CS1231S Tutorial 10

Counting 2

(Slides by Prof Aaron and Audrey Felicio Anwar, edited by Theodore Leebrant)

Recap

Theorem 9.5.1 Formula for $\binom{n}{r}$

The number of subsets of size *r* (or *r*-combinations) that can be chosen from a set of *n* elements, $\binom{n}{r}$, is given by the formula

$$\binom{n}{r} = \frac{P(n,r)}{r!}$$

or, equivalently,

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

where *n* and *r* are non-negative integers with $r \leq n$.

Recap

Theorem 9.5.2 Permutations with sets of indistinguishable objects

Suppose a collection consists of n objects of which

n_1 are of type 1 and are indistinguishable from each other
n_2 are of type 2 and are indistinguishable from each other
:
n_k are of type k and are indistinguishable from each other

and suppose that $n_1 + n_2 + \dots + n_k = n$. Then the number of distinguishable permutations of the n objects is

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\cdots-n_{k-1}}{n_k}$$

$$= \frac{n!}{n_1!n_2!n_3!\cdots n_k!}$$

Recap

Theorem 9.6.1 Number of *r*-combinations with Repetition Allowed

The number of *r*-combination with repetition allowed (multisets of size *r*) that can be selected from a set of *n* elements is:

$$\binom{r+n-1}{r}$$

This equals the number of ways *r* objects can be selected from *n* categories of objects with repetitions allowed.

	Order Matters	Order Does Not Matter
Repetition Is Allowed	n^k	$\binom{k+n-1}{k}$
Repetition Is Not Allowed	$P(n, k)$	$\binom{n}{k}$

Recap

Theorem 9.7.1 Pascal's Formula

Let n and r be positive integers, $r \leq n$. Then

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$$

Theorem 9.7.2 Binomial Theorem

Given any real numbers a and b and any non-negative integer n,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

Recap

Definition: Expected Value

Suppose the possible outcomes of an experiment, or random process, are real numbers $a_1, a_2, a_3, \cdots, a_n$ which occur with probabilities $p_1, p_2, p_3, \cdots, p_n$. The **expected value** of the process is

$$\sum_{k=1}^{n} a_k p_k = a_1 p_1 + a_2 p_2 + a_3 p_3 + \cdots + a_n p_n$$

Linearity of Expectation

The expected value of the sum of random variables is equal to the sum of their individual expected values, regardless of whether they are independent. For random variables X and Y,

$$E[X + Y] = E[X] + E[Y]$$

For random variables $X_1, X_2, \cdots, X_n$ and constants $c_1, c_2, \cdots, c_n$,

$$E\left[\sum_{i=1}^{n} c_i \cdot X_i\right] = \sum_{i=1}^{n} (c_i \cdot E[X_i])$$

Recap

Definition: Conditional Probability

Let A and B be events in a sample space S. If $P(A) \neq 0$, then the **conditional probability of B given A**, denoted $P(B|A)$, is

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad 9.9.1$$

$$P(A \cap B) = P(B|A) \cdot P(A) \quad 9.9.2$$

$$P(A) = \frac{P(A \cap B)}{P(B|A)} \quad 9.9.3$$

Theorem 9.9.1 Bayes' Theorem

Suppose that a sample space S is a union of mutually disjoint events $B_1, B_2, B_3, \ldots, B_n$. Suppose A is an event in S, and suppose A and all the B_i have non-zero probabilities. If k is an integer with $1 \leq k \leq n$, then

$$P(B_k|A) = \frac{P(A|B_k) \cdot P(B_k)}{P(A|B_1) \cdot P(B_1) + P(A|B_2) \cdot P(B_2) + \cdots + P(A|B_n) \cdot P(B_n)}$$

Q1. How many teams of 5 members can you have if:

6

Designer

12

Consultant

20

Programmer

(a) The team is made up completely of programmers.

$$\binom{20}{5} = \mathbf{15,504}$$

Q1. How many teams of 5 members can you have if:

(b) The team must have at least 1 programmer.

