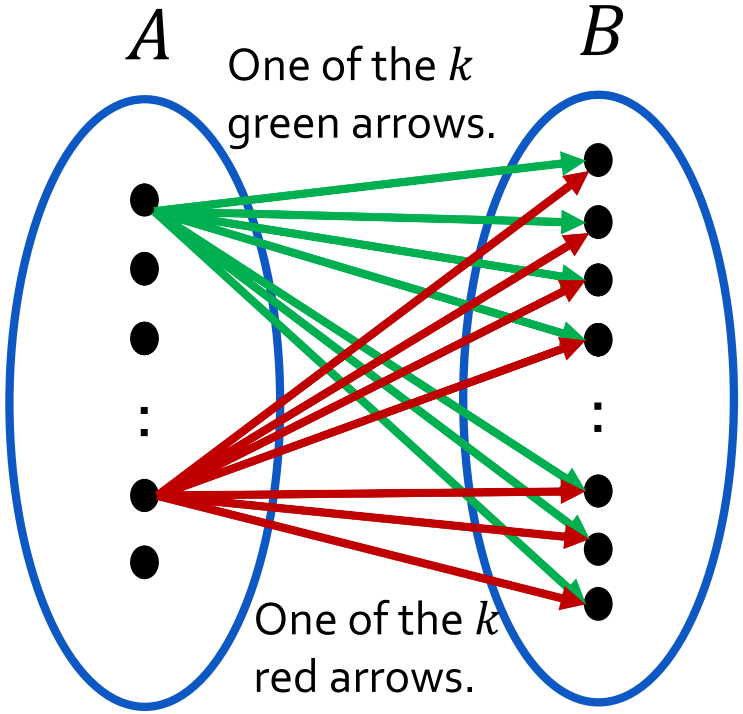
2. Therefore, total number of ways is

$$\sum_{k=1}^{n}(n-k+1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Q6.

In Tutorial #4 D1, you are asked to write down all possible functions $\{a, b, c\} \rightarrow \{1,2\}$.

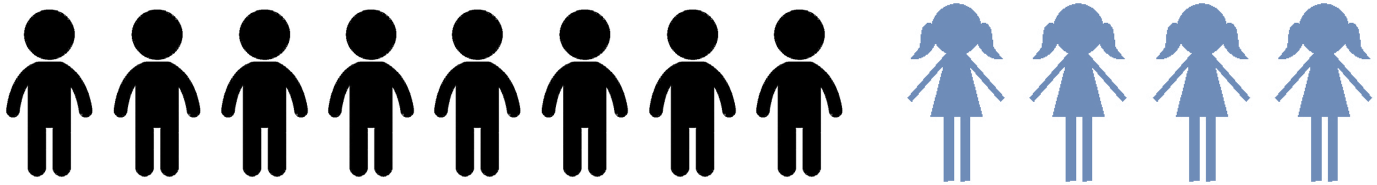
How many possible functions $f: A \rightarrow B$ are there if $|A| = n$ and $|B| = k$?



Each of the n elements in A must be mapped to one of the k elements in B.

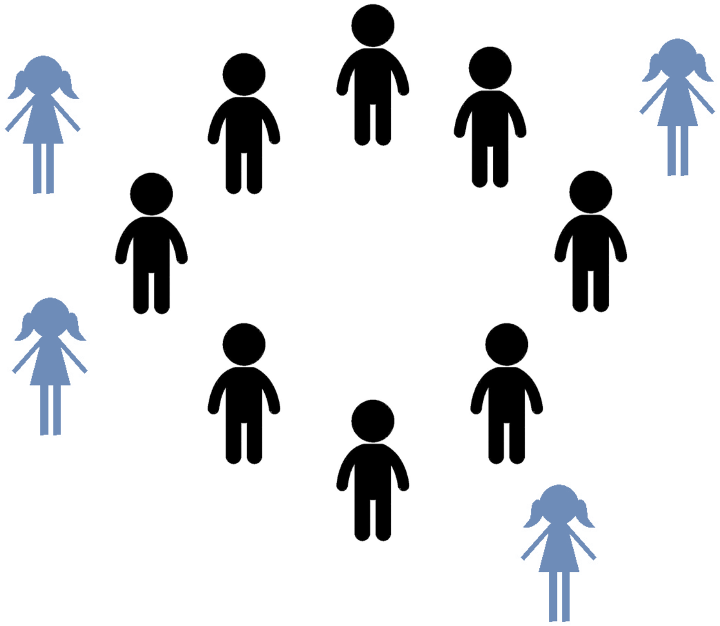
Therefore, there are k^n possible functions f.