6

Designer

12

Consultant

20

Programmer

1. Total number of possible teams: $\binom{6+12+20}{5} = 501{,}942.$
2. Teams with no programmers
   - 2.1 Teams with 5 consultants: $\binom{12}{5} = 792.$
   - 2.2 Teams with 4 consultants and 1 designer: $\binom{12}{4} \times \binom{6}{1} = 495 \times 6 = 2{,}970.$
   - 2.3 Teams with 3 consultants and 2 designers: $\binom{12}{3} \times \binom{6}{2} = 220 \times 15 = 3{,}300.$
   - 2.4 Teams with 2 consultants and 3 designers: $\binom{12}{2} \times \binom{6}{3} = 66 \times 20 = 1{,}320.$
   - 2.5 Teams with 1 consultant and 4 designers: $\binom{12}{1} \times \binom{6}{4} = 12 \times 15 = 180.$
   - 2.6 Teams with 5 designers: $\binom{6}{5} = 6.$
   - 2.7 Total #teams with no programmers: $792 + 2{,}970 + 3{,}300 + 1{,}320 + 180 + 6 = 8{,}568.$
3. Therefore, #teams with at least 1 programmer: $501{,}942 - 8{,}568 = \mathbf{493{,}374}.$

Q1. How many teams of 5 members can you have if:

(c) The team must have at least 2 programmers, at least 1 designer and at least 1 business consultant.

6

Designer

12

Consultant

20

Programmer

1. Case 1: 3 Programmers, 1 Designer, 1 Consultant

$$\binom{20}{3}\binom{6}{1}\binom{12}{1} = 1140 \times 6 \times 12 = 82080.$$

2. Case 2: 2 Programmers, 2 Designers, 1 Consultant

$$\binom{20}{2}\binom{6}{2}\binom{12}{1} = 190 \times 15 \times 12 = 34200.$$

3. Case 3: 2 Programmers, 1 Designer, 2 Consultants

$$\binom{20}{2}\binom{6}{1}\binom{12}{2} = 190 \times 6 \times 66 = 75240.$$

4. Total: $82080 + 34200 + 75240 = 191520$

Q2. \$25m of funds for 15 projects. Funding amounts in units of \$1m.

(a) How many ways can you fund the 15 projects?

Multiset problem with $n = 15, r = 25$.

$$\binom{25+15-1}{25} = \binom{39}{25} = \mathbf{15,084,503,396}.$$

(b) Must provide exactly \$3m for one project, at at least \$2m for each of 5 other projects. How many ways can you fund the 15 projects?

14 projects left to fund with $\$12m$ left.
Multiset problem with $n = 14, r = 12$.

$$\binom{12+14-1}{12} = \binom{25}{12} = \mathbf{5,200,300}.$$

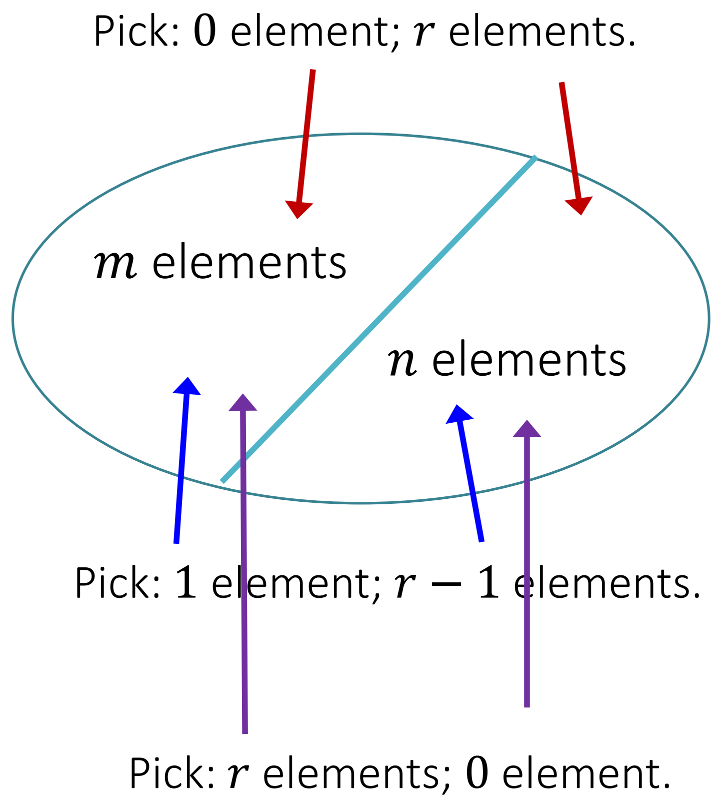
Q3. Think of a set with $m+n$ elements as composed of 2 parts, one with m elements and the other with n elements. Give a combinatorial argument to show that:

$$\binom{m+n}{r} = \boxed{\binom{m}{0}\binom{n}{r}} + \boxed{\binom{m}{1}\binom{n}{r-1}} + \cdots + \boxed{\binom{m}{r}\binom{n}{0}} \quad \ldots \text{(A)}$$

where $m, n \in \mathbb{Z}^+, r \leq m$ and $r \leq n$.

Selecting r elements from $m+n$ elements can be viewed as:

dividing into the cases of selecting k elements from the m elements and the remaining $(r-k)$ elements from the n element, for $0 \leq k \leq r$.



Q3.

$$\binom{m+n}{r} = \binom{m}{0}\binom{n}{r} + \binom{m}{1}\binom{n}{r-1} + \cdots + \binom{m}{r}\binom{n}{0} \qquad \ldots \text{(A)}$$

where $m, n \in \mathbb{Z}^+, r \le m$ and $r \le n$.

Using equation (A), prove that for all integers $n \geq 0$,

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

Let $m = n = r$, then equation (A) becomes

$$\binom{2r}{r} = \binom{r}{0}\binom{r}{r} + \binom{r}{1}\binom{r}{r-1} + \cdots + \binom{r}{r}\binom{r}{0}$$

From Lecture #11 example 8, $\binom{n}{r} = \binom{n}{n-r}$, therefore the above is equivalent to

$$\binom{2r}{r} = \binom{r}{0}\binom{r}{0} + \binom{r}{1}\binom{r}{1} + \cdots + \binom{r}{r}\binom{r}{r} = \binom{r}{0}^2 + \binom{r}{1}^2 + \cdots + \binom{r}{r}^2$$

Q4. Find the term independent of x in the expansion of

$$\left(2x^2 + \frac{1}{x}\right)^9$$

Recall the Binomial Theorem:

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

1. We have $a = 2x^2, b = \frac{1}{x}, n = 9$.
2. The general term in the expansion of $(a+b)^2$ is given by:

$$\binom{n}{r} a^{n-r} b^r = \binom{9}{r} (2x^2)^{9-r} \left(\frac{1}{x}\right)^r = \binom{9}{r} 2^{9-r} x^{18-2r} x^{-r} = \binom{9}{r} 2^{9-r} x^{18-3r}$$

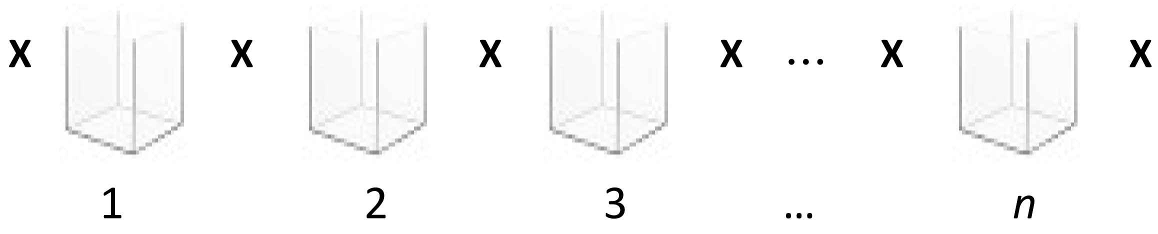
3. For this term to be independent of x, we must have $18 - 3r = 0$, or $r = 6$.
4. Therefore, the term independent of x is $\binom{9}{6} 2^{9-6} = 84 \times 2^3 = \mathbf{672}$.

Q5.