Q7.



(a) In how many ways can 8 boys and 4 girls sit around a circular table, so that no two girls sit together?

We sit the boys first, then sit each girl in between the boys so that no two girls will sit together.

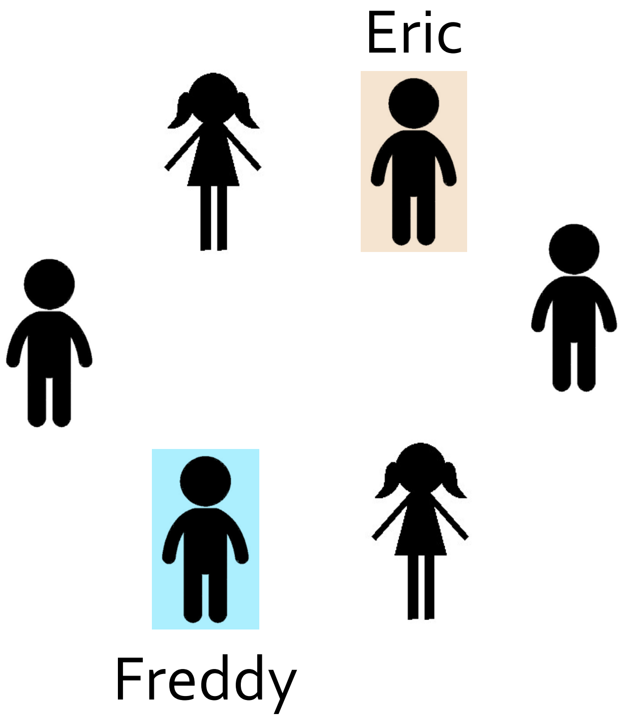


8 boys can be seated in a circle in $7!$ ways.

There are 8 spaces between the boys, which can be occupied by 4 girls in $P(8,4)$ ways.

Hence,
total number of ways = $7! \times P(8,4) =$ $5040 \times 1680 = \mathbf{8467200}$.

Q7. (b) In how many ways can 6 people sit around a circular table, but Eric would not sit next to Freddy?

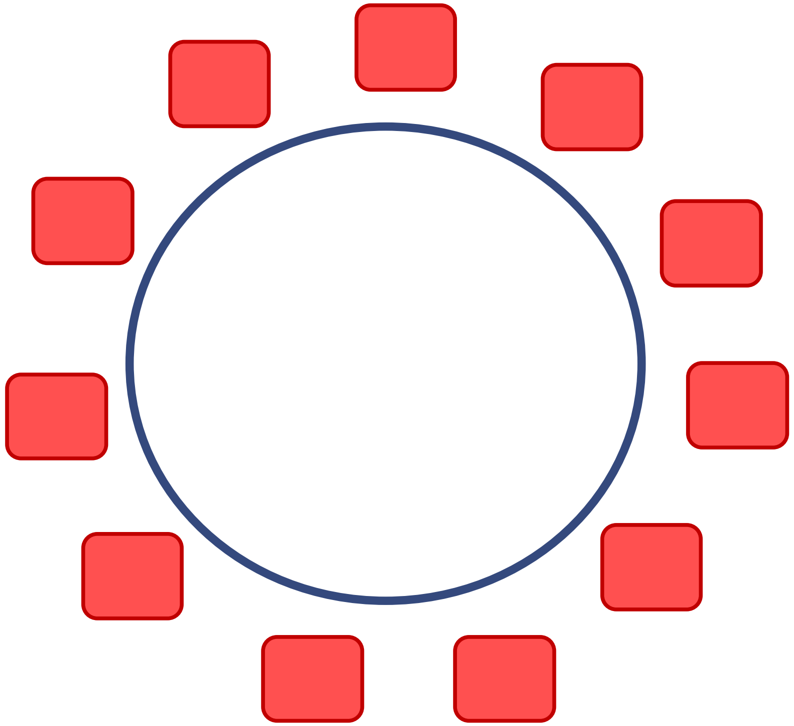


There are $5! = 120$ ways for 6 people to sit around a circular table.