Given n boxes numbered 1 to n, each box to be filled with a white ball or a blue ball. At least one box contains a white ball and boxes containing white balls must be consecutively numbered.

What is the total number of ways this can be done?

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$



The task is similar to choose 2 out of the $n+1$ crosses to mark the start and end of the consecutively numbered boxes that contain the white balls.

$$\binom{n+1}{2} = \frac{(n+1)!}{(n-1)!\,2!} = \boldsymbol{\frac{(n+1)n}{2}}$$

Q6.

Toss 3 coins once each for \$2.
3 heads → win \$10; 2 heads (not in a row) → win \$5; 2 heads (in a row) → win \$1.
If you play this game many times, how much would you win overall per game?

HHH: $0.5^3 = 0.125$

HTH: $0.5^3 = 0.125$

HHT or THH: $2 \times 0.5^3 = 0.25$

TTT, TTH, THT or HTT: $4 \times 0.5^3 = 0.5$

Therefore, expected winning per game:

$$\begin{aligned}
&0.125 \times \$(10 - 2) \\
&\quad + 0.125 \times \$(5 - 2) \\
&\quad + 0.25 \times \$(1 - 2) \\
&\quad + 0.5 \times (\$0 - 2) \\
&= \$1 + \$0.375 - \$0.25 - \$1 = \mathbf{\$0.125}
\end{aligned}$$

Q7.

2 loaded coins with probability of 0.7 of getting tails, and one fair coin.
Is there a particular arrangement of coins (eg: FLL, where F=fair and L=loaded) that he should use to maximize his profits?

3 possible arrangement of coins:

F L L, L F L, L L F

We will build a table of possible probabilities for each winning outcomes.

Expected winning for **F L L arrangement**

$$(1) \times 8 + (2) \times 3 + [(3) + (4)] \times -1$$
$$+ [(5) + (6) + (7) + (8)] \times -2 = -\$0.875$$

	Arrangement: F L L
HHH (1) Profit: 8	0.5 × 0.3 × 0.3 = 0.045
HTH (2) Profit: 3	0.5 × 0.7 × 0.3 = 0.105
HHT (3) Profit: -1	0.5 × 0.3 × 0.7 = 0.105
THH (4) Profit: -1	0.5 × 0.3 × 0.3 = 0.045
TTT (5) Profit: -2	0.5 x 0.7 x 0.7 = 0.245
TTH (6) Profit: -2	0.5 x 0.7 x 0.3 = 0.105
THT (7) Profit: -2	0.5 x 0.3 x 0.7 = 0.105
HTT (8) Profit: -2	0.5 x 0.7 x 0.7 = 0.245

Q7. 2 loaded coins with probability of 0.7 of getting tails, and one fair coin.

Is there a particular arrangement of coins (eg: FLL, where F=fair and L=loaded) that he should use to maximize his profits?

Maximize hustler's profit
= Minimize expected winning

He should use L F L.

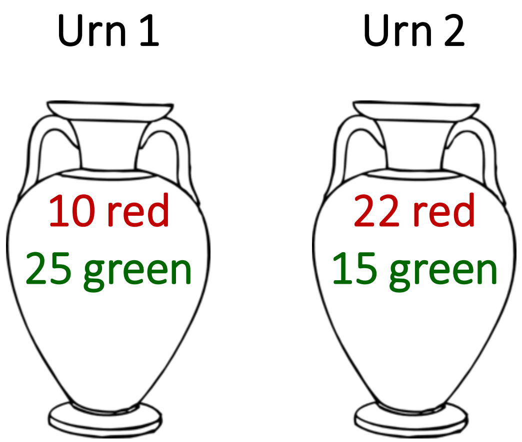
We want to maximize (3) to (8) because it reduces the overall winning, since the bettor actually loses a dollar for (3) and (4) and $2 for (5) to (8).

Doing L F L maximizes these two because the tail in both cases end up in the "loaded" position giving a large probability of 0.7.

	F L L	L F L	L L F
HHH (1) Profit: 8	0.5 × 0.3 × 0.3 = 0.045	0.3 × 0.5 × 0.3 = 0.045	0.3 × 0.3 × 0.5 = 0.045
HTH (2) Profit: 3	0.5 × 0.7 × 0.3 = 0.105	0.3 × 0.5 × 0.3 = 0.045	0.3 × 0.7 × 0.5 = 0.105
HHT (3) Profit: -1	0.5 × 0.3 × 0.7 = 0.105	0.3 × 0.5 × 0.7 = 0.105	0.3 × 0.3 × 0.5 = 0.045
THH (4) Profit: -1	0.5 × 0.3 × 0.3 = 0.045	0.7 × 0.5 × 0.3 = 0.105	0.7 × 0.3 × 0.5 = 0.105
TTT (5) Profit: -2	0.5 x 0.7 x 0.7 = 0.245	0.7 x 0.5 x 0.7 = 0.245	0.7 x 0.7 x 0.5 = 0.245
TTH (6) Profit: -2	0.5 x 0.7 x 0.3 = 0.105	0.7 x 0.5 x 0.3 = 0.105	0.7 x 0.7 x 0.5 = 0.245
THT (7) Profit: -2	0.5 x 0.3 x 0.7 = 0.105	0.7 x 0.5 x 0.7 = 0.245	0.7 x 0.3 x 0.5 = 0.105
HTT (8) Profit: -2	0.5 x 0.7 x 0.7 = 0.245	0.3 x 0.5 x 0.7 = 0.105	0.3 x 0.7 x 0.5 = 0.105
Expected Winning (1) × 8 + (2) × 3 + [(3) + (4)] × -1 + ((5) + (6) + (7) + (8)) x -2	-$0.875	-$1.115	-$0.875

Q8.

First urn chosen by tossing a loaded coin with probability of 0.4 of landing heads (H) up and probability of 0.6 of landing tails (T) up.
H → first urn is chosen; T → second urn is chosen.
Then a ball is picked at random from the chosen urn.

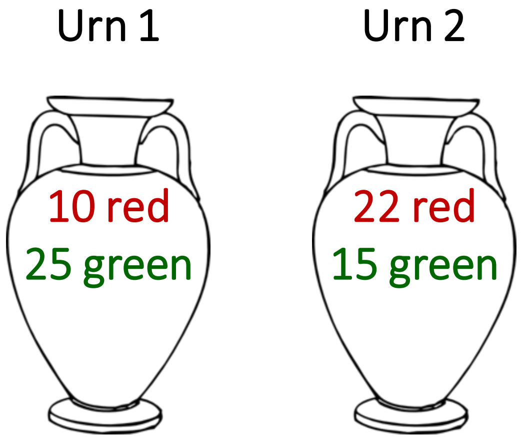


(a) What is the probability that the chosen ball is green?

$$\left(\frac{4}{10}\right)\left(\frac{25}{35}\right)+\left(\frac{6}{10}\right)\left(\frac{15}{37}\right)=\mathbf{52.9\%}$$

Q8.

First urn chosen by tossing a loaded coin with probability of 0.4 of landing heads (H) up and probability of 0.6 of landing tails (T) up.
H → first urn is chosen; T → second urn is chosen.
Then a ball is picked at random from the chosen urn.



(b) If the chosen ball is green, what is the probability that it was picked from the first urn?

Let G be the event that the chosen ball is green, U_1 the event that the ball came from the first urn, and U_2 the event that the ball came from the second urn.

$$P(U_1) = 0.4, \qquad P(U_2) = 0.6, \qquad P(G|U_1) = \frac{25}{35}, \qquad P(G|U_2) = \frac{15}{37}.$$

By Bayes' Theorem,

$$P(U_1|G) = \frac{P(G|U_1)\cdot P(U_1)}{P(G|U_1)\cdot P(U_1) + P(G|U_2)\cdot P(U_2)}$$

$$= \frac{\left(\frac{25}{35}\right)\cdot 0.4}{\left(\frac{25}{35}\right)\cdot 0.4 + \left(\frac{15}{37}\right)\cdot 0.6} = \frac{74}{137} = \mathbf{54.0\%}$$

Q9.