There are $2 \times 4! = 48$ ways for Eric and Freddy to sit together.

Therefore, the answer is $120 - 48 =$ **72** ways.

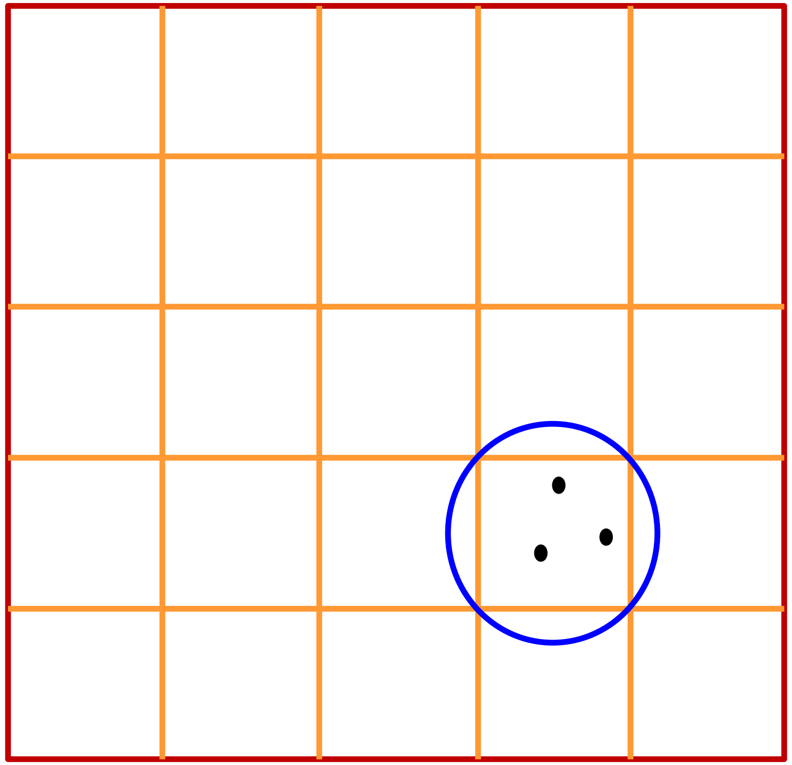
Q7. (c) In how many ways can $n-1$ people sit around a circular table with n chair?



Treat the empty chair as just another person.

Therefore, there are $(\boldsymbol{n}-\mathbf{1})!$ ways to seat $n-1$ people around a table with n chairs.