Let $A = \{1,2,3,4\}$. Since each element of $\wp(A \times A)$ is a subset of $A \times A$, it is a binary relation on A. ($\wp(S)$ denotes the power set of S.)

Assuming each relation in $\wp(A \times A)$ is equally likely to be chosen, what is the probability that a randomly chosen relation is (a) reflexive? (b) symmetric?

We will solve the general case. Let $A = \{a_1, a_2, \cdots, a_n\}$ and so $|A| = n$.
A relation R on A can be represented by an $n \times n$ matrix where the entry
$a_{i,j} = 1$ if $a_i \, R \, a_j$, or $a_{i,j} = 0$ if $a_i \not R \, a_j$.

Example:

$$\begin{array}{cc} & \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

This matrix represents this relation R on A:

$R = \{(a_1, a_2), (a_1, a_4), (a_2, a_2), (a_3, a_1), (a_4, a_3), (a_4, a_4)\}$

Q9. (a) Reflexive relations

1. For a set A with n elements, there are 2^{n^2} possible relations on A. (why?)
2. For a relation to be reflexive, $a_i \; R \; a_i \; \forall a_i \in A$. Hence, the main diagonal entries $a_{i,i}$ must be filled with 1, as shown below.

$$\begin{bmatrix} 1 & ? & ? & \cdots & ? \\ ? & 1 & ? & \cdots & ? \\ ? & ? & 1 & \cdots & ? \\ \vdots & \vdots & \vdots & \ddots & ? \\ ? & ? & ? & ? & 1 \end{bmatrix}$$

3. The remaining $n^2 - n$ entries may be filled with 0 or 1 (two choices).
4. Therefore, there are 2^{n^2-n} reflexive relations on A with n elements.
5. Hence, the probability that a randomly chosen relation on set A is reflexive is:

$$\frac{2^{n^2-n}}{2^{n^2}} = \frac{1}{2^n}$$

In particular, when $n = 4$, the probability is $\frac{1}{2^4}$ or $\frac{1}{16}$.

Q9. (b) Symmetric relations

1. For a set A with n elements, there are 2^{n^2} possible relations on A.
2. For a relation to be symmetric, for every entry $a_{i,j}$ (where $i < j$), i.e. in the upper triangular region (red triangle), its corresponding mirror image along the main diagonal, $a_{j,i}$ in the lower triangular region (blue triangle) must follow with the same value.

$$\begin{bmatrix} ? & ? & ? & \cdots & ? \\ ? & ? & ? & \cdots & ? \\ ? & ? & ? & \cdots & ? \\ \vdots & \vdots & \vdots & \ddots & ? \\ ? & ? & ? & ? & ? \end{bmatrix}$$

3. There are $\frac{n(n-1)}{2}$ entries in the upper triangle. There are n entries along the main diagonal. Therefore, there are $\frac{n(n-1)}{2} + n$, or $\frac{n(n+1)}{2}$ entries to be filled with 0 or 1.
4. Therefore, there are $2^{\frac{n(n+1)}{2}}$ symmetric relations on A with n elements.
5. Hence, the probability that a randomly chosen relation on set A is symmetric is:

$$\frac{2^{\frac{n(n+1)}{2}}}{2^{n^2}} = \frac{1}{2^{\frac{n^2-n}{2}}}$$

In particular, when $n = 4$, the probability is $\frac{1}{2^6}$ or $\frac{1}{64}$.

Q10

Let us define a function $W(a,b)$ to be the number of ways the tournament can be completed if team A has to win a more games to win, while team B has to win b more games to win.

$$W(a,b) = \begin{cases} 1, & \text{if } a = 0 \text{ or } b = 0. \\ W(a,b-1) + W(a-1,b), & \text{if } a > 0 \text{ and } b > 0. \end{cases}$$

Verify that $W(a,b)$ may be expressed as follows: $W(a,b) = \binom{a+b}{a}$.

1. $\binom{a+b}{a} = \binom{a+b}{b}$ (by lecture 11 example 8: $\binom{n}{r} = \binom{n}{n-r}$.)
2. $= \binom{a+b-1}{b-1} + \binom{a+b-1}{b}$ (by Pascal's Formula: $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.)
3. $= \binom{a+(b-1)}{b-1} + \binom{(a-1)+b}{b}$
4. $= \binom{a+(b-1)}{a} + \binom{(a-1)+b}{a-1}$ (by lecture 11 example 8: $\binom{n}{r} = \binom{n}{n-r}$.)
5. Case 1: $a = 0$, then $W(a,b) = \binom{a+b}{a} = \binom{b}{0} = 1$.
6. Case 2: $b = 0$, then $W(a,b) = \binom{a+b}{a} = \binom{a}{a} = 1$.
7. Case 3: $a > 0$ and $b > 0$, then $\binom{a+b}{a} = W(a,b)$; $\binom{a+(b-1)}{a} = W(a,b-1)$; $\binom{(a-1)+b}{a-1} = W(a-1,b)$.
8. Hence, $W(a,b) = \binom{a+b}{a}$ for all cases.

Q10

$$W(a,b) = \binom{a+b}{a}.$$

- Now, we denote the function $T(n,k)$ to be the number of ways the tournament can be completed, given that the first team to win n games wins the tournament, and team A wins the first k (where $k \leq n$) games.
Derive a simple combination formula for $T(n,k)$.

$$T(n,k) = W(n-k,n) = \binom{2n-k}{n-k} = \binom{2n-k}{n}.$$ (by lecture 11 example 8: $\binom{n}{r} = \binom{n}{n-r}$.)

Therefore, $T(4,2) = \binom{6}{4} = \mathbf{15}$.

Extra Practice

1. Prove the following identity (Try to use combinatorial argument)

$$n \cdot 2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n}$$

2. Theodore is currently at home and wants to go to school. Theodore's home is located at the point $(0, 0)$. At each step, Theodore can only move 2 units east or 3 units north. Count the number of ways Theodore can reach school under the following conditions:

(a) The school is located at point $(2^{1231} + 1, 2^{1231} + 1)$

(b) The school is located at point $(8, 9)$

(c) Theodore must go through 7-Eleven to buy breakfast. The 7-Eleven is located at $(6, 9)$ and the school is located at $(10, 21)$

Extra Practice

(d) Now there are multiple 7-Eleven stores and Theodore can buy breakfast at any one of them. It is okay for him to buy breakfast at both stores as he is very hungry. 7-Eleven stores are located at $(4, 6)$ and $(10, 15)$. School is located at $(18, 21)$.

(e) Theodore got a headache today and cannot walk properly. Now at each step, he is able to walk 2 or 3 units east or north. For example, from $(0, 0)$ he can go to $(0, 2)$ or $(0, 3)$ or $(2, 0)$ or $(3, 0)$. School is located at $(9, 9)$.

3. 6 boys and 9 girls are to be seated in a row. How many ways are there to arrange the seating such that no two boys sit next to each other?

Extra Practice

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3. 6 boys and 9 girls are to be seated in a row. How many ways are there to arrange the seating such that no two boys sit next to each other?

Extra Practice

4. How many 5-digit positive integers are there such that the digits from left to right are in non-decreasing order?

5. In a tournament with n players, everyone plays with everybody else exactly once. Prove that at any point during the tournament, there must always exist two players who have played the same number of games.

6. Let S be a set containing n positive integers. Prove that there exists a subset of S such that the sum of the elements in the subset is divisible by n.

Extra Practice

7. 33 rooks are placed on an 8×8 chessboard. Prove that no matter what the placement of the rooks are, you can always choose 5 of them such that they do not attack other.

8. Let n be a positive integer. Suppose n has its prime factorization $p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}$, where p_i are primes and $e_i \in Z_{\geq 0}$. How many positive divisors does n have?

9. Let $S = \{1, 2, 3, \dots, n\}$ where n is a positive integer. Find the minimal n such that if any 2016 elements are removed from S, it is still guaranteed to be able to find 2016 distinct numbers among the remaining elements such that their sum is n.

(Hint: consider the worst case of removing the 2016 elements)

THE END