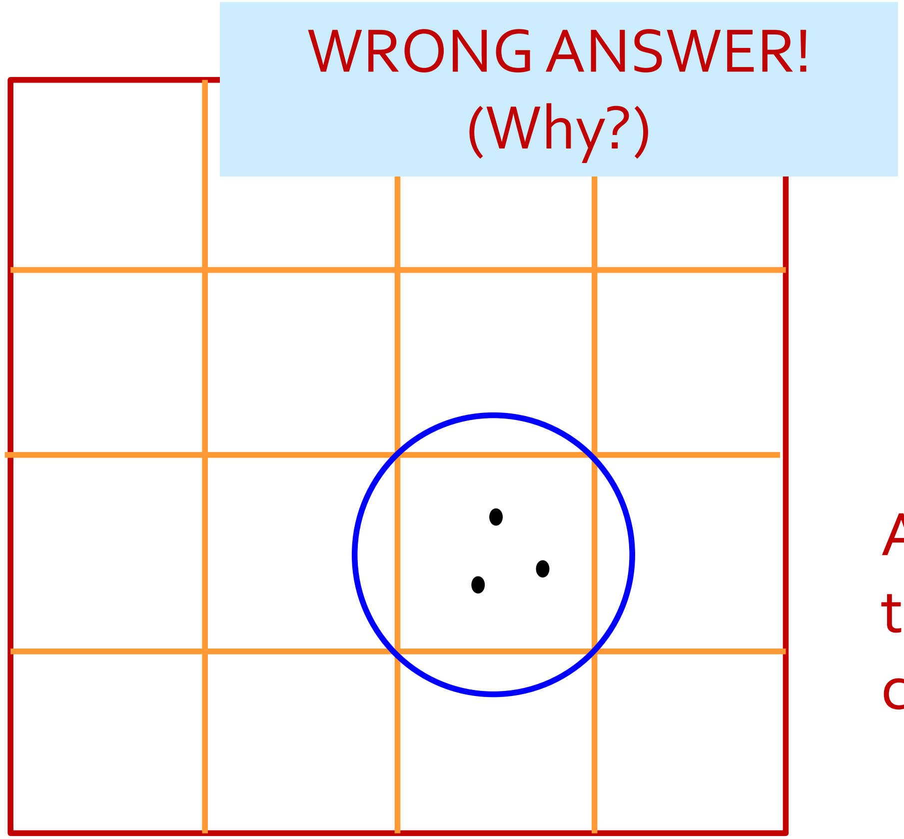
Q8. Prove that if you randomly put 51 points inside a unit square, there are always 3 points that can be covered by a circle of radius 1/7.



1. Divide the unit square into 25 equal smaller squares of side 1/5 each.
2. Now, at least one of these small squares would contain at least 3 points. (Why?)
3. Now, the circle circumvented around the small square with the 3 points inside also contains these 3 points as it has radius

$$\sqrt{\left(\frac{1}{10}\right)^2+\left(\frac{1}{10}\right)^2}=\sqrt{\frac{1}{50}}<\sqrt{\frac{1}{49}}=\frac{1}{7}$$

Q8. Prove that if you randomly put 51 points inside a unit square, there are always 3 points that can be covered by a circle of radius 1/7.



1. Divide the unit square into 16 equal smaller squares of side 1/4 each.
2. Now, at least one of these small squares would contain at least 3 points. (Why?)
3. Now, the area of a small square is 1/16=0.0625, which is smaller than the area of a circle of radius 1/7 which is 0.0641.

Although the area of the small square < area of the circle, the radius of the smallest circle that covers the small square completely is

$$\sqrt{\left(\frac{1}{8}\right)^2 + \left(\frac{1}{8}\right)^2} = \sqrt{\frac{1}{32}} > \frac{1}{7}$$

Q9. Let $S = \{3,4,5,6,7,8,9,10,11,12\}$. What is the smallest number of integers you must choose from S such that two of them sum to 15?

1. Partition the set S into the following 5 subsets, where each subset contains a pair of integers that sum to 15: $\{3,12\}, \{4,11\}, \{5,10\}, \{6,9\}, \{7,8\}$.

   5 pigeonholes.

2. For any $n \leq 5$, we can choose n elements such that each element belongs to a different subset. Then we won't be able to find two elements among them that sum to 15.
3. However, if more than 5 integers are chosen from set S, 2 of them must be from the same subset according to PHP.
4. Therefore, the smallest n is **6**.

Q10.

(Past year's exam question.)

In a city, houses are randomly assigned distinct numbers between 1 and 50 inclusive. What is the minimum number of houses to ensure that there are 5 houses numbered consecutively?

For example, the number of houses cannot be 10 because we can choose to number the houses 1, 8, 9, 15, 18, 21, 22, 23, 24, 32, hence no 5 houses are numbered consecutively.

Split the numbers into 10 pigeonholes: 1-5, 6-10, 11-15, ..., 46-50. Therefore, there must be at least $10 \times 4 + 1 =$ **41** houses (pigeons).

If there are 40 houses, then you may label the houses 1-4, 6-9, 11-14, 16-19, 21-24, 26-29, 31-34, 36-39, 41-44, and 46-50.

THE END